Preorderings in Dimension 2

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Contents

Introduction

Let S be a finite subset of the real polynomial ring $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$, then the basic closed semialgebraic set generated by S is defined as

$$K_S := \{x \in \mathbb{R}^n \mid s(x) \geq 0 \text{ for all } s \in S\}$$

and the preordering generated by S is defined as

$$PO(S) := \{ \sum_{e \in \{0,1\}^r} s_1^{e_1} \ldots s_r^{e_r} \sigma_e \mid \sigma_e \in \sum \mathbb{R}[\underline{x}]^2 \},$$

where $\sum \mathbb{R}[\underline{x}]^2$ is the set of sums of squares of elements of $\mathbb{R}[\underline{x}]$. Obviously, the elements of $PO(S)$ only take nonnegative values on K_S.
We study the question whether or not the converse holds, i.e. whether or not every $f \in \mathbb{R}[\underline{x}]$ with $f \geq 0$ on K_S lies in $PO(S)$. Such a preordering $PO(S)$ is called *saturated*.
This question originally goes back to Hilbert. He studied whether every polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $2d$, which takes only nonnegative values on $\mathbb{R}^n$, is a sum of squares of polynomials. In 1888, Hilbert ([6]) proved that, whenever $n \geq 2$ and $2d \geq 4$, there always exists a polynomial of degree $2d$ which is not a sum of squares of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$. The only exception is the case $(n, 2d) = (2, 4)$. After proving this negative result, he turned to the question whether or not every nonnegative polynomial is a sum of squares of rational functions. This question, which was also included in Hilbert's famous list of unsolved problems ([7]), was later solved by Artin ([1]).
Further, Scheiderer ([19]) showed that, whenever the dimension of the semialgebraic set is greater than 3 or the semialgebraic set contains an open cone in $\mathbb{R}^2$, the preordering can never be saturated. On the other hand, preorderings in $\mathbb{R}[x]$ are completely understood due to Kuhlmann, Marshall and Schwartz ([9]). They proved that a preordering $PO(S) \subseteq \mathbb{R}[x]$ is saturated if and only if the preordering contains the *natural generators* for the semialgebraic set $K_S \subseteq \mathbb{R}$.
So in general, for any preordering to be saturated is a very strong property and in many cases, we cannot hope to get a positive answer to the question above.
Due to this, several weaker properties of preorderings have been studied.

Let $S \subseteq \mathbb{R}[\underline{x}]$ be finite. Among other things, Kuhlmann and Marshall ([10]) introduced that a preordering $PO(S)$ satisfies ($\dagger$) if $PO(S)$ contains all polynomials that are strictly positive on the basic closed semialgebraic set K_S. This originally was a property that was studied with respect to the moment problem (see [23] and the literature mentioned there for an introduction to this topic). In terms of this weaker property, Schmüdgen ([23]) proved that $PO(S)$ satisfies ($\dagger$) whenever the semialgebraic set K_S is compact.

As for preorderings in $\mathbb{R}[x,y]$ where the semialgebraic set is not compact and does not contain an open cone, very few results are known. One of the first results in this setting was Marshall's solution of the strip conjecture ([14]). He showed that every polynomial which is nonnegative on the set $[0,1] \times \mathbb{R}$ is contained in the preordering $PO(S)$ generated by $S = \{x, 1-x\}$. Although there are other examples of saturated preorderings in $\mathbb{R}[x,y]$ where the semialgebraic set is not compact (see [21]), this was the first example where the ring $B(K_S)$ of bounded polynomials on K_S is of transcendence degree 1 over $\mathbb{R}$. In all other examples, the transcendence degree of $B(K_S)$ over $\mathbb{R}$ was always 2 and before Marshall's result, it was not known whether or not such an example even exists.

Marshall's result was then generalized by Nguyen and Powers ([16]) to arbitrary strips of the form $K \times \mathbb{R}$ (where $K \subseteq \mathbb{R}$ is a compact set) as well as half-strips of the form $K \times \{y \geq 0\}$.

Marshall's result is also the starting point for this thesis. Here, we will generalize Marshall's results in two different ways:

In the first part, which consists of chapter 2 and chapter 3, we will study two kinds of families of preorderings generated by a finite $S \subseteq \mathbb{R}[x,y]$.

Let $K \subseteq \mathbb{R}$ be compact. The first family consists of preorderings $PO(S) \subseteq \mathbb{R}[x,y]$ where the basic closed semialgebraic set is contained in a strip $K \times \mathbb{R}$ and satisfies an unboundedness condition. That is, the semialgebraic set satisfies

$$K \times \{y \in \mathbb{R} \mid y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

We refer to such a preordering as a preordering in the strip case.

The second family consists of preorderings where the basic closed semialgebraic set is contained in a half-strip $K \times \mathbb{R}_+$ and satisfies an unboundedness condition. That is, the semialgebraic set satisfies

$$K \times \{y \in \mathbb{R} \mid y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+.$$

We refer to such a preordering as a preordering in the half-strip case.
If any preordering $PO(S)$ in the strip- or in the half-strip case is saturated, then it is necessary that $PO(S \cap \mathbb{R}[x])$ is also saturated (see proposition 3.2), so we will always assume our generating set $S \subseteq \mathbb{R}[x, y]$ contains the natural generators for K ,i.e. we will always assume S is of the form

$$S = \{s_1, \ldots, s_r, g_1, \ldots, g_s\}$$

where $s_1, \ldots s_r \in \mathbb{R}[x]$ are the natural generators for the compact set K and $g_1, \ldots g_s \in \mathbb{R}[x, y]$ are some additional generators. Whether such a preordering is saturated or satisfies (†) mainly depends on the additional generators $g_1, \ldots, g_r \in \mathbb{R}[x, y]$.
To discuss these preorderings, we mainly use three methods:
The first method is described in theorem 2.10 and theorem 2.13 in section 2.1. The advantage of these theorems is that they allow us to work in a polynomial ring over a power series ring $\mathbb{R}\{x\}[y]$ instead of the polynomial ring in two variables $\mathbb{R}[x, y]$ in our setting. The first ring is a polynomial ring over a henselian discrete valuation ring where factorization into irreducible factors becomes much easier (see section 2.4.2). This turns out to be a big advantage when dealing with preorderings in the strip or the half-strip case. Unfortunately, both theorems are also very technical, so we need most of chapter 2 to provide the tools necessary to apply these two theorems to examples. This new method is mainly based on the technique used by Marshall in [14].
The second method we should mention here is an observation which enables us to study preorderings generated by a polynomial $g \in \mathbb{R}[x, y]$ with $\deg_y(g) = 2$ in the strip case (see the proof of proposition 3.27). Before, we were only able to discuss preorderings generated by a polynomial p with $\deg_y(p) \leq 1$. We use this new technique to study preorderings generated by such a polynomial g in the ring $\mathbb{R}\{x\}[y]$ (proposition 2.80) and to solve the "punctured strip problem" stated by Powers in [17] (proposition 3.27).
The third method we use is derived from a result proven by Kuhlmann, Marshall and Schwartz in [9]. From their result follows that a preordering $PO(S)$ in the strip or the half-strip case can only satisfy (†) if the generators S satisfy a fiber condition. That is, if the generators restricted to all vertical lines L contain the natural generators for $L \cap K_S$ (see section 3.1 for more details). From this, we conclude that for any such preordering, under an extra mild assumption, the semialgebraic set K_S is already described by

those $s \in S$ with $\deg_y(s) \leq 2$ (proposition 3.19 and proposition 3.20). This immediately gives a family of examples of preorderings which can never be saturated (proposition 3.40).
We use these new methods to study preorderings $PO(S) \subseteq \mathbb{R}[x, y]$ in the strip and the half-strip case in the following ways:
First, we will discuss preorderings $PO(S)$ in the strip and half-strip case where S has only one additional generator $g \in \mathbb{R}[x, y]$ in section 3.2. Under the assumption that the leading coefficient of g as a polynomial in y is strictly positive on K, we can almost completely solve this problem. That is, for every such g we can either describe the saturation of K_S if it is finitely generated, or we can prove that it is not finitely generated. There are just two sub-cases where we were only able to show (†) instead of saturation (corollary 3.30 and proposition 3.38). Whether or not the saturation is finitely generated in these sub cases is still open.
In section 3.3, we study preorderings in the strip or the half-strip case with more than one additional generator. Here we use a combination of all three methods to prove that preorderings in the strip or the half strip-case where the generators satisfy the fiber condition mentioned above as well as a small extra condition, already satisfy (†) (theorem 3.54 and theorem 3.61). This generalizes a result from Kuhlmann, Marshall and Schwartz ([9, theorem 5.3]) in the strip case and the half-strip case.
We will further show that under certain extra assumptions such preorderings are already saturated (theorem 3.74 and theorem 3.80). Although these assumptions are probably too strong, these theorems lead to a whole new family of examples of saturated preorderings.
We finish this part of the work with some more examples and open problems (section 3.5). Here, we mainly focus on examples where our new methods are not enough to show that the corresponding preordering is saturated. Further studying these examples will hopefully lead to a relaxation of the extra assumptions in theorem 3.74 and theorem 3.80.

The second part of this work consists of chapter 4 where we generalize Marshall's result in a slightly different direction.
Let C be a real smooth curve with $C(\mathbb{R})$ compact and let $V = C \times \mathbb{A}^1$. The basic idea is to replace the compact set K in Marshall's proof by the set $C(\mathbb{R})$. Thus, we study the question whether or not every regular function $f \in \mathbb{R}[V]$ on V which only takes nonnegative values on $V(\mathbb{R})$ is a sum of

squares of elements in $\mathbb{R}[V]$, in short, does psd $=$sos hold in $\mathbb{R}[V]$?
Here, we generalize the method used by Marshall in [14] to this setting to show that every psd regular function with finitely many zeros in $V(\mathbb{R})$ is indeed a sum of squares of elements of $\mathbb{R}[V]$ (theorem 4.16). Unfortunately, this is not enough to show that psd $=$ sos holds in $\mathbb{R}[V]$, since psd regular functions may have infinitely many real zeros. However, in the special case where $C = V(x^2 + y^2 - 1)$ is the circle and $V = C \times \mathbb{A}^1$, we were able to avoid this problem, which gives us a first example of a real variety V, where $\dim(V) = 2$, psd $=$ sos holds in $\mathbb{R}[V]$ and $V(\mathbb{R})$ is not compact (proposition 4.39).
The methods used for this are again based on methods used by Marshall in [14]. Unfortunately, they do not seem to generalize to arbitrary smooth curves, thus so far, V above is the only case where this works. Whether or not psd $=$ sos holds in $\mathbb{R}[V]$ for other (smooth) curves C (and $V = C \times \mathbb{A}^1$) is still open.

1 Notations and Known Results

1.1 Notations

In this section, we will introduce the necessary notations of geometric objects which are used during this work. Furthermore, we will present some well-known results in these areas of interest (See [2] for a more detailed introduction to the topics presented here).

Definition 1.1
Let A be a commutative ring with 1. A subset $P \subseteq A$ called a preordering of A, if

(i) $P + P \subseteq P$

(ii) $P \cdot P \subseteq P$

(iii) $a^2 \in P$ for all $a \in A$

Clearly for any ring A,

$$\sum A^2 := \{\sum_{i=1}^{s} a_i^2 \mid a_i \in A\},$$

the set of sums of squares of elements of A is the smallest preordering in A. Let $S \subseteq A$. We define $PO(S)$ to be the closure of S and $\sum A^2$ under addition and multiplication, the smallest preordering that contains S, especially if $S = \{s_1, \ldots, s_r\} \subseteq A$ is finite, then

$$PO(S) := \{\sum_{e \in \{0,1\}^r} s^e \sigma_e \mid \sigma_e \in \sum A^2\},$$

where $s^e := s_1^{e_1} \cdots s_n^{e_n}$.
For $S \subseteq \mathbb{R}[x_1, \ldots, x_n] \subseteq \mathbb{R}[\underline{x}]$, we define the corresponding semi-algebraic set to be

$$K_S := \{x \in \mathbb{R}^n \mid s(x) \geq 0 \text{ for all } s \in S\}.$$

If $S = \mathbb{R}[\underline{x}]$ is finite, then the set is called a basic closed semialgebraic set.

Given $S \subseteq \mathbb{R}[\underline{x}]$. From our definition it is clear that each $f \in PO(S)$ only takes nonnegative values on K_S. One of the main questions of this work is whether or not the converse holds, i.e. given $f \in \mathbb{R}[x]$ with $f \geq 0$ on K_S: Is then $f \in PO(S)$?
Since this question is in general too hard to answer, we will define certain other (weaker) properties of the preordering $PO(S)$, which were first introduced by Kuhlmann, Marshall and Schwartz in [9]:

Definition 1.2
Let $P \subseteq \mathbb{R}[\underline{x}]$ be a preordering and $K = K_P$ the corresponding semialgebraic set, then

- P is *saturated* iff $f \in P$ for every $f \in \mathbb{R}[\underline{x}]$ with $f \geq 0$ on K.
- P satisfies $(\dagger)$ iff $f + \delta \in P$ for every $\delta > 0$ and $f \in \mathbb{R}[\underline{x}]$ with $f \geq 0$ on K.
- P satisfies $(\ddagger)$ iff for every $f \in \mathbb{R}[\underline{x}]$ with $f \geq 0$ on K there exists an $q \in \mathbb{R}[\underline{x}]$ such that $f + \delta q \in P$ for every $\delta > 0$.
- P solves the moment problem (short (SMP)) iff the closure of P in the finest locally convex topology on $\mathbb{R}[\underline{x}]$ is saturated.

Remark 1.3
Let $S \subseteq \mathbb{R}[\underline{x}]$ be finite and $P := PO(S)$ the preordering generated by S.

(i) Obviously, we have

$$P \text{ saturated } \Rightarrow P \text{ satisfies } (\dagger) \Rightarrow \text{P satisfies } (\ddagger) \Rightarrow P \text{ satisfies } (SMP)$$

and in the papers [10], [9] and [15], it is shown that all these implications are strict.

(ii) In this work, we are mainly interested in the properties saturated and $(\dagger)$. The other two properties are mainly needed for reference reasons.

(iii) Given $f \in \mathbb{R}[\underline{x}]$ with $f \geq 0$ on K_S, then, for every $\delta > 0$, $f + \delta > 0$ on K_S. So if P satisfies $(\dagger)$, then P contains all $f \in \mathbb{R}[\underline{x}]$ that are strictly positive on K_S.

Note that for any $S \subseteq \mathbb{R}[\underline{x}]$, the set K_S is uniquely determined by $PO(S)$. The converse is not true in general. So whether or not a preordering P is saturated (or satisfies (†), (‡) or (SMP)) will highly depend on generating set $S \subseteq \mathbb{R}[\underline{x}]$.

Given $S \subseteq \mathbb{R}[\underline{x}]$. We are mainly interested in describing nonnegative (positive) polynomials on the basic closed semialgebraic set K_S, so another question we are going to ask is:

Does a subset $S' \subseteq \mathbb{R}[\underline{x}]$ exist such that $K_S = K_{S'}$ and $PO(S')$ is saturated (satisfies (†))?

1.2 Known Results

A lot of work has been done on this topic. A very good introduction to the topic can be found in [22], [17] and in the book [13]. We will now recall some of the most famous results in this area of interest:

Theorem 1.4 (Schmüdgen)
Let $S \subseteq \mathbb{R}[\underline{x}]$ be finite such that K_S is compact. Then $PO(S)$ has property $(\dagger)$.

Proof — [23, theorem 1] $\square$

Theorem 1.5 (Scheiderer)

(i) Let $S \subseteq \mathbb{R}[\underline{x}]$ be finite. If $\dim(K_S) \geq 3$ then $PO(S)$ is not saturated.

(ii) Let $S \subseteq \mathbb{R}[x, y]$ be finite. If K_S contains an open cone, then $PO(S)$ is not saturated.

Proof — [19, proposition 6.1 + remark 6.7] $\square$

The most remarkable fact about these three results is that they are independent of the choice of generators. For most of the other semialgebraic sets K_S, this will not be true.
The one dimensional case was completely solved by Kuhlmann, Marshall and Schwartz in their papers [10] and [9]:

Definition 1.6
Let $K \subseteq \mathbb{R}$ be a non-empty, basic closed semi-algebraic set. Define a set $N \subseteq \mathbb{R}[x]$ as follows:

- If $a \in K$ and $(-\infty, a) \cap K = \varnothing$, then $x - a \in N$.
- If $a \in K$ and $(a, \infty) \cap K = \varnothing$, then $a - x \in N$.
- If $a, b \in K$, $a < b$ and $(a, b) \cap K = \varnothing$, then $(x - a)(x - b) \in N$.
- N has no other elements except these.

We call N the *natural set of generators* for K.

Theorem 1.7
Let $S \subseteq \mathbb{R}[x]$.

a) If K_S is not compact, then the following are equivalent:

(i) $PO(S)$ is saturated.

(ii) $PO(S)$ contains the natural set of generators for K_S.

(iii) S contains the natural set of generators for K_S (up to scaling by positive reals).

b) If K_S is compact, then $PO(S)$ satisfies (†) and the following are equivalent:

(i) $PO(S)$ is saturated.

(ii) $PO(S)$ contains the natural set of generators for K_S.

Proof — [9, theorem 3.1] □

Remark 1.8
Note that in the compact case $PO(S)$ can be saturated, even if S does not contain a scaling of the natural generators for K_S. In this case, Kuhlmann, Marshall and Schwartz found a more practical criterion:

Theorem 1.9
Let $S = \{s_1, \ldots, s_r\} \subseteq \mathbb{R}[x]$ such that

$$K_S = \bigcup_{i=1}^{n} [a_i b_i]$$

with $a_i \leq b_i < a_{i+1}$ $(i = 1, \ldots, n-1)$.
Then $PO(S)$ is saturated if and only if the following two conditions hold:

(i) For each endpoint a_j $(j = 1, \ldots, n)$, there exists an $i \in \{1, \ldots s\}$ such that $s_i(a_j) = 0$ and $s_i'(a_j) \neq 0$.

(ii) For each endpoint b_j $(j = 1, \ldots, n)$, there exists an $i \in \{1, \ldots s\}$ such that $s_i(b_j) = 0$ and $s_i'(b_j) \neq 0$.

Proof — [9, theorem 3.2] □

Corollary 1.10
Let $S = \{s_1, \dots, s_r\} \subseteq \mathbb{R}[x]$ such that K_S is compact. Assume K_S has no isolated points. Then $PO(S)$ is saturated if and only if, for each endpoint $a \in K_S$, there exists $i \in \{1, \dots s\}$ such that $x - a$ divides s_i but $(x-a)^2$ does not.

Proof — [9, corollary 3.3] □

This leaves us with the case of preorderings in $\mathbb{R}[x, y]$, where the basic closed semialgebraic set is not compact and does not contain an open cone. Here the first example was given by Scheiderer in [21]. He proved that the preordering generated by $S = \{x, 1-x, 1-xy, y\}$ is saturated (compare [21, remark 3.16]).
The second example in this setting was given by Marshall. In his paper [14], he settled the following long outstanding open problem:

Theorem 1.11 (Marshall)
Let $S := \{x, 1-x\}$. Then $PO(S) \subseteq \mathbb{R}[x, y]$ is saturated.

This result was extended by Nguyen and Powers in the following ways:

Theorem 1.12 (Nguyen, Powers)
Suppose $K \subseteq \mathbb{R}$ is compact and $S \subseteq \mathbb{R}[x]$ is the natural choice of generators for K. Then $PO(S)$ is saturated.

Proof — [16, theorem 3]. □

Proposition 1.13
Given a compact $K \subseteq \mathbb{R}$ with the natural choice of generators $\{s_1, \dots, s_r\}$ and $q \in \mathbb{R}[x]$, then the preordering generated by $S := \{s_1, \dots, s_r, y - q(x)\}$ is saturated.

Proof — [16, theorem 4] □

To our knowledge, these are the only examples of saturated preorderings, generated by a finite $S \subseteq \mathbb{R}[x,y]$ such that the corresponding semialgebraic set $K_S \subseteq \mathbb{R}^2$ is not compact. In this work, we will use similar methods as in theorem 1.11 and theorem 1.12 to add more examples to this list (see section 3.2).

1.3 Sums of Squares on Real Algebraic Varieties

We now look at a more general setup. Let $V \subseteq \mathbb{A}^n$ be an affine variety defined over $\mathbb{R}$, i.e. V is the zero set of some real polynomials $f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n] = \mathbb{R}[\underline{x}]$ with coordinates in $\mathbb{C}^n$. We denote the coordinate ring of V by

$$\mathbb{R}[V] := \mathbb{R}[\underline{x}]/(f_1, \ldots, f_r)$$

and the set of real points by $V(\mathbb{R}) := V \cap \mathbb{R}^n$. Further, given any $f \in \mathbb{R}[V]$, we denote by $V_{\mathbb{R}}(f)$ the set of real zeros of f, i.e.

$$V_{\mathbb{R}}(f) = \{P \in V(\mathbb{R}) \mid f(P) = 0\}.$$

Then a $f \in \mathbb{R}[V]$ is *psd* if $f(x) \geq 0$ for all $x \in V(\mathbb{R})$ and f is *sos* if f is a sum of squares of elements in $\mathbb{R}[V]$.
In this more general setup, the question whether or not *psd* = *sos* holds was investigated by Scheiderer in several papers.

Proposition 1.14 (Scheiderer)
Suppose $\dim(V) \geq 3$. *Then there exists a psd polynomial* $f \in \mathbb{R}[V]$ *that is not contained in* $\sum \mathbb{R}[V]^2$.

Proof — [19, proposition 6.1] □

On the other hand, the case of real algebraic curves is completely understood. Among other things, Scheiderer proved the following :

Theorem 1.15 (Scheiderer)
Let C *be an irreducible affine curve over* $\mathbb{R}$. *In each of the following cases, we have psd=sos in* $\mathbb{R}[C]$:

(i) $C(\mathbb{R}) = 0$,

(ii) C *is an open subcurve of the affine line* $\mathbb{A}^1$,

(iii) C *is reduced, has a non-real point at infinity and all real singular points are ordinary multiple points with independent tangents.*

In all other cases, we have psd $\neq$ sos in $\mathbb{R}[V]$.

Proof — [20, theorem 4.17] □

Especially the following special case of theorem 1.15 will be interesting for us:

Corollary 1.16
Let C be a smooth irreducible affine curve over $\mathbb{R}$ for which $C(\mathbb{R})$ is compact. Then psd = sos holds in $\mathbb{R}[C]$.

Proof —

In dimension 2, the case where $V(\mathbb{R})$ is compact was mainly investigated:

Theorem 1.17 (Scheiderer)
Let V be a nonsingular affine surface over $\mathbb{R}$ for which $V(\mathbb{R})$ is compact. Then psd = sos holds on V.

Proof — [21, corollary 3.4] □

The case where V is a affine surface over $\mathbb{R}$ with $V(\mathbb{R})$ not compact is not yet well understood.
Let C be a smooth curve over $\mathbb{R}$ and $V := C \times \mathbb{A}^1$. We ask whether or not psd = sos holds in $\mathbb{R}[V]$.
In chapter 4 of this work, we use the method used by Marshall in [14] to show that every $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ and $V_{\mathbb{R}}(f)$ finite is a sum of squares in $\mathbb{R}[V]$.
In the special case where C is a circle, i.e.

$$V = V(x_1^2 + x_2^2 - 1) \subseteq \mathbb{A}^3,$$

we are even able to show that psd = sos holds. To our knowledge, this is the first example of an affine real variety V with $\dim(V) = 2$ where psd=sos holds in $\mathbb{R}[V]$ and $V(\mathbb{R})$ is not compact.

2 Technical Background

Throughout this chapter, $K \subseteq \mathbb{R}$ will always be a compact set without isolated points, i.e. for every $P \in K$, either $[P, P+\delta] \subseteq K$ or $[P-\delta, P] \subseteq K$ for $\delta > 0$ small enough.
In this work, we deal with 2 general setups:
We will investigate preorderings, generated by a finite $S \subseteq \mathbb{R}[x,y]$ such that there exists a $N \in \mathbb{N}$ with

- $K \times \{y \in \mathbb{R} \mid y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$ or
- $K \times \{y \in \mathbb{R} \mid y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+$

(note that $0 \in \mathbb{R}_+$). We will refer to the first case as the strip case and to the second case as the half-strip case. For simplicity, we will write

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

in the strip case and

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+$$

in the half-strip case. Whenever possible, we will handle both cases at once. In that case, we simply assume

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

If not stated otherwise, we will equip every subset of $\mathbb{R}^n$ with the euclidean topology.
Given any polynomial $f \in \mathbb{R}[x,y]$. The variable y, which corresponds to the unbounded direction in the strip and the half-strip case, will play a major role, so we will denote by $\deg_y(f)$ the degree of f as a polynomial in y over $\mathbb{R}[x]$.

In the first section of this chapter, we will prove a very helpful technical theorem in the strip and in the half-strip case. These technical theorems allow us to work in the ring of convergent power series instead of the polynomial ring, which will prove very helpful. Unfortunately, in order to apply these technical theorems, some preliminary work has to be done. This will be done in sections 2.2 and 2.3. In section 2.4, we will first review some well known facts of the ring of convergent power series and study preorderings in this ring. All this is needed to apply this new theory to certain examples in chapter 3.

2.1 Technical Theorem

In the first part, we prove two useful technical theorems, theorem 2.10 in the strip case and theorem 2.13 in the half-strip case. Both of them are based on the work [14] of Marshall and its generalisation [16] of Nguyen and Powers. First, we need several lemmas:

Proposition 2.1
Suppose $\phi, \psi : K \longrightarrow \mathbb{R}$ are continuous functions, $\phi(x) \leq \psi(x)$ for all $x \in K$, and $\phi(x) < \psi(x)$ for all but finitely many $x \in K$.
If ϕ and ψ are analytic at each point $a \in K$, where $\phi(a) = \psi(a)$, then there exists a polynomial $p \in \mathbb{R}[x]$ such that $\phi(x) \leq p(x) \leq \psi(x)$ holds for all $x \in K$.

Proof — [16, proposition 2] □

Lemma 2.2
Suppose $\varepsilon \in \mathbb{R}[x]$ and $I \subseteq \mathbb{N}^2$ is finite. Let $P_1, \ldots, P_r$ be the zeros of ε in K, and A be the ring of continuous functions from K to $\mathbb{R}$ which are analytic in a neighborhood of $P_1, \ldots, P_r$. Suppose $f \in \mathbb{R}[x, y]$ with

$$f = \sum_{(i,j)\in I} p_{i,j}^2 h_i,$$

where $p_{i,j} \in A[y]$, $h_i \in \mathbb{R}[x, y]$ and

$$d := \deg_y(f) \geq \deg_y(p_{i,j}^2 h_i)$$

for all $i, j \in I$,
then there exist $g_{i,j} \in \mathbb{R}[x, y]$, $a_i \in \mathbb{R}[x]$ with $|a_i| \leq \varepsilon^2$ on K ($(i, j) \in I$) such that

$$f = (\sum_{(i,j)\in I} g_{i,j}^2 h_i) + \sum_{i=0}^{2d} a_i y^i.$$

Proof — Write

$$p_{i,j} = \sum_{k=0}^{d_{i,j}} p_{i,j,k} y^k,$$

with $p_{i,j,k} \in A$ and $d_{i,j} = \deg_y(p_{i,j})$.
If $p_{i,j,k} = 0$, define $b_{i,j,k}^{(N)} := 0$ for all $N \in \mathbb{N}$. If not, then by proposition 2.1

for every $N \in \mathbb{N}$ and every (i,j,k), there exists a $b^{(N)}_{i,j,k} \in \mathbb{R}[x]$ such that

$$p_{i,j,k} - \frac{\varepsilon^2}{N} \leq b^{(N)}_{i,j,k} \leq p_{i,j,k} + \frac{\varepsilon^2}{N}$$

holds on K. Define

$$g^{(N)}_{i,j} := \sum_{k=0}^{d_{i,j}} b^{(N)}_{i,j,k} y^k$$

and

$$\sum_{i=0}^{d} a^{(N)}_i y^i := f - \sum_{(i,j)\in I} (g^{(N)}_{i,j})^2 h_i. \tag{2.1}$$

Note that since

$$d = \deg_y(f) \geq \deg_y(p^2_{i,j} h_i) = \deg_y((g^{(N)}_{i,j})^2 h_i) \text{ for all } (i,j) \in I,$$

this is well defined. Computing the right hand side in 2.1 and comparing coefficients, we see that there exists a $r > 0$ and polynomials $f_i \in \mathbb{R}[x, t_1, \ldots, t_r]$ such that

$$a^{(N)}_i(x) = f_i(x, b^{(N)}_{j,k,l}(x)). \tag{2.2}$$

Here (j,k,l) runs through all possible indices.
Now, write $b^{(N)}_{j,k,l} = p_{j,k,l} + c^{(N)}_{j,k,l}$ and substitute this in 2.2 to get

$$\begin{aligned} a^{(N)}_i(x) &= f_i(x, p_{j,k,l}(x) + c^{(N)}_{j,k,l}(x)) \\ &= f_i(x, p_{j,k,l}(x)) + q_i(c^{(N)}_{j,k,l}) \end{aligned} \tag{2.3}$$

where the q_i are polynomials in r variables, whose coefficients are continuous functions on K (that depend on the given $p_{j,k,l}$).
If $b^{(N)}_{j,k,l} = p_{j,k,l}$ for all (j,k,l), then the approximation above is exact, so $a_i = 0$ for all $i \in \{0, \ldots, d\}$ (see equation 2.1), thus, we conclude from equation 2.2 that

$$0 = a_i(x) = f_i(x, p_{j,k,l}(x))$$

and from equation 2.3

$$0 = a_i(x) = f_i(x, p_{j,k,l}(x)) + q_i(0, \ldots, 0) = q_i(0, \ldots, 0).$$

Let $\tilde{q}_i \in \mathbb{R}[t_1, \ldots, t_r]$ be the polynomial obtained by replacing each coefficient by the maximum of its absolute value on K. Since $q_i(0, \ldots, 0) = 0$, we still have $\tilde{q}_i(0, \ldots, 0) = 0$. Next, we define $s_i \in \mathbb{R}[t]$ by $s_i(t) = \frac{\tilde{q}_i(t,\ldots,t)}{t}$

(This is well defined since $\tilde{q}_i(0,\ldots,0)=0$) and M_i to be the maximum of s_i on $[0,\max_K(\varepsilon^2)]$. Since $|c_{i,j,k}^{(N)}| \le \frac{\varepsilon^2}{N}$ (for all (i,j,k) and $N \in \mathbb{N}$), we now compute

$$\begin{aligned} |a_i^{(N)}(x)| &= |q_i(c_{i,j,k}^{(N)})| \\ &\le \tilde{q}(|c_{i,j,k}^{(N)}|) \\ &\le \tilde{q}(\frac{\varepsilon^2}{N},\ldots,\frac{\varepsilon^2}{N}) \\ &= \frac{\varepsilon^2}{N} s_i(\frac{\varepsilon^2}{N}) \\ &\le \varepsilon^2 \frac{M_i}{N}. \end{aligned}$$

So for N big enough, we have $\frac{M_i}{N} \le 1$ and the lemma is proven. □

Remark 2.3
In the last lemma, the term $\sum a_i y^i$ can be seen as an "error" term for the approximation of the expression

$$f = \sum_{(i,j)\in I} p_{i,j}^2 h_i$$

by polynomials. In this sense, the last lemma not only allows us to approximate this expression by polynomials, but also enables us to control the error that appears due to this approximation. This fact will play a major role in the proof of theorem 2.10.

Definition 2.4
A formal power series

$$f = \sum_{i=0}^{\infty} a_i x^i \in \mathbb{R}[[x]]$$

converges if there exists a $\delta > 0$ such that the expression $\sum_{i=0}^{\infty} |a_i x^i|$ converges for all $x \in \mathbb{R}$ with $|x| < \delta$. We will denote by $\mathbb{R}\{x\}$ the ring of convergent power series over $\mathbb{R}$.

Lemma 2.5
Let $f \in \mathbb{R}[x,y]$, $S \subseteq \mathbb{R}[x,y]$, $P_1,\ldots,P_r \in K$ and A be the ring of continuous functions $\varphi : K \longrightarrow \mathbb{R}$ that are analytic in a neighborhood of $P_1,\ldots,P_r$.

Suppose for every $P \in K$ *there exist* $h_{i,P} \in \mathbb{R}\{x-P\}[y]$, $s_{i,P} \in S$ *and* $n_P \in \mathbb{N}$ *such that*

(i) $f = \sum_{i=1}^{n_P} h_{i,P}^2 s_{i,P}$,

(ii) $\deg_y(h_{i,P}^2 s_{i,P}) \leq \deg_y(f)$.

Then there exist an $n \in N$, $\phi_i \in A[y]$, $s_i \in S$ $(i = 1, \ldots, n)$ *such that*

$$f = \sum_{i=1}^{n} \phi_i^2 s_i.$$

and $\deg_y(\phi_i^2 s_i) \leq \deg_y(f)$.

Proof — For each $P \in K$, there exists an open neighborhood U_P of P in K such that each coefficient in (i) converges in U_P. Since K is compact, there exist $P_{r+1}, \ldots, P_s \in K$ such that $K = \bigcup_{i=1}^{s} U_{P_i}$. By shrinking U_{P_i}, if necessary, we may assume $P_j \notin \overline{U_{P_i}}$ for $i \neq j$.
Let

$$1 = \mu_1 + \cdots + \mu_s$$

be a continuous partition of unity such that $supp(\mu_i) \subseteq U_{P_i}$. Since $P_j \notin \bar{U}_{P_i}$ $(i \neq j)$, for every $i = 1, \ldots, s$ there exists an open neighborhood U_i of P_i such that $\mu_i|_{U_i} \equiv 1$ and $\mu_i|_{U_j} \equiv 0$ $(j \neq i)$. Now we compute

$$\mu_j f = \sum_{i=1}^{n_{P_j}} ((\sqrt{\mu_j} h_{i,P_j})^2) s_{i,P_j},$$

where the coefficients of $\phi_{i,j} := (\sqrt{\mu_j} h_{i,P_j})$ are expanded by 0 outside of U_{P_j}. Then $\phi_{i,j}$ is continuous. $\phi_{i,j}$ is also analytic in a neighborhood of $P_1, \ldots, P_s$, since $\phi_{i,j} = h_{i,P_j}$ in U_j (since $\mu_j \equiv 1$) and $\phi_{i,j} \equiv 0$ in U_k $(k \neq j)$. So $\phi_{i,j} \in A[y]$,

$$\deg_y(\phi_{i,j}^2 s_{i,P_j}) = \deg_y(h_{i,P_j}^2 s_{i,P_j}) \leq \deg_y(f)$$

by construction and we compute

$$f = \sum_{j=1}^{s} \mu_j f = \sum_{j=1}^{s} \sum_{i=1}^{n_{P_j}} \phi_{i,j}^2 s_{i,P_j}.$$

$\square$

Definition 2.6
Suppose $S \subseteq \mathbb{R}[x,y]$. For $P \in \mathbb{R}$, define $PO_P(S)$ to be the preordering in $\mathbb{R}\{x-P\}[y]$ generated by S.

Lemma 2.7
Let $N \in \mathbb{N}$ and $S \subseteq \mathbb{R}[x,y]$ be finite such that

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Let

$$s = \sum_{i=0}^{d} a_i y^i \in S$$

with $a_i \in \mathbb{R}[x]$ and $a_d \neq 0$. Then $a_d \geq 0$ on K.

Proof — Suppose there exists $P \in K$ such that $a_d(P) < 0$.
Then $s(P,y) \in \mathbb{R}[y]$ has a negative leading coefficient, so there exists a $M' \in \mathbb{R}$ such that

$$s(P,M) < 0 \text{ for all } M > M'.$$

Since $s \in S$, we have $s \geq 0$ on K_S by definition, so $(P,M) \notin K_S$ for all $M > M'$. This is a contradiction to $K \times \{y \geq N\} \subseteq K_S$. □

Lemma 2.8
Let $N \in \mathbb{N}$ and $S = \{s_1, \ldots, s_r\} \subseteq \mathbb{R}[x,y]$ be finite such that

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Suppose $P \in K$, $f \in \mathbb{R}[x,y]$ such that there exist $n \in \mathbb{N}$ and $\sigma_e \in \sum \mathbb{R}\{x-P\}[y]^2$ ($e \in \{0,1\}^r$) with

$$f = \sum_{e \in \{0,1\}^n} \sigma_e s_1^{e_1} \ldots s_r^{e_r}.$$

Then

$$\deg_y(f) \geq \deg_y(\sigma_e s_1^{e_1} \ldots s_r^{e_r})$$

for all $e \in \{0,1\}^r$.

Proof — We may assume $P = 0$. Choose $\delta > 0$ such that each coefficient in the expression above converges in $(-\delta, \delta)$. We have to show that the leading coefficient as a polynomial in y doesn't cancel out:

Since $K\times\{y>N\}\subseteq K_S$, the leading coefficient of each $s\in S$ satisfies $s\geq 0$ on K by lemma 2.7. The leading coefficient of each σ_e $(e\in\{0,1\}^r)$ is a sum of squares in $\mathbb{R}\{x\}$, thus the leading coefficient of each σ_e $(e\in\{0,1\}^r)$ is psd on $(-\delta,\delta)$ for some $\delta>0$ small enough. Since K has no isolated points, we may assume $[0,\delta]\subseteq K$. Write $a_e\in\mathbb{R}\{x\}$ for the leading coefficient of $\sigma_e s_1^{e_1}\dots s_r^{e_r}$ as a polynomial in y.
Then for each $e\in\{0,1\}^r$, there exists a $P_e\in[0,\varepsilon]$ such that $a_e(P_e)\neq 0$. Now assume

$$\deg_y(f)<\deg_y(\sigma_e s_1^{e_1}\dots s_r^{e_r})$$

for an $e\in\{0,1\}^r$. Since

$$f=\sum_{e\in\{0,1\}^n}\sigma_e s_1^{e_1}\dots s_r^{e_r},$$

there exists a subset $I\subseteq\{0,1\}^r$ such that

$$\sum_{e\in I}a_e=0.$$

Fix $e_0\in I$. Then we compute

$$\begin{aligned}0&=\sum_{e\in I}a_e(P_{e_0})\\&=a_{e_0}(P_{e_0})+\sum_{e\in I\setminus\{e_0\}}a_e(P_{e_0})\\&\geq a_{e_0}(P_{e_0})\\&>0,\end{aligned}$$

a contradiction. □

Remark 2.9
This last lemma would not hold if K has an isolated point.
For example take S$=\{x,-x,y+x^2y^2\}$. Then

$$K_S=\{0\}\times\mathbb{R}_+$$

and we compute

$$y=y+x^2y^2+x(-x)y^2\in PO(S),$$

but

$$\deg_y(x(-x)y^2)=2>1=\deg_y(y).$$

Theorem 2.10
Let $S = \{s_1, \dots, s_r\} \subseteq \mathbb{R}[x, y]$ be finite. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 > N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) S contains the natural choice of generators for K

and let $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S and $\deg_y(f) = 2d$ be even. If there exists an $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f - \varepsilon^2(1 + y^2)^d \in PO_P(S)$$

for every $P \in K$, then $f \in PO(S)$.

Proof — Fix $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(1 + y^2)^d \in PO_P(S) \text{ for every } P \in K.$$

Let $P_1, \dots, P_s$ be the zeros of ε in K and let A be the ring of continuous functions $\phi : K \longrightarrow \mathbb{R}$ that are analytic in a neighborhood of $P_1, \dots, P_s$. By lemma 2.8, the various expressions $f_1 \in PO_P(S)$ satisfy the degree condition of lemma 2.5, so there exist $r \in \mathbb{N}$ and $\phi_e \in \sum A[y]^2$ ($e \in \{0, 1\}^r$) such that

$$f_1 = \sum_{e \in \{0,1\}^r} \phi_e s^e,$$

where $s^e := s_1^{e_1} \cdots s_r^{e_r}$. By lemma 2.2, there exist $g_e \in \sum \mathbb{R}[x, y]^2$ ($e \in \{0, 1\}^r$) and $a_i \in \mathbb{R}[x]$ ($i = 0, \dots, 2d$) with $|a_i| \leq \frac{2}{5}\varepsilon^2$ on K such that

$$f_1 = \sum_{e \in \{0,1\}^r} g_e s^e + \sum_{i=0}^{2d} a_i y^i.$$

This yields

$$f = s_1 + s_2 + s_3$$

with

$$\begin{aligned}
s_1 &:= \sum_{e \in \{0,1\}^r} g_e s^e, \\
s_2 &:= \frac{2}{5}\varepsilon^2(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{d} a_i y^i, \\
s_3 &:= \varepsilon^2((1 + y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d})).
\end{aligned}$$

Clearly, we have $s_1 \in PO(S)$. We will show $s_2, s_3 \in PO(S)$ in the following lemmas: $\square$

Lemma 2.11
In the above setting, we have $s_2 \in PO(S)$.

Proof — Compare [14].
Since

$$|a_i| \leq \frac{2}{5}\varepsilon^2 \text{ on } K$$

and S contains the natural choice of generators for K, we have

$$\frac{2}{5}\varepsilon^2 \pm a_i \in PO(S)$$

by theorem 1.7, for $i = 0, \ldots, 2d$. This yields

$$\frac{2}{5}\varepsilon^2 y^i + a_i y^i = y^i(\frac{2}{5}\varepsilon^2 + a_i) \in PO(S) \text{ for i even.}$$

For i odd, i.e. $i = 2m+1$, we compute

$$\begin{aligned}
& \frac{2}{5}\varepsilon^2(y^{2m} + y^{2m+1} + y^{2m+2}) + a_{2m+1}y^{2m+1} \\
= \; & \frac{2}{5}\varepsilon^2(y^{2m} + \frac{1}{2}y^{2m}((y+1)^2 - y^2 - 1) + y^{2m+2}) + a_{2m+1}\frac{1}{2}y^{2m}((y+1)^2 - y^2 - 1) \\
= \; & \frac{1}{2}\big[(\frac{2}{5}\varepsilon^2 - a_{2m+1})y^{2m} + (\frac{2}{5}\varepsilon^2 - a_{2m+1})y^{2m+2} + (\frac{2}{5}\varepsilon^2 + a_{2m+1})y^{2m}(y+1)^2\big] \in PO(S).
\end{aligned}$$

From both cases together, we conclude $s_2 \in PO(S)$. $\square$

Lemma 2.12
In the above setting, we have $s_3 \in PO(S)$.

Proof — Compare [14]
Obviously, $\varepsilon^2 \in PO(S)$. Further, we compute

$$\begin{aligned}
& (1+y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \ldots + y^{2d-1} + 2y^{2d}) \\
= \; & (1+y^2)^d + \frac{1}{5}(1 + y^2 + \ldots + y^{2d-2})(1-y)^2 - \frac{8}{5}(y^2 + y^4 + \ldots + y^{2d-2}) - (1 + y^{2d}) \\
= \; & \frac{1}{5}(1 + y^2 + \ldots + y^{2d-2})(1-y)^2 + \sum_{i=1}^{d-1}(\binom{d}{i} - \frac{8}{5})y^{2i} \in \sum \mathbb{R}[x,y]^2.
\end{aligned}$$

Thus $s_3 \in PO(S)$. $\square$

Theorem 2.13
Let $S = \{s_1, \ldots, s_r\} \subseteq \mathbb{R}[x,y]$ *such that*

(i) there exists an $N \in \mathbb{N}$ *such that*

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) S *contains the natural choice of generators for* K*,*

(iii) $y \in S$.

Suppose $f \in \mathbb{R}[x,y]$ *with* $f \geq 0$ *on* K_S *and* $\deg(f)_y = d$*. If there exists an* $0 \neq \varepsilon \in \mathbb{R}[x]$ *such that*

$$f - \varepsilon^2 \sum_{i=0}^{d} y^i \in PO_P(S)$$

for every $P \in K$*, then* $f \in PO(S)$.

Proof — The proof is nearly the same as the proof of theorem 2.10:
Suppose $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \sum_{i=0}^{d} \varepsilon^2 y^i \in PO_P(S)$$

for every $P \in K$. Let $P_1, \ldots, P_s$ be the zeros of ε in K. Let A be the ring of continuous functions $\phi : K \longrightarrow \mathbb{R}$ which are analytic in a neighborhood of $P_1, \ldots, P_s$. By lemma 2.8, the various expressions $f_1 \in PO_P(S)$ satisfy the degree condition of lemma 2.5, so there exist $\phi_e \in \sum A^2$ $(e \in \{0,1\}^n)$ such that

$$f_1 = \sum_{e \in \{0,1\}^r} \phi_e s^e,$$

where $s^e := s_1^{e_1} \cdots s_n^{e_n}$. By lemma 2.2, there exist $g_e \in \sum \mathbb{R}[x,y]^2$ $(e \in \{0,1\}^r)$ and $a_i \in \mathbb{R}[x]$ $(i = 0, \ldots, d)$ with $|a_i| \leq \varepsilon^2$ on K such that

$$f_1 = (\sum_{e \in \{0,1\}^r} g_e s^e) + \sum_{i=0}^{2d} a_i y^i.$$

This yields

$$f = (\sum_{e \in \{0,1\}^r} g_e s^e) + (\sum_{i=0}^{2d} (\varepsilon^2 + a_i) y^i).$$

Clearly, $(\sum_{e\in\{0,1\}^r} g_e s^e) \in PO(S)$.
For the second term, we have $y \in PO(S)$ and

$$\varepsilon^2 + a_i \geq 0 \text{ on } K$$

(for $i = 0, \ldots, d$) by construction. Since S contains the natural choice of generators for K, $\varepsilon^2 + a_i \in PO(S)$ by theorem 1.7.
This yields $f \in PO(S)$. □

Remark 2.14

(i) Theorem 2.13 was not known in [14] or in [16], although it was proven with the same methods.

(ii) Theorem 2.10 and theorem 2.13 are not useful in applications yet. One reason for this is that it is not clear whether the $\varepsilon \in \mathbb{R}[x]$ above exists. This will be studied in the next chapter.

(iii) The assumptions on the semialgebraic set K_S , i.e.

$$K \times \{y^2 > N\} \subseteq K_S \; (K \times \{y > N\} \subseteq K_S)$$

and K has no isolated points, are only needed to ensure that the "localised descriptions"

$$f = \sum_{i=1}^{n_P} h_{i,P}^2 s_{i,P}$$

(in the notation of lemma 2.8) have the right degree, i.e.

$$\deg_y(h_{i,P}^2 s_{i,P}) \leq \deg_y(f)$$

(again in the notation of lemma 2.8). This assumption could be replaced by an assumption on these "localised descriptions", which would make the theorem slightly stronger but even more technical.
Unfortunately, we were not able to apply such a stronger version in examples. See chapter 3.5 for examples where such a stronger version would be helpful.

(iv) The condition that S contains the natural choice of generators for K is needed to ensure that the "error terms" a_i in lemma 2.2 can be handled. For this, the preordering $PO(S) \cap \mathbb{R}[x]$ has to be saturated.

Marshall, Kuhlmann and Schwartz gave a full description of saturated preorderings in $\mathbb{R}[x]$ (see section 1.2).
For simplicity, we assumed (and will assume) that S contains the natural generators for K.

2.2 The Existence of ε

In this chapter, we study the existence of $\varepsilon \in \mathbb{R}[x]$ in theorem 2.10 and theorem 2.13. From now on, we denote by π_1 the projection on the x-axis, i.e.

$$\begin{aligned} \pi_1 : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto x. \end{aligned}$$

Further, for any $f \in \mathbb{R}[x, y]$, we write

$$V_{\mathbb{R}}(f) := \{P \in \mathbb{R}^2 \mid f(P) = 0\}$$

for the set of real zeros of f. First some simple observations:

Lemma 2.15
Let $S \subseteq \mathbb{R}[x, y]$ such that

(i) S contains the natural generators for K,

(ii) there exists an $N \in \mathbb{N}$ with

$$K \times \{y^2 > N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Suppose $f \in \mathbb{R}[x, y]$ with $\deg_y(f) = 2d$ even such that $f \in PO_P(S)$ for every $P \in K$, then $f \geq 0$ on K_S.

Proof — Suppose there exists $P = (P_1, P_2) \in K_S$ such that $f(P) < 0$. Define $\varepsilon := (x - P_1)$ and $f_1 := f + \varepsilon^2(1 + y^2)^d$. Since

$$f_1 - \varepsilon^2(1 + y^2)^d = f \ \in PO_P(S)$$

for every $P \in K$, we have $f_1 \in PO(S)$ by theorem 2.10, so especially $f_1 \geq 0$ on K_S. But $f_1(P) = f(P) < 0$. A contradiction. □

Lemma 2.16
Let $S \subseteq \mathbb{R}[x, y]$ such that

(i) S contains the natural generators for K,

(ii) there exists an $N \in \mathbb{N}$ with

$$K \times \{y^2 > N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Let $f \in \mathbb{R}[x,y]$ with $\deg_y(f) = 2d$ such that there exists $0 \neq \varepsilon \in \mathbb{R}[x]$ with

$$f_1 := f - \varepsilon^2(1+y^2)^d \in PO_P(S)$$

for every $P \in K$. Then $\pi_1(V_{\mathbb{R}}(f) \cap K_S) \subseteq K$ is finite.

Proof — Suppose $\pi_1(V_{\mathbb{R}}(f) \cap K_S) \neq \varnothing$. Let

$$P_1 \in V := \pi_1(V_{\mathbb{R}}(f) \cap K_S) \subseteq K.$$

Then there exists a $P_2 \in \mathbb{R}$ such that

$$P = (P_1, P_2) \in (V_{\mathbb{R}}(f) \cap K_S).$$

By lemma 2.15 we conclude

$$0 \leq f_1(P) = f(P) - \varepsilon^2(P_1)(1+P_2)^d = -\varepsilon^2(P_1)(1+P_2^2)^d.$$

Thus $\varepsilon(P_1) = 0$ for every $P_1 \in V$ and since $\varepsilon \neq 0$, V is finite. □

The same also holds in the Half-strip case:

Lemma 2.17
Let $S \subseteq \mathbb{R}[x,y]$ such that

(i) S contains the natural generators for K,

(ii) $y \in S$,

(iii) there exists an $N \in \mathbb{N}$ with

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}_+.$$

Let $f \in \mathbb{R}[x,y]$ such that $f \in PO_P(S)$ for every $P \in K$,
then $f \geq 0$ on K_S.

Proof — Suppose there exists $P = (P_1, P_2) \in K_S$ such that $f(P) < 0$. Define $\varepsilon := (x - P_1)$ and $f_1 := f + \sum_{i=0}^{d} \varepsilon^2 y^i$. Since

$$f_1 - \sum_{i=0}^{d} \varepsilon^2 y^i = f \ \in PO_P(S)$$

for every $P \in K$, we have $f_1 \in PO(S)$ by theorem 2.13, so especially $f_1 \geq 0$ on K_S. But $f_1(P) = f(P) < 0$. A contradiction. $\square$

Lemma 2.18
Let $S \subseteq \mathbb{R}[x, y]$ such that

(i) S contains the natural generators for K,

(ii) $y \in S$,

(iii) there exists an $N \in \mathbb{N}$ with

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}_+.$$

Let $f \in \mathbb{R}[x, y]$ with $\deg_y(f) = d$ such that there exists $0 \neq \varepsilon \in \mathbb{R}[x]$ with

$$f_1 := f - \varepsilon^2 \sum_{i=0}^{d} y^i \ \in PO_P(S)$$

for every $P \in K$, then $\pi_1(V_{\mathbb{R}}(f) \cap K_S) \subseteq K$ is finite.

Proof — Suppose $\pi_1(V_{\mathbb{R}}(f) \cap K_S) \neq \varnothing$. Let

$$P_1 \in V := \pi_1(V_{\mathbb{R}}(f) \cap K_S) \subseteq K.$$

Then there exists a $P_2 \in \mathbb{R}_+$ such that

$$P = (P_1, P_2) \in (V_{\mathbb{R}}(f) \cap K_S).$$

By lemma 2.17 we conclude

$$0 \leq f_1(P) = f(P) - \varepsilon^2(P_1) \sum_{i=0}^{d} P_2^i = -\varepsilon^2(P_1) \sum_{i=0}^{d} P_2^i.$$

Since $P_2 \geq 0$, we conclude $\varepsilon(P_1) = 0$. So ε vanishes on V.
Since $\varepsilon \neq 0$, V is finite. $\square$

Remark 2.19

(i) The lemmas 2.16 and 2.18 show that a necessary condition for the existence of the $\varepsilon \in \mathbb{R}[x]$ in theorem 2.10 and theorem 2.13 for any given $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S is that $\pi_1(V_{\mathbb{R}}(f) \cap K_S) \subseteq K$ is finite.

In the next part of this section, we will construct an $\varepsilon \in \mathbb{R}[x]$ such that

$$f - \varepsilon^2(1+y^2)^d \geq 0 \text{ on } K_S \; (f - \varepsilon \sum_{i=0}^{d} y^i \geq 0 \text{ on } K_S)$$

(in the notation of theorem 2.10 (2.13)). By lemma 2.15 (lemma 2.17), this is necessary for the membership in the preorderings $PO_P(S)$ (for all $P \in K$).

(ii) In his paper [14] Marshall showed that in the case $S = \{x, 1-x\}$ and $K_S = [0,1] \times \mathbb{R}$, an $\varepsilon \in \mathbb{R}[x]$ as in theorem 2.10 always exists for every

$$f = \sum_{i=0}^{2d} a_i(x) y^i \in \mathbb{R}[x,y]$$

with

(i) $f \geq 0$ on K_S,

(ii) f has finitely many zeros in K_S ,

(iii) $a_{2d}(x)$ has no zeros in K,

(compare [14]). However, his method doesn't seem to work in our more general setting. Instead, we use the theory of semialgebraic functions to show this existence here (see [2] for an introduction on semialgebraic functions). This allows us to drop the assumption on the leading term of f.

Definition 2.20

Let $K \subseteq \mathbb{R}$ be a semialgebraic set. A function $f : K \to \mathbb{R}$ is called semialgebraic if

$$Graph(f) := \{(x, f(x)) \mid x \in K\} \subseteq \mathbb{R}^2$$

is a semialgebraic set.

Lemma 2.21

Let $K' \subseteq K \times \mathbb{S}_1$ be a basic closed semi-algebraic set with $\pi(K') = K$ and

$f \in \mathbb{R}[x, y, z]$. Then

$$\begin{array}{rcl} \varphi_f : K & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \min_{y \in K'} f(x, y) \end{array}$$

is a bounded semialgebraic function.

Proof — We have

$$\begin{array}{rcl} Graph(\varphi_f) & = & \{(x,t) \,|\, \forall y \in \mathbb{S}_1((x,y) \in K' \Rightarrow t \le f(x,y)), \\ & & \forall \epsilon(\epsilon \ge 0 \Rightarrow \exists y \in \mathbb{S}_1 \,:\, (x,y) \in K',\, t + \epsilon > f(x,y))\}. \end{array}$$

So φ_f is a semialgebraic function.
Suppose φ_f is not bounded. Then for every $n \in \mathbb{N}$ there exists an $x_n \in K$ with $n < \varphi(x_n)$. Since $\{x_n\} \times \mathbb{S}_1$ is compact, for every $n \in \mathbb{N}$ there exists a $P_n \in (\{x_n\} \times \mathbb{S}_1) \cap K'$ such that $\varphi_f(x_n) = f(P_n)$. By going to a subsequence we can assume that the sequence $(P_n)_{n \in \mathbb{N}}$ converges to $P \in K'$. Since f is continuous, we have $f(P) = \lim_{n \to \infty} f(P_n)$. This is a contradiction, since $f(P_n) > n$ for all $n \in \mathbb{N}$ and $f(P) < \infty$. □

Lemma 2.22
Let $\varphi : (0, r] \longrightarrow \mathbb{R}$ be a continuous bounded semialgebraic function. Then φ can be continuously extended to 0.

Proof — [2, Proposition 2.5.3.] □

Lemma 2.23
Let $K' \subseteq K \times \mathbb{S}_1$ be a basic closed semialgebraic set with $\pi_1(K') = K$. Suppose $f \in \mathbb{R}[x, y, z]$ with $f \ge 0$ on K' and $\pi_1(V_{\mathbb{R}}(f) \cap K')$ is finite. Let $U \subseteq K$ be an interval such that $\varphi := \varphi_f|_U$ is continuous.
Then φ can be continuously extended on $\bar{U}$.

Proof — By lemma 2.21 $\varphi = \varphi_f$ is a bounded semialgebraic function, so φ is also bounded and can be extended continuously to $\bar{U}$ by lemma 2.22. □

Proposition 2.24
Let X be a locally closed semialgebraic set and let $f_1, f_2 : X \longrightarrow \mathbb{R}$ be continuous semialgebraic functions such that

$$f_1^{-1}(0) \subseteq f_2^{-1}(0).$$

Then there exists a semialgebraic neighborhood U of $f_2^{-1}(0)$ and a $m \in \mathbb{N}$ such that

$$|f_2(x)|^m \leq |f_1(x)| \text{ for all } x \in U.$$

Proof — See [24, I.1.13] □

Lemma 2.25
Let $\varphi : K \longrightarrow \mathbb{R}$ be a continuous semialgebraic function with $\varphi \geq 0$ on K and only finitely many zeros in K.
Then there is an $0 \neq \varepsilon \in \mathbb{R}[x]$ such that $\varphi \geq \varepsilon^2$ on K.

Proof — Let $P_1, \ldots, P_r$ be the zeros of φ in K. Define

$$q(x) := \prod_{i=1}^{r}(x - P_i)$$

and

$$p(x) := q(x)^2.$$

By proposition 2.24 there exists an open neighborhood U of $P_1, \ldots P_r$ in K and a $s \in \mathbb{N}$ such that

$$p(x)^s \leq \varphi(x) \text{ for all } x \in U.$$

If $U = K$ the lemma is proven, if not then p attains its maximum $M > 0$ and φ attains its minimum $m > 0$ on the compact set $K \backslash U$. So we compute

$$\frac{p(x)^s}{M^s} \leq 1 \leq \frac{\varphi(x)}{m},$$

thus

$$\frac{m}{M^s} p(x)^s \leq \varphi(x)$$

for all $x \in (K \setminus U)$. Combining these two arguments, we obtain

$$\varphi \geq \min\{1, \frac{m}{M^s}\}(q^2)^s =: \varepsilon^2 \text{ on } K.$$

□

Lemma 2.26
Let $\varphi : K \longrightarrow \mathbb{R}$ be a bounded semialgebraic function with $\varphi \geq 0$ on K. Suppose φ has only finitely many zeros in K.
Then there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ with

$$\varphi \geq \varepsilon^2 \text{ on } K.$$

Proof — Since φ is a semialgebraic function, φ is continuous in all but finitely many points, so there exists a covering

$$K = \bigcup_{i=1}^{r} K_i$$

with intervals $K_i \subseteq K$ such that $\varphi_i := \varphi|_{K_i}$ is continuous. By lemma 2.22 each φ_i can be continuously extended on $\bar{K}_i$. We will denote the continuation also by φ_i. Using lemma 2.25, there exists a $0 \neq p_i \in \mathbb{R}[x]$ for each i such that

$$p_i^2 \leq \varphi_i \text{ on } \bar{K}_i.$$

By replacing p_i by $\frac{p_i}{C_i}$ if necessary, where C_i is the maximum of p_i on K, we may assume that $p_i \leq 1$ on K for each i.
Define

$$\varepsilon := \prod_{i=1}^{r} p_i.$$

Now let $P \in K$, then, without loss of generality, $P \in K_1$. Then we compute

$$\varphi(P) \geq p_1^2(P) \geq \prod_{i=1}^{r} p_i^2(P) = \varepsilon^2(P).$$

□

Lemma 2.27
Let $S \subseteq \mathbb{R}[x, y]$ such that

(i) S contains the natural generators for K,

(ii) there exists a $N \in \mathbb{N}$ with

$$K \times \{y^2 > N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Let $f \in \mathbb{R}[x, y]$ such that

a) $f \geq 0$ on K_S,

b) $\deg_y(f) = 2d$ is even and

c) $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.

Then there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f(x, y) - \varepsilon^2(x)(1 + y^2)^d \geq 0 \text{ on } K_S.$$

Proof — Write

$$f = \sum_{i=0}^{2d} a_i(x) y^i$$

with $a_i \in \mathbb{R}[x]$. Define

$$\tilde{f} := \sum_{i=0}^{2d} a_i(x) y^i z^{2d-i} = z^{2d} f(x, \frac{y}{z}).$$

We identify

$$K \times \mathbb{R} \subseteq K \times \mathbb{S}_1$$

via

$$(x, y) \longmapsto \big(x, \frac{y}{\sqrt{1+y^2}}, \frac{1}{\sqrt{1+y^2}}\big)$$

and let K' be the closure of K_S in $K \times \mathbb{S}_1$. Then K' is a basic closed semialgebraic set with $\pi_1(K') = K$ and $\varphi := \varphi_{\tilde{f}}$ is a bounded semialgebraic function from K to $\mathbb{R}$ by lemma 2.21.
For $z \neq 0$, we compute

$$\tilde{f}(x, y, z) = z^{2d} f(x, \frac{y}{z}),$$

and

$$\tilde{f}(x, 1, 0) = a_{2d}(x) \neq 0.$$

Thus φ has only finitely many zeros in K, so by lemma 2.26 there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ with $\varphi \geq \varepsilon^2$ on K. We especially have $\tilde{f} - \varepsilon^2 \geq 0$ on K'.
Now, suppose $(x_0, y_0) \in K_S$ is arbitrary. Then we have

$$(x_0, \frac{y_0}{\sqrt{1+y_0^2}}, \frac{1}{\sqrt{1+y_0^2}}) \in K'$$

and we compute

$$\begin{aligned} f(x_0, y_0) - \varepsilon(x_0)^2(1+y_0^2)^d &= (1+y_0^2)^d \tilde{f}\left(x_0, \frac{y_0}{\sqrt{1+y_0^2}}, \frac{1}{\sqrt{1+y_0^2}}\right) - \varepsilon(x_0)^2(1+y_0^2)^d \\ &= (1+y_0^2)^d \left[\tilde{f}\Big(x_0, \frac{y_0}{\sqrt{1+y_0^2}}, \frac{1}{\sqrt{1+y_0^2}}\Big) - \varepsilon(x_0)^2\right] \\ &\geq 0. \end{aligned}$$

So

$$f - \varepsilon^2(1+y^2)^d \geq 0 \text{ on } K_S.$$

□

Lemma 2.28
Let $K \subseteq \mathbb{R}$ be compact and $S \subseteq \mathbb{R}[x,y]$ such that

(i) S contains the natural generators for K,

(ii) $y \in S$,

(iii) there exists a $N \in \mathbb{N}$ with

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}_+.$$

Let $f \in \mathbb{R}[x,y]$ such that

a) $f \geq 0$ on K_S,

b) $\deg_y(f) = d$ and

c) $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.

Then there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f - \varepsilon^2 \sum_{i=0}^{d} y^i \geq 0 \text{ on } K_S.$$

Proof — Define $S' := \{s(x, y^2) \mid s \in S\}$, then $f_1 := f(x, y^2) \geq 0$ on $K_{S'}$, $\deg_y(f_1) = 2d$ is even and

$$\pi_1(V_{\mathbb{R}}(f_1) \cap K_{S'}) = \pi_1(V_{\mathbb{R}}(f) \cap K_S)$$

is finite, so by lemma 2.27, there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f_1 - \varepsilon^2(1 + y^2)^d \geq 0 \text{ on } K_{S'}.$$

Resubstituting, we compute

$$0 \leq f - \varepsilon^2(1+y)^d \leq f - \varepsilon^2 \sum_{i=0}^{d} y^i \text{ on } K_S.$$

□

Remark 2.29

(i) We have proven above that whenever $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite, there exists an $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f(x,y) - \varepsilon^2(x)(1 + y^2)^d \geq 0 \text{ on } K_S.$$

Whether or not such a f_1 lies in all preorderings $PO_P(S)$ highly depends on S. We will study preorderings in $\mathbb{R}\{x\}[y]$ in section 2.4.

(ii) Given S as in theorem 2.10 or 2.13. If we want to show that such a preordering is saturated, we would need to show an arbitrary $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_S lies in the preordering $PO(S)$. Unfortunately, theorem 2.10 and 2.13 only applies to polynomials with finitely many zeros in K_S. Fortunately, under certain assumptions, we can reduce the problem to this case. This will be discussed in the next section.

(iii) We will finish this chapter with some lemmas that give us more control over the

$$f(x,y) - \varepsilon^2(x)(1+y^2)^d \; (f(x,y) - \varepsilon^2(x)\sum_{i=0}^{d} y^i)$$

term in theorem 2.10 (theorem 2.13). These will become useful in the applications in section 3.

Lemma 2.30
Let $S \subseteq \mathbb{R}[x,y]$ such that

(i) S contains the natural generators for K,

(ii) there exists a $N \in \mathbb{N}$ with

$$K \times \{y^2 > N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Let $f \in \mathbb{R}[x,y]$ such that $f \geq 0$ on K_S, $\deg_y(f) = 2d$ is even and $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.
Then there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(1+y^2)^d \geq 0 \text{ on } K_S,$$

$\deg_y(f_1) = 2d$ *and* $\pi_1(V_{\mathbb{R}}(f_1) \cap K_S)$ *is finite .*

Proof — Write $f = \sum_{i=0}^{2d} a_i(x)y^i$ with $a_i \in \mathbb{R}[x]$. By lemma 2.27, there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(1+y^2)^d \geq 0 \text{ on } K_S.$$

Note that, since $\varepsilon^2(1+y^2)^d$ is sos in $\mathbb{R}[x,y]$, we have

$$\varepsilon^2(1+y^2)^d \geq \frac{\varepsilon^2(1+y^2)^d}{N}$$

for all $N \in \mathbb{N}$, so we conclude

$$f - \varepsilon^2(1+y^2)^d \leq f - \frac{\varepsilon^2}{N}(1+y^2)^d \leq f \text{ on } K_S$$

for every $N \in \mathbb{N}$. So by replacing ε by $\frac{\varepsilon}{2}$ if necessary, we may assume $\deg_y(f_1) = 2d$. Now, suppose $\pi_1(V_{\mathbb{R}}(f_1) \cap K_S)$ is not finite. Define

$$f_2 := f - \frac{\varepsilon}{2}(1+y^2)^d.$$

Then we compute

$$f \geq f_2 \geq f_1 \geq 0 \text{ on } K_S.$$

Let $P \in K_S$. If $\varepsilon(P) \neq 0$, then

$$f_2(P) > f_1(P) \geq 0.$$

So

$$\pi_1(V_{\mathbb{R}}(f_2) \cap K_S) \subseteq V_{\mathbb{R}}(\varepsilon) \cap K.$$

Since $\varepsilon \neq 0$, $\pi_1(V_{\mathbb{R}}(f_2) \cap K_S)$ is finite. □

Lemma 2.31
Let $S \subseteq \mathbb{R}[x,y]$ such that

(i) S contains the natural generators for K,

(ii) there exists a $N \in \mathbb{N}$ with

$$K \times \{y^2 > N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Let $f \in \mathbb{R}[x,y]$ such that $f > 0$ on K_S and $\deg_y(f) = 2d$ is even. Then there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(1+y^2)^d > 0 \text{ on } K_S$$

and $\deg_y(f_1) = 2d$.

Proof — By lemma 2.30, there exists a $0 \neq \varepsilon_1 \in \mathbb{R}[x]$ such that

$$g := f - \varepsilon_1^2(1+y^2)^d \geq 0 \text{ on } K_S,$$

$\deg_y(g) = 2d$ and

$$\pi_1(V_{\mathbb{R}}(g) \cap K_S) =: \{P_1, \dots, P_r\}$$

is finite. Let $M \in \mathbb{N}$ such that

$$M^2 > (x-P_1)^2 \dots (x-P_r)^2 \text{ on } K.$$

Then we compute

$$1 > \frac{(x-P_1)^2 \dots (x-P_r)^2}{M^2} \geq 0 \text{ on } K.$$

Define

$$\varepsilon := \varepsilon_1 \frac{(x-P_1) \dots (x-P_r)}{M}$$

and

$$f_1 := f - \varepsilon^2(1+y^2)^d.$$

Then we compute

$$f \geq f_1 > g \geq 0 \text{ on } K_S.$$

□

Lemma 2.32
Let $S \subseteq \mathbb{R}[x,y]$ such that

(i) S contains the natural generators for K,

(ii) $y \in S$,

(iii) there exists a $N \in \mathbb{N}$ with

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}_+.$$

Let $f \in \mathbb{R}[x,y]$ such that $f \geq 0$ on K_S and $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite. Then there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f_1(x,y) := f(x,y) - \varepsilon^2 \sum_{i=0}^{d} y^i \geq 0 \text{ on } K_S$$

and $\pi_1(V_{\mathbb{R}}(f_1) \cap K_S)$ is finite .

Proof — Write

$$f = \sum_{i=0}^{2d} a_i(x) y^i,$$

with $a_i \in \mathbb{R}[x]$. By lemma 2.28, there exists a $0 \neq \varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2 \sum_{i=0}^{d} y^i \geq 0 \text{ on } K_S.$$

Note that, since $\varepsilon^2 \sum_{i=0}^{d} y^i \geq 0$ on K_S, we have

$$\varepsilon^2 \sum_{i=0}^{d} y^i \geq \frac{\varepsilon^2}{N} \sum_{i=0}^{d} y^i \text{ on } K_S$$

for all $N \in \mathbb{N}$, so we conclude

$$f - \varepsilon^2 \sum_{i=0}^{d} y^i \leq f - \frac{\varepsilon^2}{N} \sum_{i=0}^{d} y^i \leq f \text{ on } K_S$$

for every $N \in \mathbb{N}$. Thus, by replacing ε by $\frac{\varepsilon}{2}$ if necessary, we may assume $\deg_y(f_1) = 2d$.
Now, suppose $\pi_1(V_{\mathbb{R}}(f_1))$ is not finite. Define

$$f_2 := f - \frac{\varepsilon}{2} \sum_{i=0}^{d} y^i.$$

Let $P \in K_S$. If $\varepsilon(P) \neq 0$, then $f_2(P) > f_1(P) \geq 0$.
So

$$\pi_1(V_{\mathbb{R}}(f_2) \cap K_S) \subseteq V_{\mathbb{R}}(\varepsilon) \cap K,$$

thus $\pi_1(V_{\mathbb{R}}(f_2) \cap K_S)$ is finite. $\square$

Lemma 2.33
Let $S \subseteq \mathbb{R}[x, y]$ such that

(i) S contains the natural generators for K,

(ii) $y \in S$,

(iii) there exists a $N \in \mathbb{N}$ with

$$K \times \{y > N\} \subseteq K_S \subseteq K \times \mathbb{R}_+.$$

Let $f \in \mathbb{R}[x,y]$ *with* $f > 0$ *on* K_S *and* $\deg_y(f) = d$. *Then there exists a* $0 \neq \varepsilon \in \mathbb{R}[x]$ *such that*

$$f_1 := f - \varepsilon^2 \sum_{i=0}^{d} y^i > 0 \text{ on } K_S.$$

Proof — By lemma 2.32, there exists a $0 \neq \varepsilon_1 \in \mathbb{R}[x]$ such that

$$g := f - \varepsilon_1^2 \sum_{i=0}^{d} y^i \geq 0 \text{ on } K_S$$

and

$$\pi_1(V_{\mathbb{R}}(g) \cap K_S) =: \{P_1, \ldots, P_r\}$$

is finite. Let $M \in \mathbb{N}$ such that

$$M^2 > (x - P_1)^2 \ldots (x - P_r)^2 \text{ on } K.$$

Then we compute

$$1 > \frac{(x - P_1)^2 \ldots (x - P_r)^2}{M^2} \geq 0 \text{ on } K.$$

Define

$$\varepsilon := \varepsilon_1 \frac{(x - P_1) \ldots (x - P_r)}{M}$$

and

$$f_1 := f - \varepsilon^2 \sum_{i=0}^{d} y^i.$$

Then we compute

$$f \geq f_1 > g \geq 0 \text{ on } K_S.$$

□

2.3 Reductions

In the last section, we have seen that a necessary condition for the existence of the $0 \neq \varepsilon \in \mathbb{R}[x]$ term in theorem 2.10 and theorem 2.13 is that $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite. Fortunately, since $K_S \subseteq \mathbb{R}^2$, we are able to make some reductions such that this condition is satisfied in certain cases. We will study them in this section.
Most of what is done here is already known and is a slight variation of the theory presented in chapter 9.4 in [13].

Lemma 2.34
Let $S \subseteq \mathbb{R}[x, y]$ and $f \in \mathbb{R}[x, y]$ such that $f \geq 0$ on K_S. Suppose there exist infinitely many $P \in \mathbb{R}$ such that there exists an $M_P \in \mathbb{N}$ with

$$\{P\} \times \{y \mid y^2 \geq M_P\} \subseteq K_S.$$

Then $\deg_y(f)$ is even.

Proof — Let $P \in \mathbb{R}$ such that there exists an $M_P \in \mathbb{N}$ with

$$\{P\} \times \{y \mid y^2 \geq M_P\} \subseteq K_S,$$

and let

$$f = \sum_{i=0}^{d} a_i y^i \geq 0 \text{ on } K_S$$

with $a_i \in \mathbb{R}[x]$, $a_d \neq 0$. Suppose d is odd. Since $f \geq 0$ on K_S and $\{P\} \times \{y \mid y^2 \geq M_P\} \subseteq K_S$, we have

$$f(P, y) = \sum_{i=0}^{d} a_i(P) y^i \geq 0 \text{ for } y^2 > M_P,$$

thus $\deg_y(f(P, y))$ is even. Since d is odd, we conclude $a_d(P) = 0$. So a_d vanishes on every such $P \in K$. Since there are infinitely many of these $P \in K$, we conclude $a_d = 0$. A contradiction. $\square$

Remark 2.35
From the last lemma follows that, whenever $f \geq 0$ on K_S, the condition that $\deg_y(f)$ is even in theorem 2.10 is always satisfied (in the notation of theorem 2.10).
Next, we need to remember some well-known facts:

Lemma 2.36

(i) If $f, g \in \mathbb{R}[x, y]$, g is irreducible and $g \nmid f$, then f, g have only finitely many common zeros.

(ii) If $f \in \mathbb{R}[x, y]$ is irreducible, then f has only finitely many singular zeros.

(iii) If $f \in \mathbb{R}[x, y]$ and $P \in \mathbb{R}^2$ is a nonsingular zero of f, then f changes sign in each neighborhood of P.

Proof — [13, lemma 9.4.1.] □

Lemma 2.37

Suppose $s_1, \ldots, s_r$ are irreducible, $S = \{s_1, \ldots, s_r\}$ and $0 \neq f \in \mathbb{R}[x, y]$. Factor f as

$$f = gsh^2,$$

where $g, s, h \in \mathbb{R}[x, y]$, s is a product of the s_i, g is squarefree and no s_i divides g.
Then $f \geq 0$ on K_S iff $g \geq 0$ on K_S.

Proof — [13, lemma 9.4.2.] □

Remark 2.38

Obviously, in the above setting, if $g \in PO(S)$, then $f \in PO(S)$.

Lemma 2.39

Let $S = \{s_1, \ldots, s_r\} \subseteq \mathbb{R}[x, y]$. Suppose

(i) $s_1, \ldots, s_r$ are irreducible,

(ii) $0 \neq f \in \mathbb{R}[x, y]$ is squarefree,

(iii) $s_i \nmid f$ $(i = 1, \ldots, r)$ and

(iv) $f \geq 0$ on K_S.

Then f has only finitely many zeros in K_S.

Proof — [13, lemma 9.4.3.] □

Remark 2.40

(i) Given any finite $S \subseteq \mathbb{R}[x, y]$ as above. If we want to show that $PO(S)$ is saturated, we have to show that any $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S lies in $PO(S)$. Using lemma 2.39 and lemma 2.37, we may at first assume that f is squarefree and that no $s \in S$ divides f, then f has only finitely many zeros in K_S. A reduction of this kind will prove very useful later for applying theorem 2.10 and theorem 2.13.

(ii) Without the irreducibility of the s_i, lemma 2.39 above becomes false: Take $S = \{x, 1 - x, y(y - 2)\}$ and $f = y(y - 1)$. Here $f \geq 0$ on K_S, but f vanishes on the set $[0, 1] \times \{0\} \subseteq K_S$, so it has infinitely many zeros in K_S.

(iii) Unfortunately, the basic closed semialgebraic sets we are going to discuss might be generated by reducible $s_i \in \mathbb{R}[x, y]$, so we need to generalize lemma 2.39. If we look closely at the proof of lemma 2.39, we can deduce the following lemmas:

Lemma 2.41

Suppose $0 \neq f \in \mathbb{R}[x, y]$ is squarefree and $f \geq 0$ on a basic closed semialgebraic set K', then f has only finitely many zeros in the interior of K'.

Proof — Suppose f has infinitely many zeros in the interior of K'.
Then f has an irreducible factor $p \in \mathbb{R}[x, y]$ with infinitely many zeros in the interior of K', so by lemma 2.36 (i) and (ii) there exists a $P \in int(K')$ which is a nonsingular zero of p, but is not a zero of any other irreducible factor $q \neq p$ of f.
By lemma 2.36 (iii) p changes its sign in each neighborhood of P. Since every other irreducible factor q of f has a constant sign in a neighborhood of P, f changes its sign in a neighborhood of P.
A contradiction, since $f \geq 0$ on K'. □

Remark 2.42

The example in remark 2.40 has shown that in the last lemma, f may have infinitely many zeros on the boundary of K'. We can, however, analyse on which part of the boundary the majority of the real zeros of f are:

Proposition 2.43

Suppose $S = \{s_1, \ldots, s_r, t_1, \ldots, t_s\} \subseteq \mathbb{R}[x, y]$ such that *(i)* $s_1, \dots s_r$ *are irreducible,*

(ii) $t_1, \dots, t_s$ *are squarefree,*

(iii) $s_i \nmid t_j$ *for all* $i = 1, \dots, r$*;* $j = 1, \dots, s$

(iv) $K_S = \overline{int(K_S)}$*.*

Let $0 \neq f \in \mathbb{R}[x, y]$*. Factor* f *as* $f = gsh^2$*, where* $g, s, h \in \mathbb{R}[x, y]$*,* s *is a product of the* s_i*,* g *is squarefree and no* s_i *divides* g*.*

a) Then $f \geq 0$ *on* K_S *iff* $g \geq 0$ *on* K_S*.*

b) If $f \geq 0$ *on* K_S*, then* g *has only finitely many zeros in*

$$K_S \setminus (\bigcup_{i=1}^{s} V_{\mathbb{R}}(t_i)).$$

c) If $g \in PO(S)$*, then* $f \in PO(S)$*.*

Proof — (i) "$\Leftarrow$": is clear.
"$\Rightarrow$: Suppose there exists a $P \in int(K_S)$ with $g(P) < 0$. Since g is continuous in the euclidean topology, there exists a neighborhood U of P in K_S such that $g < 0$ on U. Since $f = gsh^2$ is psd on $U \subseteq K_S$ and s has only finitely many zeros in U by lemma 2.41, we have $h^2 = 0$ on U, so $h = 0$ and then $f = 0$. A contradiction.
Thus $g \geq 0$ on $\overline{int(K_S)} = K_S$.

(ii) Since $f \geq 0$ on K_S, we have $g \geq 0$ on K_S by a), so by lemma 2.41 g has only finitely many zeros in $int(K_S)$. The boundary δK_S of K_S is contained in

$$(\bigcup_{i=1}^{r} V_{\mathbb{R}}(s_i)) \cup (\bigcup_{i=1}^{s} V_{\mathbb{R}}(t_i)).$$

Since the s_i are irreducible and $s_i \nmid g$, g has only finitely many zeros on $V_{\mathbb{R}}(s_i)$. Putting this together, we conclude that g has only finitely many zeros in $K_S \setminus (\bigcup_{i=1}^{s} V_{\mathbb{R}}(t_i))$.

(iii) Suppose $g \in PO(S)$. Since $h^2, s \in PO(S)$ by definition and $PO(S)$

is closed under multiplication, we have

$$f = gsh^2 \in PO(S).$$

$\square$

Remark 2.44

(i) Given $S = \{s_1, \dots s_r, t_1, \dots, t_s\} \subseteq \mathbb{R}[x, y]$ as in the last lemma. To show that $PO(S)$ is saturated, we have to show that every $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S lies in $PO(S)$. By the last proposition, it is enough to show that $PO(S)$ contains every polynomial $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S and finitely many zeros in $K_S \setminus (\bigcup_{i=1}^{s} V_{\mathbb{R}}(t_i))$. This reduction will be useful for applying theorems 2.10 and 2.13.

(ii) In example 2.40, we have $s_1 = x$, $s_2 = 1 - x$ and $t_1 = y(y-2)$ and $f = g = y(y-1)$ in the notation of lemma 2.43. This shows that g may actually have infinitely many zeros on $V_{\mathbb{R}}(t_1)$. Here no further reduction is possible.

2.4 The Ring $\mathbb{R}\{x\}[y]$

In this section, we will study some properties of the ring $\mathbb{R}\{x\}[y]$. In the first part, we will review some well-known properties of this ring. We will specially describe irreducible elements in $\mathbb{R}\{x\}[y]$ by using Puiseux's theorem (theorem 2.46). In the second part, we will describe these irreducible elements with respect to their real zero set (see section 2.4.2 for the definitions). These descriptions will become important especially in sections 3.3 and 3.4.

In section 2.4.3, we will finally discuss preorderings in the ring $\mathbb{R}\{x\}[y]$. This discussion will play a major role for applying theorems 2.10 and 2.13 to examples.

2.4.1 General Properties

We first fix the notation:

$\mathbb{R}\{x\}$, $\mathbb{C}\{x\}$ denote the rings of convergent power series over $\mathbb{R}$ resp. $\mathbb{C}$ and $\mathbb{R}(\{x\})$ resp. $\mathbb{C}(\{x\})$ denote their quotient fields. A convergent Puiseux series over $\mathbb{R}$ resp. $\mathbb{C}$ is a series of the form $f(x^{\frac{1}{r}})$ for some $r \in \mathbb{N}$ and $f \in \mathbb{R}\{x\}$ (resp. $f \in \mathbb{C}\{x\}$). We denote the ring of convergent Puiseux series over $\mathbb{R}$ (resp. $\mathbb{C}$) by $\mathbb{R}\{x^*\}$ (resp. $\mathbb{C}\{x^*\}$) and its quotient field by $\mathbb{R}(\{x^*\})$ (resp $\mathbb{C}(\{x^*\})$). Any element $\psi \in \mathbb{R}(\{x^*\})$ can be written as

$$\psi(x) = \sum_{k=n}^{\infty} a_k x^{\frac{k}{r}}$$

for some $r \in \mathbb{N}$, $n \in \mathbb{Z}$ and some $a_k \in \mathbb{R}$ (resp. $\mathbb{C}$) and $a_n \neq 0$. For a $p \in \mathbb{C}\{x\}$ (resp. $p \in \mathbb{C}\{x^*\}$, $p \in \mathbb{C}\{x\}[y]$, $p \in \mathbb{C}\{x^*\}[y]$), we denote by $\bar{p}$ the series obtained by conjugating coefficientwise. See [18] for a more detailed introduction to this topic. Further, let $E_n \subseteq \mathbb{C}$ denote the set of n-th roots in $\mathbb{C}$.

At first, we need to introduce some basic properties of power series rings, which are widely known and can be found for example in [18] and [2].

Proposition 2.45
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$, then

(i) $\mathbb{K}\{x\}$ is a factorial, henselian, local ring with maximal ideal $\mathrm{m} := (x)$.

(ii) $f \in \mathbb{K}\{x\}^*$ *iff* $f(0) \neq 0$.

(iii) *Every* $f \in \mathbb{K}\{x\}$ *can be written as* $f = x^n u$ *for some* $n \in \mathbb{N}$, $u \in \mathbb{K}\{x\}^*$.

(iv) *Every* $f \in \mathbb{K}(\{x\})$ *can be written as* $f = x^z u$ *for some* $z \in \mathbb{Z}$, $u \in \mathbb{K}\{x\}^*$

(v) *For every* $f \in \mathbb{K}\{x\}^*$ *there exists a* $u \in \mathbb{K}\{x\}^*$ *and an* $\epsilon \in \{1-, 1\}$ *such that* $f = \epsilon u^2$.

(vi) *Let* $f := x^n u$ *with* $n \in \mathbb{N}$ *and* $u \in \mathbb{R}\{x\}^*$. *Then* $f \in \mathbb{R}[x]^2$ *iff* n *is even.*

(vii) $\mathbb{K}\{x^*\}$ *is a factorial, henselian, local ring with maximal ideal*

$$\mathrm{m} := \{f \in \mathbb{K}\{x^*\} \mid f(0) = 0\}.$$

(viii) *Every* $f \in \mathbb{K}(\{x^*\})$ *can be written as* $x^{\frac{a}{b}} f(x^{\frac{1}{b}})$ *for some* $b \in \mathbb{N}, a \in \mathbb{Z}$ *and* $f \in \mathbb{K}\{x\}^*$.

Theorem 2.46 (Puiseux)
If $p \in \mathbb{C}\{x\}[y]$ *is an irreducible monic polynomial of degree* n *in* y, *then there exists a* $g \in \mathbb{C}\{x\}$ *such that*

$$p = \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})).$$

Proof — [18, Proposition III.4.4] □

Proposition 2.47
Let $p \in \mathbb{C}\{x\}[y]$ *be irreducible and monic. Then*

(i) *the degree of* p *is the smallest integer* n *such that* p *has a root of the form* $h = f(x^{\frac{1}{n}})$ *for some* $f \in \mathbb{C}\{x\}$, *and*

(ii) *any other root of* p *is then of the form* $f(\zeta x^{\frac{1}{n}})$ *for some* $\zeta \in E_n$.

Proof — [18, Proposition II.4.5 and remark II.4.6] □

Remark 2.48

Unfortunately, we need a slightly more general version of proposition 2.47, i.e. one that includes polynomials that are not monic. This is probably widely known, but because of a lack of references, we include the proofs here.

In the following, for a $p \in \mathbb{C}\{x\}[y]$ (or $\mathbb{R}\{x\}[y]$), we denote by $\deg(p)$ the degree of p as a polynomial in y over the ring $\mathbb{C}\{x\}$ (resp. $\mathbb{R}\{x\}$).

Lemma 2.49

Let $p \in \mathbb{C}\{x\}[y]$ be irreducible with $\deg(p) = n$ and leading coefficient x^r (as a polynomial in y). Then there exists $q \in \mathbb{C}\{x\}$ and $\epsilon \in \{-1, 1\}$ such that

$$p = \epsilon \prod_{\zeta \in E_n} (\zeta^r x^{\frac{r}{n}} y - q(\zeta x^{\frac{1}{n}})).$$

Proof — If $n = 1$, this is clear and if $r = 0$, this follows from the last proposition. So we may assume $r > 0$ and $n > 1$ and

$$p = x^r y^n + \sum_{i=0}^{n-1} a_i y^i$$

for some $a_i \in \mathbb{C}\{x\}$. Now, we compute

$$\begin{aligned} x^{r(n-1)} p &= x^{r(n-1)} x^r y^n + \sum_{i=0}^{n} x^{r(n-1)} a_i y^i \\ &= (x^r y)^n + \sum_{i=0}^{n-1} x^{r(n-1-i)} a_i (x^r y)^i \\ &=: \tilde{p}(x, x^r y). \end{aligned}$$

By proposition 2.47 there exists a $q \in \mathbb{C}\{x\}$ such that

$$x^{(n-1)r} p = \tilde{p}(x, x^r y) = \prod_{\zeta \in E_n} (x^r y - q(\zeta x^{\frac{1}{n}})).$$

Since $x = (-1)^{n+1} \prod_{\zeta \in E_n} \zeta x^{\frac{1}{n}}$, we compute

$$\begin{aligned} p &= x^{-(n-1)r} \prod_{\zeta \in E_n} (x^r y - q(\zeta x^{\frac{1}{n}})) \\ &= (-1)^{(n+1)(n-1)r} \prod_{\zeta \in E_n} (\zeta x^{\frac{1}{n}})^{-r(n-1)} ((\zeta x^{\frac{1}{n}})^{nr} y - q(\zeta x^{\frac{1}{n}})) \\ &= (-1)^{(n+1)(n-1)r} \prod_{\zeta \in E_n} (\zeta^r x^{\frac{r}{n}} y - (\zeta x^{\frac{1}{n}})^{-(n-1)r} q(\zeta x^{\frac{1}{n}})). \end{aligned}$$

Note that

$$(\zeta x^{\frac{1}{n}})^{-(n-1)r} q(\zeta x^{\frac{1}{n}}) = \tilde{q}(\zeta x^{\frac{1}{n}})$$

for some $\tilde{q} \in \mathbb{C}\{x\}$, since

$$\prod_{\zeta \in E_n} -(\zeta x^{\frac{1}{n}})^{-(n-1)r} q(\zeta x^{\frac{1}{n}}) = p(x, 0) \in \mathbb{C}\{x\},$$

the proof is finished. □

Proposition 2.50
Let $p \in \mathbb{C}\{x\}[y]$ be irreducible with $n = \deg_y(p)$. Then

(i) p has a root of the form $h = f(x^{\frac{1}{n}})$ for some $f \in \mathbb{C}(\{x\})$, and

(ii) any other root of p is of the form $f(\zeta x^{\frac{1}{n}})$ for some $\zeta \in E_n$.

Proof — Write

$$p = \sum_{i=0}^{n} a_i(x) y^i$$

with $a_i \in \mathbb{R}\{x\}, a_n \neq 0$ and $a_n = x^r u$ with $u \in \mathbb{R}\{x\}^*$. We may assume $u = 1$ (otherwise we work with $p' = u^{-1} p \in \mathbb{R}\{x\}$). By lemma 2.49 there exists a $q \in \mathbb{C}\{x\}$ and an $\epsilon \in \{-1, 1\}$ such that

$$p = \epsilon \prod_{\zeta \in E_n} (\zeta^r x^{\frac{r}{n}} y - q(\zeta x^{\frac{1}{n}})).$$

So the zeros of p in $\mathbb{C}(\{x^*\})$ are

$$g_\zeta(x^{\frac{1}{n}}) := \frac{q(\zeta x^{\frac{1}{n}})}{\zeta^r x^{\frac{r}{n}}} \in \mathbb{C}(\{x^{\frac{1}{n}}\})\ (\zeta \in E_n).$$

This shows (i). For (ii) note that, for all $\zeta \in E_n$, we have

$$g_\zeta(x^{\frac{1}{n}}) = g_1(\zeta x^{\frac{1}{n}}).$$

□

Lemma 2.51
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible. If p is reducible in $\mathbb{C}\{x\}[y]$, then there exists an irreducible $h \in \mathbb{C}\{x\}[y]$ and an $\epsilon \in \{-1, 1\}$ such that

$$p = \epsilon h \bar{h}.$$

Proof — Let $q_1, \dots, q_r$ be the irreducible factors of p in $\mathbb{C}\{x\}[y]$. By our assumption we have $r \geq 2$ and $q_1, \dots, q_r \in \mathbb{C}\{x\}[y] \setminus \mathbb{R}\{x\}[y]$. We compute

$$q_1 \cdots q_r = p = \bar{p} = \overline{q_1 \cdots q_r} = \bar{q}_1 \cdots \bar{q}_r.$$

So

$$q := (q_1 \bar{q}_1) \mid p$$

and since

$$\bar{q} = \overline{(q_1 \bar{q}_1)} = q_1 \bar{q}_1 = q,$$

we conclude $q \in \mathbb{R}\{x\}[y]$. Since $p \in \mathbb{R}\{x\}[y]$ is irreducible, there exists an $u \in \mathbb{R}\{x\}^*$ such that

$$p = uq.$$

By proposition 2.45 there exists a $v \in \mathbb{R}\{x\}^*$ and an $\epsilon \in \{-1, 1\}$ with

$$\epsilon v^2 = u,$$

so with $h = vq_1$ we compute

$$p = \epsilon v^2 q = \epsilon v q_1 \overline{(v q_1)} = \epsilon h \bar{h}.$$

□

Lemma 2.52
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible. If p is reducible in $\mathbb{C}\{x\}[y]$, then there exist $h_1, h_2 \in \mathbb{R}\{x\}[y]$ and $\epsilon \in \{-1, 1\}$ such that

$$p = \epsilon(h_1^2 + h_2^2).$$

Proof — By lemma 2.51 there exists an $h \in \mathbb{C}\{x\}[y]$ and an $\epsilon \in \{-1, 1\}$ such that $p = \epsilon h \bar{h}$. Write $h = h_1 + i h_2$ with $h_1, h_2 \in \mathbb{R}\{x\}[y]$, then we compute

$$p = \epsilon h \bar{h} = \epsilon(h_1 + i h_2)(h_1 - i h_2) = \epsilon(h_1^2 + h_2^2).$$

□

Lemma 2.53
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible and monic. If p is reducible in $\mathbb{C}\{x\}[y]$, then there exist $h_1, h_2 \in \mathbb{R}\{x\}[y]$ such that

$$p = h_1^2 + h_2^2.$$

Proof — By lemma 2.52 there exist $h_1, h_2 \in \mathbb{R}\{x\}[y]$ and $\epsilon \in \{-1, 1\}$ such that

$$p = \epsilon(h_1^2 + h_2^2).$$

Comparing the leading coefficients on both sides, we conclude $\epsilon = 1$. $\square$

Lemma 2.54
Suppose $\tau \in E_n$ and $g \in \mathbb{C}(\{x\})$ such that $\overline{g(x)} = g(\tau^2 x)$. Then $g(\tau x) \in \mathbb{R}(\{x\})$.

Proof — Write

$$g = \sum_{i=z}^{\infty} a_i x_i$$

with $z \in \mathbb{Z}$ and $a_i \in \mathbb{C}$. Since $\overline{g(x)} = g(\tau^2 x)$, we compute

$$\bar{a}_i = a_i \tau^{2i}$$

for $i \geq z$. Thus

$$a_i \tau^i = \bar{a}_i \tau^{-i} = \bar{a}_i \bar{\tau}^i = \overline{(a_i \tau^i)}.$$

So we compute

$$p(\tau x) = \sum_{i=z}^{\infty} (a_i \tau^i) x^i \ \in \mathbb{R}(\{x\}).$$

$\square$

Lemma 2.55
Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p) = n$ odd. Then there exists a $g \in \mathbb{R}\{x\}$ such that

$$p = \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})).$$

Proof — Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p) = n$ odd. By proposition 2.47, there exists a $h \in \mathbb{C}\{x\}$ such that

$$p = \prod_{\zeta \in E_n} (y - h(\zeta x^{\frac{1}{n}})).$$

We compute

$$\begin{aligned} \bar{p} &= \overline{\prod_{\zeta \in E_n} (y - h(\zeta x^{\frac{1}{n}})} \\ &= \prod_{\zeta \in E_n} (y - \overline{h(\zeta x^{\frac{1}{n}})}). \end{aligned}$$

So there exists an $\zeta \in E_n$ such that

$$\overline{(y - h(x^{\frac{1}{n}}))} = (y - h(\zeta x^{\frac{1}{n}})).$$

Since n is odd, there exists an $\tau \in E_n$ such that $\tau^2 = \zeta$, so by lemma 2.54

$$g(x^{\frac{1}{n}}) := h(\tau x^{\frac{1}{n}}) \in \mathbb{R}\{x^{\frac{1}{n}}\}$$

and

$$\begin{aligned} p &= \prod_{\zeta \in E_n} (y - h(\tau\zeta x^{\frac{1}{n}})) \\ &= \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})). \end{aligned}$$

□

Lemma 2.56
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p) = n$ odd and leading coefficient x^r. Then there exists a $g \in \mathbb{R}\{x\}$ and an $\epsilon \in \{-1, 1\}$ such that

$$p = \epsilon \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^r y - g(\zeta x^{\frac{1}{n}})).$$

Proof — By lemma 2.50 there exists a $h \in \mathbb{C}\{x\}$ such that the zeros of p in $\mathbb{C}(\{x^*\})$ are given by $h(\zeta x^{\frac{1}{n}})$ for $\zeta \in E_n$. Since

$$p \in \mathbb{R}\{x\}[y] \subseteq \mathbb{R}(\{x\})[y],$$

$\overline{k(x^{\frac{1}{n}})}$ is also a zero of p. So

$$\overline{h(x^{\frac{1}{n}})} = h(\zeta x^{\frac{1}{n}})$$

for some $\zeta \in E_n$. Since n is odd, there is a $\tau \in E_n$ such that $\tau^2 = \zeta$, so by lemma 2.54

$$h(\tau x^{\frac{1}{n}}) \in \mathbb{R}(\{x^{\frac{1}{n}}\}).$$

Now write

$$h(\tau x) = x^{-s} g$$

with $g \in \mathbb{R}\{x\}$ and $g(0) \neq 0$. Then we compute

$$\begin{aligned} p &= x^r \prod_{\zeta \in E_n} (y - h(\zeta x^{\frac{1}{n}})) \\ &= x^r \prod_{\zeta \in E_n} (y - h(\tau\zeta x^{\frac{1}{n}})) \\ &= x^r \prod_{\zeta \in E_n} (\zeta x^{\frac{1}{n}})^{-s}((\zeta x^{\frac{1}{n}})^s y - g(\zeta x^{\frac{1}{n}})) \\ &= (-1)^{(n+1)s} x^r x^{-s} \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^s y - g(\zeta x^{\frac{1}{n}})). \end{aligned} \tag{2.4}$$

We have to show $r = s$.
Comparing coefficients, we get

$$p(x, 0) = (-1)^{(n+1)s} x^r x^{-s} \prod_{\zeta \in E_n} -g(\zeta x^{\frac{1}{n}}) \in \mathbb{R}\{x\}.$$

So $r \geq s$. If $r > s$, then $x|p$ in $\mathbb{R}\{x^*\}$ by equation 2.4 above, so $x|p$ in $\mathbb{R}\{x\}$. A contradiction, since p is irreducible. $\square$

Lemma 2.57
Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p) = n$ even. Then there exists a $g \in \mathbb{C}\{x\}$ and $\tau \in E_{2n}$ such that

(i) $p = \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}}))$,

(ii) $g(\tau x) \in \mathbb{R}\{x\}$.

Proof — Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p) = n$ even. By proposition 2.47 there exists an $g \in \mathbb{C}\{x\}$ such that

$$p = \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})).$$

Again, we compute

$$\bar{p} = \overline{\prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}}))} = \prod_{\zeta \in E_n} (y - \overline{g(\zeta x^{\frac{1}{n}})}).$$

So there exists an $\zeta \in E_n$ such that

$$\overline{(y - g(x^{\frac{1}{n}}))} = (y - g(\zeta x^{\frac{1}{n}})).$$

Let $\tau \in E_{2n}$ be defined by $(\tau^2) = \zeta$. Then $g(\tau x) \in \mathbb{R}\{x\}$ by lemma 2.54. $\square$

Lemma 2.58
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p) = n$ even and leading coefficient x^r. Then there exists a $g \in \mathbb{C}\{x\}$, an $\epsilon \in \{-1, 1\}$ and $\tau \in E_{2n}$ such that

(i) $p = \epsilon \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^r y - g(\zeta x^{\frac{1}{n}}))$.

(ii) $(\tau x)^{-r} g(\tau x) \in \mathbb{R}(\{x\})$.

Proof — By lemma 2.50 there exists a $h \in \mathbb{C}\{x\}$ such that the zeros of p in $\mathbb{C}(\{x^*\})$ are given by $h(\zeta x^{\frac{1}{n}})$ for $\zeta \in E_n$. Write $h(x) = x^{-s} g$ with $g \in \mathbb{C}\{x\}$ and $g(0) \neq 0$. Then we compute for some $u \in \mathbb{C}(\{x^*\})[y]^* = \mathbb{C}(\{x^*\})$

$$\begin{aligned} p &= x^r \prod_{\zeta \in E_n} (y - h(\zeta x^{\frac{1}{n}})) \\ &= x^r \prod_{\zeta \in E_n} (\zeta x^{\frac{1}{n}})^{-s} ((\zeta x^{\frac{1}{n}})^s y - g(\zeta x^{\frac{1}{n}})) \\ &= (-1)^{(n+1)s} x^r x^{-s} \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^s y - g(\zeta x^{\frac{1}{n}})) \end{aligned} \tag{2.5}$$

We have to show $r = s$.
Comparing coefficients, we get

$$p(x, 0) = (-1)^{(n+1)s} x^r x^{-s} \prod_{\zeta \in E_n} -g(\zeta x^{\frac{1}{n}}) \in \mathbb{R}\{x\}.$$

So $r \geq s$. If $r > x$, then $x|p$ in $\mathbb{R}\{x^*\}$ by equation 2.5, so $x|p$ in $\mathbb{R}\{x\}$. A contradiction, since p is irreducible.
This shows (i).
For (ii): Since

$$p \in \mathbb{R}\{x\}[y] \subseteq \mathbb{R}(\{x\})[y],$$

$\overline{k(x^{\frac{1}{n}})}$ is also a zero of p. So

$$\overline{h(x^{\frac{1}{n}})} = h(\zeta x^{\frac{1}{n}})$$

for some $\zeta \in E_n$. Let $\tau \in E_{2n}$ be defined by $\tau^2 = \zeta$. Then

$$h(\tau x^{\frac{1}{n}}) = (\tau x)^{-r} g(\tau x) \in \mathbb{R}(\{x\})$$

by lemma 2.54. □

Remark 2.59
Lemma 2.55 and lemma 2.57 are probably widely known. Lemma 2.56 and 2.58 are not. We are mainly interested in the (real) zeros of any $p \in \mathbb{C}\{x\}[y]$ ($\in \mathbb{R}\{x\}[y]$) and in the factorisation of polynomial $f \in \mathbb{R}[x, y]$ in the ring $\mathbb{R}\{x\}[y]$. In the next section, we will make a connection between the set of real zeros of a polynomial $f \in \mathbb{R}[x, y]$ and its irreducible factors in $\mathbb{R}\{x\}[y]$. This will become important in the applications later.

2.4.2 Real Zeros

In this section, we will try to make a connection between the real zeros of a polynomial over the ring of convergent power series and its irreducible factors.
By definition, each $a \in \mathbb{R}\{x\}$ converges in some neighborhood of 0.
Given $g \in \mathbb{R}\{x\}[y]$. Since g has only finitely many coefficients that are $\neq 0$, there exists a $\delta_g > 0$ such that each coefficient of g converges in $(-\delta_g, \delta_g)$.
So every $g \in \mathbb{R}\{x\}[y]$ admits a continuous function

$$\begin{array}{rcl} g : (-\delta_g, \delta_g) \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & g(x, y) \end{array}$$

for δ_g as above. Now let $g \in \mathbb{R}\{x^*\}[y]$, then $g \in \mathbb{R}\{x^{\frac{1}{n}}\}[y]$ for some $n \in \mathbb{N}$, so $g(x^n, y) \in \mathbb{R}\{x\}[y]$. Thus every $g \in \mathbb{R}\{x^*\}[y]$ admits a continuous function

$$\begin{array}{rcl} g : (-\delta_g, \delta_g) \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (x, y) & \longmapsto & g(x^n, y). \end{array}$$

Our goal in this section is to characterize the irreducible elements $g \in \mathbb{R}\{x\}[y]$ with respect to the set of real zeros of g in $(-\delta_g, \delta_g) \times \mathbb{R}$.

Lemma 2.60
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible and monic. Then p has at most one zero on the line $\{0\} \times \mathbb{R}$.

Proof — Let $p \in \mathbb{R}\{x\}[y]$ be irreducible and monic with $\deg(p) = n$. If p is irreducible in $\mathbb{C}\{x\}[y]$, then by lemma 2.57 there exists a $g \in \mathbb{C}\{x\}$ such that

$$p = \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})).$$

Substituting $x = 0$, we compute

$$p(0, y) = (y - g(0))^n \in \mathbb{R}[y].$$

So if $g(0) \in \mathbb{R}$, then p has a zero; if $g(0) \in \mathbb{C} \setminus \mathbb{R}$, then p has no zero on the line $\{0\} \times \mathbb{R}$.
If p is reducible in $\mathbb{C}\{x\}[y]$, then by lemma 2.51

$$p = \epsilon h \bar{h}$$

for some $\epsilon \in \{-1, 1\}$ and some irreducible $h \in \mathbb{C}\{x\}[y]$ with $\deg(h) = \frac{n}{2} = m$. We may assume $\epsilon = 1$.
By lemma 2.49 there exists a $g \in \mathbb{C}\{x\}$ such that

$$h = \prod_{\zeta \in E_m} (y - g(\zeta x^{\frac{1}{m}})).$$

Substituting $x = 0$, we compute

$$p(0, y) = (y - g(0))^m (y - \overline{g(0)})^m.$$

Write

$$g(0) = a + ib,$$

then

$$p(0, y) = ((y - a)^2 + b^2)^m.$$

So if $b = 0$, then p vanishes at $(0, a)$, if not, then $p > 0$ on $\{0\} \times \mathbb{R}$. $\square$

Lemma 2.61
Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$. Suppose $n = \deg_y(p)$. Then

$$p(0, y) = (y - a)^n$$

for some $a \in \mathbb{R}$. Especially p has exactly one zero on the line $\{0\} \times \mathbb{R}$.

Proof — If n is odd, then there exists a $g \in \mathbb{R}\{x\}$ such that

$$p = \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})).$$

Thus

$$p(0, y) = (y - g(0))^n.$$

If n is even, then there exists a $g \in \mathbb{C}\{x\}$ such that

$$p = \prod_{\zeta \in E_n} (y - g(\zeta x)).$$

Thus

$$p(0, y) = (y - g(0))^n.$$

Since

$$g(0)^n = p(0, 0) \in \mathbb{R},$$

we conclude $g(0) \in \mathbb{R}$, so the proof is finished. $\square$

Lemma 2.62
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible with leading coefficient x^r ($r \geq 1$) and $\deg(p) \geq 1$. *Then $p(0, y) \in \mathbb{R} \setminus \{0\}$, in particular p has no zeros on the line* $\{0\} \times \mathbb{R}$.

Proof — By lemma 2.49, there exists a $g \in \mathbb{C}\{x\}$ and an $\epsilon \in \{-1, 1\}$ such that
$$p = \epsilon \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^r y - g(\zeta x^{\frac{1}{n}})).$$
So we compute
$$p(0, y) = \epsilon(-g(0))^n,$$
thus
$$p(0, y) \in \mathbb{C} \cap \mathbb{R}[y] = \mathbb{R}.$$
Since p is irreducible, we have $x \nmid p$, so
$$p(0, y) = \epsilon(-g(0))^n \neq 0.$$
□

Lemma 2.63
Let $p \in \mathbb{R}\{x\}[y]$ be irreducible. If p is reducible in $\mathbb{C}\{x\}[y]$, then there exists an $\delta > 0$ such that p has at most one zero in $(-\delta, \delta) \times \mathbb{R}$.

Proof — By lemma 2.52 there exist $g_1, g_2 \in \mathbb{R}\{x\}[y]$ and $\epsilon \in \{-1, 1\}$ such that
$$p = \epsilon(p_1^2 + p_2^2).$$
Now consider the ring extension
$$\mathbb{R}\{x\}[y] \subseteq \mathbb{R}(\{x\})[y]$$
and the Ideal
$$I = (p_1, p_2) \subseteq \mathbb{R}(\{x\})[y].$$
Since $\mathbb{R}(\{x\})[y]$ is a principal ideal domain, there exists a $h \in \mathbb{R}(\{x\})[y]$ with $I = (h)$. p_1, p_2 are irreducible in $\mathbb{R}\{x\}[y]$, so p_i is either irreducible in $\mathbb{R}(\{x\})[y]$ (if $\deg_y(p_i) \neq 0$) or a unit (if $\deg_y(p_i) = 0$) ($i = 1, 2$). In both cases we have $I = (1)$, so there exist $h_1, h_2 \in \mathbb{R}(\{x\})[y]$ such that
$$p_1 h_1 + p_2 h_2 = 1.$$

Clearing denominators, there exist $g_1, g_2 \in \mathbb{R}\{x\}[y]$ and an $n \in \mathbb{N}$ such that

$$p_1 g_1 + p_2 g_2 = x^n. \tag{2.6}$$

Now choose $\delta > 0$ such that every coefficient of p, p_1, p_2, g_2 and g_2 converges in $(-\delta, \delta)$. Then the zeros of p in $(-\delta, \delta) \times \mathbb{R}$ are the common zeros of p_1 and p_2 and by equation 2.6 those zeros are contained in $\{0\} \times \mathbb{R}$. So by lemma 2.62 and lemma 2.60 p has at most one zero in $(-\delta, \delta) \times \mathbb{R}$. □

Lemma 2.64
Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$ and $\deg(p) = n$ be odd. Then there exists a $\delta > 0$ and a $g \in \mathbb{R}\{x\}$ such that the real zero set

$$V_\delta(p) \subseteq (-\delta, \delta) \times \mathbb{R}$$

of p is given by

$$V_\delta(p) = \{(x^n, y) \mid |x^n| < \delta, y = g(x)\}.$$

Proof — By lemma 2.55 there exists a $g \in \mathbb{R}\{x\}$ such that

$$p = \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})).$$

Substituting x by x^n, we compute

$$p(x^n, y) = \prod_{\zeta \in E_n} (y - g(\zeta x)).$$

Choose $\delta > 0$ such that g and each coefficient of p converges in $(-\delta, \delta)$. Then p vanishes on

$$Z_p := \bigcup_{\zeta \in E_n} \{(x^n, p(\zeta x)) \mid x^n \in (-\delta, \delta)\} \subseteq (-\delta, \delta) \times \mathbb{C}.$$

For every $1 \neq \zeta \in E_n$, write

$$p(\zeta x) = p_{\zeta,1} + i p_{\zeta,2}$$

with $p_{\zeta,1}, p_{\zeta,2} \in \mathbb{R}\{x\}$. Since $p(\zeta x) \notin \mathbb{R}\{x\}$, we have $p_{\zeta,2} \neq 0$, thus p_2 has only finitely many zeros in $(-\delta, \delta)$. By shrinking δ if necessary, we may assume that $p_{\zeta,2}$ has no zero in $(-\delta, 0) \cup (0, \delta)$. With this we conclude

$$V_\delta(p) = \{(x^n, y) \mid |x^n| < \delta, y = g(x)\}.$$

□

Lemma 2.65
Let $p \in \mathbb{R}\{x\}[y]$ be monic, irreducible in $\mathbb{C}\{x\}[y]$ with leading coefficient x^r ($r \geq 1$) and $\deg_y(p) = n$ odd. Then there exists a $g \in \mathbb{R}\{x\}$ and a $\delta > 0$ such that the zero set of p in $(-\delta, \delta) \times \mathbb{R}$ is

$$V_\delta(p) = \{(x^n, x^{-r}g(x)|x^n \in (-\delta, \delta), x \neq 0\} \subseteq (-\delta, \delta) \times \mathbb{R}.$$

Proof — By lemma 2.56 there exists a $g \in \mathbb{R}\{x\}$ and an $\epsilon \in \{-1, 1\}$ such that

$$p = \epsilon \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^r y - g(\zeta x^{\frac{1}{n}})).$$

Choose $\delta > 0$ such that g and each coefficient of p converges in $(-\delta, \delta)$. Then p vanishes on the set

$$\bigcup_{\zeta \in E_n} = \{(x^n, (\zeta x)^{-r} g(\zeta x)|x \neq 0\} \subseteq (-\delta, \delta) \times \mathbb{C}.$$

As in the last proof, we conclude

$$V_{\mathbb{R}}(p) = \{(x^n, x^{-r}g(x)|x^n \in (-\delta, \delta), x \neq 0\} \subseteq (-\delta, \delta) \times \mathbb{R}.$$

□

Remark 2.66

(i) Let $p \in \mathbb{R}\{x\}[y]$ be irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p)$ odd, then by the last two lemmas, p vanishes on two "half-branches" (compare [14]), one defined for $x \leq 0$ ($x < 0$) and one defined for $x \geq 0$ ($x > 0$). If the leading coefficient is a unit, both half-branches are connected at a point $(0, a)$ ($a \in \mathbb{R}$), which is also a zero of p.
If the leading coefficient is not a unit, then both "half-branches" go to infinity when x goes to 0.

(ii) It can be shown that p changes its sign at each of the zeros above (although we don't need it for this work).

Lemma 2.67
Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$ with $\deg(p) = n$ even. Then there exists $\delta > 0$, $\epsilon \in \{-1, 1\}$ and $g \in \mathbb{R}\{x\}$ such that

$$V_\delta(p) = \{(\epsilon x^n, g(x)) \mid x \in (-\delta, \delta)\} \subseteq (-\delta, \delta) \times \mathbb{R}.$$

Proof — By lemma 2.57 there exists $g \in \mathbb{C}\{x\}$, an $\epsilon \in \{0,1\}$ and $\tau \in E_{2n}$ such that

- $p = \epsilon \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}}))$,

- $g(\tau x) \in \mathbb{R}\{x\}$.

Since $\tau \in E_{2n}$, we have $\tau^n \in \{-1, 1\}$.
If $\tau^n = 1$, then $\tau \in E_n$; so by replacing $g(x)$ by $g(\tau x)$ above, we may assume $\tau = 1$ and $g \in \mathbb{R}\{x\}$. The same proof as in lemma 2.64 shows that

$$V_\delta(p) = \{(x^n, g(x)) | x \in (-\delta, \delta)\} \subseteq (-\delta, \delta) \times \mathbb{R}$$

for $\delta > 0$ small enough.
If $\tau^n = -1$, then $\tau \notin E_n$ and $\tau^2 \in E_n$. Define

$$q(x) = g(\tau x) \in \mathbb{R}[x],$$

then we compute

$$\begin{aligned} p &= \prod_{\zeta \in E_n} (y - g(\zeta x^{\frac{1}{n}})) \\ &= \prod_{\zeta \in E_n} (y - g(\zeta \tau^2 x^{\frac{1}{n}})) \\ &= \prod_{\zeta \in E_n} (y - g(\tau \zeta (\tau^n x)^{\frac{1}{n}})) \\ &= \prod_{\zeta \in E_n} (y - q(\zeta (-x)^{\frac{1}{n}})). \end{aligned}$$

Substituting $x = -x^n$, we get

$$p(-x^n, y) = \prod_{\zeta \in E_n} (y - q(\zeta x)).$$

As above, we conclude

$$V_\delta(p) = \{(-x^n, q(x)) | x \in (-\delta, \delta)\} \subseteq (-\delta, \delta) \times \mathbb{R}$$

for $\delta > 0$ small enough. □

Lemma 2.68
Let $p \in \mathbb{R}\{x\}[y]$ be monic and irreducible in $\mathbb{C}\{x\}[y]$ with $\deg(p) = n$ even and leading coefficient x^r. Then there exists an $\delta > 0$, $\epsilon \in \{-1, 1\}$ and $g \in \mathbb{R}\{x\}$ such that

$$V_\delta(p) = \{(\epsilon x^n, x^{-r} g(x) | x \in (-\delta, \delta), x \neq 0\} \subseteq (-\delta, \delta) \times \mathbb{R}.$$

Proof — By lemma 2.58 there exists $g \in \mathbb{C}\{x\}$, an $\epsilon \in \{-1,1\}$ and $\tau \in E_{2n}$ such that

- $p = \epsilon \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^r y - g(\zeta x^{\frac{1}{n}}))$,
- $(\tau x)^{-r} g(\tau x) \in \mathbb{R}(\{x\})$.

Since $\tau \in E_{2n}$, we have $\tau^n \in \{-1,1\}$.
If $\tau^n = 1$, then $\tau \in E_n$, so by replacing $x^{-r}g(x)$ by $(\tau x)^{-r} g(\tau x)$ above, we may assume $\tau = 1$ and $g \in \mathbb{R}\{x\}$. The same proof as in the odd case (lemma 2.65) shows that

$$V_\delta(p) = \{(x^n, x^{-r}g(x)) | x \in (-\delta, \delta), x \neq 0\} \subseteq (-\delta, \delta) \times \mathbb{R}$$

for $\delta > 0$ small enough.
If $\tau^n = -1$, then $\tau \notin E_n$ and $\tau^2 \in E_n$. Define

$$x^{-r}q(x) := (\tau x)^{-r} g(\tau x) \in \mathbb{R}(\{x\}),$$

then we compute

$$\begin{aligned} p &= \epsilon \prod_{\zeta \in E_n} ((\zeta x^{\frac{1}{n}})^r y - g(\zeta x^{\frac{1}{n}})) \\ &= \epsilon \prod_{\zeta \in E_n} ((\zeta \tau^2 x^{\frac{1}{n}})^r y - g(\zeta \tau^2 x^{\frac{1}{n}})) \\ &= \epsilon \prod_{\zeta \in E_n} ((\zeta \tau (\tau^n x)^{\frac{1}{n}})^r y - g(\tau \zeta (\tau^n x)^{\frac{1}{n}})) \\ &= \epsilon \prod_{\zeta \in E_n} ((\zeta (-x)^{\frac{1}{n}})^r y - q(\zeta (-x)^{\frac{1}{n}})) \end{aligned}$$

Substituting $x = -x^n$, we get

$$p(-x^n, y) = \epsilon \prod_{\zeta \in E_n} ((\zeta x)^r y - q(\zeta x)).$$

As in lemma 2.65, we conclude

$$V_\delta(p) = \{(-x^n, q(x)) | x \in (-\delta, \delta), x \neq 0\} \subseteq (-\delta, \delta) \times \mathbb{R}$$

for $\delta > 0$ small enough. □

Remark 2.69

(i) Let $p \in \mathbb{R}\{x\}[y]$ be irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(p)$ even. Then by the last two lemmas, p vanishes on two "half-branches" (compare remark 2.66). In contrast to the odd case, both of these "half-branches" are either defined for $x > 0$ (if $\epsilon = 1$ above) or $x < 0$ (if $\epsilon = -1$). Again, if the leading coefficient of p is a unit, both of these half-branches are connected at a zero $(0, a)$ of p ($a \in \mathbb{R}$); if not, then both "half-branches" go to infinity, when x goes to 0.

(ii) It is again possible to show that p changes its sign at each of the zeros above (although we don't need it here).

2.4.3 Preorderings in $\mathbb{R}\{x\}[y]$

In order to apply theorems 2.10 and 2.13, we need to study preorderings in $\mathbb{R}\{x\}[y]$. We are ultimately interested in the following question:
Given $S \subseteq \mathbb{R}[x, y]$ finite such that $K_S \subseteq K \times \mathbb{R}$ and $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S.
Is $f \in PO_P(S)$ for every $P \in K$?
The Tarski transfer principle provides a big help in answering this question:

Theorem 2.70 (Tarski transfer principle)
Suppose $(F, \geq)$ is an ordered field extension of $(\mathbb{R}, \geq)$ and there exists $x = (x_1, \ldots, x_n) \in F^n$ satisfying some finite system of polynomial equations and inequalities with coefficients in $\mathbb{R}$. Then there exists $x \in \mathbb{R}^n$ satisfying the same equations and inequalities.

Proof — [13, theorem 1.4.2] □

In our case this applies the following way:

Lemma 2.71
Let $S = \{s_1, \ldots, s_k\} \subseteq \mathbb{R}[x, y]$ such that $K_S \subseteq K \times \mathbb{R}$ and $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S.
Then $f > 0$ at every ordering of

$$K_P := Quot(\mathbb{R}\{x - P\}[y])$$

that satisfies $s_1 > 0, \ldots, s_k > 0$.

Proof — Suppose there exists a point $P \in K$ and an ordering $<$ of K_P that satisfies $f < 0$ and $s_1, \ldots, s_r > 0$. By the Tarski transfer principle (lemma 2.70) there exists a point $P \in \mathbb{R}^2$ that satisfies $s_1(P), \ldots s_k(P) > 0$ and $f(P) < 0$. A contradiction, since $f \geq 0$ on K_S. □

So this leaves us with the following question: Given $f \in \mathbb{R}\{x\}[y]$ and $s_1, \ldots, s_r \in \mathbb{R}\{x\}[y]$. If $f > 0$ at every ordering of K_P that satisfies $s_1 > 0, \ldots, s_r > 0$, is $f \in PO_P(\{s_1, \ldots, s_r\})$?

A main tool to study this question will be the well known Baer-Krull theorem. To state the theorem, we first need to repeat some background on ordered fields and valuations:
Let K be a field,

$$v : K^* \longrightarrow \Gamma$$

a surjective valuation with valuation ring A. We denote the maximal ideal of A by $\mathfrak{m}$ and the residue field by $\kappa(A) = A/\mathfrak{m}$. Then v induces an isomorphism

$$v_2 : K*/K*^2 \longrightarrow \Gamma/2\Gamma.$$

Let $\pi_i \in A$ $(i \in I)$ such that $\{v(\pi_i) \mid i \in I\}$ is a basis of the $\mathbb{F}_2$ vector space $\Gamma/2\Gamma$. Such a set is called a *quadratic representation system* of K for A. Now let P be an ordering of K such that A is a convex set. Then P induces an ordering $\bar{P}$ on $\kappa(A)$ (see [8, chapter 2.6] for details).
Specifically, we have for all $a \in A^*$:

$$a >_P = 0 \Leftrightarrow \bar{a} >_{\bar{P}} 0,$$

where $\bar{a}$ is the image of a in $\kappa(A)$.

Theorem 2.72 (Baer-Krull)
Let A be a valuation ring of K, Q be an ordering of $\kappa(A)$ and $\{\pi_i \mid i \in I\}$ be a quadratic representation system of K for A. Let Y_Q be the set of all orderings P of K that make A convex and for which we have $Q = \bar{P}$. Then map

$$\begin{array}{rcl} Y_Q & \longrightarrow & \{1, -1\}^I \\ P & \longmapsto & (sign_P(\pi_i) \mid i \in I) \end{array}$$

is a bijection.

Proof — see [8, chapter 7]. □

Additional background on ordered fields and valuations can be found in [4] and [8]. In the last part, we used the same notation as in [8].

Another important tool which we are going to need is a lemma proven by Scheiderer in [20]. We state the lemma in a slightly different version, but the proof stays essentially the same:

Lemma 2.73
Let A be a ring, let $(g_i)_{i\in I}$ be a family of elements of A and let $P = PO(\{g_i : i \in I\}))$ be the preordering it generates in A. Let $I_1 \subseteq I$. We formally adjoin the square roots of the g_i ($i \in I_1$) and write

$$B = A[t_i : i \in I_1]/(t_i^2 - g_i).$$

Let T be the preordering generated by $\{g_i \mid i \in I \setminus I_1\}$ in B and let $f \in A$. Then f lies in P iff f lies in T.

Proof — Compare [20, Lemma 3.9].
If f lies in P, then obviously f lies in T, so we have to prove the other direction.
As an A-module, B is free with basis $(t_J)_{J\subseteq I_1 finite}$, where $t_J = \prod\limits_{i\in J} t_i$. Let $s : B \longrightarrow A$ be the map of A-modules defined by $s(1) = 1$ and $s(t_J) = 0$ for finite $\varnothing \neq J \subseteq I_1$. For

$$b = \sum_J a_j t_j \in B$$

with $a_j \in A$, we compute

$$s(b^2) = \sum_J a_J^2 g_J \in P,$$

where

$$g_J := \prod_{i\in J} g_i.$$

Thus $s(\sigma) \in P$ for all $\sigma \in \sum B^2$. Now, suppose $f \in T$, i.e.

$$f = \sum_{J\subseteq(I\setminus I_1)} \sigma_J g_J,$$

where $\sigma_J \in \sum B^2$. Then we compute

$$f = s(f) = s(\sum_{J\subseteq(I\setminus I_1)} \sigma_J g_J) = \sum_{J\subseteq(I\setminus I_1)} s(\sigma_J) g_J \in P.$$

$\square$

Now, we are able to investigate the question stated at the beginning of section 2.4.3. We are especially interested in the case $r = 1$ and $\deg_y(s) \in \{1, 2\}$. We first need some lemmas:

Lemma 2.74
Suppose $g \in \mathbb{R}\{x\}[y]$ is irreducible in $\mathbb{C}\{x\}[y]$ with $\deg_y(g) = d \geq 1$. Then $Quot(\frac{\mathbb{R}\{x\}[y]}{(g)})$ is a real field.

Proof — We have

$$Quot(\frac{\mathbb{R}\{x\}[y]}{(g)}) = \frac{\mathbb{R}(\{x\})[y]}{(g)_1} =: F.$$

Suppose F is not real. Then -1 is a sum of squares in F. Clearing denominators, there exist $f, f_1, \ldots, f_n \in \mathbb{R}(\{x\})[y]$ such that

$$-1 = \sum_{i=1}^{n} f_i^2 + gf.$$

By lemma 2.57 (if $\deg_y(g)$ is even) or by lemma 2.57 (if $\deg_y(g)$ is odd) g has a zero $p \in \mathbb{R}(\{\zeta x^{\frac{1}{d}}\})$ ($\zeta \in E_{2d}$). Define

$$\begin{array}{rcl} \varphi : \mathbb{R}(\{x\})[y] & \longrightarrow & \mathbb{R}(\{\zeta x^{\frac{1}{d}}\}) \\ y & \longmapsto & p. \end{array}$$

Then we compute

$$-1 = \varphi(-1) = \varphi\left(\sum_{i=1}^{n} f_i^2 + gf\right) = \sum_{i=0}^{n} \varphi(f_i)^2.$$

A contradiction, since $\mathbb{R}(\{\zeta x^{\frac{1}{d}}\})$ is a real field. □

Proposition 2.75
Suppose $f \in \mathbb{R}\{x\}[y]$ with $f \geq 0$ at every ordering of $K = Quot(\mathbb{R}\{x\}[y])$. Then $f \in \sum \mathbb{R}\{x\}[y]^2$.

Proof — We can assume $f \neq 0$, then the factorisation of f into irreducibles in $\mathbb{R}\{x\}[y]$ can be expressed in the form

$$f = ux^k p_1^{k_1} \ldots p_r^{k_r},$$

where $p_1, \ldots, p_r \in \mathbb{R}\{x\}[y] \setminus \mathbb{R}\{x\}$ are irreducible and pairwise disjunct, $\deg_y(p_i) \neq 0$ and $u \in (\mathbb{R}\{x\}[y])^* = \mathbb{R}\{x\}^*$. By proposition 2.45, there exists an $v \in \mathbb{R}\{x\}^*$ and an $\epsilon \in \{-1, 1\}$ such that $u = \epsilon v^2$.
We first show that k is even:
Consider the discrete valuation v_x on K with valuation ring $\mathbb{R}\{x\}[y]_{(x)}$. Then the residue field

$$Quot\Big(\frac{\mathbb{R}\{x\}[y]}{(x)}\Big) = Quot(\mathbb{R}[y]) = \mathbb{R}(y)$$

is real. Fix an ordering of $\mathbb{R}(y)$, then by the Baer Krull theorem 2.72 there are 2 orderings of K obtained from this ordering, one with $x > 0$ and one with $x < 0$. If k would be odd, then f would be positive at one of the orderings and negative at the other. A contradiction.
Next, we show that for each i, either k_i is even or p_i is a sum of squares in $\mathbb{R}\{x\}[y]$.
Fix $i \in \{1, \ldots, r\}$ and write $p = p_i$.
If p is reducible in $\mathbb{C}\{x\}[y]$, then there exist $a, b \in \mathbb{R}\{x\}[y]$ and $\epsilon \in \{-1, 1\}$ such that

$$p_i = \epsilon(a^2 + b^2)$$

by lemma 2.53. So we may assume p is irreducible in $\mathbb{C}\{x\}[y]$. Consider the valuation v_p with valuation ring $\mathbb{R}\{x\}[y]_{(p)}$ and residue field

$$\kappa = Quot(\frac{\mathbb{R}\{x\}[y]}{(p)}).$$

By lemma 2.74 κ is real. Fix any ordering of κ, then again by Baer Krull there exist 2 orderings of K, one with $p < 0$ and one with $p > 0$. If k would be odd, then f would be positive at one of the orderings and negative at the other. A contradiction.
Putting all these together, there exists an $\epsilon \in \{1, -1\}$ and $\sigma \in \sum \mathbb{R}\{x\}[y]^2$ such that

$$f = \epsilon\sigma.$$

If $\epsilon = -1$, then $f < 0$ for all orderings of K. A contradiction. So $\epsilon = 1$. This finishes the proof. □

Corollary 2.76
Suppose $f \in \mathbb{R}\{x\}[y]$ and $K := Quot(\mathbb{R}\{x\}[y])$.

(i) If $f > 0$ at each ordering of K that satisfies $x > 0$, then $f \in PO(\{x\}) \subseteq \mathbb{R}\{x\}[y]$.

(ii) If $f > 0$ at each ordering of K that satisfies $y > 0$, then $f \in PO(\{y\}) \subseteq \mathbb{R}\{x\}[y]$.

(iii) If $f > 0$ at each ordering of K that satisfies $x > 0$ and $y > 0$, then $f \in PO(\{y, x\}) \subseteq \mathbb{R}\{x\}[y]$.

Proof — (i) Consider the field extension

$$K \subseteq L = Quot(\mathbb{R}\{x\}[y,t]/(t^2 - x)) = Quot(\mathbb{R}\{t\}[y]).$$

Each ordering of L restricts to an ordering of K that satisfies $x > 0$. Thus $f > 0$ at every ordering of L, so

$$f \in \sum \mathbb{R}\{t\}[y]^2$$

by proposition 2.75. By lemma 2.73 we conclude

$$f \in PO(\{x\}) \subseteq \mathbb{R}\{x\}[y].$$

(ii) Consider the field extension

$$K \subseteq L = Quot(\mathbb{R}\{x\}[y,t]/(t^2 - y)) = Quot(\mathbb{R}\{x\}[t]).$$

Each ordering of L restricts to an ordering of K that satisfies $y > 0$. Thus $f > 0$ at every ordering of L, so

$$f \in \sum \mathbb{R}\{x\}[t]^2$$

by proposition 2.75. By lemma 2.73 we conclude

$$f \in PO(\{y\}) \subseteq \mathbb{R}\{x\}[y].$$

(iii) Consider the field extension

$$K \subseteq L = Quot(\mathbb{R}\{x\}[y,t_1,t_2]/(t_1^2 - x, t_2^2 - y)) = Quot(\mathbb{R}\{t_1\}[t_2]).$$

Each ordering of L restricts to an ordering of K that satisfies $x > 0$ and $y > 0$. Thus $f > 0$ at every ordering of L, so

$$f \in \sum \mathbb{R}\{t_1\}[t_2]^2$$

by proposition 2.75. By lemma 2.73 we conclude

$$f \in PO(\{x, y\}) \subseteq \mathbb{R}\{x\}[y].$$

□

Remark 2.77

(i) Note that by a linear change of coordinates, the y term above can be replaced by any linear term $p = ay - b$ with $a \in \mathbb{R}\{x\}^*$.

(ii) Proposition 2.75 and Corollary 2.76 are probably widely know, although many authors work in the ring of formal power series $\mathbb{R}[[x_1, x_2]]$ instead of a polynomial ring over a convergent power series ring and reduce the problem to a polynomial ring over a power series ring using the Weiserstraß preparation theorem (see [18, Proposition I.3.3]). The proofs of proposition 2.75 and corollary 2.76 in the setting of formal power series are essentially the same as presented above and can be found for example in [13, Lemma 9.4.4].

Lemma 2.78
Let K be a field, v a discrete valuation on K with real residue field κ, then $v(f_1^2 + \dots f_n^2) = 2 \min v(f_i)$ for all $f_1, \dots f_n \in K$.

Proof — [19, Lemma 0.2] □

Lemma 2.79
Suppose $f \in \mathbb{R}\{x\}[y]$, $0 \neq v \in \mathbb{R}\{x\}$ and $N \in \mathbb{N}$ is even such that

$$y^N f \in PO_0(\{y(y-v)\}) \subseteq \mathbb{R}\{x\}[y],$$

then

$$f \in PO_0(\{y(y-v)\}) \subseteq \mathbb{R}\{x\}[y].$$

Proof — By our assumption, there exist $\sigma_1, \sigma_2 \in \sum \mathbb{R}\{x\}[y]^2$ such that

$$y^N f = \sigma_1 + y(y-v)\sigma_2.$$

Consider the valuation v_y with valuation ring $\mathbb{R}\{x\}[y]_{(y)}$ and residue field

$$\kappa := \frac{\mathbb{R}\{x\}[y]_{(y)}}{(y)} = \mathbb{R}(\{x\}).$$

Since κ is a real field, we have $v_y(\sigma_i) = 2d_i$ by lemma 2.78. So we compute

$$N \leq v_y(y^N f) = v_y(\sigma_1 + y(y-v)\sigma_2) = \min\{2d_1, 2d_2 + 1\}.$$

Thus $y^N | \sigma_i$ $(i = 1, 2)$, so we conclude $\sigma_1 = y^N \sigma_3$ and $\sigma_2 = y^N \sigma_4$ with $\sigma_3, \sigma_4 \in \mathbb{R}\{x\}[y]^2$. Therefore,

$$f = \sigma_3 + y(y-v)\sigma_4 \in PO(\{y(y-v)\}).$$

□

Proposition 2.80
Let $f \in \mathbb{R}\{x\}[y]$, $K := Quot(\mathbb{R}\{x\}[y])$ and $v \in \mathbb{R}\{x\}$ with $v(0) < 0$.

(i) If $f \geq 0$ at each ordering of K that satisfies $y(y-v) > 0$, then $f \in PO(\{y(y-v)\}) \subseteq \mathbb{R}\{x\}[y]$.

(ii) If $f \geq 0$ at each ordering of K that satisfies $x > 0$ and $y(y-v) > 0$, then $f \in PO(\{y(y-v), x\}) \subseteq \mathbb{R}\{x\}[y]$.

Proof — (i) Let

$$f(x,y) = \sum_{i=0}^{d} a_i y^i,$$

with $a_i \in \mathbb{R}\{x\}$, $a_d \neq 0$. Then d is even. Define

$$\tilde{f}(x,y,z) := z^d f(x, \frac{y}{z}) \in \mathbb{R}\{x\}[y,z].$$

Then $\tilde{f} > 0$ at every ordering of $K_1 := Quot(\mathbb{R}\{x\}[y,z])$ that satisfies $\frac{y}{z}(\frac{y}{z} - v) > 0$, thus $\tilde{f} > 0$ at every ordering of K_1 that satisfies

$$z^2 \frac{y}{z}(\frac{y}{z} - v) = y(y - vz) > 0.$$

Substituting $y = 1$, we conclude that $\tilde{f}(x,1,z) > 0$ at every ordering of $Quot(\mathbb{R}\{x\}[y,z])$ that satisfies $(1 - vz) > 0$.
Define $z' := 1 - vz$. Since $v(0) \neq 0$, we have $v \in \mathbb{R}\{x\}^*$ and we can compute

$$z = v^{-1}(z' - 1).$$

Then $\tilde{f}(x, 1, v^{-1}(z'-1)) > 0$ at every ordering of $Quot(\mathbb{R}\{x\}[y,z'])$ that satisfies $z' > 0$. So by corollary 2.76, there exist $\sigma_1', \sigma_2' \in \sum \mathbb{R}\{x\}[z']^2$ such that

$$\tilde{f}(x, 1, v^{-1}(z'-1)) = \sigma_1(x,z') + z'\sigma_2(x,z').$$

Define

$$\sigma_i(x,z) := \sigma_i'(x, 1 - vz) \in \sum \mathbb{R}\{x\}[z]^2 \ (i = 1,2).$$

Then we compute

$$\begin{aligned}\tilde{f}(x,1,z) &= \sigma_1'(x,1-vz)+(1-vz)\sigma_2'(x,1-vz)\\ &= \sigma_1(x,z)+(1-v(x)z)\sigma_2(x,z), \qquad (2.7)\end{aligned}$$

thus

$$\begin{aligned}y^{N-d}\tilde{f}(x,y,z) &= y^N\tilde{f}(x,1,\frac{z}{y})\\ &= y^N\sigma_1(x,\frac{z}{y})+y(y-v(x)z)y^{N-2}\sigma_2(x,\frac{z}{y})\end{aligned}$$

for every $N\in\mathbb{N}$. Now choose $d\leq N\in\mathbb{N}$ big enough such that

$$y^N\sigma_1(x,\frac{1}{y}),y^{N-2}\sigma_2(x,\frac{1}{y})\in\sum\mathbb{R}\{x\}[y]^2.$$

Substituting $z=1$ in 2.7, we compute

$$y^{N-d}\tilde{f}(x,y,1)=y^{N-d}f=y^N\sigma_1(x,\frac{1}{y})+y(y-v(x))y^{N-2}\sigma_2(x,\frac{1}{y}),$$

thus $y^{N-d}f\in PO(S)$, so by lemma 2.79 $f\in PO(S)$.

(ii) Define

$$A:=\mathbb{R}\{x\}[y,t]/(t^2-x)=\mathbb{R}\{t\}[y]$$

and let P be the preordering generated by $\{y(y-v)\}$ in A. By part (i) (applied to the ring A) we have $f\in P$, so

$$f\in PO_0(\{y(y-v),x\})\subseteq\mathbb{R}\{x\}[y]$$

by lemma 2.73. □

Remark 2.81

(i) Note that by a linear change of coordinates, the $y(y-v)$ term above can be replaced by any term $(y-v_1)(y-v_2)$ with $v_1,v_2\in\mathbb{R}\{x\}$ and $v_1(0)\neq v_2(0)$.

(ii) The basic ideal of the last proof is the following:
We use a combination of homogenization and dehomogenization to transform the preordering P generated by $p=y(y-v)$ in $\mathbb{R}\{x\}[y]$ to a preordering P' generated by $p'=1-vz$ in $\mathbb{R}\{x\}[z]$ (with $v\in\mathbb{R}\{x\}$

as above). The advantage of this transformation is that it reduces degree of the generating polynomial, i.e. we have $\deg_y(p) = 2$, but $\deg_z(p') = 1$. Here, we only have to assume $v(0) \neq 0$ such that the preordering P' is one of the preorderings studied in corollary 2.76 (compare 2.77).

3 Applications

We are now able to use theorem 2.10 and theorem 2.13 to get some first examples of saturated preorderings. In the first section, we will state some necessary conditions for any preordering in the strip and the half-strip case that satisfy the properties (SAT), $(\dagger)$, $(\ddagger)$ and (SMP). In some cases, these conditions have been found by other authors, in other cases these necessary conditions arise from our very specific assumption (i.e $K \times \{y \geq N\} \subseteq K_S$, etc.).
In the second section, we will give some first examples where our new methods are helpful. Among others, we will solve the "punctured strip problem" stated in [17] (see proposition 3.27).
In section 3.3, we will prove that the necessary conditions for a preordering in the strip and the half-strip case to satisfy $(\dagger)$ are also sufficient. This generalizes a result from Kuhlmann, Marshall and Schwartz (corollary 3.7) in the strip and half-strip case.
In section 3.4, we will prove that, under certain additional assumptions such preorderings are saturated. Even if these assumptions are probably too strong, this section provides us with new examples of saturated preorderings in the strip and the half-strip case.
We finish this chapter with some examples and open problems in section 3.5. In that section, we will discuss examples where our new theory cannot be applied yet and we will discuss what has to be proven to apply it.
During this chapter, $K \subseteq \mathbb{R}$ will always be a compact set without isolated points. Further, we will denote the natural generators for K by $\{s_1, \dots, s_r\} \subseteq \mathbb{R}[x]$.

3.1 Necessary Conditions

In this first part we will state necessary conditions for any preordering $PO(S)$ ($S \subseteq \mathbb{R}[x,y]$ finite) to satisfy the properties (SAT), (†), (‡) and (SMP). First, some simple observations:

Lemma 3.1
Let $S = \{s_1, \dots, s_r\} \subseteq \mathbb{R}[x,y]$ such that there exists an $N \in \mathbb{N}$ with

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

and $f \in \mathbb{R}[x,y]$. Suppose there exist $\sigma_e \in \sum \mathbb{R}[x,y]^2$ ($e \in \{0,1\}^r$) such that

$$f = \sum_{e \in \{0,1\}^r} \sigma_e s_1^{e_1} \dots s_r^{e_r}.$$

Then

$$\deg_y(f) \geq \deg_y(\sigma_e s_1^{e_1} \dots s_r^{e_r}) \text{ for all } e \in \{0,1\}^r.$$

Proof — Since $\mathbb{R}[x,y] \subseteq \mathbb{R}\{x - P\}[y]$ for all $P \in K$, this follows directly from lemma 2.8. □

Proposition 3.2
Let $S \subseteq \mathbb{R}[x,y]$ such that there exists an $N \in \mathbb{N}$ with

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Define $S_0 := S \cap \mathbb{R}[x]$.
If $PO(S)$ is saturated, then $PO(S_0) \subseteq \mathbb{R}[x]$ is saturated.

Proof — Let $S = \{s_1, \dots, s_r\}$ with $PO(S) \subseteq \mathbb{R}[x,y]$ saturated.
Let $f \in \mathbb{R}[x]$ with $f \geq 0$ on K.
Then $f \geq 0$ on K_S, so $f \in PO(S)$. Thus there exist $\sigma_e \in \sum \mathbb{R}[x,y]^2$ ($e \in \{0,1\}^r$) with

$$f = \sum_{e \in \{0,1\}^r} \sigma_e s^{e_1} \cdots s^{e_r}.$$

By lemma 2.8 we have

$$\deg_y(\sigma_e s^{e_1} \cdots s^{e_r}) \leq \deg_y(f) = 0,$$

thus

$$\sigma_e s^{e_1} \cdots s^{e_r} \in \mathbb{R}[x] \text{ for all } e \in \{0,1\}^r$$

and

$$f \in PO(S_0) \subseteq \mathbb{R}[x].$$

Since f was arbitrary, we have $PO(S_0) \subseteq \mathbb{R}[x]$ is saturated. □

Remark 3.3

(i) So in the strip case as well as in the half-strip case, a necessary condition for $PO(S)$ to be saturated is that $PO(S_0)$ is saturated.

(ii) In [9], Kuhlmann, Marshall and Schwartz gave a full description of saturated preorderings in $\mathbb{R}[x]$ (see section 1.2 above). They showed that $PO(S_0)$ is saturated iff $PO(S_0)$ contains the natural generators for K. For simplicity, we always assume that S contains the natural generators for K.

(iii) In their paper [9], Kuhlmann Marshall and Schwartz also gave a necessary condition for any preordering $PO(S)$ with $K_S \subseteq K' \times \mathbb{R}$ (with $K' \subseteq \mathbb{R}^n$ compact) to fulfill (SMP). We will first review their results, then deduce some more geometric conditions from their work.
But first, we need to introduce some notation:

Definition 3.4
Let $K' \subseteq \mathbb{R}^n$ be compact, $S \subseteq \mathbb{R}[x_1, \ldots, x_n, y]$ such that $K_S \subseteq K' \times \mathbb{R}$. For each $P \in K'$, define

$$\begin{array}{rcl} \lambda_P : \mathbb{R} & \longrightarrow & \mathbb{R}^{n+1} \\ y & \longmapsto & (P, y). \end{array}$$

The corresponding map between the polynomial rings is given by

$$\begin{array}{rcl} \lambda_P^* : \mathbb{R}[x_1, \ldots, x_n, y] & \longrightarrow & \mathbb{R}[y] \\ \underline{x} & \longmapsto & P \\ y & \longmapsto & y. \end{array}$$

For every $P \in K'$, the fiber $\lambda_P^{-1}(K_S)$ is a basic closed semialgebraic set in $\mathbb{R}$ defined by $\lambda_P^*(S)$ (see [9] for details).

Corollary 3.5
Let $S \subseteq \mathbb{R}[x_1, \ldots, x_n, y]$. Suppose the image of K_S under the projection $(x, y) \longmapsto x$ is bounded. For each $P \in \mathbb{R}^n$, let λ_P be as above, then the following are equivalent:

(i) $PO(S)$ *satisfies* (SMP),

(ii) *for each non-compact* $\lambda_P^{-1}(K_S)$, $\lambda_P^*(S)$ *contains the natural choice of generators for* $\lambda_P^{-1}(K_S)$ *(up to scaling by positive reals).*

Proof — [9, corollary 5.2] $\square$

Theorem 3.6
Let $S \subseteq \mathbb{R}[x_1, \ldots, x_n, y]$, *suppose the following conditions hold:*

(i) *There exists an* $N \in \mathbb{N}$ *such that* $N - (\sum_{i=1}^{n} x_i^2) \in PO(S)$.

(ii) *For each nonempty* $\lambda_P^{-1}(K_S)$, $\lambda_P^*(S)$ *contains the natural generators for* $\lambda_P^{-1}(K_S)$ *(up to scaling by positive reals).*

Then $PO(S)$ *satisfies* $(\ddagger)$.

Proof — [9, theorem 5.3] $\square$

Corollary 3.7
Let $S \subseteq \mathbb{R}[x_1, \ldots, x_n]$. *Suppose the following conditions hold:*

(i) *There exists a* $N \in \mathbb{N}$ *such that* $N - \sum_{i=1}^{n} x_i^2 \in PO(S)$ *and*

(ii) *each nonempty* $\lambda_P^{-1}(K_S)$ *is unbounded.*

Then the following are equivalent:

(i) $PO(S)$ *satisfies condition* $(\ddagger)$.

(ii) *For each nonempty* $\lambda_P^{-1}(K_S)$, $\lambda_P^*(S)$ *contains the natural choice of generators for* λ_P^{-1}*(up to scaling by positive reals).*

Proof — [9, corollary 5.4] $\square$

We are mainly interested in preorderings in $\mathbb{R}[x, y]$ (so $n = 1$ above). For these preorderings we get some more geometrical conditions:

Lemma 3.8
Let $S \subseteq \mathbb{R}[x,y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) There exists a $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

(ii) $PO(S)$ satisfies (SMP).

Then $\pi_1(K_{S_2} \setminus K_S) \subseteq K$ is finite.

Proof — Since $S_2 \subseteq S$, we have $K_S \subseteq K_{S_2}$.
Suppose $A := \pi_1(K_{S_2} \setminus K_S) \subseteq K$ is not finite.
Since $PO(S)$ satisfies (SMP), $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ for every $P \in A \subseteq K$ by corollary 3.7 (up to scaling by positive reals).
By definition, we have $\lambda_P^{-1}(K_S) \neq \lambda_P^{-1}(K_{S_2})$ for all $P \in A$, so for each $P \in A$ there exists a $s_P \in S \setminus S_2$ such that $\lambda_P^*(s_P) = s_P(P,y)$ is a natural generator for $\lambda_P^{-1}(K_S)$. Especially $\deg_y(s_P) \geq 3$ (since $s_P \notin S_2$) and

$$\deg(\lambda_P^*(s_P)) = \deg_y(s_P(P,y)) \leq 2,$$

since all natural generators have degree ≤ 2. So the leading coefficient of s_P as a polynomial in y over $\mathbb{R}[x]$ vanishes at $x = P$. Since A is infinite and S is finite, there exists an $s \in S$ such that $s = s_P$ for infinitely many $P \in A$. This is a contradiction, since the leading coefficient of s only vanishes at finitely many points. $\square$

Lemma 3.9
Let $S \subseteq \mathbb{R}[x,y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ with

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

(ii) $PO(S)$ satisfies $(\dagger)$.

Then $K_{S_2} = K_S$.

Proof — Since $S_2 \subseteq S$, we have $K_S \subseteq K_{S_2}$.
Suppose $K_{S_2} \setminus K_S \neq \varnothing$. Let $P \in K_{S_2} \setminus K_S$. Since K_S is closed in the

euclidean topology, there exists a $p \in \mathbb{R}[x,y]$ with $\deg_y(p) = 2$ such that $p(P) < 0$ and $p \geq 0$ on K_S
(For $P = (P_1, P_2)$, take for example $p = (x - P_1)^2 + (y - P_2)^2 - c$ for $0 < c \in \mathbb{R}$ small enough). Fix $0 < \varepsilon < -p(P)$, then $p(P) + \varepsilon < 0$.
Since $PO(S)$ satisfies (†), there exists $s_1, \ldots s_r \in S$ and $\sigma_e \in \sum \mathbb{R}[x,y]^2$ such that

$$p + \varepsilon = \sum_{e \in \{0,1\}^r} \sigma_e s_1^{e_1} \ldots s_r^{e_r}.$$

By lemma 3.1 there are no degree cancellations, i.e.

$$\deg_y(\sigma_e s_1^{e_1} \ldots s_r^{e_r}) \leq \deg_y(p + \varepsilon) = 2$$

for all $e \in \{0,1\}^r$, thus $p + \varepsilon \in PO(S_2)$.
A contradiction, since $P \in S_2$ and $(p + \varepsilon)(P) < 0$. □

Remark 3.10
From lemma 3.8 follows that, if a finitely generated preordering in the strip or the half-strip case satisfies (SMP), then $K_{S_2} \setminus K_S$ is a finite union of (isolated) points and (open) vertical line segments.
If $PO(S)$ satisfies (†), then $K_S = K_{S_2}$. So if we want to improve corollary 3.7, we have to make some additional assumptions such that $K_S = K_{S_2}$ holds. Here, we found the following:

Proposition 3.11
Let $S \subseteq \mathbb{R}[x,y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for each $P \in K$, $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals).

Then $K_S = K_{S_2}$.

Proof — Again, we have $K_S \subseteq K_{S_2}$.
Suppose $K_{S_2} \setminus K_S \neq \varnothing$. Let $P = (P_1, P_2) \in K_{S_2} \setminus K_S$.
Then $P_2 \notin \lambda_{P_1}^{-1}(K_S)$. Since $\lambda_{P_1}^*(S_2)$ contains the natural generators for $\lambda_{P_1}^{-1}(K_S)$, there exists an $s \in S_2$ such that

$$s(P) = s(P_1, P_2) = \lambda_{P_1}^*(s)(P_2) < 0.$$

A contradiction, since $P \in K_{S_2}$. □

Lemma 3.12
Let $S \subseteq \mathbb{R}[x,y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

(ii) $\lambda_P^*(S)$ contains the natural choice of generators for λ_P^{-1}(up to scaling by positive reals).

Then $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals) for all but finitely many $P \in K$.

Proof — Fix $P \in K$. Let $s \in S$ such that $\lambda_P^*(s)$ is one of the natural generators for $\lambda_P^{-1}(K_S)$. Then

$$\deg(\lambda_P^*(s)) = \deg(s(P,y)) \leq 2.$$

So either $s \in S_2$, or $s \in S \backslash S_2$ and the leading coefficient of s as a polynomial in y vanishes at P. Since S is finite, the second case may only happen at finitely many $P \in K$. □

Remark 3.13
Note that lemma 2.36 and lemma 3.12 are only true in the very special setup we are working in. It only holds because degree cancellations cannot occur in this case (compare lemma 3.1). We have to investigate further when there might exist some $s \in S \setminus S_2$ such that $\lambda_P^*(s)$ is a natural generator for $\lambda_P^{-1}(K_S)$ and is not in $\lambda_P^*(S)$ (up to scaling, for some $P \in K$).

Lemma 3.14
Let $S \subseteq \mathbb{R}[x,y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) $\lambda_P^*(S)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x-P) \nmid s$ *for every* $s \in S_2 \setminus \mathbb{R}[x]$ *and every* $P \in K$.

Let $s \in S$ *and* $P \in K$ *such that* $\lambda_P^*(s)$ *is one of the natural generators of* $\lambda_P^{-1}(K_S)$. *Then there exists a* $c > 0$ *and a* $s' \in S_2$ *such that*

$$\lambda_P^*(s) = c \cdot \lambda_P^*(s').$$

Proof — We may assume $P = 0$ and $[0, \varepsilon] \subseteq K$ for $0 < \varepsilon \in \mathbb{R}$.
By lemma 3.12, $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ for all but finitely many $P \in K$.
By shrinking ε if necessary, we may assume $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ for all $P \in (0, \varepsilon]$.
If $s \in S_2$, we take $s = s'$ and $c = 1$. So assume $s \in S \setminus S_2$.
Since $K \times \{y^2 \geq N\} \subseteq K_S$, all natural generators for $\lambda_0^{-1}(K_S)$ have degree 2, so there exist $P_0^+, P_0^- \in \lambda_0^{-1}(K_S)$, $P_0^+ > P_0^-$ and $0 \neq c \in \mathbb{R}$ such that

$$\lambda_0^*(s) = s(0, y) = c(y - P_0^+)(y - P_0^-).$$

Especially

$$s(0, P) < 0 \text{ for all } P \in (P_0^-, P_0^+),$$

so $(0, P) \notin K_S$ for all $P \in (P_0^-, P_0^+)$. Take $P_0 \in (P_0^-, P_0^+)$.
By shrinking ε if necessary, we may assume $(P, P_0) \notin K_S$ for all $P \in [0, \varepsilon]$, so for every $0 < \delta \leq \varepsilon$, there exist $P_\delta^+, P_\delta^- \in \mathbb{R}$ such that

- $(\delta, P_\delta^+), (\delta, P_\delta^-) \in K_S$,
- $\lambda_\delta^{-1}(K_S) = K_S \cap (\{\delta\} \times \mathbb{R}) \subseteq \{\delta\} \times ((\infty, P_\delta^-] \cup [P_\delta^+, \infty))$.

Thus we have

$$-N \leq P_\delta^- < P_0 < P_\delta^+ \leq N.$$

Take a sequence $(\delta_n)_n \subseteq (0, \varepsilon]$ with $\delta_n \longrightarrow 0$.
By going to a subsequence, we may assume the sequences

$$(P_{\delta_n}^-)_n, (P_{\delta_n}^+)_n \subseteq [-N, N]$$

converge to $P^+, P^- \in \mathbb{R}$.
Then the sequences $(\delta_n, P_{\delta_n}^+)_n \subseteq K_S$ and $(\delta_n, P_{\delta_n}^-)_n \subseteq K_S$ converge to $(0, P^+) \in K_S$ and $(0, P^-) \in K_S$. Since $(0, P) \notin K_S$ for all $P \in (P_0^-, P_0^+)$, we have

$$P^- \leq P_0^- < P_0^+ \leq P^+. \tag{3.1}$$

$\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ for all $P \in (0, \varepsilon]$ (up to scaling), so for each $n \in \mathbb{N}$ there exists a $s_n \in S_2$ and a $0 \neq c_n \in \mathbb{R}$ such that

$$\lambda_{\delta_n}^*(s_n) = s_n(\delta_n, y) = c_n(y - P_{\delta_n}^+)(y - P_{\delta_n}^-).$$

Since S_2 is finite, there exists a $s' \in S_2$ such that $s' = s_n$ for infinitely many $n \in \mathbb{N}$, so by going to a subsequence of $(\delta_n)_n$, we may assume there exists a $s' \in S_2$ such that $s' = s_n$ for all $n \in \mathbb{N}$. Since s' is continuous, s' vanishes at

$$(0, P^+) = \lim_{n\to\infty}(0, P_{\delta_n}^+)_n \text{ and } (0, P^-) = \lim_{n\to\infty}(0, P_{\delta_n}^-)_n.$$

By our assumption we have $x \nmid s'$, so $s'(0, y) \neq 0$. Since

$$s'(0, N), s'(0, -N) \geq 0$$

and

$$\deg_y(s'(0, y)) \leq \deg_y(s') = 2,$$

there exists a $0 < b \in \mathbb{R}$ such that

$$s'(0, y) = b(y - P^+)(y - P^-).$$

By equation 3.1, we have $P^- < P_0^+ \leq P^+$. If $P_0^+ < P^+$, then $s'(0, P_0^+) < 0$. A contradiction, since $(0, P_0^+) \in K_S$. So $P_0^+ = P^+$.
In the same manner, we conclude $P_0^- = P^-$.
With this, we compute

$$\lambda_0^*(s') = s'(0, y) = as(0, y) = a\lambda_0^*(s)$$

for $a = \frac{c}{b} \in \mathbb{R}$. □

Lemma 3.15
Let $S \subseteq \mathbb{R}[x, y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) $\lambda_P^*(S)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ *for every* $s \in S_2 \setminus \mathbb{R}[x]$ *and every* $P \in K$.

Let $s \in S$ *and* $P \in K$ *such that* $\lambda_P^*(s)$ *is one of the natural generators of* $\lambda_P^{-1}(K_S)$*. Then there exists a* $c > 0$ *and an* $s' \in S_2$ *such that*

a) $\lambda_P^*(s) = c \cdot \lambda_P^*(s')$.

b) The leading coefficient of s' *as a polynomial in* y *over* $\mathbb{R}[x]$ *does not vanish at* P.

Proof — We may assume $P = 0$ and $[0, \varepsilon] \subseteq K$ for $0 < \varepsilon \in \mathbb{R}$.
By lemma 3.12 $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ for all but finitely many $P \in K$.
So by shrinking ε if necessary, we may assume $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ for all $P \in (0, \varepsilon]$.
Let $s \in S$ such that $\lambda_0^*(s)$ is a natural generator for $\lambda_0^{-1}(K_S)$.
If $\deg_y(\lambda_0(s)) = 2$, we may argue the exact same way as in lemma 3.14 to show that there exists a $s' \in S_2$ and a $c > 0$ such that

$$\lambda_0^*(s) = c \cdot \lambda_0^*(s').$$

Since $\deg_y(s') = 2 = \deg_y(\lambda_0(s'))$, the leading coefficient of s' does not vanish at 0.
So suppose $\deg_y(\lambda^*(s)) = 1$.
Then there exists an $P \in \mathbb{R}_+$ and an $a > 0$ such that

$$\lambda_0^*(s) = s(0, y) = a(y - P).$$

Especially $s(0, P') < 0$ for all $P' < P$. Let $(P_{1,n})_{n \in \mathbb{N}} \subseteq (0, \varepsilon)$ be a sequence with $P_{1,n} \longrightarrow 0$ $(n \longrightarrow \infty)$. Further, let $P_{2,n}$ be the smallest point in $\lambda_{P_{1,n}}^{-1}(K_S)$, i.e

$$P_n := (P_{1,n}, P_{2,n}) \in K_S$$

and $(P_{1,n}, Q) \notin K_S$ for all $Q < P_{2,n}$. Then, for every $n \in \mathbb{N}$, there exists an $s_n \in S_2$ and an $b_n > 0$ such that

$$\lambda_{P_{1,n}}^*(s_n) = s_n(P_{1,n}, y) = b_n(y - P_{2,n}).$$

Since S_2 is finite, there exists an $s' \in S_2$ such that $s' = s_n$ for infinitely many $n \in \mathbb{N}$, so by going to a subsequence of $(P_{1,N})_N$, we may assume $s' = s_n$ for all $n \in \mathbb{N}$. By again going to a subsequence, we may assume

$P_{2,n}$ converges to $Q \in \mathbb{R}$ (This is possible since $P_{2,n} \in [0, N]$ by assumption (i)) and b_n converges to b. Then we have

$$s'(0, y) = b(y - Q).$$

$b_n > 0$ for all $n \in \mathbb{N}$, so $b \geq 0$. Since $x \nmid s'$, we have $s'(0, y) \neq 0$, especially $b \neq 0$. So we conclude $b > 0$.
We have $(0, Q) \in K_S$ by construction and $(0, P) \in K_S$ by definition, so we compute

$$\begin{aligned} 0 \leq s(0, Q) &= a(Q - P), \\ 0 \leq s'(0, P) &= b(P - Q), \end{aligned}$$

so $P = Q$ (since $a, b > 0$) and

$$\lambda_P^*(s) = c \cdot \lambda_P^*(s')$$

with $c = \frac{a}{b}$. □

Lemma 3.16
Let $S \subseteq \mathbb{R}[x, y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) $\lambda_P^*(S)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ for every $s \in S_2 \setminus \mathbb{R}[x]$ and every $P \in K$.

Then for all $P \in K$, $\lambda_P^*(S_2)$ contains the natural generators for $\lambda_P^{-1}(K_S)$.

Proof — Let $P \in K$ be arbitrary. Let $s_1, \ldots s_r \in S$ such that $\{\lambda_P^*(s_1), \ldots, \lambda_P^*(s_r)\}$ are the natural generators $\lambda_P^{-1}(K_S)$ (up to scaling). By lemma 3.14 there exist $s'_1, \ldots s'_r \in S_2$ and $c_1, \ldots c_r \in \mathbb{R}_{>0}$ such that

$$\{\lambda_P^*(s_1), \ldots, \lambda_P^*(s_r)\} = \{c_1 \lambda_P^*(s'_1), \ldots, c_r \lambda_P^*(s'_r)\}.$$

Thus $\lambda_P(S_2)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling).□

Lemma 3.17
Let $S \subseteq \mathbb{R}[x,y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) $\lambda_P^*(S)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x-P) \nmid s$ for every $s \in S_2 \setminus \mathbb{R}[x]$ and every $P \in K$.

Then for all $P \in K$, $\lambda_P^*(S_2)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals).

Proof — Let $P \in K$ be arbitrary. Let $s_1, \dots s_r \in S$ such that $\{\lambda_P^*(s_1), \dots, \lambda_P^*(s_r)\}$ are the natural generators $\lambda_P^{-1}(K_S)$ (up to scaling). By lemma 3.15, there exist $s'_1, \dots s'_r \in S_2$ and $c_1, \dots c_r \in \mathbb{R}_{>0}$ such that

$$\{\lambda_P^*(s_1), \dots, \lambda_P^*(s_r)\} = \{c_1\lambda_P^*(s'_1), \dots, c_r\lambda_P^*(s'_r)\}.$$

Thus $\lambda_P(S_2)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling).□

Remark 3.18
Suppose we are either in the strip case or in the half-strip case and suppose there exists an $P \in K$ and an $s \in S \setminus S_2$ such that $\lambda_P^*(s)$ is a natural generator which is not among the $\lambda_P^{-1}(S_2)$ (up to scaling). Then the proofs of lemma 3.14 and lemma 3.15 have shown that there exists a point $P \in K$ and an $s' \in S_2$ such that s' vanishes identically at P (i.e. $s(P,y) = 0$) and $\lambda_Q^*(s')$ is a natural generator for $\lambda_Q^{-1}(K_S)$ for infinitely many $Q \in K$ close to P. In this sense, the assumption "$(x-P) \nmid s$ for all $s \in S_2$" is a little bit too strong. We may allow some $s \in S_2$ to have such a factor $(x-P)$, if s is not among the natural generators of $\lambda_Q^{-1}(K_S)$ for Q near P. But since this is a rather small condition in general, we will continue to work with this assumption.

Proposition 3.19
Let $S \subseteq \mathbb{R}[x,y]$ be finite and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

(ii) $PO(S)$ *satisfies* (SMP),

(iii) $(x-P) \nmid s$ *for every* $s \in S_2 \setminus \mathbb{R}[x]$ *and every* $P \in K$.

Then $K_S = K_{S_2}$.

Proof — By lemma 3.16, $\lambda_P^*(S_2)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling) for every $P \in K$, so $K_S = K_{S_2}$ by proposition 3.11. □

Proposition 3.20
Let $S \subseteq \mathbb{R}[x,y]$ *be finite and* $S_2 := \{s \in S \mid \deg_y(s) \le 2\}$. *Suppose*

(i) *there exists an* $N \in \mathbb{N}$ *such that*

$$K \times \{y \ge N\} \subseteq K_S \subseteq K \times \mathbb{R}_+$$

(ii) $PO(S)$ *satisfies* (SMP),

(iii) $(x-P) \nmid s$ *for every* $s \in S_2 \setminus \mathbb{R}[x]$ *and every* $P \in K$.

Then $K_S = K_{S_2}$.

Proof — By lemma 3.17, $\lambda_P^*(S_2)$ contains the natural generators for $\lambda_P^{-1}(K_S)$ (up to scaling) for every $P \in K$, so $K_S = K_{S_2}$ by proposition 3.11. □

Remark 3.21

(i) Assumption *(iii)* above is actually necessary here:
For example, take

$$f_1 := x(y^2+1),\ f_2 := (y^2+x^2+1) + x^2y^4$$

and

$$T = \{x, 1-x, y^2+1\}\ ,\ S := \{x, 1-x, f_1, f_2\}.$$

Then

$$K_T = K_S = [0,1] \times ((-\infty,-1] \cup [1,\infty))$$

is a "punctured strip". We will show later that $PO(T)$ is saturated (see proposition 3.27). Since

$$\lambda_P^*(f_1) = P(y^2+1)$$

is the natural generator for $\lambda_P^{-1}(K_S)$ for all $P \in (0,1]$ and

$$\lambda_0^*(f_2) = (y^2 + 1)$$

is the natural generator for $\lambda_0^{-1}(K_S)$, $PO(S)$ satisfies (‡) by corollary 3.7. But $\deg_y(f_2) = 4$, so we have

$$K_S \subsetneq K_S \cup (\{0\} \times \mathbb{R}) = K_{S_2},$$

thus $K_{S_2} \neq K_S$.
Especially $PO(S)$ does not satisfy (†) by lemma 3.9.

(ii) We have proven now that it might be possible to improve corollary 3.7 in the strip or half-strip case, if we additionally add assumption (iii) above. We will do this in section 3.3.

3.2 First Examples of Saturated Preorderings

In this section, we will use theorem 2.10 and theorem 2.13 to get new examples of saturated preorderings in the strip and in the half-strip case. We have seen in the last section that a necessary condition for such a preordering is that S contains the natural generators for K and that $K_{S_2} = K_S$. The setup for this section is that we discuss preorderings in $\mathbb{R}[x,y]$ generated by

$$S := \{s_1, \ldots, s_r, g\},$$

where $s_1, \ldots, s_r$ are the natural generators for K and $g \in \mathbb{R}[x,y]$ is one additional generator.
Depending on the additional generator g, we will discuss whether $PO(S)$ satisfies any of the properties listed in definition 1.2.
For this, we equip every set $A \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) with the euclidean topology and denote by $int(A)$ the interior and by δA the boundary of A in the euclidean topology.
As far as we know, proposition 1.12 and proposition 1.13 are the only examples in either the strip case or the half-strip case which were known to be saturated until now. We will add a few more examples to this list. But first, we need some helpful lemmas:

Lemma 3.22
Let $P \in K$.

(i) If $P \in int(K)$, then there exist $u_i \in \mathbb{R}\{x-P\}^2$ such that $s_i = u_i^2$ ($i \in \{1, \ldots, r\}$).

(ii) If $P \in \delta K$, then there exist $u_i \in \mathbb{R}\{x-P\}^*$ ($i = 1, \ldots, r$), $\epsilon \in \{+1, -1\}$ and exactly one $j \in \{1, \ldots, r\}$ such that

$$\begin{aligned} s_i &= u_i^2 \ (i \neq j), \\ s_j &= \epsilon(x-P)u_j^2. \end{aligned}$$

Proof — (i) Let $P \in int(K)$. Then for all $i \in \{1, \ldots, r\}$, $s_i(P) > 0$ by definition of the natural generators. Thus $s_i \in \mathbb{R}\{x-P\}^2$ by proposition 2.45.

(ii) If $P \in \delta K$ lies on the boundary of K, then there is exactly one

$j \in \{1, \ldots, r\}$ such that $s_j(P) = 0$ and $s_i(P) > 0$ for all $i \neq j$.
Again, we conclude $s_i \in (\mathbb{R}\{x-P\}^*)^2$ by proposition 2.45 for $i \neq j$.
If $\deg(s_j) = 1$, then the lemma follows directly. If $\deg(s_j) = 2$, then write

$$s_j = c(x-P)(x-Q)$$

with $c \in \mathbb{R}$ and $P \neq Q \in K$. Since $P \neq Q$, we have

$$(x-Q) = \epsilon_1 u^2$$

for some $\epsilon_1 \in \{1, -1\}$ and $u \in \mathbb{R}\{x\}^*$ by proposition 2.45. So we compute

$$c(x-Q) = \epsilon_1 u^2 sign(c)(\sqrt{|c|})^2 =: \epsilon u_j^2$$

(with $sign(c) = -1$ if $c < 0$ and $sign(c) = 1$ if $c > 0$). □

Lemma 3.23
Let

$$g := \sum_{i=0}^{d} a_i y^i$$

with $a_i \in \mathbb{R}[x]$, $a_d \neq 0$.
Define $S = \{s_1, \ldots, s_r, g\}$ *and* $S' = \{s_1, \ldots, s_r, g, y\}$.

(i) If d *is even and* $a_d > 0$ *on* K*, then there exists an* $N \in \mathbb{N}$ *such that*

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

(ii) If d *is odd and* $a_d > 0$ *on* K*, then there exists an* $N \in \mathbb{N}$ *such that*

$$K \times \{y \geq N\} \subseteq K_{S'} \subseteq K \times \mathbb{R}_+.$$

Proof — (i) The second inclusion is obvious. So we have to show the first inclusion:
Since $K \subseteq \mathbb{R}$ is compact, a_d attains its minimum value $m > 0$ on K and each $|a_i|$ takes its maximum value M_i on K. Now, define

$$N := \cdot max\left\{\sum_{i=0}^{d-1} \frac{M_i}{m}, 1\right\},$$

then we compute for $(X,Y) \in K \times \mathbb{R}$ with $|Y| \geq N$:

$$\begin{aligned} |Y| &\geq \sum_{i=0}^{d-1} \frac{M_i}{m} \\ &\geq \sum_{i=0}^{d-1} \frac{M_i}{m} |\frac{Y^i}{Y^{d-1}}| \\ &\geq \sum_{i=0}^{d-1} \frac{|a_i(X)|}{a_d(X)} \left|\frac{Y^i}{Y^{d-1}}\right|, \end{aligned}$$

so

$$0 \leq a_d(X)Y^d - \sum_{i=0}^{d-1} |a_i(X)Y^i| \leq g(X,Y),$$

thus $(X,Y) \in K_S$. The fact that

$$|Y| \geq N \Leftrightarrow Y^2 \geq N^2$$

finishes the proof.

(ii) Again, the second inclusion is obvious. So we have to show the first inclusion:
Define

$$g_1(x,y) := g(x,y^2).$$

Then $\deg_y(g_1) = 2d$ is even, so there exists an $N \in \mathbb{N}$ such that for all $(X,Y) \in K \times \mathbb{R}$ with $Y^2 \geq N$, we have

$$g_1(X,Y) = g(X,Y^2) \geq 0.$$

Replacing Y^2 by Y, we compute $g(X,Y) \geq 0$ for all $(X,Y) \in K \times \mathbb{R}$ with $Y \geq N \geq 0$. Thus $K \times \{y \geq N\} \subseteq K_{S'}$. □

Definition 3.24
For every $P \in \mathbb{R}$, we define

$$K_P = Quot(\mathbb{R}\{x - P\}[y]).$$

The first case we are going to discuss is a slight improvement of proposition 1.13:

Proposition 3.25
Let $a, b \in \mathbb{R}[x]$ with $a, b > 0$ on K, then the preordering generated by

$$S := \{s_1, \ldots, s_k, a(x)y - b(x), y\} \subseteq \mathbb{R}[x, y]$$

is saturated.

Proof — Since $a > 0$ on K, there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+$$

by lemma 3.23. Let $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S and $\deg_y(f) = d$. We have to show $f \in PO(S)$.
If

$$f = (ay - b)g^2 h$$

with $g, h \in \mathbb{R}[x, y]$ and $h \in PO(S)$, then $f \in PO(S)$, so we may assume f is squarefree and $ay - b \nmid f$ (compare lemma 2.43).
Then f has only finitely many zeros in $K_S \setminus \bigcup_{i=1}^{k} V_{\mathbb{R}}(s_i)$ by lemma 2.43. Thus $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.
So by lemma 2.32, there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2 \sum_{i=0}^{d} y^i \geq 0 \text{ on } K_S$$

with only finitely many zeros in K_S.
By theorem 2.13, we have to show $f_1 \in PO_P(S)$ for every $P \in K$.
First let $P \in int(K)$. We may assume $P = 0$.
Then $s_1, \ldots, s_r \in (\mathbb{R}\{x\}^*)^2$ by lemma 3.22.
Since $f_1 \geq 0$ on K_S, we have $f_1 > 0$ at every ordering of K_0 that satisfies $ay + b > 0$ and $s_1, \ldots, s_r > 0$ by lemma 2.71. Since $s_1, \ldots, s_r \in \mathbb{R}\{x\}^2$, $s_1, \ldots, s_r > 0$ at every ordering of K_0, so we conclude $f_1 > 0$ at every ordering of K_0 that satisfies $ay + b > 0$, thus

$$f_1 \in PO_0(\{ay - b\}) \subseteq PO_0(S)$$

by proposition 2.76 (compare remark 2.77).

Now let $P \in \delta K$. We may again assume $P = 0$.
Then by lemma 3.22, there exist $\epsilon \in \{-1, 1\}$, $u_i \in \mathbb{R}\{x\}^*$ $(i = 1, \ldots, r)$ and

exactly one $j \in \{1, \dots, r\}$ such that

$$\begin{aligned} s_i &= u_i^2 \, (i \neq j), \\ s_j &= \epsilon x u_j^2. \end{aligned}$$

We may assume $j = 1$ and by replacing x by $-x$ if necessary, we may assume $\epsilon = 1$.
By lemma 2.71 $f_1 > 0$ at every ordering of K_0 that satisfies $ay - b > 0$ and $s_1, \dots, s_r > 0$. Again, $s_2, \dots, s_r > 0$ at every ordering of K_0 and $s_1 > 0$ at every ordering of K_0 that satisfies $x > 0$. We conclude $f_1 > 0$ at every ordering of K_0 that satisfies $ay - b > 0$ and $x > 0$, so

$$f_1 \in PO_0(\{x, ay - b\}) = PO_0(\{s_1, ay - b\}) \subseteq PO_0(S)$$

by proposition 2.76. Note that $PO_0(\{x\}) = PO(\{xu_j^2\})$, since $u_j \in \mathbb{R}\{x\}^*$. Putting these two together, we conclude $f_1 \in PO_P(S)$ for every $P \in K$. $\square$

Remark 3.26

(i) Define S, a, b as in the last proposition and

$$S' := \{s_1, \dots, s_k, a(x)y - b(x)\}.$$

Then we have $K_S = K_{S'}$. If $a \in \mathbb{R}$, then $PO(S')$ is saturated by proposition 1.13, so $PO(S) = PO(S')$.

(ii) If $a \in \mathbb{R}[x] \setminus \mathbb{R}$ in the last proposition, the term y in S is actually necessary:
Suppose $y \in PO(S')$, then there exist $\sigma_1, \sigma_2 \in PO(s_1, \dots, s_k) \subseteq \mathbb{R}[x, y]$ with

$$y = (a(x)y + b(x))\sigma_1 + \sigma_2.$$

By lemma 3.1 we have no degree cancellation in y, so we conclude $\sigma_1, \sigma_2 \in \mathbb{R}[x]$. Comparing coefficients, we get $1 = a(x)\sigma_1(x)$, i.e. $a \in \mathbb{R}[x]^* = \mathbb{R}$, a contradiction.

(iii) Note that, given any $a, b \in \mathbb{R}[x]$, whenever a has a constant sign on K, the preordering can be brought to the form above by a linear change of coordinates.

(iv) If leading coefficient $a \in \mathbb{R}[x]$ in the last proposition changes its sign on K, then the corresponding semialgebraic set K_S is not contained in $K \times \mathbb{R}_+$, so theorem 2.13 does not apply. Whether or not such a preordering is saturated, is still open.

(v) If a has a zero P in K (and does not change its sign), then the semialgebraic set does not have the desired form. We would need a stronger version of theorem 2.13 to discuss this case (see remark 2.14). But even if we proved such a stronger version, it is still not clear whether $f_1 \in PO_P(S)$ (in the notation of the last proof). This is also still open. See example 3.91 in chapter 3.5 for more details.

Next, we study preorderings in the strip generated by a polynomial $g \in \mathbb{R}[x, y]$ with $\deg_y(g) = 2$.

Proposition 3.27
The preordering $PO(S)$ generated by $S = \{s_1, \ldots, s_r, y(y+1)\}$ is saturated.

Proof — Let

$$f = \sum_{i=0}^{d} a_i y^i \in \mathbb{R}[x, y]$$

with $f \geq 0$ on K_S. Then d is even by lemma 2.34. We have to show $f \in PO(S)$.
Define

$$\tilde{f}(x, y, z) := z^d f(x, \frac{y}{z}) = \sum_{i=0}^{d} a_i(x) y^i z^{d-i}.$$

Then $\tilde{f} \geq 0$ on the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid s_1(x) \geq 0, \ldots, s_r(x) \geq 0, y(y+z) \geq 0\}.$$

Substituting $y = 1$, we conclude $\tilde{f}(x, 1, z) \geq 0$ on the set

$$\{(x, z) \in \mathbb{R}^2 \mid s_1(x) \geq 0, \ldots, s_r(x) \geq 0, (1+z) \geq 0\}.$$

$PO(\{s_1, \ldots, s_r, 1+z\}) \subseteq \mathbb{R}[x, z]$ is saturated by proposition 1.13, so there exist $\sigma_i \in \sum \mathbb{R}[x, z]^2$ $(i \in \{0, 1\}^{r+1})$ with

$$\tilde{f}(x, 1, z) = \sum_{i \in \{0,1\}^{r+1}} \sigma_i(x, z)(1+z)^{i_{r+1}} \prod_{j=1}^{r} s_j^{i_j}.$$

By lemma 3.1 we conclude

$$\deg_z(\tilde{f}) \geq \deg_z(\sigma_i(x,z)(1+z)^{i_{r+1}} \prod_{j=1}^{r} s_j^{i_j})$$

for all $i \in \{0,1\}^{r+1}$. So we compute

$$\begin{aligned} f(x,y) &= \tilde{f}(x,y,1) \\ &= y^d \tilde{f}(x,1,\frac{1}{y}) \\ &= \sum_{i\in\{0,1\}^{r+1}} y^d \sigma_i(x,\frac{1}{y})(1+\frac{1}{y})^{i_{s+1}} \prod_{j=1}^{s} s_j^{i_j}(x) \\ &= \sum_{i\in\{0,1\}^{r+1}} y^{d-2} y^{2(1-i_{s+1})} \sigma_i(x,\frac{1}{y})(y(y+1))^{i_{s+1}} \prod_{j=1}^{s} s_j^{i_j}. \end{aligned}$$

Since

$$d \geq \deg_z(\tilde{f}(x,1,z)) \geq \deg_z(\sigma_i(x,y)(1+z)^{i_{s+1}}),$$

we have

$$y^{d-2} y^{2(1-i_{s+1})} \sigma_i(x,\frac{1}{y}) \in \sum \mathbb{R}[x,y]^2$$

for all $i \in \{0,1\}^{s+1}$. Thus $f \in PO(S)$. □

Remark 3.28

(i) This last proposition solves the "punctured strip problem" stated in [17].

(ii) In the case where we replace the term $y(y+1)$ by $y(y-b(x))$ with $b(x) < 0$ on K, things get more complicated:

Proposition 3.29
Let $b \in \mathbb{R}[x]$ with no zeros on K and let $S := \{s_1,\ldots,s_k, y(y-b(x))\}$. Suppose $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_S.
If $y,(y-b) \nmid f$, then $f \in PO(S)$.

Proof — Since g is monic, there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

by lemma 3.23.
By lemma 2.34 $d := \deg_y(f)$ is even, i.e. $d = 2m$. We may assume f is

squarefree, so f has only finitely many zeros in $K_S \setminus (\bigcup_{s\in S} V_{\mathbb{R}}(s))$ by lemma 2.43. Since $y, (y-b) \nmid f$, f has only finitely many zeros on $V_{\mathbb{R}}(y(y-b))$, thus $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.
So by lemma 2.30, there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(1+y^2)^m \geq 0 \text{ on } K_S.$$

By theorem 2.10, we have to show $f_1 \in PO_P(S)$ for every $P \in K$.
First let $P \in int(K)$.
Since $f_1 \geq 0$ on K_S, $f_1 > 0$ at every ordering of K_P that satisfies $y(y-b) > 0$ and $s_1, \ldots, s_r > 0$ by lemma 2.71. $s_1, \ldots, s_k \in \mathbb{R}\{x-P\}^2$ by lemma 3.22, so $f_1 > 0$ at every ordering of K_P that satisfies $y(y-b) > 0$ and

$$f_1 \in PO_P(\{y(y-b)\}) \subseteq PO_P(S)$$

by proposition 2.80.
Suppose $P \in \delta K$. We may assume $P = 0$.
By lemma 2.43, we may assume $s_2, \ldots, s_r \in \mathbb{R}\{x\}^2$, $s_1 = \epsilon x u^2$ for an $u \in \mathbb{R}\{x\}^*$ and $\epsilon \in \{-1, 1\}$. By replacing x by $-x$ if necessary, we may assume $\epsilon = 1$.
By lemma 2.71, $f_1 > 0$ at every ordering of K_0 that satisfies $ay - b > 0$ and $s_1, \ldots, s_r > 0$. Again, $s_2, \ldots, s_r > 0$ at every ordering of K_0 and $s_1 > 0$ at every ordering of K_0 that satisfies $x > 0$. So $f_1 > 0$ at every ordering of K_0 that satisfies $x > 0$ and $y(y-b) > 0$, thus

$$f_1 \in PO_0(\{x, y(y-b)\}) = PO_0(\{s_1, y(y-b)\}) \subseteq PO_0(S)$$

by proposition 2.80. □

Corollary 3.30
Suppose $b \in \mathbb{R}[x]$ has no zeros in K and let

$$S := \{s_1, \ldots, s_k, y(y-b(x))\}.$$

Then $PO(S)$ satisfies (†).

Proof — Let $\delta > 0$ and $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_S. Then

$$f_1 := f + \delta > 0 \text{ on } K_S,$$

especially f_1 has no zeros in K_S and $y, (y-b) \nmid f_1$. Thus $f_1 \in PO(S)$ by proposition 3.29. □

Remark 3.31
Unfortunately, proposition 3.29 is not enough to prove that $PO(S)$ is saturated (with S as in the proposition). The reason for this is that we can only reduce to the case where f has finitely many zeros in $K_S \setminus V_{\mathbb{R}}(y(y-b(x))$ by lemma 2.43. If $y|f$ or $(y-b(x))|f$ and $y(y-b(x)) \nmid f$, then we cannot reduce to the case where $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite, thus there exists no $0 \neq \varepsilon \in \mathbb{R}[x]$ which satisfies the conditions of theorem 2.13 (see lemma 2.18 above). In this case, f might not be in the preordering. Indeed, we get the following lemma and proposition:

Lemma 3.32
Let $b \in \mathbb{R}[x] \setminus \mathbb{R}$ with $b < -1$ on K and let

$$S := \{s_1, \dots, s_k, y(y - b(x))\}.$$

Then

(i) $y(y+1) \notin PO(S)$,

(ii) $(y+1)(y-b(x)) \notin PO(S)$.

Proof — (i) Suppose $y(y+1) \in PO(S)$. Then there exist $\sigma_1, \sigma_2 \in PO(\{s_1, \dots, s_k\}) \subseteq \mathbb{R}[x, y]$ such that

$$y(y+1) = \sigma_1 + y(y - b(x))\sigma_2.$$

By lemma 3.1 we conclude $\deg_y(\sigma_2) = 0$ and $\deg_y(\sigma_1) \leq 2$. Since y divides the left hand side, we conclude that $y|\sigma_1$, thus $\sigma_1 = y^2\sigma_3$ for some $\sigma_3 \in \mathbb{R}[x]$. Comparing the coefficients of y on both sides, we get

$$1 = -\sigma_2 b(x).$$

So $b(x) \in (\mathbb{R}[x])^* = \mathbb{R}$. A contradiction.

(ii) Suppose $(y+1)(y-b(x)) \in PO(S)$.
Then there exist $\sigma_1, \sigma_2 \in PO(\{s_1, \dots, s_k\}) \subseteq \mathbb{R}[x, y]$ such that

$$(y+1)(y-b(x)) = \sigma_1 + y(y-b(x))\sigma_2.$$

By lemma 3.1 we have $\deg_y(\sigma_2) = 0$ and $\deg_y(\sigma_1) \leq 2$.
Since $(y - b(x))$ divides the left hand side, we conclude that

$$(y - b(x))|\sigma_1,$$

so $\sigma_1 = (y - b(x))^2\sigma_3$ for some $\sigma_3 \in \mathbb{R}[x]$. Comparing the constant coefficients in the equation, we conclude

$$\begin{aligned} -b(x) &= b(x)^2\sigma_3 \\ \Leftrightarrow 1 &= -b(x)\sigma_3, \end{aligned}$$

so $b \in \mathbb{R}[x]^* = \mathbb{R}$. A contradiction. $\square$

Proposition 3.33
Suppose $b \in \mathbb{R}[x]$ satisfies $b < -1$ on K. Define

$$S := \{s_1, \ldots, s_k, y(y - b(x)), y(y + 1), (y + 1)(y - b(x))\}.$$

Then $PO(S)$ is saturated.

Proof — Define
$$S' := \{s_1, \ldots, s_k, y(y - b(x))\}.$$
Since $b(x) < -1$, we conclude $y(y+1) \geq 0$ on $K_{S'}$ and $(y+1)(y-b(x)) \geq 0$ on $K_{S'}$, thus $K_{S'} = K_S$. Let

$$f = \sum_{i=0}^{d} a_i y^i \in \mathbb{R}[x, y]$$

with $f \geq 0$ on $K_S = K_{S'}$. Then d is even by lemma 2.34. We have to show $f \in PO(S)$.
If $d = 0$, then $f \in \mathbb{R}[x]$ and $f \geq 0$ on K. Since S contains the natural choice of generators for K, we have $f \in PO(S)$, so we may assume $d \geq 2$.
Define
$$\tilde{f}(x, y, z) := z^d f(x, \frac{y}{z}) = \sum_{i=0}^{d} a_i(x) y^i z^{d-i} \in \mathbb{R}[x, y, z].$$
Then $\tilde{f} \geq 0$ on the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid s_1(x) \geq 0, \ldots, s_k(x) \geq 0, y(y - b(x)z) \geq 0\}.$$

Substituting $y = 1$, we get $\tilde{f}(x, 1, z) \geq 0$ on the set

$$\{(x, z) \in \mathbb{R}^2 \mid s_1(x) \geq 0, \ldots, s_k(x) \geq 0, (1 - b(x)z) \geq 0\} =: A.$$

Since $b < -1$ on K, b has no zeros in K and $1+z \geq 0$ on A. By proposition 3.25 the preordering generated by $\{s_1, \ldots, s_k, 1-b(x)z, z+1\}$ is saturated (see remark 3.26), so there exist $\sigma_i \in \sum \mathbb{R}[x,z]^2$ ($i \in \{0,1\}^{s+2}$) with

$$\tilde{f}(x,1,z) = \sum_{i \in \{0,1\}^{s+2}} \sigma_i(x,z)(1-b(x)z)^{i_{s+1}}(z+1)^{s+2} \prod_{j=1}^{s} s_j^{i_j}.$$

So we compute

$$\begin{aligned}
f(x,y) &= \tilde{f}(x,y,1) \\
&= y^d \tilde{f}(x,1,\frac{1}{y}) \\
&= \sum_{i \in \{0,1\}^{s+2}} y^d \sigma_i(x,\frac{1}{y})(1-b(x)\frac{1}{y})^{i_{s+1}}(1+\frac{1}{y})^{i_{s_2}} \prod_{j=1}^{s} s_j^{i_j}(x) \\
&= \sum_{i \in \{0,1\}^{s}} y^d \sigma_{(i,0,0)}(x,\frac{1}{y}) \\
&+ \sum_{i \in \{0,1\}^{s}} y^{d-2} \sigma_{(i,1,0)}(x,\frac{1}{y}) y(y-b(x)) \\
&+ \sum_{i \in \{0,1\}^{s}} y^{d-2} \sigma_{(i,0,1)}(x,\frac{1}{y}) y(y+1) \\
&+ \sum_{i \in \{0,1\}^{s}} y^{d-2} \sigma_{(i,1,1)}(x,\frac{1}{y})(y+1)(y-b(x)).
\end{aligned}$$

Since

$$d \geq \deg_z(\tilde{f}(x,1,z)) \geq \deg_z(\sigma_i(1-b(x)z)^{i_{s+1}})(1+z)^{i_{s+2}} \; (\forall i \in \{0,1\}^{s+2})$$

by lemma 3.1, we conclude

$$y^d \sigma_{(i,0,0)}(x,\frac{1}{y}), y^{d-2}\sigma_{(i,1,0)}(x,\frac{1}{y}), y^{d-2}\sigma_{(i,0,1)}(x,\frac{1}{y}), y^{d-2}\sigma_{(i,1,1)}(x,\frac{1}{y}) \in \sum \mathbb{R}[x,y]^2$$

for all $i \in \{0,1\}^s$, so $f \in PO(S)$. □

Remark 3.34

(i) By lemma 3.32, the preordering $PO(S')$ in the proof above is not saturated. So it is necessary to add at least one of the terms above.

(ii) Given any $a, b \in \mathbb{R}[x]$. Whenever $a > b$ (or $a < b$) on K, the preordering generated by $S = \{s_1 \ldots, s_r, (y-a)(y-b)\}$ can be brought to the form above ($a = 0$ and $b < -1$) by a linear change of coordinates.

(iii) If we only have $0 = a \geq b$ above, then in the proof above the corresponding preordering in $\mathbb{R}[x, z]$ is generated by

$$S' = \{s_1, \ldots, s_k, 1 - b(x)z\}.$$

It is not known whether or not this preordering is saturated (see remark 3.26).

(iv) If b has no zero in K, but changes its sign on K, then the corresponding preordering in $\mathbb{R}[x, z]$ is generated by $S' = \{s_1, \ldots, s_k, 1 - b(x)z\}$; therefore, it is not of our desired form. Again, it is not known if such a preordering is saturated (see remark 3.26).

Definition 3.35
For $f = ay^2 + by + c$ with $a, b, c \in \mathbb{R}[x]$, define

$$d(x) := b^2 - 4ac$$

the discriminant of f.

Lemma 3.36
Let $f := ay^2 + by + c$ with $a, b, c \in \mathbb{R}[x]$, $a(0) > 0$. Suppose f is irreducible in $\mathbb{R}[x, y]$. Let d be the discriminant of f.

(i) If $d(0) > 0$, then f is reducible in $\mathbb{R}\{x\}[y]$.

(ii) If $d(0) < 0$, then $f \in \sum \mathbb{R}\{x\}[y]^2$.

Proof — Since $a(0) > 0$, $a \in \mathbb{R}\{x\}^*$ by proposition 2.45.

(i) If $d(0) > 0$, then there exists an $u \in \mathbb{R}\{x\}$ such that $d = u^2$ by proposition 2.45. So we compute

$$\begin{aligned}(y - \frac{-b+u}{2a})(y - \frac{-b-u}{2a}) &= (y - \frac{-b}{2a})^2 - \frac{u^2}{4a^2} \\ &= y^2 + \frac{b}{a}y + \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} \\ &= y^2 + \frac{b}{a}y + \frac{c}{a} \\ &= \frac{1}{a}f.\end{aligned}$$

Thus f is reducible in $\mathbb{R}\{x\}[y]$.

(ii) If $d(0) < 0$, then $-d = u^2$ for some $u \in \mathbb{R}\{x\}^*$ by proposition 2.45, so $d = (iu)^2$. The same calculation as before shows that f is reducible in $\mathbb{C}\{x\}[y]$. That f is irreducible in $\mathbb{R}\{x\}[y]$ is obvious, since $f(0, y)$ is irreducible in $\mathbb{R}[y]$. □

Proposition 3.37
Let $g := ay^2 + by + c$ *with* $a, b, c \in \mathbb{R}[x]$. *Suppose*

(i) $a > 0$ *on* K,

(ii) g *is irreducible in* $\mathbb{R}[x, y]$

(iii) the discriminant $d(x) = b^2 - 4ac$ *of* g *has no zeros in* K.

Then the preordering generated by $S := \{s_1, \ldots, s_k, g\} \subseteq \mathbb{R}[x, y]$ *is saturated.*

Proof — Since $a > 0$, there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

by lemma 3.23.
Let $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S. By lemma 2.34 $\deg_y(f)$ is even, say $\deg_y(f) = 2d$.
We may assume f is squarefree and $g \nmid f$. Then f has only finitely many zeros in $K_S \backslash (\bigcup_{i=1}^{k} V_{\mathbb{R}}(s_i))$ (since g is irreducible) by lemma 2.43, so $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.
By lemma 2.30, there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(1 + y^2)^d \geq 0 \text{ on } K_S$$

with only finitely many zeros in K_S.
By theorem 2.10, we have to show $f_1 \in PO_P(S)$ for every $P \in K$.
Let $P \in K$. Without loss of generality, we may assume $P = 0$. Since $a(0) > 0$, we have $a \in (\mathbb{R}\{x\}^*)^2$ by proposition 2.45. So we may assume $a = 1$.

(i) Case 1: $d(0) < 0$:
Then $g \in \sum \mathbb{R}\{x\}[y]^2$ by lemma 3.36.

First let $0 = P \in int(K)$.
By lemma 2.71, $f_1 > 0$ at every ordering of K_0 that satisfies $s_i > 0$ $(i = 1, \ldots, r)$ and $g > 0$. Since $s_i \in \mathbb{R}\{x\}^2$ $(i = 1, \ldots, r)$ by lemma 3.22 and $g \in \sum \mathbb{R}\{x\}[y]^2$, $f_1 > 0$ at every ordering of K_0 and we conclude

$$f_1 \in \sum \mathbb{R}\{x\}[y]^2 \subseteq PO_0(S)$$

by proposition 2.76.
Now let $0 = P \in \delta K$.
By lemma 2.43, we may assume $s_2, \ldots, s_r \in \mathbb{R}\{x\}^2$, $s_1 = \epsilon x u^2$ for an $u \in \mathbb{R}\{x\}^*$ and $\epsilon \in \{-1, 1\}$. Replacing x by $-x$ if necessary, we may assume $\epsilon = 1$.
By lemma 2.71, $f_1 > 0$ at every ordering of K_0 that satisfies $s_i > 0$ $(i = 1, \ldots, r)$ and $g > 0$. Again, $s_2, \ldots, s_r, g > 0$ at every ordering of K_0 and $s_1 > 0$ at every ordering of K_0 that satisfies $x > 0$. So $f_1 > 0$ at every ordering of K_0 that satisfies $x > 0$, thus

$$f_1 \in PO_0(\{x\}) = PO_0(\{s_1\}) \subseteq PO_0(S)$$

by corollary 2.76.

(ii) Case 2: $d(0) > 0$.
Then g is reducible in $\mathbb{R}\{x\}[y]$ by lemma 3.36, i.e.

$$g = (y - v_1)(y - v_2)$$

with $v_1, v_2 \in \mathbb{R}\{x\}$. Since $d(0) > 0$, we have

$$v_1(0) \neq v_2(0),$$

so we may assume $v_1(0) > v_2(0)$.
First let $0 = P \in int(K)$.
By lemma 2.71, $f_1 > 0$ at every ordering of K_0 that satisfies $s_i > 0$ $(i = 1, \ldots, r)$ and $g > 0$. Since $s_i \in \mathbb{R}\{x\}^2$ $(i = 1, \ldots, r)$ by lemma 3.22, $f_1 > 0$ at every ordering of K_0 that satisfies $g > 0$, so

$$f_1 \in PO_0(\{g\}) \subseteq PO_0(S)$$

by proposition 2.80.
Now let $0 = P \in \delta K$.

By lemma 2.43, we may assume $s_2, \ldots, s_r \in \mathbb{R}\{x\}^2$, $s_1 = \epsilon x u^2$ for an $u \in \mathbb{R}\{x\}^*$ and $\epsilon \in \{-1, 1\}$. Replacing x by $-x$ if necessary, we may assume $\epsilon = 1$.
By lemma 2.71, $f_1 > 0$ at every ordering of K_0 that satisfies $s_i > 0$ $(i = 1, \ldots, r)$ and $g > 0$. Again, $s_2, \ldots, s_r > 0$ at every ordering of K_0 and $s_1 > 0$ at every ordering of K_0 that satisfies $x > 0$. So $f_1 > 0$ at every ordering of K_0 that satisfies $x > 0$ and $g > 0$. Thus

$$f_1 \in PO_0(\{x, g\}) = PO_0(\{s_1, g\}) \subseteq PO_0(S)$$

by proposition 2.80. □

In the notation of the last proposition, if the discriminant d vanishes at some point $P \in K$ we have the following partial result:

Proposition 3.38
Let $g := ay^2 + by + c \in \mathbb{R}[x, y]$ with $a, b, c \in \mathbb{R}[x]$. Suppose

(i) $a > 0$ on K,

(ii) g is irreducible in $\mathbb{R}[x, y]$.

Then the preordering generated by

$$S := \{s_1, \ldots, s_k, g\} \subseteq \mathbb{R}[x, y]$$

has property (†).

Proof — Let $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S. By lemma 2.34 $\deg_y(f)$ is even, say $\deg_y(f) = 2d$. By lemma 3.23 there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}.$$

Now let $\delta > 0$ and define

$$f_1 := f + \delta.$$

Then $f_1 > 0$ on K_S. So by lemma 2.31, there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_2 := f_1 - \varepsilon^2(1 + y^2)^d > 0 \text{ on } K_S.$$

By theorem 2.10 we have to show that $f_2 \in PO_P(S)$ for every $P \in K$.
Let $P \in K$. If $d(P) \neq 0$ we can argue exactly the same as in proposition

3.37 to show that $f_2 \in PO_P(S)$. So assume $d(P) = 0$.
Without loss of generality, we may assume $P = 0$.
Since $d(0) = 0$, we have $g(0, y) \in \mathbb{R}[y]^2$, thus

$$\{0\} \times \mathbb{R} \subseteq K_S.$$

Since $f_2 > 0$ on K_S, f_2 has no zeros on the line $\{0\}\times\mathbb{R}$. So every irreducible factor of f_2 in $\mathbb{R}\{x\}[y]$ is reducible in $\mathbb{C}\{x\}[y]$ (since every irreducible factor of f, which is irreducible in $\mathbb{C}\{x\}[y]$, has a zero on the line $\{0\}\times\mathbb{R}$ by lemma 2.61). So each irreducible factor of f_2 is a sum of squares by lemma 2.52. Thus

$$f_2 \in \sum \mathbb{R}\{x\}[y]^2 \subseteq PO_0(S).$$

□

Remark 3.39

(i) Whether or not such a preordering (with $d(P) = 0$ for some $P \in K$) is saturated is not known. In this situation, one can show that g is also irreducible in $\mathbb{C}\{x - P\}[y]$ for any P with $d(P) = 0$, so this case isn't covered by proposition 2.80. See example 3.85 for more details.

(ii) In the case where $S = \{s_1, \ldots, s_k, g\}$ and $\deg_y(g) \geq 3$, we have the following negative result:

Proposition 3.40
Let

$$g := \sum_{i=0}^{d} a_i y^i$$

with $a_i \in \mathbb{R}[x]$ *(*$i = 1, \ldots, d$*),* $a_d > 0$ *on* K*.*
Define $S := \{s_1, \ldots, s_n, g\}$*. If* $d \geq 3$ *and* $g \notin PO(\{s_1, \ldots, s_k\})$*, then* $PO(S)$ *does not satisfy* (SMP)*.*

Proof — Suppose $P = PO(S)$ satisfies (SMP). Since $a_d > 0$ on K, we are either in the strip case or in the half-strip case by lemma 3.23. In the notation of proposition 3.19, we have $S = \{s_1, \ldots, s_n, g\}$ and $S_2 = \{s_1, \ldots, s_n\}$.

$$A := K_{S_2} \setminus K_S = (K \times \mathbb{R}) \setminus K_S$$

is an open set in $K \times \mathbb{R}$. If $A \neq \varnothing$, then $\pi_1(A) \subseteq K$ is open. Since K contains no isolated points, $\pi_1(A) \subseteq K$ is not finite. A contradiction to proposition 3.8.
Thus $A = \varnothing$, $K_{S_2} = K_S$ and

$$g \geq 0 \text{ on } K_{S_2}.$$

By proposition 1.12 $PO(S_2)$ is saturated, so $g \in PO(S_2)$. A contradiction.□

Whether or not the saturation is finitely generated depends on the set K_S. Let A be the euclidean boundary of K_S and let $Z_1, \ldots, Z_s$ be the irreducible components of the Zariski closure of A. Each of these is either a point or the zero set of some polynomial $g \in \mathbb{R}[x, y]$. We get the following results:

Proposition 3.41
In the above setting, if there exists an i such that $Z_i = V(f)$ for an $f \in \mathbb{R}[x, y]$ with $\deg_y(f) \geq 3$, then there exists no finite S' with $K_{S'} = K_S$ such that $PO(S')$ satisfies (SMP).

Proof — Let $Z_1 = V(f)$ with $f \in \mathbb{R}[x, y]$ irreducible and $\deg_y(f) \geq 3$. Suppose there exists an $S' \subseteq \mathbb{R}[x, y]$ finite with $K_{S'} = K_S$ such that $P := PO(S')$ satisfies (SMP). Let

$$S'_2 := \{s' \in S' \ : \ \deg_2(s') \leq 2\}.$$

Then

$$K_{S'_2} \setminus K_{S'} = K_1 \subseteq K$$

is finite by lemma 3.8. Since

$$A \subseteq \bigcup_{s' \in S'_2} V_{\mathbb{R}}(s')$$

and since S'_2 is finite, there exists an $s' \in S'_2$ such that

$$V_{\mathbb{R}}(s') \cap Z_i = V_{\mathbb{R}}(s') \cap V(f)$$

is not finite. Then $f|s'$ by lemma 2.36, since f is irreducible. So

$$3 \leq \deg_y(f) \leq \deg_y(s') = 2,$$

a contradiction. □

Lemma 3.42
Let $g \in \mathbb{R}[x,y]$ *be irreducible with* $\deg_y(g) \geq 3$ *and* $S := \{s_1, \dots, s_n, g\}$. *Write*

$$g = \sum_{i=0}^{d} a_i y^i$$

with $a_0, \dots, a_d \in \mathbb{R}[x]$. *Suppose* $a_d > 0$ *on* K *and* $g \notin PO(\{s_1, \dots, s_k\})$. *Let* A *be the euclidean boundary of* K_S, *then* $V_{\mathbb{R}}(g) \cap A$ *is not finite.*

Proof — Define $S' = \{s_1, \dots, s_r\}$. Since $a_d > 0$ on K, there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq \mathbb{R} \times K$$

by lemma 3.23. $PO(S')$ is saturated by 1.12 and $g \notin PO(S')$, so there exists an $P \in int(K) \times \mathbb{R}$ with $g(P) < 0$. Let $I \subseteq K$ be an interval nonempty interior such that $P \in I \times \mathbb{R}$.
Then we have $g(i, N) \geq 0$ for all $i \in I$ and $g(P) < 0$, so g has a zero on the line l_i between P and (i, N) for each $i \in I$. Especially g has infinitely many zeros in $int(K \times \mathbb{R})$. g changes its sign on each of these zeros by lemma 2.36, so they are on the boundary of K_S, thus $A \cap V_{\mathbb{R}}(g)$ is not finite. □

Corollary 3.43
Let $g \in \mathbb{R}[x,y]$ *be irreducible with* $\deg_y(g) \geq 3$ *and* $S := \{s_1, \dots, s_n, g\}$. *Write*

$$g = \sum_{i=0}^{d} a_i y^i$$

with $a_0, \dots, a_d \in \mathbb{R}[x]$ *and suppose* $a_d > 0$. *If* $g \notin PO(\{s_1, \dots, s_k\})$, *then there exists no finite* $S' \subseteq \mathbb{R}[x,y]$ *with* $K_{S'} = K_S$ *such that* $PO(S')$ *satisfies* (SMP).

Proof — By lemma 3.23 there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R} = K_{\{s_1, \dots, s_r\}}.$$

Let A be the euclidean boundary of K_S and let $Z_1, \dots, Z_s$ be the irreducible components of the Zariski closure of A. By lemma 3.42, $V_{\mathbb{R}}(g) \cap A$ is not finite, so there exists an $i \in \{1, \dots, s\}$ such that $V_{\mathbb{R}}(g) \cap Z_i$ is not finite, so $V(g) = Z_i$. So the corollary follows from proposition 3.41. □

3.3 Preorderings that Satisfy (†)

In this section, we will discuss preorderings in a strip generated by more than one extra element. Again, $K \subseteq \mathbb{R}$ will be a compact set without isolated points,

$$\begin{aligned} \pi_1 : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto x \end{aligned}$$

the projection and
$$K_P := Quot(\mathbb{R}\{x-P\}[y]).$$

The goal of this section is to improve the results of Kuhlmann, Marshall and Schwartz (corollary 3.7) in our more specific setup by using theorem 2.10 and theorem 2.13.
In section 3.1, we have already proven a necessary condition for such an improvement, namely that $K_S = K_{S_2}$ (see lemma 3.9).
In section 3.1, we also proved that for this, it is sufficient to assume that

$$(x-P) \nmid s \text{ for all } s \in S_2 \setminus \mathbb{R}[x].$$

So given any $S \subseteq \mathbb{R}[x,y]$. If we want to prove that $PO(S)$ satisfies (†), we have to prove that any $f \in \mathbb{R}[x,y]$ with $f > 0$ on K_S lies in $PO(S)$. The basic idea we are going to use here, is to first use theorem 2.10 or theorem 2.13 to work in the ring of convergent power series $\mathbb{R}\{x\}[y]$ instead of the polynomial ring $\mathbb{R}[x,y]$, then use the theory in section 2.4.2 to factorize f in $\mathbb{R}\{x\}[y]$ and handle each irreducible factor of f in $\mathbb{R}\{x\}[y]$ separately.
Therefore, we first need several lemmas:

Lemma 3.44
Let $v \in \mathbb{R}\{x\}$ and $p \in \mathbb{R}\{x\}[y]$ be monic with

$$p(0,y) > 0 \text{ on } K_{\{y-v(0)\}} \subseteq \mathbb{R},$$

then $p > 0$ at every ordering of K_0 that satisfies $y - v > 0$.

Proof — By a linear change of coordinates, we may assume $v = 0$.
Factorize
$$p = \prod_{i=1}^{s} p_i$$

with $p_i \in \mathbb{R}\{x\}[y]$ irreducible and monic. It is enough to show that $p_i \in PO_0(\{y\})$, so we may assume $s = 1$ and p is irreducible in $\mathbb{R}\{x\}[y]$.
If p is reducible in $\mathbb{C}\{x\}[y]$, then

$$p \in \sum \mathbb{R}\{x\}[y]^2 \subseteq PO_0(S)$$

by lemma 2.52. So assume p is irreducible in $\mathbb{C}\{x\}[y]$. Then

$$p(0, y) = (y - a)^n \tag{3.2}$$

with $n = \deg_y(p)$ and $a \in \mathbb{R}$ by lemma 2.61. By our assumption we have $a < 0$, so $-a > 0$.
Now write

$$p = \sum_{i=0}^{n} a_i y^i$$

with $a_i \in \mathbb{R}\{x\}$. From equation 3.2 we conclude

$$a_i(0) = \binom{i}{n}(-a)^i > 0,$$

thus $a_i \in \mathbb{R}\{x\}^2$ by proposition 2.45 and

$$p = \sum_{i=0}^{n} a_i y^i \in PO_0(\{y\}).$$

Thus, $p > 0$ at every ordering of K_0 that satisfies $y > 0$. □

Lemma 3.45
Let $v \in \mathbb{R}\{x\}$ and $p \in \mathbb{R}\{x\}[y]$ be monic with

$$p(0, y) > 0 \text{ on } K_{\{v(0)-y\}} \subseteq \mathbb{R},$$

then $p > 0$ at every ordering of K_0 that satisfies $y - v < 0$.

Proof — Let $<$ be an ordering of K_0 that satisfies $y - v < 0$ and let $<_1$ be the ordering of K_0 defined by

$$p(x, y) <_1 q(x, y) \Leftrightarrow p(x, -y) < q(x, -y).$$

Then we have

$$y + v = -(-y - v) >_1 0.$$

Further, we have

$$p(0,-y) > 0 \text{ on } K_{\{v(0)-(-y)\}} = K_{\{y+v(0)\}} \subseteq \mathbb{R},$$

so $p(x,-y) >_1 0$ by lemma 3.44, thus $p > 0$. Since $<$ was arbitrary, the proof is finished. □

Lemma 3.46
Let

$$s = (y - v_1)(y - v_2) \in \mathbb{R}\{x\}[y]$$

with $v_1, v_2 \in \mathbb{R}\{x\}$ *and* $v_1(0) > v_2(0)$. *Suppose* $p \in \mathbb{R}\{x\}[y]$ *with*

(i) $p > 0$ *at every ordering of* K_0 *that satisfies* $y - v_1 > 0$ *and*

(ii) $p > 0$ *at every ordering of* K_0 *that satisfies* $y - v_2 < 0$.

Then $p > 0$ *at every ordering* K_0 *that satisfies* $s > 0$.

Proof — Let $<$ be an ordering of K_0 that satisfies $s > 0$. Since

$$s = (y - v_1)(y - v_2),$$

we have either $(y - v_1), (y - v_2) > 0$ or $(y - v_1), (y - v_2) < 0$. By our assumption we have $p > 0$ in both cases. □

Lemma 3.47
Let

$$s = (y - v_1)(y - v_2) \in \mathbb{R}\{x\}[y]$$

with $v_1, v_2 \in \mathbb{R}\{x\}$ *and* $v_1(0) > v_2(0)$. *Let* $p \in \mathbb{R}\{x\}[y]$ *be monic. Define*

$$s_0 := s(0, y) \in \mathbb{R}[y].$$

If

$$p(0, y) > 0 \textit{ on } K_{\{s_0\}} \subseteq \mathbb{R},$$

then $p > 0$ *at every ordering of* K_0 *that satisfies* $s > 0$.

Proof —

$$s(0, y) = (y - v_1(0))(y - v_2(0)),$$

so

$$K_{\{s_0\}} = K_{\{y - v_1(0)\}} \cup K_{\{v_2(0) - y\}},$$

thus we conclude $p > 0$ at every ordering of K_0 that satisfies $y - v_1 > 0$ by lemma 3.44 and $p > 0$ at every ordering of K_0 that satisfies $y - v_2 < 0$ by lemma 3.45. So $p > 0$ at every ordering of K_0 that satisfies $s > 0$ by lemma 3.46. $\square$

We are now able to improve corollary 3.7 in the strip and the half-strip case. As said before, the main idea of the proof is to factorize $f \in \mathbb{R}[x, y]$ into irreducibles in $\mathbb{R}\{x\}[y]$ and handle each irreducible factor separately. We will do this in the next several lemmas:

Lemma 3.48
Let $S \subseteq \mathbb{R}[x, y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x, y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $f > 0$ on K_S,

Let $P \in K$ and $p \in \mathbb{R}\{x - P\}[y]$ be an irreducible factor of f with $\deg_y(p) \geq 1$. Suppose

a) p is irreducible in $\mathbb{C}\{x - P\}[y]$,

b) p is monic,

c) $\deg_y(p)$ is even.

Then $p \in PO_P(S)$.

Proof — We may assume $P = 0$.
By lemma 2.61 we have

$$p(0, y) = (y - a)^n$$

for some $a \in \mathbb{R}$. Since $f > 0$ on K_S, we conclude $a \notin \lambda_0^{-1}(K_S)$. By lemma 3.16 $\lambda_0^*(S_2)$ contains the natural generators for $\lambda_0^{-1}(K_S)$, so there exists an $s \in S_2$ such that

- $\lambda_0^*(s) = s(0, y)$ is one of the natural generators for $\lambda_0^{-1}(K_S)$,

- $s(0, a) < 0$.

Then $s(0, y)$ has exactly 2 distinct zeros in $\lambda_0^{-1}(K_S) \subseteq \mathbb{R}$, so s is reducible in $\mathbb{R}\{x\}[y]$ and the leading coefficient of s as a polynomial y over $\mathbb{R}\{x\}$ is a unit. Write

$$s = v(y - u_1)(y - u_2),$$

with $u_1, u_2 \in \mathbb{R}\{x\}$, $v \in \mathbb{R}\{x\}^*$. By definition of the natural generators, we have $v(0) > 0$ and $u_1(0) \neq u_2(0)$, thus we may assume $u_1(0) < u_2(0)$. Since $v(0) > 0$, we have $v = u^2$ for some $u \in \mathbb{R}\{x\}^*$ and since

$$PO_0(\{s\}) = PO_0(\{u^{-2}s\}),$$

we may assume $u = v = 1$.
Since $a \notin \lambda_0^{-1}(K_S)$, we conclude $a \in (u_1(0), u_2(0))$ and

$$(y - a)^n = p(0, y) > 0 \text{ on } K_{\{s(0,y)\}}.$$

So $p > 0$ at every ordering of K_0 that satisfies $s > 0$ by lemma 3.47 and

$$p \in PO_0(\{s\}) \subseteq PO_0(S)$$

by proposition 2.80. □

Lemma 3.49
Let $S \subseteq \mathbb{R}[x, y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x, y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $f > 0$ on K_S,

Let $P \in K$ and $p \in \mathbb{R}\{x-P\}[y]$ be an irreducible factor of f. Suppose

a) p is irreducible in $\mathbb{C}\{x-P\}[y]$,

b) p is monic,

c) $n = \deg_y(p)$ is odd.

Then there exists another monic irreducible factor $q \in \mathbb{R}\{x-P\}[y]$ of f such that $pq \in PO_P(S)$.

Proof — We may assume $P = 0$.
By lemma 2.61

$$p(0,y) = (y-a)^n$$

for some $a \notin \lambda_0^{-1}(K_S)$.
By lemma 3.16 $\lambda_0^*(S_2)$ contains the natural generators for $\lambda_0^{-1}(K_S)$, so there exists an $s \in S_2$ such that

- $\lambda_0^*(s) = s(0,y)$ is one of the natural generators for $\lambda_0^{-1}(K_S)$

- $s(0,a) < 0$.

So again, s is reducible in $\mathbb{R}\{x\}[y]$.
Write

$$s = v(y-u_1)(y-u_2),$$

with $u_1, u_2 \in \mathbb{R}\{x\}$, $v \in \mathbb{R}\{x\}^*$. By definition of the natural generators, we have $u_1(0) \neq u_2(0)$ and $v(0) > 0$, so we may again assume $u_1(0) > u_2(0)$ and $v = 1$ (compare the last proof).
Since $n = \deg_y(p)$ is odd, $p(0,y) = (y-a)^n$ changes its sign at a. But

$$f(0,u_1(0)), f(0,u_2(0)) > 0$$

and $u_1(0) > a > u_2(0)$ by construction, so there exists another irreducible factor $q \in \mathbb{R}\{x\}[y]$ of f such that $q(0,y)$ changes its sign at $b \in (u_2(0), u_1(0))$. Then the leading coefficient of q is a unit in $\mathbb{R}\{x\}$ (compare lemma 2.60 and lemma 2.62).
So we may assume q is monic and

$$q(0,y) = (y-b)^m$$

with $m = \deg_y(q)$ odd. Then

$$p(0,y)q(0,y) > 0 \text{ on } K_{s(0,y)}$$

by construction, so $pq > 0$ at every ordering of K_0 that satisfies $s > 0$ by lemma 3.47, and

$$pq \in PO_0(\{s\}) \subseteq PO_0(S) \subseteq \mathbb{R}\{x\}[y]$$

by proposition 2.80. □

Remark 3.50

(i) Note that the proof of the last lemma shows even more about the second irreducible factor:
Assume $P = 0$ and let $p \in \mathbb{R}\{x\}[y]$ as above with $p(0,y) = (y-a)^n$. Let $c, d \in \lambda_0^{-1}(K_S)$ such that

- $a \in (c,d)$,

- $\lambda_0^{-1}(K_S) \subseteq (\infty, c] \cup [d, \infty)$.

Then the other irreducible factor $q \in \mathbb{R}\{x\}[y]$ satisfies

$$q(0,y) = (y-b)^m,$$

where $m = \deg_y(q)$ is odd and $b \in (c,d)$. Since $f(0,c), f(0,d) > 0$, there exists an even number of such irreducible factors with odd degree. So the last lemma shows inductively that the product of all irreducible factors of f of odd degree lies in $PO_0(S)$.

(ii) Here it is a huge advantage that we work in the ring $\mathbb{R}\{x\}[y]$ instead of the ring $\mathbb{R}[x,y]$. In $\mathbb{R}\{x\}[y]$ we always need only one element of S to show that an irreducible factor p of f in $\mathbb{R}\{x-P\}[y]$ lies in $PO_P(S)$ $(P \in K)$. In the polynomial ring $\mathbb{R}[x,y]$, this is not true in general.

(iii) Also note that the natural generators $\{s_1, \ldots, s_r\}$ are not needed to show that a monic irreducible factor p of f in $\mathbb{R}\{x-P\}[y]$ lies in $PO_0(S)$. However, we will need them to handle irreducible factors whose leading coefficient are not a unit. These irreducible factors will appear whenever the leading coefficient of f as a polynomial in y over $\mathbb{R}[x]$ vanishes at a point $P \in K$.

(iv) In his paper [14], Marshall used a method for $K = [0,1]$ to reduce to the case where the leading coefficient of f has no zeros in K. His method, however, does not seem to work if K is a compact set which is not an interval (compare lemma 2.43).

We will now handle those irreducible factors whose leading coefficients are not monic. Since we will need these lemmas again in the next section, we will prove a slightly more general version than we actually need here:

Lemma 3.51
Let $S \subseteq \mathbb{R}[x,y]$ and $f \in \mathbb{R}[x,y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) $f \geq 0$ on K_S,

(iii) $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ is finite.

Let $P \in int(K)$ and $p \in \mathbb{R}\{x-P\}[y]$ be an irreducible factor of f. If p is irreducible in $\mathbb{C}\{x-P\}[y]$, then the leading coefficient of p is a unit.

Proof — We may assume $P = 0$. Suppose the leading coefficient of p is a not unit, then by lemma 2.58 and lemma 2.56 p (and also f) vanishes near 0 on a set of the form

$$V_\delta = \{(\epsilon(x)^n, (x)^{-r}g(x)) \mid x \neq 0,\ |x| < \delta\}$$

for $\epsilon \in \{-1,1\}$, $g \in \mathbb{R}\{x\}[y]$ and $\delta > 0$. Since $K \times \{y^2 \geq N\} \subseteq K_S$, this set lies in K_S for δ small enough.
A contradiction, since $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ is finite. □

Lemma 3.52
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$, $f \in \mathbb{R}[x,y]$ and $P \in int(K)$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) *for every* $P \in K$, $\lambda_P^*(S)$ *contains the natural choice of generators for* $\lambda_P^{-1}(K_S)$ *(up to scaling by positive reals),*

(iii) $(x - P) \nmid s$ *for all* $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $f > 0$ *on* K_S.

Then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$. We have to show that $f \in PO_0(S)$.
Factorize

$$f = u(\prod_{i=1}^{s} p_i)(\prod_{j=1}^{r} q_i),$$

where

- $p_i \in \mathbb{R}\{x\}[y]$ are irreducible in $\mathbb{C}\{x\}[y]$ $(i = 1, \ldots, s)$,

- $q_j \in \mathbb{R}\{x\}[y]$ are irreducible in $\mathbb{R}\{x\}[y]$ and reducible in $\mathbb{C}\{x\}[y]$ $(j = 1, \ldots, r)$,

- $u \in \mathbb{R}\{x\}^*$.

By lemma 3.51 the leading coefficient of every p_i is a unit in $\mathbb{R}\{x\}$, so we may assume all p_i are monic. Since $PO_0(S)$ is closed under multiplication, we conclude

$$p = \prod_{i=1}^{s} p_i \in PO_0(S)$$

by lemma 3.48 and lemma 3.49 (compare remark 3.50).
Since $u \in \mathbb{R}\{x\}^*$, we have either $u \in \mathbb{R}\{x\}^2$ or $-u \in \mathbb{R}\{x\}^2$ by lemma 2.45. For each q_j $(j = 1, \ldots, r)$, we have either $q_j \in \sum \mathbb{R}\{x\}[y]^2$ or $-q_j \in \sum \mathbb{R}\{x\}[y]^2$ by lemma 2.52. So, since $PO_0(S)$ is closed under multiplication, we conclude that either $f \in PO_0(S)$ or $-f \in PO_0(S)$.
Since $-f < 0$ on K_S, we conclude $f \in PO_0(S)$. □

Lemma 3.53
Let $S \subseteq \mathbb{R}[x, y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$, $f \in \mathbb{R}[x, y]$ *and* $P \in \delta K$. *Suppose*

(i) there exists an $N \in \mathbb{N}$ *such that*

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ *contains the natural choice of generators for* $\lambda_P^{-1}(K_S)$ *(up to scaling by positive reals),*

(iii) $(x - P) \nmid s$ *for all* $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $S \cap \mathbb{R}[x]$ *contains the natural choice of generators for* K,

(v) $f > 0$ *on* K_S,

Then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$ and $[0, \varepsilon] \subseteq K$ for $0 < \varepsilon \in \mathbb{R}$.
By lemma 3.22, there exists an $s \in S$ such that $s = xv^2$ for a $v \in \mathbb{R}\{x\}^*$. Since $PO(\{x\}) = PO(\{xv^2\})$, we may assume $v = 1$.
Define

$$A = \mathbb{R}\{x\}[y,t]/(t^2 - x) = \mathbb{R}\{t\}[y]$$

and let P be the preordering generated by $S \setminus \{s\}$ in A. By lemma 2.73 we have to show $f(t^2, y) \in P$.
Note that P in $\mathbb{R}\{t\}[y]$ is generated by

$$S' = \{s(t^2, y) \mid s \in S\} \subseteq \mathbb{R}[t, y].$$

Now, work in the polynomial ring $\mathbb{R}[t, y]$ and let $P' := PO(S') \subseteq \mathbb{R}[t, y]$. Then $0 \in int(\pi_1(K_{S'}))$, so

$$f(t^2, y) \in PO_0(S') = P \subseteq \mathbb{R}\{t\}[y]$$

by lemma 3.52. □

Theorem 3.54
Let $S \subseteq \mathbb{R}[x, y]$ *and* $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. *Suppose*

(i) there exists an $N \in \mathbb{N}$ *such that*

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) *for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),*

(iii) $(x - P) \nmid s$ *for all* $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) *S contains the natural choice of generators for K,*

then $PO(S)$ satisfies (†).

Proof — Suppose $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S. Then $\deg_y(f)$ is even by lemma 2.34, say $\deg_y(f) = 2d$. Fix $\delta > 0$, then $f_1 := f + \delta > 0$ on K_S. By lemma 2.31 there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_2 := f_1 - \varepsilon^2(y^2 + 1)^d > 0 \text{ on } K_S.$$

By lemma 3.52 and lemma 3.53 we have $f_2 \in PO_P(S)$ for every $P \in K$, so $f_1 \in PO(S)$ by theorem 2.10. □

Now, the same in the half-strip case:

Lemma 3.55
Let $S \subseteq \mathbb{R}[x, y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x, y]$. Suppose

(i) *there exists an $N \in \mathbb{N}$ such that*

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) *for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),*

(iii) $(x - P) \nmid s$ *for all* $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $f > 0$ *on* K_S,

Let $P \in K$ and $p \in \mathbb{R}\{x - P\}[y]$ be an irreducible factor of f. Suppose

a) *p is irreducible in $\mathbb{C}\{x - P\}[y]$,*

b) *p is monic,*

c) $\deg_y(p)$ *is even.*

Then $p \in PO_P(S)$.

Proof — We may assume $P = 0$.
Since $f > 0$ on K_S, we have $x \nmid f$, so $p \notin \mathbb{R}\{x\}$. Then by lemma 2.61

$$p(0, y) = (y - a)^n$$

for some $a \notin \lambda_0^{-1}(K_S)$. Since $\lambda_0^*(S)$ contains the natural generators for $\lambda_0^{-1}(K_S)$, there exists an $s \in S$ such that

(i) $\lambda_0^*(s) = s(0, y)$ is one of the natural generators for $\lambda_0^{-1}(K_S)$,

(ii) $s(0, a) < 0$.

Using lemma 3.15, we may assume that the leading coefficient of s as a polynomial in y over $\mathbb{R}[x]$ does not vanish at 0.
Then we have

$$\deg_y(s) = \deg_y(s(0, y)) \in \{1, 2\}.$$

If $\deg_y(s) = 2$, then we can factorize

$$s = v(y - u_1)(y - u_2),$$

with $v \in \mathbb{R}\{x\}^*$ and $u_1, u_2 \in \mathbb{R}\{x\}$ with $u_1(0) \neq u_2(0)$. Now we may argue the exact same way as in lemma 3.48 to show that

$$p \in PO_0(\{s\}) \subseteq PO_0(S).$$

If $\deg_y(s) = 1$, then

$$s = v(y - u),$$

with $u \in \mathbb{R}\{x\}$ and $v \in \mathbb{R}\{x\}^*$. $\lambda_0^*(s)$ is a natural generator for $\lambda_0^{-1}(K_S)$, thus

$$\lambda_0^*(s) = s(0, y) = c(y - b),$$

with $c > 0$ and $b \in \lambda_0^{-1}(K_S)$. We conclude $v(0) = c > 0$, so $v \in (\mathbb{R}\{x\}^*)^2$ and we may again assume $v = 1$.
Since

$$0 > s(0, a) = a - u(0),$$

we have $a < u(0)$, so $p > 0$ at every ordering of K_0 that satisfies $s > 0$ by lemma 3.44 and

$$p \in PO_0(\{s\}) \subseteq PO_0(S)$$

by proposition 2.76. $\square$

Lemma 3.56
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x,y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x-P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $f > 0$ on K_S,

Let $P \in K$ and $p \in \mathbb{R}\{x-P\}[y]$ be an irreducible factor of f. Suppose

a) p is irreducible in $\mathbb{C}\{x-P\}[y]$,

b) p is monic,

c) $n = \deg_y(p)$ is odd.

Then either $p \in PO_P(S)$ or there exists another monic irreducible factor $q \in \mathbb{R}\{x-P\}[y]$ of f such that $pq \in PO_P(S)$.

Proof — We may assume $P = 0$.
By lemma 2.61

$$p(0,y) = (y-a)^n$$

for an $a \notin \lambda_0^{-1}(K_S)$.
Since $\lambda_0^*(S)$ contains the natural generators for $\lambda_0^{-1}(K_S)$, there exists an $s \in S$ such that

(i) $\lambda_0^*(s) = s'(0,y)$ is one of the natural generators for $\lambda_0^{-1}(K_S)$,

(ii) $s(0,a) < 0$.

Using lemma 3.15, we may assume that the leading coefficient of s as a polynomial in y over $\mathbb{R}[x]$ does not vanish at 0.

If $\deg_y(s) = 2$, then

$$s = v(y - u_1)(y - u_2)$$

with $v \in \mathbb{R}\{x\}^*$ and $u_1, u_2 \in \mathbb{R}\{x\}$ with $u_1(0) = u_2(0)$. So we may argue the exact same way as in lemma 3.49 to show that there exists another irreducible factor $q \in \mathbb{R}\{x\}[y]$ of f such that

$$pq \in PO_0(\{s\}) \subseteq PO_0(S).$$

If $\deg_y(s) = 1$, we can argue the exact same way as in lemma 3.55 to show that

$$p \in PO_0(\{s\}) \subseteq PO_0(S).$$

□

Remark 3.57
In the situation of the last lemma, we can be even more precise:
We assume $P = 0$ and $p = (y - a)^n$ with $a \notin \lambda^{-1}(K_S)$. If there exist $c, d \in \lambda_0^{-1}(K_S)$ such that $a \in (c, d)$, then there exists another irreducible factor q of odd degree with a zero $b \in (c, d)$ such that $pq \in PO_0(S)$ (compare remark 3.50). Again, since $f(0, c), f(0, d) > 0$, f has an even number of such irreducible factors. If there exists an $c > a$ such that $\lambda_0^{-1}(K_S) \subseteq (c, \infty)$, then $p \in PO_0(S)$. So we can again conclude from the last lemma, that the product of all monic irreducible factors of odd degree lies in $PO_0(S)$.

We will now handle those irreducible factors whose leading coefficients are not monic. Since we will need these lemmas again in the next section, we will again prove a slightly more general version than we actually need here:

Lemma 3.58
Let $S \subseteq \mathbb{R}[x, y]$ and $f \in \mathbb{R}[x, y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) $y \in S$

(iii) $f \geq 0$ on K_S,

(iv) $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ *is finite.*

Let $P \in int(K)$ *and* $p \in \mathbb{R}\{x - P\}[y]$ *be an irreducible factor of* f. *If* p *is irreducible in* $\mathbb{C}\{x - P\}[y]$ *and the leading coefficient of* p *in* $\mathbb{R}\{x\}$ *is not a unit, then either* $p \in PO_P(S)$ *or* $-p \in PO_P(S)$.

Proof — We may assume $P = 0$.
By lemma 2.62 we have $p(0, y) \in \mathbb{R} \setminus 0$. We may assume $p(0, y) > 0$ (otherwise we work with $-p$).
Define

$$A = \mathbb{R}\{x\}[y, t]/(t^2 - y) = \mathbb{R}\{x\}[t]$$

and let P be the preordering generated by $S \setminus \{s\}$ in A. By lemma 2.73 we have to show $p(x, t^2) \in P$.
Note that P in $\mathbb{R}\{x\}[t]$ is generated by

$$S' = \{s(x, t^2) \mid s \in S\} \subseteq \mathbb{R}[x, t].$$

Now, work in the polynomial ring $\mathbb{R}[x, t]$ and define

$$P' := PO(S') \subseteq \mathbb{R}[x, t].$$

Then

$$K \times \{t^2 \geq N\} \subseteq K_{S'} \subseteq K \times \mathbb{R},$$

so S' and $f(x, t^2)$ satisfy the assumptions of lemma 3.51.
Let $p' \in \mathbb{R}\{x\}[t]$ be an irreducible factor of $p(x, t^2)$. Then p' is also an irreducible factor of $f(x, t^2)$. Since

$$p(0, y) = p(0, t^2) \in \mathbb{R},$$

we conclude $p'(0, t) \in \mathbb{R}$, so the leading coefficient of p' is not a unit in $\mathbb{R}\{x\}$, thus p' is reducible in $\mathbb{C}\{x\}[t]$ by lemma 3.51. Using lemma 2.52, there exists an $\epsilon \in \{-1, 1\}$ such that

$$\epsilon p(0, t^2) \in \sum \mathbb{R}\{x\}[t]^2.$$

Since $p(0, t^2) \in \mathbb{R}_{>0}$, we conclude $\epsilon = 1$. Thus,

$$p(0, t^2) \in \sum \mathbb{R}\{x\}[t]^2 \subseteq P.$$

$\square$

Lemma 3.59
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$, $f \in \mathbb{R}[x,y]$ *and* $P \in int(K)$. *Suppose*

(i) there exists an $N \in \mathbb{N}$ *such that*

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ *contains the natural choice of generators for* $\lambda_P^{-1}(K_S)$ *(up to scaling by positive reals),*

(iii) $(x - P) \nmid s$ *for all* $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $y \in S$,

(v) $f > 0$ *on* K_S.

Then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$. We have to show that $f \in PO_0(S)$.
Factorize

$$f = u(\prod_{i=1}^{s} p_i)(\prod_{j=1}^{r} q_i),$$

where

(i) $p_i \in \mathbb{R}\{x\}[y]$ are irreducible in $\mathbb{C}\{x\}[y]$ $(i = 1, \dots, s)$,

(ii) $q_j \in \mathbb{R}\{x\}[y]$ are irreducible in $\mathbb{R}\{x\}[y]$ and reducible in $\mathbb{C}\{x\}[y]$ $(j = 1, \dots, r)$,

(iii) $u \in \mathbb{R}\{x\}^*$.

Define

$$p := \prod_{i=1}^{s} p_i.$$

Then we have either $p \in PO_0(S)$ or $-p \in PO_0(S)$ by lemma 3.55, lemma 3.56 and lemma 3.58 (compare remark 3.57).
Since $u \in \mathbb{R}\{x\}^*$, we have either $u \in \mathbb{R}\{x\}^2$ or $-u \in \mathbb{R}\{x\}^2$ by lemma 2.45.
For each q_j $(j = 1 \dots, r)$, we have either $q_j \in \sum \mathbb{R}\{x\}[y]^2$ or

$-q_j \in \sum \mathbb{R}\{x\}[y]^2$ by lemma 2.52.
If we put this together, we have either $f \in PO_0(S)$ or $-f \in PO_0(S)$.
Since $-f < 0$ on K_S, we conclude $f \in PO_0(S)$. □

Lemma 3.60
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$, $f \in \mathbb{R}[x,y]$ and $P \in \delta K)$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) $y \in S$,

(v) $f > 0$ on K_S,

then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$ and $[0, \varepsilon] \subseteq K$ for $0 < \varepsilon \in \mathbb{R}$.
By lemma 3.22, there exists an $s \in S$ such that $s = xv^2$ for an $v \in \mathbb{R}\{x\}^*$.
Since $PO(\{x\}) = PO(\{xv^2\}$, we may assume $v = 1$.
Define

$$A = \mathbb{R}\{x\}[y,t]/(t^2 - x) = \mathbb{R}\{t\}[y]$$

and let P be the preordering generated by $S \setminus \{s\}$ in A. By lemma 2.73 we have to show $f(t^2, y) \in P$.
Note that P in $\mathbb{R}\{t\}[y]$ is generated by

$$S' = \{s(t^2, y) \mid s \in S\} \subseteq \mathbb{R}[t,y].$$

Now, work in the polynomial ring $\mathbb{R}[t,y]$ and define $P' := PO(S') \subseteq \mathbb{R}[t,y]$.
Then $0 \in int(\pi_1(K_{S'}))$, so

$$f(t^2, y) \in PO_0(S') = P \subseteq \mathbb{R}\{t\}[y]$$

by lemma 3.59. □

Now we are able to improve corollary 3.7 in the half-strip case:

Theorem 3.61
Let $S \subseteq \mathbb{R}[x, y]$ *and* $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. *Suppose*

(i) there exists an $N \in \mathbb{N}$ *such that*

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ *contains the natural choice of generators for* $\lambda_P^{-1}(K_S)$ *(up to scaling by positive reals),*

(iii) $(x - P) \nmid s$ *for all* $s \in S_2 \setminus \mathbb{R}[x]$,

(iv) S *contains the natural choice of generators for* K,

(v) $y \in S$,

then $PO(S)$ *satisfies* (†).

Proof — Suppose $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S and $\deg_y(f) = d$. Fix $\delta > 0$, then $f_1 := f + \delta > 0$ on K_S.
By lemma 2.33 there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_2 := f_1 - \varepsilon^2 \sum_{i=0}^{d} > 0 \text{ on } K_S.$$

By lemma 3.59 and lemma 3.60 we have $f_2 \in PO_P(S)$ for every $P \in K$, so $f_1 \in PO(S)$ by theorem 2.13. $\square$

3.4 Saturated Preorderings

In this subsection, we want to improve theorem 3.54 and theorem 3.61 to get examples of saturated preorderings. Unfortunately, we were not able to prove that a preordering as in theorem 3.54 (3.61) is saturated (see section 3.5 for some of the problems that occur here). However, we were able to prove saturatedness under certain extra assumptions. Those extra assumptions are:

- any $s \in S \setminus \mathbb{R}[x]$ is irreducible,
- any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,
- for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K

(for $S \subseteq \mathbb{R}[x, y]$ and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$).
Here we added the first assumption to be able to apply lemma 2.43. Without this assumption, the preordering $PO(S)$ will probably not be saturated (compare the examples of punctured strips in corollary 3.30 and proposition 3.33 above).
To explain why we added the other two assumptions, we prove several lemmas:

Lemma 3.62
Let $S \subseteq \mathbb{R}[x, y]$ and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ with

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) $K_S = K_{S_2}$,

(iii) $S \cap \mathbb{R}[x]$ contains the natural choice of generators for K,

(iv) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S.

Let $P = (P_1, P_2) \in \delta K_S$.

a) If $P_1 \in int(K)$, then there exists an open neighborhood U of P and a *unique* $s \in S_2 \setminus \mathbb{R}[x]$ *such that*

$$U \cap K_S = U \cap \{P \in \mathbb{R}^2 \mid s(P) \geq 0\}.$$

b) If $P_1 \in \delta K$*, then there exists an open neighborhood* U *of* P*, a unique* $s_1 \in S_2 \setminus \mathbb{R}[x] \cup \{1\}$ *and an* $s_2 \in (S \cap \mathbb{R}[x])$ *such that*

$$U \cap K_S = U \cap \{P \in \mathbb{R}^2 \mid s_1(P) \geq 0, s_2(P) \geq 0\}.$$

Proof — a) Let $P_1 \in int(K)$. Since $K_S = K_{S_2}$, there exists an $s \in S_2$ which changes its sign at P, especially $s(P) = 0$. We also have $s' \geq 0$ on K for all $s' \in S \cap \mathbb{R}[x]$, so we conclude $s \in S_2 \setminus \mathbb{R}[x]$. Since $s'(P) > 0$ for all $s \neq s' \in S_2$ by assumption (iv), this s is unique. Now define

$$U := (int(K) \times \mathbb{R}) \cap \bigcap_{s \neq \tilde{s} \in S_2 \setminus \mathbb{R}[x]} \tilde{s}^{-1}(0, \infty) \subseteq \mathbb{R}^2.$$

Then U and s are as desired.

b) Now let $P_1 \in \delta K$. Then there exists a natural generator $s_2 \in S$ which changes its sign at P_1, especially $s_2(P_1) = 0$. Now let $U_1 \subseteq K$ such that

$$U_1 \cap K = U_1 \cap \{P \in \mathbb{R} \mid s_2(p) \geq 0\}.$$

Such a U_1 does exist, since K has no isolated points. Now there are 2 cases:

Case 1:

Suppose for every $s \in S_2 \setminus \mathbb{R}[x]$, there exists an open neighborhood $U_s \subseteq \mathbb{R}^2$ of P such that $s \geq 0$ on U_s. Then define $s_1 = 1$ and

$$U := (U_1 \times \mathbb{R}) \cap \bigcap_{s \in S_2 \setminus \mathbb{R}[x]} U_s.$$

Since $K_S = K_{S_2}$ by assumption (ii), U, s_1 and s_2 are as desired.

Case 2:

Now, suppose there exists an $s_1 \in S_2 \setminus \mathbb{R}[x]$ that changes its sign at P, specifically we have $s_1(P) = 0$. By assumption (iv) we have $s'(P) > 0$ for all $s' \in S_2 \setminus \mathbb{R}[x]$. Define

$$U := (U_1 \times \mathbb{R}) \cap \bigcap_{s \neq s' \in S_2 \setminus \mathbb{R}[x]} s'^{-1}(0, \infty) \subseteq \mathbb{R}^2,$$

then we have $P \in U$ and U, s_1 and s_2 are as desired. $\square$

Remark 3.63

(i) Note that $K_S = K_{S_2}$ holds in the strip and the half-strip case by lemma 3.19 and lemma 3.20, so the last lemma deals with both cases at once.

(ii) Due to our new assumptions, the semialgebraic set is locally described by at most one polynomial $s \in S_2$ with $\deg_y(s) \geq 1$. This will become important later.

Lemma 3.64

Let $S \subseteq \mathbb{R}[x,y]$ and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ with

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $S \cap \mathbb{R}[x]$ contains the natural choice of generators for K,

(iv) $(x-P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$ and all $P \in K$,

(v) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(vi) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K.

Let $P = (P_1, P_2) \in \delta K_S$ and $s \in S_2 \setminus \mathbb{R}[x]$ with $s(P) = 0$.
If $s \notin \mathbb{R}\{x - P_1\}[y]^2$, then s factors

$$s = u(y - v_1)(y - v_2)$$

with $u \in \mathbb{R}\{x - P_1\}^*$; $v_1, v_2 \in \mathbb{R}\{x - P_1\}$ and $v_1(P_1) \neq v_2(P_1)$.

Proof — We may assume $P_1 = 0$.
By lemma 2.34 $\deg_y(s)$ is even, so $\deg_y(s) = 2$ (since $s \notin \mathbb{R}[x]$). Thus

$$s = ay^2 + by + c$$

with $a, b, c \in \mathbb{R}\{x\}$. We have $x \nmid s$ by assumption (iv), so $s(0,y) \neq 0$. Further, P_2 is a zero of $s(0,y)$, so $\deg_y(s(0,y)) \geq 1$.

If $\deg_y(s(0,y)) = 1$, then $s(0,y)$ changes its sign at P_2 and nowhere else. A contradiction, since

$$s \geq 0 \text{ on } \{0\} \times \{y^2 \geq N\}.$$

Thus, $\deg_y(s(0,y)) = 2$, especially $a(0) \neq 0$ and $a \in \mathbb{R}\{x\}^*$.
Then s is reducible in $\mathbb{R}\{x\}[y]$ by lemma 3.36, thus

$$s = u(y - v_1)(y - v_2)$$

with $u \in \mathbb{R}\{x\}^*$; $v_1, v_2 \in \mathbb{R}\{x\}$. If $v_1(0) = v_2(0)$, then

$$\begin{aligned} d(0) &= u(0)^2[(v_1(0) + v_2(0))^2 - 4v_1(0)v_2(0)] \\ &= a(0)^2[4v_1(0) - 4v_1(0)] = 0, \end{aligned}$$

so $v_1(0) \neq v_2(0)$. □

Lemma 3.65
Let $S \subseteq \mathbb{R}[x,y]$ and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ with

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) for every $P \in K$, $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) S contains the natural choice of generators for K,

(iv) $(x - P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$ and all $P \in K$,

(v) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(vi) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K.

Let $P = (P_1, P_2) \in \delta K_S$ and $s \in S_2 \setminus \mathbb{R}[x]$ with $s(P) = 0$.

a) If $\deg_y(s) = 2$ and $s \notin \mathbb{R}\{x - P_1\}[y]^2$, then s factors

$$s = u(y - v_1)(y - v_2)$$

with $u \in \mathbb{R}\{x - P_1\}^*$; $v_1, v_2 \in \mathbb{R}\{x - P_1\}$ and $v_1(P_1) \neq v_2(P_1)$.

b) If $\deg_y(s) = 1$*, then* s *factors*

$$s = u(y - v_1)$$

with $u \in \mathbb{R}\{x - P_1\}^*$ *and* $v_1, \in \mathbb{R}\{x - P_1\}$.

Proof — We may assume $P_1 = 0$ and $[0, \varepsilon] \subseteq K$ for $\varepsilon > 0$.

a) Let $\deg_y(s) = 2$. Then

$$s = ay^2 + by + c$$

with $a, b, c \in \mathbb{R}\{x\}$. We have $x \nmid s$ by assumption (iv), thus $s(0, y) \neq 0$. Further, P_2 is a zero of $s(0, y)$, so $\deg_y(s(0, y)) \geq 1$. If $\deg_y(s(0, y)) = 1$, then $a(0) = 0$ and

$$\lambda_0^{-1}(K_S) \subseteq [P_2, \infty),$$

since $s(0, P') < 0$ for all $P' < P_2$. So by lemma 3.65 there exists a $c > 0$ and another $s \neq s' \in S_2$ whose leading coefficient does not vanish at 0 such that

$$s'(0, y) = \lambda_0^*(s') = c\lambda_0(s) = cs(0, y).$$

In particular $s'(0, P_2) = 0$. A contradiction to assumption (v).
Thus, $\deg_y(s(0, y)) = 2$, especially $a(0) \neq 0$ and $a \in \mathbb{R}\{x\}^*$.
Then s is reducible in $\mathbb{R}\{x\}[y]$ by lemma 3.36, thus

$$s = u(y - v_1)(y - v_2)$$

with $u \in \mathbb{R}\{x\}^*$; $v_1, v_2 \in \mathbb{R}\{x\}$. If $v_1(0) = v_2(0)$, then

$$\begin{aligned} d(0) &= u(0)^2[(v_1(0) + v_2(0))^2 - 4v_1(0)v_2(0)] \\ &= a(0)^2[4v_1(0)^2 - 4v_1(0)^2] = 0, \end{aligned}$$

so $v_1(0) \neq v_2(0)$.

b) Let $\deg_y(s) = 1$. Write

$$s = ay + b$$

with $a, b \in \mathbb{R}\{x\}$. We have $x \nmid s$ by assumption (iv), thus $s(0, y) \neq 0$. Further, P_2 is a zero of $s(0, y)$, so $\deg_y(s(0, y)) = 1$, especially $a(0) \neq 0$. So $a \in \mathbb{R}\{x\}^*$ and

$$s = a(y + a^{-1}b).$$

□

Remark 3.66

(i) Note that the assumption

$$(x - P) \nmid s \text{ for all } s \in S_2 \setminus \mathbb{R}[x]$$

is automatically satisfied if every $s \in S \setminus \mathbb{R}[x]$ is irreducible.

(ii) From the last three lemmas, we conclude that, under our new assumptions, our semialgebraic set K_S is "locally" described by exactly one $s \in S$ (plus eventually one of the natural generators of K) which is reducible in $\mathbb{R}\{x\}[y]$ if $\deg_y(s) = 2$. This will become important later.

We are now able to prove that a preordering, which satisfies our new assumptions, is saturated. As before, theorem 2.10 lets us work in the ring $\mathbb{R}\{x - P\}[y]$ $(P \in K)$. Here we have to show that $f \in PO_P(S)$ for some $f \in \mathbb{R}[x, y]$. The main idea is again to handle each irreducible factor of f in $\mathbb{R}\{x - P\}[y]$ separately.

Lemma 3.67

Let $S \subseteq \mathbb{R}[x, y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x, y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ for all $s \in S \setminus \mathbb{R}[x]$ and all $P \in K$,

(iv) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(v) for any $s \in S_2$ *with* $\deg_y(s) = 2$*, the discriminant* $d(x)$ *doesn't vanish on* K*,*

(vi) $f \geq 0$ *on* K_S*,*

(vii) $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ *is finite.*

Let $P \in int(K)$ *and* $p \in \mathbb{R}\{x - P\}[y]$ *be an irreducible factor of* f*. Suppose*

a) p *is irreducible in* $\mathbb{C}\{x - P\}[y]$*,*

b) p *is monic,*

c) $\deg_y(p)$ *is even.*

Then $p \in PO_P(S)$*.*

Proof — We may assume $P = 0$.
By lemma 2.57 p (and f) vanishes on

$$V_\delta(p) = \{(\epsilon x^n, g(x)) \mid |x| < \delta\}$$

for some $g \in \mathbb{R}\{x\}$, $\epsilon \in \{-1, 1\}$ and $\delta > 0$. Define

$$a := g(0).$$

If $(0, a) \in int(K_S)$, then p (and f) would have infinitely many zeros in K_S, a contradiction.
So $(0, a) \notin int(K_S)$.
If $(0, a) \notin K_S$, we can argue the same way as in lemma 3.48 to show that

$$p \in PO_0(S).$$

So assume $(0, a) \in \delta K_S$. By lemma 3.62 there exists an open neighborhood U of P and a unique $s \in S_2 \setminus \mathbb{R}[x]$ such that

$$U \cap K_S = U \cap \{P \in \mathbb{R}^2 \mid s(P) \geq 0\}.$$

By lemma 3.64 we can factorize

$$s = u(y - v_1)(y - v_2)$$

with $u \in \mathbb{R}\{x - P_1\}^*$; $v_1, v_2 \in \mathbb{R}\{x - P_1\}$ and $v_1(P_1) \neq v_2(P_1)$. Without loss of generality, we assume $v_1(0) > v_2(0)$ and $u = 1$.
It is enough to show that

$$p \in PO_0(\{s\}) \subseteq PO_0(S).$$

For this, we have to show that $p > 0$ at every ordering of K_0 that satisfies $s > 0$ by lemma 2.80. By lemma 3.46 it is enough to show that

- $p > 0$ at every ordering of K_0 that satisfies $y - v_1 > 0$ and
- $p > 0$ at every ordering of K_0 that satisfies $y - v_2 < 0$.

Assume $a = v_1(0)$ (otherwise work with the ordering $<_1$ from lemma 3.45). Then $p > 0$ at every ordering of K_0 that satisfies $y - v_2 < 0$ by lemma 3.45. So we have to show that $p > 0$ at every ordering that satisfies $y - v_1 > 0$. For this, it is enough to show that $p \in PO_0(\{y - v_1\})$.
By a linear change of coordinates, we may assume $v_1 = 0$. Then by lemma 2.73 we have to show that $p \in \sum A^2$, where

$$A := \mathbb{R}\{x\}[y, t]/(t^2 - y) = \mathbb{R}\{x\}[t].$$

Note that $p(x, y) = p(x, t^2)$. Let $q \in \mathbb{R}\{x\}[t]$ be an irreducible factor of p. Since p is monic as a polynomial in t over $\mathbb{R}\{x\}$ and

$$p(0, t^2) = t^{2n},$$

we may assume q is monic and

$$q(0, t) = t^m$$

with $m < 2n$. If q would be irreducible in $\mathbb{C}\{x\}[t]$, then by lemma 2.67 and lemma 2.64 q would vanish on some set

$$V = \{(\epsilon x^m, g(x)) \mid |x| < \delta\}$$

for $m \in \mathbb{N}$, $\epsilon \in \{-1, 1\}$, $g \in \mathbb{R}\{x\}$ with $g(0) = 0$ and $\delta > 0$. Thus $p(x, y)$ would vanish on

$$V' = \{(\epsilon x^m, g(x)^2) \mid |x| < \delta\}.$$

This set lies in $U \cap K_S$ for δ small enough (since $g(x)^2 \geq 0$). A contradiction, since $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ is finite.

So every irreducible factor of $p(x,t^2)$ in A is monic and reducible in $\mathbb{C}\{x\}[t]$, thus a sum of squares in A by lemma 2.53. So we conclude

$$p \in \sum A^2.$$

□

Lemma 3.68
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x,y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x-P) \nmid s$ for all $s \in S \setminus \mathbb{R}[x]$ and all $P \in K$,

(iv) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(v) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K,

(vi) $f \geq 0$ on K_S,

(vii) $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ is finite.

Let $P \in int(K)$ and $p \in \mathbb{R}\{x-P\}[y]$ be an irreducible factor of f. Suppose

a) p is irreducible in $\mathbb{C}\{x-P\}[y]$,

b) p is monic,

c) $n := \deg_y(p)$ is odd.

Then there exists another irreducible factor $q \in \mathbb{R}\{x\}[y]$ of f such that $pq \in PO_P(S)$.

Proof — We may assume $P = 0$.
By lemma 2.57 p (and f) vanishes on

$$V_\delta(p) = \{(x^n, g(x)) \mid |x| < \delta\}$$

for some $g \in \mathbb{R}\{x\}$ and $\delta > 0$. Define

$$a := g(0).$$

If $(0, a) \in int(K_S)$, then p (and f) would have infinitely many zeros in K_S, a contradiction.
So $(0, a) \notin int(K_S)$.
If $(0, a) \in \delta K$, then there exists an open neighborhood U of P and a unique $s \in S_2 \setminus \mathbb{R}[x]$ such that

$$U \cap K_S = U \cap \{P \in \mathbb{R}^2 \mid s(P) \geq 0\}.$$

By lemma 3.64 s is reducible in $\mathbb{R}\{x\}[y]$ and the leading coefficient is a unit.
If $(0, a) \notin K_S$, there exists an $s \in S_2$ such that

- $\lambda_0^*(s) = s(0, y)$ is one of the natural generators for $\lambda_0^{-1}(K_S)$,
- $s(0, y)$ has exactly 2 distinct zeros in $\lambda_0^{-1}(K_S) \subseteq \mathbb{R}$,
- $s(0, a) < 0$.

Since s has two zeros in $\{0\} \times \mathbb{R}$, s is reducible in $\mathbb{R}\{x\}[y]$.
So in both cases, there exists an $s \in S$ such that

$$s = u(y - v_1)(y - v_2),$$

where $u \in \mathbb{R}\{x\}^*$, $v_1, v_2 \in \mathbb{R}\{x\}$,$v_1(0) > v_2(0)$ and $a \in [v_2(0), v_1(0)]$. Without loss of generality, we may assume $u = 1$.
Since $n = \deg_y(p)$ is odd, $p(0, y) = (y - a)^n$ changes its sign at a. Since

$$f(0, v_1(0)), f(0, v_2(0)) \geq 0$$

and $v_1(0) \geq a \geq v_2(0)$ by construction, there exists another irreducible factor $q \in \mathbb{R}\{x\}[y]$ of f such that $q(0, y)$ changes its sign at $b \in [v_2(0), v_1(0)]$. Thus the leading coefficient of q is a unit in $\mathbb{R}\{x\}$. So we may assume q is monic and

$$q(0, y) = (y - b)^m$$

with $m = \deg_y(q)$ odd. Now it is enough to show that

$$pq \in PO_0(\{s\}) \subseteq PO_0(S).$$

For this, we have to show $pq > 0$ at every ordering of K_0 that satisfies $s > 0$ by lemma 2.80. By lemma 3.46 it is enough to show that

- $pq > 0$ at every ordering of K_0 that satisfies $y - v_1 > 0$ and

- $pq > 0$ at every ordering of K_0 that satisfies $y - v_2 < 0$.

We will show this in the next lemma. This finishes the proof. □

Lemma 3.69
In the situation above, we have

(i) $pq > 0$ *at every ordering of* K_0 *that satisfies* $y - v_1 > 0$ *and*

(ii) $pq > 0$ *at every ordering of* K_0 *that satisfies* $y - v_2 < 0$.

Proof — (i) By a linear change of coordinates we may assume $v_1 = 0$. It is enough to show that $pq \in PO_0(\{y\})$, so by lemma 2.73 we have to show

$$pq \in \sum A^2,$$

where

$$A := \mathbb{R}\{x\}[y,t]/(t^2 - y) = \mathbb{R}\{x\}[t].$$

Note that $pq = p(x,t^2)q(x,t^2)$ in A.
So pq is monic as a polynomial in t over $\mathbb{R}\{x\}$. Factorize

$$pq = \prod_{i=1}^{s} p_i$$

with $p_i \in \mathbb{R}\{x\}[t]$ irreducible and monic ($i = 1, \ldots, s$). Suppose there exists a p_i that is irreducible in $\mathbb{C}\{x\}[y]$, then $p_i(0,t) = (t - a_i)^{n_i}$ for $a_i \in \mathbb{R}$ and $n_i = \deg_t(p_i)$, especially p_i vanishes at $(0, a_i)$, thus $pq \in \mathbb{R}\{x\}[y]$ vanishes at $(0, a_i^2)$, so we conclude $a_i = 0$. Then by lemma 2.67 and lemma 2.64 p_i vanishes on a set of the form

$$V = \{(\epsilon x^{n_i}, g(x)) \mid |x| < \varepsilon\}$$

with $\epsilon \in \{1, -1\}$ and $g \in \mathbb{R}\{x\}$ with $g(0) = 0$. Thus $p(x,y)$ would vanish on

$$V' = \{(\epsilon x^m, g(x)^2) \mid |x| < \varepsilon\}.$$

Since $g(0) = 0$, p (and thus f) would have infinitely many zeros in K_S. A contradiction.
So every irreducible factor p_i of pq in A is monic and reducible in $\mathbb{C}\{x\}[t]$, thus

$$p \in \sum A^2.$$

(ii) It is again enough to prove

$$pq \in PO_0(\{-(y-v_2)\}).$$

We introduce a new variable

$$y' := -(y-v_2) = v_2 - y$$

and define

$$h(x,y') := p(x, v_2 - y')q(x, v_2 - y').$$

Since

$$p(x,y)q(x,y) = h(x, -(y-v_2)),$$

we have to show

$$h \in PO_0(\{y'\}) \subseteq \mathbb{R}\{x\}[y'].$$

The proof is now the same as in (i):
We again have to show that

$$h \in \sum B^2,$$

where

$$B := \mathbb{R}\{x\}[y',t]/(t^2 - y') = \mathbb{R}\{x\}[t].$$

$h(x,t^2) \in \mathbb{R}\{x\}[t]$ is monic, so we can factorize

$$h = \prod_{i=1}^{s} p_i$$

with $p_i \in \mathbb{R}\{x\}[t]$ irreducible and monic $(i = 1, \ldots, s)$. As in (i), we conclude each irreducible factor p_i of h in B satisfies $p_i \in \sum B^2$, thus

$$h \in \sum B^2.$$

□

Remark 3.70

As from lemma 3.49, we can also conclude from this lemma that the product of all irreducible factors of f, which are of odd degree, lies in $PO_0(S)$ (compare remark 3.50).

Lemma 3.71

Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ *and* $f \in \mathbb{R}[x,y]$. *Suppose* *(i)* *there exists an* $N \in \mathbb{N}$ *such that*

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) *for every* $P \in K$, $\lambda_P^*(S)$ *contains the natural choice of generators for* $\lambda_P^{-1}(K_S)$ *(up to scaling by positive reals),*

(iii) $(x - P) \nmid s$ *for all* $s \in S \setminus \mathbb{R}[x]$ *and all* $P \in K$,

(iv) *any* $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ *with* $s_1 \neq s_2$ *have no common zero in* K_S,

(v) *for any* $s \in S_2$ *with* $\deg_y(s) = 2$, *the discriminant* $d(x)$ *doesn't vanish on* K,

(vi) $f \geq 0$ *on* K_S,

(vii) $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ *is finite,*

(viii) $P \in int(K)$,

then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$. We have to show that $f \in PO_0(S)$.
Factorize

$$f = u(\prod_{i=1}^{s} p_i)(\prod_{j=1}^{r} q_i),$$

where

(i) $p_i \in \mathbb{R}\{x\}[y]$ is irreducible in $\mathbb{C}\{x\}[y]$ $(i = 1, \ldots, s)$,

(ii) $q_j \in \mathbb{R}\{x\}[y]$ is irreducible, but reducible in $\mathbb{C}\{x\}[y]$ $(j = 1, \ldots, r)$,

(iii) $u \in \mathbb{R}\{x\}^*$.

By lemma 3.51 the leading coefficient of every p_i is a unit in $\mathbb{R}\{x\}$, so we may assume all p_i are monic. Then

$$\prod_{i=1}^{s} p_i \in PO_0(S)$$

by lemma 3.67 and lemma 3.68 (compare remark 3.70).
Since $u \in \mathbb{R}\{x\}^*$, we have either $u \in \mathbb{R}\{x\}^2$ or $-u \in \mathbb{R}\{x\}^2$ by lemma 2.45.
For each q_i, we have either $q_i \in \sum \mathbb{R}\{x\}[y]^2$ or $-q_i \in \sum \mathbb{R}\{x\}[y]^2$ by lemma 2.52. If we put this together, we have either $f \in PO_0(S)$ or $-f \in PO_0(S)$.
Since $-f < 0$ on K_S, we conclude $f \in PO_0(S)$. $\square$

Lemma 3.72
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x,y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x-P) \nmid s$ for all $s \in S \setminus \mathbb{R}[x]$ and all $P \in K$,

(iv) S contains the natural choice of generators for K,

(v) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(vi) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K,

(vii) $f \geq 0$ on K_S,

(viii) $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite,

(ix) $P \in \delta K$.

Then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$ and $[0, \varepsilon] \subseteq K$ for $0 < \varepsilon \in \mathbb{R}$.
By lemma 3.22, there exists an $s \in S$ such that $s = xv^2$ for an $v \in \mathbb{R}\{x\}^*$.
Since $PO(\{x\}) = PO(\{xv^2\})$, we may assume $v = 1$.
Define

$$A = \mathbb{R}\{x\}[y,t]/(t^2 - x) = \mathbb{R}\{t\}[y]$$

and let P be the preordering generated by $S \setminus \{s\}$ in A. By lemma 2.73 we

have to show $f(t^2, y) \in P$.
Note that P in $\mathbb{R}\{t\}[y]$ is generated by

$$S' = \{s(t^2, y) \mid s \in S\} \subseteq \mathbb{R}[t, y].$$

Now, work in the polynomial ring $\mathbb{R}[t, y]$ and let $P' := PO(S') \subseteq \mathbb{R}[t, y]$. Then $0 \in int(\pi_1(K_{S'}))$ and S' satisfies all the assumptions of lemma 3.71, so

$$f(t^2, y) \in PO_0(S') = P \subseteq \mathbb{R}\{t\}[y]$$

by lemma 3.71. □

Remark 3.73
Suppose that, in the situation of the last lemma, every $s \in S \setminus \mathbb{R}[x]$ is irreducible. Then the elements of S' might not be irreducible in $\mathbb{R}[t, y]$. This is why we used the assumption

$$(x - P) \nmid s \text{ for all } s \in S \setminus \mathbb{R}[x] \text{ and } P \in K$$

instead of irreducibility.

Theorem 3.74
Let $S \subseteq \mathbb{R}[x, y]$ and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $S \cap \mathbb{R}[x]$ contains the natural choice of generators for K,

(iv) any $s \in S \setminus \mathbb{R}[x]$ is irreducible,

(v) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(vi) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K.

Then $PO(S)$ is saturated.

Proof — Let $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_S. Then $\deg_y(f)$ is even by lemma 2.34, say $\deg_y(f) = 2d$.
We may assume f is squarefree and no $s \in S$ divides f. Then f has only finitely many zeros in $K_S \setminus V_{\mathbb{R}}(S \cap \mathbb{R}[x])$ by proposition 2.43, thus $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.
By lemma 2.30, there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(y^2+1)^d \geq 0 \text{ on } K_S$$

and $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite. By lemma 3.71 and lemma 3.72, we have $f_1 \in PO_P(S)$ for every $P \in K$, so $f \in PO(S)$ by theorem 2.10. □

In the case where the semialgebraic set is contained in a half-strip, the proof is essentially the same:

Lemma 3.75
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x,y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x-P) \nmid s$ for all $s \in S \setminus \mathbb{R}[x]$ and all $P \in K$,

(iv) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(v) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K,

(vi) $f \geq 0$ on K_S,

(vii) $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ is finite.

Let $P \in int(K)$ and $p \in \mathbb{R}\{x-P\}[y]$ be an irreducible factor of f. Suppose

a) p is irreducible in $\mathbb{C}\{x-P\}[y]$,

b) p is monic,

c) $\deg_y(p)$ *is even.*

Then $p \in PO_P(S)$.

Proof — We may assume $P = 0$.
By lemma 2.57 p (and f) vanishes on

$$V_\delta(p) = \{(\epsilon x^n, g(x)) \mid |x| < \delta\}$$

for some $g \in \mathbb{R}\{x\}$ and $\epsilon \in \{-1, 1\}$. Define

$$a := g(0).$$

If $(0, a) \in int(K_S)$, then $\pi_1(\mathbb{V}_\mathbb{R}(f) \cap K_S)$ would not be finite, a contradiction.
So $(0, a) \notin int(K_S)$.
If $(0, a) \notin K_S$, we can argue the same way as in lemma 3.55 to show that

$$p \in PO_0(S).$$

So suppose $(0, a) \in \delta K_S$. By lemma 3.62 there exists an open neighborhood U of P and an unique $s \in S_2 \setminus \mathbb{R}[x]$ such that

$$U \cap K_S = U \cap \{P \in \mathbb{R}^2 \mid s(P) \geq 0\}.$$

If $\deg_y(s) = 2$, then s factorizes

$$s = u(y - v_1)(y - v_2)$$

with $u \in \mathbb{R}\{x - P_1\}^*$; $v_1, v_2 \in \mathbb{R}\{x - P_1\}$ and $v_1(P_1) \neq v_2(P_1)$ by lemma 3.65. Now, we can argue the exact same way as in lemma 3.67 to show that

$$p \in PO_0(S).$$

So assume $\deg_y(s) = 1$. Then s factors

$$s = u(y - v)$$

with $u \in \mathbb{R}\{x - P_1\}^*$ and $v, \in \mathbb{R}\{x - P_1\}$ by lemma 3.65.
Since $PO(\{s\}) = PO(\{u^{-1}s\})$, we may assume $u = 1$ and, by a linear change of coordinates, we may assume $v = 0$.
Then by lemma 2.73 we have to show that $p \in \sum A^2$, where

$$A := \mathbb{R}\{x\}[y, t]/(t^2 - y) = \mathbb{R}\{x\}[t].$$

Note that $p(x,y) = p(x,t^2)$ in A. Let $q \in \mathbb{R}\{x\}[t]$ be an irreducible factor of p. Since p is monic as a polynomial in t over $\mathbb{R}\{x\}$ and

$$p(0,t^2) = t^{2n},$$

we may assume q is monic and

$$q(0,t) = t^m$$

with $m \leq 2n$. If q would be irreducible in $\mathbb{C}\{x\}[t]$, then by lemma 2.67 and lemma 2.64 q would vanish on some set

$$V = \{(\epsilon x^m, g(x)) \mid |x| < \delta\}$$

for $\epsilon \in \{-1,1\}$, $g \in \mathbb{R}\{x\}$ with $g(0) = 0$ and $\delta > 0$. Thus $p(x,y)$ would vanish on

$$V' = \{(\epsilon x^m, g(x)^2) \mid |x| < \delta\}.$$

Since $g(x)^2 \geq 0$, this set lies in $U \cap K_S$ for δ small enough. A contradiction, since $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ is finite.
So every irreducible factor of $p(x,t^2)$ in A is monic and reducible in $\mathbb{C}\{x\}[t]$, thus a sum of squares in A. So we conclude

$$p \in \sum A^2.$$

□

Lemma 3.76
Let $S \subseteq \mathbb{R}[x,y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x,y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R},$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x-P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$ and all $P \in K$,

(iv) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(v) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K,

(vi) $f \geq 0$ *on* K_S,

(vii) $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ *is finite.*

Let $P \in int(K)$ *and* $p \in \mathbb{R}\{x-P\}[y]$ *be an irreducible factor of* f. *Suppose*

a) p *is irreducible in* $\mathbb{C}\{x-P\}[y]$,

b) p *is monic,*

c) $\deg_y(p)$ *is odd.*

Then either $p \in PO_P(S)$ *or there exists another monic irreducible factor* $q \in \mathbb{R}\{x-P\}[y]$ *of* f *such that* $pq \in PO_P(S)$.

Proof — We may assume $P = 0$.
By lemma 2.57 p (and f) vanishes on

$$V_\delta(p) = \{(x^n, g(x)) \mid |x| < \delta\}$$

for some $g \in \mathbb{R}\{x\}$ and $\delta > 0$. Define

$$a := g(0).$$

Then

$$p(0, y) = (y-a)^n$$

by lemma 2.61. If $(0, a) \in int(K_S)$, then p (and f) would have infinitely many zeros in K_S, a contradiction.
So $(0, a) \notin int(K_S)$.
If $(0, a) \in \delta K$, then there exists an open neighborhood U of P and a unique $s \in S_2 \setminus \mathbb{R}[x]$ such that

$$U \cap K_S = U \cap \{P \in \mathbb{R}^2 \mid s(P) \geq 0\}.$$

If $(0, a) \notin K_S$, then there exists an $s \in S_2$ such that

- $\lambda_0^*(s) = s(0, y)$ is one of the natural generators for $\lambda_0^{-1}(K_S)$,
- the leading coefficient does not vanish at 0,
- $s(0, a) < 0$.

So in either case, we fix this $s \in S$.
If $\deg_y(s) = 2$, we argue exactly the same way as in lemma 3.68 to show that there exists another irreducible factor $q \in \mathbb{R}\{x\}[y]$ such that

$$pq \in PO_0(\{s\}) \subseteq PO_0(S).$$

So assume $\deg_y(s) = 1$.
If $(0, a) \notin K_S$ we can argue the exact same way as in lemma 3.56 to show that

$$p \in PO_0(\{s\}) \subseteq PO_0(S).$$

So assume $(0, a) \in \delta K_S$ and $\deg_y(s) = 1$. In this case we have

$$s = u(y - v),$$

with $u \in \mathbb{R}\{x\}^*$ and $v \in \mathbb{R}\{x\}$ by lemma 3.65. We may again assume $u = 1$ and $v = 0$ by a linear change of coordinates. Then we have $a = v(0) = 0$ and $s = y$. It is enough to show that $p \in PO_0(\{y\})$, so by lemma 2.73 we have to show

$$p \in \sum A^2,$$

where

$$A := \mathbb{R}\{x\}[y, t]/(t^2 - y) = \mathbb{R}\{x\}[t].$$

Note that $p = p(x, t^2)$ in A.
So p is monic as a polynomial in t over $\mathbb{R}\{x\}$ and

$$p(0, t) = t^{2n}.$$

Factorize

$$p = \prod_{i=1}^{s} p_i$$

with $p_i \in \mathbb{R}\{x\}[t]$ irreducible and monic ($i = 1, \dots, s$). Suppose there exists a p_i that is irreducible in $\mathbb{C}\{x\}[y]$, then $p_i(0, t) = t^{n_i}$ for $n_i = \deg_t(p_i)$, especially p_i vanishes at $(0, 0)$, thus $p \in \mathbb{R}\{x\}[y]$ vanishes at $(0, a_i^2)$, so we conclude $a_i = 0$. Then by lemma 2.67 and lemma 2.64, p_i vanishes on a set of the form

$$V = \{(\epsilon x^{n_i}, g(x)) \mid |x| < \delta\}$$

with $\epsilon \in \{1, -1\}$, $g \in \mathbb{R}\{x\}$ with $g(0) = 0$ and $\delta > 0$. Thus $p(x, y)$ would vanish on

$$V' = \{(\epsilon x^m, g(x)^2) \mid |x| < \delta\}.$$

Since $g(x)^2 \geq 0$, this set lies in $U \cap K_S$ for δ small enough. A contradiction, since $\pi_1(\mathbb{V}_{\mathbb{R}}(f) \cap K_S)$ is finite.
So every irreducible factor p_i of $p(x, t^2)$ in A is monic and reducible in $\mathbb{C}\{x\}[t]$, thus p_i is a sum of squares in A. So we conclude

$$p \in \sum A^2.$$

□

Remark 3.77
As from lemma 3.56, we can also conclude from this lemma that the product of all irreducible factors of f of odd degree lies in $PO_0(S)$ (compare remark 3.57).

Lemma 3.78
Let $S \subseteq \mathbb{R}[x, y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x, y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) for every $P \in K$, $\lambda_P^*(S_2)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$ and all $P \in K$,

(iv) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(v) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K.

(vi) $y \in S$,

(vii) $f \geq 0$ on K_S,

(viii) $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite,

(ix) $P \in int(K)$.

Then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$. We have to show that $f \in PO_0(S)$.
Factorize
$$f = u(\prod_{i=1}^{s} p_i)(\prod_{j=1}^{r} q_i),$$
where

(i) $p_i \in \mathbb{R}\{x\}[y]$ are irreducible in $\mathbb{C}\{x\}[y]$ $(i = 1, \ldots, s)$,

(ii) $q_j \in \mathbb{R}\{x\}[y]$ are irreducible, but reducible in $\mathbb{C}\{x\}[y]$ $(j = 1, \ldots, r)$,

(iii) $u \in \mathbb{R}\{x\}^*$.

Define
$$p := \prod_{i=1}^{s} p_i.$$
Then either $p \in PO_0(S)$ or $-p \in PO_0(S)$ by lemma 3.58, lemma 3.75 and lemma 3.76 (compare 3.77).
Since $u \in \mathbb{R}\{x\}^*$, we have either $u \in \mathbb{R}\{x\}^2$ or $-u \in \mathbb{R}\{x\}^2$ by lemma 2.45. For each q_i, we have either $q_i \in \sum \mathbb{R}\{x\}[y]^2$ or $-q_i \in \sum \mathbb{R}\{x\}[y]^2$ by lemma 2.52. If we put this together, we have either $f \in PO_0(S)$ or $-f \in PO_0(S)$. Since $-f < 0$ on K_S, we conclude $f \in PO_0(S)$. □

Lemma 3.79
Let $S \subseteq \mathbb{R}[x, y]$, $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$ and $f \in \mathbb{R}[x, y]$. Suppose

(i) there exists an $N \in \mathbb{N}$ such that
$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),

(iii) $(x - P) \nmid s$ for all $s \in S_2 \setminus \mathbb{R}[x]$ and all $P \in K$,

(iv) any $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ with $s_1 \neq s_2$ have no common zero in K_S,

(v) for any $s \in S_2$ with $\deg_y(s) = 2$, the discriminant $d(x)$ doesn't vanish on K.

(vi) *S contains the natural generators for K,*

(vii) *$f \geq 0$ on K_S,*

(viii) *$\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite,*

(ix) *$P \in \delta K$.*

Then $f \in PO_P(S)$.

Proof — Without loss of generality, we may assume $P = 0$ and $[0, \varepsilon] \subseteq K$ for $0 < \varepsilon \in \mathbb{R}$.
By lemma 3.22, there exists an $s \in S$ such that $s = xv^2$ for a $v \in \mathbb{R}\{x\}^*$.
Since $PO(\{x\}) = PO(\{xv^2\}$, we may assume $v = 1$.
Define

$$A = \mathbb{R}\{x\}[y,t]/(t^2 - x) = \mathbb{R}\{t\}[y]$$

and let P be the preordering generated by $S \setminus \{s\}$ in A. By lemma 2.73 we have to show $f(t^2, y) \in P$.
Note that P in $\mathbb{R}\{t\}[y]$ is generated by

$$S' = \{s(t^2, y) \mid s \in S\} \subseteq \mathbb{R}[t, y].$$

Now, work in the polynomial ring $\mathbb{R}[t, y]$ and let $P' := PO(S') \subseteq \mathbb{R}[t, y]$. Then $0 \in int(\pi_1(K_{S'})$ and S' satisfies all the assumptions of lemma 3.78, so

$$f(t^2, y) \in PO_0(S') = P \subseteq \mathbb{R}\{t\}[y]$$

by lemma 3.78. □

Theorem 3.80
Let $S \subseteq \mathbb{R}[x, y]$ and $S_2 := \{s \in S \mid \deg_y(s) \leq 2\}$. Suppose

(i) *there exists an $N \in \mathbb{N}$ such that*

$$K \times \{y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+,$$

(ii) *for every $P \in K$, $\lambda_P^*(S)$ contains the natural choice of generators for $\lambda_P^{-1}(K_S)$ (up to scaling by positive reals),*

(iii) *$S \cap \mathbb{R}[x]$ contains the natural choice of generators for K,* *(iv)* $y \in S$,

(v) *any* $s \in S \setminus \mathbb{R}[x]$ *is irreducible,*

(vi) *any* $s_1, s_2 \in S_2 \setminus \mathbb{R}[x]$ *with* $s_1 \neq s_2$ *have no common zero in* K_S,

(vii) *for any* $s \in S_2$ *with* $\deg_y(s) = 2$, *the discriminant* $d(x)$ *doesn't vanish on* K.

Then $PO(S)$ *is saturated.*

Proof — Let $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_S and $\deg_y(f) = d$.
We may assume f is squarefree. Then f has only finitely many zeros in $K_S \setminus V_{\mathbb{R}}(S \cap \mathbb{R}[x])$ by proposition 2.43, so $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite.
By lemma 2.32 there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2 \sum_{i=0}^{d} y^i \geq 0 \text{ on } K_S,$$

and $\pi_1(V_{\mathbb{R}}(f) \cap K_S)$ is finite. By lemma 3.78 and lemma 3.79, we have $f_1 \in PO_P(S)$ for every $P \in K$, so $f \in PO(S)$ by theorem 2.13. $\square$

Remark 3.81
Unfortunately, theorems 3.74 and 3.80 have a lot of assumptions. At least some of them might not be necessary. In the next section, we will give examples where these theorems can be applied and discuss where a generalisation of these two theorems fails.

3.5 Examples and Open Problems

We will start this section with an example where theorem 3.74 can be applied.

Proposition 3.82
Let $K, U \subseteq \mathbb{R}$ be basic closed semi-algebraic sets. Suppose K is compact and has no isolated points and U is not compact. Let $S' \subseteq \mathbb{R}[x]$ be the natural generators for K, $S'' \subseteq \mathbb{R}[y]$ be the natural generators for U and

$$S = S' \cup S''.$$

Then $PO(S)$ satisfies (†).

Proof — Since U is unbounded, we may assume there exists an $N \in \mathbb{N}$ such that $[N, \infty) \subseteq U$. If U has a lowest point P, we may assume $P = 0$. Then $K_S \subseteq K \times \mathbb{R}_+$ and y is one of the natural generators for U, so $y \in S$. Thus $PO(S)$ satisfies (†) by theorem 3.61.
If U has no lowest point, then $PO(S)$ satisfies (†) by theorem 3.54. □

Remark 3.83
In their paper [16], Nguyen an Powers conjectured that the preordering above is saturated. Unfortunately, we could only prove that it satisfies (†). If the U above is connected, then $PO(S)$ above is saturated by proposition 1.11 or proposition 1.13 and if U consists of two unbounded intervals, then $PO(S)$ is saturated by 3.27. In all other cases, it is still open whether or not $PO(S)$ is saturated.

The next example is an application of theorem 3.74.

Proposition 3.84
Let $P_1, \dots P_r \in \mathbb{R}$ with $|P_i - P_j| > 2$ for all $i, j \in \{1, \dots, r\}$ with $i \neq j$. Define

$$f_i = (y - P_i)^2 + x^2 - 2,$$

then the preordering generated by

$$S := \{x - 1, 1 - x, f_1, \dots, f_r\} \subseteq \mathbb{R}[x, y]$$

is saturated.

Proof — We want to use theorem 3.74. Here, we have $K = [-1, 1]$. Clearly all the conditions of theorem 3.74 are satisfied:

(i) Since all f_i are monic as polynomials in y, this follows from lemma 3.23.

(ii) For $P \in [0, 1]$, we have

$$\lambda_P^*(f_i) = (y - P_i)^2 - (2 - P^2) = (y - P_i)^2 - c$$

with $c \in [1, 2]$. These are the natural generators for $\lambda_P^{-1}(K_S)$.

(iii) $\{x - 1, 1 - x\}$ are the natural generators for $K = [-1, 1]$.

(iv) Every f_i is irreducible in $\mathbb{R}[x, y]$.

(v) Suppose $P \in \mathbb{R}^2$ is a common zero of f_i and f_j $(i \neq j)$. Then we compute

$$0 = f_i(P)+f_j(P) = \|P-(0, P_i)\|^2-2+\|P-(0, P_j)\|^2-2 \geq \|(0, P_i)-(0, P_j)\|^2-4 > 0,$$

where $\|\cdot\|$ is the euclidean norm. A contradiction.

(vi) For the discriminant d_i of f_i, we compute

$$d_i = 4P_i^2 - 4(P_i^2 - 2 + x^2) = 4(2 - x^2) > 0$$

for $x \in [-1, 1] = K$.

So $PO(S)$ is saturated by theorem 3.74. □

The next two examples explain why we had to add the assumption on the discriminant and the assumption on the common zeros in theorem 3.74 and theorem 3.80:

Example 3.85
Let

$$S_1 := \{1 - x, x - 1, y^2 - x\}.$$

We have proven earlier that $PO(S_1)$ satisfies (†) (see proposition 3.38 or theorem 3.74). We would like to prove that $PO(S_1)$ is saturated.
So let $f \in \mathbb{R}[x, y]$ with $f \geq 0$ on K_{S_1} and $\deg_y(f) = 2d$.

We may assume f is squarefree and $(1-x),(x-1),(y^2-x) \nmid f$. Then $\pi_1(V_{\mathbb{R}}(f) \cap K_{S_1})$ is finite by proposition 2.43. So by lemma 2.30 there exists an $\varepsilon \in \mathbb{R}[x]$ such that

$$f_1 := f - \varepsilon^2(1+y^2)^d \geq 0 \text{ on } K_{S_1}$$

and f_1 has only finitely many zeros in K_{S_1}. By theorem 2.10 we now have to prove that $f_1 \in PO_P(S_1)$ for every $P \in [-1,1]$.
For $P \neq 0$ this is possible (compare the proof of proposition 3.37). For $P = 0$, we have the following partial result:

Lemma 3.86
Let $S_1 := \{1-x, x-1, y^2-x\}$ and $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_{S_1} with $\pi_1(V_{\mathbb{R}}(f) \cap K_{S_1})$ finite. If $f(0,0) \neq 0$, then $f \in PO_0(S_1)$.

Proof — Factorize

$$f = u \prod_{i=1}^{s} p_i$$

with $u \in \mathbb{R}\{x\}^*$ and $p_i \in \mathbb{R}\{x\}[y]$ irreducible such that each p_i has the leading coefficient x^{r_i} as a polynomial in y over $\mathbb{R}\{x\}$.
Suppose there exists a $p = p_i$ which is irreducible in $\mathbb{C}\{x\}[y]$. Then the leading coefficient of p is a unit by lemma 3.51, so we may assume p is monic. Then p vanishes on a set of the form

$$V_\delta(p) = \{(\epsilon x^n, g(x)) \mid |x| < \delta\}$$

for some $g \in \mathbb{R}\{x\}$, $\epsilon \in \{-1,1\}$, $\delta > 0$ and $n = \deg_y(p)$. Since $f(0,0) \neq 0$, we have $g(0) \neq 0$, thus $V_\delta(p) \subseteq K_{S_1}$ for δ small enough. A contradiction.
So each p_i is reducible in $\mathbb{C}\{x\}[y]$ and therefore, a sum of squares in $\mathbb{R}\{x\}[y]$ by lemma 2.53. Since $u \in \mathbb{R}\{x\}^2$ by proposition 2.45, we conclude

$$f \in \sum \mathbb{R}\{x\}[y]^2 \subseteq PO_0(S_1).$$

□

Remark 3.87
With this, it would be possible to prove that $PO(S_1)$ contains every $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_{S_1} and $f(0,0) \neq 0$. Unfortunately, this is not enough to prove that $PO(S_1)$ is saturated. To achieve this, we would need to investigate the preordering generated by $y^2 - x$ in $\mathbb{R}\{x\}[y]$. Note that

$y^2 - x$ is irreducible in $\mathbb{R}\{x\}[y]$, so this case is not covered by proposition 2.80. This is why we had to include the assumption on the discriminant in section 3.4.

Example 3.88
Let
$$S_2 := \{1-x, x-1, y, y-x\}.$$
Then $PO(S)$ satisfies (†) by theorem 3.61. Unfortunately, since y and $y-2x$ both vanish at $(0,0)$, theorem 3.80 does not apply. So it is not known whether or not $PO(S_2)$ is saturated. To prove this, we would have to show again that $f \in PO_P(S_2)$ for every $P \in K$ and every $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_{S_2} and $\pi_1(V_{\mathbb{R}}(f) \cap K_{S_2})$ finite (compare the last example). For $P \neq 0$, this is possible (compare the proof of theorem 3.61). For $P = 0$, we have the following partial results:

Lemma 3.89
Let $S_2 := \{1-x, x-1, y, y-x\}$ and $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_{S_2} with $\pi_1(V_{\mathbb{R}}(f) \cap K_{S_2})$ finite. If $f(0,0) \neq 0$, then $f \in PO_0(S_2)$.

Proof — It is enough to prove that every irreducible factor p of f lies in $PO(S_2)$. If p is reducible in $\mathbb{C}\{x\}[y]$, then $\epsilon p \in \sum \mathbb{R}\{x\}[y]^2$ with $\epsilon \in \{-1,1\}$ by lemma 2.52.
If p is irreducible in $\mathbb{C}\{x\}[y]$ and the leading coefficient is not a unit in $\mathbb{R}\{x\}$, then $\epsilon p \in PO_0(S_2)$ for $\epsilon \in \{-1,1\}$ by lemma 3.58.
If p is irreducible in $\mathbb{C}\{x\}[y]$ and the leading coefficient is a unit in $\mathbb{R}\{x\}$, then we may assume p is monic. Then we have
$$p(0,y) = (y-a)^n,$$
where $n = \deg_y(f)$ and $a \in \mathbb{R}$ by lemma 2.61. If $(0,a) \in int(K_S)$, then f would have infinitely many zeros in K_{S_2}, a contradiction (compare lemma 3.75). So $a \leq 0$.
We have $a \neq 0$ by our assumption, thus $a < 0$. Then we conclude
$$p \in PO_0(\{y\}) \subseteq PO_0(S_2)$$
by lemma 3.44.
This shows that either $f \in PO_0(S_2)$ or $-f \in PO_0(S_2)$. Since $-f \leq 0$ on K_{S_2}, we conclude $f \in PO_0(S_2)$. □

Remark 3.90
Unfortunately, this is again not enough to prove that $PO(S_2)$ is saturated. To achieve this, we would need to show that an arbitrary f with $f \geq 0$ on K_S (and possibly $f(0,0) = 0$) lies in $PO_0(S_2)$. This is also still open.
Note that in the proof of theorem 3.80, we always used a single $s \in S$ to prove that an irreducible factor p of f lies in the preordering (compare lemma 3.62). This might not be possible if $s_1, s_2 \in S \setminus \mathbb{R}[x]$ have a common zero. This is why we had to add the assumption on the common zeros in section 3.4.

The next example is a case where a generalisation of theorem 2.10 and 2.13 would be helpful:

Example 3.91
Compare example 1 in [16].
Let

$$S_3 = \{x, 1-x, yx-1, y\} \subseteq \mathbb{R}[x,y].$$

We would like to prove that $PO(S_3)$ is saturated (or satisfies (†)). Note that the term y is necessary for this (compare remark 3.26).
Since $\{0\} \times \{y \geq N\} \nsubseteq K_S$ for all $N \in \mathbb{N}$, our technical theorem 2.13 cannot be applied. But it is possible to prove that no degree cancellations can happen in this case, similar to lemma 2.8. So a stronger version of the theorem could be applied (compare remark 2.14). This, together with proposition 2.43, would reduce the problem to the question whether any $f \in \mathbb{R}[x,y]$ with $f \geq 0$ on K_{S_3} and finitely many zeros in K_{S_3} satisfies $f \in PO_P(S)$ for every $P \in K$. For $0 \neq P \in K$, we can use the same proof as in proposition 3.25, thus we would be reduced to show that $f \in PO_0(S_3)$. This is still open, but we have the following partial result:

Lemma 3.92
$(y - Z) \in PO_0(S_3)$ for every $Z \in \mathbb{Z}$.

Proof — Since $y \in PO(S_3)$, we have $y + N \in PO_0(S_3)$ for every $N \in \mathbb{N}$, so we have to show $(y - N) \in PO_0(S_3)$ for every $N \in \mathbb{N}$.
By proposition 2.45 there exists an $u \in \mathbb{R}\{x\}$ such that $u^2 = 1 - Nx$. Now

we compute

$$N(xy-1)+u^2y=(1-Nx+Nx)y-N=y-N\in PO_0(S_3).$$

$\square$

Proposition 3.93
Let $f\in\mathbb{R}[x,y]$, $f\geq 0$ *on* K_{S_3}, $f=\sum_{i=0}^{d}a_iy^i$ $a_i\in\mathbb{R}[x]$. *If* $a_d(0)\neq 0$, *then* $f\in PO_0(S_3)$.

Proof — Since $a_d(0)\neq 0$, we have $a_d(0)>0$ and $a_d=u^2$ for $u\in\mathbb{R}\{x\}$. Factorize

$$p=a_d(\prod p_i)(\prod q_j),$$

where

(i) $p_1,\ldots p_s\in\mathbb{R}\{x\}[y]$ are monic and irreducible in $\mathbb{C}\{x\}[y]$,

(ii) $q_1,\ldots q_r\in\mathbb{R}\{x\}[y]$ are monic, irreducible in $\mathbb{R}\{x\}[y]$ and reducible in $\mathbb{C}\{x\}[y]$.

Then

$$q_j\in\sum\mathbb{R}\{x\}[y]^2\ (j=1,\ldots,r)$$

by lemma 2.53. On the other hand, we have

$$p_i(0,y)=(y-a_i)^{n_i}$$

$(a_i\in\mathbb{R},n_i=\deg_y(p_i))$ by lemma 2.61. Let $N\in\mathbb{N}$ with $N>a_i$, then $y-N\in PO_0(S_3)$ by lemma 3.92 and

$$p_i\in PO_0(\{y-N\})\subseteq PO_0(S_3)$$

by lemma 3.44. Since $PO_0(S_3)$ is closed under multiplication, we conclude

$$f\in PO_0(S_3).$$

$\square$

Remark 3.94
With all of the above, it would be possible to prove that $PO(S_3)$ satisfies (‡) (since $f+\delta y^{\deg_y(f)}$ has a strictly positive leading term for every $\delta>0$).

Unfortunately, we can use theorem 3.6 from Kuhlmann, Marshall and Schwartz to prove that $PO(S')$ with

$$S' := \{x, 1-x, yx-1, \} = S_3 \setminus \{y\}$$

already satisfies (‡):

Proposition 3.95
$PO(S')$ satisfies (‡).

Proof — Let $K := [0,1]$, then $K_{S'} \subseteq K \times \mathbb{R}$. We compute

$$\lambda_P^*(xy-1) = P(y - \frac{1}{P})$$

is the natural generator for

$$\lambda_P^{-1}(K_{S'}) = [\frac{1}{P}, \infty)$$

for all $P \in (0,1]$ and $\lambda_0^{-1}(K_{S'}) = \varnothing$.
So $PO(S')$ satisfies (‡) by theorem 3.6. □

We finish this chapter with an example where our new theory cannot be applied:

Example 3.96
Let $S = \{1-x, 1+x, 1-xy\}$. Again, it would be possible to prove that no degree cancellation happens in this case (similar to lemma 2.8). Unfortunately, K_S is not contained in a half-strip (not even after a linear change of variables). So even a general version of theorem 2.13 could not be applied here. A general version of 2.10 could only deal with polynomials of even degree. So we would need a whole new method in this case. Therefore, this is also still open.

4 Psd = sos in coordinate rings

From now on, we work in a slightly different setup:
We will keep the following notion throughout this chapter:
Let C be an irreducible smooth affine curve over $\mathbb{R}$ with $C(\mathbb{R})$ compact and let $V = C \times \mathbb{A}^1$. We will write

$$\mathbb{R}[C] = \mathbb{R}[x_1, \dots x_n]/I(C)$$

for the coordinate ring of C and

$$\mathbb{R}[V] = \mathbb{R}[C][y]$$

for the coordinate ring of V. The goal of this chapter is to prove that every $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ and finitely many real zeros is a sum of squares in $\mathbb{R}[V]$ (see theorem 4.16). To prove this we will use similar methods as in chapter 2.
In the first section 4.1, we will generalize proposition 2.27 to coordinate rings, i.e. we will construct a $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ such that

$$f - \varepsilon(1+y^2)^d \geq 0 \text{ on } V(\mathbb{R}) \ (2d = \deg_y(f)).$$

In the second section, we will generalize proposition 2.1 to the coordinate ring $\mathbb{R}[C]$ (see proposition 4.9). We will prove the following: Given two continuous functions

$$\psi, \varphi : C(\mathbb{R}) \longrightarrow \mathbb{R}$$

with $\psi \leq \varphi$ on $C(\mathbb{R})$, we will prove that, under certain assumptions, there exists a $p \in \mathbb{R}[C]$ such that $\psi \leq p \leq \varphi$ holds on $C(\mathbb{R})$. This is also a generalization of the WeierstraßApproximation Theorem (see [3, Theorem 4.6.4]).
In third section, we will finally state the theorem mentioned above and prove it. Unfortunately, the theorem is not enough to show that psd=sos holds in $\mathbb{R}[V]$ (since there may exist psd regular functions with infinitely many zeros in $V(\mathbb{R})$).
In section 4.4, we will make reductions similar to those in section 2.3 to show that theorem 4.16 is enough to show that psd=sos holds in the case

where $C = V(x_1^2 + x_2^2 - 1)$ is the circle. Unfortunately, the methods used there do not seem to generalize to arbitrary smooth curves (with $C(\mathbb{R})$ compact).

4.1 The Existence of $\varepsilon \in \mathbb{R}[C]$

In this section we will give a construction for the term $\varepsilon \in \sum \mathbb{R}[C]^2$ mentioned above. For this, we will use again the theory of semialgebraic functions (compare section 2.2). If not mentioned otherwise, we fix the euclidean topology on the set of real points $V(\mathbb{R})$ and $C(\mathbb{R})$.

Lemma 4.1
Let $g \in \mathbb{R}[C][y,z]$. Then

$$\begin{array}{rcl} \varphi_g : C(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \min\limits_{(y,z)\in\mathbb{S}_1} g(x,y,z). \end{array}$$

is a bounded semialgebraic function.

Proof — It is clear that φ_g is a semialgebraic function (compare lemma 2.21).

Suppose φ_g is not continuous, then, for all $N \in \mathbb{N}$, there exists $P \in C(\mathbb{R})$, $\delta > 0$ and $P_N \in C(\mathbb{R})$ such that

$$\begin{array}{rcl} \|P - P_N\| & < & \dfrac{1}{N} \\ |\varphi_g(P) - \varphi_g(P_N)| & > & \delta. \end{array}$$

Since $\{P_N\}\times\mathbb{S}_1$ is compact for every $N \in \mathbb{N}$, there exists an $Q_N \in \{P_N\}\times\mathbb{S}_1$ such that

$$\varphi_g(P_N) = g(Q_N).$$

$C(\mathbb{R}) \times \mathbb{S}_1$ is compact, so by going to a subsequence, we may assume Q_N converges to $Q_1 \in \{P\} \times \mathbb{S}_1$. Since g is continuous, $g(Q_N)$ converges to $g(Q_1)$. On the other hand, there exists a $Q_2 \in \{P\} \times \mathbb{S}_1$ such that

$$\varphi_g(P) = g(Q_2).$$

By definition, we have

$$g(Q_2) = \varphi_g(P) \leq g(Q_1).$$

If $g(Q_2) = g(Q_1)$, then $\varphi_g(P_N) = g(Q_N)$ would converge to $g(Q_1) = g(Q_2) = \varphi_g(P)$. A contradiction since $|\varphi_g(P) - \varphi_g(P_N)| > \delta$.

So $g(Q_2) < g(Q_1)$. Now let $Q'_N \in P_N \times \mathbb{S}_1$ such that $(Q'_N)_N \longrightarrow Q_2$. Then $g(Q'_N) \longrightarrow g(Q_2)$(since g is continuous) and we have

$$g(Q'_{N_0}) < g(Q_{N_0}) = \varphi_g(P_{N_0})$$

for $N_0 \in \mathbb{N}$ big enough. A contradiction. $\square$

Remark 4.2
The main difference between lemma 4.1 and lemma 2.21 above is that here the semialgebraic function is continuous. With this, the whole construction of the $\varepsilon \in \mathbb{R}[C]$ term becomes much easier (compared to section 2.2).

Lemma 4.3
Let $\varphi : C(\mathbb{R}) \longrightarrow \mathbb{R}$ be a continuous semialgebraic function with $\varphi \geq 0$ on $C(\mathbb{R})$. Suppose φ has only finitely many zeros in $C(\mathbb{R})$.
Then there exists a $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ such that

$$\varphi \geq \varepsilon \text{ on } C(\mathbb{R}).$$

Proof — Let $P_1, \ldots, P_r$ be the zeros of φ in $C(\mathbb{R})$. For $i = 1, \ldots, r$, take $t_i \in \mathbb{R}[C]$ with $t_i(P_i) = 0$. Define

$$q(x) := \prod_{i=1}^{r} t_i$$

and

$$p(x) := q(x)^2.$$

By proposition 2.24 there exists an open neighborhood U of $P_1, \ldots P_r$ in $C(\mathbb{R})$ and an $s \in \mathbb{N}$ such that

$$p(x)^s \leq \varphi(x) \text{ for all } x \in U.$$

If $U = C(\mathbb{R})$, the lemma is proven (with $\varepsilon := p^s$), if not, then

$$K := C(\mathbb{R}) \setminus U \neq \varnothing$$

is a compact set. So p attains its maximum $M > 0$ and φ attains its minimum $m > 0$ on the compact set K. We compute

$$\frac{p(x)^s}{M^s} \leq 1 \leq \frac{\varphi(x)}{m}$$

and
$$\frac{m}{M^s}p(x)^s \leq \varphi(x)$$
for all $x \in K$. So with both parts together, we compute
$$\varepsilon := \min\{1, \frac{m}{M^s}\}p^s \leq \varphi \text{ on } C(\mathbb{R}).$$

□

Lemma 4.4
Let $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ and $\deg_y(f) = 2d$. Suppose f has only finitely many real zeros in V. Then there exists a $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ such that
$$f - \varepsilon(1+y^2)^d \geq 0 \text{ on } V(\mathbb{R}).$$

Proof — Compare lemma 2.27.
Write
$$f = \sum_{i=0}^{2d} a_i y^i$$
with $a_i \in \mathbb{R}[C]$. Define
$$\tilde{f} := \sum_{i=0}^{2d} a_i y^i z^{2d-i} = z^{2d} f(x, \frac{y}{z}).$$
We identify
$$C(\mathbb{R}) \times \mathbb{R} \subseteq C(\mathbb{R}) \times \mathbb{S}_1$$
via
$$(x,y) \longmapsto \big(x, \frac{y}{\sqrt{1+y^2}}, \frac{1}{\sqrt{1+y^2}}\big).$$
Then $\varphi := \varphi_{\tilde{f}}$ is a bounded semialgebraic function from $C(\mathbb{R})$ to $\mathbb{R}$ by lemma 4.1.
For $z \neq 0$, we compute
$$\tilde{f}(x,y,z) = z^{2d} f(x, \frac{y}{z}),$$
and
$$\tilde{f}(x,1,0) = a_{2d}(x) \neq 0.$$
Thus $\varphi \geq 0$ on $C(\mathbb{R})$ with only finitely many zeros in $C(\mathbb{R})$, so by lemma 4.3 there exists a $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ with
$$\varphi \geq \varepsilon \text{ on } C(\mathbb{R}).$$

Especially, we have

$$\tilde{f} - \varepsilon \geq 0 \text{ on } C(\mathbb{R}) \times \mathbb{S}_1.$$

Now, suppose $(x_0, y_0) \in C(\mathbb{R}) \times \mathbb{R} = V(\mathbb{R})$ is arbitrary. Then we have

$$\left(x_0, \frac{y_0}{\sqrt{1+y_0^2}}, \frac{1}{\sqrt{1+y_0^2}}\right) \in C(\mathbb{R}) \times \mathbb{S}_1$$

and we compute

$$\begin{aligned} f(x_0, y_0) - \varepsilon(x_0)(1+y_0^2)^d &= (1+y_0^2)^d \tilde{f}\left(x_0, \frac{y_0}{\sqrt{1+y_0^2}}, \frac{1}{\sqrt{1+y_0^2}}\right) - \varepsilon(x_0)(1+y_0^2)^d \\ &= (1+y_0^2)^d \left[\tilde{f}\Big(x_0, \frac{y_0}{\sqrt{1+y_0^2}}, \frac{1}{\sqrt{1+y_0^2}}\Big) - \varepsilon(x_0)\right] \\ &\geq 0. \end{aligned}$$

So

$$f - \varepsilon(1+y^2)^d \geq 0 \text{ on } V(\mathbb{R}).$$

□

Remark 4.5
Similar to lemma 2.16 in section 2.2 above, it is also here possible to show that such a $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ can only exist if f has finitely many real zeros. So an attempt to generalize lemma 4.4 to all $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ fails.

4.2 Approximation in Manifolds

In this chapter, we will generalize proposition 2.1 to smooth curves (see proposition 4.9). The proof is essentially the same as in [14], we only need the language of analytic manifolds and analytic sets to describe analytic functions and their relation to regular (polynomial) functions on $C(\mathbb{R})$. First, we introduce some basic notations (see [11] and [12] for a more complete introduction):

A topological manifold M of dimension n is a second countable Hausdorff space, which is locally homeomorphic to $\mathbb{R}^n$, so for every $P \in M$, there exists an open set $U_P \subseteq M$ and a homeomorphism

$$\delta : U_P \longrightarrow \delta(U_P) \subseteq \mathbb{R}^n.$$

Here, (U_P, δ) is called a chart of M around P.
If $\delta(P) = 0$, the chart is centered at P.
A family of charts $\mathcal{A}$ is called an analytic atlas on M, if

(i) $\bigcup\limits_{(U,\delta)\in\mathcal{A}} U = M$ and

(ii) for any $(U, \delta), (U', \delta') \in \mathcal{A}$ with $U \cap U' \neq \varnothing$, the transition map

$$\delta \circ (\delta')^{-1} : \delta'(U \cap U') \longrightarrow \delta(U \cap U')$$

is an analytic homeomorphism.

An analytic atlas $\mathcal{A}$ on M is maximal if it is not contained in a strictly larger analytic atlas. Such a maximal atlas is called an analytic structure on M and an analytic manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a analytic structure. Whenever the structure is fixed, we will just say M is a manifold.

A subset $N \subseteq M$ is called a submanifold of $(M, \mathcal{A})$, if for every point $P \in N$, there exists a $m \in \mathbb{N}$ a chart (U, δ) of M around P such that $\delta(U \cap N) = \delta(U) \cap \mathbb{R}^m$. Then $\mathcal{A}$ induces an analytic structure on N. It is a well known fact, that the real points of a smooth affine real curve $C(\mathbb{R}) \subseteq \mathbb{R}^n$ form a analytic submanifold of $\mathbb{R}^n$ of dimension 1 (the proof is nearly the same as in the smooth case. See [11, chapter 8] for a proof in the smooth case or compare [12, chapter C.3]).

So we equip $C(\mathbb{R})$ with the topology and the structure of an analytic manifold.

A continuous map $\varphi : C(\mathbb{R}) \longrightarrow \mathbb{R}$ is called analytic in a neighborhood of $P \in C(\mathbb{R})$, if it is analytic in local coordinates, i.e. if there exists a chart (U, δ) around P such that

$$\varphi \circ \delta^{-1} : \delta(U) \longrightarrow \mathbb{R}$$

is analytic.
φ is analytic, if it is analytic in a neighborhood of every point $P \in C(\mathbb{R})$.
We denote the by $\mathcal{A}_C$ the ring of analytic functions from $C(\mathbb{R})$ to $\mathbb{R}$. Especially, every $f \in \mathbb{R}[C]$ is a analytic function from $C(\mathbb{R})$ to $\mathbb{R}$, since it is the restriction of a polynomial function to a submanifold of $\mathbb{R}^n$ (compare [12, chapter 1.1]).
Let $P \in C(\mathbb{R})$ be a point. We denote by $\mathcal{A}_P$ the germs of analytic functions at P (compare [12, chapter B.4]). Especially the elements of $\mathcal{A}_P$ are analytic functions from U to $\mathbb{R}$, where U is a neighborhood of P in $C(\mathbb{R})$. So we have $\mathbb{R}[C] \subseteq \mathcal{A}_P$ for every $P \in C(\mathbb{R})$. $\mathcal{A}_P$ is a local ring and the maximal ideal of $\mathcal{A}_P$ is given by

$$\mathfrak{m}_P = \{f \in \mathcal{A}_P \mid f(P) = 0\}.$$

It is a principal ideal (since it is generated by $x \circ \delta$, where (U, δ) is a chart centered at P).
The generator can be chosen to be a polynomial $f \in \mathbb{R}[C]$ (compare [12, chapter VII.15]). Indeed, any $f \in \mathbb{R}[C]$ that has a zero of order 1 at P generates this maximal ideal.
Now, we are able to generalize proposition 2.1, but we first need some lemmas:

Lemma 4.6
Let $P \in C(\mathbb{R})$ and (U, δ) be a chart centered at P. Then there exists a $g \in \mathbb{R}[C]$ such that

(i) $g > 0$ on $C(\mathbb{R}) \setminus \{P\}$;

(ii) $g \circ \delta^{-1} = x^2 g_0$ with $g_0 \in \mathbb{R}\{x\}^*$.

Proof — Let $t \in \mathbb{R}[C]$ such that t generates the maximal ideal $\mathfrak{m}$ of $\mathcal{A}_P$. Let $t_1, \dots, t_r \in \mathbb{R}[C]$ such that $V_{\mathbb{R}}(t_1, \dots, t_r) = \{P\}$ and define

$$g := t^2 + t_1^2 + \cdots + t_r^2.$$

Then $g > 0$ on $C(\mathbb{R}) \setminus \{P\}$. Further, we have for the description in local coordinates $\tilde{t} := t \circ \delta^{-1}$:

$$\tilde{t} = xh_0$$

for $h_0 \in \mathbb{R}\{x\}^*$. For all the other t_i, we compute

$$\tilde{t_i} = xh_i \, (i = 1, \dots, r)$$

with $h_i \in \mathbb{R}\{x\}$. So we finally compute for g:

$$\tilde{g} = x^2(h_0^2 + h_1^2 + \dots + h_r^2) =: x^2 g_0.$$

Since

$$g_0(0) \geq h_0(0)^2 > 0,$$

we conclude $g_0 \in \mathbb{R}\{x\}^*$ by proposition 2.45. $\square$

Lemma 4.7
Let $P \in C(\mathbb{R})$, $n \in \mathbb{N}$ even and (U, δ) be a chart centered at P. Then there exists a $g \in \mathbb{R}[C]$ such that

- *$g > 0$ on $C(\mathbb{R}) \setminus \{P\}$;*
- *$g \circ \delta^{-1} = x^n g_0$ with $g_0 \in \mathbb{R}\{x\}^*$.*

Proof — $n \in \mathbb{N}$ is even, say $n = 2m$. Take $g' \in \mathbb{R}[C]$ from lemma 4.6 and define

$$g := (g')^m.$$

Then g is as desired. $\square$

Lemma 4.8
Let $f \in \mathbb{R}[x]$ with $n = \deg(f)$, $P \in C(\mathbb{R})$ and (U, δ) be a chart centered at P. Then there exists a $p \in \mathbb{R}[C]$ and a $g \in \mathbb{R}\{x\}$ such that

$$p \circ \delta^{-1} = f + x^{n+1}g.$$

Proof — We show this by an induction on the degree of f.
If $\deg(f) = 0$, then the constant function $p = f \in \mathbb{R} \subseteq \mathbb{R}[C]$ is as desired.
So assume the lemma is proven for polynomials of degree $< s$ and assume $deg(f) = s$. Let $t \in \mathbb{R}[C]$ such that t generates the maximal ideal $\mathfrak{m}_P$ of $\mathcal{A}_P$. We denote by $\tilde{t} = t \circ \delta^{-1}$ the description in local coordinates, i.e.

$$\tilde{t} = xh_0(x),$$

where $h_0 \in \mathbb{R}\{x\}^*$. Now write

$$f = ax^s + f'$$

with $a \in \mathbb{R}$ and $f' \in \mathbb{R}[x]$ with $\deg(f') < s$. By our induction, there exists a $p' \in \mathbb{R}[C]$ such that

$$p' \circ \delta^{-1} = f' + x^s g'$$

for some $g' \in \mathbb{R}\{x\}$. Now define

$$p := p' + \frac{(a - g'(0))}{h_0(0)^s} t^s.$$

Then we compute for the description in local coordinates:

$$\tilde{p} = f' + x^s g'(0) + x^s(g' - g'(0)) + \frac{(a - g'(0))}{h_0(0)^s} h_0^s x^s$$

and for the coefficient of x^s in this equation we compute

$$g'(0) + \frac{(a - g'(0))}{h_0(0)^s} h_0(0)^s = a$$

(Note that $h_0(0) \neq 0$ since $h_0 \in \mathbb{R}\{x\}^*$). So $\tilde{p}$ has the desired form. □

Proposition 4.9
Let C be a smooth real curve with $C(\mathbb{R})$ compact. Let

$$\varphi, \psi : C(\mathbb{R}) \longrightarrow \mathbb{R}$$

two continuous functions (as maps between topological manifolds) with $\varphi(P) \leq \psi(P)$ for all $P \in C(\mathbb{R})$. Suppose $\varphi(P) < \psi(P)$ for all but finitely many $P \in C(\mathbb{R})$ and both φ and ψ are analytic in a neighborhood of each $P \in C(\mathbb{R})$ with $\varphi(P) = \psi(P)$ (as a map between analytic manifolds). Then there exists a $p \in \mathbb{R}[C]$ such that

$$\varphi \leq p \leq \psi \text{ holds on } C(\mathbb{R}).$$

Proof — Let $Z := \{P_1, \ldots, P_s\} \subseteq C(\mathbb{R})$ be the points where $\varphi = \psi$. We show the proposition by an induction on the number of points s.
If there are no such points, then such a p exists by the Weierstraß Approximation Theorem ([3, Theorem 4.6.4]).
So assume the proposition is proven for all Z with $|Z| < s$ and suppose

$Z = \{P_1, \ldots, P_s\}$. Fix $P = P_s \in Z$. Since φ and ψ are analytic at P, there exists a neighborhood U of P in $C(\mathbb{R})$ such that

$$\varphi, \psi : U \longrightarrow \mathbb{R}$$

are analytic functions. Let (U, δ) be a chart centered at P and let $\tilde{\varphi} := \varphi \circ \delta^{-1}$ and $\tilde{\psi} := \psi \circ \delta^{-1}$ be the description of φ and ψ in local coordinates. Then we have

$$\begin{aligned} \tilde{\varphi} &= f + x^n g_1 \\ \tilde{\psi} &= f + x^n g_2, \end{aligned}$$

where $f \in \mathbb{R}[x]$, $g_1, g_2 \in \mathbb{R}\{x\}$, $g_1(0) \neq g_2(0)$ and $n \in \mathbb{N}$. Since $\tilde{\varphi} \leq \tilde{\psi}$ in a neighborhood of 0, we conclude n is even and $g_1(0) < g_2(0)$. By lemma 4.8 there exist an $f_0 \in \mathbb{R}[C]$ such that

$$\tilde{f}_0 = f + x^n g_0$$

with $g_0 \in \mathbb{R}\{x\}$. So we compute

$$\begin{aligned} \tilde{\varphi} - \tilde{f}_0 &= x^n(g_1 + g_0) \\ \tilde{\psi} - \tilde{f}_0 &= x^n(g_2 + g_0). \end{aligned}$$

Since n is even, there exists an $h \in \mathbb{R}[C]$ such that

- $h > 0$ on $C(\mathbb{R}) \setminus \{P\}$;

- $\tilde{h} = x^n h_1(x)$ with $h_1 \in \mathbb{R}\{x\}^*$

by lemma 4.7. Then

$$\varphi_1 := \frac{\varphi - f_0}{h} : C(\mathbb{R}) \setminus \{P\} \to \mathbb{R}$$

and

$$\psi_1 := \frac{\psi - f_0}{h} : C(\mathbb{R}) \setminus \{P\} \to \mathbb{R}$$

are continuous functions, which are analytic in a neighborhood of $P_1, \ldots, P_{s-1}$. By construction, φ_1, ψ_1 can be extended continuously to $C(\mathbb{R})$ by defining

$$\varphi_1(P) := (g_1(0) + g_0(0))h_1(0)^{-1}$$

and

$$\psi_1(P) := (g_2(0) + g_0(0))h_1(0)^{-1}.$$

Note that $h_1 \in \mathbb{R}\{x\}^*$, so $h_1(0) \neq 0$. Further we have $\varphi_1(P) < \psi_1(P)$. So by our induction, there exists a $k \in \mathbb{R}[C]$ such that

$$\varphi_1 \leq k \leq \psi_1 \text{ on } C(\mathbb{R}).$$

Define

$$p := f_0 + hk \in \mathbb{R}[C],$$

then we compute

$$\varphi \leq p \leq \psi \text{ on } C(\mathbb{R}).$$

□

Remark 4.10

(i) The proof of the last proposition is essentially the same as the proof of proposition 2.1 in [14] and [16]. We only needed to ensure that polynomials with the essential properties exist in this case (see lemma 4.7 and lemma 4.8).

(ii) The proposition above can be easily generalized to arbitrary curves C', if we assume that the points $\{P_1, \ldots, P_s\}$ in the proof above are all regular.

(iii) At a singular point $P \in C'(\mathbb{R})_{sing}$, it is not even clear how to define analyticity in a way that it helps us to generalize proposition 4.16 below.

4.3 Proof of the Theorem

Now we can prove the main theorem of this chapter. As before, C is an irreducible smooth affine real curve over $\mathbb{R}$ with $C(\mathbb{R})$ compact and $V := C \times \mathbb{A}^1$. Further we fix $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ and $V_{\mathbb{R}}(f)$ finite for this section.
Our goal is to prove that

$$f \in \sum \mathbb{R}[V]^2.$$

We keep the notation of the last section:
$C(\mathbb{R})$ forms an analytic submanifold of $\mathbb{R}^n$. For $P \in \mathbb{R}[C]$, we denote by $\mathcal{A}_P$ the germs of analytic functions at P, especially every $p \in \mathcal{A}_P$ is an analytic function from an open neighborhood U_P of P to $\mathbb{R}$. Now, given $g \in \mathbb{R}[V]$ and a chart (U, δ), we identify $\mathbb{R}[V] = \mathbb{R}[C][y]$ and write each coefficient of g as a polynomial in y over $\mathbb{R}[C]$ in local coordinates. Then we obtain $\tilde{g} \in \mathbb{R}\{x\}[y]$. If the chart (U, δ) of $C(\mathbb{R})$ is fixed, we will call $\tilde{g}$ the description of g in local coordinates (Note that $(U \times \mathbb{R}, \delta \times id)$ is a chart of the analytic manifold $V(\mathbb{R})$).

Lemma 4.11
Let $P \in C(\mathbb{R})$. Then there exist functions $g_1 \dots g_r \in \mathcal{A}_P[y]$ such that

$$f = \sum_{i=1}^{s} g_i^2 \text{ in } \mathcal{A}_P[y]$$

Proof — Let (U, δ) be a chart centered at P. Let $\tilde{f} \in \mathbb{R}\{x\}[y]$ be the description of f in local coordinates. Then $\tilde{f} \in \mathbb{R}\{x\}[y]$ is a psd function (since $f \geq 0$ on $V(\mathbb{R}) = C(\mathbb{R}) \times \mathbb{R}$). So $\tilde{f} \in \sum \mathbb{R}\{x\}[y]^2$ by proposition 2.75, i.e. there exist $h_1 \dots, h_s \in \mathbb{R}\{x\}[y]$ such that

$$f = h_1^2 + \cdots + h_s^2.$$

Now let g_i be the element obtained by pullback of the coefficients via δ, i.e. for

$$h_i = \sum a_i y^i$$

with $a_i \in \mathbb{R}\{x\}$, we define

$$g_i := \sum (a_i \circ \delta) y^i.$$

Especially the coefficients of the g_i are analytic functions in a neighborhood of P, hence $g_i \in \mathcal{A}_P[y]$ and we have

$$f = \sum_{i=1}^{s} g_i^2 \text{ in } \mathcal{A}_P[y].$$

□

Definition 4.12
For any $P \in C(\mathbb{R})$, we denote by A_P the set of continuous functions

$$\varphi : C(\mathbb{R}) \longrightarrow \mathbb{R},$$

that are analytic in a neighborhood of P (as a map between analytic manifolds).

Lemma 4.13
Let $P_1 \ldots, P_r \in C(\mathbb{R})$ and $A := \bigcap_{i=1}^{r} A_{P_i}$. Then $f \in \sum A[y]^2$.

Proof — For every $P \in C(\mathbb{R})$, there exist an open set $U_P \subseteq C(\mathbb{R})$ and analytic functions

$$p_{P,i} : U_P \times \mathbb{R} \longmapsto \mathbb{R}\ (i = 1, \ldots, s_P)$$

such that

$$f = \sum_{i=1}^{r_P} p_{P,i}^2$$

by lemma 4.11. Since $C(\mathbb{R})$ is compact, there exist $P_{r+1}, \ldots, P_s \in C(\mathbb{R})$ such that

$$(\bigcup_{i=1}^{s} U_{P_i}) = C(\mathbb{R}).$$

By shrinking U_{P_i} if necessary, we may assume $P_j \notin U_{P_i}$ for $i \neq j$. Let $1 = \mu_1 + \cdots + \mu_s$ be a continuous partition of unity such that $supp(\mu_i) \subseteq U_{P_i}$, i.e

$$\mu_i : \mathbb{C}(\mathbb{R}) \longrightarrow \mathbb{R}\ (i = 1, \ldots, s)$$

are continuous and $\mu_i(P) = 0$ for $P \in C(\mathbb{R}) \backslash U_{P_i}$. Since $P_j \notin U_{P_i} \supseteq supp(\mu_i)$ $((i \neq j))$ by construction, there exists for $i = 1, \ldots, s$ an open neighborhood U_i of P_i such that

$$\mu_i|_{U_i} \equiv 1$$

and

$$\mu_i|_{U_j} \equiv 0 \; (j \neq i).$$

Now we compute

$$\mu_j f = \sum_{j=1}^{s} \sum_{i=1}^{r_j} (\sqrt{\mu_j} p_{P_j,i})^2,$$

where the coefficients of $\phi_{i,j} := \sqrt{\mu_j} p_{P_j,i}$ are expanded by 0 outside of U_{P_j}. Then $\phi_{i,j}$ is continuous and analytic a neighborhood of $P_1, \ldots, P_s$ since $\phi_{i,j} = h_{i,P_j}$ in U_j (since $\mu_j \equiv 1$) and $\phi_{i,j} \equiv 0$ in U_k $(k \neq j)$. So $\phi_{i,j} \in A[y]$. Then we compute

$$f = \sum_{j=1}^{s} \mu_j f = \sum_{i=1}^{s} (\sqrt{\mu_j} p_{P_j,i})^2.$$

□

Remark 4.14
Note that in the last lemma, we could have also constructed a description $f = \sum g_i^2$ with continuous replaced by C^∞. We just have to take a C^∞ partition of unity instead of a continuous one. Unfortunately, the g_i above can never be analytic on $C(\mathbb{R})$, since the partition of unity in the proof above can never be analytic. This is the main reason why we had to formulate proposition 4.9 above in such a specific way.

Lemma 4.15
Let $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ with finitely many real zeros $P_1 \ldots, P_r \in C(\mathbb{R})$,

$$A := \bigcap_{i=1}^{r} A_{P_i}$$

and $f \in \sum A[y]^2$ with $d = \deg_y(f)$.
Then there exist $\sigma \in \sum \mathbb{R}[V]^2$ and $b_0, \ldots, b_d \in \mathbb{R}[C]$, such that

(i) $f = \sigma + \sum\limits_{i=1}^{2d} b_i(x) y^i$,

(ii) $|b_i| \leq \varepsilon$ *on* $C(\mathbb{R})$.

Proof — Compare lemma 2.2 above.
Since $f \in \sum A[y]^2$, there exist $f_1, \ldots f_s \in A[y]$ such that

$$f = \sum_{i=1}^{s} f_i^2.$$

Write

$$f_i = \sum_{i=0}^{d_i} a_{i,j} y^j$$

with $a_{i,j} \in A$. Clearly, $2d_i \leq d$ for $i = 1, \dots s$. By lemma 4.9 we can approximate $a_{i,j} \in A$ by $c_{i,j}^{(N)} \in \mathbb{R}[C]$ such that

$$-\frac{\varepsilon}{N} \leq a_{i,j} - c_{i,j}^{(N)} \leq \frac{\varepsilon}{N} \text{ on } C(\mathbb{R}).$$

Define

$$g_i^{(N)} = \sum_{j=0}^{d_i} c_{i,j}^{(N)} y^j \in \mathbb{R}[C][y] = \mathbb{R}[V]$$

and

$$\sum_{i=0}^{d} b_i^{(N)} y^i := f - \sum_{i=1}^{s} (g_i^{(N)})^2. \tag{4.1}$$

We claim that

$$|b_i^{(N)}| \leq \varepsilon \text{ on } C(\mathbb{R})$$

for $N \in \mathbb{N}$ big enough:
If we compute the right hand side in equation 4.1, we conclude that there exists $q_i \in \mathbb{R}[C][t_1, \dots, t_r]$ such that

$$b_i^{(N)} = q_i(x, c_{j,k}^{(N)}), \tag{4.2}$$

where the indices on the right hand side run through all possible (j, k). Now write $c_{j,k}^{(N)} = a_{j,k} + d_{j,k}^{(N)}$ and substitute this in 4.2 to get

$$\begin{aligned} b_i^{(N)}(x) &= q_i(x, a_{j,k}(x) + c_{j,k}^{(N)}(x)) \qquad (4.3)\\ &= q_i(x, a_{j,k}(x)) + h_i(c_{j,k}^{(N)}), \end{aligned}$$

where the h_i are polynomials in r variables, whose coefficients are continuous functions on $C(\mathbb{R})$ (that depend on the given $a_{j,k}$).
If $c_{j,k}^{(N)} = a_{j,k}$ for all (j, k), then the approximation above is exact, thus $a_i = 0$ for all $i \in \{0, \dots, d\}$ (see equation 4.1), thus, we conclude from equation 4.2, that

$$0 = b_i(x) = q_i(x, a_{j,k}(x))$$

and from equation 4.3

$$0 = b_i(x) = q_i(x, a_{j,k}(x)) + h_i(0, \dots, 0) = h_i(0, \dots, 0).$$

Let $\tilde{h}_i$ be the polynomial obtained by replacing each coefficient of h_i by its maximum absolute value on $C(\mathbb{R})$. Since $h_i(0,\ldots,0) = 0$, we still have $\tilde{h}_i(0,\ldots,0) = 0$. Next, we define $s_i \in \mathbb{R}[t]$ by

$$s_i(t) = \frac{\tilde{h}_i(t,\ldots,t)}{t}$$

(This is well defined since $\tilde{h}_i(0,\ldots,0) = 0$). Further, let M be the maximum of ε on $C(\mathbb{R})$ and M_i to be the Maximum of s_i on $[0,M]$. Since

$$|d_{j,k}^{(N)}| \leq \frac{\varepsilon}{N}$$

for all (j,k) and $N \in \mathbb{N}$, we now compute

$$\begin{aligned} |b_i^{(N)}(x)| &= |q_i(x, a_{j,k}(x)) + h_i(c_{j,k}^{(N)})| \\ &\leq \tilde{h}_i(|c_{j,k}^{(N)}|) \\ &\leq \tilde{h}_i(\frac{\varepsilon}{N},\ldots,\frac{\varepsilon}{N}) \\ &= \frac{\varepsilon}{N} s_i(\frac{\varepsilon^2}{N}) \\ &\leq \varepsilon\frac{M_i}{N} \text{ on } C(\mathbb{R}) \end{aligned}$$

So for N big enough, we have $\frac{M_i}{N} \leq 1$ and the lemma is proven. □

Theorem 4.16
Let $f \in \mathbb{R}[V] = \mathbb{R}[C][y]$ with $f \geq 0$ on $V(\mathbb{R})$ and $V_{\mathbb{R}}(f)$ finite.
Then $f \in \sum \mathbb{R}[V]^2$.

Proof — Write

$$f = \sum_{i=0}^{n} a_i y^i,$$

with $n \in \mathbb{N}$ and $a_i \in \mathbb{R}[C]$. Since $f \geq 0$ on $V(\mathbb{R})$, n is even, i.e $n = 2d$. By lemma 4.4, there exists a $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ such that

$$f_1 := f - \varepsilon(y^2+1)^d \geq 0 \text{ on } V(\mathbb{R}).$$

By lemma 4.15 there exist $\sigma_1 \in \sum \mathbb{R}[V]^2$ and $b_i \in \mathbb{R}[C]$ $(i = 1,\ldots,2d)$ such that

- $f_1 = \sigma_1 + \sum_{i=1}^{2d} b_i(x)y^i$,

- $|b_i| \leq \frac{2}{5}\varepsilon$ on $C(\mathbb{R})$.

Then we compute

$$f = \sigma_1 + \sigma_2 + \sigma_3,$$

where

$$\begin{aligned}\sigma_2 &:= \frac{2}{5}\varepsilon(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} b_i y^i,\\ \sigma_3 &:= \varepsilon[(1+y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + \cdots + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} b_i y^i)].\end{aligned}$$

It is enough to show that $\sigma_i \in \sum \mathbb{R}[V]^2$ (i=1,2,3).
By definition, we have $\sigma_1 \in \sum \mathbb{R}[x,y]^2$. We show $\sigma_2, \sigma_3 \in \sum \mathbb{R}[V]^2$ in the next lemmas. This finishes the proof. □

Lemma 4.17
In the above setting, we have $\sigma_2 \in \sum \mathbb{R}[V]^2$.

Proof — compare lemma 2.11 above.
Since $|b_i| \leq \frac{2}{5}\varepsilon$ on $C(\mathbb{R})$, we have

$$\frac{2}{5}\varepsilon^2 \pm a_i \in \sum \mathbb{R}[V]^2$$

by theorem 1.16, for $i = 0, \ldots, 2d$. This yields

$$\frac{2}{5}\varepsilon y^i + a_i y^i \in \sum \mathbb{R}[V]^2 \text{ for i even.}$$

For i odd, i.e. $i = 2m+1$, we compute

$$\begin{aligned}&\frac{2}{5}\varepsilon(y^{2m} + y^{2m+1} + y^{2m+2}) + a_{2m+1}y^{2m+1}\\ =\ &\frac{2}{5}\varepsilon(y^{2m} + \frac{1}{2}y^{2m}((y+1)^2 - y^2 - 1) + y^{2m+2}) + a_{2m+1}\frac{1}{2}y^{2m}((y+1)^2 - y^2 - 1)\\ =\ &\frac{1}{2}[(\frac{2}{5}\varepsilon - a_{2m+1})y^{2m} + (\frac{2}{5}\varepsilon - a_{2m+1})y^{2m+2} + (\frac{2}{5}\varepsilon + a_{2m+1})y^{2m}(y+1)^2].\end{aligned}$$

From both cases together, we conclude $\sigma_2 \in \sum \mathbb{R}[V]^2$. □

Lemma 4.18
In the above setting, we have $\sigma_3 \in \sum \mathbb{R}[V]^2$.

Proof — Compare lemma 2.12 above.
We compute

$$\begin{aligned}
& (1+y^2)^d - \frac{2}{5}(2+y+3y^2+y^3+3y^4+...+y^{2d-1}+2y^{2d}) \\
= \;& (1+y^2)^d + \frac{1}{5}(1+y^2+...+y^{2d-2})(1-y)^2 \\
& -\frac{8}{5}(y^2+y^4+...+y^{2d-2}) - (1+y^{2d}) \\
= \;& \frac{1}{5}(1+y^2+\cdots+y^{2d-2})(1-y)^2 + \sum_{i=1}^{d-1}(\binom{d}{i} - \frac{8}{5})y^{2i} \in \sum \mathbb{R}[V]^2.
\end{aligned}$$

Thus

$$\sigma_3 \in \sum \mathbb{R}[V]^2.$$

□

Remark 4.19

(i) The last theorem was proven with the same methods used by Marshall in his paper [14]. These are the same methods presented in section 2.1. We only had to generalize certain steps in the proof to coordinate rings (i.e. section 4.1 and section 4.2).

(ii) Theorem 4.16 does not apply to all $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$. The reason is that the term $0 \neq \varepsilon \in \sum \mathbb{R}[C]^2$ may not exist (compare remark 4.5). Without this term, we have no method to handle the error terms of the approximation b_i above. So the theorem does not generalize to all $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$. However, in the special case where $C = V(x_1^2 + x_2^2 - 1)$, we can avoid this problem. See the next section for details.

(iii) It should also be possible in the last theorem to allow the curves to have singular points, if we additionally assume that f has no zeros in $\{P\} \times \mathbb{R}$ for all $P \in C(\mathbb{R}) \cap C_{sing}$. In this case, we could construct the $\varepsilon \in \sum \mathbb{R}[C]^2$ in section 4.1 without a zero in C_{sing}, so we would only need polynomials over continuous functions in lemma 4.11 for the singular points $P \in C(\mathbb{R}) \cap C_{sing}$. The approximation in section 4.2 should also work in this more general case (compare remark 4.10).

(iv) If we allow f to have a zero in $\{P\} \times \mathbb{R}$ for a $P \in C(\mathbb{R}) \cap C_{sing}$, then lemma 4.11 and proposition 4.9 would fail. It is not clear how to generalize both such that we can prove a generalisation of theorem 4.16.

4.4 The Cylinder Case

Let $C = V(x_1^2 + x_2^2 - 1) \subseteq \mathbb{A}^2$ and $V = C \times A^1$. We keep the same notation as before, i.e.

$$\mathbb{R}[C] = \mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2 - 1)$$

and

$$\mathbb{R}[V] = \mathbb{R}[C][y] = \mathbb{R}[x_1, x_2, y]/(x_1^2 + x_2^2 - 1).$$

So C is an irreducible smooth real curve with $C(\mathbb{R})$ compact.
Also note that every $f \in \mathbb{R}[V]$ can be seen as polynomial in y over $\mathbb{R}[C]$, so we will denote by $\deg_y(f)$ the degree f as a polynomial in y over $\mathbb{R}[C]$.
The goal of this section is to prove that psd $=$ sos holds in $\mathbb{R}[V]$.
So given $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$, we have to show that $f \in \sum \mathbb{R}[V]^2$.
The main idea is to first reduce to the case where f has only finitely many zeros in $V(\mathbb{R})$, then use theorem 4.16 to show that $f \in \sum \mathbb{R}[V]^2$. To do this, we need to make reductions similar to those in section 2.3.

Note that the methods presented here do not seem to work for arbitrary smooth curves C' (with $C'(\mathbb{R})$ compact). So far, this (with C and V as above) is the only case where the methods used here seem to work. The reductions presented in this are a generalisation of those in section 2.3.

4.4.1 Finitely Many Zeros

Before we can make the reductions mentioned above, we first need to introduce some notations:
Let K be a field and X be a nonsingular K-variety. A prime divisor of X is a irreducible subset $P \subseteq X$ of codimension 1. We write $\mathfrak{P}(X)$ for the set prime divisors of X. The group of divisors on X, denoted by $Div(X)$, is the free abelian group generated by $\mathfrak{P}(X)$, i.e.

$$Div(X) = \left\{ \sum_{P \in \mathfrak{P}(X)} n_P P \mid n_P \in \mathbb{Z}\,,\ n_P = 0 \quad \text{for all but finitely many } P \in \mathfrak{P} \right\}.$$

A divisor $D = \sum n_P P \in Div(X)$ is effective, if $n_P \geq 0$ for all $P \in \mathfrak{P}(X)$.
Every prime divisor $P \in \mathfrak{P}(X)$ induces a discrete valuation

$$v_P : K(X) \longrightarrow \mathbb{Z}$$

on $K(X)$. We denote the corresponding valuation ring by $\mathcal{O}_P$ and the residue field by $\kappa(P) = \mathcal{O}_P/\mathfrak{m}_P$, where $\mathfrak{m}_P$ is the maximal ideal of $\mathcal{O}_P$.
If X is a curve, then $\kappa(P)$ is a finite field extension of K, so we define the degree of P to be the degree of the field extension, i.e. $\deg(P) = [\kappa(P) : K]$.
In particular if X is a nonsingular curve over $\mathbb{R}$ and $P \in \mathfrak{P}(X)$,
then $\deg(P) = 1$ if P consist of a real point and $\deg(P) = 2$ if P consist of two complex points.
For any divisor of a nonsingular variety

$$D = \sum_{P \in \mathfrak{P}(X)} n_P P \ \in Div(X)$$

we define

$$\deg(D) := \sum_{P \in \mathfrak{P}(X)} n_P \deg(P),$$

so $\deg : Div(X) \to \mathbb{Z}$ is an homomorphism of groups.
For every $f \in K(X)^*$, we can define

$$(f) = \sum_{P \in \mathfrak{P}(X)} v_P(f) P \in Div(X)$$

(Note that, since $f \neq 0$, we have $v_P(f) = 0$ for all but finitely many $P \in \mathfrak{P}(X)$, so this is well defined; compare [5, chapter II.6]). Such a divisor is called a principal divisor. We have $(\frac{f}{g}) = (f) - (g)$ for all $f, g \in K(X)^*$, so the set

$$Div_0(X) := \{(f) \mid f \in K(X),\ f \neq 0\}$$

forms a subgroup of $Div(X)$, called the group of principal divisors. The Quotient

$$Cl(X) := \mathrm{Div}(X)/\mathrm{Div}_0(X)$$

is called the divisor class group. Two divisor $D, D' \in Div(X)$ are called equivalent, if $[D] = [D']$ in $Cl(X)$, i.e. $D - D' \in D_0(X)$.
Let L/K be a field extension. Let X_L be the L variety defined by X. Then each K-irreducible set can be written as a unit of L-irreducible sets. In this way, we can see $Div(X)$ as a subgroup of $Div(X_L)$ (compare [25, chapter II.3]). In this sense, results on groups on divisors of varieties over an algebraically closed field can often be directly generalized to a field that is not algebraically closed.
We fix, for this section, the two real varieties

$$\begin{aligned} C &:= V(x_1^2 + x_2^2 - 1) \subseteq \mathbb{A}^2, \\ V &:= C \times \mathbb{A}^1 \subseteq \mathbb{A}^3. \end{aligned}$$

We identify the affine space $\mathbb{A}^n$ with the subspace $D(x_0)$ of the projective space $\mathbb{P}^n$ and we denote by $\tilde{C}$ be the Zariski-closure of C in $\mathbb{P}^2$. Then $\tilde{C}$ is an irreducible nonsingular projective curve. In particular, we have $\tilde{C} = V_+(x_1^2 + x_2^2 - x_0^2) \subseteq \mathbb{P}^2$. So we compute

$$\tilde{C} = C \cup L,$$

where L denotes the 2 complex points at infinity, i.e.

$$L = \{(0 : 1 : i), (0 : 1 : -i)\} \subseteq \mathbb{P}^2.$$

L is irreducible in the $\mathbb{R}-$ Zariski topology of $\mathbb{P}^2$.
Further, we denote by $\tilde{V}$ the Zariski-closure of V in $\mathbb{P}^2 \times \mathbb{P}^1$. Then $\tilde{V}$ is a nonsingular surface in $\mathbb{P}^2 \times \mathbb{P}^1$ and we compute

$$\begin{aligned} \tilde{V} &= V \cup (L \times \mathbb{P}^1) \cup (\tilde{C} \times M) \\ &= (\tilde{C} \times \mathbb{A}^1) \cup (\tilde{C} \times M), \end{aligned}$$

where $M = (0 : 1) \in \mathbb{P}^1 \setminus \mathbb{A}^1$ denotes the point at infinity of $\mathbb{A}^1$.
Before the next step, we first need to describe $Cl(V)$, $Cl(C)$, $Cl(\tilde{C})$ and

$Cl(\tilde{V})$.
Note that $\tilde{C}$ is isomorphic to $\mathbb{P}^1$ as a real algebraic variety (see [27, example 6.5.8]). Hence, we have

$$Cl(\tilde{C}) = Cl(\mathbb{P}^1) = \mathbb{Z}.$$

Indeed,

$$\begin{array}{rcl} \deg : Cl(\tilde{C}) & \longrightarrow & \mathbb{Z} \\ [X] & \longmapsto & \deg(X) \end{array}$$

is a well defined isomorphism of groups (see [26, theorem 1.4.11] or [5, Proposition II.6.4] or [27, 14.2.8]).
Removing a subset of codimension 1 from a nonsingular variety X changes the divisor class group in a controllable way:

Lemma 4.20
Let X be a nonsingular variety, $Z \subseteq X$ a irreducible subvariety of codimension 1, then there is an exact sequence

$$\mathbb{Z} \longrightarrow Cl(X) \longrightarrow Cl(X \setminus Z) \longrightarrow 0.$$

Proof — see [5, proposition II.6.5] for a proof over an algebraically closed field, or [27, 14.2.8]. □

Note that, since Z is irreducible, we have $Z \in \mathfrak{P}(X)$ and the first map is defined by

$$\mathbb{Z} \ni z \longmapsto [zZ] \in Cl(X).$$

The second map is defined by

$$[D = \sum n_P P] \longmapsto [\sum n_P (P \cap U)],$$

where $U = X \setminus Z$ and all P with $P \cap U = \varnothing$ are omitted. Note that $P \cap U$ is irreducible in U of codimension 1, so if $P \cap U \neq \varnothing$, then $P \cap U \in \mathfrak{P}(U)$.
With this, we can describe $Cl(C)$:

Proposition 4.21
We have $Cl(C) = \{0, [1 \cdot P]\}$, where $P \in \mathfrak{P}(X)$ with $P \subseteq C(\mathbb{R})$.

Proof — By lemma 4.20 there is a group homomorphism

$$\psi : Cl(\tilde{C}) \to Cl(C)$$

with kernel $\{[zL] \mid z \in \mathbb{Z}\}$. Further,

$$\deg : Cl(\tilde{C}) \to \mathbb{Z}$$

is an isomorphism and since $\deg(L) = 2$, it induces an isomorphism

$$\deg : Cl(C) \to \mathbb{Z}/2\mathbb{Z}.$$

Especially $Cl(C)$ has exactly 2 elements namely $[0]$ and $[D]$, where $D \in Div(C)$ with $\deg(D) = 1$. Now let $P \in \mathfrak{P}$ with $P \subseteq C(\mathbb{R})$.
Then $\deg(P) = 1$, so we can choose $D = 1 \cdot P$. □

Remark 4.22
Especially we have seen in the last proof, that a divisor $D \in Div(C)$ is a principal divisor if and only if $\deg(D) \in 2\mathbb{Z}$.

Lemma 4.23
Let X be a smooth affine normal $\mathbb{R}$-variety, $f \in \mathbb{R}(X)$ with $v_{\mathfrak{p}}(f) \geq 0$ for all prime ideals $\mathfrak{p} \subseteq \mathbb{R}[X]$ with $ht(\mathfrak{p}) = 1$. Then $f \in \mathbb{R}[X]$.

Proof — see [5, Proposition II.6.3A] for a proof over an algebraically closed field. It also holds over $\mathbb{R}$ since $\mathbb{R}(X) \cap \mathbb{C}[X] = \mathbb{R}[X]$. □

In our case, this applies in the following way:

Lemma 4.24
Let $f \in \mathbb{R}(C)$ such that $(f) \in Div(C)$ is effective, then $f \in \mathbb{R}[C]$.

Proof — C is a smooth affine normal $\mathbb{R}$-variety. Since C is affine, the prime divisors of C are in 1-1 correspondence to the ideals $\mathfrak{p} \subseteq \mathbb{R}[X]$ with $ht(\mathfrak{p}) = 1$. Since (f) is effective, we have $v_{\mathfrak{p}}(f) \geq 0$ for all prime ideals $\mathfrak{p} \subseteq \mathbb{R}[X]$ with $ht(\mathfrak{p}) = 1$. So $f \in \mathbb{R}[C]$ by lemma 4.23. □

Lemma 4.25
Let X be a smooth affine $\mathbb{R}$ variety and $f \in \mathbb{R}[X]$ with $f \geq 0$ on $X(\mathbb{R})$. Let $P \in \mathfrak{P}(X)$ such that $\kappa(P)$ is real. Then $v_P(f)$ is even.

Proof — Since $f \geq 0$ on $X(\mathbb{R})$, there exists an equation

$$f = \sum_{i=1}^{n} f_i^2$$

with $f_i \in \mathbb{R}(X)$. Thus we have

$$v_P(f) = v_P(\sum_{i=1}^{n} f_i) = \min_i v_P(f_i^2) = 2 \min_i v_P(f_i)$$

by lemma 2.78. $\square$

Proposition 4.26
Suppose $0 \neq a \in \mathbb{R}[C]$ such that $a \geq 0$ on $C(\mathbb{R})$, then there exist $a_1, a_2 \in \mathbb{R}[C]$ such that

(i) $a = a_1 a_2$,

(ii) every zero of a_1 in C is real,

(iii) $a_2 > 0$ *on* $C(\mathbb{R})$.

Proof — Since C is irreducible and $a \neq 0$, a has finitely many zeros in C. Let $P_1, \ldots, P_s$ the zeros of a. Then $\{P_i\} \in \mathfrak{P}(C)$ with $\deg(\{P_i\}) = 1$. Define

$$D = \sum_{i=1}^{s} v_{P_i}(a)\{P_i\} \in Div(C).$$

Then

$$\deg(D) = \sum_{i=1}^{s} v_{P_i}(a)$$

is even by lemma 4.25, so $D \in Div_0(C)$ (compare remark 4.22).
Let $a_1 \in \mathbb{R}(C)$ with $(a_1) = D$. Then $a_1 \in \mathbb{R}[X]$ by lemma 4.24. Now define

$$a_2 = \frac{a}{a_1} \in \mathbb{R}(C).$$

By construction $(a_2) = (a) - (a_1)$ is effective, so $a_2 \in \mathbb{R}[C]$ by lemma 4.24. Since a_2 has no zero in $C(\mathbb{R})$ by construction, a_2 has a constant sign on $C(\mathbb{R})$ (since $C(\mathbb{R})$ is connected in the euclidean topology). So we may assume $a_2 > 0$ on $C(\mathbb{R})$ (Otherwise we work with $-a_1$ and $-a_2$). Note that, by construction, all zeros of a_1 are real, so a_1 and a_2 are as desired. $\square$

Remark 4.27

(i) The last proposition is the first goal of this section. It will be a crucial step in showing that $psd = sos$ holds in $\mathbb{R}[V]$.

(ii) Note that the proof of the last proposition relies heavily on the fact that every divisor of degree 2 is a principal divisor. For arbitrary smooth curves, this is not true. This is why an attempt to generalize proposition 4.26 to arbitrary curves fails. It is also the first (but not the last) reason why we cannot generalise proposition 4.39 later to arbitrary smooth curves.

For the next step, we first need to compute $Div(V)$ and $Div(\tilde{V})$. The following will be helpful (compare the proof of [5, Proposition II.6.6]):
Let X be a smooth $\mathbb{R}$-variety, $Y = X \times \mathbb{A}^1$ and $\pi : Y \to X$ the projection. If $P \subseteq X$ is a irreducible subset of codimension 1, then $\pi^{-1}(X) \subseteq Y$ is a irreducible subset of codimension 1. Thus π induces a morphism

$$\pi^* : \mathfrak{P}(X) \to \mathfrak{P}(Y),$$

which expands to a morphism

$$\pi^* : Div(X) \to Div(Y).$$

Further, we have $\mathbb{R}(Y) = \mathbb{R}(X)(t)$, and for $f \in \mathbb{R}(X)^*$ we have

$$\pi^*((f)_X) = (f)_Y,$$

where $(f)_X$ (resp. $(f)_Y$) is the principal divisor defined by f in $Div(X)$ (resp. $Div(Y)$). Thus π^* induces an homomorphism (which we also denote by π^*)

$$\begin{array}{rcl} \pi^* : Cl(X) & \longrightarrow & Cl(X \times \mathbb{A}^1) \\ [D] & \longmapsto & [\pi^*(D)]. \end{array}$$

Indeed, π^* is an isomorphism (see for example [5, Proposition II.6.6.]. The exact same proof holds when X is a variety over $\mathbb{R}$). Especially we get

Lemma 4.28
Let $D \in Div(V)$ and let $P \in \mathfrak{P}(C)$ consist of a real point. Further, let $\pi : V \to C$ the projection. Then either $D \in Div_0(V)$ or $[D] = [1 \cdot \pi^*(P)]$ in $Cl(V)$.

Proof —
$$\pi^* : Cl(C) \to Cl(V)$$
is an isomorphism, so there exists a $D' \in Div(C)$ such that
$$\pi^*([D']) = [\pi^*(D')] = [D].$$
By lemma 4.21 we have either $[D'] = 0$ or $[D'] = [1 \cdot P]$. So, in the first case, we have $[D] = 0$ and in the second case we have $[D] = [1 \cdot \pi^*(P)]$. □

Lemma 4.29
Let $Z \in \mathfrak{P}(V)$ with $1 \cdot Z \notin Div_0(V)$. Let $P \in C$ consist of a real point and let $Z_{\tilde{V}}$ be the closure of Z in $\tilde{V}$. Define
$$P_{\tilde{V}} := P \times \mathbb{P}^1 \in \mathfrak{P}(\tilde{V}),$$
$$L_{\tilde{V}} := L \times \mathbb{P}^1 \in \mathfrak{P}(\tilde{V})$$
and
$$\tilde{M} := \tilde{C} \times \{(0 : 1)\} \in \mathfrak{P}(\tilde{V}).$$
Then $\tilde{M}$, $Z_{\tilde{V}}$, $P_{\tilde{V}} \in \mathfrak{P}(\tilde{V})$ and there exist $z_1, z_2 \in \mathbb{Z}$ such that
$$[1 \cdot Z_{\tilde{V}}] = [1 \cdot P_{\tilde{V}} + z_1\tilde{M} + z_2 L_{\tilde{V}}] \text{ in } Cl(\tilde{V}).$$

Proof — $Z_{\tilde{V}}$, $P_{\tilde{V}}$ and $\tilde{M}$ are irreducible subsets of $\tilde{V}$ of codimension 1, hence $\tilde{M}$, $Z_{\tilde{V}}$, $P_{\tilde{V}} \in \mathfrak{P}(\tilde{V})$.
Define
$$P_V := P \times \mathbb{A}^1 = P_{\tilde{V}} \cap V \in \mathfrak{P}(V).$$
We have $Z \notin Div_0(V)$, so $[1 \cdot Z] = [1 \cdot P_V]$ in $Cl(V)$ by lemma 4.28. Define
$$P_{V'} := P \times \mathbb{A}^1 = P_{\tilde{V}} \cap V'$$
and
$$L_{V'} := L \times \mathbb{A}^1.$$
Let $Z_{V'}$ be the closure of Z in V'. Then $L_{V'}, P_{V'}, Z_{V'} \in \mathfrak{P}(V')$. We have
$$V = V' \setminus L_{V'},$$
hence
$$P_{V'} \cap V = P_V \in \mathfrak{P}(V)$$

and

$$\begin{array}{rcl} \varphi_1 : Cl(V') & \to & Cl(V) \\ [D = \sum n_Q Q] & \longmapsto & [D = \sum n_Q (Q \cap V)] \end{array}$$

is a homomorphism with kernel $ker(\varphi_1) = \{[zL_{V'}] \mid z \in \mathbb{Z}\}$ (see lemma 4.20 above). Further, we compute

$$\varphi_1([1 \cdot P_{V'}]) = [1 \cdot P_V] = [1 \cdot Z] = \varphi_1([1 \cdot Z_{V'}]),$$

so there exists a $z_1 \in \mathbb{Z}$ such that

$$[1 \cdot Z_{V'}] = [1 \cdot P_{V'} + z_1 \cdot L_{V'}].$$

Now we have

$$V' = \tilde{V} \setminus \tilde{M},$$

so

$$\begin{array}{rcl} \varphi_2 : Cl(\tilde{V}) & \to & Cl(V') \\ [D = \sum n_Q Q] & \longmapsto & [D = \sum n_Q (Q \cap V')] \end{array}$$

is a homomorphism with kernel $ker(\varphi_2) = \{[z\tilde{M}] \mid z \in \mathbb{Z}\}$ (see lemma 4.20 above). Again, we compute

$$\varphi_2([1 \cdot Z_{\tilde{V}}]) = [1 \cdot Z_{V'}] = [1 \cdot P_{V'} + z_1 \cdot L_{V'}] = \varphi_2([1 \cdot P_{\tilde{V}}] + z_1[L_{\tilde{V}}]),$$

thus there exists a $z_2 \in \mathbb{Z}$ such that

$$[1 \cdot Z_{\tilde{V}}] = [1 \cdot P_{\tilde{V}} + z_1 L_{\tilde{V}} + z_2 \tilde{M}] \text{ in } Cl(\tilde{V}).$$

□

So for the next step, we need some intersection theory. Here, we use the notation introduced in [5]:

Theorem 4.30
Let X be a nonsingular projective surface over $\mathbb{R}$. Then there exists a unique pairing $Div(X) \times Div(X) \longrightarrow \mathbb{Z}$, denoted by $C.D$ for any two divisors $C, D \in Div(X)$ such that

(i) if C and D are nonsingular curves meeting transversally, then $C.D = \sharp(C \cap D)$ the number of points in $C \cap D$,

(ii) it is symmetric, i.e. $C.D = D.C$,

(iii) it is additive $(C_1 + C_2).D = C_1.D + C_2.D$,

(iv) it only depends on the linear equivalence classes: if $[C_1] = [C_2]$ in $Cl(X)$, then $C_1.D = C_2.D$.

Proof — In [5, theorem V.1.1] this is proven for a nonsingular surface over $\mathbb{C}$. It also holds over $\mathbb{R}$ (see [27, chapter 20 ff], or use the fact that $Div(X) \subseteq Div(X_{\mathbb{C}})$). $\square$

Remark 4.31

(i) Here, a curve on X is a very ample effective divisor, and two curves C, D meet transversally in $P \in V$ if the local equations of C and D generate the maximal ideal of $\mathcal{O}_{P,X}$.

(ii) The reason we needed to define the projective closures $\tilde{C}$ and $\tilde{V}$ is that the intersection multiplicity only depends on the equivalence class of a divisor if X is a projective nonsingular surface.

Lemma 4.32
Let $D, M \in \mathfrak{P}(\tilde{V})$ such that $(1 \cdot D).(1 \cdot M)$ is odd, then $D(\mathbb{R}) \cap M(\mathbb{R}) \neq \varnothing$.

Proof — We may assume they meet transversally (compare [5, lemma V.1.2]). For any $P \in \mathbb{P}^n$, let $\bar{P}$ be the point obtained by complex conjugation of the coordinates. If $P \in C \cap D$, then also $\bar{P} \in C \cap D$. So if $D(\mathbb{R}) \cap M(\mathbb{R}) = \varnothing$, then $(1 \cdot D).(1 \cdot M) = \sharp C \cap D$ would be even. A contradiction. $\square$

Proposition 4.33
Let $Z \in \mathfrak{P}(V)$ with $Z(\mathbb{R})$ compact. Then $Z \in Div_0(V)$.

Proof — Suppose $Z \notin Div_0(V)$. Then by lemma 4.29 there exist $z_1, z_2 \in \mathbb{Z}$ such that

$$[1 \cdot Z_{\tilde{V}}] = [1 \cdot P_{\tilde{V}} + z_1\tilde{M} + z_2 L_{\tilde{V}}] \text{ in } Cl(\tilde{V}).$$

(in the notion of the lemma). Now choose $Q \in \mathbb{A}^1$ such that

$$(\tilde{C} \times \{Q\}) \cap Z(\mathbb{R}) = \varnothing.$$

This is possible since $Z(\mathbb{R})$ compact. Define

$$M = \tilde{C} \times \{Q\} \in \mathfrak{P}(\tilde{V}).$$

Then

$$M(\mathbb{R}) \cap Z_{\tilde{V}} = \varnothing,$$

but

$$(1 \cdot M).(1 \cdot Z_{\tilde{V}}) = (1 \cdot M).(1 \cdot P_{\tilde{V}} + z_1 \tilde{M} + z_2 L_{\tilde{V}}) = 1 + 2z_2$$

is odd; a contradiction to lemma 4.32. □

Remark 4.34

(i) Proposition 4.33 was the second goal of this section. This will be the other main ingredient in showing that psd = sos holds in $\mathbb{R}[V]$.

(ii) Note that, in the last proof, we again relied heavily on the very specific form of $Div(C)$. The same argument will probably not work for other smooth irreducible curves C (and $V = C \times \mathbb{A}^1$).

4.4.2 psd = sos in $\mathbb{R}[C]$

Now we are able to prove that psd = sos holds in $\mathbb{R}[V]$. We first need several lemmas:

Lemma 4.35
Suppose $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$. Write

$$f = \sum_{i=0}^{d} a_i y^i$$

with $a_i \in \mathbb{R}[C]$ and $a_d \neq 0$. If $a_d > 0$ on $C(\mathbb{R})$, then $V_{\mathbb{R}}(f) \subseteq V(\mathbb{R})$ is compact.

Proof — We have to show that $V_{\mathbb{R}}(f) \subseteq V(\mathbb{R})$ is bounded. The proof is essentially the same as the proof of lemma 3.23:
Since $f \geq 0$ on $V(\mathbb{R})$, d is even. Since $C(\mathbb{R})$ is compact, a_d attains its minimum value $m > 0$ and every a_i $(i < d)$ attains its maximum value M_i on $C(\mathbb{R})$. Now, define

$$N := max\left\{\sum_{i=0}^{d-1} \frac{M_i}{m}, 1\right\} + 1,$$

then we compute for $(X, Y) \in C(\mathbb{R}) \times \mathbb{R} = V(\mathbb{R})$ with $|Y| \geq N$:

$$\begin{aligned} |Y| &> \sum_{i=0}^{d-1} \frac{M_i}{m} \\ &\geq \sum_{i=0}^{d-1} \frac{M_i}{m} |\frac{Y^i}{Y^{d-1}}| \\ &\geq \sum_{i=0}^{d-1} \frac{|a_i(X)|}{a_d(X)} \left|\frac{Y^i}{Y^{d-1}}\right|, \end{aligned}$$

so

$$0 < a_d(X)Y^d - \sum_{i=0}^{d-1} |a_i(X)Y^i| \leq f(X, Y),$$

thus $f(X, Y) > 0$. □

Lemma 4.36
Let $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ and $V_{\mathbb{R}}(f)$ compact. If f is squarefree, then $V_{\mathbb{R}}(f)$ is finite.

Proof — Suppose $V_{\mathbb{R}}(f)$ is not finite. Then there exists an irreducible $Z \subseteq V(f) \subseteq V$ such that $Z(\mathbb{R})$ is not finite. Since $Z(\mathbb{R}) \subseteq V_{\mathbb{R}}(f)$, we conclude $Z(\mathbb{R})$ compact, so $1 \cdot Z \in Div_0(V')$ by lemma 4.33, i.e. $1 \cdot Z = (f_Z)$ for a $f_Z \in \mathbb{R}[V]$. v_Z is a real valuation (since $Z(\mathbb{R})$ is not finite), so $v_Z(f)$ is even by lemma 4.25, i.e. $v_Z(f) = 2m$. Then

$$(\frac{f}{f_Z^{2m}}) = (f) - (f_Z^{2m}) = \sum_{P \in \mathfrak{P}(V)} v_P(f)P - 2mZ = \sum_{P \in \mathfrak{P}(V), P \neq Z} v_P(f)$$

is effective, so $g := \frac{f}{f_Z^{2n}} \in \mathbb{R}[V]$ by lemma 4.24 and we compute

$$f = (f_Z^n)^2 g.$$

A contradiction, since f is squarefree. □

Lemma 4.37
Let $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$ and $V_{\mathbb{R}}(f)$ compact. Then $f \in \sum \mathbb{R}[V]^2$.

Proof — We may assume f is squarefree, then $V_{\mathbb{R}}(f)$ is finite by lemma 4.36. So $f \in \sum \mathbb{R}[V]^2$ by theorem 4.16. □

Lemma 4.38
Let $f \in \mathbb{R}[V]$ and $a \in \mathbb{R}[C]$ with $a \geq 0$ on $C(\mathbb{R})$. Suppose all zeros of a are real. If $af \in \sum \mathbb{R}[V]^2$, then $f \in \sum \mathbb{R}[V]^2$.

Proof — Since $a \geq 0$ on $C(\mathbb{R})$, we have $a \in \sum \mathbb{R}[C]^2$ by corollary 1.16, thus

$$a^2 f = a(af) \in \sum \mathbb{R}[V]^2.$$

So suppose

$$a^2 f = \sum_{i=1}^{r} f_i^2$$

for $f_i \in \mathbb{R}[V]$ $(i = 1, \ldots, r)$. Then we compute

$$f = \sum_{i=1}^{r} (\frac{f_i}{a})^2$$

and it is enough to show $\frac{f_i}{a} \in \mathbb{R}[V]$ $(i = 1, \ldots r)$. By lemma 4.23 it is enough to show that $v_{\mathfrak{p}}(\frac{f_i}{a}) \geq 0$ for all $i \in \{1, \ldots, s\}$ and all prime ideals with $ht(\mathfrak{p}) = 1$.

So fix $i \in \{1, \dots, s\}$ and let $\mathfrak{p} \subseteq \mathbb{R}[V]$ be a prime ideal of height 1 with discrete valuation $v = v_{\mathfrak{p}}$ and residue field $\kappa = \kappa(\mathfrak{p})$.
If κ is not real, then $v(a) = 0$ by our assumption, so we compute

$$v(\frac{f_i}{a}) = v(f_i) - v(a) = v(f_i) \geq 0.$$

If κ is real, then we compute with lemma 2.78

$$2v(a) \leq 2v(a) + v(f) = v(a^2 f) = 2 \min_j v(f_j) \leq 2v(f_i).$$

So in both cases, we conclude

$$v(\frac{f_i}{a}) = v(f_i) - v(a) \geq 0.$$

□

Proposition 4.39
psd = sos holds in $\mathbb{R}[V]$.

Proof — Let $f \in \mathbb{R}[V]$ with $f \geq 0$ on $V(\mathbb{R})$. We have $\mathbb{R}[V] = (\mathbb{R}[C])[y]$, so we can write

$$f = \sum_{i=0}^{\deg_y(f)} a_i y^i$$

with $a_i \in \mathbb{R}[C]$. Since $f \geq 0$ on $V(\mathbb{R})$, $\deg_y(f)$ is even, say $\deg_y(f) = 2d$ and $a_{2d} \geq 0$ on $C(\mathbb{R})$. Using lemma 4.26, we can factorize

$$a_{2d} = a_1 a_2$$

with $a_1, a_2 \in \mathbb{R}[C]$ such that all zeros of a_1 are real and $a_2 > 0$ on $C(\mathbb{R})$. Note that since $a_2 > 0$ on $C(\mathbb{R})$ and $a_{2d} \geq 0$ on $C(\mathbb{R})$, we have $a_1 \geq 0$ on $C(\mathbb{R})$. Now, we compute

$$\begin{aligned} a_1^{d-1} f &= a_2 a_1^d y^d + \sum_{i=0}^{d-1} a_1^{d-1-i} a_i a_1^i y^i \\ &= a_2 (a_1 y)^d + \sum_{i=0}^{d-1} a_1^{d-1-i} a_i (a_1 y)^i \\ &=: f'(x, a_1 y). \end{aligned}$$

Now we have $f' \in \mathbb{R}[V]$, $f' \geq 0$ on $V(\mathbb{R})$ and the leading coefficient $a_2 \in \mathbb{R}[C]$ satisfies $a_2 > 0$ on $C(\mathbb{R})$, so $V_{\mathbb{R}}(f')$ is compact by lemma 4.35. Then $f' \in \sum \mathbb{R}[V]$ by lemma 4.37, i.e there exist $g_1 \dots g_r \in \mathbb{R}[V]$ such that

$$f' = g_1^2 + \cdots + g_r^2.$$

Then we compute

$$a_1^{d-1} f = f'(x, a_1 y) = g_1(x, a_1 y) + \cdots + g_r(x, a_1 y).$$

So $a_1^{d-1} f \in \sum \mathbb{R}[V]^2$ and by lemma 4.38 we conclude $f \in \sum \mathbb{R}[V]^2$. $\square$

Remark 4.40

(i) Proposition 4.39 gives us an example of an affine real variety V with $\dim(V) = 2$ where psd=sos holds in $\mathbb{R}[V]$ and $V(\mathbb{R})$ is not compact. To our knowledge, this is the first example of such a variety that is known .

(ii) The main arguments in the last proof were proposition 4.26 and proposition 4.33. These do not hold for other irreducible real smooth curves (compare remark 4.27 and remark 4.34). So an attempt to generalize proposition 4.39 to other curves fails here.

5 Zusammenfassung auf Deutsch

Sei S eine endliche Teilmenge des Polynomrings $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$, dann definieren wir die von S erzeugte basisch abgeschlossene Menge als

$$K_S := \{x \in \mathbb{R}^n \mid s(x) \geq 0 \text{ for all } s \in S\}$$

und die von S erzeugte Präordnung als

$$PO(S) := \{ \sum_{e \in \{0,1\}^r} s_1^{e_1} \ldots s_r^{e_r} \sigma_e \mid \sigma_e \in \sum \mathbb{R}[\underline{x}]^2 \},$$

wobei $\sum \mathbb{R}[\underline{x}]^2$ die Menge der Quadratsummen in $\mathbb{R}[\underline{x}]$ bezeichnet. Offensichtlich ist jedes Element der Präordnung $PO(S)$ auf der Menge K_S nichtnegativ. Wir beschäftigen uns mit der Frage ob die Umkehrung gilt, d.h. ob jedes Polynom, welches auf der Menge K_S nichtnegativ ist, auch in der Prärdnung $PO(S)$ liegt. Eine Präordnung mit dieser Eigenschaft heißt *saturiert.*

Diese Fragestellung geht zurück auf Hilbert. Er studierte die Frage, ob jedes Polynom $f \in \mathbb{R}[x_1, \ldots, x_n]$ von Grad $2d$, das auf ganz $\mathbb{R}^n$ nichtnegativ ist, eine Summe von Quadraten von Polynomen ist. 1888 bewies Hilbert ([6]) dann, dass für $n \geq 2$ und $2d \geq 4$ immer ein auf ganz $\mathbb{R}^n$ nichtnegatives Polynom von Grad $2d$ existiert, welches keine Summe von Quadraten von Polynomen in $\mathbb{R}[x_1, \ldots, x_n]$ ist. Die einzige Ausnahme bildet der Fall $(n, 2d) = (2, 4)$. Nachdem er dieses negative Resultat bewiesen hat, beschäftigte Hilbert sich damit, ob jedes auf ganz $\mathbb{R}^n$ nichtnegative Polynom eine Summe von Quadraten von rationalen Funktionen ist. Diese Frage wurde von Hilbert in seine berühmte Liste mathematischer Probleme ([7]) aufgenommen und wurde später von Artin ([1]) gelöst.

Des Weiteren bewies Scheiderer ([19]), dass eine Präordnung niemals saturiert sein kann, wenn die Dimension der semialgebraischen Menge größer als 3 ist oder die semialgebraische Menge einen offenen Kegel in $\mathbb{R}^2$ enthält. Andererseits wurde die Frage in einer Variablen durch Kuhlmann, Marshall und Schwartz ([9]) komplett gelöst. Sie haben gezeigt, dass eine Präordnung $PO(S) \subseteq \mathbb{R}[x]$ genau dann saturiert ist, wenn sie gewisse *natürliche Erzeuger* für die semialgebraische Menge $K_S \subseteq \mathbb{R}$ enthält. Generell kann man sagen, dass Saturiertheit eine sehr starke Eigenschaft ist und dass man in sehr vielen Fällen keine Chance hat, eine positive Antwort auf die obige Frage zu bekommen. Deshalb wurden im Laufe der Zeit einige schwächere Eigenschaften untersucht. Sei nun wieder $S \subseteq \mathbb{R}[\underline{x}]$ endlich. Unter anderem haben Kuhlmann und Marshall ([10]) eingeführt, dass eine Präordnung $PO(S)$ $(\dagger)$ erfüllt, falls sie alle auf K_S strikt positiven Polynome enthält. Im Bezug auf diese schwächere Eigenschaft zeigte Schmüdgen ([23]), dass eine Präordnung $PO(S)$ $(\dagger)$ erfüllt, wenn die zugehörige semialgebraische Menge K_S kompakt ist.

Über Präordnungen im Polynomring in zwei Variablen $PO(S) \subseteq \mathbb{R}[x, y]$, bei der die zugehörige semialgebraische Menge K_S nicht kompakt ist und keinen offenen Kegel enthält, gibt es bisher nur wenige Resultate. Eins der ersten Resultate in dieser Situation war die Lösung der *strip conjecture* von Marshall ([14]). Er bewies, dass jedes Polynom, das auf der Menge $[0, 1] \times \mathbb{R}$ nichtnegativ ist, in der von $S = \{x, 1 - x\}$ erzeugten Präordnung $PO(S) \subseteq \mathbb{R}[x, y]$ liegt. Der Beweis von Marshall wurde später von Nguyen und Powers ([16]) auf beliebige Streifen der Form $K \times \mathbb{R}$ (mit $K \subseteq \mathbb{R}$ kompakt) und Halbstreifen der Form $K \times \{y \geq 0\}$ verallgemeinert.

Marshalls Beweis bildet auch den Ausgangspunkt dieser Arbeit. Wir verallgemeinern seine Ergebnisse auf zweierlei Weisen:

Im ersten Teil, bestehend aus Kapitel 2 und 3, studieren wir zwei Arten von Präordnungen, welche beide von einer endlichen Menge $S \subseteq \mathbb{R}[x, y]$ erzeugt werden.

Sei $K \subseteq \mathbb{R}$ kompakt. Wir untersuchen einerseits Präordnungen $PO(S)$, bei denen die semialgebraische Menge der Bedingung

$$K \times \{y \in \mathbb{R} \mid y^2 \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}$$

genügt. Eine solche Präordnung bezeichnen wir als Präordnung im Streifenfall. Andererseits untersuchen wir Präordnungen, deren semialgebraische Menge die Bedingung

$$K \times \{y \in \mathbb{R} \mid y \geq N\} \subseteq K_S \subseteq K \times \mathbb{R}_+$$

erfüllt. Eine solche Präordnung bezeichnen wir als Präordnung im Halbstreifenfall.

Falls eine Präordnung $PO(S)$ im Streifen- oder Halbstreifenfall saturiert ist, so muss bereits die Präordnung $PO(S \cap \mathbb{R}[x])$ saturiert sein (siehe Proposition 3.2). Deshalb gehen wir immer davon aus, dass schon S die natürlichen

Erzeuger von K enthält, d.h. wir nehmen an, S habe die Form

$$S = \{s_1, \ldots, s_r, g_1, \ldots, g_s\},$$

wobei $s_1, \ldots, s_r$ die natürlichen Erzeuger von K und $g_1, \ldots, g_s$ weitere beliebige zusätzliche Erzeuger sind. Ob die entsprechende Präordnung saturiert ist (oder (†) erfüllt), hängt vor allem von diesen zusätzlichen Erzeugern ab. Um diese Präordnungen zu untersuchen, benutzen wir hauptsächlich drei Methoden:
Die erste Methode wird im Abschnitt 2.1 in Theorem 2.10 und Theorem 2.13 erklärt. Der Vorteil hierbei ist, dass diese beiden Theoreme uns ermöglichen, im Polynomring über einem Potenzreihenring $\mathbb{R}\{x\}[y]$ zu arbeiten, anstatt in dem Polynomring in zwei Variablen $\mathbb{R}[x, y]$. Der erste Ring ist ein Polynomring über einem henselschen diskreten Bewertungsring, in welchem vor allem die Faktorisierung eines Polynoms in irreduzible Faktoren sehr viel einfacher wird (siehe Abschnitt 2.4.2). Dies stellt sich als sehr vorteilhaft heraus, wenn man mit Präordnungen im Streifen- oder Halbstreifenfall arbeitet. Leider sind diese beiden Theoreme auch sehr technisch, weshalb wir den größten Teil von Kapitel 2 damit verbringen, die notwendigen mathematischen Werkzeuge bereitzustellen, um diese Theoreme anzuwenden. Diese Methode basiert hauptsächlich auf dem Beweis von Marshall in [14].
Die zweite Methode ist eine Beobachtung, welche es uns ermöglicht, Präordnungen im Streifenfall zu studieren, die von einem Polynom $g \in \mathbb{R}[x, y]$ mit $\deg_y(g) = 2$ erzeugt werden. Bisher wurden nur Präordnungen studiert, deren Erzeuger p $\deg_y(p) \leq 1$ erfüllen. Einerseits benutzen wir diese neue Technik, um Präordnungen, die von einem solchen g im Ring $\mathbb{R}\{x\}[y]$ erzeugt werden, zu analysieren (Propositon 2.80), andererseits um ein von Powers in [17] gestelltes offenes Problem zu lösen (Propositon 3.27).
Die dritte Methode basiert auf einem Resultat von Kuhlmann, Marshall und Schwartz in [9]. Aus deren Arbeit folgt, dass eine Präordnung $PO(S)$ im Streifen- oder im Halbstreifenfall nur dann (†) erfüllen kann, wenn die zugehörige Menge der Erzeuger $S \subseteq \mathbb{R}[x, y]$ eine gewisse Fasernbedingung erfüllt, nämlich dass die Menge der Erzeuger, eingeschränkt auf alle vertikalen Geraden L, die natürlichen Erzeuger für $L \cap K_S$ enthält (siehe Abschnitt 3.1). Hieraus folgt, dass die semialgebraische Menge K_S einer solchen Präordnung $PO(S)$, unter einer kleinen zusätzlichen Annahme, bereits von denjenigen Elementen in $s \in S$ erzeugt wird, die $\deg_y(s) \leq 2$ erfüllen (Proposition 3.19 und Proposition 3.20). Dies gibt uns sofort eine neue Familie

von Präordnungen, die niemals saturiert sein können (Proposition 3.40).

Wir verwenden diese 3 Methoden, um Präordnungen im Streifen- und Halbstreifenfall folgendermaßen zu untersuchen: Zunächst betrachten wir in Abschnitt 3.2 Präordnungen im Streifen- und Halbstreifenfall, bei denen die erzeugende Menge S nur einen zusätzlichen Erzeuger $g \in \mathbb{R}[x, y]$ hat. Unter der Voraussetzung, dass der Leitkoeffizient von g als Polynom in y über $\mathbb{R}[x]$ strikt positiv auf K ist, können wir diese Problem fast komplett lösen. Das heißt, wir können entweder die Saturierung beschreiben, falls sie endlich erzeugt ist, oder zeigen, dass die Saturierung nicht endlich erzeugt ist. Lediglich in zwei Unterfällen können wir nur (†) zeigen anstatt Saturiertheit (corollary 3.30 und Proposition 3.38). Ob in diesen beiden Fällen die Saturierung endlich erzeugt ist, bleibt offen.
Danach betrachten wir Präordnungen im Streifen- und Halbstreifenfall mit mehr als einem zusätzlichen Erzeuger. Hier benutzen wir eine Kombination aus allen drei oben genannten Methoden um zu beweisen, dass eine Präordnung im Streifen- oder Halbstreifenfall, deren Erzeuger der oben genannten Fasernbedingung sowie einer weiteren kleinen technischen Bedingung genügen, bereits (†) erfüllen (Theorem 3.54 and Theorem 3.61). Dies Verallgemeinert ein Resultat von Kuhlmann, Marshall und Schwartz ([9, Theorem 5.3]) im Streifen- und Halbstreifenfall.
Des Weiteren zeigen wir, dass solche Präordnungen unter gewissen zusätzlichen Bedingungen bereits saturiert sind (Theorem 3.74 und Theorem 3.80). Obwohl diese Bedingungen wahrscheinlich zu stark sind, liefert uns dies eine völlig neue Familie von Beispielen saturierter Präordnungen im Streifen- und Halbstreifenfall.
Wir beenden diesen Teil der Arbeit mit einigen weiteren Beispielen und offenen Problemen (Abschnitt 3.5). Wir konzentrieren uns hierbei weitestgehend auf Beispiele, bei denen unsere neuen Methoden nicht ausreichen, um zu zeigen, dass die entsprechende Präordnung saturiert ist. Durch weitere Untersuchungen dieser Beispiele hoffen wir, die zusätzlichen Bedingungen in Theorem 3.74 und Theorem 3.80 abschwächen oder sogar ganz weglassen zu können.

Der zweite Teil der Arbeit besteht aus Kapitel 4. Hier verallgemeinern wir Marshalls Beweis in einer etwas anderen Richtung:
Sei C eine glatte reelle Kurve mit $C(\mathbb{R})$ kompakt und sei $V = C \times \mathbb{A}^1$. Die

Idee ist, die kompakte Menge K in Marshalls Beweis durch die kompakte Menge $C(\mathbb{R})$ zu ersetzen. Das heißt, wir beschäftigen uns mit der Frage, ob jede reguläre Funktion $f \in \mathbb{R}[V]$, die auf $V(\mathbb{R})$ nur nichtnegative Werte annimmt, eine Summe von Quadraten in $\mathbb{R}[V]$ ist. In anderen Worten: Gilt psd = sos in $\mathbb{R}[V]$?
Hierbei verallgemeinern wir die Methode von Marshall, um sie in dieser Situation anzuwenden und zeigen, dass jede auf $V(\mathbb{R})$ nichtnegative reguläre Funktion mit endlich viele Nullstellen in $V(\mathbb{R})$ eine Summe von Quadraten in $\mathbb{R}[V]$ ist (Theorem 4.16). Leider reicht dies nicht aus, um psd = sos in $\mathbb{R}[V]$ zu zeigen, da es auch nichtnegative reguläre Funktionen mit unendlich vielen Nullstellen gibt. In dem Spezialfall $C = V(x^2 + y^2 - 1)$ und $V = C \times \mathbb{A}^1$ ist es uns jedoch gelungen, dieses Problem zu umgehen und zu zeigen, dass psd = sos in $\mathbb{R}[V]$ gilt (Propositon 4.39). Dies gibt uns ein erstes Beispiel einer Varietät V mit $\dim(V) = 2$, wo psd = sos in $\mathbb{R}[V]$ gilt und $V(\mathbb{R})$ nicht kompakt ist. Die Methoden, die hierfür verwendet wurden, basieren wieder auf der Arbeit [14] von Marshall.
Leider scheinen sich diese Methoden nicht auf andere (glatte) Kurven anwenden zu lassen, sodass das obige V das einzige Beispiel bleibt, bei dem dies funktioniert. Ob auch bei anderen (glatten) Kurven C (und $V = C \times \mathbb{A}^1$) psd = sos in $\mathbb{R}[V]$ gilt, bleibt offen.

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