Boundary value problems in spaces of mixed smoothness and applications to plate-membrane systems

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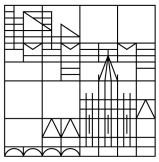

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Contents

Chapter 1

Introduction

In this thesis, we study systems of linear parameter-elliptic boundary value problems of the form

$$\begin{aligned} (\lambda - A)w &= f \qquad \text{in } \Omega, \\ Bw &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{1-1}$$

in L^p-based Sobolev spaces, where Ω is either the half-space $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$ or a domain in $\mathbb{R}^n$ with compact and sufficiently smooth boundary $\partial\Omega$. Moreover, A is a mixed order $N \times N$-matrix of Douglis–Nirenberg type with differential operators as entries. While the orders may differ, we assume them to be constantly $\mu \in \mathbb{N}$ on the diagonal of A. In addition, we suppose μN to be even and consider $M \times N$ boundary systems B, where $M := \frac{\mu N}{2}$. In the classical theory, $g = (g_1, \dots, g_M)$ belongs to a product of Besov spaces of strictly positive order. We focus on Besov spaces, where also zero or negative orders are allowed.

In those spaces even the formulation of the boundary conditions needs justification. It is known that the classical trace $u \mapsto u|_{\partial\Omega}$, first defined for smooth functions, has a continuous extension to an operator $\gamma_0 \colon H^s_p(\Omega) \to B^{s-1/p}_{pp}(\partial\Omega)$ if and only if $s > \frac{1}{p}$, see [45]. Nevertheless, it is possible to define a continuous trace on subspaces of $H^s_p(\Omega)$ for $s \leqslant \frac{1}{p}$.

In the literature, one can find several approaches to generalized traces and corresponding boundary value problems with rough boundary data: By considering the dual boundary value problem as by Lions and Magenes [52], [53], one obtains unique solvability in some negative order spaces. However, these spaces depend on the boundary conditions, which is the reason for introducing the universal but less natural spaces $\Xi^s(\Omega)$ in the beginning of Section 6.3 in [52]. By Roitberg [68], [69], generalized traces were defined using the completion of smooth functions. This concept leads to isomorphism results, but the considered spaces are non-standard and in general not even spaces of distributions on Ω. The Roitberg spaces are described in more detail in Remark 3.1.8. Another approach to rough boundary data was developed for instance by Hummel and Lindemulder [41], [43], where weighted Sobolev spaces (with respect to some distance function to the boundary) lead to a priori estimates. The spaces are natural and do not depend on the operators, but the order at the boundary is still restricted to the non-negative scale, see Theorem 6.2 of [43]. Spaces of arbitrary negative tangential order can be obtained as combinations of weighted spaces and spaces of dominating mixed smoothness, see Theorem 6.1 of [41].

In the present work, we will introduce a class of Sobolev spaces $H^{s,\sigma}_p(\mathbb{R}^n)$ of anisotropic type, for which the trace exists as a continuous operator, following the ideas of Grubb, see [35] and [36]. These spaces differ from anisotropic Sobolev spaces in the sense of [45] and [78] and from spaces with dominating mixed smoothness in the sense of [71] and [76]. For this class of Sobolev spaces, both the existence of a continuous trace and the unique solvability of parameter-elliptic model problems in the whole space and in the half-space follow immediately from known results. How-

ever, the passage from model problems, that is constant coefficients and no lower-order terms, to variable coefficients is not standard. It requires the application of an elaborate localization procedure even for problems on the half-space. In domains the definition of the Sobolev spaces with mixed smoothness is not canonical. Therefore, we work with classical Sobolev spaces in domains but employ local embeddings into our spaces of mixed smoothness. The necessity to estimate certain commutators leads to restrictions on the orders of the involved spaces, see Lemma 3.3.4.

With the help of the one-dimensional Fourier transform, we pass over to time-dependent systems of the form

$$\begin{aligned} (\partial_t - A)w &= f \qquad \text{in } (0,T)\times\Omega, \\ Bw &= g \qquad \text{on } (0,T)\times\partial\Omega, \\ w(0) &= 0 \end{aligned} \tag{1-2}$$

for $T > 0$. In some cases, this allows us to estimate the solution w in suitable Sobolev spaces with $\|g\|_{L^2((0,T),L^2(\partial\Omega))}$. We restrict ourselves to the Hilbert space case $p = 2$ since it is sufficient for our purpose.

As an application, we discuss a plate-membrane transmission problem with different variants of damping. The geometric situation is illustrated in Figure 1.1.

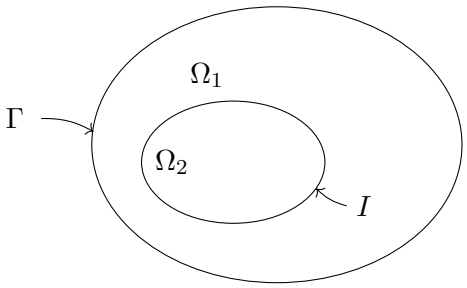

Figure 1.1: Geometric situation of the plate-membrane system in the case $n = 2$.

To be more precise, we consider a thermoelastic plate in Ω_1. The classical linear model of thermoelastic plates due to Kirchhoff is given by

$$\begin{aligned} u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + \Delta\theta &= 0 \quad \text{in } (0,\infty)\times\Omega_1, \\ \theta_t - \Delta u_t - \Delta\theta &= 0 \quad \text{in } (0,\infty)\times\Omega_1, \end{aligned} \tag{1-3}$$

where $\gamma \geqslant 0$ represents the rotational inertia of the filaments of the plate. It is proportional to the square of the plate thickness. The case $\gamma = 0$ corresponds to a thin plate. In this case, (1-3) can be written as a time-dependent Douglis–Nirenberg system of the form (1-2).

Here, $u(t,x)$ represents the transverse displacement, and $\theta(t,x)$ is the difference of temperature on the plate of the point x at time t with respect to a reference temperature. For the physical model and the deduction of these equations, see Chapter I of [46] and Chapter 2 of [47].

In the region Ω_2, we have an elastic membrane which is characterised by the equation

$$v_{tt} - \Delta v + m v_t = 0 \quad \text{in } (0,\infty)\times\Omega_2, \tag{1-4}$$

where $m \geqslant 0$ describes the damping (or the abscence thereof) for the wave equation. Again, $v(t,x)$ is the vertical displacement of the point with coordinates x at time t. We endow the equations with suitable boundary and transmission conditions as well as initial conditions and study the well-posedness, regularity and asymptotic behaviour of the solution, in dependence on the parameters m and γ.

In the literature, there are various results on thermoelastic plates in both cases $\gamma = 0$ and $\gamma > 0$, and on plate-membrane systems. For the case $\gamma = 0$, we refer to [7], [12], [38], [39] and [54], where structures formed by a plate and a membrane were studied. In [39], the author models a system composed by an elastic thin plate coupled with an elastic membrane and shows existence and uniqueness of weak solutions. In [54], the authors study undamped plate-membrane systems, where the plate and the membrane are two layers occupying the same region in the plane. In [7], a coupled system of a wave equation and a plate equation with damping on the boundary without thermal effects is analysed. In [38], the author considers a transmission problem with the configuration presented in Figure 1.1, but with the plate being surrounded by the membrane. In [12] the plate is not thermoelastic but structurally damped. Most of these references study some kind of stability for the solution but only a few of them deal with regularity. Regarding $\gamma > 0$, we mention for instance [9], [14], [15], [31], [48], [49], [60] and [62]. From these, only in [60] a transmission problem is analysed and it is of the thermoelastic plate-plate type. There seem to be few results for the structure (1-3) and (1-4), even for the case $\gamma = 0$.

The main results of this work state the existence and uniqueness of the solution of problem (1-3) and (1-4) with appropriate boundary and transmission conditions as well as initial conditions, and its continuous dependence on the data. It is also proved that the solution has higher regularity. In particular, the boundary and transmission conditions hold in the strong sense of traces if the initial values are smooth enough. Furthermore, we study the asymptotic behaviour of the solution in terms of the stability of the associated semigroup in different situations. For a damped membrane $m > 0$, we show exponential stability of the solution in Theorem 4.3.1. For an undamped membrane $m = 0$, there is a lack of exponential stability but we still get polynomial stability under some geometric condition, see Theorem 4.3.14 and 4.3.5. To prove that there is a lack of exponential stability for $m = 0$, we use the theory for time-dependent systems of the form (1-2).

The thesis is structured as follows:

In **Chapter 2**, we give a short introduction to the notation used throughout this thesis. Additionally, we present several tools needed for the succeeding chapters. We start with some observations concerning interpolation and extrapolation scales. In particular, we analyse the interaction between complex interpolation spaces and isomorphisms which depend on the interpolation parameter, see Proposition 2.2.1.

Afterwards, we introduce mild solutions and various types of stability of strongly continuous semigroups. The Gearhart–Prüss Theorem as well as a generalization of the results of Borichev and Tomilov yield an idea how to prove the exponential or polynomial stability of a C_0-semigroup, see Proposition 2.3.6 and 2.3.7.

As preparation for the analysis of the long-time behaviour of the solution of system (1-3) and (1-4), we deduce some integral equalities and inequalities. In doing so, we get an expression for the $L^2(\partial\Omega)$-norm at each point of time of the solution of the wave equation in Ω,

see Corollary 2.4.4. Furthermore, we derive another presentation of the $L^2(\Omega)$-norm using an identity of Rellich's type, see Lemma 2.4.6.

In the last part of Chapter 2, we explain the structure of Douglis–Nirenberg systems and mention the classical a priori estimates of parameter-elliptic boundary value problems. As example, we show that the operator Δ^m with boundary conditions of the form $\partial_n^{k_1}, \dots, \partial_n^{k_{m-1}}$ is elliptic in the half-space $\mathbb{R}^n_+$ for each $m \in \mathbb{N}$ and $k_1, \dots, k_{m-1} \in \{0, \dots, 2m-1\}$, see Example 2.5.4. We also discuss completely and not completely elliptic boundary conditions in Example 2.5.5. In Proposition 2.5.10, we suggest an improvement of the classical a priori estimate in some special cases. As an illustration, we discuss the statement of the before mentioned proposition for the structurally damped plate equation and various types of boundary conditions.

Chapter 3 is based on a joint work with Robert Denk, David Ploß and Jörg Seiler [19]. In the first part, we introduce the spaces $H_p^{s,\sigma}(\mathbb{R}^n)$ for $s, \sigma \in \mathbb{R}$ and $p \in (1, \infty)$, which have different smoothness in normal and in tangential direction. We also discuss what the complex interpolation of those spaces looks like, see Lemma 3.1.3. In case that $s > \frac{1}{p}$, we can form the continuous trace

$$\gamma_0 \colon H_p^{s,\sigma}(\Omega) \to B_{pp}^{s+\sigma-1/p}(\partial\Omega), \quad u \mapsto u\big|_{\partial\Omega}.$$

To obtain a priori estimates for the solution of (1-1), where the constants do not depend on the parameter λ, we deal with parameter-dependent spaces $H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)$. In Lemma 3.1.11, we analyse the continuity of appropriate multiplication operators in $H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)$.

Having introduced the spaces $H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)$, we solve system (1-1) for $\Omega = \mathbb{R}^n_+$ in those spaces of mixed smoothness. The main result of Section 3.2 is Theorem 3.2.14, where we show that

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} : \mathbb{E}_\lambda^{r,\sigma}(\mathbb{R}^n_+) \to \mathbb{F}_\lambda^{r,\sigma}(\mathbb{R}^n_+)$$

is an isomorphism for sufficiently large λ. The solution space $\mathbb{E}_\lambda^{r,\sigma}(\mathbb{R}^n_+)$ is a product of spaces of mixed smoothness. In $\mathbb{F}_\lambda^{r,\sigma}(\mathbb{R}^n_+)$ we encode the suitable space for the right hand side f and g of problem (1-1), which is also a product of spaces of mixed smoothness as well as Besov spaces.

In Section 3.3, we study problem (1-1) in domains Ω with compact and sufficiently smooth boundary. With localization procedures, we can use the observations of the half-space case, but since it is not canonical to transfer the definition of $H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)$ to domains, we restrict ourselves to classical spaces in Ω. This leads to the restriction $\sigma > -1$ to preserve suitable commutator estimates, see Lemma 3.3.4. In Theorem 3.3.5, we guarantee the unique solvability in larger spaces and formulate a priori estimates in classical Sobolev spaces of lower order.

Lastly, we use the previous findings and the one-dimensional Fourier transform to get corresponding results for the time-dependent problem (1-2). Here, we only investigate the Hilbert space setting, as in this case Besov spaces and Bessel potential spaces coincide and we can use the Theorem of Paley–Wiener in our arguments. The main result Theorem 3.4.8 is applied to the thermoelastic plate equation (1-3), which leads to the a priori estimate in Theorem 3.5.3. The considerations concerning time-dependent systems are not included in [19]. Moreover, [19] is formulated for scalar equations, which are transformed to systems of Douglis–Nirenberg structure in this work.

In **Chapter 4**, we keep on studying the thermoelastic plate equation (1-3) but in combination with the wave equation (1-4). It is based on a joint work with Bienvenido Barráza Martínez, Robert Denk, Jairo Hernández Monzón and Jonathan González Ospino [11]. In a first step, we show the well-posedness of system (1-3) and (1-4) for $\gamma, m \geqslant 0$ with appropriate boundary and transmission conditions as well as initial conditions. To this end, we reformulate (1-3) and (1-4) as a first-order Cauchy problem

$$\partial_t w - \mathcal{A}w = 0 \quad \text{in } (0,\infty)$$

with an operator $\mathcal{A}\colon \mathcal{H}_\gamma \supset D(\mathcal{A}) \to \mathcal{H}_\gamma$ in a suitable Hilbert space $\mathcal{H}_\gamma$. The most challenging part is the rotational term γ, which complicates the representation of $\mathcal{A}$. Next, we prove $0 \in \rho(\mathcal{A})$ and higher regularity of the domain $D(\mathcal{A})$, which allows us to conclude the compactness of $\mathcal{A}^{-1}\colon \mathcal{H}_\gamma \to \mathcal{H}_\gamma$.

Section 4.3 is about the long-time behaviour of the solution of the plate-membrane system, where we can observe a big difference between a positive damping parameter $m > 0$ and the absence of damping $m = 0$. In the case of a damped membrane, the solution is exponentially stable. Although there is a lack of exponential stability for $m = 0$, we can still find polynomial stability under certain geometric conditions. This extends the stability results of [11] by some combinations of γ and m.

The last section is an excursion to resolvent estimates of structurally damped plate-plate transmission problems in the one-dimensional case. It can be seen as a short outlook as there are no rigorous proofs of the results therein but an illustration of numerical simulations.

Chapter 2

Tools for the proofs of the main results

We start with some notation frequently used in the following. Parts of the notation are adapted from [11] and [19]. Throughout the whole thesis $C > 0$ stands for a generic constant, which may vary in each time of appearance.

2.1 Function spaces and some notation

For two $\mathbb{C}$-Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, let $L(X,Y)$ be the space of bounded linear operators from X to Y, $L(X) := L(X,X)$, and let $X' := L(X,\mathbb{C})$ be the dual space of X. We write $X \subset Y$ if X is a subset of Y and the inclusion map $X \to Y$ is bounded. This induces in particular that $X = Y$ if both spaces have the same elements and equivalent norms. To express that the norms on X and Y are equivalent, we sometimes use the notation $\|\cdot\|_X \approx \|\cdot\|_Y$. In case the inclusion map is compact, we write $X \overset{c}{\subset} Y$. For a linear, densely defined operator $A\colon X \supset D(A) \to Y$ with domain $D(A)$ and range $R(A)$, $A'\colon Y' \supset D(A') \to X'$ describes the adjoint operator. For a linear, closed operator $A\colon X \supset D(A) \to X$, we denote by $\sigma(A)$ the spectrum and by $\rho(A)$ the resolvent set of A.

In Lemma 2.1.1 we elaborate that subsets in many cases automatically possess a bounded inclusion map. While the result is formulated for Hilbert spaces in Lemma 3.5 of [11], we give it here in the more general context of Banach spaces. However, the proof works analogously as the Hilbert space property is not used therein.

2.1.1 Lemma. [11, Lemma 3.5]. *Let $A\colon X \supset D(A) \to X$ be a linear, closed operator on the Banach space X, and let V be a Banach space. If $D(A)$ is a subset of V, then we have the continuous embedding $D(A) \subset V$, where we endow $D(A)$ with the graph norm.*

Proof. As A is closed, $D(A)$ with the graph norm is a Banach space. We show that the identity $\mathrm{id}\colon D(A) \to V \cap X$ is a closed operator. For this, let $(x_k)_{k\in\mathbb{N}} \subset D(A)$ be a sequence with $x_k \to x$ in $D(A)$ and $x_k \to y$ in $V \cap X$. Then we obtain $x_k \to x$ in X by the definition of the graph norm, and also $x_k \to y$ in X by the definition of the norm $\|\cdot\|_{V\cap X} := \|\cdot\|_V + \|\cdot\|_X$ in $V \cap X$. This yields $x = y$. Consequently, $\mathrm{id}\colon D(A) \to V \cap X$ is closed and, by the closed graph theorem, continuous. As the embedding $V \cap X \to V$ is continuous by the definition of the norms, we obtain the continuity of $\mathrm{id}\colon D(A) \to V$. □

Next, we introduce the function spaces we are dealing with throughout the thesis. The space dimension is denoted by $n \in \mathbb{N}$. In the following, $\mathrm{BUC}^s(\mathbb{R}^n)$ denotes the space of all complex valued functions which are s-times continuously differentiable in $\mathbb{R}^n$ and for which all derivatives

up to order $s \in \mathbb{N}_0$ are bounded and uniformly continuous. In addition, we write $\mathscr{D}(\mathbb{R}^n)$ for the space of infinitely differentiable functions f on $\mathbb{R}^n$ with compact support $\operatorname{supp} f$ and $\mathscr{D}'(\mathbb{R}^n)$ for the space of distributions. Analogously, we write $\mathscr{S}(\mathbb{R}^n)$ for the space of Schwartz functions on $\mathbb{R}^n$ and $\mathscr{S}'(\mathbb{R}^n)$ for the space of tempered distributions. For $f \in \mathscr{S}(\mathbb{R}^n)$ we define the n-dimensional Fourier transform $\mathscr{F}_n f$ of f by

$$(\mathscr{F}_n f)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx \quad (\xi \in \mathbb{R}^n).$$

Here, $x\cdot\xi := \sum_{j=1}^n x_j\xi_j$ denotes the inner product on the Euclidean space $\mathbb{R}^n$. Since the mapping $\mathscr{F}_n\colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ is well defined and isomorphic, the Fourier transform can be canonically extended to a mapping $\mathscr{F}_n\colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$, see Corollary 2.2.15 and Definition 2.3.7 of [33]. If the dimension is clear from the context, we omit the index n and only write $\mathscr{F}$ for the Fourier transform.

Before we give the definition of L^p-based Bessel potential spaces, we explain our notation of multi indices. For $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}_0^n$ we write

$$\xi^\alpha := \prod_{j=1}^n \xi_j^{\alpha_j} \quad \text{and} \quad |\alpha| := \sum_{j=1}^n \alpha_j.$$

Additionally, let $\beta \in \mathbb{N}_0^n$. If $\alpha_j \leqslant \beta_j$ for every $j = 1, \dots, n$, we use the abbreviation $\alpha \leqslant \beta$.

For the half-space we write

$$\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}.$$

Similarly, we introduce $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

We fix $\mu \in \mathbb{N}$ (which will be specified later) and define

$$\langle \xi, \lambda\rangle := (1 + |\xi|^2 + |\lambda|^{2/\mu})^{1/2} \quad \text{and} \quad \langle \xi', \lambda\rangle := (1 + |\xi'|^2 + |\lambda|^{2/\mu})^{1/2} \tag{2-1}$$

for $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$. Moreover, we write $\langle \xi\rangle := \langle \xi, 0\rangle$, $\langle \xi'\rangle := \langle \xi', 0\rangle$ and $\langle \lambda\rangle := \langle 0, \lambda\rangle$. Even though the notation might not be very rigorous as we use the same description for different things, the meaning will be clear from the context.

For (suitable) functions $\varphi = \varphi(\xi)$ defined on $\mathbb{R}^n$, we denote by $\varphi(D)$ its associated Fourier multiplier, which is defined by $\varphi(D) := \mathscr{F}^{-1}(\varphi\mathscr{F})$. In particular, we have $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$ for $\alpha \in \mathbb{N}_0^n$, where ∂^α is the α-th distributional derivative. In case $\varphi(\xi) = \varphi(\xi')$ is independent of ξ_n, the associated Fourier multiplier is also denoted by $\varphi(D')$.

For $p \in (1, \infty)$ and $s \in \mathbb{R}$, we define the Bessel potential space $H_p^s(\mathbb{R}^n)$ via

$$H_p^s(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D\rangle^s u \in L^p(\mathbb{R}^n)\}$$

with the canonical norm $\|u\|_{H_p^s(\mathbb{R}^n)} := \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)}$ for $u \in H_p^s(\mathbb{R}^n)$. Notice that for $s \in \mathbb{N}_0$ the spaces $H_p^s(\mathbb{R}^n)$ coincide with the standard L^p-based Sobolev spaces

$$W_p^s(\mathbb{R}^n) := \{u \in L^p(\mathbb{R}^n) : \partial^\alpha u \in L^p(\mathbb{R}^n) \text{ for } |\alpha| \leqslant s\}$$

of order s. This follows from Mikhlin's Theorem, see Section IV.3.2 of [73]. In the Hilbert space case $p = 2$ we sometimes omit the index p. On $L^2(\mathbb{R}^n)$ we use the inner product

$$\langle u, v\rangle_{L^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} u(x)\overline{v(x)}\,dx \quad (u, v \in L^2(\mathbb{R}^n)).$$

Besides the Bessel potential spaces, we also need to introduce Besov spaces which appear canonically as trace spaces in the context of boundary value problems. There are various ways to define Besov spaces. Here, we choose the definition via dyadic decompositions.

2.1.2 Definition. [77, Section 2.3.1]. A series $(\varphi_k)_{k\in\mathbb{N}_0} \subset \mathscr{D}(\mathbb{R}^n)$ is called a *dyadic decomposition* if the following three properties hold:

(i) $\varphi_k \geqslant 0$ for all $k \in \mathbb{N}_0$, $\operatorname{supp}\varphi_0 \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}$ and

$$\operatorname{supp}\varphi_k \subset \{\xi \in \mathbb{R}^n : 2^{k-1} < |\xi| < 2^{k+1}\} \quad (k \in \mathbb{N}).$$

(ii) There exists a $C > 0$ such that for all $\xi \in \mathbb{R}^n$ we have $\sum_{k\in\mathbb{N}_0} \varphi_k(\xi) \geqslant C$.

(iii) For every $\alpha \in \mathbb{N}_0^n$ there is a constant $C_\alpha > 0$ with $|\xi|^{|\alpha|}|\partial^\alpha\varphi_k(\xi)| \leqslant C_\alpha$ for all $\xi \in \mathbb{R}^n$ and $k \in \mathbb{N}_0$.

With the help of the definition above, we are now in the situation to describe Besov spaces.

2.1.3 Definition. [77, Section 2.3.2]. Let $(\varphi_k)_{k\in\mathbb{N}_0} \subset \mathscr{D}(\mathbb{R}^n)$ be a fixed dyadic decomposition. For $s \in \mathbb{R}$ and $p, q \in (1, \infty)$, we call $B^s_{pq}(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) : \|u\|_{B^s_{pq}(\mathbb{R}^n)} < \infty\}$ the *Besov space of order s with parameters p and q*. The norm on $B^s_{pq}(\mathbb{R}^n)$ is given by

$$\|u\|_{B^s_{pq}(\mathbb{R}^n)} := \left(\sum_{k\in\mathbb{N}_0} 2^{skq}\|\varphi_k(D)u\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}.$$

Up to equivalence of norms, this definition is independent of the concrete choice of the sequence $(\varphi_k)_{k\in\mathbb{N}_0} \subset \mathscr{D}(\mathbb{R}^n)$.

If X is one of the function spaces mentioned above, that is, $X \in \{\mathscr{S}, \mathscr{S}', H^s_p, B^s_{pq}, \ldots\}$, we define

$$X(\Omega) := \{u\big|_\Omega : u \in X(\mathbb{R}^n)\}$$

for a domain $\Omega \subset \mathbb{R}^n$, where the restriction is understood in the distributional sense, with the canonical norm

$$\|v\|_{X(\Omega)} := \inf\{\|u\|_{X(\mathbb{R}^n)} : u \in X(\mathbb{R}^n),\, u\big|_\Omega = v\},$$

see for instance Definition 4.1 in [78]. Moreover, for a closed subset $A \subset \mathbb{R}^n$ we define

$$\dot{X}(A) := \{u \in X(\mathbb{R}^n) : \operatorname{supp} u \subset A\}.$$

By definition we see that $X(\Omega)$ can be identified with the quotient space $X(\mathbb{R}^n)/\dot{X}(\mathbb{R}^n\backslash\Omega)$, see Section 2.2 of [19].

Some of the spaces will also appear in the vector-valued setting. The definitions adapt in an obvious way.

2.1.4 Remark. In this remark, we want to collect some important characterizations concerning Besov spaces.

(i) Another quite common description of Besov spaces can be given by the method of real interpolation. For $s_0, s_1 \in \mathbb{R}$, $p, q \in (1,\infty)$ and $\theta \in (0,1)$ it holds

$$B^s_{pq}(\mathbb{R}^n) = (H^{s_0}_p(\mathbb{R}^n), H^{s_1}_p(\mathbb{R}^n))_{\theta,q},$$

where $(\cdot,\cdot)_{\theta,q}$ denotes the real interpolation functor and $s \in \mathbb{R}$ is given by $s := (1-\theta)s_0 + \theta s_1$, see Remark 4 in Section 2.4 of [77].

(ii) In case $p = q$, the Besov spaces can be seen as trace spaces of $H^s_p(\mathbb{R}^n_+)$ in the following sense, see Section 2.3 of [19]. The trace

$$\gamma_0 \colon H^s_p(\mathbb{R}^n_+) \to B^{s-1/p}_{pp}(\mathbb{R}^{n-1}), \quad u \mapsto \gamma_0 u := u\big|_{x_n=0} \tag{2-2}$$

exists and is continuous if and only if $s > \frac{1}{p}$. In fact, it was shown in Theorem 2.4 of [45] that for $s \leqslant \frac{1}{p}$ the map $u \mapsto \gamma_0 u$ is not even continuous from $H^s_p(\mathbb{R}^n_+)$ to $\mathscr{D}'(\mathbb{R}^{n-1})$. If $s > \frac{1}{p}$, then (2-2) is a retraction, and γ_0 is the unique extension of the classical boundary trace $u \mapsto u\big|_{x_n=0}$ for smooth functions $u \in \mathscr{S}(\mathbb{R}^n_+)$. We will also consider the higher-order traces $\gamma_j \colon H^s_p(\mathbb{R}^n_+) \to B^{s-j-1/p}_{pp}(\mathbb{R}^{n-1})$, $u \mapsto \gamma_0 \partial_n^j u$ for $j \in \mathbb{N}_0$ and $s > j + \frac{1}{p}$.

Both the Bessel potential spaces and the Besov spaces will also appear in a parameter-dependent version later on. For this, we follow the approach of Grubb–Kokholm [36].

2.1.5 Definition. [19, Definition 2.8]. Let X_λ and Y_λ be families of Banach spaces (parametrized by λ from some index set). A family of linear operators $T(\lambda) \colon X_\lambda \to Y_\lambda$ is said to be *continuous* if $T(\lambda) \in L(X_\lambda, Y_\lambda)$ for every fixed λ and the operator norm $\|T(\lambda)\|_{L(X_\lambda,Y_\lambda)}$ is uniformly bounded in λ. A continuous family is called an *isomorphism* if each $T(\lambda)$ is invertible and $T(\lambda)^{-1} \colon Y_\lambda \to X_\lambda$ is continuous, too.

In our context, the occurring families of spaces X_λ will consist of a fixed vector space X equipped with a norm depending on the parameter λ.

2.1.6 Definition. [19, Definition 2.9]. For $p \in (1,\infty)$ and $\lambda \in \mathbb{C}$, let κ_λ denote the homeomorphism on $\mathscr{S}'(\mathbb{R}^n)$ given on $\mathscr{S}(\mathbb{R}^n)$ by

$$(\kappa_\lambda u)(x) := u(\langle\lambda\rangle x) \quad (u \in \mathscr{S}(\mathbb{R}^n),\, x \in \mathbb{R}^n).$$

Then, we define the parameter-dependent norms by

$$\begin{aligned} \|u\|_{H^s_{p,\lambda}(\mathbb{R}^n)} &:= \langle\lambda\rangle^{s-n/p} \|\kappa_\lambda^{-1} u\|_{H^s_p(\mathbb{R}^n)}, \\ \|u\|_{B^s_{pp,\lambda}(\mathbb{R}^{n-1})} &:= \langle\lambda\rangle^{s-(n-1)/p} \|\kappa_\lambda^{-1} u\|_{B^s_{pp}(\mathbb{R}^{n-1})} \end{aligned}$$

for $s \in \mathbb{R}$. Analogously, we define $H^s_{p,\lambda}(\mathbb{R}^n_+)$.

Notice that $\|\cdot\|_{H^s_{p,0}(\mathbb{R}^n)} = \|\cdot\|_{H^s_p(\mathbb{R}^n)}$ and $\|\cdot\|_{B^s_{pp,0}(\mathbb{R}^{n-1})} = \|\cdot\|_{B^s_{pp}(\mathbb{R}^{n-1})}$ for $s \in \mathbb{R}$.

To get a better feeling about how the parameter-dependent norms behave, we make the following observation. The result was already shown in [36], which is the reason why we omit the proof here.

2.1.7 Lemma. [36, Section 1.1, (1.6)]. *For $p \in (1,\infty)$, $s \geqslant 0$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$ it holds*

$$\|u\|_{H^s_{p,\lambda}(\mathbb{R}^n)} \approx \|u\|_{H^s_p(\mathbb{R}^n)} + |\lambda|^{s/\mu}\|u\|_{L^p(\mathbb{R}^n)} \quad (u \in H^s_p(\mathbb{R}^n)).$$

In the case of Besov spaces, we get a similar result for $s - \frac{1}{p}$, where $s > 0$ denotes an integer. This follows from the structure of the trace space of $H^s_{p,\lambda}(\mathbb{R}^n_+)$.

2.1.8 Lemma. *For $p \in (1,\infty)$, $s \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$ it holds*

$$\|u\|_{B^{s-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})} \approx \|u\|_{B^{s-1/p}_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{(s-1/p)/\mu}\|u\|_{L^p(\mathbb{R}^{n-1})} \quad (u \in B^{s-1/p}_{pp}(\mathbb{R}^{n-1})).$$

Proof. Due to (1.21) of [36] as well as Proposition 2.2 and Proposition 2.3 of [2], the mappings $\widetilde{\gamma}_1 \colon H^s_{p,\lambda}(\mathbb{R}^n_+) \to B^{s-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})$ and

$$\widetilde{\gamma}_2 \colon H^s_{p,\lambda}(\mathbb{R}^n_+) \to \big(B^{s-1/p}_{pp}(\mathbb{R}^{n-1}), \|\cdot\|_{B^{s-1/p}_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{(s-1/p)/\mu}\|\cdot\|_{L^p(\mathbb{R}^{n-1})}\big)$$

with $\widetilde{\gamma}_j v := v\big|_{x_n=0}$ for $j = 1,2$ are retractions. The appearing constants do not depend on λ. In consequence, there are continuous extension operators $e_1 \colon B^{s-1/p}_{pp,\lambda}(\mathbb{R}^{n-1}) \to H^s_{p,\lambda}(\mathbb{R}^n_+)$ and

$$e_2 \colon \big(B^{s-1/p}_{pp}(\mathbb{R}^{n-1}), \|\cdot\|_{B^{s-1/p}_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{(s-1/p)/\mu}\|\cdot\|_{L^p(\mathbb{R}^{n-1})}\big) \to H^s_{p,\lambda}(\mathbb{R}^n_+)$$

such that $\widetilde{\gamma}_j e_j = \mathrm{id}$ for $j = 1,2$. Altogether, we obtain

$$\|u\|_{B^{s-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})} = \|\widetilde{\gamma}_1 e_2 u\|_{B^{s-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})} \leqslant C\big(\|u\|_{B^{s-1/p}_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{(s-1/p)/\mu}\|u\|_{L^p(\mathbb{R}^{n-1})}\big)$$

and analogously

$$\|u\|_{B^{s-1/p}_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{(s-1/p)/\mu}\|u\|_{L^p(\mathbb{R}^{n-1})} \leqslant C\|u\|_{B^{s-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}$$

for $u \in B^{s-1/p}_{pp}(\mathbb{R}^{n-1})$. □

2.1.9 Remark. Using (1.16) of [36] and Proposition 5.1 of [50], real interpolation and Lemma 2.1.7 lead to

$$\|u\|_{B^s_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{s/\mu}\|u\|_{L^p(\mathbb{R}^{n-1})} \leqslant C\|u\|_{B^s_{pp,\lambda}(\mathbb{R}^{n-1})} \quad (u \in B^s_{pp}(\mathbb{R}^{n-1}))$$

for arbitrary $s > 0$, where $C > 0$ does not depend on $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant 1$.

On the other hand, we have

$$\|u\|_{B^s_{pp}(\mathbb{R}^{n-1})} \approx \|u\|_{L^p(\mathbb{R}^{n-1})} + \Big(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+sp}}\,dx\,dy\Big)^{1/p}$$

for $s \in (0,1)$ according to Remark 4 in Section 2.5.1 of [77]. Integration by substitution shows

$$\|u\|_{B^s_{pp,\lambda}(\mathbb{R}^{n-1})} \leqslant C\big(\|u\|_{B^s_{pp}(\mathbb{R}^{n-1})} + |\lambda|^{s/\mu}\|u\|_{L^p(\mathbb{R}^{n-1})}\big) \quad (u \in B^s_{pp}(\mathbb{R}^{n-1}))$$

for $s \in (0,1)$. Since $C > 0$ is independent of λ, this generalizes Lemma 2.1.8.

2.2 Interpolation and extrapolation spaces

In this section, we give a short introduction to the method of complex interpolation, which is based on Section 2.2 of [19] and references therein. Afterwards, we briefly recall the theory of extrapolation spaces.

Let X_0 and X_1 be an interpolation couple of complex Banach spaces and

$$S := \{z \in \mathbb{C} : 0 < \operatorname{Re}(z) < 1\}.$$

Denote by $\mathcal{F}(X_0, X_1)$ the space of all continuous functions $f\colon \overline{S} \to X_0 + X_1$, defined on the closure $\overline{S}$ of S, such that $f\big|_S$ is holomorphic as an $(X_0 + X_1)$-valued function on S and the mapping $\mathbb{R} \to X_j$, $b \mapsto f(j + ib)$ is bounded and continuous for $j \in \{0, 1\}$. $\mathcal{F}(X_0, X_1)$ becomes a Banach space with the norm

$$\|f\|_{\mathcal{F}(X_0,X_1)} := \max_{j=0,1} \sup_{b\in\mathbb{R}} \|f(j + ib)\|_{X_j}.$$

For $\theta \in (0, 1)$, the complex interpolation space $[X_0, X_1]_\theta$ is defined as the space of all $x \in X_0 + X_1$ for which $x = f(\theta)$ for some $f \in \mathcal{F}(X_0, X_1)$, endowed with the norm

$$\|x\|_{[X_0,X_1]_\theta} := \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} \,:\, f(\theta) = x\}.$$

With this norm, the complex interpolation space becomes a Banach space satisfying

$$X_0 \cap X_1 \subset [X_0, X_1]_\theta \subset X_0 + X_1.$$

In the definition of the interpolation space and the norm, the space $\mathcal{F}(X_0, X_1)$ can be replaced by the subspace $\mathcal{F}_0(X_0, X_1)$ which consists of all $f \in \mathcal{F}(X_0, X_1)$ with $\lim_{|b|\to\infty} \|f(j + ib)\|_{X_j} = 0$ for $j \in \{0, 1\}$, see Proposition C.2.4 of [44]. We will also consider the space $\mathcal{F}_0(X_0, X_1; X_0 \cap X_1)$ consisting of all $f \in \mathcal{F}_0(X_0, X_1)$ for which $f(z) \in X_0 \cap X_1$ for all $z \in \overline{S}$ and where f is holomorphic on S and continuous on $\overline{S}$ as a function with values in $X_0 \cap X_1$. Note that the definition of this space differs from the one in [44]. By Theorem 1.9.1 of [77], $\mathcal{F}_0(X_0, X_1; X_0 \cap X_1)$ is dense in $\mathcal{F}_0(X_0, X_1)$.

The following proposition is a generalized version of Lemma 2.3 in [19]. The ideas of the proof are adapted to the slightly different context.

2.2.1 Proposition. *Let X_θ, Y_θ be Banach spaces, where X_0, X_1 and Y_0, Y_1 are interpolation couples of complex Banach spaces with $[X_0, X_1]_\theta = X_\theta$ for each $\theta \in (0, 1)$. In addition, let Z be a Hausdorff space with $Y_0 + Y_1 \subset Z$ and $Y_\theta \subset Z$ for $\theta \in (0, 1)$. Let $T_z\colon X_0 + X_1 \to Z$ be a linear operator such that $T_z\colon X_{\operatorname{Re}(z)} \to Y_{\operatorname{Re}(z)}$ is an isomorphism for all $z \in \overline{S}$.*

(i) Let $Y_0 \cap Y_1 \subset Y_\theta$ for $\theta \in (0, 1)$ and let the mapping $\overline{S} \to X_0 + X_1$, $z \mapsto T_z^{-1} y$ be holomorphic on S and continuous on $\overline{S}$ for all $y \in Y_0 \cap Y_1$. Moreover, we assume $\|T_{j+ib}^{-1}\|_{L(Y_j,X_j)} \leqslant Ce^{b^2}$ for $j \in \{0, 1\}$, $b \in \mathbb{R}$, and a constant $C > 0$. Then we have $[Y_0, Y_1]_\theta \subset Y_\theta$ for $\theta \in (0, 1)$.

(ii) Let the mapping $\overline{S} \to Y_0 + Y_1$, $z \mapsto T_z x$ be holomorphic on S and continuous on $\overline{S}$ for all $x \in X_0 \cap X_1$ and $\|T_{j+ib}\|_{L(X_j,Y_j)} \leqslant Ce^{b^2}$ for $j \in \{0, 1\}$, $b \in \mathbb{R}$, and a constant $C > 0$. Then we have $Y_\theta \subset [Y_0, Y_1]_\theta$ for $\theta \in (0, 1)$.

Proof. We prove the first statement in detail. The second one works similarly. Let $\theta \in (0,1)$.

(i) Let $y \in [Y_0, Y_1]_\theta$ and $g \in \mathcal{F}_0(Y_0, Y_1)$ with $g(\theta) = y$. By density, there exists a sequence $(g_k)_{k\in\mathbb{N}} \subset \mathcal{F}_0(Y_0, Y_1; Y_0 \cap Y_1)$ such that $g_k \to g$ in $\mathcal{F}(Y_0, Y_1)$. It follows that $g_k(\theta) \to g(\theta)$ in $[Y_0, Y_1]_\theta$. For $k \in \mathbb{N}$ let us define

$$f_k(z) := e^{z^2-\theta^2} T_z^{-1} g_k(z) \quad (z \in \overline{S}).$$

We notice that $f_k(z) \in X_0 + X_1$ for each $z \in \overline{S}$ with continuous and holomorphic dependence on $z \in \overline{S}$ and $z \in S$, respectively. In addition, we get

$$\|f_k(j+ib)\|_{X_j} \leqslant C\|g_k(j+ib)\|_{Y_j} \quad (b \in \mathbb{R})$$

for $j \in \{0,1\}$. In particular, we have

$$\sup_{b\in\mathbb{R}} \|f_k(j+ib)\|_{X_j} \leqslant C \sup_{b\in\mathbb{R}} \|g_k(j+ib)\|_{Y_j}$$

for $j \in \{0,1\}$. Summing up, we have shown that $f_k \in \mathcal{F}(X_0, X_1)$ with

$$\|f_k\|_{\mathcal{F}(X_0,X_1)} \leqslant C\|g_k\|_{\mathcal{F}(Y_0,Y_1)}.$$

As $(g_k)_{k\in\mathbb{N}}$ is convergent and therefore a Cauchy sequence, we obtain that $(f_k(\theta))_{k\in\mathbb{N}}$ is a Cauchy sequence in $[X_0, X_1]_\theta = X_\theta$. By completeness, there exists $x \in X_\theta$ with $f_k(\theta) \to x$ in X_θ and consequently $g_k(\theta) = T_\theta f_k(\theta) \to T_\theta x$ in Y_θ for $k \to \infty$. On the other hand, $g_k(\theta) \to y$ in $[Y_0, Y_1]_\theta$ for $k \to \infty$. Because of the uniqueness of limits in Z we get $y = T_\theta x \in Y_\theta$ with

$$\|y\|_{Y_\theta} \leqslant C\|x\|_{X_\theta} \leqslant C \lim_{k\to\infty} \|f_k\|_{\mathcal{F}(X_0,X_1)} \leqslant C \lim_{k\to\infty} \|g_k\|_{\mathcal{F}(Y_0,Y_1)} = C\|g\|_{\mathcal{F}(Y_0,Y_1)}.$$

As $g \in \mathcal{F}_0(Y_0, Y_1)$ was arbitrary with $g(\theta) = y$, we obtain $\|y\|_{Y_\theta} \leqslant C\|y\|_{[Y_0,Y_1]_\theta}$ which finishes the proof of (i).

(ii) Let $y \in Y_\theta$ and $x := T_\theta^{-1} y \in X_\theta = [X_0, X_1]_\theta$, and let $f \in \mathcal{F}_0(X_0, X_1)$ with $f(\theta) = x$. We approximate f by a sequence $(f_k)_{k\in\mathbb{N}} \subset \mathcal{F}_0(X_0, X_1; X_0 \cap X_1)$ and set $g_k(z) := e^{z^2-\theta^2} T_z f_k(z)$ for $z \in \overline{S}$ and $k \in \mathbb{N}$. We have $g_k(z) \in Y_0 + Y_1$ for each $z \in \overline{S}$ and can argue as above to show the embedding $Y_\theta \subset [Y_0, Y_1]_\theta$. □

The next lemma shows the compatibility of interpolation and the dependence of a parameter in the case of Bessel potential spaces.

2.2.2 Lemma. [34, Theorem 1.4.3.3]. *Let $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. For $s_1 > s_2 > s_3 \geqslant 0$ there exists a constant $C > 0$ (depending on s_1, s_2, s_3, Ω and p) such that*

$$|\lambda|^{\frac{s_1-s_2}{s_1-s_3}} \|u\|_{H_p^{s_2}(\Omega)} \leqslant C\left(\|u\|_{H_p^{s_1}(\Omega)} + |\lambda|\|u\|_{H_p^{s_3}(\Omega)}\right)$$

for all $u \in H_p^{s_1}(\Omega)$ and $\lambda \in \mathbb{C}$. An analogue inequality also holds for $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{R}^n_+$.

Given two Banach spaces, the main idea of interpolation theory is to create new Banach spaces "in between". In contrast, the concept of extrapolation spaces is to generate larger Banach spaces out of only one Banach space. Hereby, the new spaces depend on a closed operator A. We give a short summary of the results and refer to Chapter V of Amann's book [5] for more details.

2.2.3 Definition. [5, Subsection V.1.3]. Let X_0 be a Banach space and $A\colon X_0 \supset D(A) \to X_0$ a linear, closed and densely defined operator with $0 \in \rho(A)$. Let X_{-1} be the completion of X_0 with respect to the norm $\|\cdot\|_{-1} := \|A^{-1}\cdot\|$. Then it holds $X_0 \subset X_{-1}$ and we call X_{-1} the *extrapolation space of X_0 generated by A.*

2.2.4 Remark. [5, Corollary V.1.4.7]. Under the assumptions of Defintion 2.2.3, we can identify the space X_{-1} with the dual space of $D(A')$, where $A'\colon X_0' \supset D(A') \to X_0'$ denotes the adjoint operator of A.

The following proposition connects the interpolation and extrapolation theory and gives both a density and compactness result in the interpolation-extrapolation scale.

2.2.5 Proposition. [5, Theorem V.1.5.1]. *Under the assumptions of Definition 2.2.3, we set $X_{-\theta} := [X_0, X_{-1}]_\theta$ for $\theta \in (0,1)$. Then the embedding $X_{-\alpha} \subset X_{-\beta}$ is dense for $0 \leqslant \alpha < \beta \leqslant 1$. The embedding is also compact if and only if $A^{-1} \in L(X_0)$ is a compact operator.*

2.3 Mild solutions and various types of stability of a semigroup

The Cauchy problem

$$\begin{aligned} \partial_t w - Aw &= 0 \quad \text{in } (0,\infty),\\ w(0) &= w_0, \end{aligned}$$

where the closed operator $A\colon H \supset D(A) \to H$ on a Hilbert space H and $w_0 \in D(A)$ are known, is classically treated by semigroup approaches. It possesses a unique solution

$$w \in C^1([0,\infty), H) \cap C([0,\infty), D(A))$$

if A is the generator of a strongly continuous semigroup $(S(t))_{t\geqslant 0}$. The solution is thus given by $w(t) = S(t)w_0$ for $t \geqslant 0$, see Proposition II.6.2 of [26].

We recall that a *strongly continuous semigroup* or *C_0-semigroup* is a family $(S(t))_{t\geqslant 0} \subset L(H)$ of linear, continuous operators on a Hilbert space H with

(i) $S(t+s) = S(t)S(s)$ for all $t, s \geqslant 0$,

(ii) $\lim\limits_{t\searrow 0} S(t)x = x$ for all $x \in H$.

If there is a constant $C > 0$ such that $\|S(t)\|_{L(H)} \leqslant C$ for all $t \geqslant 0$, we call $(S(t))_{t\geqslant 0}$ a *bounded C_0-semigroup*. In the case where we can choose $C = 1$, we speak of a *C_0-semigroup of contractions*.

The *generator* A of the C_0-semigroup $(S(t))_{t\geqslant 0}$ is defined by $A\colon H \supset D(A) \to H$ with

$$D(A) = \left\{ x \in H : \lim_{h \searrow 0} \frac{S(h)x - x}{h} \text{ exists} \right\}$$

and $Ax := \lim_{h\searrow 0} \frac{S(h)x-x}{h}$ for $x \in D(A)$, see Definition I.5.1 and Definition II.1.2 of [26].

In addition to the classical solution introduced above, we also need a weaker concept of solutions for a Cauchy problem. The following definition is adapted from Definition II.6.3 in [26] to finite time intervals.

2.3.1 Definition. [26, Definition II.6.3]. Let $T \in (0,\infty)$, H be a Hilbert space, $w_0 \in H$ and $A\colon H \supset D(A) \to H$ a linear, closed operator. A continuous function $w\colon [0,T] \to H$ is called a *mild solution* of the abstract Cauchy problem

$$\begin{aligned} \partial_t w - Aw &= 0 \quad \text{in } (0,T), \\ w(0) &= w_0 \end{aligned} \tag{2-3}$$

if $\int_0^t w(s)\,ds \in D(A)$ and

$$w(t) = A\int_0^t w(s)\,ds + w_0$$

for all $t \in [0,T]$.

It is of particular interest to us that the mild solution of an abstract Cauchy problem is unique. The proof works analogously as the one of Proposition II.6.4 in [26] but is modified for finite time intervals.

2.3.2 Proposition. [26, Proposition II.6.4]. *Let $T \in (0,\infty)$ and $A\colon H \supset D(A) \to H$ be the generator of a C_0-semigroup $(S(t))_{t\geqslant 0}$ on a Hilbert space H. Then $w\colon [0,T] \to H$ with $w(t) := S(t)w_0$ is the unique mild solution of (2-3) for every $w_0 \in H$.*

Proof. Without loss of generality, we assume $w\colon [0,T] \to H$ to be a mild solution of (2-3) with $w_0 = 0$ and take $t \in (0,T]$. Then we obtain

$$\frac{d}{ds}\left(S(t-s)\int_0^s w(r)\,dr \right) = S(t-s)w(s) - S(t-s)A\int_0^s w(r)\,dr = 0$$

for each $s \in (0,t)$. Integration from 0 to t of the equality above leads to

$$\int_0^t w(r)\,dr = \int_0^t \frac{d}{ds}\left(S(t-s)\int_0^s w(r)\,dr \right) ds = 0$$

and hence $w(t) = 0$. □

Besides the notion of mild solutions, we focus on the asymptotic behaviour of a semigroup. Therefore, we introduce the definition of polynomially and exponentially stable semigroups. In the literature the last mentioned concept is also called uniformly exponential stability but we omit the term uniformly in this context.

2.3.3 Definition. [26, Definition V.1.1]. Let H be a Hilbert space and $(S(t))_{t\geqslant 0}$ a strongly continuous semigroup with generator $A\colon H \supset D(A) \to H$. We call $(S(t))_{t\geqslant 0}$ *exponentially stable* if there are constants $C > 0$ and $\delta > 0$ such that

$$\|S(t)\|_{L(H)} \leqslant Ce^{-\delta t}$$

for all $t > 0$.

2.3.4 Definition. Let H be a Hilbert space and $(S(t))_{t\geqslant 0}$ a strongly continuous semigroup with generator $A\colon H \supset D(A) \to H$. We say that $(S(t))_{t\geqslant 0}$ is *polynomially stable* or *decays polynomially* if there are constants $C > 0$ and $\alpha > 0$ such that

$$\|S(t)w_0\|_H \leqslant Ct^{-\alpha}\big(\|w_0\|_H + \|Aw_0\|_H\big)$$

for all $t > 0$ and $w_0 \in D(A)$.

2.3.5 Remark. The exponential stability is a condition on the operator norm in $L(H)$ whereas the polynomial stability is a condition on the operator norm in $L(D(A), H)$ of the semigroup. If we asked the operator norm in $L(H)$ to tend polynomially to zero this would already be equivalent to the exponential stability of the semigroup. For a proof we refer to Proposition V.1.2 of [26].

An approach to showing that a semigroup is exponentially stable is the following characterization. It is known as the Theorem of Gearhart and Prüss, see Corollary 4 of [66] and Theorem V.1.11 of [26]. We only mention the special case of bounded C_0-semigroups here as it is sufficient for our purpose.

2.3.6 Proposition. [16, Theorem 1.1]. *Let H be a Hilbert space and $(S(t))_{t\geqslant 0}$ a bounded C_0-semigroup with generator $A\colon H \supset D(A) \to H$. Then, $(S(t))_{t\geqslant 0}$ is exponentially stable if and only if the imaginary axis $i\mathbb{R} := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = 0\}$ is a subset of $\rho(A)$ with the resolvent satisfying*

$$\sup_{\lambda\in\mathbb{R}}\{\|(i\lambda - A)^{-1}\|_{L(H)}\} < \infty.$$

Similarly, there is also a relation between the polynomial stability of a semigroup and some condition on the norm of the resolvent of the corresponding generator. Here, we mention a generalization of the result from Borichev and Tomilov, see Theorem 2.4 of [13].

2.3.7 Proposition. [61, Lemma 5.2]. *Let $\alpha \in \mathbb{N}_0$, $\beta > 0$, H be a Hilbert space and $(S(t))_{t\geqslant 0}$ a C_0-semigroup of contractions with generator $A\colon H \supset D(A) \to H$ and $i\mathbb{R} \subset \rho(A)$. Then, the following two conditions are equivalent.*

(i) There is a constant $C > 0$ and a $\lambda_0 > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant \lambda_0$ and all $f \in D(A^\alpha)$ the inequality

$$\|(i\lambda - A)^{-1}f\|_H \leqslant C|\lambda|^\beta\|A^\alpha f\|_H$$

holds true.

(ii) There exists a constant $C > 0$ such that for all $t > 0$ and $w_0 \in D(A)$ we have

$$\|S(t)w_0\|_H \leqslant Ct^{-1/(\alpha+\beta)}\|Aw_0\|_H.$$

2.4 Integral equalities and inequalities

This section might be seen as a groundwork for the last chapter. To investigate the stability of the plate-membrane transmission problem given in Chapter 4 later on, we need some integral equalities and inequalities. Our focus is especially on estimates of the $L^2(\partial\Omega)$-norm of the boundary of a domain $\Omega \subset \mathbb{R}^n$.

2.4.1 Lemma. [56, Theorem 1.4.4]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1-boundary. Then, there is a constant $C > 0$ such that for any function $u \in H^1(\Omega)$ the following estimate holds:*

$$\|u\|_{L^2(\partial\Omega)} \leqslant C\|u\|^{1/2}_{H^1(\Omega)}\|u\|^{1/2}_{L^2(\Omega)}.$$

Proof. We only sketch the proof to give a rough idea how one could prove the result and omit the details. We restrict ourselves to the half-space case. For $u \in C^1(\mathbb{R}^n_+)$ with bounded support it holds

$$\begin{aligned}
\|u\|^2_{L^2(\mathbb{R}^{n-1})} &= \int_{\mathbb{R}^{n-1}} |u(x',0)|^2\,dx' \\
&= -\int_{\mathbb{R}^{n-1}}\int_0^\infty \partial_n|u(x',x_n)|^2\,dx_n\,dx' \\
&\leqslant \int_{\mathbb{R}^{n-1}}\int_0^\infty 2|\partial_n u(x',x_n)\cdot u(x',x_n)|\,dx_n\,dx' \\
&\leqslant 2\|u\|_{H^1(\mathbb{R}^n_+)}\|u\|_{L^2(\mathbb{R}^n_+)}.
\end{aligned}$$

By density and partition of the unity, one also gets the result for $u \in H^1(\mathbb{R}^n_+)$ and general domains $\Omega \subset \mathbb{R}^n$. □

2.4.2 Remark. Notice that we are in a kind of borderline situation in the previous lemma. It is known that the trace mapping $\gamma_0 \colon H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ with $\gamma_0(u) := u\big|_{\partial\Omega}$ is continuous for every $s > \frac{1}{2}$. In the case of $s = \frac{1}{2}$, the continuity of the trace function is lost, see Section 1.10 of [3] for a counterexample. On the other hand, we have $\|u\|_{H^{1/2}(\Omega)} \leqslant C\|u\|^{1/2}_{H^1(\Omega)}\|u\|^{1/2}_{L^2(\Omega)}$ for every $u \in H^1(\Omega)$ due to interpolation theory. Lemma 2.4.1 shows that although we cannot estimate $\|u\|_{L^2(\partial\Omega)}$ from above with $\|u\|_{H^{1/2}(\Omega)}$, as the trace mapping is discontinuous in this case, it is possible to skip this intermediate (and wrong) step and still get the estimate $\|u\|_{L^2(\partial\Omega)} \leqslant C\|u\|^{1/2}_{H^1(\Omega)}\|u\|^{1/2}_{L^2(\Omega)}$.

2.4 a) Integral equalities for solutions of the wave equation

As we deal with a plate-membrane transmission problem in Chapter 4, it is not too surprising that there is a reasonable interest in integral equalities for solutions of the wave equation. The following two results are a modification of Lemma 4.1 in [67] to a one-dimensional version. While the first lemma is rather general, we highlight the meaning for solutions of the wave equation in Corollary 2.4.4.

2.4.3 Lemma. *Let $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1 with outward pointing normal vector ν. Let $\sigma \in C^1(\overline{\Omega})^n$ be a vector field satisfying $\sigma = \nu$ on $\partial\Omega$. For*

$$v = v(t,x) \in C([0,T],H^2(\Omega)) \cap C^1([0,T],H^1(\Omega)) \cap C^2([0,T],L^2(\Omega))$$

and $s \in (0,T)$ it holds

$$\begin{aligned}\operatorname{Re}\int_\Omega v_{tt}(s)(\sigma\cdot\overline{\nabla v(s)})\,dx = &-\frac{1}{2}\|v_t(s)\|^2_{L^2(\partial\Omega)}+\frac{1}{2}\int_\Omega |v_t(s)|^2\operatorname{div}(\sigma)\,dx\\ &+\operatorname{Re}\frac{d}{ds}\int_\Omega v_t(s)(\sigma\cdot\overline{\nabla v(s)})\,dx\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re}\int_\Omega \Delta v(s)(\sigma\cdot\overline{\nabla v(s)})\,dx = &\frac{1}{2}\|\partial_\nu v(s)\|^2_{L^2(\partial\Omega)}-\frac{1}{2}\sum_{j=1}^{n-1}\|\partial_{\tau_j} v(s)\|^2_{L^2(\partial\Omega)}+\frac{1}{2}\int_\Omega |\nabla v(s)|^2\operatorname{div}(\sigma)\,dx\\ &-\operatorname{Re}\int_\Omega \nabla v(s)\cdot(D\sigma\overline{\nabla v(s)})\,dx.\end{aligned}$$

Here, $D\sigma$ denotes the total derivative of σ and $\tau_j=\tau_j(x)$ are tangential to the boundary for $j=1,\dots,n-1$ in each point $x\in\partial\Omega$ such that $(\nu,\tau_1,\dots,\tau_{n-1})$ form an orthonormal basis.

Proof. For the first equality, we use the product rule and get

$$\frac{d}{ds}\int_\Omega v_t(s)(\sigma\cdot\overline{\nabla v(s)})\,dx=\int_\Omega v_{tt}(s)(\sigma\cdot\overline{\nabla v(s)})\,dx+\int_\Omega v_t(s)(\sigma\cdot\overline{\nabla v_t(s)})\,dx.$$

Integration by parts leads to

$$\begin{aligned}\int_\Omega v_t(s)(\sigma\cdot\overline{\nabla v_t(s)})\,dx &= \int_{\partial\Omega}\overline{v_t(s)}(v_t(s)\sigma\cdot\nu)\,dS-\int_\Omega \overline{v_t(s)}\operatorname{div}(v_t(s)\sigma)\,dx\\ &= \|v_t(s)\|^2_{L^2(\partial\Omega)}-\int_\Omega\overline{v_t(s)}\operatorname{div}(v_t(s)\sigma)\,dx\\ &= \|v_t(s)\|^2_{L^2(\partial\Omega)}-\int_\Omega |v_t(s)|^2\operatorname{div}(\sigma)+\overline{v_t(s)}(\sigma\cdot\nabla v_t(s))\,dx.\end{aligned}$$

Therefore, we have

$$\operatorname{Re}\int_\Omega v_t(s)(\sigma\cdot\overline{\nabla v_t(s)})\,dx=\frac{1}{2}\|v_t(s)\|^2_{L^2(\partial\Omega)}-\frac{1}{2}\int_\Omega |v_t(s)|^2\operatorname{div}(\sigma)\,dx.$$

Consequently,

$$\begin{aligned}\operatorname{Re}\int_\Omega v_{tt}(s)(\sigma\cdot\overline{\nabla v(s)})\,dx = &-\frac{1}{2}\|v_t(s)\|^2_{L^2(\partial\Omega)}+\frac{1}{2}\int_\Omega |v_t(s)|^2\operatorname{div}(\sigma)\,dx\\ &+\operatorname{Re}\frac{d}{ds}\int_\Omega v_t(s)(\sigma\cdot\overline{\nabla v(s)})\,dx.\end{aligned}$$

For the second equality, we use integration by parts again and get

$$\begin{aligned}\int_\Omega \Delta v(s)(\sigma\cdot\overline{\nabla v(s)})\,dx &= \int_\Omega \operatorname{div}(\nabla v(s))(\sigma\cdot\overline{\nabla v(s)})\,dx\\ &= \int_{\partial\Omega}(\sigma\cdot\overline{\nabla v(s)})(\nabla v(s)\cdot\nu)\,dS-\int_\Omega \nabla v(s)\cdot\nabla(\sigma\cdot\overline{\nabla v(s)})\,dx\\ &= \int_{\partial\Omega}|\partial_\nu v(s)|^2\,dS-\int_\Omega\nabla v(s)\cdot\nabla(\sigma\cdot\overline{\nabla v(s)})\,dx\\ &= \int_{\partial\Omega}|\partial_\nu v(s)|^2\,dS-\int_\Omega\nabla v(s)\cdot(D\sigma\overline{\nabla v(s)})+\sigma\cdot(\operatorname{Hess}(\overline{v(s)})\nabla v(s))\,dx,\end{aligned}$$

where $\mathrm{Hess}(\overline{v(s)})$ denotes the Hessian matrix of $\overline{v(s)}$. It holds

$$\begin{aligned}
\mathrm{Re}\int_\Omega \sigma\cdot(\mathrm{Hess}(\overline{v(s)})\nabla v(s))\,dx &= \mathrm{Re}\int_\Omega \sum_{k,l=1}^n \partial_l v(s)\sigma_k\partial_l\partial_k\overline{v(s)}\,dx\\
&= \int_\Omega \frac{1}{2}\sum_{k,l=1}^n \sigma_k\partial_k(\partial_l v(s)\partial_l\overline{v(s)})\,dx\\
&= \frac{1}{2}\sum_{l=1}^n\int_\Omega \sigma\cdot\nabla(\partial_l v(s)\partial_l\overline{v(s)})\,dx\\
&= \frac{1}{2}\sum_{l=1}^n\left(\int_{\partial\Omega}(\partial_l v(s)\partial_l\overline{v(s)})(\sigma\cdot\nu)\,dS - \int_\Omega(\partial_l v(s)\partial_l\overline{v(s)})\,\mathrm{div}(\sigma)\,dx\right)\\
&= \frac{1}{2}\left(\int_{\partial\Omega}|\nabla v(s)|^2\,dS - \int_\Omega|\nabla v(s)|^2\,\mathrm{div}(\sigma)\,dx\right)\\
&= \frac{1}{2}\left(\int_{\partial\Omega}|\partial_\nu v(s)|^2 + \sum_{j=1}^{n-1}|\partial_{\tau_j}v(s)|^2\,dS - \int_\Omega|\nabla v(s)|^2\,\mathrm{div}(\sigma)\,dx\right).
\end{aligned}$$

In consequence, we have

$$\begin{aligned}
\mathrm{Re}\int_\Omega \Delta v(s)(\sigma\cdot\overline{\nabla v(s)})\,dx = \frac{1}{2}\int_{\partial\Omega}|\partial_\nu v(s)|^2 - \sum_{j=1}^{n-1}|\partial_{\tau_j}v(s)|^2\,dS + \frac{1}{2}\int_\Omega|\nabla v(s)|^2\,\mathrm{div}(\sigma)\,dx\\
- \mathrm{Re}\int_\Omega \nabla v(s)\cdot(D\sigma\overline{\nabla v(s)})\,dx.
\end{aligned}$$

□

2.4.4 Corollary. *Let Ω and σ be as in Lemma 2.4.3. Then every solution*

$$v = v(t,x) \in C([0,T],H^2(\Omega))\cap C^1([0,T],H^1(\Omega))\cap C^2([0,T],L^2(\Omega))$$

of the wave equation

$$v_{tt} - \Delta v = 0 \quad \text{in } (0,T)\times\Omega$$

fulfils

$$\begin{aligned}
\|\partial_\nu v(s)\|^2_{L^2(\partial\Omega)} + \|v_t(s)\|^2_{L^2(\partial\Omega)} = \sum_{j=1}^{n-1}\|\partial_{\tau_j}v(s)\|^2_{L^2(\partial\Omega)} + \int_\Omega \mathrm{div}(\sigma)(|v_t(s)|^2 - |\nabla v(s)|^2)\,dx\\
+ 2\,\mathrm{Re}\int_\Omega\nabla v(s)\cdot(D\sigma\overline{\nabla v(s)})\,dx + 2\,\mathrm{Re}\,\frac{d}{ds}\int_\Omega v_t(s)(\sigma\cdot\overline{\nabla v(s)})\,dx
\end{aligned}$$

for all $s\in(0,T)$.

Proof. Multiplying the differential equation with $\sigma\cdot\overline{\nabla v(s)}$ and integrating, we obtain

$$\int_\Omega v_{tt}(s)(\sigma\cdot\overline{\nabla v(s)})\,dx - \int_\Omega\Delta v(s)(\sigma\cdot\overline{\nabla v(s)})\,dx = 0.$$

Lemma 2.4.3 completes the proof. □

2.4 b) Rellich's identity

In the last part of this subsection, we state a special case of Rellich's identity. A more general version can be found in Corollary 2.1 of [59]. Later on, it will be useful to prove the polynomial stability of the plate-membrane transmission problem analysed in Chapter 4 in the case of an undamped membrane.

2.4.5 Lemma. [59, Corollary 2.1]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1 with outward pointing normal vector ν, $u \in H^2(\Omega)$ and $x_0 \in \mathbb{R}^n$. We define $q\colon \mathbb{R}^n \to \mathbb{R}^n$ via $q(x) = x - x_0$ for $x \in \mathbb{R}^n$. Then it holds*

$$2\int_\Omega \Delta u(q\cdot\overline{\nabla u})\,dx = \int_{\partial\Omega} 2\partial_\nu u(q\cdot\overline{\nabla u}) - |\nabla u|^2(q\cdot\nu)\,dS + (n-2)\|\nabla u\|^2_{L^2(\Omega)}.$$

In particular, we obtain

$$\int_\Omega \Delta u(q\cdot\overline{\nabla u})\,dx = \int_{\partial\Omega} \partial_\nu u(q\cdot\overline{\nabla u}) - \frac{1}{2}|\nabla u|^2(q\cdot\nu)\,dS$$

for $n = 2$.

In the following lemma, we state another representation of the $L^2(\Omega)$-norm of a function. The two lemmas of this subsection are especially useful for estimates if the sign of $q\cdot\nu$ is known.

2.4.6 Lemma. *Let again $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1 and ν, u, x_0, q as in Lemma 2.4.5. Then it holds*

$$\|u\|^2_{L^2(\Omega)} = \frac{1}{n}\int_{\partial\Omega}(q\cdot\nu)|u|^2\,dS - \frac{2}{n}\operatorname{Re}\int_\Omega u(q\cdot\overline{\nabla u})\,dx.$$

Proof. Using integration by parts and the identity $q\cdot\overline{\nabla u} = \operatorname{div}(q\overline{u}) - n\overline{u}$, we get

$$\begin{aligned}
\int_\Omega u(q\cdot\overline{\nabla u})\,dx &= \int_\Omega u\left(\operatorname{div}(q\overline{u}) - n\overline{u}\right)\,dx \\
&= \int_\Omega u\operatorname{div}(q\overline{u})\,dx - n\,\|u\|^2_{L^2(\Omega)} \\
&= -\int_\Omega \nabla u\cdot q\overline{u}\,dx + \int_{\partial\Omega} u(q\overline{u}\cdot\nu)\,dS - n\,\|u\|^2_{L^2(\Omega)} \\
&= -\int_\Omega \nabla u\cdot q\overline{u}\,dx + \int_{\partial\Omega}(q\cdot\nu)|u|^2\,dS - n\,\|u\|^2_{L^2(\Omega)}
\end{aligned}$$

and therefore

$$\int_\Omega u(q\cdot\overline{\nabla u})\,dx + \overline{\int_\Omega u(q\cdot\overline{\nabla u})\,dx} = \int_{\partial\Omega}(q\cdot\nu)|u|^2\,dS - n\,\|u\|^2_{L^2(\Omega)}\,.$$

Thus,

$$\frac{2}{n}\operatorname{Re}\int_\Omega u(q\cdot\overline{\nabla u})\,dx = \frac{1}{n}\int_{\partial\Omega}(q\cdot\nu)|u|^2\,dS - \|u\|^2_{L^2(\Omega)}\,.$$

□

2.5 Parameter-elliptic systems of boundary value problems

In this section, we investigate so-called Douglis–Nirenberg systems of boundary value problems in domains and in the half-space. In the following, let $\Omega \subset \mathbb{R}^n$ be a domain with compact and sufficiently smooth boundary Γ, or let $\Omega = \mathbb{R}^n_+$ with boundary $\Gamma = \mathbb{R}^{n-1}$. Since we deal with systems of partial differential equations, we introduce a notation for products of Sobolev spaces as well as for products of Besov spaces.

2.5.1 Definition. For $N \in \mathbb{N}$, $p \in (1,\infty)$, $\lambda \in \mathbb{C}$, $r \in \mathbb{R}$ and $\mathbf{s} = (s_1,\dots,s_N) \in \mathbb{R}^N$ we write

$$H^{\mathbf{s}}_{p,\lambda}(\mathbb{R}^n) := \prod_{j=1}^{N} H^{s_j}_{p,\lambda}(\mathbb{R}^n)$$

endowed with the canonical norm $\|w\|_{H^{\mathbf{s}}_{p,\lambda}(\mathbb{R}^n)} := \Big(\sum_{j=1}^N \|w_j\|^p_{H^{s_j}_{p,\lambda}(\mathbb{R}^n)}\Big)^{1/p}$. In addition, we use the abbreviation $H^{\mathbf{s}+r}_{p,\lambda}(\mathbb{R}^n) := H^{\mathbf{s}+(r,\dots,r)}_{p,\lambda}(\mathbb{R}^n)$. In the case of Besov spaces, we define

$$B^{\mathbf{m}}_{pp,\lambda}(\mathbb{R}^{n-1}) := \prod_{j=1}^{M} B^{m_j}_{pp,\lambda}(\mathbb{R}^{n-1})$$

and $B^{\mathbf{m}+\sigma}_{pp,\lambda}(\mathbb{R}^{n-1}) := B^{\mathbf{m}+(\sigma,\dots,\sigma)}_{pp,\lambda}(\mathbb{R}^{n-1})$ for $M \in \mathbb{N}$, $\sigma \in \mathbb{R}$ and $\mathbf{m} = (m_1,\dots,m_M) \in \mathbb{R}^M$. For the parameter-independent spaces $H^{\mathbf{s}}_p(\mathbb{R}^n) := H^{\mathbf{s}}_{p,0}(\mathbb{R}^n)$ and $B^{\mathbf{m}}_{pp}(\mathbb{R}^{n-1}) := B^{\mathbf{m}}_{pp,0}(\mathbb{R}^{n-1})$ the definitions are modified in an obvious way.

First, we introduce the vectors needed for the orders of the appearing differential operators and boundary operators. We fix $N \in \mathbb{N}$ and $\mathbf{s},\mathbf{t} \in \mathbb{Z}^N$ with

$$\max_{i=1,\dots,N} s_i = 0 \leqslant \min_{j=1,\dots,N} t_j. \tag{2-4}$$

In addition, we assume $s_i + t_i =: \mu \in \mathbb{N}$ to be constant for $i \in \{1,\dots,N\}$, μN to be even and set $M := \frac{\mu N}{2} \in \mathbb{N}$. Let $\mathbf{m} \in \mathbb{Z}^M$ with

$$\max_{\ell=1,\dots,M} m_\ell < 0. \tag{2-5}$$

Notice that μ appears in the definition of $\langle \xi,\lambda\rangle$ and $\langle \xi',\lambda\rangle$ in (2-1).

In the following, let $\Lambda \subset \mathbb{C}$ be a closed sector in the complex plane with vertex at the origin.

We consider the system of boundary value problems

$$\begin{aligned} (\lambda - A)w &= f \quad \text{in } \Omega,\\ Bw &= g \quad \text{on } \Gamma, \end{aligned} \tag{2-6}$$

where $\lambda \in \Lambda$, $f = (f_1,\dots,f_N)$ and $g = (g_1,\dots,g_M)$. $A = (A_{ij})_{1\leqslant i,j\leqslant N}$ denotes an $N\times N$-matrix whose entries are linear differential operators of order $\operatorname{ord} A_{ij} \leqslant s_i + t_j$ if $s_i + t_j \geqslant 0$ and $A_{ij} = 0$, whenever $s_i + t_j < 0$ and $B = (B_{\ell j})_{1\leqslant \ell\leqslant M, 1\leqslant j\leqslant N}$ is an $M \times N$-matrix whose entries are linear

boundary operators of order $\operatorname{ord} B_{\ell j} \leqslant m_\ell + t_j$ if $m_\ell + t_j \geqslant 0$ and $B_{\ell j} = 0$, whenever $m_\ell + t_j < 0$. The entries are of the form

$$A_{ij} = A_{ij}(x, D) = \sum_{|\alpha| \leqslant s_i + t_j} a_{ij}^\alpha(x) D^\alpha,$$
$$B_{\ell j} = B_{\ell j}(x, D) = \sum_{|\beta| \leqslant m_\ell + t_j} b_{\ell j}^\beta(x) \gamma_0 D^\beta$$

for $1 \leqslant i, j \leqslant N$ and $1 \leqslant \ell \leqslant M$ with given functions $a_{ij}^\alpha \colon \overline{\Omega} \to \mathbb{C}$ and $b_{\ell j}^\beta \colon \Gamma \to \mathbb{C}$. For the sake of readability, we skip the concrete smoothness assumptions at this point and refer to Chapter 3 for details.

By $A_0(x, \xi) = (A_{ij}^0(x, \xi))_{1 \leqslant i, j \leqslant N}$ we denote the principal symbol of A, where

$$A_{ij}^0(x, \xi) := \sum_{|\alpha| = s_i + t_j} a_{ij}^\alpha(x) \xi^\alpha.$$

Similarly, we define the principal symbol $B_0(x, \xi) = (B_{\ell j}^0(x, \xi))_{1 \leqslant \ell \leqslant M, 1 \leqslant j \leqslant N}$ via

$$B_{\ell j}^0(x, \xi) := \sum_{|\beta| = m_\ell + t_j} b_{\ell j}^\beta(x) \xi^\beta.$$

2.5.2 Definition. The family $\lambda - A(x, D)$ is called *parameter-elliptic* in Λ if the principal symbol $A_0(x, \xi)$ satisfies

$$\det(\lambda - A_0(x, \xi)) \neq 0 \quad (x \in \overline{\Omega},\ \lambda \in \Lambda,\ \xi \in \mathbb{R}^n,\ (\xi, \lambda) \neq 0). \tag{2-7}$$

If (2-7) holds for $\lambda = 0$, we say that $A(x, D)$ is *elliptic*.

The boundary value problem $(\lambda - A(x, D), B(x, D))$ is called *parameter-elliptic* in Λ if $\lambda - A(x, D)$ is parameter-elliptic in Λ and if the following Shapiro–Lopatinskii condition holds:

Let $x_0 \in \partial\Omega$ be an arbitrary point of the boundary, and rewrite the boundary value problem $(\lambda - A_0(x_0, D), B_0(x_0, D))$ in the coordinate system associated with x_0, which is obtained from the original one by a rotation after which the positive x_n-axis has the direction of the interior normal to $\partial\Omega$ at x_0. Then the trivial solution $\widetilde{w} = 0$ is the only stable solution of the ordinary differential equation on the half-line

$$(\lambda - A_0(x_0, \xi', D_n))\widetilde{w}(x_n) = 0 \quad (x_n > 0),$$
$$B_0(x_0, \xi', D_n)\widetilde{w}(0) = 0$$

for all $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \Lambda$ with $(\xi', \lambda) \neq 0$. If the condition above is fulfilled for $\lambda = 0$, we call the boundary value problem *elliptic*.

2.5.3 Remark. If $A(x, D)$ satisfies the slightly stronger condition that $\xi_n \mapsto \det A_0(x, \xi', \xi_n)$ possesses $\frac{\mu N}{2}$ roots with positive and $\frac{\mu N}{2}$ roots with negative imaginary part for all $x \in \overline{\Omega}$ and $0 \neq \xi' \in \mathbb{R}^n$, we call $A(x, D)$ *regular elliptic*, see Section 7.1 of [3]. For $n > 2$ ellipticity already implies that μN is even and every elliptic operator is regular elliptic, see Remark 6.1.1 of [3]. If $\lambda - A(x, D)$ is parameter-elliptic in Λ, the natural number μN is already even for $n \geqslant 2$ due to Proposition 2.2 of [4].

2.5.4 Example. For $m \in \mathbb{N}$ we consider the boundary value problem $(\Delta^m, \gamma_{k_0}, \dots, \gamma_{k_{m-1}})$ in the half-space, where $k_0, \dots, k_{m-1} \in \{0, \dots, 2m-1\}$ with $k_i < k_j$ for $i < j$. In particular, in this case we have $N = 1$. To show ellipticity of the boundary value problem, we first notice that the symbol of Δ^m is given by $(-|\xi|^2)^m$, which does not vanish for $0 \neq \xi \in \mathbb{R}^n$. Thus Δ^m fulfils condition (2-7). The ordinary differential equation of interest for the Shapiro–Lopatinskii condition is

$$(\partial_n^2 - |\xi'|^2)^m \widetilde{w}(x_n) = 0 \tag{2-8}$$

with $\widetilde{w}^{(k_\ell)}(0) = 0$ for $0 \leqslant \ell \leqslant m-1$. The stable solutions of (2-8) are functions of the form $\widetilde{w}(x_n) = \sum_{j=0}^{m-1} c_j x_n^j \exp(-|\xi'|x_n)$ where $c_0, \dots, c_{m-1} \in \mathbb{C}$. The initial condition leads to

$$\widetilde{w}^{(k_\ell)}(0) = \sum_{j=0}^{k_\ell} \binom{k_\ell}{j} j!(-|\xi'|)^{k_\ell - j} c_j,$$

where $c_m = \ldots = c_{2m-1} := 0$. Consequently, we have to show that the matrix $L = (l_{\ell j})_{0 \leqslant \ell, j \leqslant m-1}$ with

$$l_{\ell j} = \begin{cases} \binom{k_\ell}{j} j!(-|\xi'|)^{k_\ell - j}, & j \leqslant k_\ell, \\ 0, & j > k_\ell \end{cases}$$

is invertible. By multiplying the ℓ-th row with $(-|\xi'|)^{-k_\ell}$ and the j-th column with $\frac{(-|\xi'|)^j}{j!}$ we obtain the matrix $\widetilde{L} = (\widetilde{l}_{\ell j})_{0 \leqslant \ell, j \leqslant m-1}$ with

$$\widetilde{l}_{\ell j} = \begin{cases} \binom{k_\ell}{j}, & j \leqslant k_\ell, \\ 0, & j > k_\ell \end{cases}$$

for $0 \leqslant \ell, j \leqslant m-1$. Now we consider the upper triangular matrix $U = (u_{ji})_{0 \leqslant j, i \leqslant m-1}$ with

$$u_{ji} = \begin{cases} \binom{i}{j}, & j \leqslant i, \\ 0, & j > i \end{cases}$$

for $0 \leqslant j, i \leqslant m-1$. It holds $\widetilde{L}U = \widetilde{S}$, where $\widetilde{S} = (\widetilde{s}_{\ell i})_{0 \leqslant \ell, i \leqslant m-1}$ with

$$\widetilde{s}_{\ell i} = \sum_{j=0}^{m-1} \widetilde{l}_{\ell j} u_{ji} = \sum_{j=0}^{\min\{i, k_\ell\}} \binom{k_\ell}{j} \binom{i}{j} = \binom{k_\ell + i}{i}$$

for $0 \leqslant \ell, i \leqslant m-1$. The matrix $\widetilde{S}$ is a quadratic submatrix of the $2m \times 2m$-Pascal matrix $S = (s_{\ell i})_{0 \leqslant \ell, i \leqslant 2m-1}$ with $s_{\ell i} = \binom{\ell + i}{i}$ for $0 \leqslant \ell, i \leqslant 2m-1$. As the determinant of every quadratic submatrix of S is positive due to Example 0.1.6 of [30], we have $\det(\widetilde{S}) > 0$. Because of $\det(U) = 1$ and $\widetilde{S} = \widetilde{L}U$ we finally obtain $\det(\widetilde{L}) > 0$ and thus $\det(L) \neq 0$. Therefore, the boundary value problem $(\Delta^m, \gamma_{k_0}, \dots, \gamma_{k_{m-1}})$ is elliptic.

2.5.5 Example. We again restrict ourselves to the scalar case $N = 1$ in the half-space. There are some boundary conditions $(B_1(x,D),\dots,B_m(x,D))$ such that the boundary value problem $(A(x,D),B_1(x,D),\dots,B_m(x,D))$ is elliptic for every regular elliptic operator A of order $2m$. Boundary conditions with this property are called *completely elliptic*, see Definition 3.1 of [40]. The prime example is given in the generalized Dirichlet boundary conditions $(\gamma_0,\dots,\gamma_{m-1})$, see Remark 7.1.2 of [3].

This example can easily be extended to the boundary conditions $(\gamma_s,\dots,\gamma_{m-1+s})$ for a fixed $s \in \{0,\dots,m-1\}$. Let $\widetilde{w}$ be a stable solution of the corresponding initial value problem

$$A_0(x_0,\xi',D_n)\widetilde{w}(x_n) = 0 \quad (x_n > 0) \tag{2-9}$$

with $\widetilde{w}^{(s+j)}(0) = 0$ for $j \in \{0,\dots,m-1\}$. We apply ∂_n^s to (2-9). Since ∂_n^s and $A_0(x_0,\xi',D_n)$ commute, $\widetilde{w}^{(s)}$ is a solution of the problem with generalized Dirichlet boundary conditions and thus $\widetilde{w}^{(s)} = 0$. As $\widetilde{w}$ is a stable solution, we obtain $\widetilde{w} = 0$. For $s = m-1$ we get the generalized Neumann boundary conditions.

For $m = 2$ we can also show that (γ_0,γ_2) as well as (γ_1,γ_3) are completely elliptic. The second claim follows, as before, from the first one. To see that the conditions (γ_0,γ_2) are completely elliptic, let $A(x,D)$ be a regular elliptic differential operator of order 4. We assume that the roots of the function $\xi_n \mapsto A_0(x_0,\xi',-i\xi_n)$ with negative real part are $\tau_j := \tau_j(x_0,\xi')$ for $j = 1,2$ and fixed $x_0,\xi' \in \mathbb{R}^{n-1}$ with $\xi' \neq 0$. We distinguish between the cases $\tau_1 \neq \tau_2$ and $\tau_1 = \tau_2$.

For $\tau_1 \neq \tau_2$ the stable solutions of the ordinary differential equation of interest for the Shapiro–Lopatinskii condition are of the form $\widetilde{w}(x_n) = c_1\exp(\tau_1 x_n) + c_2\exp(\tau_2 x_n)$ for $x_n > 0$ and $c_1,c_2 \in \mathbb{C}$. As the matrix

$$\begin{pmatrix} 1 & 1 \\ \tau_1^2 & \tau_2^2 \end{pmatrix}$$

is invertible since $\tau_1 \neq \tau_2$ both have negative real part, the trivial solution is the only stable solution with $\widetilde{w}(0) = \widetilde{w}''(0) = 0$.

For $\tau := \tau_1 = \tau_2$ the stable solutions are of the form $\widetilde{w}(x_n) = c_1\exp(\tau x_n) + c_2 x_n\exp(\tau x_n)$ for $x_n > 0$ and $c_1,c_2 \in \mathbb{C}$. Again, the only stable solution with

$$\begin{aligned} 0 &= \widetilde{w}(0) = c_1, \\ 0 &= \widetilde{w}''(0) = \tau^2 c_1 + 2\tau c_2 \end{aligned}$$

is the trivial one. Thus (γ_0,γ_2) are completely elliptic boundary conditions.

In view of the last observations one could suppose that the boundary conditions $(\gamma_{k_0},\dots,\gamma_{k_{m-1}})$ considered in Example 2.5.4 might also be completely elliptic in general. Actually, this is not the case. We analyse for instance

$$A(D) := \left(D_n^2 + \zeta^2\sum_{j=1}^{n-1} D_j^2\right)\left(D_n^2 + \zeta^4\sum_{j=1}^{n-1} D_j^2\right) = (\partial_n^2 + \zeta^2\Delta')(\partial_n^2 + \zeta^4\Delta'),$$

where $\zeta := \exp\left(\frac{2\pi i}{3}\right)$ denotes a third root of unity. Because of

$$A(\xi) = (\xi_n + \zeta|\xi'|i)(\xi_n - \zeta|\xi'|i)(\xi_n + \zeta^2|\xi'|i)(\xi_n - \zeta^2|\xi'|i)$$

for $\xi \in \mathbb{R}^n$ the operator A is regular elliptic. As for a fixed $\xi' \in \mathbb{R}^{n-1}$ with $|\xi'| = 1$ the function $\widetilde{w}(x_n) = \exp(\zeta x_n) - \exp(\zeta^2 x_n)$ is a non-trivial stable solution of

$$A(\xi', D_n)\widetilde{w}(x_n) = (\zeta^2 - \partial_n^2)(\zeta^4 - \partial_n^2)\widetilde{w}(x_n) = 0 \quad (x_n > 0)$$

fulfilling $\widetilde{w}(0) = 0$ and $\widetilde{w}^{(3)}(0) = \zeta^3 - \zeta^6 = 0$, the boundary value problem (A, γ_0, γ_3) is not elliptic.

We mention here the classical a priori estimate of parameter-elliptic boundary value problems. A proof in the case $r = 0$ can also be found in Theorem 2.6 of [29] and in Theorem 3.3 of [28].

2.5.6 Proposition. [68, Theorem 1.5.1]. *Let $p \in (1, \infty)$, $r > -1 + \frac{1}{p}$ and $(\lambda - A(x, D), B(x, D))$ be parameter-elliptic in Λ with orders as described in* (2-4) *and* (2-5)*. Then there is a $\lambda_0 > 0$ such that the boundary value problem* (2-6) *possesses a unique solution $w \in H_p^{\mathbf{t}+r}(\Omega)$ for every $f \in H_p^{-\mathbf{s}+r}(\Omega)$, $g \in B_{pp}^{-\mathbf{m}+r-1/p}(\Gamma)$ and $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$. The solution w satisfies*

$$\|w\|_{H_{p,\lambda}^{\mathbf{t}+r}(\Omega)} \leqslant C\Big(\|f\|_{H_{p,\lambda}^{-\mathbf{s}+r}(\Omega)} + \|g\|_{B_{pp,\lambda}^{-\mathbf{m}+r-1/p}(\Gamma)}\Big)$$

for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$ and a constant $C > 0$ not depending upon f, g and λ. In the case of constant coefficients and no lower order terms, we can choose $\lambda_0 > 0$ arbitrary small.

The situation becomes easier in the whole space case, where an analogue statement follows from Mikhlin's Theorem, see Theorem 3.5 of [28].

2.5.7 Lemma. *Let $p \in (1, \infty)$, $r \in \mathbb{R}$ and $\lambda - A(x, D)$ be parameter-elliptic in Λ with orders as described in* (2-4)*. Then there is a $\lambda_0 > 0$ such that the problem*

$$(\lambda - A(x, D))w = f \quad \text{in } \mathbb{R}^n$$

possesses a unique solution $w \in H_p^{\mathbf{t}+r}(\mathbb{R}^n)$ for every $f \in H_p^{-\mathbf{s}+r}(\mathbb{R}^n)$ and $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$ satisfying

$$\|w\|_{H_{p,\lambda}^{\mathbf{t}+r}(\mathbb{R}^n)} \leqslant C\|f\|_{H_{p,\lambda}^{-\mathbf{s}+r}(\mathbb{R}^n)}$$

with a constant $C > 0$ not depending upon f and λ. In the case of constant coefficients and no lower order terms, we can choose $\lambda_0 > 0$ arbitrary small.

By using Lemma 2.1.7, we can write the dependency on λ in the previous lemma explicitly in the case $r \geqslant 0$.

2.5.8 Corollary. *In the situation of Lemma 2.5.7 with $r \geqslant 0$, we can rewrite the a priori estimate in the form*

$$\|w\|_{H_p^{\mathbf{t}+r}(\mathbb{R}^n)} + \sum_{j=1}^{N} |\lambda|^{(t_j+r)/\mu}\|w_j\|_{L^p(\mathbb{R}^n)} \leqslant C\left(\|f\|_{H_p^{-\mathbf{s}+r}(\mathbb{R}^n)} + \sum_{i=1}^{N} |\lambda|^{(-s_i+r)/\mu}\|f_i\|_{L^p(\mathbb{R}^n)}\right) \quad (2\text{-}10)$$

for $\lambda \in \Lambda$ with $|\lambda| \geqslant \max\{1, \lambda_0\}$. In particular, we get

$$\|w\|_{H_p^{\mathbf{t}+r}(\mathbb{R}^n)} + |\lambda|\|w\|_{H_p^{-\mathbf{s}+r}(\mathbb{R}^n)} \leqslant C\|f\|_{H_{p,\lambda}^{-\mathbf{s}+r}(\mathbb{R}^n)}.$$

Proof. The first statement follows directly from Lemma 2.5.7 and Lemma 2.1.7. For the second assertion notice that

$$|\lambda|\|w_j\|_{H_p^{-s_j+r}(\mathbb{R}^n)} = |\lambda|^{(t_j+r+s_j-r)/\mu}\|w_j\|_{H_p^{-s_j+r}(\mathbb{R}^n)} \leqslant C\big(\|w_j\|_{H_p^{t_j+r}(\mathbb{R}^n)} + |\lambda|^{(t_j+r)/\mu}\|w_j\|_{L^p(\mathbb{R}^n)}\big)$$

for $j = 1, \dots, N$ due to Lemma 2.2.2. □

Under some additional smoothness assumptions, we can improve the a priori estimate from Lemma 2.5.7 in the whole space case.

2.5.9 Lemma. [20, Theorem 4.1]. *Under the additional assumption that A has constant coefficients, we obtain*

$$\|w\|_{H_p^{\mathbf{t}+r}(\mathbb{R}^n)} + |\lambda|\|w\|_{H_p^{-\mathbf{s}+r}(\mathbb{R}^n)} \leqslant C\|f\|_{H_p^{-\mathbf{s}+r}(\mathbb{R}^n)} \tag{2-11}$$

for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$ in the situation of Lemma 2.5.7.

The following proposition improves the λ-dependence in the a priori estimate of Proposition 2.5.6 in the half-space case with $r = 0$ for appropriate orders of boundary conditions.

2.5.10 Proposition. *We assume that A and B both have constant coefficients. Let the boundary value problem $(\lambda - A(D), B(D))$ be parameter-elliptic with orders as described in* (2-4) *and* (2-5). *We also assume*

$$\gamma := \max_{\ell=1,\dots,M}\left\{-m_\ell - \mu - \frac{1}{p}\right\} > 0. \tag{2-12}$$

Thus, there is a $\lambda_0 \geqslant 1$ such that the boundary value problem

$$\begin{aligned} (\lambda - A)w &= f \qquad \text{in } \mathbb{R}^n_+, \\ Bw &= g \qquad \text{on } \mathbb{R}^{n-1} \end{aligned} \tag{2-13}$$

possesses a unique solution $w \in H_p^{\mathbf{t}}(\mathbb{R}^n_+)$ for every $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$, $f \in H_p^{-\mathbf{s}}(\mathbb{R}^n_+)$ and $g \in B_{pp}^{-\mathbf{m}-1/p}(\mathbb{R}^{n-1})$. Furthermore, there exists a constant $C > 0$, independent of f, g and λ, such that

$$\|w\|_{H_p^{\mathbf{t}}(\mathbb{R}^n_+)} + |\lambda|\|w\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \leqslant C\Big(|\lambda|^{\gamma/\mu}\|f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} + \|g\|_{B_{pp,\lambda}^{-\mathbf{m}-1/p}(\mathbb{R}^{n-1})}\Big).$$

Proof. We follow the proof of Theorem 3.6 in [22]. Let f be in $H_p^{-\mathbf{s}}(\mathbb{R}^n_+)$ and $g \in B_{pp}^{-\mathbf{m}-1/p}(\mathbb{R}^{n-1})$. Since $(\lambda - A, B)$ is parameter-elliptic, we already know unique solvability of the boundary value problem (2-13) for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$ and a suitable $\lambda_0 > 0$. We write the solution w in the form $w = w_f + w_g + w_0$. Here, $w_f := r_{\mathbb{R}^n_+}\widetilde{w}_f$ with $\widetilde{w}_f$ being the solution of

$$(\lambda - A)\widetilde{w}_f = e_{\mathbb{R}^n_+}f \quad \text{in } \mathbb{R}^n,$$

where $r_{\mathbb{R}^n_+} : H_p^{\mathbf{t}}(\mathbb{R}^n) \to H_p^{\mathbf{t}}(\mathbb{R}^n_+)$, $w \mapsto w\big|_{\mathbb{R}^n_+}$ denotes the restriction of a function to $\mathbb{R}^n_+$ and $e_{\mathbb{R}^n_+} : H_p^{-\mathbf{s}}(\mathbb{R}^n_+) \to H_p^{-\mathbf{s}}(\mathbb{R}^n)$ is a continuous extension operator, see Remark 3.1.5 for details. The function w_g solves the boundary value problem

$$\begin{aligned} (\lambda - A)w_g &= 0 \quad \text{in } \mathbb{R}^n_+, \\ Bw_g &= g \quad \text{on } \mathbb{R}^{n-1}. \end{aligned}$$

Therefore, w_0 is the solution to

$$\begin{aligned}(\lambda - A)w_0 &= 0 && \text{in } \mathbb{R}^n_+,\\ Bw_0 &= -Bw_f && \text{on } \mathbb{R}^{n-1}.\end{aligned}$$

Due to Proposition 2.5.6, there exists a constant $C > 0$ such that

$$\|w_g\|_{H^{\mathbf{t}}_{p,\lambda}(\mathbb{R}^n_+)} \leqslant C\|g\|_{B^{-\mathbf{m}-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}.$$

Using the interpolation inequality of Lemma 2.2.2, we get

$$|\lambda|\|w_g\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)} \leqslant C\|w_g\|_{H^{\mathbf{t}}_{p,\lambda}(\mathbb{R}^n_+)}.$$

Hence, we find a $C > 0$ such that

$$\|w_g\|_{H^{\mathbf{t}}_p(\mathbb{R}^n_+)} + |\lambda|\|w_g\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)} \leqslant C\|g\|_{B^{-\mathbf{m}-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}.$$

Similarly, we get for w_0 the estimate

$$\|w_0\|_{H^{\mathbf{t}}_p(\mathbb{R}^n_+)} + |\lambda|\|w_0\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)} \leqslant C\|Bw_f\|_{B^{-\mathbf{m}-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}.$$

Now, we use Lemma 2.1.8 and consider every term of the right-hand side separately. We find

$$\|Bw_f\|_{B^{-\mathbf{m}-1/p}_{pp}(\mathbb{R}^{n-1})} \leqslant C\|w_f\|_{H^{\mathbf{t}}_p(\mathbb{R}^n_+)} \overset{(2\text{-}11)}{\leqslant} C\|e_{\mathbb{R}^n_+}f\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n)} \leqslant C\|f\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)}.$$

For the λ-weighted L^p-norms we distinguish between two cases.

Case 1: If $-m_\ell > \mu + \frac{1}{p}$ for $\ell \in \{1, \dots, M\}$, we get

$$\begin{aligned}|\lambda|^{(-m_\ell-1/p)/\mu}\left\|(Bw_f)_\ell\right\|_{L^p(\mathbb{R}^{n-1})} &\leqslant C|\lambda|^{(-m_\ell-1/p)/\mu}\left\|(Bw_f)_\ell\right\|_{B^{-m_\ell-\mu-1/p}_{pp}(\mathbb{R}^{n-1})}\\ &\leqslant C|\lambda|^{(-m_\ell-1/p)/\mu}\|w_f\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)}\\ &\overset{(2\text{-}11)}{\leqslant} C|\lambda|^{(-m_\ell-\mu-1/p)/\mu}\|f\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)}\\ &\leqslant C|\lambda|^{\gamma/\mu}\|f\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)}.\end{aligned}$$

Case 2: If $-m_\ell < \mu + \frac{1}{p}$ for $\ell \in \{1, \dots, M\}$, we can estimate as follows:

$$\begin{aligned}|\lambda|^{(-m_\ell-1/p)/\mu}\left\|(Bw_f)_\ell\right\|_{L^p(\mathbb{R}^{n-1})} &\leqslant C|\lambda|^{(-m_\ell-1/p)/\mu}\left\|(Bw_f)_\ell\right\|_{B^{1-1/p}_{pp}(\mathbb{R}^{n-1})}\\ &\leqslant C|\lambda|^{(-m_\ell-1/p)/\mu}\|w_f\|_{H^{\mathbf{t}+1+m_\ell}_p(\mathbb{R}^n_+)}\\ &= C|\lambda|^{\gamma/\mu}|\lambda|^{(-m_\ell-\gamma-1/p)/\mu}\|w_f\|_{H^{\mathbf{t}+1+m_\ell}_p(\mathbb{R}^n_+)}\\ &\leqslant C|\lambda|^{\gamma/\mu}|\lambda|^{(-m_\ell-1)/\mu}\|w_f\|_{H^{\mathbf{t}+1+m_\ell}_p(\mathbb{R}^n_+)}\\ &\leqslant C|\lambda|^{\gamma/\mu}\Big(\|w_f\|_{H^{\mathbf{t}}_p(\mathbb{R}^n_+)} + |\lambda|\|w_f\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)}\Big)\\ &\overset{(2\text{-}11)}{\leqslant} C|\lambda|^{\gamma/\mu}\|f\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)}.\end{aligned}$$

In the second to last step we used Lemma 2.2.2 again and the inequality

$$-s_j < 1 - s_j - \frac{1}{p} = 1 - \mu - \frac{1}{p} + t_j < 1 + m_\ell + t_j \leqslant t_j$$

for $j = 1, \dots, N$. Altogether, we obtain the desired result. □

2.5.11 Remark. Adapting the proof of Corollary 2.5.8 in the case $r = 0$ to boundary value problems in the half-space, we get the estimate

$$\|w\|_{H^{\mathbf{t}}_p(\mathbb{R}^n_+)} + |\lambda|\|w\|_{H^{-\mathbf{s}}_p(\mathbb{R}^n_+)} \leqslant C\Big(\|f\|_{H^{-\mathbf{s}}_{p,\lambda}(\mathbb{R}^n_+)} + \|g\|_{B^{-\mathbf{m}-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}\Big).$$

In Proposition 2.5.10 the power of λ on the right-hand side concerning f is given by

$$|\lambda|^{\gamma/\mu} = |\lambda|^{(-(t_j+m_\ell)-s_j-1/p)/\mu} \leqslant |\lambda|^{-s_j/\mu}$$

for suitable $\ell \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$. Consequently, the estimate in Proposition 2.5.10 is at least better in some components of f, particularly if the orders of the boundary conditions are quite high.

2.5.12 Example. As an example we investigate the stationary plate equation with structural damping, that is

$$A(D) := \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta \end{pmatrix}$$

for $\rho > 0$. We give a short summary of the results in Section 2 of [21]. The operator $A(D)$ is of Douglis–Nirenberg structure with $\mathbf{t} = (4, 2)$, $\mathbf{s} = (-2, 0)$ and $\mu = 2$. In addition, we have

$$\lambda - A(\xi) = \begin{pmatrix} \lambda & -1 \\ |\xi|^4 & \lambda + \rho|\xi|^2 \end{pmatrix}$$

such that

$$\det(\lambda - A(\xi)) = \lambda^2 + \rho|\xi|^2\lambda + |\xi|^4 = (\alpha_+\lambda + |\xi|^2)(\alpha_-\lambda + |\xi|^2),$$

where

$$\alpha_\pm := \begin{cases} \frac{\rho}{2} \pm i\sqrt{1 - \frac{\rho^2}{4}}, & 0 < \rho < 2, \\ \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} - 1}, & \rho \geqslant 2. \end{cases}$$

For

$$\vartheta := \begin{cases} \arctan\left(\frac{2}{\rho}\sqrt{1 - \frac{\rho^2}{4}}\right), & 0 < \rho < 2, \\ 0, & \rho \geqslant 2, \end{cases}$$

$0 < \varphi < \pi - \vartheta$, $\Sigma_\varphi := \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \varphi\}$ and $(\xi, \lambda) \in (\mathbb{R}^n \times \overline{\Sigma_\varphi})\backslash\{0\}$ it holds $\det(\lambda - A(\xi)) \neq 0$. Thus, $\lambda - A(D)$ is parameter-elliptic in $\overline{\Sigma_\varphi}$, see Section 2 of [21].

As we are interested in the parameter-ellipticity of $(\lambda - A(D), B(D))$ for various boundary conditions $B(D)$, we take a closer look at the stable solutions of

$$\begin{pmatrix} \lambda & -1 \\ (|\xi'|^2 - \partial_n^2)^2 & \lambda + \rho(|\xi'|^2 - \partial_n^2) \end{pmatrix} \begin{pmatrix} w_1(x_n) \\ w_2(x_n) \end{pmatrix} = 0 \quad (x_n > 0). \tag{2-14}$$

From the first row we know that $w_2(x_n) = \lambda w_1(x_n)$, which reduces the second row to

$$\lambda^2 w_1(x_n) + \lambda\rho(|\xi'|^2 - \partial_n^2)w_1(x_n) + (|\xi'|^2 - \partial_n^2)^2 w_1(x_n) = 0 \quad (x_n > 0).$$

The stable solutions of (2-14) are of the form

$$c_1 \begin{pmatrix} w_1^1(x_n) \\ w_2^1(x_n) \end{pmatrix} + c_2 \begin{pmatrix} w_1^2(x_n) \\ w_2^2(x_n) \end{pmatrix},$$

where $w_1^j(x_n) = \exp(\tau_j x_n)$ and $w_2^j(x_n) = \lambda w_1^j(x_n)$ for $\rho \neq 2$ with

$$\tau_1 = -\sqrt{|\xi'|^2 + \alpha_+ \lambda} \quad \text{and} \quad \tau_2 = -\sqrt{|\xi'|^2 + \alpha_- \lambda}$$

for $j = 1, 2$. The root $\sqrt{\cdot}\colon \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ denotes the principal square root, that is the one with positive real part. In the case $\rho = 2$ the values $\tau := \tau_1 = \tau_2$ coincide and the stable solutions are given by $w_1^1(x_n) = \exp(\tau x_n)$, $w_1^2(x_n) = x_n \exp(\tau x_n)$ and again $w_2^j(x_n) = \lambda w_1^j(x_n)$ for $j = 1, 2$, see Lemma 3.1 of [21].

For the Dirichlet boundary conditions $B_1(D) = \begin{pmatrix} 1 & 0 \\ \partial_n & 0 \end{pmatrix}$ the resulting initial conditions are $w_1(0) = w_1'(0) = 0$. We easily check that the only stable solution with these initial conditions is the trivial one. The same holds for the Neumann boundary conditions $B_2(D) = \begin{pmatrix} \partial_n & 0 \\ \partial_n^2 & 0 \end{pmatrix}$. Lastly, we consider the boundary conditions $B_3(D) = \begin{pmatrix} 1 & 0 \\ \Delta & 0 \end{pmatrix}$, that is, the initial conditions $w_1(0) = w_1''(0) - |\xi'|^2 w_1(0) = 0$. They directly simplify to $w_1(0) = w_1''(0) = 0$, where again the trivial solution is the only stable solution.

Notice that we sometimes omit the traces in the definition of the boundary conditions as it is clear from the context how to read them.

We see that, even though we deal with a system of equations, the problem reduces to a scalar initial value problem, where the initial conditions are similar to those we already took into consideration in Example 2.5.5.

Now, we apply Proposition 2.5.10 in the Hilbert space case $p = 2$ to the operator $A(D)$ of the stationary structurally damped plate equation, defined in Example 2.5.12, to point out the differences in the power of λ for several boundary conditions.

2.5.13 Example. We consider the boundary conditions $B_1(D) = \begin{pmatrix} 1 & 0 \\ \partial_n & 0 \end{pmatrix}$. As we have seen in Example 2.5.12, the problem $(\lambda - A(D), B_1(D))$ is parameter-elliptic. Particularly due to Proposition 2.5.10, we get the estimate

$$|\lambda|^{1/4} \|w\|_{H^{-\mathbf{s}}(\mathbb{R}^n_+)} \leqslant C \|f\|_{H^{-\mathbf{s}}(\mathbb{R}^n_+)}$$

for $p = 2$, $g = 0$, $\lambda \in \overline{\Sigma_\varphi}$ with $|\lambda| \geqslant \lambda_0$ and $0 < \varphi < \pi - \vartheta$ since the orders corresponding to $B_1(D)$ fulfil $\mathbf{m}_1 = (-4, -3)$. The notation for w and u is adopted from Proposition 2.5.10. We obtain the same result for $B_3(D) = \begin{pmatrix} 1 & 0 \\ \Delta & 0 \end{pmatrix}$, where $\mathbf{m}_3 = (-4, -2)$.

Together with the Neumann boundary condition $B_2(D) = \begin{pmatrix} \partial_n & 0 \\ \partial_n^2 & 0 \end{pmatrix}$, the system $(\lambda - A(D), B_2(D))$ also becomes parameter-elliptic, see Example 2.5.12. We have $\mathbf{m}_2 = (-3, -2)$ and therefore obtain

$$|\lambda|^{3/4} \|w\|_{H^{-\mathbf{s}}(\mathbb{R}^n_+)} \leqslant C \|f\|_{H^{-\mathbf{s}}(\mathbb{R}^n_+)}$$

for p, g and λ as before.

Now, we use the Yosida-approximation in a slightly different form to gain a better estimate than in Proposition 2.5.10 if f satisfies several additional assumptions, see Lemma II.3.4 of [26].

2.5.14 Proposition. *In the situation of Proposition 2.5.10, we define*

$$\mathcal{A} \colon H_p^{-\mathbf{s}}(\mathbb{R}^n_+) \supset D(\mathcal{A}) \to H_p^{-\mathbf{s}}(\mathbb{R}^n_+)$$

by

$$D(\mathcal{A}) := \big\{ w \in H_p^{\mathbf{t}}(\mathbb{R}^n_+) : B(D)w = 0 \big\} \quad \text{and} \quad \mathcal{A}w := Aw \quad (w \in D(\mathcal{A})).$$

Then $\lambda - \mathcal{A}$ is invertible for $\lambda \in \Lambda$ with sufficiently large absolute value. For $f \in D(\mathcal{A})$, $g = 0$ and $\gamma < \mu$ we obtain $\lambda w \to f$ in $H_p^{-\mathbf{s}}(\mathbb{R}^n_+)$ for $\lambda \to \infty$. Particularly, there is a $\lambda_0 > 0$ such that

$$|\lambda| \|w\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \leqslant C \|f\|_{H_p^{\mathbf{t}}(\mathbb{R}^n_+)} \tag{2-15}$$

for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$.

Proof. First, we observe that $\lambda - \mathcal{A}$ is invertible due to Proposition 2.5.6 for $\lambda \in \Lambda$ with sufficiently large absolute value. Let f be in $D(\mathcal{A})$ and $\gamma < \mu$. According to Proposition 2.5.10 there is a $\lambda_0 > 0$ such that

$$\begin{aligned} \|\lambda w - f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} &= \|\lambda(\lambda - \mathcal{A})^{-1} f - f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \\ &= \|(\lambda - \mathcal{A})^{-1}(\lambda - \mathcal{A} + \mathcal{A}) f - f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \\ &= \|(\lambda - \mathcal{A})^{-1} \mathcal{A} f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \\ &\leqslant C |\lambda|^{\gamma/\mu - 1} \|\mathcal{A} f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \end{aligned}$$

for all $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$. The right-hand side converges to zero for $\lambda \to \infty$. Inequality (2-15) follows from

$$\|\lambda w\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \leqslant \|f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} + C \|\mathcal{A} f\|_{H_p^{-\mathbf{s}}(\mathbb{R}^n_+)} \leqslant C \|f\|_{H_p^{\mathbf{t}}(\mathbb{R}^n_+)}.$$

□

Chapter 3

Systems of boundary value problems in spaces of mixed smoothness

This chapter is based on [19], which treats scalar parameter-elliptic boundary value problems with rough boundary data. Here, we generalize the results to systems of parameter-elliptic boundary value problems. While [19] focusses on the application to semigroup theory with ground space $L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$, our main interests are a priori estimates in rougher Sobolev spaces. Therefore, the studies of the linearised Cahn–Hilliard equation in $L^p(\mathbb{R}^n_+) \times L^p(\mathbb{R}^{n-1})$ are omitted here but can be found in Chapter 5 of [19]. Throughout this chapter, we assume $p \in (1, \infty)$.

3.1 Sobolev spaces of mixed smoothness and traces

3.1 a) Some function spaces

In this first section of Chapter 3 we want to introduce spaces of mixed smoothness, which have different regularity in normal and tangential direction. The advantage of those spaces is that we are able to define traces for functions, where the regularity in tangential direction is rather bad and might even get negative. For $p = 2$, the following definition can also be found in Appendix A.3 of [35].

3.1.1 Definition. For $s, \sigma \in \mathbb{R}$ and $p \in (1, \infty)$ we define the Bessel potential space of mixed smoothness $H_p^{s,\sigma}(\mathbb{R}^n)$ as

$$H_p^{s,\sigma}(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D' \rangle^\sigma u \in H_p^s(\mathbb{R}^n)\} = \{u \in \mathscr{S}'(\mathbb{R}^n) : \langle D \rangle^s \langle D' \rangle^\sigma u \in L^p(\mathbb{R}^n)\}$$

with canonical norm $\|u\|_{H_p^{s,\sigma}(\mathbb{R}^n)} := \|\langle D' \rangle^\sigma u\|_{H_p^s(\mathbb{R}^n)} = \|\langle D \rangle^s \langle D' \rangle^\sigma u\|_{L^p(\mathbb{R}^n)}$, where $\langle \cdot \rangle$ is as described in (2-1).

In the previous definition, $\langle D' \rangle^\sigma$ acts only on the x'-variable. Therefore, the spaces above have smoothness $s + \sigma$ in x'-direction and smoothness s in x_n-direction. We will often assume $\sigma < 0$ because in our applications we need higher regularity in normal direction than in tangential direction to define traces.

Clearly, $H_p^{s,\sigma}(\mathbb{R}^n)$ is a Banach space. Since $\langle D' \rangle^\sigma$ and $\langle D \rangle^s$ leave $\mathscr{S}(\mathbb{R}^n)$ invariant, the rapidly decreasing functions are a dense subset of $H_p^{s,\sigma}(\mathbb{R}^n)$. For every $t, \tau \in \mathbb{R}$ the map

$$\langle D \rangle^t \langle D' \rangle^\tau \colon H_p^{s,\sigma}(\mathbb{R}^n) \to H_p^{s-t,\sigma-\tau}(\mathbb{R}^n)$$

is an isometric isomorphism with inverse $\langle D \rangle^{-t} \langle D' \rangle^{-\tau}$.

We remark that the scale $H_p^{s,\sigma}(\mathbb{R}^n)$ for $s,\sigma\in\mathbb{R}$ is different from the scale of anisotropic spaces $H_p^{s,\vec{a}}(\mathbb{R}^n)$ in the sense of Proposition 2.10 in [45], see also Section 5.1.3 of [78]. In particular, for $s>0$ and $\sigma<-s$, we have positive smoothness with respect to x_n but negative smoothness in x', which is not allowed for the anisotropic spaces. $H_p^{s,\sigma}(\mathbb{R}^n)$ is also different from the space of dominating mixed smoothness, see for instance Section 1.1.2 of [76]. Spaces of dominating mixed smoothness are defined similarly as above, but with $\langle\xi\rangle$ being replaced by $\prod_{j=1}^n(1+\xi_j^2)^{1/2}$. We refer to Section 1 of [71] and Subsection 2.2 of [42] for further information on spaces of dominating mixed smoothness and applications to boundary value problems.

We state some elementary properties of the Bessel potential spaces with mixed smoothness, which we shall use frequently later on.

3.1.2 Proposition. *Let $s,\sigma\in\mathbb{R}$.*

(i) $H_p^{s,0}(\mathbb{R}^n)=H_p^s(\mathbb{R}^n)$ *and* $H_p^{0,\sigma}(\mathbb{R}^n)=L^p(\mathbb{R},H_p^\sigma(\mathbb{R}^{n-1}))$.

(ii) $H_p^{t,\tau}(\mathbb{R}^n)\subset H_p^{s,\sigma}(\mathbb{R}^n)$ *whenever* $s\leqslant t$ *and* $\sigma\leqslant\tau$.

(iii) For $\sigma\geqslant 0$ *we have*

$$H_p^{s+\sigma}(\mathbb{R}^n)\subset H_p^{s,\sigma}(\mathbb{R}^n)\subset H_p^s(\mathbb{R}^n)\subset H_p^{s,-\sigma}(\mathbb{R}^n)\subset H_p^{s-\sigma}(\mathbb{R}^n).$$

(iv) Let q be the dual coefficient to p, that is $\frac{1}{p}+\frac{1}{q}=1$. Then the standard bilinear pairing $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\to\mathbb{C}$ induces an identification of the dual space of $H_p^{s,\sigma}(\mathbb{R}^n)$ with $H_q^{-s,-\sigma}(\mathbb{R}^n)$.

(v) In case of $s\geqslant 0$ *we have*

$$H_p^{s,\sigma}(\mathbb{R}^n)=L^p(\mathbb{R},H_p^{s+\sigma}(\mathbb{R}^{n-1}))\cap H_p^s(\mathbb{R},H_p^\sigma(\mathbb{R}^{n-1})).$$

(vi) For $\alpha\in\mathbb{N}_0^n$, the derivative $\partial^\alpha\colon H_p^{s,\sigma}(\mathbb{R}^n)\to H_p^{s-|\alpha_n|,\sigma-|\alpha'|}(\mathbb{R}^n)$ is continuous.

(vii) If $s\in\mathbb{N}_0$ is an integer, $u\in\mathscr{S}'(\mathbb{R}^n)$ belongs to $H_p^{s,\sigma}(\mathbb{R}^n)$ if and only if $\partial_n^j u\in H_p^{0,s+\sigma-j}(\mathbb{R}^n)$ for all $j=0,\dots,s$. Moreover, $\|u\|:=\sum_{j=0}^s\|\partial_n^j u\|_{H_p^{0,s+\sigma-j}(\mathbb{R}^n)}$ defines an equivalent norm on $H_p^{s,\sigma}(\mathbb{R}^n)$.

Proof. The main idea of the proof is to go back to the case $\sigma=0$, where the results are well-known.

(i) This statement is clear from the definition of the spaces.

(ii) This embedding holds true, since $\langle D\rangle^{s-t}\langle D'\rangle^{\sigma-\tau}$ is a bounded operator in $L^p(\mathbb{R}^n)$ due to Mikhlin's Theorem.

(iii) By Mikhlin's Theorem, both $\langle D\rangle^{-\sigma}\langle D'\rangle^{\sigma}$ and $\langle D'\rangle^{-\sigma}$ are bounded operators in $L^p(\mathbb{R}^n)$ for $\sigma\geqslant 0$. This yields the first and the second inclusion. The other two are verified analogously.

(iv) As $\langle D' \rangle^{\sigma} \colon H_p^{s,\sigma}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$ is an isometric isomorphism, the adjoint operator

$$(\langle D' \rangle^{\sigma})' \colon (H_p^s(\mathbb{R}^n))' \to (H_p^{s,\sigma}(\mathbb{R}^n))'$$

is an isometric isomorphism in the dual spaces. In the bilinear pairing, this adjoint operator is again $\langle D' \rangle^{\sigma}$, and the dual space of $H_p^s(\mathbb{R}^n)$ is given by $H_q^{-s}(\mathbb{R}^n)$. Hence, the dual space of $H_p^{s,\sigma}(\mathbb{R}^n)$ is identified with $\langle D' \rangle^{\sigma}(H_q^{-s}(\mathbb{R}^n)) = H_q^{-s,-\sigma}(\mathbb{R}^n)$.

(v) The equality is known to be true in case $\sigma = 0$. Applying $\langle D' \rangle^{-\sigma}$ to both sides of this equality yields the claim.

(vi) This holds because $\partial^{\alpha} \langle D' \rangle^{-|\alpha'|} \langle D \rangle^{-\alpha_n}$ is bounded in $L^p(\mathbb{R}^n)$ due to Mikhlin's Theorem.

(vii) Again this is known in case $\sigma = 0$. The general case follows because $\langle D' \rangle^{\sigma}$ commutes with ∂_n. □

3.1 b) Interpolation spaces

In this subsection we show that complex interpolation of the spaces $H_p^{s,\sigma}(\mathbb{R}^n)$ works as one would expect. The following result is an application of Proposition 2.2.1. In difference to Lemma 2.3 in [19], we can also vary the parameter p here.

3.1.3 Lemma. *Let $s_0, s_1, \sigma_0, \sigma_1 \in \mathbb{R}$, $p_0, p_1 \in (1, \infty)$, and $\theta \in (0, 1)$. Then*

$$[H_{p_0}^{s_0,\sigma_0}(\mathbb{R}^n), H_{p_1}^{s_1,\sigma_1}(\mathbb{R}^n)]_{\theta} = H_{p_\theta}^{s_\theta,\sigma_\theta}(\mathbb{R}^n) \tag{3-1}$$

with $s_\theta = (1-\theta)s_0 + \theta s_1$, $\sigma_\theta = (1-\theta)\sigma_0 + \theta\sigma_1$, and $\frac{1}{p_\theta} = \frac{(1-\theta)}{p_0} + \frac{\theta}{p_1}$.

Proof. Because of $[X_0, X_1]_{\theta} = [X_1, X_0]_{1-\theta}$, we may assume $s_1 \leqslant s_0$. Applying the operator $\langle D \rangle^{s_1} \langle D' \rangle^{\sigma_0}$ and setting $s := s_0 - s_1 \geqslant 0$ and $\sigma := \sigma_1 - \sigma_0 \in \mathbb{R}$, by the retraction argument from Proposition 2.3.2 of [5], it remains to show that

$$[H_{p_0}^{s}(\mathbb{R}^n), H_{p_1}^{0,\sigma}(\mathbb{R}^n)]_{\theta} = H_{p_\theta}^{(1-\theta)s,\theta\sigma}(\mathbb{R}^n). \tag{3-2}$$

First, we observe

$$\begin{aligned} [H_{p_0}^{s}(\mathbb{R}^n), H_{p_1}^{0,\sigma}(\mathbb{R}^n)]_{\theta} &= [H_{p_0}^{0,s}(\mathbb{R}^n) \cap H_{p_0}^{s}(\mathbb{R}, L^{p_0}(\mathbb{R}^{n-1})), H_{p_1}^{0,\sigma}(\mathbb{R}^n)]_{\theta} \\ &\subset [H_{p_0}^{0,s}(\mathbb{R}^n), H_{p_1}^{0,\sigma}(\mathbb{R}^n)]_{\theta} \cap [H_{p_0}^{s}(\mathbb{R}, L^{p_0}(\mathbb{R}^{n-1})), H_{p_1}^{0,\sigma}(\mathbb{R}^n)]_{\theta} \\ &= L^{p_\theta}(\mathbb{R}, H_{p_\theta}^{(1-\theta)s+\theta\sigma}(\mathbb{R}^{n-1})) \cap H_{p_\theta}^{(1-\theta)s}(\mathbb{R}, H_{p_\theta}^{\theta\sigma}(\mathbb{R}^{n-1})) \\ &= H_{p_\theta}^{(1-\theta)s,\theta\sigma}(\mathbb{R}^n) \end{aligned} \tag{3-3}$$

according to Proposition 3.1.2 and Theorem 3.18 of [51].

For the other inclusion, we apply Proposition 2.2.1 (ii) to the spaces

$$X_\theta := H_{p_\theta}^{(1-\theta)s}(\mathbb{R}^n) \quad \text{and} \quad Y_\theta := H_{p_\theta}^{(1-\theta)s,\theta\sigma}(\mathbb{R}^n)$$

for $\theta \in [0, 1]$. It is known from Theorem 3.18 of [51] that $[H_{p_0}^{s}(\mathbb{R}^n), L^{p_1}(\mathbb{R}^n)]_{\theta} = H_{p_\theta}^{(1-\theta)s}(\mathbb{R}^n)$. We notice that $\langle D' \rangle^{-z\sigma} \colon H_{p_0}^{s}(\mathbb{R}^n) + L^{p_1}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is linear such that

$$\langle D' \rangle^{-z\sigma} \colon H_{p_\theta}^{(1-\operatorname{Re}(z))s}(\mathbb{R}^n) \to H_{p_\theta}^{(1-\operatorname{Re}(z))s,\operatorname{Re}(z)\sigma}(\mathbb{R}^n)$$

is an isomorphism for all $z \in \overline{S}$ with

$$\begin{aligned}\|\langle D'\rangle^{-ib\sigma}\|_{L(H^s_{p_0}(\mathbb{R}^n),H^s_{p_0}(\mathbb{R}^n))} &\leqslant \|\langle D'\rangle^{-ib\sigma}\|_{L(L^{p_0}(\mathbb{R}^n),L^{p_0}(\mathbb{R}^n))} \leqslant C(1+|b\sigma|),\\ \|\langle D'\rangle^{-(1+ib)\sigma}\|_{L(L^{p_1}(\mathbb{R}^n),H^{0,\sigma}_{p_1}(\mathbb{R}^n))} &\leqslant \|\langle D'\rangle^{-ib\sigma}\|_{L(L^{p_1}(\mathbb{R}^n),L^{p_1}(\mathbb{R}^n))} \leqslant C(1+|b\sigma|)\end{aligned}$$

for all $b \in \mathbb{R}$ due to inequality (5.53) of [44]. If $\sigma \geqslant 0$, the mapping

$$\overline{S} \to H^s_{p_0}(\mathbb{R}^n) \subset H^s_{p_0}(\mathbb{R}^n) + H^{0,\sigma}_{p_1}(\mathbb{R}^n), \quad z \mapsto \langle D'\rangle^{-z\sigma}x = \langle D\rangle^{-s}\langle D'\rangle^{-z\sigma}\langle D\rangle^{s}x$$

is holomorphic on S and continuous on $\overline{S}$ for all $x \in H^s_{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n) \subset H^s_{p_0}(\mathbb{R}^n)$ because of Lemma 5.6.8 in [44]. In case $\sigma < 0$, the mapping

$$\overline{S} \to H^{0,\sigma}_{p_1}(\mathbb{R}^n) \subset H^s_{p_0}(\mathbb{R}^n) + H^{0,\sigma}_{p_1}(\mathbb{R}^n), \quad z \mapsto \langle D'\rangle^{-z\sigma}x = \langle D'\rangle^{-\sigma}\langle D'\rangle^{(1-z)\sigma}x$$

fulfils again the desired smoothness assumptions for $x \in H^s_{p_0}(\mathbb{R}^n) \cap L^{p_1}(\mathbb{R}^n) \subset L^{p_1}(\mathbb{R}^n)$ according to Lemma 5.6.8 in [44]. An application of Proposition 2.2.1 (ii) finishes the proof. □

3.1.4 Remark. Because of Lemma 3.1.3 and the inclusion (3-3), we have

$$\begin{aligned}&\big[H^{0,s}_{p_0}(\mathbb{R}^n) \cap H^s_{p_0}(\mathbb{R}, L^{p_0}(\mathbb{R}^{n-1})), H^{0,\sigma}_{p_1}(\mathbb{R}^n)\big]_\theta\\ &= \big[H^{0,s}_{p_0}(\mathbb{R}^n), H^{0,\sigma}_{p_1}(\mathbb{R}^n)\big]_\theta \cap \big[H^s_{p_0}(\mathbb{R}, L^{p_0}(\mathbb{R}^{n-1})), H^{0,\sigma}_{p_1}(\mathbb{R}^n)\big]_\theta\end{aligned} \tag{3-4}$$

for $s \geqslant 0$ and $\sigma \in \mathbb{R}$. Therefore, we can interchange complex interpolation and intersection for those spaces in the given way. For general spaces, it is not clear in which setting this equality is true, see for instance [65]. A counterexample can be found in [74]. If there was an argument to see equality (3-4) independently of the proof of Lemma 3.1.3, one could show Lemma 3.1.3 with only (3-3) and (3-4).

3.1.5 Remark. In this remark we collect some properties which are still true if we replace $\mathbb{R}^n$ by $\mathbb{R}^n_+$ or a domain $\Omega \subset \mathbb{R}^n$ with smooth boundary.

(i) Let $\Omega = \mathbb{R}^n_+$. Then the restriction $r_{\mathbb{R}^n_+} : X(\mathbb{R}^n) \to X(\mathbb{R}^n_+)$, $u \mapsto u\big|_{\mathbb{R}^n_+}$ is a retraction for $X \in \{\mathscr{S}, \mathscr{S}', H^s_p, B^s_{pp}, H^{s,\sigma}_p, \ldots\}$. This follows from the fact that there exists a restriction-extension pair (R, E) for $(\mathscr{S}'(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n_+))$ in the sense of Theorem VI.1.2.3 in [6] which yields the restriction-extension pair $(r_{\mathbb{R}^n_+}, e_{\mathbb{R}^n_+})$ on $(X(\mathbb{R}^n), X(\mathbb{R}^n_+))$ by Lemma VII.2.8.1 of [6]. In particular, the extension operator $e_{\mathbb{R}^n_+}$ is universal for all considered spaces. Later, we will also consider e^0_Ω, the canonical extension from Ω to $\mathbb{R}^n$ by zero.

(ii) Similarly, if $\Omega \subset \mathbb{R}^n$ has smooth boundary, the map $u \mapsto u\big|_\Omega$ is a retraction from $H^s_p(\mathbb{R}^n)$ to $H^s_p(\Omega)$ and from $B^s_{pp}(\mathbb{R}^n)$ to $B^s_{pp}(\Omega)$, and for all $N \in \mathbb{N}$ there exists a common coretraction, that is, a continuous right-inverse, for all $|s| < N$, see Theorem 3.3.4 of [79].

(iii) Due to (i) and standard retraction-coretraction arguments, see Section I.2.3 of [5], all statements of Proposition 3.1.2 and Lemma 3.1.3 remain valid if we replace $\mathbb{R}^n$ by $\mathbb{R}^n_+$, with the exception of Proposition 3.1.2 (iv) which has to be modified in the following form: For all $s, \sigma \in \mathbb{R}$ the dual space of $H^{s,\sigma}_p(\mathbb{R}^n_+)$ with respect to the standard pairing is given by $(H^{s,\sigma}_p(\mathbb{R}^n_+))' = \dot{H}^{-s,-\sigma}_q(\overline{\mathbb{R}^n_+})$. Noting that $\langle D'\rangle^\sigma u$ has support in $\overline{\mathbb{R}^n_+}$ if u does, this follows in the same way as in the proof of Proposition 3.1.2 (iv) from $(H^s_p(\mathbb{R}^n_+))' = \dot{H}^{-s}_q(\overline{\mathbb{R}^n_+})$, see Theorem VII.4.4.2 of [6].

3.1 c) Boundary traces

To define the trace space of $H_p^{s,\sigma}(\mathbb{R}^n_+)$ on the boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, we recall that for the standard space $H_p^s(\mathbb{R}^n_+)$ the j-th order trace

$$\gamma_j \colon H_p^s(\mathbb{R}^n_+) \to B_{pp}^{s-j-1/p}(\mathbb{R}^{n-1}), \quad u \mapsto \gamma_j u := \partial_n^j u|_{\mathbb{R}^{n-1}}$$

is a continuous retraction for $s > j + \frac{1}{p}$, see Remark 2.1.4 (ii).

3.1.6 Definition. Let $j \in \mathbb{N}_0$, $s > j + \frac{1}{p}$, and $\sigma \in \mathbb{R}$. Then we define the j-th order trace $\widetilde{\gamma}_j$ on $H_p^{s,\sigma}(\mathbb{R}^n_+)$ as

$$\widetilde{\gamma}_j \colon H_p^{s,\sigma}(\mathbb{R}^n_+) \to B_{pp}^{s+\sigma-j-1/p}(\mathbb{R}^{n-1}), \quad u \mapsto \langle D' \rangle^{-\sigma} \gamma_j \langle D' \rangle^{\sigma} u. \tag{3-5}$$

3.1.7 Remark. Note that $\widetilde{\gamma}_j$ is well defined as $\langle D' \rangle^{\sigma} u \in H_p^s(\mathbb{R}^n_+)$ and the unique continuous extension of the classical trace which is defined on the dense subspace $\mathscr{S}(\mathbb{R}^n_+)$. The fact that γ_j is a retraction on the classical space $H_p^s(\mathbb{R}^n_+)$ immediately implies that (3-5) is a retraction, too. In fact, if e_j is a coretraction to γ_j, then $\widetilde{e}_j := \langle D' \rangle^{-\sigma} e_j \langle D' \rangle^{\sigma}$ is a coretraction to $\widetilde{\gamma}_j$. As $\widetilde{\gamma}_j$ is (for $\sigma \leqslant 0$) the unique continuous extension of γ_j to $H_p^{s,\sigma}(\mathbb{R}^n_+)$, we will write γ_j instead of $\widetilde{\gamma}_j$ again.

3.1.8 Remark (Roitberg spaces)**.** There is a theory of generalized boundary value problems in spaces of negative regularity due to Roitberg [69]. In this theory, for $s \in \mathbb{R}$ and $\ell \in \mathbb{N}_0$, the space $\widetilde{H}_p^{s,(\ell)}(\mathbb{R}^n_+)$ is defined as the set of all tuples $(u, g_0, \ldots, g_{\ell-1})$ such that there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n_+)$ satisfying $(u_k, \gamma_0 u_k, \ldots, \gamma_{\ell-1} u_k) \to (u, g_0, \ldots, g_{\ell-1})$, where the convergence takes place in the space

$$\begin{aligned} &H_p^s(\mathbb{R}^n_+) \times \prod_{j=0}^{\ell-1} B_{pp}^{s-j-1/p}(\mathbb{R}^{n-1}) && \text{if } s \geqslant 0, \\ &\dot{H}_p^s(\overline{\mathbb{R}^n_+}) \times \prod_{j=0}^{\ell-1} B_{pp}^{s-j-1/p}(\mathbb{R}^{n-1}) && \text{if } s < 0. \end{aligned}$$

For $s > \ell - 1 + \frac{1}{p}$, the space $\widetilde{H}_p^{s,(\ell)}(\mathbb{R}^n_+)$ can be identified with the standard Sobolev space $H_p^s(\mathbb{R}^n_+)$, and we have $g_j = \gamma_j u$ for $j = 0, \ldots, \ell - 1$ in this case, see Section 2.1 of [69].

Let $\ell \in \mathbb{N}_0$, $s > \ell - 1 + \frac{1}{p}$, $\sigma \leqslant 0$, and $u \in H_p^{s,\sigma}(\mathbb{R}^n_+)$. By density, there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset \mathscr{S}(\mathbb{R}^n_+)$ with $\|u_k - u\|_{H_p^{s,\sigma}(\mathbb{R}^n_+)} \to 0$ for $k \to \infty$. The continuity of (3-5) yields

$$\|\gamma_j u_k - \gamma_j u\|_{B_{pp}^{s+\sigma-j-1/p}(\mathbb{R}^{n-1})} \to 0 \quad (k \to \infty)$$

for $j = 0, \ldots, \ell - 1$. From this and Lemma 3.1.2 (i), we obtain the continuous embeddings

$$\begin{aligned} &H_p^{s,\sigma}(\mathbb{R}^n_+) && \subset \widetilde{H}_p^{s+\sigma,(\ell)}(\mathbb{R}^n_+) && \text{if } s + \sigma \geqslant 0, \\ &H_p^{s,\sigma}(\mathbb{R}^n_+) \cap \dot{H}_p^{s+\sigma}(\overline{\mathbb{R}^n_+}) && \subset \widetilde{H}_p^{s+\sigma,(\ell)}(\mathbb{R}^n_+) && \text{if } s + \sigma < 0, \end{aligned}$$

where we identify $u \in H_p^{s,\sigma}(\mathbb{R}^n_+)$ with the tuple $(u, \gamma_0 u, \ldots, \gamma_{\ell-1} u)$.

3.1 d) Parameter-dependent spaces

In a next step we extend the parameter-dependent norms from Definition 2.1.6 to the spaces of mixed smoothness $H_p^{s,\sigma}(\mathbb{R}^n)$.

3.1.9 Definition. For $\lambda \in \mathbb{C}$ let κ_λ again denote the homeomorphism on $\mathscr{S}'(\mathbb{R}^n)$ given on $\mathscr{S}(\mathbb{R}^n)$ by $(\kappa_\lambda u)(x) = u(\langle\lambda\rangle x)$. Then, we define the parameter-dependent norms by

$$\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} := \langle\lambda\rangle^{s+\sigma-n/p}\|\kappa_\lambda^{-1}u\|_{H_p^{s,\sigma}(\mathbb{R}^n)} \quad (s,\sigma\in\mathbb{R}).$$

Analogously, we define $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)$.

3.1.10 Lemma. *We observe the following properties of the parameter-dependent spaces.*

(i) The statements from Proposition 3.1.2 (i)–(vi), Lemma 3.1.3, Remark 3.1.5 and Remark 3.1.7 remain valid in the spaces $H^{s,\sigma}_{p,\lambda}$ for $\lambda \in \mathbb{C}$ with respect to the parameter-dependent norms.

(ii) For all $s,\sigma\in\mathbb{R}$ and $\lambda\in\mathbb{C}$, we have

$$\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} = \|\langle D',\lambda\rangle^\sigma u\|_{H^s_{p,\lambda}(\mathbb{R}^n)} = \|\langle D,\lambda\rangle^s\langle D',\lambda\rangle^\sigma u\|_{L^p(\mathbb{R}^n)}.$$

(iii) For $s,\sigma\geqslant 0$ and $\lambda\in\mathbb{C}$, we have

$$\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} \approx \|u\|_{H_p^{s,\sigma}(\mathbb{R}^n)} + \langle\lambda\rangle^\sigma\|u\|_{H_p^s(\mathbb{R}^n)} + \langle\lambda\rangle^s\|u\|_{H_p^{0,\sigma}(\mathbb{R}^n)} + \langle\lambda\rangle^{s+\sigma}\|u\|_{L^p(\mathbb{R}^n)}.$$

(iv) (Interpolation inequality) Let $s_0 < s < s_1$ and $\sigma\in\mathbb{R}$. For every $\varepsilon>0$ there exists a constant $C(\varepsilon)>0$ such that

$$\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} \leqslant \varepsilon\|u\|_{H^{s_1,\sigma}_{p,\lambda}(\mathbb{R}^n)} + C(\varepsilon)\langle\lambda\rangle^{s-s_0}\|u\|_{H^{s_0,\sigma}_{p,\lambda}(\mathbb{R}^n)}$$

for every $\lambda\in\mathbb{C}$ and $u\in H^{s_1,\sigma}_{p,\lambda}(\mathbb{R}^n)$. The analogue statement holds for $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$.

Proof. (i) We can apply the above results in the parameter-independent norms to the function $\kappa_\lambda^{-1}u$ and obtain constants independent of λ, noting also that κ_λ commutes with taking the trace on the boundary $\mathbb{R}^{n-1}$.

(ii) For $\sigma=0$, the statement follows from Formula (1.9) in [36]. For general σ, we use the identity

$$\langle\lambda\rangle^\sigma\kappa_\lambda\langle D'\rangle^\sigma\kappa_\lambda^{-1} = \kappa_\lambda\big\langle\langle\lambda\rangle D',\lambda\big\rangle^\sigma\kappa_\lambda^{-1} = \langle D',\lambda\rangle^\sigma,$$

which is obtained by straightforward calculation. This yields

$$\begin{aligned}\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} &= \langle\lambda\rangle^{s+\sigma-n/p}\big\|\langle D'\rangle^\sigma\kappa_\lambda^{-1}u\big\|_{H_p^s(\mathbb{R}^n)} = \langle\lambda\rangle^\sigma\big\|\kappa_\lambda\langle D'\rangle^\sigma\kappa_\lambda^{-1}u\big\|_{H^s_{p,\lambda}(\mathbb{R}^n)}\\ &= \big\|\langle D',\lambda\rangle^\sigma u\big\|_{H^s_{p,\lambda}(\mathbb{R}^n)} = \big\|\langle D,\lambda\rangle^s\langle D',\lambda\rangle^\sigma u\big\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

(iii) Due to Mikhlin's Theorem we obtain

$$\begin{aligned}\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} &= \|\langle D,\lambda\rangle^s\langle D',\lambda\rangle^\sigma u\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C\|(\langle D\rangle^s+\langle\lambda\rangle^s)\langle D',\lambda\rangle^\sigma u\|_{L^p(\mathbb{R}^n)}\\ &\leqslant C(\|\langle D',\lambda\rangle^\sigma u\|_{H^s_p(\mathbb{R}^n)}+\langle\lambda\rangle^s\|\langle D',\lambda\rangle^\sigma u\|_{L^p(\mathbb{R}^n)})\\ &\leqslant C(\|u\|_{H^{s,\sigma}_p(\mathbb{R}^n)}+\langle\lambda\rangle^\sigma\|u\|_{H^s_p(\mathbb{R}^n)}+\langle\lambda\rangle^s\|u\|_{H^{0,\sigma}_p(\mathbb{R}^n)}+\langle\lambda\rangle^{s+\sigma}\|u\|_{L^p(\mathbb{R}^n)}).\end{aligned}$$

For the other inequality, we use $s,\sigma\geqslant 0$ and again Mikhlin's Theorem to get

$$\|u\|_{H^{s,\sigma}_p(\mathbb{R}^n)}+\langle\lambda\rangle^\sigma\|u\|_{H^s_p(\mathbb{R}^n)}+\langle\lambda\rangle^s\|u\|_{H^{0,\sigma}_p(\mathbb{R}^n)}+\langle\lambda\rangle^{s+\sigma}\|u\|_{L^p(\mathbb{R}^n)}\leqslant C\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)}.$$

(iv) An application of the standard interpolation inequality gives

$$\begin{aligned}\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} &= \langle\lambda\rangle^{s+\sigma-n/p}\big\|\langle D'\rangle^\sigma\kappa_\lambda^{-1}u\big\|_{H^s_p(\mathbb{R}^n)}\\ &\leqslant \langle\lambda\rangle^{s+\sigma-n/p}\Big(\varepsilon\|\langle D'\rangle^\sigma\kappa_\lambda^{-1}u\|_{H^{s_1}_p(\mathbb{R}^n)}+C(\varepsilon)\|\langle D'\rangle^\sigma\kappa_\lambda^{-1}u\|_{H^{s_0}_p(\mathbb{R}^n)}\Big)\\ &\leqslant \varepsilon\|u\|_{H^{s_1,\sigma}_{p,\lambda}(\mathbb{R}^n)}+C(\varepsilon)\langle\lambda\rangle^{s-s_0}\|u\|_{H^{s_0,\sigma}_{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

□

Let us remark that a statement analogous to Lemma 3.1.10 (ii) does not hold for the parameter-dependent Besov spaces (with $L^p(\mathbb{R}^n)$ being replaced by $B^0_{pp}(\mathbb{R}^n)$). Although

$$\langle D',\lambda\rangle^s\colon B^s_{pp,\lambda}(\mathbb{R}^{n-1})\to B^0_{pp,\lambda}(\mathbb{R}^{n-1})$$

is an isomorphism, the norm in the Besov space $B^0_{pp,\lambda}(\mathbb{R}^{n-1})$ still depends on λ, in contrast to $\|\cdot\|_{H^0_{p,\lambda}(\mathbb{R}^n)}=\|\cdot\|_{L^p(\mathbb{R}^n)}$. This was observed in Section 1.1 of [36].

3.1 e) Multiplication operators

We finish this section with some considerations concerning multiplication operators. For a sufficiently smooth function $a\colon\mathbb{R}^n\to\mathbb{C}$, we define the multiplication operator M_a by $M_au:=au$ whenever the function u belongs to some Sobolev space of positive order. For negative order spaces, we define the multiplication operator by duality with respect to the canonical pairing $L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)$, where $\frac{1}{p}+\frac{1}{q}=1$.

3.1.11 Lemma. *Let $s,\sigma\in\mathbb{R}$, $\rho'=\rho'(s,\sigma):=\max\{|s|,|\sigma|,|s+\sigma|\}$, and $a\in\mathrm{BUC}^{\lceil\rho'\rceil}(\mathbb{R}^n)$.*

(i) There is a constant $C=C(\rho')=C(s,\sigma)>0$ such that for all $\lambda\in\mathbb{C}$

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))}\leqslant C\|a\|_{\mathrm{BUC}^{\lceil\rho'\rceil}(\mathbb{R}^n)}. \tag{3-6}$$

(ii) For every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon,s,\sigma)>0$ and a $\lambda_0=\lambda_0\big(\|a\|_{\mathrm{BUC}^{\lceil\rho'\rceil}(\mathbb{R}^n)},\varepsilon\big)>0$ such that

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))}<\varepsilon$$

whenever $\|a\|_\infty<\delta$ and $\lambda\in\mathbb{C}$ with $|\lambda|\geqslant\lambda_0$.

(iii) The results in (i) and (ii) hold analogously for $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$ with $a\in\mathrm{BUC}^{\lceil\rho'\rceil}(\mathbb{R}^n_+)$.

(iv) The results in (i) and (ii) also hold if we replace $\lceil\rho'\rceil$ by $\lfloor|s|\rfloor+1$ and $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)$ by $B^s_{pp,\lambda}(\mathbb{R}^n)$.

Proof. We set $\rho := \lceil \rho' \rceil$.

(i) Consider the hexagon which is the convex hull of the vertex set

$$\mathcal{H} := \{(\rho,0),(0,\rho),(-\rho,\rho),(-\rho,0),(0,-\rho),(\rho,-\rho)\},$$

see Figure 3.1. In a first step we are going to show that for all $P \in \mathcal{H}$ we can deduce the bound

$$\|M_a\|_{L(H_p^P(\mathbb{R}^n))} \leqslant C(\rho)\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}.$$

For the first two vertices this follows from the fact that $H_p^{0,\rho}(\mathbb{R}^n) = L^p(\mathbb{R}, H_p^\rho(\mathbb{R}^{n-1}))$ and $H_p^{\rho,0}(\mathbb{R}^n) = H_p^\rho(\mathbb{R}^n)$ due to Lemma 3.1.2 (i) as well as the product rule. Their counterparts $(0,-\rho)$ and $(-\rho,0)$ can be treated by a duality argument. For the space $H_p^{\rho,-\rho}(\mathbb{R}^n)$ we use Lemma 3.1.2 (vii) to obtain

$$\begin{aligned}
\|au\|_{H_p^{\rho,-\rho}(\mathbb{R}^n)} &\leqslant C\sum_{j=0}^{\rho}\|\partial_n^j(au)\|_{H_p^{0,-j}(\mathbb{R}^n)} \\
&\leqslant C\sum_{j=0}^{\rho}\sum_{l=0}^{j}\|(\partial_n^{j-l}a)(\partial_n^l u)\|_{H_p^{0,-j}(\mathbb{R}^n)} \\
&\leqslant C\sum_{j=0}^{\rho}\sum_{l=0}^{j}\|(\partial_n^{j-l}a)(\partial_n^l u)\|_{H_p^{0,-l}(\mathbb{R}^n)},
\end{aligned}$$

where we used $H_p^{0,-l}(\mathbb{R}^n) \subset H_p^{0,-j}(\mathbb{R}^n)$. Furthermore, we have $\partial_n^{j-l}a \in \mathrm{BUC}^l(\mathbb{R}^n)$ as $j \leqslant \rho$. So we may apply the boundedness of $M_{\partial_n^{j-l}a}$ on $H_p^{0,-l}(\mathbb{R}^n)$ and find

$$\begin{aligned}
\|au\|_{H_p^{\rho,-\rho}(\mathbb{R}^n)} &\leqslant C\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}\sum_{j=0}^{\rho}\sum_{l=0}^{j}\|\partial_n^l u\|_{H_p^{0,-l}(\mathbb{R}^n)} \\
&= C\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}\sum_{l=0}^{\rho}(\rho-l+1)\|\partial_n^l u\|_{H_p^{0,-l}(\mathbb{R}^n)} \\
&\leqslant C\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}\sum_{l=0}^{\rho}\|\partial_n^l u\|_{H_p^{0,-l}(\mathbb{R}^n)} \\
&\leqslant C\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}\|u\|_{H_p^{\rho,-\rho}(\mathbb{R}^n)}
\end{aligned}$$

for all $u \in H_p^{\rho,-\rho}(\mathbb{R}^n)$. The last vertex follows again by duality.

In a second step we interpolate along the edges of the hexagon, which is precisely the domain $\{(t,\tau) : \max\{|t|,|\tau|,|t+\tau|\} = \rho\}$. For any point $P_\theta = (1-\theta)P_0 + \theta P_1$, where $0 < \theta < 1$ and $P_0, P_1 \in \mathcal{H}$, we obtain by interpolation $H_p^{P_\theta}(\mathbb{R}^n) = [H_p^{P_0}(\mathbb{R}^n), H_p^{P_1}(\mathbb{R}^n)]_\theta$ and thus

$$\|M_a\|_{L(H_p^{P_\theta}(\mathbb{R}^n))} \leqslant C\|M_a\|^{1-\theta}_{L(H_p^{P_0}(\mathbb{R}^n))}\|M_a\|^{\theta}_{L(H_p^{P_1}(\mathbb{R}^n))} \leqslant C\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}. \tag{3-7}$$

Finally, we interpolate along a straight line that starts in the origin, passes through (s, σ) and hits the boundary of the hexagon in the point $\left(\frac{\rho s}{\rho'}, \frac{\rho\sigma}{\rho'}\right)$. More precisely, we use the interpolation

$$H_p^{s,\sigma}(\mathbb{R}^n) = \left[L^p(\mathbb{R}^n), H_p^{\rho s/\rho', \rho\sigma/\rho'}(\mathbb{R}^n)\right]_{\rho'/\rho}$$

to find

$$\|M_a\|_{L(H_p^{s,\sigma}(\mathbb{R}^n))} \leqslant C\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}.$$

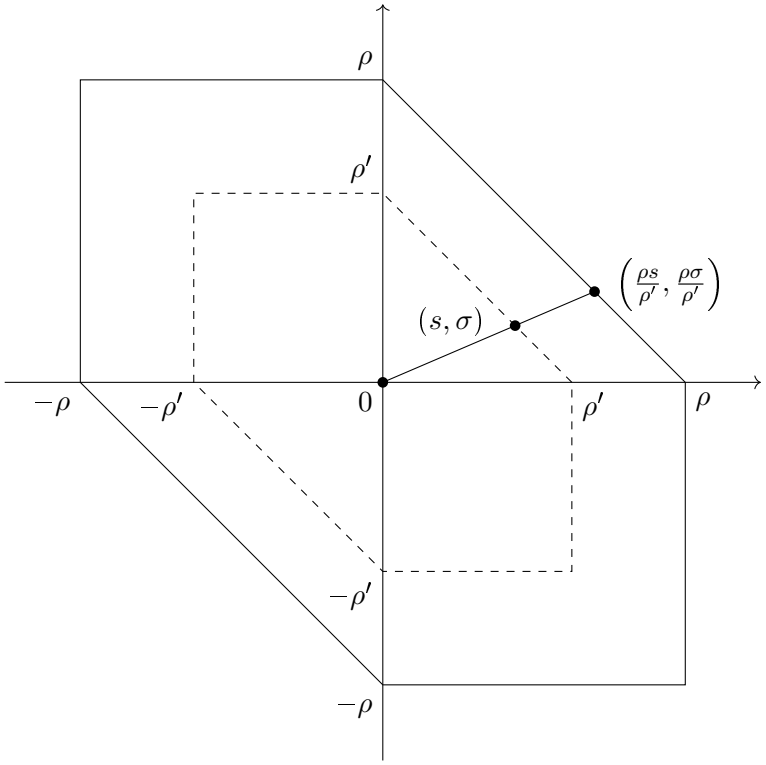

Figure 3.1: In a first step, we see that the operator M_a is continuous on $H_p^P(\mathbb{R}^n)$ for every vertex $P \in \mathcal{H}$ of the outer hexagon and therefore by interpolation continuous on $H_p^{P_\theta}(\mathbb{R}^n)$ for every P_θ on its boundary. Finally, we interpolate between that boundary and the origin to get the continuity on $H_p^{s,\sigma}(\mathbb{R}^n)$ for every (s, σ) on the boundary of the dashed hexagon.

In order to carry the result over to the parameter-dependent norms, we observe the following for any bounded operator T in $H_p^{s,\sigma}(\mathbb{R}^n)$: By Definition 3.1.9 we have

$$\begin{aligned}\|Tu\|_{H_{p,\lambda}^{s,\sigma}(\mathbb{R}^n)} &= \langle\lambda\rangle^{s+\sigma-n/p}\|\kappa_\lambda^{-1}(Tu)\|_{H_p^{s,\sigma}(\mathbb{R}^n)} \\ &= \langle\lambda\rangle^{s+\sigma-n/p}\|(\kappa_\lambda^{-1}T\kappa_\lambda)(\kappa_\lambda^{-1}u)\|_{H_p^{s,\sigma}(\mathbb{R}^n)}.\end{aligned}$$

Dividing by $\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)} = \langle\lambda\rangle^{s+\sigma-n/p}\|\kappa_\lambda^{-1}u\|_{H^{s,\sigma}_p(\mathbb{R}^n)}$ and passing to the supremum over all $0 \neq u \in H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)$, we conclude that

$$\|T\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} = \|\kappa_\lambda^{-1}T\kappa_\lambda\|_{L(H^{s,\sigma}_p(\mathbb{R}^n))}.$$

Since we have

$$\kappa_\lambda^{-1}M_a\kappa_\lambda u(x) = \kappa_\lambda^{-1}[a(x)u(\langle\lambda\rangle x)] = a(\langle\lambda\rangle^{-1}x)u(x) = (\kappa_\lambda^{-1}a)(x)u(x),$$

it holds

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} = \|M_{\kappa_\lambda^{-1}a}\|_{L(H^{s,\sigma}_p(\mathbb{R}^n))}.$$

Thus

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} \leqslant C\|\kappa_\lambda^{-1}a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)} \leqslant C(\|a\|_\infty + \langle\lambda\rangle^{-1}\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}) \tag{3-8}$$

since $\|\partial^\alpha(\kappa_\lambda^{-1}a)\|_\infty = \langle\lambda\rangle^{-|\alpha|}\|\partial^\alpha a\|_\infty$ for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leqslant \rho$.

(ii) Let $\varepsilon > 0$ and choose $\delta < \frac{\varepsilon}{2C}$. Let $a \in \mathrm{BUC}^\rho(\mathbb{R}^n)$ with $\|a\|_\infty < \delta$ and $\lambda_0 > 0$ such that $\langle\lambda_0\rangle^{-1}\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)} < \frac{\varepsilon}{2C}$. According to (3-8), we obtain

$$\|M_a\|_{L(H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n))} < \varepsilon.$$

(iii) There exists a bounded extension operator $E_{\mathbb{R}^n_+}\colon \mathrm{BUC}^\rho(\mathbb{R}^n_+) \to \mathrm{BUC}^\rho(\mathbb{R}^n)$ for any $\rho \in \mathbb{N}_0$, see for instance the construction in Theorem 5.19 of [1]. Then for $\mathbb{R}^n_+$ part (i) and (ii) follow from

$$\begin{aligned}\|M_a u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)} &\leqslant \|M_{E_{\mathbb{R}^n_+}a}(e_{\mathbb{R}^n_+}u)\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)}\\ &\leqslant C\big(\|E_{\mathbb{R}^n_+}a\|_\infty + \langle\lambda\rangle^{-1}\|E_{\mathbb{R}^n_+}a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n)}\big)\|e_{\mathbb{R}^n_+}u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n)}\\ &\leqslant C\big(\|a\|_\infty + \langle\lambda\rangle^{-1}\|a\|_{\mathrm{BUC}^\rho(\mathbb{R}^n_+)}\big)\|u\|_{H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)}\end{aligned}$$

for all $u \in H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)$.

(iv) Taking $\sigma = 0$, we use the result from (i) and (ii) for the spaces $H^{s+\gamma}_{p,\lambda}(\mathbb{R}^n)$ and $H^{s-\gamma}_{p,\lambda}(\mathbb{R}^n)$ for a sufficiently small $\gamma > 0$ such that $|s \pm \gamma| \leqslant \lfloor|s|\rfloor + 1$ still holds. Then the result follows by real interpolation of the λ-dependent, but classical Sobolev spaces, which was established in (1.16) of [36]. □

3.1.12 Remark. In the case of $s \in \mathbb{N}_0$ and $\sigma = \lambda = 0$ in Lemma 3.1.11 (i), we directly get back the classical results for the usual Sobolev spaces $H^s_p(\mathbb{R}^n)$. Furthermore, we would like to note that pointwise multipliers in Besov spaces with $\lambda = 0$ are described for instance in [58] and [81]. In particular, it is known that functions which are Hölder continuous with Hölder index larger than $|s|$ are multipliers in $B^s_{pp}(\mathbb{R}^n)$, see Theorem 4.7.1 (ii) of [70]. For our purposes, however, Lemma 3.1.11 (iv) is sufficient.

Notice that we need slightly higher smoothness assumptions for the multiplier a in the Besov space case. Even though it might be natural to guess $M_a \in L(B^0_{pp,\lambda}(\mathbb{R}^n))$ for $a \in \mathrm{BUC}^0(\mathbb{R}^n)$, this seems to be unclear. Since we use interpolation methods for the proof, we need some variability in the smoothness, which is the reason why we are not able to handle the borderline case.

3.2 Systems of boundary value problems in the half-space

We now deal with Douglis–Nirenberg systems of boundary value problems in domains and in the half-space. In the following, let $\Omega \subset \mathbb{R}^n$ be a domain with compact and sufficiently smooth boundary Γ, or let $\Omega = \mathbb{R}^n_+$ with boundary $\Gamma = \mathbb{R}^{n-1}$.

First, we extend Definition 2.5.1 to Sobolev spaces of mixed smoothness.

3.2.1 Definition. For $p \in (1,\infty)$, $\lambda \in \mathbb{C}$, $r, \sigma \in \mathbb{R}$ and $\mathbf{s} = (s_1, \dots, s_N) \in \mathbb{R}^N$ we write

$$H^{\mathbf{s},\sigma}_{p,\lambda}(\mathbb{R}^n) := \prod_{j=1}^{N} H^{s_j,\sigma}_{p,\lambda}(\mathbb{R}^n)$$

with canonical norm $\|w\|_{H^{\mathbf{s},\sigma}_{p,\lambda}(\mathbb{R}^n)} := \Big(\sum_{j=1}^N \|w_j\|^p_{H^{s_j,\sigma}_{p,\lambda}(\mathbb{R}^n)}\Big)^{1/p}$. Additionally, we use the abbreviation $H^{\mathbf{s}+r,\sigma}_{p,\lambda}(\mathbb{R}^n) := H^{\mathbf{s}+(r,\dots,r),\sigma}_{p,\lambda}(\mathbb{R}^n)$. For $H^{\mathbf{s},\sigma}_{p}(\mathbb{R}^n) := H^{\mathbf{s},\sigma}_{p,0}(\mathbb{R}^n)$ the definition is modified in an obvious way.

We fix $N, \mu \in \mathbb{N}$, $\mathbf{s}, \mathbf{t} \in \mathbb{Z}^N$, and $\mathbf{m} \in \mathbb{Z}^M$ with the conditions from (2-4) and (2-5) and consider again the system of boundary value problems

$$\begin{aligned} (\lambda - A)w &= f \quad \text{in } \Omega, \\ Bw &= g \quad \text{on } \Gamma. \end{aligned} \tag{3-9}$$

As in Section 2.5 the entries of A and B are of the form

$$A_{ij} = A_{ij}(x,D) = \sum_{|\alpha| \leqslant s_i + t_j} a^\alpha_{ij}(x) D^\alpha, \tag{3-10}$$

$$B_{\ell j} = B_{\ell j}(x,D) = \sum_{|\beta| \leqslant m_\ell + t_j} b^\beta_{\ell j}(x) \gamma_0 D^\beta \tag{3-11}$$

for $1 \leqslant i,j \leqslant N$ and $1 \leqslant \ell \leqslant M$. Here, $a^\alpha_{ij} \colon \overline{\Omega} \to \mathbb{C}$ and $b^\beta_{\ell j} \colon \Gamma \to \mathbb{C}$ are sufficiently smooth functions.

More precisely, we will consider the following smoothness assumptions, depending on $(r,\sigma) \in \mathbb{R}^2$.

(S1) For each $i \in \{1,\dots,N\}$, let $\rho_i' := \max\{|r - s_i|, |\sigma|, |r - s_i + \sigma|\}$ and $\rho_i := \lceil \rho_i' \rceil$. We assume $a^\alpha_{ij} \in \mathrm{BUC}^{\rho_i}(\Omega)$ for all $|\alpha| \leqslant s_i + t_j$, where $1 \leqslant j \leqslant N$.

(S2) If Ω is unbounded, then $a^\alpha_{ij}(\infty) := \lim_{x \in \Omega,\, |x| \to \infty} a^\alpha_{ij}(x)$ exists for all $|\alpha| \leqslant s_i + t_j$. In addition, all derivatives of the function

$$x \mapsto a^\alpha_{ij}\left(\frac{x}{|x|^2}\right) \quad (x \neq 0)$$

up to order ρ_i possess a continuous extension to $x = 0$ for $1 \leqslant i,j \leqslant N$ and $|\alpha| \leqslant s_i + t_j$.

(S3) For each $\ell \in \{1,\dots,M\}$, let $k_\ell' := |r - m_\ell + \sigma - \frac{1}{p}|$ and $k_\ell := \lfloor k_\ell' \rfloor + 1$. We assume $b^\beta_{\ell j} \in \mathrm{BUC}^{k_\ell}(\Gamma)$ for all $|\beta| \leqslant m_\ell + t_j$, where $1 \leqslant j \leqslant N$.

(S4) If $\Omega = \mathbb{R}^n_+$, then $b^\beta_{\ell j}(\infty) := \lim_{x\in\mathbb{R}^{n-1},\,|x|\to\infty} b^\beta_{\ell j}(x)$ exists for $|\beta| \leqslant m_\ell + t_j$. In addition, all derivatives of the function

$$x \mapsto b^\beta_{\ell j}\left(\frac{x}{|x|^2}\right) \quad (x \neq 0)$$

up to order k_ℓ possess a continuous extension to $x = 0$ for $1 \leqslant j \leqslant N$, $1 \leqslant \ell \leqslant M$ and $|\beta| \leqslant m_\ell + t_j$.

(S5) The domain Ω is of class $C^{\max_j t_j + \lceil \rho' \rceil}$, where $\rho' := \max\{|r|, |\sigma|, |r+\sigma|\}$.

In this section we show that parameter-elliptic problems induce an isomorphism between parameter-dependent spaces (in the sense of Definition 2.1.5). We focus on the case of the half-space. In the following, let $\Lambda \subset \mathbb{C}$ be a closed sector in the complex plane with vertex at the origin.

3.2 a) Model problems and small perturbations

3.2.2 Lemma (Model problem in $\mathbb{R}^n$)**.** *Let $A_0(D) = (A^0_{ij}(D))_{1\leqslant i,j\leqslant N}$ have constant coefficients, that is, $A^0_{ij}(D) = \sum_{|\alpha|=s_i+t_j} a^\alpha_{ij} D^\alpha$, where $a^\alpha_{ij} \in \mathbb{C}$ for $1 \leqslant i,j \leqslant N$, and let $\lambda - A_0(D)$ be parameter-elliptic in Λ. Then, for every $r, \sigma \in \mathbb{R}$ and every $\lambda_0 > 0$, the operator family*

$$\lambda - A_0 \colon H^{\mathbf{t}+r,\sigma}_{p,\lambda}(\mathbb{R}^n) \to H^{-\mathbf{s}+r,\sigma}_{p,\lambda}(\mathbb{R}^n) \tag{3-12}$$

is an isomorphism for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$.

Proof. The result is well known in case $\sigma = 0$, see Lemma 2.5.7. Let us denote by $(\lambda - A_0)^{-1}_{(r,0)}$ the corresponding inverse operator. We use the description of the parameter-dependent norm from Lemma 3.1.10 (ii).

Since A_0 commutes with $\langle D', \lambda\rangle^\sigma$ and $\langle D', \lambda\rangle^\sigma \colon H^{r,\sigma}_{p,\lambda}(\mathbb{R}^n) \to H^r_{p,\lambda}(\mathbb{R}^n)$ is an isometric isomorphism for all $r \in \mathbb{R}$, the inverse to (3-12) is

$$(\lambda - A_0)^{-1}_{(r,\sigma)} = \operatorname{diag}(\langle D', \lambda\rangle^{-\sigma}, \dots, \langle D', \lambda\rangle^{-\sigma})(\lambda - A_0)^{-1}_{(r,0)} \operatorname{diag}(\langle D', \lambda\rangle^{\sigma}, \dots, \langle D', \lambda\rangle^{\sigma}),$$

which then has the same uniform bound as $(\lambda - A_0)^{-1}_{(r,0)}$. □

Let us now turn to the situation in the half-space, where we consider the following boundary problem.

3.2.3 Theorem (Model problem in $\mathbb{R}^n_+$)**.** *Let $(\lambda - A_0, B_0)$ be parameter-elliptic in Λ. Here again, we have $A_0(D) = (A^0_{ij}(D))_{1\leqslant i,j\leqslant N}$, where $A^0_{ij}(D) = \sum_{|\alpha|=s_i+t_j} a^\alpha_{ij} D^\alpha$ with constant coefficients $a^\alpha_{ij} \in \mathbb{C}$, as well as $B_0(D) = (B^0_{\ell j}(D))_{1\leqslant \ell\leqslant M, 1\leqslant j\leqslant N}$, where $B^0_{\ell j}(D) = \sum_{|\beta|=m_\ell+t_j} b^\beta_{\ell j} \gamma_0 D^\beta$ with constant coefficients $b^\beta_{\ell j} \in \mathbb{C}$ for $1 \leqslant i,j \leqslant N$ and $1 \leqslant \ell \leqslant M$. Then, for every $r > -1 + \frac{1}{p}$, $\sigma \in \mathbb{R}$, and $\lambda_0 > 0$, the family of operators*

$$\begin{pmatrix} \lambda - A_0 \\ B_0 \end{pmatrix} : H^{\mathbf{t}+r,\sigma}_{p,\lambda}(\mathbb{R}^n_+) \to H^{-\mathbf{s}+r,\sigma}_{p,\lambda}(\mathbb{R}^n_+) \times B^{-\mathbf{m}+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1}) \tag{3-13}$$

is an isomorphism for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$.

Proof. The proof is similar to that of Lemma 3.2.2. The result is known for $\sigma = 0$, see Theorem 2.5.6. Let $L_{(r,0)}(\lambda)$ be the corresponding inverse. All involved operators commute with $\langle D', \lambda\rangle^{\sigma}$. Hence the inverse operator $L_{(r,\sigma)}(\lambda)$ for general σ is given by

$$\operatorname{diag}(\langle D', \lambda\rangle^{-\sigma}, \dots, \langle D', \lambda\rangle^{-\sigma}) L_{(r,0)}(\lambda) \operatorname{diag}(\langle D', \lambda\rangle^{\sigma}, \dots, \langle D', \lambda\rangle^{\sigma}),$$

where the left diagonal matrix is of size $N \times N$ and the right one is of size $(N+M) \times (N+M)$. Hence $L_{(r,\sigma)}(\lambda)$ has the same uniform norm-bound as $L_{(r,0)}(\lambda)$. □

Motivated by the last two results, we define the parameter-dependent spaces

$$\mathbb{E}^{r,\sigma}_{\lambda}(\Omega) := H^{\mathbf{t}+r,\sigma}_{p,\lambda}(\Omega) \qquad (\text{with } \Omega = \mathbb{R}^n \text{ or } \Omega = \mathbb{R}^n_+), \tag{3-14}$$

as well as

$$\begin{aligned} \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n) &:= H^{-\mathbf{s}+r,\sigma}_{p,\lambda}(\mathbb{R}^n), \\ \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+) &:= H^{-\mathbf{s}+r,\sigma}_{p,\lambda}(\mathbb{R}^n_+) \times B^{-\mathbf{m}+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1}) \end{aligned} \tag{3-15}$$

for $r, \sigma \in \mathbb{R}$. Then

$$\lambda - A_0 \colon \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n) \to \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n) \quad \text{and} \quad \begin{pmatrix} \lambda - A_0 \\ B_0 \end{pmatrix} \colon \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+) \to \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)$$

are both isomorphisms. Below, we will consider the case of variable coefficients which are close to constant coefficients in an appropriate sense. As a preparation, we show some auxiliary continuity results.

3.2.4 Lemma. *Let $(r, \sigma) \in \mathbb{R}^2$.*

(i) Let A be a system of differential operators in $\mathbb{R}^n$ as in (3-10) *and assume* (S1) *to hold. Let*

$$M_A := \max_{\substack{|\alpha| \leqslant s_i + t_j, \\ 1 \leqslant i,j \leqslant N}} \|a^{\alpha}_{ij}\|_{\mathrm{BUC}^{\rho_i}(\mathbb{R}^n)}$$

for $1 \leqslant i, j \leqslant N$. Then for every $\varepsilon > 0$ there exist constants $\delta = \delta(\varepsilon, r, \sigma) > 0$ and $\lambda_0 = \lambda_0(M_A, \varepsilon) > 0$ such that

$$\|A\|_{L(\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n), \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n))} < \varepsilon$$

holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant \lambda_0$ provided $\max_{|\alpha| = s_i + t_j} \|a^{\alpha}_{ij}\|_{\infty} < \delta$ for $1 \leqslant i, j \leqslant N$.

(ii) Let (A, B) be a system of boundary value problems of the form (3-10)–(3-11) *in $\mathbb{R}^n_+$ and assume* (S1) *and* (S3) *to hold with $r > -1 + \frac{1}{p}$. Let*

$$M_{A,B} := \max_{\substack{|\alpha| \leqslant s_i + t_j, \\ 1 \leqslant i,j \leqslant N}} \|a^{\alpha}_{ij}\|_{\mathrm{BUC}^{\rho_i}(\mathbb{R}^n_+)} + \max_{\substack{|\beta| \leqslant m_\ell + t_j, \\ 1 \leqslant \ell \leqslant M,\, 1 \leqslant j \leqslant N}} \|b^{\beta}_{\ell j}\|_{\mathrm{BUC}^{k_\ell}(\mathbb{R}^{n-1})}.$$

Then for every $\varepsilon > 0$ there exist constants $\delta = \delta(\varepsilon, r, \sigma) > 0$ and $\lambda_0 = \lambda_0(M_{A,B}, \varepsilon) > 0$ such that

$$\left\| \begin{pmatrix} A \\ B \end{pmatrix} \right\|_{L(\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+), \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+))} < \varepsilon$$

holds for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant \lambda_0$ provided

$$\max_{|\alpha|=s_i+t_j} \|a^{\alpha}_{ij}\|_{\infty} + \max_{|\beta|=m_\ell+t_j} \|b^{\beta}_{\ell j}\|_{\infty} < \delta$$

for $1 \leqslant i, j \leqslant N$ and $1 \leqslant \ell \leqslant M$.

Proof. The most important ingredients for this proof are the results on multiplication operators from Lemma 3.1.11.

(i) We set $A_0 := (A^0_{ij})_{1 \leqslant i,j \leqslant N}$ with $A^0_{ij} = \sum_{|\alpha|=s_i+t_j} a^{\alpha}_{ij}(x) D^{\alpha}$ and $\widetilde{A} := A - A_0$. Let $\varepsilon > 0$ be fixed and $w \in H^{\mathbf{t},\sigma}_{p,\lambda}(\mathbb{R}^n)$ arbitrary. Then, due to Lemma 3.1.11 (ii), there exist $\delta(\varepsilon, r, \sigma) > 0$ and $\lambda_0(M_A, \varepsilon) > 0$ such that for $|\lambda| \geqslant \lambda_0$ and appropriate $\varepsilon' > 0$ we have

$$\begin{aligned} \|(A_0 w)_i\|_{H^{-s_i+r,\sigma}_{p,\lambda}(\mathbb{R}^n)} &\leqslant \sum_{j=1}^{N} \sum_{|\alpha|=s_i+t_j} \|a^{\alpha}_{ij}(\cdot) D^{\alpha} w_j\|_{H^{-s_i+r,\sigma}_{p,\lambda}(\mathbb{R}^n)} \\ &\leqslant \varepsilon' \sum_{j=1}^{N} \sum_{|\alpha|=s_i+t_j} \|D^{\alpha} w_j\|_{H^{-s_i+r,\sigma}_{p,\lambda}(\mathbb{R}^n)} \\ &\leqslant \frac{\varepsilon}{2N} \|w\|_{\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n)} \end{aligned}$$

for $1 \leqslant i \leqslant N$ given $\max_{|\alpha|=s_i+t_j} \|a^{ij}_{\alpha}\|_{\infty} < \delta$ for $1 \leqslant i, j \leqslant N$. For $\widetilde{A}w$ we use Lemma 3.1.11 (i), as we only rely on the fact that the coefficients are multipliers. Thus, we obtain the estimate

$$\begin{aligned} \|(\widetilde{A} w)_i\|_{H^{-s_i+r,\sigma}_{p,\lambda}(\mathbb{R}^n)} &\leqslant \sum_{j=1}^{N} \sum_{|\alpha|<s_i+t_j} \|a^{\alpha}_{ij}(\cdot) D^{\alpha} w_j\|_{H^{-s_i+r,\sigma}_{p,\lambda}(\mathbb{R}^n)} \\ &\leqslant C M_A \sum_{j=1}^{N} \sum_{|\alpha|<s_i+t_j} \|D^{\alpha} w_j\|_{H^{-s_i+r,\sigma}_{p,\lambda}(\mathbb{R}^n)} \qquad (3\text{-}16) \\ &\leqslant C M_A \|w\|_{\mathbb{E}^{r-1,\sigma}_{\lambda}(\mathbb{R}^n)} \\ &\leqslant C M_A \langle\lambda\rangle^{-1} \|w\|_{\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n)}. \end{aligned}$$

The last inequality holds true because we have

$$\|w_j\|_{H^{t_j+r-1,\sigma}_{p,\lambda}(\mathbb{R}^n)} = \|\langle D, \lambda\rangle^{t_j+r-1} \langle D', \lambda\rangle^{\sigma} w_j\|_{L^p(\mathbb{R}^n)} \leqslant C \langle\lambda\rangle^{-1} \|\langle D, \lambda\rangle^{t_j+r} \langle D', \lambda\rangle^{\sigma} w_j\|_{L^p(\mathbb{R}^n)}$$

uniformly in λ for $1 \leqslant j \leqslant N$, since $\langle\lambda\rangle\langle\xi, \lambda\rangle^{-1}$ is a Mikhlin multiplier with symbol estimates that are uniform in λ. As $\langle\lambda\rangle^{-1}$ vanishes for $|\lambda| \to \infty$, we can choose λ_0 so large that $C M_A \langle\lambda\rangle^{-1} < \frac{\varepsilon}{2N}$ whenever $|\lambda| \geqslant \lambda_0$.

(ii) The calculations for A are similar to (i), we just replace the whole space estimates by the half-space estimates. For the boundary operator B we use Lemma 3.1.11 (iv) instead, noting that (S3) yields the required smoothness. Hence for B we split off the lower order terms again. Then for a given $\varepsilon > 0$ there exist $\delta(\varepsilon, s, \sigma) > 0$ and $\lambda_0(M_{A,B}, \varepsilon) > 0$ such that for $|\lambda| \geqslant \lambda_0$ and appropriate $\varepsilon' > 0$ we obtain

$$\begin{aligned}
\|(B_0 w)_\ell\|_{B^{-m_\ell+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})} &\leqslant \sum_{j=1}^{N} \sum_{|\beta|=m_\ell+t_j} \|b^\beta_{\ell j}(\cdot)\gamma_0 D^\beta w_j\|_{B^{-m_\ell+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})} \\
&\leqslant \varepsilon' \sum_{j=1}^{N} \sum_{|\beta|=m_\ell+t_j} \|\gamma_0 D^\beta w_j\|_{B^{-m_\ell+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})} \\
&\leqslant C\varepsilon' \sum_{j=1}^{N} \sum_{|\beta|=m_\ell+t_j} \|D^\beta w_j\|_{H^{-m_\ell+r,\sigma}_{p,\lambda}(\mathbb{R}^n_+)} \\
&\leqslant \frac{\varepsilon}{2N} \|w\|_{\mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+)}
\end{aligned}$$

for $1 \leqslant \ell \leqslant M$ given $\max_{|\alpha|=s_i+t_j} \|a^\alpha_{ij}\|_\infty + \max_{|\beta|=m_\ell+t_j} \|b^\beta_{\ell j}\|_\infty < \delta$ for $1 \leqslant i, j \leqslant N$ and $1 \leqslant \ell \leqslant M$. Here, we also used the continuity of the trace from Definition 3.1.6. The lower order terms can be handled as in (i) by applying Lemma 3.1.11 (iv) once more. □

3.2.5 Lemma (Small perturbation in $\mathbb{R}^n$)**.** *Let $\lambda - A_0$ be as in Lemma* 3.2.2*, and let $A = A_0 + \widetilde{A}$, where $\widetilde{A} = (\widetilde{A}_{ij})_{1\leqslant i,j\leqslant N}$ with*

$$\widetilde{A}_{ij} = \widetilde{A}_{ij}(x, D) = \sum_{|\alpha|\leqslant s_i+t_j} \tilde{a}^\alpha_{ij}(x) D^\alpha$$

for $1 \leqslant i, j \leqslant N$. Moreover, let $r, \sigma \in \mathbb{R}$ and assume (S1) *to hold. Define $\delta = \delta(\frac{1}{2\rho}, r, \sigma)$ and $\lambda_0 = \max\{\lambda_0(M_{\widetilde{A}}, \frac{1}{2\rho}), 1\}$ as in Lemma 3.2.4 (i), where*

$$\rho := \sup_{|\lambda|\geqslant 1} \|(\lambda - A_0)^{-1}\|_{L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n), \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n))}.$$

Then, if $\max_{|\alpha|=s_i+t_j} \|\tilde{a}^\alpha_{ij}\|_\infty < \delta$ for $1 \leqslant i, j \leqslant N$, the operator family

$$\lambda - A \colon \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n) \to \mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n)$$

is an isomorphism for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$.

Proof. Using Lemma 3.2.2, we can write

$$\lambda - A = (\lambda - A_0)\big(1 - (\lambda - A_0)^{-1}\widetilde{A}\big) \quad (\lambda \neq 0).$$

Choosing δ and λ_0 as stated and applying Lemma 3.2.4 (i) to $\widetilde{A}$, we obtain

$$\|\widetilde{A}\|_{L(\mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n), \mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n))} < \frac{1}{2\rho}$$

whenever $|\lambda| \geqslant \lambda_0$ and $\max_{|\alpha|=s_i+t_j} \|\tilde{a}^\alpha_{ij}\|_\infty < \delta$ for $1 \leqslant i, j \leqslant N$. Therefore,

$$\|(\lambda - A_0)^{-1}\widetilde{A}\|_{L(\mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n))} < \frac{1}{2}.$$

By the usual Neumann series argument, $1-(\lambda-A_0)^{-1}\widetilde{A}$ is invertible with

$$\big\|\big(1-(\lambda-A_0)^{-1}\widetilde{A}\big)^{-1}\big\|_{L(\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n))}<2$$

for every $|\lambda|\geqslant\lambda_0$. We conclude that, for such λ,

$$(\lambda-A)^{-1}=\big(1-(\lambda-A_0)^{-1}\widetilde{A}\big)^{-1}(\lambda-A_0)^{-1}$$

with

$$\|(\lambda-A)^{-1}\|_{L(\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n),\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n))}<2\|(\lambda-A_0)^{-1}\|_{L(\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n),\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n))}.$$

Using again Lemma 3.2.2 completes the proof. □

3.2.6 Theorem (Small perturbation in $\mathbb{R}^n_+$)**.** *Consider the boundary value problem $(\lambda-A,B)$ with $A=A_0+\widetilde{A}$ and $B=B_0+\widetilde{B}$, where $(\lambda-A_0,B_0)$ is as in Theorem* 3.2.3*, $\widetilde{A}=(\widetilde{A}_{ij})_{1\leqslant i,j\leqslant N}$ with*

$$\widetilde{A}_{ij}=\widetilde{A}_{ij}(x,D)=\sum_{|\alpha|\leqslant s_i+t_j}\widetilde{a}^{\alpha}_{ij}(x)D^{\alpha}$$

and $\widetilde{B}=(\widetilde{B}_{\ell j})_{1\leqslant\ell\leqslant M,\,1\leqslant j\leqslant N}$ with

$$\widetilde{B}_{\ell j}=\widetilde{B}_{\ell j}(x,D)=\sum_{|\beta|\leqslant m_\ell+t_j}\widetilde{b}^{\beta}_{\ell j}(x)\gamma_0D^{\beta}$$

for $1\leqslant i,j\leqslant N$ and $1\leqslant\ell\leqslant M$. Moreover, let $r,\sigma\in\mathbb{R}$ with $r>-1+\frac{1}{p}$, and assume (S1) *and* (S3) *to hold. Define*

$$\rho:=\sup_{|\lambda|\geqslant1}\|L(\lambda)\|_{L(\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+),\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+))},$$

where $L(\lambda)$ denotes the inverse operator of the map in (3-13)*, and choose $\delta=\delta\big(\frac{1}{2\rho},r,\sigma\big)$ and $\lambda_0=\max\big\{\lambda_0\big(M_{\widetilde{A},\widetilde{B}},\frac{1}{2\rho}\big),1\big\}$ as in Lemma 3.2.4 (ii). If*

$$\max_{|\alpha|=s_i+t_j}\|\widetilde{a}^{\alpha}_{ij}\|_{\infty}+\max_{|\beta|=m_\ell+t_j}\|\widetilde{b}^{\beta}_{\ell j}\|_{\infty}<\delta$$

for $1\leqslant i,j\leqslant N$ and $1\leqslant\ell\leqslant M$, then

$$\begin{pmatrix}\lambda-A\\ B\end{pmatrix}:\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)\to\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)\tag{3-17}$$

is an isomorphism for $\lambda\in\Lambda$ with $|\lambda|\geqslant\lambda_0$.

Proof. We first note that by Theorem 3.2.3, the inverse $L(\lambda)$ of the map in (3-13) exists, so we can write

$$\begin{pmatrix}\lambda-A\\ B\end{pmatrix}=\begin{pmatrix}\lambda-A_0\\ B_0\end{pmatrix}+\begin{pmatrix}-\widetilde{A}\\ \widetilde{B}\end{pmatrix}=\begin{pmatrix}\lambda-A_0\\ B_0\end{pmatrix}\bigg(1+L(\lambda)\begin{pmatrix}-\widetilde{A}\\ \widetilde{B}\end{pmatrix}\bigg).$$

Choosing δ and λ_0 as stated and applying Lemma 3.2.4 (ii), we see that

$$\left\| \begin{pmatrix} -\widetilde{A} \\ \widetilde{B} \end{pmatrix} \right\|_{L(\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+), \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+))} < \frac{1}{2\rho}$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| \geqslant \lambda_0$ provided

$$\max_{|\alpha|=s_i+t_j} \|\widetilde{a}^{\alpha}_{ij}\|_{\infty} + \max_{|\beta|=m_\ell+t_j} \|\widetilde{b}^{\beta}_{\ell j}\|_{\infty} < \delta$$

for $1 \leqslant i, j \leqslant N$ and $1 \leqslant \ell \leqslant M$. Therefore

$$\left\| L(\lambda) \begin{pmatrix} -\widetilde{A} \\ \widetilde{B} \end{pmatrix} \right\|_{L(\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+))} < \frac{1}{2},$$

which allows us to use the Neumann series as above, yielding the desired isomorphism. □

3.2 b) General boundary value problems

The analysis of the general case of variable coefficients is based on the classical method of freezing the coefficients. In the following, let $(\lambda - A, B)$ be a system of boundary value problems in $\mathbb{R}^n_+$ of the form (3-10)–(3-11) which is parameter-elliptic in Λ for all $x \in \overline{\mathbb{R}^n_+} \cup \{\infty\}$. Let $(r, \sigma) \in \mathbb{R}^2$, and assume the validity of (S1)–(S4).

For every point $x_0 \in \mathbb{R}^n_+$, we consider the model problem $\lambda - A_0(x_0, D)$ with frozen coefficients $a^{\alpha}_{ij}(x_0) \in \mathbb{C}$ and without lower-order terms. By the assumption of parameter-ellipticity, we can apply Lemma 3.2.2 and obtain the existence of the inverse operator

$$(\lambda - A_0(x_0, D))^{-1} \in L(\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n), \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n)).$$

In the same way, for $x_0 \in \mathbb{R}^{n-1} \cup \{\infty\}$ and $r > -1 + \frac{1}{p}$, we obtain from Theorem 3.2.3 the existence of the inverse operator

$$L_{x_0}(\lambda) := \begin{pmatrix} \lambda - A_0(x_0, D) \\ B_0(x_0, D) \end{pmatrix}^{-1} \in L(\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+), \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)).$$

3.2.7 Lemma. *With the above notation, we have*

$$\begin{aligned} \rho_{A,B} := & \sup_{\substack{x_0 \in \mathbb{R}^n_+ \\ \lambda \in \Lambda,\, |\lambda| \geqslant 1}} \|(\lambda - A_0(x_0, D))^{-1}\|_{L(\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n), \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n))} \\ & + \sup_{\substack{x_0 \in \mathbb{R}^{n-1} \\ \lambda \in \Lambda,\, |\lambda| \geqslant 1}} \|L_{x_0}(\lambda)\|_{L(\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+), \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+))} < \infty. \end{aligned}$$

Proof. Let us consider the first supremum and assume this supremum to be infinite. Then there exist sequences $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^n_+$ and $(\lambda_k)_{k\in\mathbb{N}} \subset \Lambda$ with $|\lambda_k| \geqslant 1$ such that

$$\|(\lambda_k - A_0(x_k, D))^{-1}\|_{L(\mathbb{F}^{r,\sigma}_{\lambda_k}(\mathbb{R}^n), \mathbb{E}^{r,\sigma}_{\lambda_k}(\mathbb{R}^n))} \to \infty \quad (k \to \infty).$$

By passing to a subsequence we may assume that $x_k \to x^*$ for $k \to \infty$ where either $x^* \in \overline{\mathbb{R}^n_+}$ or $x^* = \infty$. Now write

$$\lambda_k - A_0(x_k, D) = \lambda_k - A_0(x^*, D) - \widetilde{A}^k(D), \qquad \widetilde{A}^k(D) := A_0(x_k, D) - A_0(x^*, D).$$

Since $A_0(x^*,D)$ satisfies the assumptions of Lemma 3.2.2, we get

$$\lambda_k - A_0(x_k,D) = (\lambda_k - A_0(x^*,D))\big[1 - (\lambda_k - A_0(x^*,D))^{-1}\widetilde{A}^k(D)\big].$$

Now let

$$\rho^* = \sup_{\lambda\in\Lambda,\,|\lambda|\geqslant 1} \|(\lambda - A_0(x^*,D))^{-1}\|_{L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n),\mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n))}$$

which is finite due to Lemma 3.2.2. Using Lemma 3.1.11 (i), we observe similar as in the proof of Lemma 3.2.4 that

$$\sup_{\lambda\in\Lambda,\,|\lambda|\geqslant 1} \|\widetilde{A}^k(D)\|_{L(\mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n),\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n))} \leqslant \frac{1}{2\rho^*}$$

for sufficiently large k, since the (constant) coefficients of $\widetilde{A}^k(D)$ tend to zero for $k\to\infty$. It follows that

$$\sup_{\lambda\in\Lambda,\,|\lambda|\geqslant 1} \|(\lambda - A_0(x^*,D))^{-1}\widetilde{A}^k(D)\|_{L(\mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n))} \leqslant \frac{1}{2}$$

for all sufficiently large k. As above, using the Neumann series, we conclude that

$$\|(\lambda_k - A_0(x_k,D))^{-1}\|_{L(\mathbb{F}^{r,\sigma}_{\lambda_k}(\mathbb{R}^n),\mathbb{E}^{r,\sigma}_{\lambda_k}(\mathbb{R}^n))} \leqslant 2\|(\lambda_k - A_0(x^*,D))^{-1}\|_{L(\mathbb{F}^{r,\sigma}_{\lambda_k}(\mathbb{R}^n),\mathbb{E}^{r,\sigma}_{\lambda_k}(\mathbb{R}^n))} \leqslant 2\rho^*$$

for all sufficiently large k. This is a contradiction.

To prove that also the second supremum appearing in the definition of $\rho_{A,B}$ is finite works analogously. □

3.2.8 Remark. In the following, we construct a finite covering of $\mathbb{R}^n_+$ consisting of balls and the complement of a ball centered in the origin. Afterwards, we need to extend the coefficients of the localized problems to $\mathbb{R}^n$, $\mathbb{R}^n_+$, and $\mathbb{R}^{n-1}$, respectively. To this end, we will use a general extension function. We fix $\chi\in C^\infty([0,\infty))$ with $0\leqslant\chi\leqslant 1$, $\chi(z)=1$ for $0\leqslant z\leqslant 1$ and $\chi(z)=0$ for $z\geqslant 2$ and define the function $\chi_U\colon\mathbb{R}^n\to\mathbb{R}^n\cup\{\infty\}$ via

$$\chi_U(x) := \begin{cases} \frac{x}{|x|^2} + \chi\big(\frac{r'}{|x|}\big)\big(x - \frac{x}{|x|^2}\big) & \text{if there exists } r'>0:\ U=\mathbb{R}^n\backslash B(0,r'),\\ y + \chi\big(\frac{|x-y|}{r'}\big)(x-y) & \text{if there exist } r'>0,\ y\in\mathbb{R}^n:\ U=B(y,r'). \end{cases}$$

The function χ_U coincides with the identity on U and is later compatible with the parameter-ellipticity of the local operators. Since we use reflection techniques for the construction of χ_U, it is crucial that our covering consists of balls and the complement of a ball centered in the origin.

For the localization, we first apply Lemma 3.2.4 with $\varepsilon := \frac{1}{2\rho_{A,B}}$, where $\rho_{A,B}$ is taken from Lemma 3.2.7. We fix

$$\delta_0 := \delta\Big(\frac{1}{2\rho_{A,B}}, r, \sigma\Big) > 0 \tag{3-18}$$

as being defined in Lemma 3.2.4. Let $x_0 := \infty$ and $U_0 := \{x\in\mathbb{R}^n : |x| > r_0\}$ where r_0 is sufficiently large such that

$$\max_{|\alpha|=s_i+t_j} \|a^\alpha_{ij}(\cdot) - a^\alpha_{ij}(x_0)\|_{L^\infty(\tilde{U}_0\cap\mathbb{R}^n_+)} + \max_{|\beta|=m_\ell+t_j} \|b^\beta_{\ell j}(\cdot) - b^\beta_{\ell j}(x_0)\|_{L^\infty(\tilde{U}_0\cap\mathbb{R}^{n-1})} < \delta_0 \tag{3-19}$$

for $1 \leqslant i,j \leqslant N$ and $1 \leqslant \ell \leqslant M$ with $\widetilde{U}_0 := \left\{x \in \mathbb{R}^n : |x| > \frac{r_0}{2}\right\}$. This is possible due to (S2) and (S4). As the coefficients of A and B are continuous and $\overline{B(0,r_0)} \cap \mathbb{R}^{n-1}$ is compact, there exists a finite covering $\mathbb{R}^{n-1} \subset \bigcup_{k=0}^{K_0} U_k$ with $U_k := B(x_k, r_k) \subset \mathbb{R}^n$ for $k = 1, \dots, K_0$. Here, $x_k \in \mathbb{R}^{n-1}$ and $r_k > 0$ are chosen such that

$$\max_{|\alpha|=s_i+t_j} \|a_{ij}^\alpha(\cdot) - a_{ij}^\alpha(x_k)\|_{L^\infty(\widetilde{U}_k \cap \mathbb{R}^n_+)} + \max_{|\beta|=m_\ell+t_j} \|b_{\ell j}^\beta(\cdot) - b_{\ell j}^\beta(x_k)\|_{L^\infty(\widetilde{U}_k \cap \mathbb{R}^{n-1})} < \delta_0 \tag{3-20}$$

with $\widetilde{U}_k := B(x_k, 2r_k)$ for $k = 1, \dots, K_0$, $1 \leqslant i,j \leqslant N$ and $1 \leqslant \ell \leqslant M$. We set

$$\delta_{\max} := \sup\left\{\delta > 0 : \mathbb{R}^{n-1} \times [0,\delta] \subset \bigcup_{k=0}^{K_0} U_k\right\}.$$

Similarly, as $\mathbb{R}^n_+ \setminus \bigcup_{k=0}^{K_0} U_k$ is compact, we can choose $x_k \in \mathbb{R}^n_+$ and $0 < r_k < \frac{\delta_{\max}}{2}$ such that $U_k := B(x_k, r_k) \subset \left\{z \in \mathbb{R}^n \,|\, z_n > \frac{\delta_{\max}}{2}\right\}$ as well as $\overline{\mathbb{R}^n_+} \subset \bigcup_{k=0}^{K} U_k$ and

$$\max_{|\alpha|=s_i+t_j} \|a_{ij}^\alpha(\cdot) - a_{ij}^\alpha(x_k)\|_{L^\infty(\widetilde{U}_k)} < \delta_0 \tag{3-21}$$

with $\widetilde{U}_k := B(x_k, 2r_k)$ for $k = K_0+1, \dots, K$ and $1 \leqslant i,j \leqslant N$.

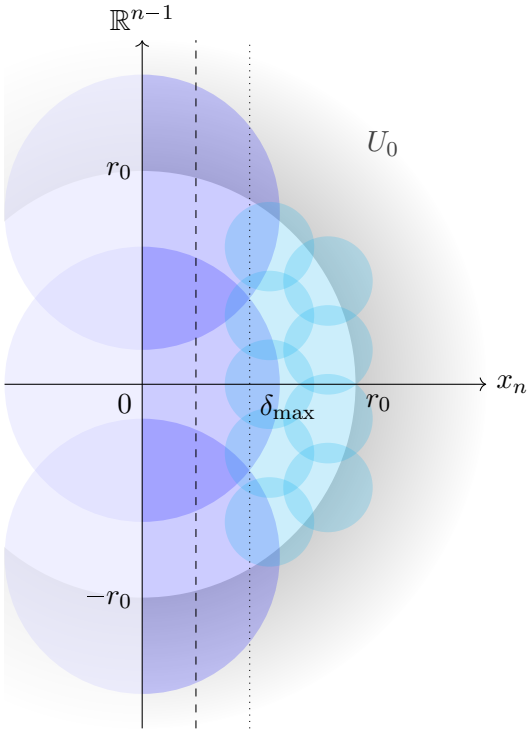

Figure 3.2: Example of a finite covering of $\overline{\mathbb{R}^n_+}$ with $K_0 = 3$ and $K = 12$. The grey part visualizes U_0 while the dark blue circles with centre points on $\mathbb{R}^{n-1}$ form a covering of the part of the boundary $\mathbb{R}^{n-1}$ which is not already covered by U_0. The light blue circles are a finite covering of the remaining part of $\overline{\mathbb{R}^n_+}$. Notice that they are contained in $\left\{z \in \mathbb{R}^n : z_n > \frac{\delta_{\max}}{2}\right\}$. Additionally, the radii of those circles are at most $\frac{\delta_{\max}}{2}$.

3.2.9 Remark (Local operators and extensions)**.** Let $x_0, \dots, x_K$ be chosen as above. Starting from the coefficient functions a_{ij}^{α} and $b_{\ell j}^{\beta}$ let us define

$$a_{ij}^{\alpha k}(x) := a_{ij}^{\alpha}(\chi_{U_k}(x)) \qquad (x \in \overline{\mathbb{R}^n_+}),$$
$$b_{\ell j}^{\beta k}(x) := b_{\ell j}^{\beta}(\chi_{U_k}(x)) \qquad (x \in \mathbb{R}^{n-1})$$

for $k = 0, \dots, K_0$ and, for $k = K_0 + 1, \dots, K$,

$$a_{ij}^{\alpha k}(x) := a_{ij}^{\alpha}(\chi_{U_k}(x)) \qquad (x \in \mathbb{R}^n),$$

where $1 \leqslant i, j \leqslant N$ and $1 \leqslant \ell \leqslant M$. Here, the function χ_{U_k} is defined as in Remark 3.2.8. These new coefficients have the same smoothness as before. The function $a_{ij}^{\alpha k}$ coincides with a_{ij}^{α} on $U_k \cap \overline{\mathbb{R}^n_+}$ and U_k, respectively, $b_{\ell j}^{\beta k}$ coincides with $b_{\ell j}^{\beta}$ on $U_k \cap \mathbb{R}^{n-1}$. By (3-19)–(3-21), we have

$$\begin{aligned} \big\| a_{ij}^{\alpha k}(\cdot) - a_{ij}^{\alpha}(x_k) \big\|_{L^\infty(\mathbb{R}^n_+)} &< \delta_0 \quad \text{for } k = 0, \dots, K_0, \\ \big\| a_{ij}^{\alpha k}(\cdot) - a_{ij}^{\alpha}(x_k) \big\|_{L^\infty(\mathbb{R}^n)} &< \delta_0 \quad \text{for } k = K_0 + 1, \dots, K, \\ \big\| b_{\ell j}^{\beta k}(\cdot) - b_{\ell j}^{\beta}(x_k) \big\|_{L^\infty(\mathbb{R}^{n-1})} &< \delta_0 \quad \text{for } k = 0, \dots, K_0. \end{aligned} \tag{3-22}$$

With the new coefficient functions we associate the operator matrices $A^k = (A_{ij}^k)_{1 \leqslant i,j \leqslant N}$ and $B^k = (B_{\ell j}^k)_{1 \leqslant \ell \leqslant M, 1 \leqslant j \leqslant N}$ via

$$A_{ij}^k = A_{ij}^k(x, D) := \sum_{|\alpha| \leqslant s_i + t_j} a_{ij}^{\alpha k}(x) D^{\alpha}, \qquad B_{\ell j}^k = B_{\ell j}^k(x, D) := \sum_{|\beta| \leqslant m_\ell + t_j} b_{\ell j}^{\beta k}(x) \gamma_0 D^{\beta}.$$

We remark that the localization procedure contains a subtlety concerning the constants δ and λ_0 in Lemma 3.2.4, Lemma 3.2.5 and Theorem 3.2.6. We defined the neighbourhoods U_k and the radii r_k in dependence of δ_0 which depends only on $\rho_{A,B}$, r, and σ, see (3-18). For the new coefficients $a_{ij}^{\alpha k}$, $b_{\ell j}^{\beta k}$, the $\| \cdot \|_\infty$-norm still satisfies the desired smallness conditions, as seen in (3-22). However, as χ_{U_k} appears in the definition of the new coefficients, the BUC^{ρ_i}-norm and BUC^{k_ℓ}-norm of the new coefficients, respectively, depend on U_k and therefore on the radius r_k. Here, it is important that δ_0 does not depend on the BUC^{ρ_i}-norm (in contrast to λ_0, see Lemma 3.2.4). Due to this, the above modification of the coefficients might lead to a larger constant λ_0, but we do not have to redefine the radii r_k, which prevents a circular reasoning in the definition of U_k.

3.2.10 Remark. The additional assumption on the derivatives of the functions $a_{ij}^{\alpha}\big(\frac{x}{|x|^2}\big)$ and $b_{\ell j}^{\beta}\big(\frac{x}{|x|^2}\big)$ in (S2) and (S4) has technical reasons to preserve the ellipticity of the local operators. It might be possible to omit this assumption by using the same extension $a_{ij}^{\alpha 0}$ of the local coefficients as before, that is $a_{ij}^{\alpha 0}(x) = a_{ij}^{\alpha}(\chi_{U_0}(x))$, but only for $x \notin B(0, r')$ with $0 < r' < \frac{r_0}{2}$ sufficiently small such that we are close to $a_{ij}^{\alpha}(\infty)$ on the boundary of $B(0, r')$. As the ellipticity condition is open, we can use (smoothed) convex combinations of the coefficients on the sphere $a_{ij}^{\alpha 0}\big(\frac{r'x}{|x|}\big)$ and $a_{ij}^{\alpha}(\infty)$ for $x \in B(0, r') \cap \overline{\mathbb{R}^n_+}$ to preserve the ellipticity. In difference to the approach presented above, this changes in general the range of the coefficient functions.

3.2.11 Lemma. *Let $r, \sigma \in \mathbb{R}$ with $r > -1 + \frac{1}{p}$, and assume* (S1)–(S4) *to hold. Then there exists a $\lambda_0 \geqslant 1$ such that the operators*

$$\begin{pmatrix} \lambda - A^k \\ B^k \end{pmatrix} : \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+) \to \mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+) \quad (k = 0, \dots, K_0),$$
$$\lambda - A^k : \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n) \to \mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n) \quad (k = K_0 + 1, \dots, K)$$

defined in Remark 3.2.9 are isomorphisms for every $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$. We denote the inverse operators by $L_k(\lambda)$ for $k = 0, \dots, K$.

Proof. We split the operators into a part with constant coefficients and a perturbation, that is, $A^k = A^k(x, D) = A^k_0(x_k, D) + \widetilde{A}^k(x, D)$. The operator B^k can be decomposed in a similar way. Due to the smallness property (3-22), the considered operators thus fit into the setting of Lemma 3.2.5 and Theorem 3.2.6, respectively. This yields the assertion. □

In the following, we will fix a smooth partition of unity $\varphi_k \in C^\infty(\mathbb{R}^n)$, $k = 0, \dots, K$, with $\operatorname{supp} \varphi_k \subset U_k$, $0 \leqslant \varphi_k \leqslant 1$, and $\sum_{k=0}^K \varphi_k = 1$ on $\overline{\mathbb{R}^n_+}$. In addition, we fix functions $\psi_k \in C^\infty(\mathbb{R}^n)$ with $0 \leqslant \psi_k \leqslant 1$, $\operatorname{supp} \psi_k \subset U_k$ and $\psi_k = 1$ on $\operatorname{supp} \varphi_k$. We can solve (3-9) locally in U_k, using the extended local operators in the half-space and in the whole space and their inverses $L_k(\lambda)$. However, the solution operators $L_k(\lambda)$ are not local, so we have to multiply the half-space solution by ψ_k. In this way, commutators appear, which are estimated in the following lemma. We write $[\cdot, \cdot]$ for the standard commutator and use the notation ψ_k also for the operator of multiplication by ψ_k. For the boundary operators, the commutator $[B^k, \psi_k]$ is defined as

$$[B^k, \psi_k]w = B^k(\psi_k w) - (\gamma_0 \psi_k) B^k w.$$

3.2.12 Lemma. *Let $r, \sigma \in \mathbb{R}$ with $r > -1 + \frac{1}{p}$, and assume* (S1)–(S4) *to hold. Let $R_0(\lambda)$ be defined on $\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+)$ by*

$$R_0(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} := \sum_{k=0}^{K_0} \psi_k L_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} + \sum_{k=K_0+1}^{K} \psi_k L_k(\lambda)(\varphi_k f). \tag{3-23}$$

Then

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} R_0(\lambda) = 1 + C(\lambda), \tag{3-24}$$

where $C(\lambda) \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+))$, and there exists a $\lambda_0 \geqslant 1$ such that $1 + C(\lambda) \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+))$ is invertible for all $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$.

Proof. As a first step of the proof, we show that the operators

$$C_k(\lambda) := \begin{pmatrix} -[A^k, \psi_k] \\ [B^k, \psi_k] \end{pmatrix} L_k(\lambda) \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+)) \qquad (k = 0, \dots, K_0),$$
$$C_k(\lambda) := -[A^k, \psi_k] L_k(\lambda) \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n)) \qquad (k = K_0 + 1, \dots, K)$$

have arbitrary small operator norm for sufficiently large λ. In the following, we only consider the case $k = 0, \dots, K_0$, since the proof for $k = K_0 + 1, \dots, K$ is analogous (and simpler).

To this end, let $w \in \mathbb{E}^{r-1,\sigma}_{\lambda}(\mathbb{R}^n_+)$. For $1 \leqslant i \leqslant N$ we have

$$([A^k, \psi_k]w)_i = \sum_{j=1}^{N} [A^k_{ij}, \psi_k] w_j.$$

The operator $[A^k_{ij}, \psi_k]$ is a differential operator of order not greater than $s_i + t_j - 1$. Therefore,

$$[A^k, \psi_k] \colon \mathbb{E}^{r-1,\sigma}_{\lambda}(\mathbb{R}^n_+) \to H^{-\mathbf{s}+r,\sigma}_{p,\lambda}(\mathbb{R}^n_+)$$

is continuous. Similarly, for the boundary operators, it holds

$$([B^k, \psi_k]w)_\ell = \sum_{j=1}^{N} [B^k_{\ell j}, \psi_k] w_j$$

for $1 \leqslant \ell \leqslant M$. Since the operator $[B^k_{\ell j}, \psi_k]$ is a boundary operator of order not greater than $m_\ell + t_j - 1$, $[B^k, \psi_k]$ is continuous as an operator

$$[B^k, \psi_k] \colon \mathbb{E}^{r-1,\sigma}_{\lambda}(\mathbb{R}^n_+) \to B^{-\mathbf{m}+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1}).$$

Similar as in (3-16), we get

$$\begin{aligned}
\left\| C_k(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)} &\leqslant C \left\| L_k(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathbb{E}^{r-1,\sigma}_{\lambda}(\mathbb{R}^n_+)} \\
&\leqslant C\langle\lambda\rangle^{-1} \left\| L_k(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)} \\
&< \varepsilon \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)}
\end{aligned}$$

for given $\varepsilon > 0$, $(f, g) \in \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)$, $k = 0, \dots, K_0$, and sufficiently large λ, since $L_k(\lambda)$ is bounded from $\mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)$ to $\mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)$ by Lemma 3.2.11.

Now, we compute

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} R_0(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} = \sum_{k=0}^{K_0} \begin{pmatrix} \lambda - A \\ B \end{pmatrix} \psi_k L_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} + \sum_{k=K_0+1}^{K} \begin{pmatrix} \lambda - A \\ B \end{pmatrix} \psi_k L_k(\lambda)(\varphi_k f)$$

and treat each term separately. For $k = 0, \dots, K_0$, we obtain

$$\begin{aligned}
\begin{pmatrix} \lambda - A \\ B \end{pmatrix} \psi_k L_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} &= \begin{pmatrix} \lambda - A^k \\ B^k \end{pmatrix} \psi_k L_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} \\
&= \left[\begin{pmatrix} \psi_k(\lambda - A^k) \\ (\gamma_0 \psi_k) B^k \end{pmatrix} L_k(\lambda) + \begin{pmatrix} -[A^k, \psi_k] \\ [B^k, \psi_k] \end{pmatrix} L_k(\lambda) \right] \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} \\
&= \begin{pmatrix} \psi_k \varphi_k f \\ (\gamma_0 \psi_k)(\gamma_0 \varphi_k) g \end{pmatrix} + C_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} \\
&= \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} + C_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix}.
\end{aligned}$$

For $k = K_0 + 1, \dots, K$, we obtain in the same way

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} \psi_k L_k(\lambda)(\varphi_k f) = \begin{pmatrix} \varphi_k f + C_k(\lambda)(\varphi_k f) \\ 0 \end{pmatrix}.$$

Summing up over k yields

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} R_0(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} = (1 + C(\lambda)) \begin{pmatrix} f \\ g \end{pmatrix}$$

with

$$C(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} := \sum_{k=0}^{K_0} C_k(\lambda) \begin{pmatrix} \varphi_k f \\ (\gamma_0 \varphi_k) g \end{pmatrix} + \sum_{k=K_0+1}^{K} \begin{pmatrix} C_k(\lambda)(\varphi_k f) \\ 0 \end{pmatrix}.$$

Note that for sake of readability we have dropped the extensions and restrictions from our notation, here. More precisely, the upper entry in the last term above would be $r_{\mathbb{R}^n_+} C_k(\lambda) e^0_{\mathbb{R}^n_+} \varphi_k f$.

From the above commutator estimates and the fact that multiplication by φ_k preserves the smoothness, we obtain $C(\lambda) \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+))$.

Using the Neumann series as in Lemma 3.2.5, we obtain that the operator $1 + C(\lambda) \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+))$ is invertible for sufficiently large λ, and the norm of its inverse is not greater than 2. □

The last result provides a solution operator for the boundary value problem (3-9). To show uniqueness, the following observation will be useful.

3.2.13 Lemma. *Let E, F be Banach spaces, and let $T \in L(E, F)$ be a retraction, that is, there exists $R \in L(F, E)$ with $TR = \mathrm{id}_F$. Let E_0 be a dense subset of E. If $T\big|_{E_0} : E_0 \to F$ is injective, then T is injective.*

Proof. Let $f \in F$ and $w \in E$ with $Tw = f$. We choose a sequence $(w_l)_{l \in \mathbb{N}} \subset E_0$ with $w_l \to w$ in E for $l \to \infty$. As $T\big|_{E_0}$ is injective, we have $w_l = Rf_l$, where $f_l := Tw_l$. With the continuity of T, we see $f_l = Tw_l \to Tw = f$ in F, and from the continuity of R we get $w_l = Rf_l \to Rf$ in E. As the limit is unique, this yields $w = Rf$, which shows the injectivity of T. □

The following theorem is the key result of this section.

3.2.14 Theorem. *Let $p \in (1, \infty)$ and $r, \sigma \in \mathbb{R}$ with $r > -1 + \frac{1}{p}$. Let $(\lambda - A, B)$ be a system of boundary value problems in $\mathbb{R}^n_+$ of the form* (3-10)–(3-11) *which is parameter-elliptic in Λ for all $x \in \overline{\mathbb{R}^n_+} \cup \{\infty\}$, and assume* (S1)–(S4) *to hold. Then, there exists a $\lambda_0 \geqslant 1$ such that for every $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$, the operator*

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} : \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+) \to \mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+) \tag{3-25}$$

is an isomorphism. Its inverse is given by

$$R(\lambda) = R_0(\lambda)(1 + C(\lambda))^{-1} \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+), \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+)),$$

where $R_0(\lambda)$ and $C(\lambda)$ are defined in Lemma 3.2.12.

Proof. Let λ_0 be as in Lemma 3.2.12. We have

$$R(\lambda) = R_0(\lambda)(1 + C(\lambda))^{-1} \in L(\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+), \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+))$$

by Lemma 3.2.11 and Lemma 3.2.12. From (3-24) we obtain

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} R(\lambda) = (1 + C(\lambda))(1 + C(\lambda))^{-1} = \mathrm{id}_{\mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+)}\,.$$

Therefore, the operator in (3-25) is surjective. To show injectivity, that is, uniqueness of the solution, we remark that $\mathbb{F}^{r,0}_\lambda(\mathbb{R}^n_+)$ and $\mathbb{E}^{r,0}_\lambda(\mathbb{R}^n_+)$ are classical spaces. Thus, we obtain unique solvability in these spaces, see Theorem 2.5.6. In particular, the restriction of the operator (3-25) to $\mathscr{S}(\mathbb{R}^n_+)$ is injective. Now, we can apply Lemma 3.2.13 with $T = \binom{\lambda - A}{B}$ and $R = R(\lambda)$ in the spaces $E = \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+)$, $F = \mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+)$, and $E_0 = \mathscr{S}(\mathbb{R}^n_+)$. □

3.2.15 Corollary. *In the situation of Theorem 3.2.14, let additionally $\sigma \in (-\infty, 0]$. Then, there exists a $\lambda_0 \geqslant 1$ such that for every $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$ and*

$$(f, g) \in H^{-\mathbf{s}+r}_{p,\lambda}(\mathbb{R}^n_+) \times B^{-\mathbf{m}+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})$$

the boundary value problem (3-9) *has a unique solution $w \in \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+)$. In particular, we have $w \in H^{\mathbf{t}+r+\sigma}_{p,\lambda}(\mathbb{R}^n_+)$ and*

$$\|w\|_{H^{\mathbf{t}+r+\sigma}_{p,\lambda}(\mathbb{R}^n_+)} \leqslant C\Big(\|f\|_{H^{-\mathbf{s}+r}_{p,\lambda}(\mathbb{R}^n_+)} + \|g\|_{B^{-\mathbf{m}+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})}\Big)$$

with a constant C independent of λ.

Proof. This follows immediately from Theorem 3.2.14 and the continuous embeddings

$$H^{-\mathbf{s}+r}_{p,\lambda}(\mathbb{R}^n_+) \subset H^{-\mathbf{s}+r,\sigma}_{p,\lambda}(\mathbb{R}^n_+) \quad \text{and} \quad \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+) = H^{\mathbf{t}+r,\sigma}_{p,\lambda}(\mathbb{R}^n_+) \subset H^{\mathbf{t}+r+\sigma}_{p,\lambda}(\mathbb{R}^n_+).$$

□

In Theorem 3.2.14, we considered the half-space case. For an operator A acting in the whole space, the analogue results hold, where the proofs are similar but much simpler, due to the absence of boundary operators. We obtain the following result.

3.2.16 Lemma. *Let $A = A(x, D)$ be an operator matrix of the form* (3-10) *with coefficients $a^\alpha_{ij} \colon \mathbb{R}^n \to \mathbb{C}$, and assume that $\lambda - A$ is parameter-elliptic in Λ. Let $r, \sigma \in \mathbb{R}$, and assume* (S1) *and* (S2) *to hold. Then, there exists a $\lambda_0 \geqslant 1$ such that for every $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$, the operator*

$$\lambda - A \colon \mathbb{E}^{r,\sigma}_\lambda(\mathbb{R}^n) \to \mathbb{F}^{r,\sigma}_\lambda(\mathbb{R}^n)$$

is an isomorphism.

3.3 Systems of boundary value problems in domains

We now consider (3-9) in a bounded or exterior domain. Throughout this section, we assume Ω to be a domain with compact boundary Γ, and $(\lambda - A, B)$ to be a system of boundary value problems which is parameter-elliptic in some closed sector $\Lambda \subset \mathbb{C}$ with vertex at the origin. Moreover, we assume (S1)–(S3) and (S5) to hold.

As the definition of the spaces $H^{s,\sigma}_{p,\lambda}$ is non-canonical in domains, we will only consider standard Sobolev spaces on Ω. For the construction of the solution operators, we will use local coordinates where the spaces $H^{s,\sigma}_{p,\lambda}(\mathbb{R}^n_+)$ are available.

We start with some remarks concerning the localization technique: Let $x_0 \in \Gamma$. Since the domain Ω has a $C^{\max_j t_j + \lceil \rho' \rceil}$-boundary, there is an open set $\widetilde{U}_{x_0}$ containing x_0, a radius $r_{x_0} > 0$ and a $C^{\max_j t_j + \lceil \rho' \rceil}$-diffeomorphism $\vartheta_{x_0} \colon \widetilde{V}_{x_0} \to \widetilde{U}_{x_0}$, where $\widetilde{V}_{x_0} = B(0, 2r_{x_0})$, such that

$$\vartheta_{x_0}(\widetilde{V}_{x_0} \cap \mathbb{R}^n_+) = \widetilde{U}_{x_0} \cap \Omega \quad \text{and} \quad \vartheta_{x_0}(0) = x_0.$$

We set $V_{x_0} := B(0, r_{x_0})$ and $U_{x_0} := \vartheta_{x_0}(V_{x_0})$. By compactness of Γ, there are $x_1, \dots, x_{K_0} \in \Gamma$ and open sets $U_{x_1}, \dots, U_{x_{K_0}}$ as above such that $\Gamma \subset \bigcup_{k=1}^{K_0} U_{x_k}$. For the sake of simplicity, we shall use k instead of x_k as index.

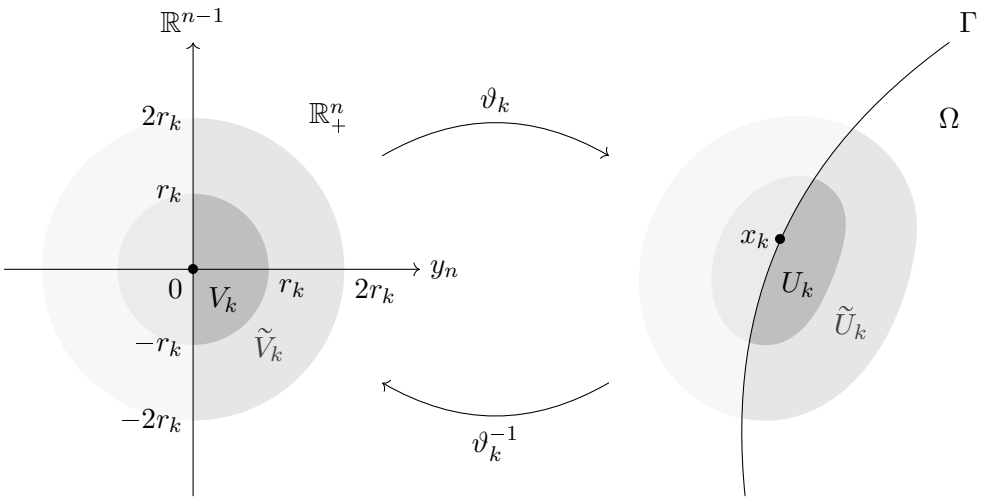

Figure 3.3: Visualisation of the diffeomorphism ϑ_k and its inverse ϑ_k^{-1} for $k \in \{1, \dots, K_0\}$.

We proceed similarly as in the half-space case. Hence, we define

$$\delta_{\max} := \sup \left\{ \delta > 0 : \left\{ x \in \Omega : \inf_{z \in \Gamma} |x - z| \leqslant \delta \right\} \subset \bigcup_{k=1}^{K_0} U_k \right\}.$$

If Ω is bounded, $\Omega \setminus \bigcup_{k=1}^{K_0} U_k$ is compact, and we can choose x_k in Ω and $0 < r_k < \frac{\delta_{\max}}{2}$ such that

$$U_k := B(x_k, r_k) \subset \left\{ x \in \Omega : \inf_{z \in \Gamma} |x - z| > \frac{\delta_{\max}}{2} \right\} \tag{3-26}$$

for $k = K_0 + 1, \dots, K$ and $\overline{\Omega} \subset \bigcup_{k=1}^{K} U_k$.

In the case of an exterior domain, this construction has to be slightly modified. We first define $U_{K_0+1} := \mathbb{R}^n \backslash \overline{B(0, r_{K_0+1})}$, where the radius r_{K_0+1} is chosen such that $\mathbb{R}^n \backslash \Omega \subset B\big(0, \frac{r_{K_0+1}}{2}\big)$. Now $\Omega \backslash \bigcup_{k=1}^{K_0+1} U_k$ is compact, and we choose x_k and r_k with (3-26) for $k = K_0 + 2, \dots, K$ such that again $\overline{\Omega} \subset \bigcup_{k=1}^{K} U_k$. For formal reasons, we define $V_k := U_k$ and $\vartheta_k := \mathrm{id}_{V_k}$ for $k = K_0+1, \dots, K$.

3.3.1 Remark (Local operators and extensions)**.** Let $x_1, \dots, x_K$ be chosen as above. For $k = 1, \dots, K_0$, we define the local operator $\widetilde{A}^k = (\widetilde{A}^k_{ij})_{1 \leqslant i,j \leqslant N}$ as the pullback of the operator A by ϑ_k. More precisely, for $v \in C^\infty(\widetilde{V}_k)$, we write

$$(\widetilde{A}^k_{ij} v)(y) := A_{ij}(v \circ \vartheta_k^{-1})(\vartheta_k(y)) =: \sum_{|\alpha| \leqslant s_i + t_j} \widetilde{a}^{\alpha k}_{ij}(y) D^\alpha v(y) \quad (y \in \widetilde{V}_k \cap \overline{\mathbb{R}^n_+}).$$

The explicit description of the coefficients $\widetilde{a}^{\alpha k}_{ij}$ (Faà di Bruno's formula, see Formula B of [32]) shows that $\widetilde{a}^{\alpha k}_{ij}$ contains the function $a^\alpha_{ij} \circ \vartheta_k$ as well as derivatives of ϑ_k^{-1} up to order $s_i + t_j + 1 - |\alpha|$ for $|\alpha| \geqslant 1$ (and no derivative for $|\alpha| = 0$), concatenated with ϑ_k. Hence, we always need at most $s_i + t_j$ derivatives of ϑ_k^{-1}, which ensures $\widetilde{a}^{\alpha k}_{ij} \in \mathrm{BUC}^{\lceil \rho'_i \rceil}$ because of $\rho' \geqslant s_i + \rho'_i$ for $1 \leqslant i \leqslant N$. Consequently, condition (S5) implies that (S1) also holds for $\widetilde{a}^{\alpha k}_{ij}$.

In the same way, we define the local operator $\widetilde{B}^k = (\widetilde{B}^k_{\ell j})_{1 \leqslant \ell \leqslant M, 1 \leqslant j \leqslant N}$ via

$$(\widetilde{B}^k_{\ell j} v)(y) := B_{\ell j}(v \circ \vartheta_k^{-1})(\vartheta_k(y)) =: \sum_{|\beta| \leqslant m_\ell + t_j} \widetilde{b}^{\beta k}_{\ell j}(y) \gamma_0 D^\beta v(y) \quad (y \in \widetilde{V}_k \cap \mathbb{R}^{n-1})$$

for $k = 1, \dots, K_0$. By an easy case-by-case analysis, we get $\lceil \rho' \rceil \geqslant m_\ell + \lfloor k'_\ell \rfloor + 1 = m_\ell + k_\ell$ and thus (S5) also implies that the coefficients of the transformed operator $\widetilde{B}^k_{\ell j}$ satisfy (S3).

For $k = K_0 + 1, \dots, K$ and $y \in B(x_k, 2r_k)$, we set $\widetilde{a}^{\alpha k}_{ij}(y) := a^\alpha_{ij}(y)$ with some obvious modifications for $k = K_0 + 1$ in the case of an exterior domain.

Again with the general extension function from Remark 3.2.8, we extend the coefficients $\widetilde{a}^{\alpha k}_{ij}$ and $\widetilde{b}^{\beta k}_{\ell j}$ to $\mathbb{R}^n_+$, $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively. We set

$$\begin{aligned}
a^{\alpha k}_{ij}(y) &:= \widetilde{a}^{\alpha k}_{ij}(\chi_{V_k}(y)) \quad (y \in \overline{\mathbb{R}^n_+}) && \text{for } k = 1, \dots, K_0,\\
a^{\alpha k}_{ij}(y) &:= \widetilde{a}^{\alpha k}_{ij}(\chi_{V_k}(y)) \quad (y \in \mathbb{R}^n) && \text{for } k = K_0 + 1, \dots, K,\\
b^{\beta k}_{\ell j}(y) &:= \widetilde{b}^{\beta k}_{\ell j}(\chi_{V_k}(y)) \quad (y \in \mathbb{R}^{n-1}) && \text{for } k = 1, \dots, K_0.
\end{aligned}$$

Finally, we define

$$\begin{aligned}
A^k_{ij} v(y) &:= \sum_{|\alpha| \leqslant s_i + t_j} a^{\alpha k}_{ij}(y) D^\alpha v(y) && (y \in \overline{\mathbb{R}^n_+}) && \text{for } k = 1, \dots, K_0,\\
A^k_{ij} v(y) &:= \sum_{|\alpha| \leqslant s_i + t_j} a^{\alpha k}_{ij}(y) D^\alpha v(y) && (y \in \mathbb{R}^n) && \text{for } k = K_0 + 1, \dots, K,\\
B^k_{\ell j} v(y) &:= \sum_{|\beta| \leqslant m_\ell + t_j} b^{\beta k}_{\ell j}(y) \gamma_0 D^\beta v(y) && (y \in \mathbb{R}^{n-1}) && \text{for } k = 1, \dots, K_0.
\end{aligned}$$

The extended local operators A^k and B^k satisfy the above smoothness and ellipticity assumptions, see Remark 3 in Chapter 4 of [25]. Therefore, we can apply the results from Section 3.2. However, as we do not have the spaces $H^{s,\sigma}_{p,\lambda}$ in domains, we use the standard Sobolev spaces as in Corollary 3.2.15.

Thus, we additionally consider the spaces

$$\mathcal{E}^{r,\sigma}_{\lambda}(\Omega) := H^{\mathbf{t}+r+\sigma}_{p,\lambda}(\Omega), \tag{3-27}$$

$$\mathcal{F}^{r,\sigma}_{\lambda}(\Omega) := H^{-\mathbf{s}+r}_{p,\lambda}(\Omega) \times B^{-\mathbf{m}+r+\sigma-1/p}_{pp,\lambda}(\Gamma) \tag{3-28}$$

and the analogue spaces with Ω being replaced by $\mathbb{R}^n_+$. We also set $\mathcal{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n) := H^{\mathbf{t}+r+\sigma}_{p,\lambda}(\mathbb{R}^n)$ and $\mathcal{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n) := H^{-\mathbf{s}+r}_{p,\lambda}(\mathbb{R}^n)$. Note that for $\sigma \leqslant 0$ we have the continuous embeddings

$$\mathcal{F}^{r,\sigma}_{\lambda} \subset \mathbb{F}^{r,\sigma}_{\lambda} \quad \text{and} \quad \mathbb{E}^{r,\sigma}_{\lambda} \subset \mathcal{E}^{r,\sigma}_{\lambda}, \tag{3-29}$$

see Proposition 3.1.2 (iii).

3.3.2 Lemma. *Let $r, \sigma \in \mathbb{R}$ with $r > -1 + \frac{1}{p}$, and let A^k, B^k denote the extended local operators. Then there exists a $\lambda_0 \geqslant 1$ such that the operator families*

$$\begin{aligned} \begin{pmatrix} \lambda - A^k \\ B^k \end{pmatrix} &: \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+) \to \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+) \quad (k = 1, \dots, K_0), \\ \lambda - A^k &: \mathbb{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n) \to \mathbb{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n) \quad (k = K_0 + 1, \dots, K) \end{aligned} \tag{3-30}$$

are isomorphisms for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$. We denote the inverse operator by $L_k(\lambda)$. For $\sigma \leqslant 0$, the restrictions of $L_k(\lambda)$ to $\mathcal{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)$ and $\mathcal{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n)$, respectively, yield bounded operator families

$$\begin{aligned} L_k(\lambda) &\in L(\mathcal{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+), \mathcal{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n_+)) \quad (k = 1, \dots, K_0), \\ L_k(\lambda) &\in L(\mathcal{F}^{r,\sigma}_{\lambda}(\mathbb{R}^n), \mathcal{E}^{r,\sigma}_{\lambda}(\mathbb{R}^n)) \quad (k = K_0 + 1, \dots, K). \end{aligned} \tag{3-31}$$

Proof. We have already seen that A^k, B^k satisfy conditions (S1) and (S3). Conditions (S2) and (S4) follow directly from the fact that the extended coefficients are constant far away from the origin by construction. Hence the statement follows for $k \in \{1, \dots, K_0\}$ from Theorem 3.2.14 and Corollary 3.2.15 and for $k \in \{K_0 + 1, \dots, K\}$ from Lemma 3.2.16 and the embeddings (3-29). □

To solve (3-9) in Ω, we first construct an approximate solution operator $R_0(\lambda)$, using the local solution operators $L_k(\lambda)$ from Lemma 3.3.2 and the local coordinate maps ϑ_k for $k = 1, \dots, K$. Setting $\Theta_k v := v \circ \vartheta_k^{-1}$, the $C^{\max_j t_j + \lceil \rho' \rceil}$-diffeomorphisms ϑ_k induce isomorphisms

$$\begin{aligned} \Theta_k &: H^r_{p,\lambda}(V_k \cap \mathbb{R}^n_+) \to H^r_{p,\lambda}(U_k \cap \Omega) \quad (k = 1, \dots, K_0), \\ \Theta_k &: H^r_{p,\lambda}(V_k) \to H^r_{p,\lambda}(U_k) \quad (k = K_0 + 1, \dots, K) \end{aligned} \tag{3-32}$$

for $r \in [0, \max_j t_j + \lceil \rho' \rceil]$. Since we have $\Theta_k(\dot{H}^r_{p,\lambda}(\overline{V_k \cap \mathbb{R}^n_+})) = \dot{H}^r_{p,\lambda}(\overline{U_k \cap \Omega})$, we even get (3-32) for all $|r| \leqslant \max_j t_j + \lceil \rho' \rceil$ via duality, see Theorem 3.5 of [75].

We fix a smooth partition of unity $(\varphi^{\Omega}_k)_{1 \leqslant k \leqslant K} \subset C^{\infty}(\mathbb{R}^n)$ with $\operatorname{supp} \varphi^{\Omega}_k \subset U_k$, $0 \leqslant \varphi^{\Omega}_k \leqslant 1$ and $\sum_{k=1}^{K} \varphi^{\Omega}_k = 1$ on $\overline{\Omega}$. Additionally, let $(\psi^{\Omega}_k)_{1 \leqslant k \leqslant K} \subset C^{\infty}(\mathbb{R}^n)$ with $0 \leqslant \psi^{\Omega}_k \leqslant 1$, $\operatorname{supp} \psi^{\Omega}_k \subset U_k$ and $\psi^{\Omega}_k = 1$ on $\operatorname{supp} \varphi^{\Omega}_k$. We set $\psi_k := \Theta_k^{-1} \psi^{\Omega}_k = \psi^{\Omega}_k \circ \vartheta_k$ for $k = 1, \dots, K$, where here and in the following, we identify functions with compact support and their trivial extensions for sake of readability. Without this identification, we have for instance $\psi_k = e^0_{V_k} \Theta_k^{-1}(r_{U_k} \psi^{\Omega}_k)$ for $k = K_0 + 1, \dots, K$, where again r_{U_k} stands for the restriction to U_k and $e^0_{V_k}$ for the trivial extension to $\mathbb{R}^n$ by zero.

Note that for $r \in [0, \max_j t_j + \lceil \rho' \rceil]$ the mappings $\Theta_k \colon H_p^{-r}(V_k \cap \mathbb{R}^n_+) \to H_p^{-r}(U_k \cap \Omega)$ and $e^0_{V_k} \colon \mathscr{D}'(V_k) \to \mathscr{D}'(\mathbb{R}^n)$ are defined via

$$\begin{aligned} (\Theta_k v)(u) &= v(\Theta_k^{-1} u) = v(u \circ \vartheta_k) \quad && (v \in H_p^{-r}(V_k \cap \mathbb{R}^n_+),\ u \in \dot{H}_q^r(\overline{U_k \cap \Omega})), \\ (e^0_{V_k} v)(\varphi) &= v((\varphi \psi_v)\big|_{V_k}) && (v \in \mathscr{D}'(V_k),\ \varphi \in \mathscr{D}(\mathbb{R}^n)) \end{aligned}$$

for $\psi_v \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp} \psi_v \subset V_k$, $\psi_v = 1$ on $\operatorname{supp} v$ and $\frac{1}{p} + \frac{1}{q} = 1$. This definition does not depend on the concrete choice of ψ_v.

Moreover, by the definition of Besov spaces on the closed $C^{\max_j t_j + \lceil \rho' \rceil}$-manifold Γ, the restriction $\vartheta_k\big|_{V_k \cap \mathbb{R}^{n-1}} \colon V_k \cap \mathbb{R}^{n-1} \to U_k \cap \Gamma$ induces the continuous mappings

$$\begin{aligned} \Theta_k \psi_k \colon B^r_{pp,\lambda}(\mathbb{R}^{n-1}) &\to B^r_{pp,\lambda}(\Gamma), \\ \Theta_k^{-1} \varphi_k^\Omega \colon B^r_{pp,\lambda}(\Gamma) &\to B^r_{pp,\lambda}(\mathbb{R}^{n-1}) \end{aligned} \tag{3-33}$$

for $k = 1, \dots, K_0$ and all $|r| < \max_j t_j + \lceil \rho' \rceil$.

In the following let $\lambda_0 \geqslant 1$ be as in Lemma 3.3.2. The approximate solution operator $R_0(\lambda)$ is formally defined as

$$R_0(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} := \sum_{k=1}^{K_0} L_k^\Omega(\lambda) \begin{pmatrix} \varphi_k^\Omega f \\ (\gamma_0 \varphi_k^\Omega) g \end{pmatrix} + \sum_{k=K_0+1}^{K} L_k^\Omega(\lambda)(\varphi_k^\Omega f) \tag{3-34}$$

for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$. Here, $L_k^\Omega(\lambda)$ is given by

$$\begin{aligned} L_k^\Omega(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} &:= \Theta_k \psi_k L_k(\lambda) \Theta_k^{-1} \begin{pmatrix} f \\ g \end{pmatrix} && (k = 1, \dots, K_0), \\ L_k^\Omega(\lambda) f &:= \Theta_k \psi_k L_k(\lambda) \Theta_k^{-1} f && (k = K_0 + 1, \dots, K). \end{aligned} \tag{3-35}$$

3.3.3 Lemma. *Let $r, \sigma \in \mathbb{R}$ with $r > -1 + \frac{1}{p}$ and $\sigma \leqslant 0$. Then the operator $R_0(\lambda)$ in (3-34) is well defined on $\mathcal{F}_\lambda^{r,\sigma}(\Omega)$ for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$ and yields a bounded operator family*

$$R_0(\lambda) \in L(\mathcal{F}_\lambda^{r,\sigma}(\Omega), \mathcal{E}_\lambda^{r,\sigma}(\Omega)). \tag{3-36}$$

Proof. The continuity of $R_0(\lambda)$ in the corresponding spaces follows from (3-32), (3-33) and Lemma 3.3.2. □

In the following, we modify $R_0(\lambda)$ to get a solution. For this, we compute $(\lambda - A, B) R_0(\lambda) \binom{f}{g}$, where we may choose (f, g) sufficiently smooth such that the classical theory can be applied. Therefore, we introduce r', and assume from now on that $r, \sigma, r' \in \mathbb{R}$ satisfy

$$r > -1 + \frac{1}{p}, \quad -1 < \sigma \leqslant 0, \quad r' \geqslant r. \tag{3-37}$$

Moreover, we assume (S1)–(S3) and (S5) for (r, σ) (as before) and also for $(r', 0)$. The conditions with respect to $(r', 0)$ collapse to $\rho_i' = r' - s_i$ and $k_\ell' = r' - m_\ell - \frac{1}{p}$.

In contrast to the half-space situation, we have a restriction on σ in (3-37). This is caused by the commutator estimates and the fact that we only consider standard Sobolev spaces in Ω.

3.3.4 Lemma. *Let r, σ and r' satisfy* (3-37).

(i) For $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_0$, we define the operator $C_k(\lambda)$ by

$$C_k(\lambda) := \begin{pmatrix} -[A^k, \psi_k] \\ [B^k, \psi_k] \end{pmatrix} L_k(\lambda) \quad (k = 1, \dots, K_0),$$
$$C_k(\lambda) := -[A^k, \psi_k] L_k(\lambda) \qquad (k = K_0 + 1, \dots, K).$$

Then

$$C_k(\lambda) \in L(\mathcal{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+)) \quad (k = 1, \dots, K_0),$$
$$C_k(\lambda) \in L(\mathcal{F}^{r,\sigma}_\lambda(\mathbb{R}^n)) \quad (k = K_0 + 1, \dots, K)$$

with arbitrary small operator norm for sufficiently large λ.

(ii) Let $(f, g) \in \mathcal{F}^{r',0}_\lambda(\Omega)$. For $R_0(\lambda)$ as in (3-34), *we get*

$$\begin{pmatrix} \lambda - A \\ B \end{pmatrix} R_0(\lambda) \begin{pmatrix} f \\ g \end{pmatrix} = (1 + C(\lambda)) \begin{pmatrix} f \\ g \end{pmatrix} \tag{3-38}$$

with an operator $C(\lambda) \in L(\mathcal{F}^{r,\sigma}_\lambda(\Omega))$. Additionally, there exists a $\lambda_1 \geqslant \lambda_0$ such that $1 + C(\lambda) \in L(\mathcal{F}^{r,\sigma}_\lambda(\Omega))$ is invertible for all $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_1$.

Proof. We adopt the ideas of the proof of Lemma 3.2.12.

(i) Without loss of generality we assume $k \in \{1, \dots, K_0\}$. Let $0 < \varepsilon < 1 + \sigma$. First, we observe that the mappings

$$[A^k, \psi_k] \colon \mathcal{E}^{r-\varepsilon,\sigma}_\lambda(\mathbb{R}^n_+) = H^{\mathbf{t}+r-\varepsilon+\sigma}_{p,\lambda}(\mathbb{R}^n_+) \to H^{-\mathbf{s}+r}_{p,\lambda}(\mathbb{R}^n_+)$$

and

$$[B^k, \psi_k] \colon \mathcal{E}^{r-\varepsilon,\sigma}_\lambda(\mathbb{R}^n_+) \to B^{-\mathbf{m}+r+\sigma-1/p}_{pp,\lambda}(\mathbb{R}^{n-1})$$

are continuous. To see the continuity of the commutator of the boundary operator, we notice that

$$[B^k_{\ell j}, \psi_k] \colon H^{t_j+r-\varepsilon+\sigma}_{p,\lambda}(\mathbb{R}^n_+) \to B^{-m_\ell+r-\varepsilon+\sigma+1-1/p}_{pp,\lambda} \subset B^{-m_\ell+r+\sigma-1/p}_{pp,\lambda}$$

is continuous for $j \in \{1, \dots, N\}$ and $\ell \in \{1, \dots, M\}$ with $-m_\ell + r + \sigma - 1/p \leqslant 0$. If $-m_\ell + r + \sigma - 1/p > 0$ we directly obtain the boundedness of

$$[B^k_{\ell j}, \psi_k] \colon H^{t_j+r-\varepsilon+\sigma}_{p}(\mathbb{R}^n_+) \to B^{-m_\ell+r+\sigma-1/p}_{pp}$$

as the appearing traces exist in the classical sense.

In addition, we have $L_k(\lambda) \in L(\mathcal{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+), \mathcal{E}^{r,\sigma}_\lambda(\mathbb{R}^n_+))$ by Lemma 3.3.2. This and the above mapping properties for the commutators show $C_k(\lambda) \in L(\mathcal{F}^{r,\sigma}_\lambda(\mathbb{R}^n_+))$ with arbitrary small operator norm for sufficiently large λ, where we can argue as in the proof of Lemma 3.2.12.

(ii) It holds $R_0(\lambda)\binom{f}{g} \in \mathcal{E}_\lambda^{r',0}(\Omega)$ due to Lemma 3.3.3, and, as $r' \geqslant 0$, the boundary operator B can be applied to $R_0(\lambda)\binom{f}{g}$ in the classical sense. Using calculations similar to the ones in the proof of Lemma 3.2.12 and the equality $\Theta_k(\psi_k\Theta_k^{-1}(\varphi_k^\Omega f)) = \psi_k^\Omega\varphi_k^\Omega f$, we get

$$\begin{pmatrix}\lambda - A\\ B\end{pmatrix} R_0(\lambda)\begin{pmatrix}f\\ g\end{pmatrix} = (1 + C(\lambda))\begin{pmatrix}f\\ g\end{pmatrix}$$

with

$$C(\lambda)\begin{pmatrix}f\\ g\end{pmatrix} := \sum_{k=1}^{K_0} \Theta_k C_k(\lambda)\Theta_k^{-1}\begin{pmatrix}\varphi_k^\Omega f\\ (\gamma_0\varphi_k^\Omega)g\end{pmatrix} + \sum_{k=K_0+1}^{K}\begin{pmatrix}\Theta_k C_k(\lambda)\Theta_k^{-1}(\varphi_k^\Omega f)\\ 0\end{pmatrix}.$$

From (i) and the fact that multiplication by φ_k^Ω and the coordinate transformations Θ_k, Θ_k^{-1} preserve the smoothness, we obtain $C(\lambda) \in L(\mathcal{F}_\lambda^{r,\sigma}(\Omega))$ with arbitrary small operator norm for sufficiently large λ. Proceeding as in the proof of Lemma 3.2.12, we conclude that the operator $1 + C(\lambda) \in L(\mathcal{F}_\lambda^{r,\sigma}(\Omega))$ is invertible for sufficiently large λ, and the norm of its inverse is not greater than 2. □

The following theorem is the key result of this section and gives an a priori estimate for the solution operator of (3-9) in spaces of rough regularity. Note that we first consider smooth functions, where the boundary operators are defined in a classical way and where we know unique solvability by classical results. However, the a priori estimate gives a continuous extension of the solution operator to larger spaces.

3.3.5 Theorem. *Let $(\lambda - A, B)$ be parameter-elliptic in the sector Λ, and let $r, \sigma, r' \in \mathbb{R}$ satisfy* (3-37). *Assume* (S1)–(S3), (S5) *to hold for (r,σ) and $(r',0)$. Then for all $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_1$ and every $(f,g) \in \mathcal{F}_\lambda^{r',0}(\Omega)$, the unique solution $w \in \mathcal{E}_\lambda^{r',0}(\Omega)$ of* (3-9) *is given by*

$$w = R(\lambda)\begin{pmatrix}f\\ g\end{pmatrix} := R_0(\lambda)(1 + C(\lambda))^{-1}\begin{pmatrix}f\\ g\end{pmatrix},$$

where λ_1 is as in Lemma 3.3.4 (ii). Additionally, we have the a priori estimate

$$\left\|R(\lambda)\begin{pmatrix}f\\ g\end{pmatrix}\right\|_{\mathcal{E}_\lambda^{r,\sigma}(\Omega)} \leqslant C\left\|\begin{pmatrix}f\\ g\end{pmatrix}\right\|_{\mathcal{F}_\lambda^{r,\sigma}(\Omega)} \tag{3-39}$$

with a constant $C > 0$ not depending on f, g or λ. In particular, the solution operator $R(\lambda)$ extends uniquely to a continuous operator family

$$R(\lambda) \in L(\mathcal{F}_\lambda^{r,\sigma}(\Omega), \mathcal{E}_\lambda^{r,\sigma}(\Omega)).$$

Proof. First, we remark that $\mathcal{F}_\lambda^{r',0}(\Omega)$ and $\mathcal{E}_\lambda^{r',0}(\Omega)$ are classical spaces, and therefore we obtain unique solvability in these spaces, see Proposition 2.5.6. By Lemma 3.3.4 (ii), the operator $1 + C(\lambda)$ is invertible in $L(\mathcal{F}_\lambda^{r',0}(\Omega))$, and from Lemma 3.3.3 we get $R_0(\lambda) \in L(\mathcal{F}_\lambda^{r',0}(\Omega), \mathcal{E}_\lambda^{r',0}(\Omega))$. Therefore, $w := R(\lambda)\binom{f}{g} \in \mathcal{E}_\lambda^{r',0}(\Omega)$. As

$$\begin{pmatrix}\lambda - A\\ B\end{pmatrix} w = (1 + C(\lambda))(1 + C(\lambda))^{-1}\begin{pmatrix}f\\ g\end{pmatrix} = \begin{pmatrix}f\\ g\end{pmatrix}$$

by Lemma 3.3.4 (ii), w is the unique solution of (3-9). Finally, the a priori estimate (3-39) follows from Lemma 3.3.3 and $(1 + C(\lambda))^{-1} \in L(\mathcal{F}_\lambda^{r,\sigma}(\Omega))$. □

3.3.6 Remark. As shown in Remark 4.4 of [12], we also get suitable a priori estimates for systems of boundary value problems with different boundary operators on disjoint and not connected parts of the boundary, given that each part of the boundary fulfils the Shapiro–Lopatinskii condition. To be more precise, let $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are compact, disjoint parts of the boundary of Ω. In addition, let $(\lambda - A, B_1)$ and $(\lambda - A, B_2)$ be parameter-elliptic, where the orders are encoded in $\mathbf{t}, \mathbf{s}, \mathbf{m}_1$ and $\mathbf{m}_2$. In the setting of Theorem 3.3.5, the solution of

$$\begin{aligned} (\lambda - A)w &= f \quad \text{in } \Omega, \\ B_1 w &= g \quad \text{on } \Gamma_1, \\ B_2 w &= g \quad \text{on } \Gamma_2 \end{aligned}$$

is unique and satisfies

$$\|w\|_{H^{\mathbf{t}+r+\sigma}_{p,\lambda}(\Omega)} \leqslant C\Big(\|f\|_{H^{-\mathbf{s}+r}_{p,\lambda}(\Omega)} + \|g\|_{B^{-\mathbf{m}_1+r+\sigma-1/p}_{pp,\lambda}(\Gamma_1)} + \|g\|_{B^{-\mathbf{m}_2+r+\sigma-1/p}_{pp,\lambda}(\Gamma_2)}\Big). \tag{3-40}$$

Basically, the reason is that we use methods of localization, where (A^k, B^k) only notices one of the compact, disjoint parts of the boundary such that the order $\mathbf{m}$ might also depend continuously on $x \in \Gamma$. However, this means that $\mathbf{m}$ is constant on each compact, disjoint part of Γ. This fact is rather useful in the context of transmission problems.

3.3.7 Remark. Notice that we used the classical regularity for f and g to get estimates in rougher spaces. It is also possible to extend this theory to functions f and g with slightly less regularity. Here, the main issue is to guarantee the existence of the appearing traces. As Theorem 3.3.5 is sufficient for the application given in Chapter 4, we skip the precise formulation of the stronger result and refer to Theorem 4.9 of [19] for details.

3.4 Time-dependent systems in the Hilbert space setting

In this section, we discuss the connection between parameter-dependent and time-dependent systems. The rough idea is that the Fourier transform converts a time-dependent system into a parameter-dependent system, such that we can apply the results of the previous section also to time-dependent problems. For the sake of simplicity, we only consider parameter-elliptic systems in the Hilbert space setting. One of the advantages in the case $p = 2$ is that Besov- and Bessel potential spaces coincide and that we can make use of Plancherel's Theorem as well as the Theorem of Paley–Wiener.

During the whole section we assume $p = 2$ and omit the index p in the appearing spaces. Similar to [37], we introduce the spaces

$$H^{\mathbf{s},\mathbf{t}}(\mathcal{I} \times \Omega) := L^2(\mathcal{I}, H^{\mathbf{s}}(\Omega)) \cap H^{\mathbf{t}}(\mathcal{I}, L^2(\Omega)) = \prod_{j=1}^{N} L^2(\mathcal{I}, H^{s_j}(\Omega)) \cap H^{t_j}(\mathcal{I}, L^2(\Omega))$$

for $N \in \mathbb{N}$, $\mathbf{s}, \mathbf{t} \in \mathbb{R}^N$, an interval $\mathcal{I} \subset \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$.

We start with an observation on how the parameter-dependent norms fit into the time-dependent spaces.

3.4.1 Lemma. *Let $\Omega \subset \mathbb{R}^n$ be open and $\mathbf{t} \in \mathbb{R}^N$ with $\mathbf{t} \geqslant 0$. Then*

$$H^{\mathbf{t},\mathbf{t}/\mu}(\mathbb{R}\times\Omega) = \left\{ w \in L^2(\mathbb{R}, H^{\mathbf{t}}(\Omega)) : \int_{\mathbb{R}} \|(\mathscr{F}_1 w)(\lambda)\|^2_{H^{\mathbf{t}}_\lambda(\Omega)}\, d\lambda < \infty \right\}.$$

An analogue statement holds for the boundary Γ of Ω.

Proof. Let $w \in H^{\mathbf{t},\mathbf{t}/\mu}(\mathbb{R}\times\Omega)$. According to Lemma 2.1.7 and Plancherel's Theorem it holds

$$\begin{aligned}
\int_{\mathbb{R}} \|(\mathscr{F}_1 w_j)(\lambda)\|^2_{H^{t_j}_\lambda(\Omega)}\, d\lambda &\approx \int_{\mathbb{R}} \|(\mathscr{F}_1 w_j)(\lambda)\|^2_{H^{t_j}(\Omega)} + \|\langle\lambda\rangle^{t_j}(\mathscr{F}_1 w_j)(\lambda)\|^2_{L^2(\Omega)}\, d\lambda \\
&= \|\mathscr{F}_1 w_j\|^2_{L^2(\mathbb{R},H^{t_j}(\Omega))} + \|(1+|\cdot|^{2/\mu})^{t_j/2}\mathscr{F}_1 w_j\|^2_{L^2(\mathbb{R},L^2(\Omega))} \\
&\approx \|w_j\|^2_{L^2(\mathbb{R},H^{t_j}(\Omega))} + \|(1+|\cdot|^2)^{t_j/(2\mu)}\mathscr{F}_1 w_j\|^2_{L^2(\mathbb{R},L^2(\Omega))} \\
&= \|w_j\|^2_{L^2(\mathbb{R},H^{t_j}(\Omega))} + \|w_j\|^2_{H^{t_j/\mu}(\mathbb{R},L^2(\Omega))}
\end{aligned}$$

for $j = 1,\dots,N$. Consequently,

$$\int_{\mathbb{R}} \|(\mathscr{F}_1 w)(\lambda)\|^2_{H^{\mathbf{t}}_\lambda(\Omega)}\, d\lambda \approx \|w\|^2_{H^{\mathbf{t},\mathbf{t}/\mu}(\mathbb{R}\times\Omega)}. \tag{3-41}$$

□

As we mainly deal with functions which have support in the positive half-line, we need a generalization of the previous lemma for functions f with $f\big|_{(-\infty,0)} = 0$. One of the main ingredients is the Theorem of Paley–Wiener. It describes what the Fourier transform of a function with support in the positive half-line looks like and vice versa. We only mention the statement and refer, for instance, to Theorem 1.8.3 of [8] or Theorem 4.2 of [80] for a proof.

3.4.2 Proposition (Paley–Wiener)**.** *Let H be a Hilbert space.*

(i) Let $f \in L^2(\mathbb{R}, H)$ with $f\big|_{(-\infty,0)} = 0$. Then $(\mathscr{F}_1 f)(-i\cdot)\colon \mathbb{C}_+ \to H$ is holomorphic with

$$\|f\|^2_{L^2(\mathbb{R},H)} = \sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f)(y-ix)\|^2_H\, dy,$$

where $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re}(z) > 0\}$.

(ii) Let $h\colon \mathbb{C}_+ \to H$ be holomorphic with $\sup_{x>0}\int_{\mathbb{R}} \|h(x+iy)\|^2_H\, dy < \infty$. Then it holds $(\mathscr{F}_1^{-1}h)(i\cdot) \in L^2(\mathbb{R},H)$ with $(\mathscr{F}_1^{-1}h)(i\cdot)\big|_{(-\infty,0)} = 0$.

In the following lemma we present another type of shift operator in the Sobolev scale. The advantage of $(1+i\cdot)^{-\beta}$ in contrast to $(1+|\cdot|^2)^{-\beta/2}$ as a shift function is its holomorphy, which allows us to apply the Theorem of Paley–Wiener to preserve the support of the shifted function.

3.4.3 Lemma. *Let H be a Hilbert space and $s \in \mathbb{R}$. For $\beta \in \mathbb{R}$ and $f \in H^s(\mathbb{R},H)$ we define*

$$S_\beta f := \mathscr{F}_1^{-1}(1+i\cdot)^{-\beta}\mathscr{F}_1 f.$$

The operator S_β has the following properties.

(i) $S_\beta\colon H^s(\mathbb{R},H) \to H^{s+\beta}(\mathbb{R},H)$ is an isometric isomorphism for $\beta \in \mathbb{R}$.

(ii) If $s,\beta \geqslant 0$ and $f \in H^s(\mathbb{R},H)$ with $f\big|_{(-\infty,0)} = 0$, we also get $S_\beta f\big|_{(-\infty,0)} = 0$.

Proof. We show the two statements separately.

(i) Because of $|1 + i\lambda|^2 = 1 + |\lambda|^2$ for $\lambda \in \mathbb{R}$ we obtain

$$\begin{aligned}\|S_\beta f\|_{H^{s+\beta}(\mathbb{R},H)} &= \|(1 + |\cdot|^2)^{(s+\beta)/2}(1 + i\cdot)^{-\beta}\mathscr{F}_1 f\|_{L^2(\mathbb{R},H)} \\ &= \|(1 + |\cdot|^2)^{s/2}\mathscr{F}_1 f\|_{L^2(\mathbb{R},H)} \\ &= \|f\|_{H^s(\mathbb{R},H)}.\end{aligned}$$

One can easily see that S_β is invertible with $S_\beta^{-1} = S_{-\beta}$.

(ii) Now, let $s, \beta \geqslant 0$ and $f \in H^s(\mathbb{R}, H)$ with $f\big|_{(-\infty,0)} = 0$. As $s \geqslant 0$ we have $f \in L^2(\mathbb{R}, H)$ and obtain that $\mathscr{F}_1 f(-i\cdot)\colon \mathbb{C}_+ \to H$ is holomorphic with

$$\|f\|^2_{L^2(\mathbb{R},H)} = \sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f)(y - ix)\|_H^2 \, dy.$$

In addition, the function $(1 + \cdot)^{-\beta}$ is holomorphic and bounded because of

$$|1 + \lambda|^{-\beta} = |\exp(-\beta \log(1 + \lambda))| = \exp(-\beta \operatorname{Re}(\log(1 + \lambda))) = \exp(-\beta \log|1 + \lambda|) \leqslant 1$$

for $\lambda \in \mathbb{C}_+$. Therefore, we can apply the Theorem of Paley–Wiener to the function

$$h := (1 + \cdot)^{-\beta}\mathscr{F}_1 f(-i\cdot)\colon \mathbb{C}_+ \to H$$

to get $S_\beta f\big|_{(-\infty,0)} = (\mathscr{F}_1^{-1} h)(i\cdot)\big|_{(-\infty,0)} = 0$. □

The next lemma gives another characterization of the norm in $H^s(\mathbb{R}, H)$ whenever the support of the function of interest lies in the positive half-line. The result is based on ideas of the proof of Theorem 6.3 in [37] but it is extended to non-integer numbers $s \geqslant 0$.

3.4.4 Lemma. *Let H be a Hilbert space, $s \geqslant 0$ and $f \in H^s(\mathbb{R}, H)$ with $f\big|_{(-\infty,0)} = 0$. Then*

$$\|f\|^2_{H^s(\mathbb{R},H)} \eqsim \sup_{x>0} \int_{\mathbb{R}} (1 + |x + iy|^2)^s \|(\mathscr{F}_1 f)(y - ix)\|_H^2 \, dy.$$

Proof. First, we assume $s = k$ to be a natural number or zero. Thus, we can apply the Theorem of Paley–Wiener to the j-th derivative $f^{(j)}$ of f for $j = 0, \dots, k$. We get

$$\begin{aligned}\|f\|^2_{H^k(\mathbb{R},H)} &= \sum_{j=0}^{k} \|f^{(j)}\|^2_{L^2(\mathbb{R},H)} \\ &= \sum_{j=0}^{k} \sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f^{(j)})(y - ix)\|_H^2 \, dy \\ &= \sum_{j=0}^{k} \sup_{x>0} \int_{\mathbb{R}} |x + iy|^{2j} \|(\mathscr{F}_1 f)(y - ix)\|_H^2 \, dy \\ &\eqsim \sup_{x>0} \int_{\mathbb{R}} (1 + |x + iy|^2)^k \|(\mathscr{F}_1 f)(y - ix)\|_H^2 \, dy.\end{aligned}$$

For arbitrary $s > 0$ we set $\beta := \lceil s \rceil - s$ and obtain

$$\begin{aligned}
\|f\|^2_{H^s(\mathbb{R},H)} &= \|S_\beta f\|^2_{H^{\lceil s\rceil}(\mathbb{R},H)} \\
&\approx \sup_{x>0} \int_{\mathbb{R}} (1+|x+iy|^2)^{\lceil s\rceil} \|(\mathscr{F}_1 S_\beta f)(y-ix)\|^2_H \, dy \\
&= \sup_{x>0} \int_{\mathbb{R}} (1+|x+iy|^2)^{\lceil s\rceil} |1+x+iy|^{-2\beta} \|(\mathscr{F}_1 f)(y-ix)\|^2_H \, dy \\
&\approx \sup_{x>0} \int_{\mathbb{R}} (1+|x+iy|^2)^{s} \|(\mathscr{F}_1 f)(y-ix)\|^2_H \, dy
\end{aligned}$$

due to Lemma 3.4.3. In the last step we used $|1+\lambda| \leqslant 1+|\lambda| \leqslant C(1+|\lambda|^2)^{1/2}$ and

$$1+|\lambda|^2 = 1+\operatorname{Re}(\lambda)^2+\operatorname{Im}(\lambda)^2 \leqslant (1+\operatorname{Re}(\lambda))^2+\operatorname{Im}(\lambda)^2 = |1+\lambda|^2$$

for $\lambda \in \mathbb{C}_+$. □

3.4.5 Lemma. *Let $\Omega \subset \mathbb{R}^n$ be open. If $\mathbf{t} \in \mathbb{R}^N$ with $\mathbf{t} \geqslant 0$ and $f \in H^{\mathbf{t},\mathbf{t}/\mu}(\mathbb{R}\times\Omega)$ with $f\big|_{(-\infty,0)} = 0$, we get*

$$\sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f)(y-ix)\|^2_{H^{\mathbf{t}}_{x+iy}(\Omega)} \, dy \approx \|f\|^2_{H^{\mathbf{t},\mathbf{t}/\mu}(\mathbb{R}\times\Omega)}.$$

Proof. For a function $f \in H^{\mathbf{t},\mathbf{t}/\mu}(\mathbb{R}\times\Omega)$ with $f\big|_{(-\infty,0)} = 0$ it holds

$$\begin{aligned}
&\sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f_j)(y-ix)\|^2_{H^{t_j}_{x+iy}(\Omega)} \, dy \\
\approx &\sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f_j)(y-ix)\|^2_{H^{t_j}(\Omega)} + \|\langle x+iy\rangle^{t_j} (\mathscr{F}_1 f_j)(y-ix)\|^2_{L^2(\Omega)} \, dy \\
\approx &\sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f_j)(y-ix)\|^2_{H^{t_j}(\Omega)} + (1+|x+iy|^2)^{t_j/\mu} \|(\mathscr{F}_1 f_j)(y-ix)\|^2_{L^2(\Omega)} \, dy \\
\leqslant &C\Big(\|f_j\|^2_{L^2(\mathbb{R},H^{t_j}(\Omega))} + \|f_j\|^2_{H^{t_j/\mu}(\mathbb{R},L^2(\Omega))} \Big)
\end{aligned}$$

for $j = 1,\dots,N$ due to Lemma 2.1.7 and Lemma 3.4.4. The second inequality

$$\|f\|^2_{H^{\mathbf{t},\mathbf{t}/\mu}(\mathbb{R}\times\Omega)} \leqslant C \sup_{x>0} \int_{\mathbb{R}} \|(\mathscr{F}_1 f)(y-ix)\|^2_{H^{\mathbf{t}}_{x+iy}(\Omega)} \, dy$$

follows from (3-41). □

In the following, we assume $\Omega \subset \mathbb{R}^n$ to be a domain with compact boundary Γ, and $(\lambda - A, B)$ to be a system of boundary value problems in Ω of the form (3-10)–(3-11), which is parameter-elliptic in a closed sector $\Lambda \supset \overline{\mathbb{C}_+}$. Moreover, we assume $r \geqslant 0$ and (S1)–(S3) and (S5) to hold for $(r,0)$.

3.4.6 Lemma. *Let $\mathcal{I} \subset \mathbb{R}$ be an interval and $r \geqslant 0$. Then, it holds*

$$H^{\mathbf{t}+r,(\mathbf{t}+r)/\mu}(\mathcal{I}\times\Omega) \subset H^1(\mathcal{I}, H^{-\mathbf{s}+r}(\Omega)). \tag{3-42}$$

Proof. Let $j \in \{1, \dots, N\}$. Because of $s_j + t_j = \mu \in \mathbb{N}$ we have $t_j > 0$. If $s_j = 0$ and $r = 0$, it holds $t_j + r = \mu$ and the claim follows trivially from the definition of the space $H^{t_j,1}(\mathcal{I} \times \Omega)$. So we may assume $r > 0$ or $s_j < 0$ such that $\frac{\mu}{t_j+r} \in (0,1)$. We get

$$[L^2(\mathcal{I}, H^{t_j+r}(\Omega)), H^{(t_j+r)/\mu}(\mathcal{I}, L^2(\Omega))]_{\mu/(t_j+r)} = H^1(\mathcal{I}, H^{t_j+r-\mu}(\Omega)) = H^1(\mathcal{I}, H^{-s_j+r}(\Omega))$$

by complex interpolation, see Theorem 3.18 of [51]. Since the intersection of the spaces is always a subset of the corresponding interpolation space, this shows (3-42). □

3.4.7 Proposition. *Let $r \geqslant 0$ and $\lambda_1 > 0$, $R(\lambda)$ be as in Theorem 3.3.5 for $\lambda \in \Lambda$ with $|\lambda| \geqslant \lambda_1$, where $\Lambda \supset \overline{\mathbb{C}_+}$. In addition, we assume*

$$f \in H^{-\mathbf{s}+r,(-\mathbf{s}+r)/\mu}(\mathbb{R} \times \Omega) \quad \text{and} \quad g \in H^{-\mathbf{m}+r-1/2,(-\mathbf{m}+r-1/2)/\mu}(\mathbb{R} \times \Gamma)$$

with $f\big|_{(-\infty,0)} = g\big|_{(-\infty,0)} = 0$. Then we get $\mathscr{F}_1^{-1} R(\lambda_1 + i\cdot)\mathscr{F}_1 \binom{f}{g} \in H^{\mathbf{t}+r,(\mathbf{t}+r)/\mu}(\mathbb{R} \times \Omega)$ with $\mathscr{F}_1^{-1} R(\lambda_1 + i\cdot)\mathscr{F}_1 \binom{f}{g}\big|_{(-\infty,0)} = 0$. In particular, we have

$$\mathscr{F}_1^{-1} R(\lambda_1 + i\cdot)\mathscr{F}_1 \begin{pmatrix} f \\ g \end{pmatrix}(0) = 0. \tag{3-43}$$

Proof. We define $h\colon \mathbb{C}_+ \to L^2(\Omega)$ via $h(\lambda) := R(\lambda_1 + \lambda)\mathscr{F}_1 \binom{f}{g}(-i\lambda)$ for $\lambda \in \mathbb{C}_+$. The function h is holomorphic due to the Theorem of Paley–Wiener and fulfils

$$\begin{aligned}
\sup_{x>0} \int_{\mathbb{R}} \|h(x+iy)\|^2_{L^2(\Omega)}\, dy &\leqslant C \sup_{x>0} \int_{\mathbb{R}} \left\| R(\lambda_1 + x + iy)\mathscr{F}_1 \begin{pmatrix} f \\ g \end{pmatrix}(y - ix) \right\|^2_{\mathcal{E}^{r,0}_{\lambda_1+x+iy}(\Omega)} dy \\
&\leqslant C \sup_{x>0} \int_{\mathbb{R}} \left\| \mathscr{F}_1 \begin{pmatrix} f \\ g \end{pmatrix}(y - ix) \right\|^2_{\mathcal{F}^{r,0}_{\lambda_1+x+iy}(\Omega)} dy \\
&\leqslant C \sup_{x>0} \int_{\mathbb{R}} \left\| \mathscr{F}_1 \begin{pmatrix} f \\ g \end{pmatrix}(y - ix) \right\|^2_{\mathcal{F}^{r,0}_{x+iy}(\Omega)} dy \\
&\leqslant C \Big(\|f\|^2_{H^{-\mathbf{s}+r,(-\mathbf{s}+r)/\mu}(\mathbb{R}\times\Omega)} + \|g\|^2_{H^{-\mathbf{m}+r-1/2,(-\mathbf{m}+r-1/2)/\mu}(\mathbb{R}\times\Gamma)} \Big)
\end{aligned}$$

because of Lemma 3.4.5. Thus, we can apply the Theorem of Paley–Wiener to the function h again to obtain $\mathscr{F}_1^{-1} R(\lambda_1 + i\cdot)\mathscr{F}_1 \binom{f}{g}\big|_{(-\infty,0)} = 0$. In addition, it holds

$$\begin{aligned}
&\left\| \mathscr{F}_1^{-1} R(\lambda_1 + i\cdot)\mathscr{F}_1 \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2_{H^{\mathbf{t}+r,(\mathbf{t}+r)/\mu}(\mathbb{R}\times\Omega)} \\
&\leqslant C \int_{\mathbb{R}} \|h(i\lambda)\|^2_{\mathcal{E}^{r,0}_{\lambda}(\Omega)}\, d\lambda \\
&\leqslant C \int_{\mathbb{R}} \left\| \mathscr{F}_1 \begin{pmatrix} f \\ g \end{pmatrix}(\lambda) \right\|^2_{\mathcal{F}^{r,0}_{\lambda}(\Omega)} d\lambda \\
&\leqslant C \Big(\|f\|^2_{H^{-\mathbf{s}+r,(-\mathbf{s}+r)/\mu}(\mathbb{R}\times\Omega)} + \|g\|^2_{H^{-\mathbf{m}+r-1/2,(-\mathbf{m}+r-1/2)/\mu}(\mathbb{R}\times\Gamma)} \Big)
\end{aligned}$$

according to (3-41). In particular, we get that

$$\mathscr{F}_1^{-1} R(\lambda_1 + i\cdot)\mathscr{F}_1 \begin{pmatrix} f \\ g \end{pmatrix} \in H^{\mathbf{t}+r,(\mathbf{t}+r)/\mu}(\mathbb{R} \times \Omega) \subset H^1(\mathbb{R}, H^{-\mathbf{s}+r}(\Omega))$$

is continuous due to Lemma 3.4.6, which shows (3-43). □

With the help of Theorem 3.3.5 we are now in the situation to formulate an a priori estimate for the solution of the time-dependent system. For the sake of readability we restrict ourselves to the case $r = 0$.

3.4.8 Theorem. *Let $(\lambda - A, B)$ be parameter-elliptic in $\Lambda \supset \overline{\mathbb{C}_+}$, and let $\sigma \in (-1, 0]$, $T > 0$. Assume* (S1)–(S3), (S5) *to hold for $(0, \sigma)$ and $(0, 0)$. We set*

$$\begin{aligned} \mathcal{H} &:= \{w \in H^{-\mathbf{s}}(\Omega) : (Bw)_\ell = 0 \text{ for } \ell \in \{1, \dots, M\} \text{ with } -m_\ell > \mu\}, \\ D(\mathcal{A}) &:= \{w \in H^{\mathbf{t}}(\Omega) : Aw \in \mathcal{H},\ Bw = 0\} \end{aligned}$$

and suppose that $\mathcal{A}\colon \mathcal{H} \supset D(\mathcal{A}) \to \mathcal{H}$ with $\mathcal{A}w = Aw$ for all $w \in D(\mathcal{A})$ generates a C_0-semigroup on $\mathcal{H}$. In addition, we assume that the time-dependent problem

$$\begin{aligned} \partial_t w - Aw &= f && \text{in } (0,T) \times \Omega, \\ Bw &= g && \text{on } (0,T) \times \Gamma, \\ w(0) &= 0 && \text{in } \Omega \end{aligned} \tag{3-44}$$

possesses a solution $w \in C([0,T], H^{\mathbf{t}}(\Omega)) \cap C^1([0,T], H^{-\mathbf{s}}(\Omega))$ for given

$$\begin{aligned} f &\in H^{-\mathbf{s},-\mathbf{s}/\mu}((-\infty, T) \times \Omega), \\ g &\in H^{-\mathbf{m}-1/2,(-\mathbf{m}-1/2)/\mu}((-\infty, T) \times \Gamma), \end{aligned}$$

where $f\big|_{(-\infty,0)} = g\big|_{(-\infty,0)} = 0$. Then, we have

$$\|w\|^2_{H^{\mathbf{t}+\sigma,(\mathbf{t}+\sigma)/\mu}((0,T)\times\Omega)} \leqslant C \left(\|f\|^2_{H^{-\mathbf{s},-\mathbf{s}/\mu}((0,T)\times\Omega)} + \|g\|^2_{H^{-\mathbf{m}+\sigma-1/2,(-\mathbf{m}+\sigma-1/2)/\mu}((0,T)\times\Gamma)} \right).$$

Proof. We fix $\lambda_1 \geqslant 1$ as in Theorem 3.3.5. Let E be a continuous extension operator, which extends the appearing functions from $(-\infty, T)$ to $\mathbb{R}$ while preserving the smoothness such that $Ef\big|_{(2T,\infty)} = Eg\big|_{(2T,\infty)} = 0$. We set $\hat{f}(\lambda) := (\mathscr{F}_1(e^{-\lambda_1 \cdot} Ef))(\lambda)$ and $\hat{g}(\lambda) := (\mathscr{F}_1(e^{-\lambda_1 \cdot} Eg))(\lambda)$. Due to Lemma 3.4.1 we have $(\hat{f}(\lambda), \hat{g}(\lambda)) \in \mathcal{F}^{0,0}_{\lambda_1+i\lambda}(\Omega)$ for almost every $\lambda \in \mathbb{R}$. According to Theorem 3.3.5 the parameter-dependent system

$$\begin{aligned} (\lambda_1 + i\lambda - A)(\hat{w}(\lambda)) &= \hat{f}(\lambda) && \text{in } \Omega, \\ B(\hat{w}(\lambda)) &= \hat{g}(\lambda) && \text{on } \Gamma \end{aligned}$$

possesses a unique solution $\hat{w}(\lambda) \in \mathcal{E}^{0,0}_{\lambda_1+i\lambda}(\Omega)$ with

$$\|\hat{w}(\lambda)\|_{\mathcal{E}^{0,\sigma}_\lambda(\Omega)} \leqslant C\|\hat{w}(\lambda)\|_{\mathcal{E}^{0,\sigma}_{\lambda_1+i\lambda}(\Omega)} \leqslant C\left\|\begin{pmatrix}\hat{f}(\lambda)\\ \hat{g}(\lambda)\end{pmatrix}\right\|_{\mathcal{F}^{0,\sigma}_{\lambda_1+i\lambda}(\Omega)} \leqslant C\left\|\begin{pmatrix}\hat{f}(\lambda)\\ \hat{g}(\lambda)\end{pmatrix}\right\|_{\mathcal{F}^{0,\sigma}_\lambda(\Omega)}$$

for almost every $\lambda \in \mathbb{R}$. Since

$$\begin{aligned} \partial_t(e^{\lambda_1 \cdot} \mathscr{F}_1^{-1}\hat{w}(\cdot))(t) - A(e^{\lambda_1 \cdot} \mathscr{F}_1^{-1}\hat{w}(\cdot))(t) &= e^{\lambda_1 t}(\mathscr{F}_1^{-1}((\lambda_1 + i \cdot - A)\hat{w}(\cdot))(t) \\ &= e^{\lambda_1 t}(\mathscr{F}_1^{-1}\hat{f}(\cdot))(t) \\ &= (Ef)(t) \end{aligned}$$

and $B(e^{\lambda_1\cdot}\mathscr{F}_1^{-1}\hat{w}(\cdot))(t) = (Eg)(t)$ for $t \in \mathbb{R}$, the function $w_E := e^{\lambda_1\cdot}(\mathscr{F}_1^{-1}\hat{w}(\cdot))$ is a solution of

$$\begin{aligned} \partial_t w_E - A w_E &= Ef \qquad \text{in } \mathbb{R}\times\Omega, \\ B w_E &= Eg \qquad \text{on } \mathbb{R}\times\Gamma. \end{aligned} \tag{3-45}$$

Because of $f\big|_{(-\infty,0)} = g\big|_{(-\infty,0)} = 0$ and Proposition 3.4.7, we obtain $w_E(0) = 0$. In particular, we have $w - w_E \in C([0,T],\mathcal{H})$ with

$$\begin{aligned} (\partial_t - \mathcal{A})(w - w_E) &= 0 \quad \text{in } (0,T), \\ (w - w_E)(0) &= 0. \end{aligned}$$

This means that $w - w_E$ is a mild solution of the Cauchy problem in the sense of Definition 2.3.1. As the mild solution is unique due to Proposition 2.3.2, we obtain $w_E\big|_{(0,T)} = w$. Using that $e^{\lambda_1\cdot}$ is a bounded multiplication operator in $H^{\mathbf{t}+\sigma,(\mathbf{t}+\sigma)/\mu}((0,T)\times\Omega)$, we get

$$\begin{aligned} \|w\|^2_{H^{\mathbf{t}+\sigma,(\mathbf{t}+\sigma)/\mu}((0,T)\times\Omega)} &\leqslant C\|\mathscr{F}_1^{-1}(\hat{w}(\cdot))\|^2_{H^{\mathbf{t}+\sigma,(\mathbf{t}+\sigma)/\mu}(\mathbb{R}\times\Omega)} \\ &\leqslant C\int_{\mathbb{R}} \|\hat{w}(\lambda)\|^2_{\mathcal{E}^{0,\sigma}_\lambda(\Omega)}\,d\lambda \\ &\leqslant C\int_{\mathbb{R}} \left\|\begin{pmatrix}\hat{f}(\lambda)\\ \hat{g}(\lambda)\end{pmatrix}\right\|^2_{\mathcal{F}^{0,\sigma}_\lambda(\Omega)} d\lambda \\ &\leqslant C\left(\|\mathscr{F}_1^{-1}\hat{f}\|^2_{H^{-\mathbf{s},-\mathbf{s}/\mu}(\mathbb{R}\times\Omega)} + \|\mathscr{F}_1^{-1}\hat{g}\|^2_{H^{-\mathbf{m}+\sigma-1/2,(-\mathbf{m}+\sigma-1/2)/\mu}(\mathbb{R}\times\Gamma)}\right) \\ &\leqslant C\left(\|Ef\|^2_{H^{-\mathbf{s},-\mathbf{s}/\mu}(\mathbb{R}\times\Omega)} + \|Eg\|^2_{H^{-\mathbf{m}+\sigma-1/2,(-\mathbf{m}+\sigma-1/2)/\mu}(\mathbb{R}\times\Gamma)}\right) \\ &\leqslant C\left(\|f\|^2_{H^{-\mathbf{s},-\mathbf{s}/\mu}((0,T)\times\Omega)} + \|g\|^2_{H^{-\mathbf{m}+\sigma-1/2,(-\mathbf{m}+\sigma-1/2)/\mu}((0,T)\times\Gamma)}\right). \end{aligned}$$

□

3.4.9 Remark. As the solution w in the theorem above only exists in the finite time interval $[0,T]$, we cannot transform the equation to a parameter-dependent problem via the Fourier transform without extending the right-hand sides. Thus, we have to show $w_E = w$ in $(0,T)$ in the time-dependent setting. Both solutions w and w_E solve the problem in the Sobolev sense. We only use the theory of mild solutions to guarantee the uniqueness of the solution. This is also the reason why we assume that $\mathcal{A}$ generates a C_0-semigroup. Actually, this assumption seems to be rather strong as we already know the existence of a solution. Anyway, in many cases the parameter-ellipticity implies that $\mathcal{A}$ is the generator of a C_0-semigroup. An example is discussed in the next section.

3.5 The thermoelastic plate equation

As an example for a parameter-elliptic system in the sense of Douglis–Nirenberg, we consider the thermoelastic plate equation

$$\begin{aligned} u_{tt} + \Delta^2 u + \Delta\theta &= 0 \quad \text{in } (0,T)\times\Omega, \\ \theta_t - \Delta u_t - \Delta\theta &= 0 \quad \text{in } (0,T)\times\Omega \end{aligned}$$

in a finite time interval $(0,T)$ and a sufficiently smooth domain Ω with various boundary conditions.

Written as a first-order system in time, we obtain

$$\partial_t w - Aw = 0,$$

where $w = (w_1, w_2, w_3)$ correlates with (u, u_t, θ) and A acts in the form of the matrix

$$A = A(D) = \begin{pmatrix} 0 & 1 & 0 \\ -\Delta^2 & 0 & -\Delta \\ 0 & \Delta & \Delta \end{pmatrix}. \tag{3-46}$$

The operator A is a Douglis–Nirenberg system with $\mathbf{t} = (4, 2, 2)$ and $\mathbf{s} = (-2, 0, 0)$. We take a closer look at the two variants

$$B_1(D) = \begin{pmatrix} 1 & 0 & 0 \\ -\partial_n & 0 & 0 \\ 0 & 0 & -\partial_n + 1 \end{pmatrix} \quad \text{and} \quad B_2(D) = \begin{pmatrix} -\partial_n & 0 & 0 \\ 0 & 0 & 1 \\ \partial_n\Delta & 0 & \partial_n \end{pmatrix} \tag{3-47}$$

of boundary conditions. The problems $(\lambda - A, B_1)$ and $(\lambda - A, B_2)$ fit into the Douglis–Nirenberg setting with $\mathbf{m}_1 = (-4, -3, -1)$ and $\mathbf{m}_2 = (-3, -2, -1)$.

We check that the systems are indeed parameter-elliptic and construct a suitable operator $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is the generator of a C_0-semigroup of contractions. Finally, we apply Theorem 3.4.8 to gain generalized a priori estimates.

3.5.1 Lemma. *The boundary value problems $(\lambda - A, B_1)$ and $(\lambda - A, B_2)$ are parameter-elliptic in the closed sector $\overline{\mathbb{C}_+}$.*

Proof. First, we observe

$$\begin{aligned} \det(\lambda - A(\xi)) &= \det \begin{pmatrix} \lambda & -1 & 0 \\ |\xi|^4 & \lambda & -|\xi|^2 \\ 0 & |\xi|^2 & \lambda + |\xi|^2 \end{pmatrix} \\ &= \lambda^2(\lambda + |\xi|^2) + |\xi|^4(\lambda + |\xi|^2) + |\xi|^4\lambda \\ &= \lambda^3 + |\xi|^2\lambda^2 + 2|\xi|^4\lambda + |\xi|^6 \end{aligned}$$

for $\xi \in \mathbb{R}^n$ and $\lambda \in \overline{\mathbb{C}_+}$. It is known from Lemma 2.3 of [63] that there are $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{C}$ with $\gamma_1 \in (0, 1)$, $\operatorname{Re}(\gamma_2) \in \left(0, \frac{1}{2}\right)$, $\operatorname{Im}(\gamma_2) > 0$ and $\gamma_2 = \overline{\gamma_3}$ such that

$$\lambda^3 + \lambda^2 + 2\lambda + 1 = (\lambda + \gamma_1)(\lambda + \gamma_2)(\lambda + \gamma_3).$$

Therefore, we have

$$\det(\lambda - A(\xi)) = \prod_{j=1}^{3} (\lambda + |\xi|^2\gamma_j).$$

To show that the Shapiro–Lopatinskii condition is satisfied, we consider the system of ordinary differential equations

$$\begin{pmatrix} \lambda^2 + (\partial_n^2 - |\xi'|^2)^2 & \partial_n^2 - |\xi'|^2 \\ \lambda(|\xi'|^2 - \partial_n^2) & \lambda + |\xi'|^2 - \partial_n^2 \end{pmatrix} \begin{pmatrix} u \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3-48}$$

for $0 \neq (\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\mathbb{C}_+}$.

Let (u,θ) be a stable solution of (3-48) with $u'(0)=0$. Similar to Lemma 4.1 of [63] and [23], we multiply the first row in (3-48) by $\overline{\lambda}\overline{u}$, the second one by $\overline{\theta}$, sum up and integrate over $(0,\infty)$. This yields

$$\begin{aligned}0 &= \lambda|\lambda|^2\|u\|^2 + \overline{\lambda}\langle(\partial_n^2-|\xi'|^2)^2u,u\rangle + \overline{\lambda}\langle(\partial_n^2-|\xi'|^2)\theta,u\rangle\\ &\quad + \lambda\langle(|\xi'|^2-\partial_n^2)u,\theta\rangle + \langle(\lambda+|\xi'|^2-\partial_n^2)\theta,\theta\rangle,\end{aligned}$$

where $\|\cdot\| := \|\cdot\|_{L^2((0,\infty))}$ and $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2((0,\infty))}$. We use $u'(0)=0$ and integration by parts to obtain

$$\begin{aligned}0 &= \lambda|\lambda|^2\|u\|^2 + \overline{\lambda}(s-u'''(0)\overline{u(0)}) - \overline{\lambda}(\langle\theta',u'\rangle + \theta'(0)\overline{u(0)} + |\xi'|^2\langle\theta,u\rangle)\\ &\quad + \lambda(|\xi'|^2\langle u,\theta\rangle + \langle u',\theta'\rangle) + (\lambda+|\xi'|^2)\|\theta\|^2 + \|\theta'\|^2 + \theta'(0)\overline{\theta(0)}\end{aligned} \tag{3-49}$$

with $s := \|u''\|^2 + 2|\xi'|^2\|u'\|^2 + |\xi'|^4\|u\|^2$.

(i) In addition, let $u(0)=0$ and $\theta'(0)=0$. Taking the real part of (3-49) leads to

$$0 = \operatorname{Re}(\lambda)(|\lambda|^2\|u\|^2+s) + (\operatorname{Re}(\lambda)+|\xi'|^2)\|\theta\|^2 + \|\theta'\|^2. \tag{3-50}$$

Because of $\operatorname{Re}(\lambda)\geqslant 0$ we get $\theta'=0$. Since the solution is stable, we get $\theta=0$. In case $\lambda\neq 0$, we know $(|\xi'|^2-\partial_n^2)u=0$ from the second row in (3-48) and thus $(\partial_n^2-|\xi'|^2)^2u=0$. The first row of (3-48) finally shows $u=0$. If $\lambda=0$, we have $u=0$ due to Example 2.5.4. Therefore, $(\lambda-A,B_1)$ is parameter-elliptic in $\overline{\mathbb{C}_+}$.

(ii) With the additional initial conditions $\theta(0)=0$ and $u'''(0)=-\theta'(0)$, the real part of (3-49) is also given by (3-50). Similar as before we obtain that $(\lambda-A,B_2)$ is parameter-elliptic in $\overline{\mathbb{C}_+}$. □

In preparation for the plate-membrane system considered in the next Chapter, we show that the following operator $\mathcal{A}$ generates a C_0-semigroup. Recall that the orders of interest are determined by $\mathbf{t}=(4,2,2)$, $\mathbf{s}=(-2,0,0)$, $\mathbf{m}_1=(-4,-3,-1)$ and $\mathbf{m}_2=(-3,-2,-1)$.

3.5.2 Proposition. *Let $\Omega\subset\mathbb{R}^n$ be a C^4-domain with $\partial\Omega=\Gamma_1\cup\Gamma_2$, where Γ_1 and Γ_2 are compact, disjoint parts of the boundary of Ω. We define the operator $\mathcal{A}\colon \mathcal{H}\supset D(\mathcal{A})\to\mathcal{H}$ by*

$$\begin{aligned}\mathcal{H} &:= \{w\in H^{-\mathbf{s}}(\Omega) : w_1=\partial_\nu w_1=0 \text{ on } \Gamma_1,\ \partial_\nu w_1=0 \text{ on } \Gamma_2\},\\ D(\mathcal{A}) &:= \{w\in H^{\mathbf{t}}(\Omega) : A(D)w\in\mathcal{H},\ B_1(D)w=0 \text{ on } \Gamma_1,\ B_2(D)w=0 \text{ on } \Gamma_2\}\end{aligned}$$

and $\mathcal{A}w := A(D)w$ for $w\in D(\mathcal{A})$, where $A(D)$, $B_1(D)$ and $B_2(D)$ are as in (3-46) *and* (3-47). *On $\mathcal{H}$ we use the inner product induced by*

$$\|w\|_{\mathcal{H}}^2 := \|\Delta w_1\|_{L^2(\Omega)}^2 + \|w_2\|_{L^2(\Omega)}^2 + \|w_3\|_{L^2(\Omega)}^2 \quad (w\in\mathcal{H}),$$

which is equivalent to the canonical norm on $\mathcal{H}$, see Lemma 4.1.1. Then $\mathcal{A}$ is the generator of a C_0-semigroup of contractions on $\mathcal{H}$.

Proof. We follow the standard approach and show that $\mathcal{A}$ is disspative and that there is a $\lambda_0>0$ such that $\lambda_0-\mathcal{A}$ is surjective. Integration by parts leads to

$$\begin{aligned}\operatorname{Re}\langle\mathcal{A}w,w\rangle_{\mathcal{H}} &= \operatorname{Re}\left(\langle\Delta w_2,\Delta w_1\rangle_{L^2(\Omega)} + \langle-\Delta^2w_1-\Delta w_3,w_2\rangle_{L^2(\Omega)} + \langle\Delta w_2+\Delta w_3,w_3\rangle_{L^2(\Omega)}\right)\\ &= -\|\nabla w_3\|_{L^2(\Omega)}^2 - \|w_3\|_{L^2(\Gamma_1)}^2\end{aligned}$$

for all $w\in D(\mathcal{A})$.

Now, let $f \in \mathcal{H}$. Because of Remark 3.3.6 and Lemma 3.5.1 we know that the system

$$\begin{aligned} (\lambda - A(D))w &= f \quad \text{in } \Omega, \\ B_1(D)w &= 0 \quad \text{on } \Gamma_1, \\ B_2(D)w &= 0 \quad \text{on } \Gamma_2 \end{aligned}$$

is parameter-elliptic in $\overline{\mathbb{C}_+}$. In particular, there is a $\lambda_0 > 0$ such that there is a $w \in D(\mathcal{A})$ with $(\lambda_0 - \mathcal{A})w = f$. This shows that $\lambda_0 - \mathcal{A}$ is surjective. An application of the Theorem of Lumer–Phillips yields that $\mathcal{A}$ is the generator of a C_0-semigroup of contractions on $\mathcal{H}$. □

3.5.3 Proposition. *Let Ω and $\mathcal{A}$ be as before but with the slightly stronger assumption that Ω possesses C^5-boundary. In addition, let $T > 0$ and*

$$g \in L^2((0,T), H^{1/2}(\Gamma_2)) \cap H^{1/4}((0,T), L^2(\Gamma_2)).$$

We assume that the time-dependent problem

$$\begin{aligned} \partial_t w - A(D)w &= 0 && \text{in } (0,T) \times \Omega, \\ B_1(D)w &= 0 && \text{on } (0,T) \times \Gamma_1, \\ B_2(D)w &= \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} && \text{on } (0,T) \times \Gamma_2 \end{aligned}$$

possesses a solution $w \in C([0,T], H^{\mathbf{t}}(\Omega)) \cap C^1([0,T], H^{-\mathbf{s}}(\Omega))$. Then it holds

$$\|w\|_{L^2((0,T),H^{\mathbf{t}+\sigma}(\Omega))\cap H^{(\mathbf{t}+\sigma)/2}((0,T),L^2(\Omega))} \leqslant C\|g\|_{L^2((0,T),H^{1/2+\sigma}(\Gamma_2))\cap H^{(1+2\sigma)/4}((0,T),L^2(\Gamma_2))} \tag{3-51}$$

for every $\sigma \in (-1,0]$. In particular, we obtain

$$\|w\|_{L^2((0,T),H^{\mathbf{t}-1/2}(\Omega))\cap H^{(2\mathbf{t}-1)/4}((0,T),L^2(\Omega))} \leqslant C\|g\|_{L^2((0,T),L^2(\Gamma_2))}. \tag{3-52}$$

Proof. Because of Lemma 3.5.1 we know that our problem is parameter-elliptic in $\overline{\mathbb{C}_+}$. Due to Proposition 3.5.2, $\mathcal{A}$ is the generator of a C_0-semigroup on $\mathcal{H}$. As $\frac{1}{4} < \frac{1}{2}$, we can extend g by zero to a function in

$$L^2((-\infty,T), H^{1/2}(\Gamma_2)) \cap H^{1/4}((-\infty,T), L^2(\Gamma_2)),$$

see Theorem 1 of Section 4.3.2 in [77]. All in all, we are in the situation of Theorem 3.4.8, which yields inequality (3-51). The estimate in (3-52) is (3-51) in the case $\sigma = -\frac{1}{2}$. □

Chapter 4

A plate-membrane transmission problem

This chapter gives a summary of the main results in [11]. We study the coupling between a thermoelastic damped plate and an elastic membrane. After showing the well-posedness and regularity results of the given problem, we investigate the asymptotic behaviour of the solution. In contrast to [11], we usually consider arbitrary space dimensions $n \geqslant 2$ because the generalization does not affect the proofs. We only use space dimension $n = 2$ to show polynomial stability of the plate-membrane system in the case of an undamped membrane.

4.1 Formulation and well-posedness of the problem

To fix the geometric situation of the problem, we consider two non-empty, open, connected and bounded subsets Ω and Ω_2 of $\mathbb{R}^n$ with boundary of class C^4 such that $\overline{\Omega_2} \subset \Omega$. We set $\Omega_1 := \Omega \backslash \overline{\Omega_2}$, $\Gamma := \partial\Omega$ and $I := \partial\Omega_2$. Note that $\partial\Omega_1 = \Gamma \cup I$. As visualized in Figure 4.1, we denote the outward pointing unit normal vector to the boundary of Ω_1 by ν and consequently, $-\nu$ is the outward pointing unit normal vector to the boundary of Ω_2. The plate-membrane system of interest consists of a thermoelastic plate in Ω_1 and a membrane in the region Ω_2.

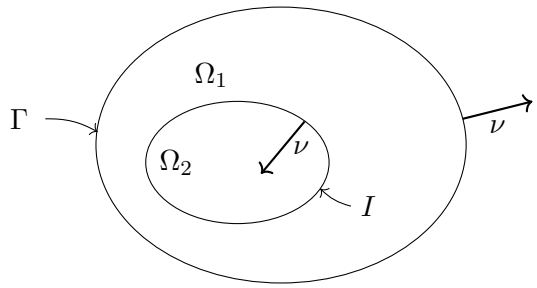

Figure 4.1: Geometric situation of the plate-membrane system in the case $n = 2$.

Let $u(t,x)$ and $v(t,x)$ be the vertical displacement of the point x at time t on the plate and the membrane, respectively. The temperature difference to a given reference temperature on the plate is denoted by $\theta(t,x)$. Then, the mathematical model for the structure can be described by the equations

$$\begin{aligned} u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + \Delta\theta = 0 \quad & \text{in } (0,\infty)\times\Omega_1, \\ \theta_t - \Delta u_t - \Delta\theta = 0 \quad & \text{in } (0,\infty)\times\Omega_1, \\ v_{tt} - \Delta v + m v_t = 0 \quad & \text{in } (0,\infty)\times\Omega_2 \end{aligned} \tag{4-1}$$

with the boundary conditions

$$\begin{aligned} u = \partial_\nu u = 0 \quad & \text{on } (0,\infty)\times\Gamma, \\ \partial_\nu \theta + \theta = 0 \quad & \text{on } (0,\infty)\times\Gamma, \\ \partial_\nu u = 0 \quad & \text{on } (0,\infty)\times I, \\ \theta = 0 \quad & \text{on } (0,\infty)\times I, \end{aligned} \tag{4-2}$$

the transmission conditions

$$\begin{aligned} u = v \qquad & \text{on } (0,\infty)\times I, \\ -\partial_\nu \Delta u - \partial_\nu \theta = \partial_\nu v \qquad & \text{on } (0,\infty)\times I \end{aligned} \tag{4-3}$$

and the initial conditions

$$u(0,\cdot) = u_0,\ u_t(0,\cdot) = u_1,\ \theta(0,\cdot) = \theta_0,\ v(0,\cdot) = v_0,\ v_t(0,\cdot) = v_1. \tag{4-4}$$

The coefficient $\gamma \geqslant 0$ represents the rotational inertia term, while $m \geqslant 0$ prescribes the damping (or the absence thereof) for the wave equation. Notice that we consider clamped boundary conditions for u on Γ. Due to the lack of thermal effects in the membrane, we assume that the difference of temperature on the interface is zero. We also suppose that the plate satisfies Newton's cooling law, which leads to the second line in (4-2). The condition $\partial_\nu u = 0$ on I has the following interpretation: the transversal force caused by the tension and the one originated by the shear stress between the plate and the membrane cancel each other, which forces the horizontal displacements on the interface to be zero, see [38].

For the sake of readability, we do not include the case of a structurally damped plate equation here. The additional damping improves the long-time behaviour as one would expect it from [12]. For details we refer to [11].

Following the standard approach, we will formulate problem (4-1)–(4-4) as an abstract Cauchy problem and study the associated semigroup. We define $w = (w_1, \dots, w_5) := (u, u_t, \theta, v, v_t)$ and rewrite problem (4-1) with the initial conditions (4-4) formally as the first-order system

$$\begin{aligned} M(D)\partial_t w - A(D)w = 0 \quad & \text{in } (0,\infty), \\ w(0) = w_0, & \end{aligned} \tag{4-5}$$

where

$$M(D) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1-\gamma\Delta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(D) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\Delta^2 & 0 & -\Delta & 0 & 0 \\ 0 & \Delta & \Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \Delta & -m \end{pmatrix}$$

and $w_0 := (u_0, u_1, \theta_0, v_0, v_1)$.

To define a suitable ground space and domain for the related operators, we set

$$H^{2,1} := \left\{ (u,v) \in H^2(\Omega_1) \times H^1(\Omega_2) : u = \partial_\nu u = 0 \text{ on } \Gamma,\ \partial_\nu u = u - v = 0 \text{ on } I \right\}.$$

On $H^{2,1}$ we use the inner product induced by

$$\|(u,v)\|^2_{H^{2,1}} := \|\Delta u\|^2_{L^2(\Omega_1)} + \|\nabla v\|^2_{L^2(\Omega_2)} \quad ((u,v) \in H^{2,1}).$$

4.1.1 Lemma. *The norm in $H^{2,1}$ is equivalent to the standard norm in $H^2(\Omega_1) \times H^1(\Omega_2)$.*

Proof. Obviously, all terms in the norm $\|\cdot\|_{H^{2,1}}$ can be estimated by the standard norm in $H^2(\Omega_1) \times H^1(\Omega_2)$, so we only have to show $\|\cdot\|_{H^2(\Omega_1)\times H^1(\Omega_2)} \leqslant C\|\cdot\|_{H^{2,1}}$.

Let $(u, v) \in H^{2,1}$. Then u is a solution of the boundary value problem

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega_1, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega_1 \end{aligned}$$

with $f := \Delta u \in L^2(\Omega_1)$. By elliptic regularity, we get

$$\|u\|_{H^2(\Omega_1)} \leqslant C\big(\|\Delta u\|_{L^2(\Omega_1)} + \|u\|_{L^2(\Omega_1)}\big), \tag{4-6}$$

see Theorem 7.1.3 of [3]. Due to the boundary and transmission conditions $u = 0$ on Γ and $u = v$ on I, the function $\chi_{\Omega_1} u + \chi_{\Omega_2} v$ belongs to $\dot{H}^1(\overline{\Omega})$. Here, χ_{Ω_j} denotes the characteristic function on Ω_j for $j = 1, 2$. With Poincaré's inequality, we obtain

$$\|u\|^2_{L^2(\Omega_1)} + \|v\|^2_{L^2(\Omega_2)} = \|\chi_{\Omega_1} u + \chi_{\Omega_2} v\|^2_{L^2(\Omega)} \leqslant C\big(\|\nabla u\|^2_{L^2(\Omega_1)} + \|\nabla v\|^2_{L^2(\Omega_2)}\big). \tag{4-7}$$

As $u = 0$ on Γ and $\partial_\nu u = 0$ on $\partial\Omega_1$, we can apply Poincaré's inequality, integration by parts, and Young's inequality to see that for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$\|u\|^2_{L^2(\Omega_1)} \leqslant C\|\nabla u\|^2_{L^2(\Omega_1)} = C\langle -\Delta u, u\rangle_{L^2(\Omega_1)} \leqslant \varepsilon\|u\|^2_{L^2(\Omega_1)} + C_\varepsilon\|\Delta u\|^2_{L^2(\Omega_1)}.$$

Choosing $\varepsilon \leqslant \frac{1}{2}$, we can estimate $\|u\|^2_{L^2(\Omega_1)} \leqslant C\|\Delta u\|^2_{L^2(\Omega_1)}$. Combining this with (4-6) and (4-7), we obtain

$$\|u\|^2_{H^2(\Omega_1)} + \|v\|^2_{H^1(\Omega_2)} \leqslant C\big(\|\Delta u\|^2_{L^2(\Omega_1)} + \|\nabla v\|^2_{L^2(\Omega_2)}\big).$$

□

For the definition of the related operators in a weak Hilbert space setting, we have to distinguish the case $\gamma > 0$ from the case $\gamma = 0$. For $\gamma > 0$, the Hilbert space $\mathcal{H}_\gamma$ is defined as

$$\begin{aligned} \mathcal{H}_\gamma &:= \big\{w \in L^2(\Omega_1)^3 \times L^2(\Omega_2)^2 : (w_1, w_4) \in H^{2,1},\ w_2 \in H^1(\Omega_1),\ w_2 = 0 \text{ on } \Gamma\big\} \\ &\quad \subset H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2). \end{aligned}$$

In case $\gamma = 0$, we modify this definition by replacing the condition $w_2 \in H^1(\Omega_1)$ by $w_2 \in L^2(\Omega_1)$ and omitting the boundary condition $w_2 = 0$ on Γ, that is,

$$\begin{aligned} \mathcal{H}_0 &:= \big\{w \in L^2(\Omega_1)^3 \times L^2(\Omega_2)^2 : (w_1, w_4) \in H^{2,1}\big\} \\ &\quad \subset H^2(\Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2). \end{aligned}$$

We endow $\mathcal{H}_\gamma$ with the inner product induced by

$$\|w\|^2_{\mathcal{H}_\gamma} := \|(w_1, w_4)\|^2_{H^{2,1}} + \|w_2\|^2_{L^2(\Omega_1)} + \gamma\|\nabla w_2\|^2_{L^2(\Omega_1)} + \|w_3\|^2_{L^2(\Omega_1)} + \|w_5\|^2_{L^2(\Omega_2)} \quad (w \in \mathcal{H}_\gamma)$$

for $\gamma \geqslant 0$.

To formulate the transmission problem (4-5) in a weak setting, we formally apply the operator Δ^2 to the first component and $-\Delta$ to the fourth component. We obtain

$$\begin{aligned} \widetilde{M}(D)\partial_t w - \widetilde{A}(D)w &= 0 \quad \text{in } (0,\infty), \\ w(0) &= w_0 \end{aligned}$$

with

$$\widetilde{M}(D) := \begin{pmatrix} \Delta^2 & 0 & 0 & 0 & 0 \\ 0 & 1-\gamma\Delta & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\Delta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \widetilde{A}(D) := \begin{pmatrix} 0 & \Delta^2 & 0 & 0 & 0 \\ -\Delta^2 & 0 & -\Delta & 0 & 0 \\ 0 & \Delta & \Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Delta \\ 0 & 0 & 0 & \Delta & -m \end{pmatrix}.$$

In this way, the weak formulation is adapted to the definition of the Hilbert space $\mathcal{H}_\gamma$ and to the boundary and transmission conditions. Let $\mathcal{H}_\gamma'$ denote the antidual space of $\mathcal{H}_\gamma$, that is, the space of all continuous, conjugate linear functionals on $\mathcal{H}_\gamma$. Even though we also sometimes use the same notation for the dual space, this should not cause any confusion as one can easily check whether the functionals are linear or conjugate linear. We define the operator $\mathbb{M}\colon \mathcal{H}_\gamma \to \mathcal{H}_\gamma'$ by

$$\langle \mathbb{M}w, \varphi\rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} := \langle w, \varphi\rangle_{\mathcal{H}_\gamma} \quad (w, \varphi \in \mathcal{H}_\gamma).$$

To define the operator related to $\widetilde{A}(D)$ in a weak setting, we introduce the subspace $\mathcal{V} \subset \mathcal{H}_\gamma$ with

$$\begin{aligned} \mathcal{V} &:= \{w \in \mathcal{H}_\gamma : (w_2, w_5) \in H^{2,1},\ w_3 \in H^1(\Omega_1),\ w_3 = 0 \text{ on } I\} \\ &\subset H^2(\Omega_1) \times H^2(\Omega_1) \times H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\Omega_2), \end{aligned}$$

where the inner product on $\mathcal{V}$ is induced by

$$\|w\|_{\mathcal{V}}^2 := \|(w_1, w_4)\|_{H^{2,1}}^2 + \|(w_2, w_5)\|_{H^{2,1}}^2 + \|\nabla w_3\|_{L^2(\Omega_1)}^2 + \|w_3\|_{L^2(\Gamma)}^2 \quad (w \in \mathcal{V}).$$

Using Remark 2.1.4 (ii), Lemma 4.1.1 and a generalized version of Poincaré's inequality, which can be found for instance in Section 4.5 of [82], we see that the norm in $\mathcal{V}$ is equivalent to the standard norm in $H^2(\Omega_1) \times H^2(\Omega_1) \times H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\Omega_2)$.

Now we can define $\mathbb{A}\colon \mathcal{V} \to \mathcal{V}'$ by

$$\begin{aligned} \langle \mathbb{A}w, \varphi\rangle_{\mathcal{V}' \times \mathcal{V}} := {} & \big\langle (w_2, w_5), (\varphi_1, \varphi_4)\big\rangle_{H^{2,1}} - \big\langle (w_1, w_4), (\varphi_2, \varphi_5)\big\rangle_{H^{2,1}} - \langle w_3, \Delta\varphi_2\rangle_{L^2(\Omega_1)} \\ & + \langle \Delta w_2, \varphi_3\rangle_{L^2(\Omega_1)} - \langle \nabla w_3, \nabla\varphi_3\rangle_{L^2(\Omega_1)} - \langle w_3, \varphi_3\rangle_{L^2(\Gamma)} \\ & - m\langle w_5, \varphi_5\rangle_{L^2(\Omega_2)} \end{aligned} \tag{4-8}$$

for $w, \varphi \in \mathcal{V}$. From the definition we immediately see that $\mathbb{M} \in L(\mathcal{H}_\gamma, \mathcal{H}_\gamma')$ and $\mathbb{A} \in L(\mathcal{V}, \mathcal{V}')$. Moreover, $\mathbb{M}$ is defined as the inner product in the Hilbert space $\mathcal{H}_\gamma$. Therefore, $\mathbb{M}$ is an isometric isomorphism from $\mathcal{H}_\gamma$ to $\mathcal{H}_\gamma'$. This allows us to define the $\mathcal{H}_\gamma$-realization of the transmission problem as the operator $\mathcal{A}\colon \mathcal{H}_\gamma \supset D(\mathcal{A}) \to \mathcal{H}_\gamma$ with

$$D(\mathcal{A}) := \{w \in \mathcal{V} : \mathbb{A}w \in \mathcal{H}_\gamma'\} \quad \text{and} \quad \mathcal{A}w := \mathbb{M}^{-1}\mathbb{A}w \quad (w \in D(\mathcal{A})).$$

We consider the abstract Cauchy problem

$$\begin{aligned} \partial_t w - \mathcal{A} w &= 0 \quad \text{in } (0,\infty), \\ w(0) &= w_0 \end{aligned} \tag{4-9}$$

with $w_0 := (u_0, u_1, \theta_0, v_0, v_1)$.

4.1.2 Remark. The following remark shows that this Cauchy problem is in fact the weak formulation of the transmission problem (4-1)–(4-4).

(i) We have $\mathbb{A} w = \widetilde{A}(D) w$ for all $w \in D(\mathcal{A})$ and $\mathbb{M} w = \widetilde{M}(D) w$ for all $w \in \mathcal{H}_\gamma$ in the sense of distributions. This follows from the definition of the operators and integration by parts by choosing $\varphi \in \mathscr{D}(\Omega_1)^3 \times \mathscr{D}(\Omega_2)^2$. Recall that $\mathscr{D}(\Omega_j)$ stands for functions in $\mathscr{D}(\mathbb{R}^n)$ with compact support in Ω_j for $j = 1, 2$. Consequently, a function $w \in C^1([0,\infty), D(\mathcal{A}))$ is a classical solution of (4-9) if and only if w satisfies (4-1) in the distributional sense.

(ii) Let $w \in D(\mathcal{A})$ be of the higher regularity $w \in H^4(\Omega_1) \times H^2(\Omega_1)^2 \times H^2(\Omega_2) \times H^1(\Omega_2)$. Then w satisfies the boundary and transmission conditions (4-2)–(4-3) in the strong sense, that is, as equality of the traces of the functions on Γ and I, respectively.

To see this, we only have to show that the second equality in (4-2) and in (4-3) holds, as the other conditions are already included in the definition of $\mathcal{V}$. Because of (i), the higher regularity of w and the density $\mathscr{D}(\Omega_j) \overset{d}{\subset} L^2(\Omega_j)$ for $j = 1, 2$, we get

$$\langle \mathbb{A} w, \varphi \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle \widetilde{A}(D) w, \varphi \rangle_{L^2(\Omega_1)^3 \times L^2(\Omega_2)^2} \quad (\varphi \in \mathcal{V}).$$

Setting $\varphi = (0, 0, \varphi_3, 0, 0)$, we obtain

$$\langle \mathbb{A} w, \varphi \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle \Delta w_2 + \Delta w_3, \varphi_3 \rangle_{L^2(\Omega_1)}$$

for all $\varphi_3 \in H^1(\Omega_1)$ with $\varphi_3 = 0$ on I. Integration by parts leads to

$$\int_\Gamma (w_3 + \partial_\nu w_3) \overline{\varphi_3}\, dS = 0$$

for all $\varphi_3 \in H^1(\Omega_1)$ with $\varphi_3 = 0$ on I. Therefore, $w_3 + \partial_\nu w_3 = 0$ holds on Γ in the strong sense, that is, as equality in the trace space $H^{1/2}(\Gamma)$. In the same way, one can prove that the second equality of (4-3) is satisfied in the strong sense.

To show well-posedness, we will also need the following result.

4.1.3 Lemma. *The space $\mathcal{V}$ is dense in $\mathcal{H}_\gamma$, and therefore we have the dense embeddings*

$$\mathcal{V} \subset \mathcal{H}_\gamma \subset L^2(\Omega_1)^3 \times L^2(\Omega_2)^2 \subset \mathcal{H}_\gamma' \subset \mathcal{V}'.$$

Proof. We divide the proof into two steps.

Step 1: In a first step, we show that

$$V(\Omega_1) := \{\varphi \in H^2(\Omega_1) : \varphi = \partial_\nu \varphi = 0 \text{ on } \Gamma,\ \partial_\nu \varphi = 0 \text{ on } I\}$$

is dense in $H^1_\Gamma(\Omega_1) := \{u \in H^1(\Omega_1) : u = 0 \text{ on } \Gamma\}$.

For this, let $u \in H^1_\Gamma(\Omega_1)$. We choose a function $\widetilde{\varphi} \in C^\infty(\Omega_1)$ with $\widetilde{\varphi} = 1$ near I, $\widetilde{\varphi} = 0$ near Γ, and $0 \leqslant \widetilde{\varphi} \leqslant 1$ in Ω_1. We set $\varphi := \widetilde{\varphi}^2$. Note that $(1-\varphi)u \in \dot{H}^1(\overline{\Omega_1})$ and $\widetilde{\varphi}u \in H^1(\Omega_1)$. As the test functions are dense in $\dot{H}^1(\overline{\Omega_1})$, there exists a sequence $\big(\varphi_k^{(1)}\big)_{k\in\mathbb{N}} \subset \mathscr{D}(\Omega_1)$ such that $\varphi_k^{(1)} \to (1-\varphi)u$ in $H^1(\Omega_1)$ for $k \to \infty$. Moreover, as the domain of the Neumann Laplacian

$$D(\Delta_N) := \{u \in H^2(\Omega_1) : \partial_\nu u = 0 \text{ on } \partial\Omega_1\}$$

is dense in $H^1(\Omega_1)$, see Lemma 1.25 of [64], there exists a sequence $\big(\widetilde{\varphi}_k^{(2)}\big)_{k\in\mathbb{N}} \subset D(\Delta_N)$ with $\widetilde{\varphi}_k^{(2)} \to \widetilde{\varphi}u$ in $H^1(\Omega_1)$ for $k \to \infty$. By setting $\varphi_k^{(2)} := \widetilde{\varphi}\widetilde{\varphi}_k^{(2)} \in V(\Omega_1)$ for $k \in \mathbb{N}$, we get $\varphi_k := \varphi_k^{(1)} + \varphi_k^{(2)} \to (1-\varphi)u + \widetilde{\varphi}^2 u = u$ in $H^1(\Omega_1)$ for $k \to \infty$.

Step 2: Now we prove that $\mathcal{V}$ is dense in $\mathcal{H}_\gamma$. Comparing the definitions of $\mathcal{V}$ and $\mathcal{H}_\gamma$ and noting that test functions are dense in L^2-spaces, we only have to consider the case $\gamma > 0$ and show that the embedding

$$H^{2,1} \subset H^1_\Gamma(\Omega_1) \times L^2(\Omega_2)$$

is dense. Therefore, we fix $u \in H^1_\Gamma(\Omega_1)$, $v \in L^2(\Omega_2)$ and $\varepsilon > 0$. Using the first step, we find a function $\varphi_1 \in V(\Omega_1)$ with $\|u - \varphi_1\|_{H^1(\Omega_1)} < \frac{\varepsilon}{2}$. Moreover, let $\widetilde{\varphi}_1 \in H^1(\Omega_2)$ such that $\varphi_1 = \widetilde{\varphi}_1$ on I, and choose $\varphi_2, \varphi_3 \in \mathscr{D}(\Omega_2)$ with $\|\widetilde{\varphi}_1 - \varphi_2\|_{L^2(\Omega_2)} < \frac{\varepsilon}{4}$ and $\|v - \varphi_3\|_{L^2(\Omega_2)} < \frac{\varepsilon}{4}$. Then we obtain $(\varphi_1, \widetilde{\varphi}_1 - \varphi_2 + \varphi_3) \in H^{2,1}$ and

$$\|(u,v) - (\varphi_1, \widetilde{\varphi}_1 - \varphi_2 + \varphi_3)\|_{H^1(\Omega_1)\times L^2(\Omega_2)} < \varepsilon.$$

This shows that the embedding $\mathcal{V} \subset \mathcal{H}_\gamma$ is dense. The same holds for the second embedding $\mathcal{H}_\gamma \subset (L^2(\Omega_1))^3 \times (L^2(\Omega_2))^2$. Therefore, all embeddings stated in the lemma are dense. □

4.1.4 Theorem. *For $\gamma, m \geqslant 0$, the operator $\mathcal{A}$ generates a C_0-semigroup $(\mathcal{S}(t))_{t\geqslant 0}$ of contractions on $\mathcal{H}_\gamma$. Therefore, for any $w_0 \in D(\mathcal{A})$ there exists a unique classical solution in*

$$C^1([0,\infty), \mathcal{H}_\gamma) \cap C([0,\infty), D(\mathcal{A}))$$

of the Cauchy problem (4-9).

Proof. We show that $\mathcal{A}$ is dissipative and $1 - \mathcal{A}$ is surjective and apply the Theorem of Lumer–Phillips, see Theorem 3.4.5 of [8].

As $\mathbb{M}\colon \mathcal{H}_\gamma \to \mathcal{H}'_\gamma$ is defined by $\langle \mathbb{M}w, \varphi\rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = \langle w, \varphi\rangle_{\mathcal{H}_\gamma}$, we get

$$\langle \mathcal{A}w, w\rangle_{\mathcal{H}_\gamma} = \langle \mathbb{M}^{-1}\mathbb{A}w, w\rangle_{\mathcal{H}_\gamma} = \langle \mathbb{A}w, w\rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} \quad (w \in D(\mathcal{A})). \tag{4-10}$$

By the definition of $\mathbb{A}$ in (4-8), we immediately obtain

$$\operatorname{Re}\langle \mathbb{A}w, w\rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = -\|\nabla w_3\|^2_{L^2(\Omega_1)} - \|w_3\|^2_{L^2(\Gamma)} - m\|w_5\|^2_{L^2(\Omega_2)} \leqslant 0 \quad (w \in D(\mathcal{A})), \tag{4-11}$$

which shows that $\mathcal{A}$ is dissipative.

To prove that $1 - \mathcal{A}$ is surjective, it suffices to show that $\mathbb{M} - \mathbb{A}\colon D(\mathcal{A}) \to \mathcal{H}'_\gamma$ is surjective since $\mathbb{M} \in L(\mathcal{H}_\gamma, \mathcal{H}'_\gamma)$ is an isomorphism. Let $f \in \mathcal{H}'_\gamma$. We have to find $w \in D(\mathcal{A})$ such that

$$(\mathbb{M} - \mathbb{A})w = f \quad \text{ in } \mathcal{H}'_\gamma. \tag{4-12}$$

In particular, we have to find $w \in \mathcal{V}$ with

$$\begin{aligned}
&\langle w, \varphi\rangle_{\mathcal{H}_\gamma} - \big\langle (w_2, w_5), (\varphi_1, \varphi_4)\big\rangle_{H^{2,1}} + \big\langle (w_1, w_4), (\varphi_2, \varphi_5)\big\rangle_{H^{2,1}} + \langle w_3, \Delta\varphi_2\rangle_{L^2(\Omega_1)} \\
&\quad - \langle \Delta w_2, \varphi_3\rangle_{L^2(\Omega_1)} + \langle \nabla w_3, \nabla\varphi_3\rangle_{L^2(\Omega_1)} + \langle w_3, \varphi_3\rangle_{L^2(\Gamma)} + m\langle w_5, \varphi_5\rangle_{L^2(\Omega_2)} \\
&= \langle (\mathbb{M} - \mathbb{A})w, \varphi\rangle_{\mathcal{V}'\times\mathcal{V}} \\
&= \langle f, \varphi\rangle_{\mathcal{V}'\times\mathcal{V}}
\end{aligned} \tag{4-13}$$

for $\varphi \in \mathcal{V}$. Choosing $\varphi = (\varphi_1, 0, 0, \varphi_4, 0) \in \mathcal{V}$ yields

$$\big\langle (w_1, w_4), (\varphi_1, \varphi_4)\big\rangle_{H^{2,1}} - \big\langle (w_2, w_5), (\varphi_1, \varphi_4)\big\rangle_{H^{2,1}} = \big\langle (f_1, f_4), (\varphi_1, \varphi_4)\big\rangle_{(H^{2,1})'\times H^{2,1}} \tag{4-14}$$

for all $(\varphi_1, \varphi_4) \in H^{2,1}$. In a first step, we only take a look at problem (4-13) in the projection of $\mathcal{V}$ to the components (w_2, w_3, w_5), that is,

$$\mathcal{V}_0 := \{(w_2, w_3, w_5) : (0, w_2, w_3, 0, w_5) \in \mathcal{V}\}$$

endowed with the canonical norm $\|\cdot\|_{\mathcal{V}_0}$. Now, we choose $\varphi = (0, \varphi_2, \varphi_3, 0, \varphi_5) \in \mathcal{V}$ and replace (4-14), applied to $(\varphi_2, \varphi_5) \in H^{2,1}$, into (4-13). This leads to

$$\begin{aligned}
&b((w_2, w_3, w_5), (\varphi_2, \varphi_3, \varphi_5)) \\
&:= \langle w_2, \varphi_2\rangle_{L^2(\Omega_1)} + \gamma\langle \nabla w_2, \nabla\varphi_2\rangle_{L^2(\Omega_1)} + \langle w_3, \varphi_3\rangle_{L^2(\Omega_1)} + \big\langle (w_2, w_5), (\varphi_2, \varphi_5)\big\rangle_{H^{2,1}} \\
&\quad - \langle \nabla w_3, \nabla\varphi_2\rangle_{L^2(\Omega_1)} + \langle \nabla w_2, \nabla\varphi_3\rangle_{L^2(\Omega_1)} + \langle \nabla w_3, \nabla\varphi_3\rangle_{L^2(\Omega_1)} + \langle w_3, \varphi_3\rangle_{L^2(\Gamma)} \\
&\quad + (1 + m)\langle w_5, \varphi_5\rangle_{L^2(\Omega_2)} \\
&= \big\langle (f_2 - f_1, f_3, f_5 - f_4), (\varphi_2, \varphi_3, \varphi_5)\big\rangle_{\mathcal{V}_0'\times\mathcal{V}_0} \\
&=: \big\langle \widetilde{f}, (\varphi_2, \varphi_3, \varphi_5)\big\rangle_{\mathcal{V}_0'\times\mathcal{V}_0}
\end{aligned} \tag{4-15}$$

for $(\varphi_2, \varphi_3, \varphi_5) \in \mathcal{V}_0$. Obviously, $b\colon \mathcal{V}_0 \times \mathcal{V}_0 \to \mathbb{C}$ is continuous, and a short computation shows

$$\operatorname{Re} b((w_2, w_3, w_5), (w_2, w_3, w_5)) \geqslant C\|(w_2, w_3, w_5)\|_{\mathcal{V}_0}^2 \quad ((w_2, w_3, w_5) \in \mathcal{V}_0).$$

Thus, we may apply the Theorem of Lax–Milgram to obtain a unique solution $(w_2, w_3, w_5) \in \mathcal{V}_0$ of

$$b((w_2, w_3, w_5), (\varphi_2, \varphi_3, \varphi_5)) = \big\langle \widetilde{f}, (\varphi_2, \varphi_3, \varphi_5)\big\rangle_{\mathcal{V}_0'\times\mathcal{V}_0} \quad ((\varphi_2, \varphi_3, \varphi_5) \in \mathcal{V}_0),$$

see Theorem 2.1 of [10]. One can argue similarly in $H^{2,1}$ to get a unique solution $(w_1, w_4) \in H^{2,1}$ of (4-14).

Altogether, we have found $w \in \mathcal{V}$ with

$$(\mathbb{M} - \mathbb{A})w = f \quad \text{in } \mathcal{V}'.$$

As the right-hand side belongs to $\mathcal{H}_\gamma'$ and $\mathcal{V}$ is dense in $\mathcal{H}_\gamma$ by Lemma 4.1.3, also the left-hand side belongs to $\mathcal{H}_\gamma'$, which shows (4-12). In particular, $\mathbb{A}w = \mathbb{M}w - f \in \mathcal{H}_\gamma'$, which ensures $w \in D(\mathcal{A})$. Therefore, $1 - \mathcal{A}$ is surjective, and an application of the Theorem of Lumer–Phillips finishes the proof. □

4.2 Spectral properties and regularity results

In this section, we study properties of the spectrum of the operator $\mathcal{A}$ defined above and show that functions in its domain have higher regularity. Note that due to Theorem 4.1.4, the operator $\mathcal{A}$ is closed and densely defined.

4.2.1 Proposition. *For $\gamma, m \geqslant 0$ we have $0 \in \rho(\mathcal{A})$.*

Proof. We show that $\mathcal{A}\colon \mathcal{H}_\gamma \supset D(\mathcal{A}) \to \mathcal{H}_\gamma$ is bijective. For this purpose, let $f \in \mathcal{H}_\gamma$. The equality $\mathcal{A}w = f$ is equivalent to

$$\langle \mathbb{A}w, \varphi\rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} = \langle \mathbb{M}f, \varphi\rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} \quad (\varphi \in \mathcal{H}_\gamma). \tag{4-16}$$

Similar to (4-14), we obtain

$$\big\langle (w_2, w_5), (\varphi_1, \varphi_4)\big\rangle_{H^{2,1}} = \big\langle (f_1, f_4), (\varphi_1, \varphi_4)\big\rangle_{H^{2,1}} \quad ((\varphi_1, \varphi_4) \in H^{2,1}),$$

which has the unique solution $(w_2, w_5) := (f_1, f_4) \in H^{2,1}$. By choosing $\varphi = (0, 0, \varphi_3, 0, 0) \in \mathcal{V}$ in (4-16), we get

$$\langle \nabla w_3, \nabla \varphi_3\rangle_{L^2(\Omega_1)} + \langle w_3, \varphi_3\rangle_{L^2(\Gamma)} = \langle \Delta f_1 - f_3, \varphi_3\rangle_{L^2(\Omega_1)} \tag{4-17}$$

for all $\varphi_3 \in H^1(\Omega_1)$ with $\varphi_3 = 0$ on I. As $\Delta f_1 - f_3 \in L^2(\Omega_1)$, the right-hand side is a continuous, conjugate linear functional of φ_3. Let us denote the left-hand side of (4-17) by $b(w_3, \varphi_3)$. Then b is a continuous sesquilinear form in the Hilbert space $\{w_3 \in H^1(\Omega_1) : w_3 = 0 \text{ on } I\}$. Since $b(\cdot, \cdot)$ is equivalent to the $H^1(\Omega_1)$-norm, the function $b(\cdot, \cdot)$ is coercive. An application of the Theorem of Lax–Milgram yields the existence of a unique solution $w_3 \in H^1(\Omega_1)$ of equation (4-17) with $w_3 = 0$ on I.

For the remaining components w_1 and w_4, we choose $\varphi = (0, \varphi_2, 0, 0, \varphi_5) \in \mathcal{V}$ in (4-16) and obtain

$$\begin{aligned} -\big\langle (w_1, w_4), (\varphi_2, \varphi_5)\big\rangle_{H^{2,1}} &= \langle w_3, \Delta\varphi_2\rangle_{L^2(\Omega_1)} + m\langle w_5, \varphi_5\rangle_{L^2(\Omega_2)} \\ &\quad + \langle f_2, \varphi_2\rangle_{L^2(\Omega_1)} + \gamma\langle \nabla f_2, \nabla\varphi_2\rangle_{L^2(\Omega_1)} + \langle f_5, \varphi_5\rangle_{L^2(\Omega_2)} \\ &=: R(\varphi_2, \varphi_5) \end{aligned} \tag{4-18}$$

for $(\varphi_2, \varphi_5) \in H^{2,1}$. The conjugate linear functional $R\colon H^{2,1} \to \mathbb{C}$ is well defined and continuous. By the Riesz representation Theorem, there exists a unique solution $(w_1, w_4) \in H^{2,1}$ of (4-18). In total, we have $w := (w_1, \dots, w_5) \in \mathcal{V}$ and, again by using the density $\mathcal{V} \overset{d}{\subset} \mathcal{H}_\gamma$, w is a solution of (4-16). In particular, $\mathbb{A}w \in \mathcal{H}_\gamma'$ by construction such that $w \in D(\mathcal{A})$. This shows that $\mathcal{A}$ is surjective. As the solution w constructed above is unique, we also obtain the injectivity of $\mathcal{A}$. As $\mathcal{A}\colon \mathcal{H}_\gamma \supset D(\mathcal{A}) \to \mathcal{H}_\gamma$ is bijective and closed, we get $0 \in \rho(\mathcal{A})$. □

4.2.2 Remark. For the proof of higher regularity of the solution, we recall that $\lambda - \Delta$ is parameter-elliptic with Dirichlet boundary condition and with Neumann boundary condition, see Example 2.5.5. As only the principal part is involved in the definition of parameter-ellipticity, also $\lambda - \Delta$ with the mixed boundary condition $\partial_\nu u + u$ is parameter-elliptic. The same holds for $\lambda + \Delta^2$ with boundary conditions u and $\partial_\nu u$, see Example 2.5.5.

In the following, we will show that $D(\mathcal{A})$ is embedded into a tuple of Sobolev spaces of higher regularity. The a priori estimate for parameter-elliptic systems from Proposition 2.5.6 together with the results above are the key for the strong solvability of the transmission problem, that is, for higher regularity of the weak solution.

4.2.3 Theorem. *Let $\gamma, m \geqslant 0$. Then the following embeddings are continuous.*

(i) $D(\mathcal{A}) \subset H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2)$,

(ii) $D(\mathcal{A}^2) \subset H^4(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2)$,

(iii) $D(\mathcal{A}) \subset H^4(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2)$ *for* $\gamma = 0$.

In consequence, if $w_0 \in D(\mathcal{A}^2)$, then $\mathcal{S}(t)w_0$ is the unique solution of problem (4-1)–(4-4) for $t \geqslant 0$ and satisfies the boundary and transmission conditions in the strong sense of traces. In the case $\gamma = 0$, we get the same result even for $w_0 \in D(\mathcal{A})$. Here, $(\mathcal{S}(t))_{t\geqslant 0}$ denotes the C_0-semigroup generated by $\mathcal{A}$.

Proof. The main idea of the proof is to set up suitable parameter-elliptic boundary value problems, where we already know that the solution is of higher regularity. Hence, we show that this solution coincides with an appropriate component of our element in the domain.

(i) Let $w \in D(\mathcal{A})$ and $f := \mathcal{A}w$. Due to the definition of $D(\mathcal{A}) \subset \mathcal{V}$, we only have to show $w_3 \in H^2(\Omega_1)$ and $w_4 \in H^2(\Omega_2)$. First, we show $w_3 \in H^2(\Omega_1)$. Similar to (4-16), we get

$$\langle \mathbb{A}w, \varphi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = \langle \mathbb{M}f, \varphi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} \quad (\varphi \in \mathcal{H}_\gamma). \tag{4-19}$$

As in (4-17), we have

$$\langle \nabla w_3, \nabla \varphi_3 \rangle_{L^2(\Omega_1)} + \langle w_3, \varphi_3 \rangle_{L^2(\Gamma)} = \langle g, \varphi_3 \rangle_{L^2(\Omega_1)} \tag{4-20}$$

for $\varphi_3 \in H^1(\Omega_1)$ with $\varphi_3 = 0$ on I, where $g := \Delta w_2 - f_3$. By Proposition 2.5.6, Remark 3.3.6 and Remark 4.2.2, we know that there exists some $\lambda_0 > 0$ such that the problem

$$\begin{aligned} (\lambda_0 - \Delta)\,\widetilde{w}_3 &= \lambda_0 w_3 + g && \text{in} \quad \Omega_1, \\ \partial_\nu \widetilde{w}_3 &= -w_3 && \text{on} \quad \Gamma, \\ \widetilde{w}_3 &= 0 && \text{on} \quad I \end{aligned}$$

has a unique solution $\widetilde{w}_3 \in H^2(\Omega_1)$. Integration by parts and (4-20) show that $z := \widetilde{w}_3 - w_3$ satisfies

$$0 = \langle (\lambda_0 - \Delta)\widetilde{w}_3 - \lambda_0 w_3 - g, \varphi_3 \rangle_{L^2(\Omega_1)} = \lambda_0 \langle z, \varphi_3 \rangle_{L^2(\Omega_1)} + \langle \nabla z, \nabla \varphi_3 \rangle_{L^2(\Omega_1)}$$

for all $\varphi_3 \in H^1(\Omega_1)$ with $\varphi_3 = 0$ on I. Choosing $\varphi_3 = z$, we get $w_3 = \widetilde{w}_3 \in H^2(\Omega_1)$.

Now, we prove $w_4 \in H^2(\Omega_2)$. We choose $\varphi = (0, 0, 0, 0, \varphi_5) \in \mathcal{V}$ in (4-19). As in (4-18), we obtain

$$\langle \nabla w_4, \nabla \varphi_5 \rangle_{L^2(\Omega_2)} = \langle g, \varphi_5 \rangle_{L^2(\Omega_2)} \quad (\varphi_5 \in \dot{H}^1(\overline{\Omega_2})),$$

where $g := -mw_5 - f_5$. Again, by parameter-elliptic theory and Remark 4.2.2, there exists a unique $\widetilde{w}_4 \in H^2(\Omega_2)$ with

$$\begin{aligned} -\Delta \widetilde{w}_4 &= g && \text{in} \quad \Omega_2 \\ \widetilde{w}_4 &= w_1 && \text{on} \quad I. \end{aligned}$$

Therefore $z := \widetilde{w}_4 - w_4 \in \dot{H}^1(\overline{\Omega_2})$ fulfils

$$0 = \langle -\Delta \widetilde{w}_4 - g, \varphi_5 \rangle_{L^2(\Omega_2)} = \langle \nabla z, \nabla \varphi_5 \rangle_{L^2(\Omega_2)} \quad (\varphi_5 \in \dot{H}^1(\overline{\Omega_2})).$$

By choosing $\varphi_5 = z$, we obtain $w_4 = \widetilde{w}_4 \in H^2(\Omega_2)$.

(ii) Now, let $w \in D(\mathcal{A}^2)$ and again $f := \mathcal{A}w$. We show $w_1 \in H^4(\Omega_1)$. Integration by parts and (4-18) leads to

$$\begin{aligned}-\big\langle (w_1, w_4), (\varphi_2, \varphi_5)\big\rangle_{H^{2,1}} &= \langle w_3, \Delta\varphi_2\rangle_{L^2(\Omega_1)} + m\langle w_5, \varphi_5\rangle_{L^2(\Omega_2)} \\ &\quad + \langle f_2, \varphi_2\rangle_{L^2(\Omega_1)} + \gamma\langle \nabla f_2, \nabla\varphi_2\rangle_{L^2(\Omega_1)} + \langle f_5, \varphi_5\rangle_{L^2(\Omega_2)} \\ &= -\langle g_1, \varphi_2\rangle_{L^2(\Omega_1)} + \langle h, \varphi_2\rangle_{L^2(I)} - \langle g_4, \varphi_5\rangle\end{aligned}$$

for $(\varphi_2, \varphi_5) \in H^{2,1}$, where $g_1 := \Delta(-w_3+\gamma f_2)-f_2$, $h := \partial_\nu(-w_3+\gamma f_2)$ and $g_4 := -mw_5-f_5$. As $f = \mathcal{A}w \in D(\mathcal{A})$, we have $\partial_\nu f_2 = 0$ such that $h = -\partial_\nu w_3 \in H^{1/2}(I)$. Due to parameter-elliptic theory, see Theorem 4 of [24], and Remark 3.3.6, there is a $\lambda_0 > 0$ such that there exists a unique solution $(\widetilde{w}_1, \widetilde{w}_4) \in H^4(\Omega_1) \times H^2(\Omega_2)$ of the transmission problem

$$\begin{aligned}(\lambda_0 + \Delta^2)\widetilde{w}_1 &= \lambda_0 w_1 + g_1 && \text{in} \quad \Omega_1,\\ (\lambda_0 - \Delta)\widetilde{w}_4 &= \lambda_0 w_4 + g_4 && \text{in} \quad \Omega_2,\\ \widetilde{w}_1 = \partial_\nu\widetilde{w}_1 &= 0 && \text{on} \quad \Gamma,\\ \widetilde{w}_1 - \widetilde{w}_4 = \partial_\nu\widetilde{w}_1 &= 0 && \text{on} \quad I,\\ \partial_\nu\Delta\widetilde{w}_1 + \partial_\nu\widetilde{w}_4 &= h && \text{on} \quad I.\end{aligned}$$

Using integration by parts, we see that $(z_1, z_4) := (\widetilde{w}_1 - w_1, \widetilde{w}_4 - w_4)$ fulfils

$$\begin{aligned}0 &= \langle(\lambda_0 + \Delta^2)\widetilde{w}_1 - \lambda_0 w_1 - g_1, \varphi_2\rangle_{L^2(\Omega_1)} + \langle(\lambda_0 - \Delta)\widetilde{w}_4 - \lambda_0 w_4 - g_4, \varphi_5\rangle_{L^2(\Omega_2)} \\ &= \lambda_0\langle z_1, \varphi_2\rangle_{L^2(\Omega_1)} + \langle\Delta z_1, \Delta\varphi_2\rangle_{L^2(\Omega_1)} + \lambda_0\langle z_4, \varphi_5\rangle_{L^2(\Omega_2)} + \langle\nabla z_4, \nabla\varphi_5\rangle_{L^2(\Omega_2)}\end{aligned}$$

for all $(\varphi_2, \varphi_5) \in H^{2,1}$. By choosing $(\varphi_2, \varphi_5) = (z_1, z_4) \in H^{2,1}$, we get $w_1 = \widetilde{w}_1 \in H^4(\Omega_1)$.

(iii) Let $\gamma = 0$ and $w \in D(\mathcal{A})$. Following the proof of (ii), we obtain $w_1 \in H^4(\Omega_1)$.

According to Lemma 2.1.1 all embeddings are continuous. □

4.2.4 Corollary. *For $\gamma, m \geqslant 0$ we have the continuous embedding*

$$D(\mathcal{A}) \subset H^3(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2).$$

Proof. From Theorem 4.1.4 we know that $\mathcal{A}\colon \mathcal{H}_\gamma \supset D(\mathcal{A}) \to \mathcal{H}_\gamma$ is the generator of a strongly continuous semigroup of contractions on $\mathcal{H}_\gamma$. So, $-\mathcal{A}$ is an m-accretive operator, see Section 4.3 of [57]. By Corollary 4.30 and Corollary 4.37 of [57], we obtain $D(\mathcal{A}) = (\mathcal{H}_\gamma, D(\mathcal{A}^2))_{1/2,2}$, where $(\cdot,\cdot)_{1/2,2}$ denotes the real interpolation functor with parameter $\frac{1}{2}$ and 2. Due to the definition of $\mathcal{H}_\gamma$ and Theorem 4.2.3 (ii) it holds

$$\begin{aligned}\mathcal{H}_\gamma &\subset H^2(\Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2),\\ D(\mathcal{A}^2) &\subset H^4(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2).\end{aligned}$$

Interpolation together with Theorem 4.2.3 (i) shows

$$D(\mathcal{A}) \subset (H^2(\Omega_1), H^4(\Omega_1))_{1/2,2} \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2).$$

By Theorem 1 of Subsection 4.3.1 in [77], we get $(H^2(\Omega_1), H^4(\Omega_1))_{1/2,2} = H^3(\Omega_1)$. □

The following result allows us to affirm that the spectrum $\sigma(\mathcal{A})$ coincides with its point spectrum $\sigma_p(\mathcal{A})$.

4.2.5 Lemma. *The operator $\mathcal{A}^{-1}\colon \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is compact for $\gamma, m \geqslant 0$.*

Proof. First, let $\gamma > 0$. By Corollary 4.2.4 and the Rellich–Kondrachov Theorem, we have

$$\begin{aligned} D(\mathcal{A}) &\subset H^3(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2) \\ &\overset{c}{\subset} H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2). \end{aligned}$$

Since $\mathcal{H}_\gamma$ is a closed subspace of $H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$, we get the compact embedding $D(\mathcal{A}) \overset{c}{\subset} \mathcal{H}_\gamma$. Therefore the identity operator $\mathrm{id}\colon D(\mathcal{A}) \to \mathcal{H}_\gamma$ is compact. Proposition 4.2.1 implies the continuity of the operator $\mathcal{A}^{-1}\colon \mathcal{H}_\gamma \to D(\mathcal{A})$. In consequence, $\mathcal{A}^{-1} = \mathrm{id} \circ \mathcal{A}^{-1}\colon \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is a compact operator. One can argue analogously for $\gamma = 0$. □

4.2.6 Lemma. *If $\gamma \geqslant 0$ and $m > 0$, then $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. Let us suppose $\gamma \geqslant 0$ and $m > 0$. Since $\mathcal{A}^{-1}$ is compact, the spectrum of $\mathcal{A}$ consists of eigenvalues only. Thus, we have to establish that there are no purely imaginary eigenvalues. Let $0 \neq \lambda \in \mathbb{R}$ and $w \in D(\mathcal{A})$ with $\mathcal{A}w = i\lambda w$, that is,

$$\langle \mathbb{A}w, \varphi \rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} = i\lambda \langle w, \varphi \rangle_{\mathcal{H}_\gamma} \quad (\varphi \in \mathcal{H}_\gamma). \tag{4-21}$$

Similar as in the proof of Proposition 4.2.1, we see that $i\lambda w_1 = w_2$ and $i\lambda w_4 = w_5$. Choosing $\varphi = w$ in (4-21), we obtain

$$\lambda \mathrm{Im} \langle \mathbb{A}w, w \rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} = \lambda^2 \|w\|^2_{\mathcal{H}_\gamma}, \tag{4-22}$$

and

$$0 = \mathrm{Re}\, \langle \mathbb{A}w, w \rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} = -\|\nabla w_3\|^2_{L^2(\Omega_1)} - \|w_3\|^2_{L^2(\Gamma)} - m\|w_5\|^2_{L^2(\Omega_2)}$$

as in (4-11). Hence, we have $w_3 = w_5 = 0$ and therefore $w_4 = 0$, that is, $w = (w_1, w_2, 0, 0, 0)$. Equality (4-22) leads to

$$2\|\Delta w_2\|^2_{L^2(\Omega_1)} = \lambda \mathrm{Im} \langle \mathbb{A}w, w \rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} = \|\Delta w_2\|^2_{L^2(\Omega_1)} + \lambda^2 \left(\|w_2\|^2_{L^2(\Omega_1)} + \gamma \|\nabla w_2\|^2_{L^2(\Omega_1)} \right),$$

therefore, we have

$$\|\Delta w_2\|^2_{L^2(\Omega_1)} = \lambda^2 \left(\|w_2\|^2_{L^2(\Omega_1)} + \gamma \|\nabla w_2\|^2_{L^2(\Omega_1)} \right). \tag{4-23}$$

Now, we choose $\varphi = (0, 0, \varphi_3, 0, 0) \in \mathcal{H}_\gamma$ in (4-21) and obtain $0 = \langle \Delta w_2, \varphi_3 \rangle_{L^2(\Omega_1)}$ for all functions $\varphi_3 \in H^1(\Omega_1)$ with $\varphi_3 = 0$ on I. In consequence, we get $\Delta w_2 = 0$. It follows from (4-23) that $w_2 = 0$ and thus $w_1 = 0$. □

4.2.7 Remark. We will see in the proof of Theorem 4.3.14 that $i\mathbb{R} \subset \rho(\mathcal{A})$ also holds for $m = 0$.

4.3 Asymptotic behaviour

In this section we investigate the asymptotic behaviour of the semigroup generated by $\mathcal{A}$. Here, we distinguish between the case of a damped membrane, that is $m > 0$, and an undamped membrane, that is $m = 0$.

4.3 a) Exponential stability in the damped case

For $m > 0$ we can show that the semigroup generated by $\mathcal{A}$ is exponentially stable in the sense of Definition 2.3.3. In difference to Theorem 4.1 of [11] we can also cover the case $\gamma > 0$. The proof is similar to the one of Theorem 4.1 in [11], but additionally uses some ideas of the proof of Theorem 2.2 in [55].

4.3.1 Theorem. *The semigroup generated by $\mathcal{A}$ is exponentially stable for $\gamma \geqslant 0$ and $m > 0$.*

Proof. For the proof, we use the characterization of exponential stability by Gearhart and Prüss which tells us that the semigroup is exponentially stable if $i\mathbb{R} \subset \rho(\mathcal{A})$ and there is a constant $C > 0$, which does not depend on $\lambda \in \mathbb{R}$, such that

$$\left\|(i\lambda - \mathcal{A})^{-1}\right\|_{L(\mathcal{H}_\gamma)} \leqslant C \quad (\lambda \in \mathbb{R}),$$

see Proposition 2.3.6. Assume that this is not true. As we already know $i\mathbb{R} \subset \rho(\mathcal{A})$ from Lemma 4.2.6, there are series $(\lambda_k)_k \subset \mathbb{R}\backslash\{0\}$ and $(w^k)_k \subset D(\mathcal{A})$ with $\|w^k\|_{\mathcal{H}_\gamma} = 1$ for every $k \in \mathbb{N}$ such that

$$f^k := (i\lambda_k - \mathcal{A})w^k \to 0 \quad \text{in } \mathcal{H}_\gamma \quad (k \to \infty).$$

Since $(\cdot - \mathcal{A})^{-1} \colon \rho(\mathcal{A}) \to L(\mathcal{H}_\gamma)$ is continuous, it holds $|\lambda_k| \to \infty$ for $k \to \infty$. Additionally, we have higher regularity due to Corollary 4.2.4 such that

$$|\lambda_k|^{-1}\|w^k\|_{H^3(\Omega_1)\times H^2(\Omega_1)^2\times H^2(\Omega_2)\times H^1(\Omega_2)} \leqslant C|\lambda_k|^{-1}(\|w^k\|_{\mathcal{H}_\gamma} + \|f^k - i\lambda_k w^k\|_{\mathcal{H}_\gamma}) \leqslant C \tag{4-24}$$

for $k \in \mathbb{N}$. Using (4-11), we obtain

$$\|\nabla w_3^k\|^2_{L^2(\Omega_1)} + \|w_3^k\|^2_{L^2(\Gamma)} + m\|w_5^k\|^2_{L^2(\Omega_2)} = \operatorname{Re}\langle(i\lambda_k - \mathcal{A})w^k, w^k\rangle_{\mathcal{H}_\gamma} \to 0 \quad (k \to \infty).$$

Applying Poincaré's inequality, we get

$$w_3^k \to 0 \quad \text{in } H^1(\Omega_1), \tag{4-25}$$

$$w_5^k \to 0 \quad \text{in } L^2(\Omega_2) \tag{4-26}$$

for $k \to \infty$. In consequence, it holds

$$\|(w_1^k, w_4^k)\|^2_{H^{2,1}} + \|w_2^k\|^2_{L^2(\Omega_1)} + \gamma\|\nabla w_2^k\|^2_{L^2(\Omega_1)} \to 1 \quad (k \to \infty). \tag{4-27}$$

The equality $f^k = (i\lambda_k - \mathcal{A})w^k$ implies

$$(i\lambda_k w_1^k - w_2^k, i\lambda_k w_4^k - w_5^k) = (f_1^k, f_4^k), \tag{4-28}$$

$$\langle i\lambda_k w^k - f^k, \varphi\rangle_{\mathcal{H}_\gamma} = \langle \mathbb{A}w^k, \varphi\rangle_{\mathcal{H}_\gamma'\times\mathcal{H}_\gamma} \quad (\varphi \in \mathcal{H}_\gamma). \tag{4-29}$$

Choosing $\varphi = (0, w_1^k, 0, 0, w_4^k) \in \mathcal{V}$ in equation (4-29) leads to

$$\begin{aligned} &\langle i\lambda_k w_2^k - f_2^k, w_1^k\rangle_{L^2(\Omega_1)} + \gamma\langle\nabla(i\lambda_k w_2^k - f_2^k), \nabla w_1^k\rangle_{L^2(\Omega_1)} + \langle i\lambda_k w_5^k - f_5^k, w_4^k\rangle_{L^2(\Omega_2)} \\ &= -\|(w_1^k, w_4^k)\|_{H^{2,1}}^2 - \langle w_3^k, \Delta w_1^k\rangle_{L^2(\Omega_1)} - m\langle w_5^k, w_4^k\rangle_{L^2(\Omega_2)}. \end{aligned} \tag{4-30}$$

Taking (4-28) into account, we obtain

$$\begin{aligned} &\langle i\lambda_k w_2^k, w_1^k\rangle_{L^2(\Omega_1)} + \gamma\langle i\lambda_k \nabla w_2^k, \nabla w_1^k\rangle_{L^2(\Omega_1)} + \langle i\lambda_k w_5^k, w_4^k\rangle_{L^2(\Omega_2)} \\ &= -\langle w_2^k, w_2^k + f_1^k\rangle_{L^2(\Omega_1)} - \gamma\langle\nabla w_2^k, \nabla(w_2^k + f_1^k)\rangle_{L^2(\Omega_1)} - \langle w_5^k, w_5^k + f_4^k\rangle_{L^2(\Omega_2)} \\ &= -\|w_2^k\|_{L^2(\Omega_1)}^2 - \gamma\|\nabla w_2^k\|_{L^2(\Omega_1)}^2 - \|w_5^k\|_{L^2(\Omega_2)}^2 \\ &\quad - \langle w_2^k, f_1^k\rangle_{L^2(\Omega_1)} - \gamma\langle\nabla w_2^k, \nabla f_1^k\rangle_{L^2(\Omega_1)} - \langle w_5^k, f_4^k\rangle_{L^2(\Omega_2)}. \end{aligned}$$

We use this fact in equality (4-30) to see

$$\begin{aligned} &\|(w_1^k, w_4^k)\|_{H^{2,1}}^2 - \|w_2^k\|_{L^2(\Omega_1)}^2 - \gamma\|\nabla w_2^k\|_{L^2(\Omega_1)}^2 \\ &= \|w_5^k\|_{L^2(\Omega_2)}^2 + \langle w_2^k, f_1^k\rangle_{L^2(\Omega_1)} + \gamma\langle\nabla w_2^k, \nabla f_1^k\rangle_{L^2(\Omega_1)} + \langle w_5^k, f_4^k\rangle_{L^2(\Omega_2)} + \langle f_2^k, w_1^k\rangle_{L^2(\Omega_1)} \\ &\quad + \gamma\langle\nabla f_2^k, \nabla w_1^k\rangle_{L^2(\Omega_1)} + \langle f_5^k, w_4^k\rangle_{L^2(\Omega_2)} - \langle w_3^k, \Delta w_1^k\rangle_{L^2(\Omega_1)} - m\langle w_5^k, w_4^k\rangle_{L^2(\Omega_2)}. \end{aligned} \tag{4-31}$$

The right-hand side converges to 0 for $k \to \infty$ because of $\|w^k\|_{\mathcal{H}_\gamma} = 1$, $\|f^k\|_{\mathcal{H}_\gamma} \to 0$ and the convergences (4-25) and (4-26). In consequence, we get

$$\|(w_1^k, w_4^k)\|_{H^{2,1}}^2 - \|w_2^k\|_{L^2(\Omega_1)}^2 - \gamma\|\nabla w_2^k\|_{L^2(\Omega_1)}^2 \to 0 \quad (k \to \infty). \tag{4-32}$$

Summing up the convergences (4-27) and (4-32) results in

$$\|(w_1^k, w_4^k)\|_{H^{2,1}}^2 \to \frac{1}{2} \quad (k \to \infty). \tag{4-33}$$

We split the remaining part of the proof into two parts.

Step 1: We show that $\|\nabla w_4^k\|_{L^2(\Omega_2)}^2 \to 0$ for $k \to \infty$. Using integration by parts, equation (4-28), the convergence (4-26) and the boundedness (4-24), we see that

$$\begin{aligned} i\|\nabla w_4^k\|_{L^2(\Omega_2)}^2 + \lambda_k^{-1}\langle w_5^k, \partial_\nu w_4^k\rangle_{L^2(I)} &= \lambda_k^{-1}\langle i\lambda_k \nabla w_4^k - \nabla w_5^k, \nabla w_4^k\rangle_{L^2(\Omega_2)} - \lambda_k^{-1}\langle w_5^k, \Delta w_4^k\rangle_{L^2(\Omega_2)} \\ &= \lambda_k^{-1}\langle\nabla f_4^k, \nabla w_4^k\rangle_{L^2(\Omega_2)} - \lambda_k^{-1}\langle w_5^k, \Delta w_4^k\rangle_{L^2(\Omega_2)} \\ &\to 0 \end{aligned}$$

for $k \to \infty$. Plugging $\varphi = (0, 0, 0, 0, \varphi_5) \in \mathcal{V}$ with $\varphi_5 \in \mathscr{D}(\Omega_2)$ into (4-29), we get

$$\langle i\lambda_k w_5^k - f_5^k, \varphi_5\rangle_{L^2(\Omega_2)} = -\langle\nabla w_4^k, \nabla\varphi_5\rangle_{L^2(\Omega_2)} - m\langle w_5^k, \varphi_5\rangle_{L^2(\Omega_2)}.$$

Since $\mathscr{D}(\Omega_2)$ is dense in $L^2(\Omega_2)$, we have

$$-\Delta w_4^k + (m + i\lambda_k) w_5^k = f_5^k.$$

Together with (4-26) we obtain

$$\lambda_k^{-1}\Delta w_4^k \to 0 \quad \text{in } L^2(\Omega_2) \quad (k \to \infty).$$

To show $\lambda_k^{-1}\langle w_5^k, \partial_\nu w_4^k\rangle_{L^2(I)} \to 0$ we use Lemma 2.4.1, $w^k \in D(\mathcal{A})$, Lemma 2.2.2, the boundedness in (4-24) and elliptic regularity. This leads to

$$\begin{aligned}
|\lambda_k^{-1}\langle w_5^k, \partial_\nu w_4^k\rangle_{L^2(I)}| &\leqslant C|\lambda_k|^{-1}\|w_2^k\|_{H^1(\Omega_1)}\|w_4^k\|^{1/2}_{H^2(\Omega_2)}\|w_4^k\|^{1/2}_{H^1(\Omega_2)}\\
&\leqslant C|\lambda_k|^{-1/2}\|w_2^k\|_{H^1(\Omega_1)}\cdot|\lambda_k|^{-1/2}\|w_4^k\|^{1/2}_{H^2(\Omega_2)}\\
&\leqslant C\big(|\lambda_k|^{-1}\|w_2^k\|_{H^2(\Omega_1)} + \|w_2^k\|_{L^2(\Omega_1)}\big)\cdot|\lambda_k|^{-1/2}\|w_4^k\|^{1/2}_{H^2(\Omega_2)}\\
&\leqslant C|\lambda_k|^{-1/2}\|w_4^k\|^{1/2}_{H^2(\Omega_2)}\\
&\leqslant C\Big(|\lambda_k|^{-1}\big(\|\Delta w_4^k\|_{L^2(\Omega_2)} + \|w_4^k\|_{L^2(\Omega_2)} + \|w_1^k\|_{H^2(\Omega_1)}\big)\Big)^{1/2}\\
&\to 0
\end{aligned}$$

for $k \to \infty$. In total, we have shown $\|\nabla w_4^k\|^2_{L^2(\Omega_2)} \to 0$ for $k \to \infty$.

Step 2: In the next part we prove $\|\Delta w_1^k\|^2_{L^2(\Omega_1)} \to 0$ for $k \to \infty$. Plugging $\varphi = (0,0,\varphi_3,0,0)$ with $\varphi_3 \in \mathscr{D}(\Omega_1)$ into (4-29) we get

$$\langle i\lambda_k w_3^k - f_3^k, \varphi_3\rangle_{L^2(\Omega_1)} = \langle \Delta w_2^k, \varphi_3\rangle_{L^2(\Omega_1)} - \langle \nabla w_3^k, \nabla\varphi_3\rangle_{L^2(\Omega_1)}.$$

In consequence, we have

$$i\lambda_k w_3^k - \Delta w_2^k - \Delta w_3^k = f_3^k.$$

Together with (4-25) it holds

$$\lambda_k^{-1}(\Delta w_2^k + \Delta w_3^k) \to 0 \quad \text{ in } L^2(\Omega_1) \quad (k \to \infty).$$

Combined with (4-28) we obtain

$$i\Delta w_1^k + \lambda_k^{-1}\Delta w_3^k \to 0 \quad \text{ in } L^2(\Omega_1) \quad (k \to \infty). \tag{4-34}$$

In particular, integration by parts leads to

$$\begin{aligned}
&i\|\Delta w_1^k\|^2_{L^2(\Omega_1)} - \lambda_k^{-1}\Big(\langle \nabla w_3^k, \nabla\Delta w_1^k\rangle_{L^2(\Omega_1)} - \langle \partial_\nu w_3^k, \Delta w_1^k\rangle_{L^2(\partial\Omega_1)}\Big)\\
=\,&i\|\Delta w_1^k\|^2_{L^2(\Omega_1)} + \lambda_k^{-1}\langle \Delta w_3^k, \Delta w_1^k\rangle_{L^2(\Omega_1)}\\
=\,&\langle i\Delta w_1^k + \lambda_k^{-1}\Delta w_3^k, \Delta w_1^k\rangle_{L^2(\Omega_1)}\\
\to\,&0
\end{aligned} \tag{4-35}$$

for $k \to \infty$. In total, we can estimate the inner products with Lemma 2.4.1 and the boundedness in (4-24) as follows

$$\begin{aligned}
&|\lambda_k|^{-1}|\langle \nabla w_3^k, \nabla\Delta w_1^k\rangle_{L^2(\Omega_1)} - \langle \partial_\nu w_3^k, \Delta w_1^k\rangle_{L^2(\partial\Omega_1)}|\\
\leqslant\,&C|\lambda_k|^{-1}\Big(\|w_3^k\|_{H^1(\Omega_1)}\|w_1^k\|_{H^3(\Omega_1)} + \|w_3^k\|^{1/2}_{H^2(\Omega_1)}\|w_3^k\|^{1/2}_{H^1(\Omega_1)}\|w_1^k\|^{1/2}_{H^3(\Omega_1)}\|w_1^k\|^{1/2}_{H^2(\Omega_1)}\Big)\\
\leqslant\,&C\Big(\|w_3^k\|_{H^1(\Omega_1)} + \|w_3^k\|^{1/2}_{H^1(\Omega_1)}\Big)\\
\to\,&0
\end{aligned}$$

for $k \to \infty$ according to convergence (4-25). Combined with (4-35) we get $\|\Delta w_1^k\|^2_{L^2(\Omega_1)} \to 0$ for $k \to \infty$. Step 1 and 2 together are in contradiction to (4-33). □

4.3 b) Lack of exponential stability in the undamped case

If the membrane is undamped, that is $m = 0$, there is a lack of exponential stability. For the proof, we follow the ideas of Theorem 3.5 in [61] and use energy methods as well as compact perturbations. To this end, we need the following compactness result of vector-valued L^2-spaces. It is known as the Lemma of Aubin–Lions. We state a formulation thereof in the Hilbert space case.

4.3.2 Lemma (Aubin–Lions)**.** [72, Proposition III.1.3]. *Let X_0, X, X_1 be Hilbert spaces with $X_0 \overset{c}{\subset} X \subset X_1$, and $T > 0$. Then it holds*

$$\{u \in L^2((0,T), X_0) : u_t \in L^2((0,T), X_1)\} \overset{c}{\subset} L^2((0,T), X).$$

To apply the Lemma of Aubin–Lions later on, we make the following observation concerning the time derivative of the solution of an abstract Cauchy problem in the larger extrapolation space, see Definition 2.2.3.

4.3.3 Lemma. *Let X_0 be a Hilbert space and $A\colon X_0 \supset D(A) \to X_0$ be the generator of a C_0-semigroup $(S(t))_{t\geqslant 0}$ with $0 \in \rho(A)$. Then for all $T > 0$ there is a constant $C > 0$ such that*

$$\sup_{t\in(0,T)} \|\partial_t(S(t)w_0)\|_{-1} \leqslant C\|w_0\|_{X_0} \quad (w_0 \in D(A)),$$

where $\|\cdot\|_{-1}$ denotes the norm of the extrapolation space X_{-1} in the sense of Definition 2.2.3.

Proof. By the definition of $\|\cdot\|_{-1}$ and by using that A is the generator of $(S(t))_{t\geqslant 0}$, we obtain

$$\sup_{t\in(0,T)} \|\partial_t(S(t)w_0)\|_{-1} = \sup_{t\in(0,T)} \|AS(t)w_0\|_{-1} = \sup_{t\in(0,T)} \|S(t)w_0\|_{X_0} \leqslant C\|w_0\|_{X_0} \quad (w_0 \in D(A))$$

as the operator norm of the semigroup $(S(t))_{t\geqslant 0}$ is at least exponentially bounded in time and therefore bounded on finite time intervals, see Proposition I.5.5 of [26]. □

In view of Remark 2.2.4, we analyse the domain of the adjoint operator $\mathcal{A}$ to get a better idea of the extrapolation space corresponding to $\mathcal{A}$.

4.3.4 Lemma. *For $\gamma, m \geqslant 0$ it holds*

$$\dot{H}^4(\overline{\Omega_1}) \times \dot{H}^2(\overline{\Omega_1}) \times \dot{H}^2(\overline{\Omega_1}) \times \dot{H}^2(\overline{\Omega_2}) \times \dot{H}^1(\overline{\Omega_2}) \subset D(\mathcal{A}').$$

Proof. Let $\varphi \in Y_0 := \dot{H}^4(\overline{\Omega_1}) \times \dot{H}^2(\overline{\Omega_1}) \times \dot{H}^2(\overline{\Omega_1}) \times \dot{H}^2(\overline{\Omega_2}) \times \dot{H}^1(\overline{\Omega_2}) \subset \mathcal{V}$, and let $w \in D(\mathcal{A})$. By definition, we have

$$\langle \mathcal{A}w, \varphi\rangle_{\mathcal{H}_\gamma} = \langle \mathbb{M}^{-1}\mathbb{A}w, \varphi\rangle_{\mathcal{H}_\gamma} = \langle \mathbb{A}w, \varphi\rangle_{\mathcal{H}_\gamma' \times \mathcal{H}_\gamma} = \langle \mathbb{A}w, \varphi\rangle_{\mathcal{V}' \times \mathcal{V}},$$

where the last equality follows from the fact that $\varphi \in \mathcal{V}$. Using the definition of $\mathbb{A}$ from equality (4-8) and integration by parts, we obtain

$$\begin{aligned}
\langle \mathcal{A}w, \varphi\rangle_{\mathcal{H}_\gamma} = & -\big\langle (w_1, w_4), (\varphi_2, \varphi_5)\big\rangle_{H^{2,1}} + \langle \Delta w_2, \Delta\varphi_1\rangle_{L^2(\Omega_1)} + \langle \nabla w_5, \nabla\varphi_4\rangle_{L^2(\Omega_2)} \\
& - \langle w_3, \Delta\varphi_2\rangle_{L^2(\Omega_1)} + \langle \Delta w_2, \varphi_3\rangle_{L^2(\Omega_1)} - \langle \nabla w_3, \nabla\varphi_3\rangle_{L^2(\Omega_1)} - m\langle w_5, \varphi_5\rangle_{L^2(\Omega_2)} \\
= & -\big\langle (w_1, w_4), (\varphi_2, \varphi_5)\big\rangle_{H^{2,1}} + \langle w_2, \Delta^2\varphi_1\rangle_{L^2(\Omega_1)} - \langle w_5, \Delta\varphi_4\rangle_{L^2(\Omega_2)} \\
& - \langle w_3, \Delta\varphi_2\rangle_{L^2(\Omega_1)} + \langle w_2, \Delta\varphi_3\rangle_{L^2(\Omega_1)} + \langle w_3, \Delta\varphi_3\rangle_{L^2(\Omega_1)} - m\langle w_5, \varphi_5\rangle_{L^2(\Omega_2)}.
\end{aligned}$$

Hence, we get $\left|\langle \mathcal{A}w, \varphi\rangle_{\mathcal{H}_\gamma}\right| \leqslant C\|\varphi\|_{Y_0}\|w\|_{\mathcal{H}_\gamma}$. Consequently, the map

$$(D(\mathcal{A}), \|\cdot\|_{\mathcal{H}_\gamma}) \to \mathbb{C}, \quad w \mapsto \langle \mathcal{A}w, \varphi\rangle_{\mathcal{H}_\gamma}$$

is a continuous, linear functional and its continuous extension to $\mathcal{H}_\gamma$ is an element of $\mathcal{H}_\gamma'$. This shows $\varphi \in D(\mathcal{A}')$, and therefore $Y_0 \subset D(\mathcal{A}')$. □

We can now prove that in the undamped case $m = 0$ there is a lack of exponential stability for $\gamma \geqslant 0$. The case $\gamma = 0$ is not included in [11].

4.3.5 Theorem. *If the boundary $\partial\Omega_1$ is of class C^5, the semigroup generated by $\mathcal{A}$ is not exponentially stable for $\gamma \geqslant 0$ and $m = 0$.*

Proof. We set $\widetilde{\mathcal{H}} := \{0\} \times \{0\} \times \{0\} \times \dot{H}^1(\overline{\Omega_2}) \times L^2(\Omega_2)$. Note that $\widetilde{\mathcal{H}}$ is a Hilbert subspace of $\mathcal{H}_\gamma$. We define the operator $\widetilde{\mathcal{A}}\colon \widetilde{\mathcal{H}} \supset D(\widetilde{\mathcal{A}}) \to \widetilde{\mathcal{H}}$ given by

$$D(\widetilde{\mathcal{A}}) := \{0\} \times \{0\} \times \{0\} \times (H^2(\Omega_2) \cap \dot{H}^1(\overline{\Omega_2})) \times \dot{H}^1(\overline{\Omega_2}) \subset \widetilde{\mathcal{H}}$$

and

$$\widetilde{\mathcal{A}}\tilde{w} := (0, 0, 0, \tilde{w}_5, \Delta\tilde{w}_4) \quad (\tilde{w} \in D(\widetilde{\mathcal{A}})).$$

With respect to the fourth and fifth component, $\widetilde{\mathcal{A}}$ is the first-order system related to the non-damped wave equation for $\tilde{v} := \tilde{w}_4$, that is,

$$\begin{aligned} \tilde{v}_{tt} - \Delta\tilde{v} &= 0 \quad \text{in } (0,\infty)\times\Omega_2, \\ \tilde{v} &= 0 \quad \text{on } (0,\infty)\times I \end{aligned} \tag{4-36}$$

with appropriate initial values. Let $(\widetilde{\mathcal{S}}(t))_{t\geqslant 0}$ be the C_0-semigroup generated by $\widetilde{\mathcal{A}}$ on $\widetilde{\mathcal{H}}$. As (4-36) contains no damping term, $(\widetilde{\mathcal{S}}(t))_{t\geqslant 0}$ is a unitary semigroup. Thus, the essential spectral radius $r_{\mathrm{ess}}(\widetilde{\mathcal{S}}(t))$ is equal to 1.

We will show that $\mathcal{S}(t) - \widetilde{\mathcal{S}}(t)\colon \widetilde{\mathcal{H}} \to \mathcal{H}_\gamma$ is compact for every $t > 0$, where $(\mathcal{S}(t))_{t\geqslant 0}$ stands for the C_0-semigroup generated by $\mathcal{A}$ as in Theorem 4.1.4. It is enough to prove that

$$\mathcal{S}(t) - \widetilde{\mathcal{S}}(t)\colon \mathcal{W} \to \mathcal{H}_\gamma$$

is compact for some dense subspace $\mathcal{W}$ of $\widetilde{\mathcal{H}}$. We define

$$\mathcal{W} := \{0\} \times \{0\} \times \{0\} \times \mathscr{D}(\Omega_2) \times \mathscr{D}(\Omega_2).$$

Then $\mathcal{W}$ is dense in $\widetilde{\mathcal{H}}$, and obviously $\mathcal{W} \subset D(\mathcal{A}) \cap D(\widetilde{\mathcal{A}})$. For $w_0 \in \mathcal{W}$, we consider

$$E(t) := \frac{1}{2}\|w(t) - \tilde{w}(t)\|^2_{\mathcal{H}_\gamma},$$

where $w(t) := \mathcal{S}(t)w_0$ and $\tilde{w}(t) := \widetilde{\mathcal{S}}(t)w_0$ for $t \geqslant 0$.

Step 1: We show the inequality

$$E(t) \leqslant \operatorname{Re} \int_0^t \langle \partial_\nu \widetilde{w}_4(s), w_2(s) \rangle_{L^2(I)}\, ds \quad (t \geqslant 0). \tag{4-37}$$

For this purpose, we notice

$$\begin{aligned} E'(t) &= \frac{1}{2} \tfrac{d}{dt} \|w(t) - \widetilde{w}(t)\|_{\mathcal{H}_\gamma}^2 \\ &= \operatorname{Re} \langle w'(t) - \widetilde{w}'(t), w(t) - \widetilde{w}(t) \rangle_{\mathcal{H}_\gamma} \\ &= \operatorname{Re} \langle \mathcal{A} w(t) - \widetilde{\mathcal{A}} \widetilde{w}(t), w(t) - \widetilde{w}(t) \rangle_{\mathcal{H}_\gamma} \\ &= \operatorname{Re} \langle \mathcal{A} w(t), w(t) \rangle_{\mathcal{H}_\gamma} + \operatorname{Re} \langle \widetilde{\mathcal{A}} \widetilde{w}(t), \widetilde{w}(t) \rangle_{\mathcal{H}_\gamma} \\ &\quad - \operatorname{Re} \langle \mathcal{A} w(t), \widetilde{w}(t) \rangle_{\mathcal{H}_\gamma} - \operatorname{Re} \langle \widetilde{\mathcal{A}} \widetilde{w}(t), w(t) \rangle_{\mathcal{H}_\gamma}. \end{aligned} \tag{4-38}$$

From (4-11) we know $\operatorname{Re} \langle \mathcal{A} w(t), w(t) \rangle_{\mathcal{H}_\gamma} \leqslant 0$, and for the undamped wave equation, we obtain $\operatorname{Re} \langle \widetilde{\mathcal{A}} \widetilde{w}(t), \widetilde{w}(t) \rangle_{\mathcal{H}_\gamma} = 0$. Moreover, by the definition of $\mathcal{A}$, we see that

$$\begin{aligned} \langle \mathcal{A} w(t), \widetilde{w}(t) \rangle_{\mathcal{H}_\gamma} &= \langle \mathbb{A} w(t), \widetilde{w}(t) \rangle_{\mathcal{V}' \times \mathcal{V}} \\ &= -\langle \nabla w_4(t), \nabla \widetilde{w}_5(t) \rangle_{L^2(\Omega_2)} + \langle \nabla w_5(t), \nabla \widetilde{w}_4(t) \rangle_{L^2(\Omega_2)}. \end{aligned}$$

Integration by parts leads to

$$\begin{aligned} \langle \widetilde{\mathcal{A}} \widetilde{w}(t), w(t) \rangle_{\mathcal{H}_\gamma} &= \langle \nabla \widetilde{w}_5(t), \nabla w_4(t) \rangle_{L^2(\Omega_2)} + \langle \Delta \widetilde{w}_4(t), w_5(t) \rangle_{L^2(\Omega_2)} \\ &= \langle \nabla \widetilde{w}_5(t), \nabla w_4(t) \rangle_{L^2(\Omega_2)} - \langle \nabla \widetilde{w}_4(t), \nabla w_5(t) \rangle_{L^2(\Omega_2)} - \langle \partial_\nu \widetilde{w}_4(t), w_5(t) \rangle_{L^2(I)}. \end{aligned}$$

Taking the real part in the last two equalities and inserting this into (4-38) results in

$$E'(t) \leqslant \operatorname{Re} \langle \partial_\nu \widetilde{w}_4(t), w_5(t) \rangle_{L^2(I)} = \operatorname{Re} \langle \partial_\nu \widetilde{w}_4(t), w_2(t) \rangle_{L^2(I)} \quad (t \geqslant 0),$$

where we used $w_5(t) = w_2(t)$ on I because $w(t) \in D(\mathcal{A})$. Since we have $E(0) = 0$, integration finishes the proof of Step 1.

Now, let $(w_0^k)_{k \in \mathbb{N}} \subset \mathcal{W}$ be a bounded sequence in $\widetilde{\mathcal{H}}$ and $t > 0$. We define $w^k(s) := \mathcal{S}(s) w_0^k$ and $\widetilde{w}^k(s) := \widetilde{\mathcal{S}}(s) w_0^k$ for $s \geqslant 0$. In the next two steps, we estimate the terms appearing on the right-hand side of (4-37) separately.

Step 2: We show that the sequence $(\partial_\nu \widetilde{w}_4^k)_{k \in \mathbb{N}} \subset L^2((0,t), L^2(I))$ is uniformly bounded. For this, we first note that $\widetilde{v}^k := \widetilde{w}_4^k$ is a solution of (4-36). We fix a vector field $\sigma \in C^1(\overline{\Omega_2})^n$ satisfying $\sigma = -\nu$ on I. By Corollary 2.4.4 and $\widetilde{v}^k(s) = 0$ on I, we get

$$\begin{aligned} \|\partial_\nu \widetilde{v}^k(s)\|_{L^2(I)}^2 &= \int_{\Omega_2} (|\widetilde{v}_t^k(s)|^2 - |\nabla \widetilde{v}^k(s)|^2) \operatorname{div}(\sigma)\, dx + 2 \operatorname{Re} \int_{\Omega_2} \nabla \widetilde{v}^k(s) \cdot (D\sigma \overline{\nabla \widetilde{v}^k(s)})\, dx \\ &\quad + 2 \operatorname{Re} \frac{d}{ds} \int_{\Omega_2} \widetilde{v}_t^k(s) (\sigma \cdot \overline{\nabla \widetilde{v}^k(s)})\, dx \end{aligned}$$

for $s \geqslant 0$. We integrate the last equality with respect to $s \in (0,t)$ and obtain

$$\begin{aligned} \|\partial_\nu \widetilde{v}^k\|_{L^2((0,t),L^2(I))}^2 &\leqslant C \sup_{s \in [0,t]} \big(\|\nabla \widetilde{v}^k(s)\|_{L^2(\Omega_2)}^2 + \|\widetilde{v}_t^k(s)\|_{L^2(\Omega_2)}^2 \big) \\ &\leqslant C \sup_{s \in [0,t]} \|\widetilde{w}^k(s)\|_{\widetilde{\mathcal{H}}}^2 \\ &\leqslant C. \end{aligned} \tag{4-39}$$

Notice that $C = C(t)$ may depend on $t > 0$.

Step 3: In the next step, we show that there exists a subsequence of $(w_2^k|_I)_{k\in\mathbb{N}}$ which converges in $L^2((0,t),L^2(I))$. Here, we distinguish between the cases $\gamma > 0$ and $\gamma = 0$. In both cases the extrapolation theory introduced in Section 2.2 as well as the Aubin–Lions Lemma 4.3.2 are crucial for the proof.

Case 1: Let $\gamma > 0$ and X_{-1} be the extrapolation space of $X_0 := \mathcal{H}_\gamma$ generated by $\mathcal{A}$. Then Proposition 2.2.5 and Lemma 4.2.5 tell us that we have the dense and compact embeddings $\mathcal{H}_\gamma = X_0 \subset X_{-\alpha} \subset X_{-1}$ for all $\alpha \in (0,1)$, where $X_{-\alpha} := [X_0, X_{-1}]_\alpha$ as in Proposition 2.2.5. As X_{-1} is the dual space of $D(\mathcal{A}')$ due to Remark 2.2.4, Lemma 4.3.4 yields

$$X_{-1} \subset H^{-4}(\Omega_1) \times H^{-2}(\Omega_1) \times H^{-2}(\Omega_1) \times H^{-2}(\Omega_2) \times H^{-1}(\Omega_2). \tag{4-40}$$

On the other hand, by definition of $\mathcal{H}_\gamma$, we have

$$X_0 = \mathcal{H}_\gamma \subset H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2). \tag{4-41}$$

With complex interpolation, we obtain from (4-40) and (4-41)

$$X_{-\alpha} \subset H^{2-6\alpha}(\Omega_1) \times H^{1-3\alpha}(\Omega_1) \times H^{-2\alpha}(\Omega_1) \times H^{1-3\alpha}(\Omega_2) \times H^{-\alpha}(\Omega_2). \tag{4-42}$$

Now, we apply the Aubin–Lions Lemma to the compact embedding $\mathcal{H}_\gamma = X_0 \subset X_{-\alpha} \subset X_{-1}$. For this, we first note that $(w^k)_{k\in\mathbb{N}}$ is bounded in $C([0,t],\mathcal{H}_\gamma)$ and therefore in $L^2((0,t),\mathcal{H}_\gamma)$. The time derivative $(\partial_t w^k)_{k\in\mathbb{N}}$ is bounded in $L^2((0,t),X_{-1})$ due to Lemma 4.3.3.

The application of the Aubin–Lions Lemma yields that there exists a subsequence $(w^{k_j})_{j\in\mathbb{N}}$ which is convergent in $L^2((0,t),X_{-\alpha})$. From embedding (4-42) we see that the second component $(w_2^{k_j})_{j\in\mathbb{N}}$ converges in $L^2((0,t),H^{1-3\alpha}(\Omega_1))$. By choosing $\alpha < \frac{1}{6}$ and taking the trace on I, we obtain convergence in $L^2((0,t),H^{1/2-3\alpha}(I))$ and therefore in $L^2((0,t),L^2(I))$ for $(w_2^{k_j}|_I)_{j\in\mathbb{N}}$.

Case 2: Let $\gamma = 0$ and $k \in \mathbb{N}$. We consider the system of boundary value problems

$$\begin{aligned} u_{tt} + \Delta^2 u + \Delta\theta &= 0 \quad \text{in } (0,\infty)\times\Omega_1,\\ \theta_t - \Delta\theta - \Delta u_t &= 0 \quad \text{in } (0,\infty)\times\Omega_1 \end{aligned}$$

with the boundary conditions

$$\begin{aligned} u = \partial_\nu u &= 0 \quad \text{on } (0,\infty)\times\Gamma,\\ \partial_\nu\theta + \theta &= 0 \quad \text{on } (0,\infty)\times\Gamma,\\ \partial_\nu u &= 0 \quad \text{on } (0,\infty)\times I,\\ \theta &= 0 \quad \text{on } (0,\infty)\times I,\\ -\partial_\nu\Delta u - \partial_\nu\theta &= \partial_\nu w_4^k \quad \text{on } (0,\infty)\times I \end{aligned}$$

for u and θ. By construction, this problem is solved by (w_1^k, w_3^k). Note that $w_0^k \in D(\mathcal{A}^2)$ such that $w^k \in C^1([0,\infty),D(\mathcal{A}))$, see Theorem II.5.5 of [26]. In particular, we have

$$\partial_\nu v^k \in C^1([0,t],H^{1/2}(I)) \subset L^2((0,t),H^{1/2}(I)) \cap H^{1/4}((0,t),L^2(I)).$$

According to Proposition 3.5.3 we obtain

$$\|w_2^k\|_{L^2((0,t),\,H^{3/2}(\Omega_1))} \leqslant C\|\partial_\nu w_4^k\|_{L^2((0,t),\,L^2(I))}.$$

Similar as in (4-39) we get $\|\partial_\nu w_4^k\|_{L^2((0,t),\,L^2(I))} \leqslant C$ because of Corollary 2.4.4, since the tangential derivatives can be estimated by using the transmission condition $w_1^k = w_4^k$ on I. With the help of Lemma 4.3.3 and embedding (4-40), we see that the time derivative $(\partial_t w_2^k)_{k\in\mathbb{N}}$ is bounded in $L^2((0,t), H^{-2}(\Omega_1))$. Due to the Lemma of Aubin–Lions, applied to the compact embeddings $H^{3/2}(\Omega_1) \subset H^1(\Omega_1) \subset H^{-2}(\Omega_1)$, there is a subsequence $(w_2^{k_j})_{j\in\mathbb{N}}$ which converges in $L^2((0,t), H^1(\Omega_1))$. After taking the trace we also get convergence of $(w_2^{k_j}|_I)_{j\in\mathbb{N}}$ in $L^2((0,t), L^2(I))$.

Step 4: Due to Step 2 and Step 3 we have that, after passing to a subsequence, $(\partial_\nu \widetilde{w}_4^k)_{k\in\mathbb{N}}$ is bounded in $L^2((0,t), L^2(I))$ and $(w_2^k|_I)_{k\in\mathbb{N}}$ converges in $L^2((0,t), L^2(I))$. We define

$$E^{k\ell}(t) := \frac{1}{2}\big\|w^{k\ell}(t) - \widetilde{w}^{k\ell}(t)\big\|_{\mathcal{H}_\gamma}^2,$$

where $w^{k\ell}(s) := \mathcal{S}(s)(w_0^k - w_0^\ell)$ and $\widetilde{w}^{k\ell}(s) := \widetilde{\mathcal{S}}(s)(w_0^k - w_0^\ell)$ for $k, \ell \in \mathbb{N}$ and $s \geqslant 0$. By (4-37) we have

$$E^{k\ell}(t) \leqslant \big|\langle \partial_\nu \widetilde{w}_4^{k\ell}, w_2^{k\ell}\rangle_{L^2((0,t),L^2(I))}\big| \to 0 \quad (k,\ell \to \infty).$$

Thus, $((\mathcal{S}(t) - \widetilde{\mathcal{S}}(t))w_0^k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_\gamma$ and therefore convergent. This shows the compactness of the operator $\mathcal{S}(t) - \widetilde{\mathcal{S}}(t)\colon \mathcal{W} \to \mathcal{H}_\gamma$. Consequently, also $\mathcal{S}(t) - \widetilde{\mathcal{S}}(t)\colon \widetilde{\mathcal{H}} \to \mathcal{H}_\gamma$ is compact. As in addition $r_{\mathrm{ess}}(\widetilde{\mathcal{S}}(t)) = 1$, Theorem 3.3 of [61] implies that $(\mathcal{S}(t))_{t\geqslant 0}$ is not exponentially stable. □

4.3.6 Remark. It is easier to prove that there is a lack of exponential stability for $\gamma > 0$ because in this case the space $\mathcal{H}_\gamma$ is better in the second component. For $\gamma = 0$ we have to make more efforts to show the existence of a convergent subsequence of $(w_2^k|_I)_{k\in\mathbb{N}} \subset L^2((0,t), L^2(I))$. The additional smoothness assumption on $\partial\Omega_1$ is only used in the case $\gamma = 0$ to apply Proposition 3.5.3.

4.3 c) Polynomial stability in the undamped case

Although the last result tells us that there is no exponential stability in the case of an undamped membrane, we will show that the system still decays polynomially under a certain geometric condition.

4.3.7 Definition. Let $G \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^1, $x_0 \in \mathbb{R}^n$ and $q\colon \mathbb{R}^n \to \mathbb{R}^n$, $q(x) := x - x_0$ for all $x \in \mathbb{R}^n$. We say that G fulfils the *geometric condition (with respect to $x_0 \in \mathbb{R}^n$)* if

$$q(x) \cdot \nu(x) \geqslant 0 \quad (x \in \partial G),$$

where $\nu(x)$ denotes the outward pointing unit normal vector of G in the point $x \in \partial G$.

4.3.8 Remark. The geometric condition means that the angle between the vector $x - x_0$, which points from x_0 to x, and the outward pointing unit normal vector $\nu(x)$ is less or equal than 90 degree. It is not possible that a domain fulfils the geometric condition with respect to a reference point $x_0 \in \mathbb{R}^n \backslash \overline{G}$.

As a short side note we want to point out the connection between star-shapedness and the geometric condition above.

4.3.9 Lemma. *Let $G \subset \mathbb{R}^n$ be bounded such that there is an $x_0 \in G$ with $x_0 + \mu(x - x_0) \in G$ for all $x \in \partial G$ and $\mu \in (0,1)$. Then G is star-shaped with respect to $x_0 \in G$, that is, $x_0 + \mu(x - x_0) \in G$ for all $x \in G$ and $\mu \in (0,1)$.*

Proof. Let $\mu \in (0,1)$ and $x \in G$. Since the domain G is bounded, there is an $s \geqslant 1$ such that $x_0 + s(x - x_0) \in \partial G$. In particular, we get $x_0 + \mu(x - x_0) \in G$. Therefore, G is star-shaped with respect to $x_0 \in G$. □

4.3.10 Remark. There are unbounded sets which fulfil the remaining assumptions of the last lemma but which are not star-shaped, for example $\mathbb{R}^n \backslash \{0\}$.

The following proposition gives an answer to the relation between star-shaped domains and domains which fulfil the geometric condition. Parts of the proof can also be found in Section 9.4.2 of [27].

4.3.11 Proposition. *Let G be a bounded domain with C^1-boundary. Then G fulfils the geometric condition if and only if $\overline{G}$ is a star-shaped set.*

Proof. Let the geometric condition be fulfilled for G with respect to $x_0 \in \overline{G}$. Assume that $\overline{G}$ is not star-shaped with respect to x_0. Due to Lemma 4.3.9 there exists a point $x \in \partial G$ and $\mu \in (0,1)$ such that $x_0 + \mu(x - x_0) \notin \overline{G}$. We choose the smallest $\mu_1 > \mu$ with $x_1 := x_0 + \mu_1(x - x_0) \in \overline{G}$. In particular, we have $x_1 \in \partial G$. Since $x_0 + s(x - x_0) \notin \overline{G}$ for $s \in [\mu, \mu_1)$, it holds $(x_1 - x_0) \cdot \nu(x_1) = 0$. On the other hand, there has to be an $x_2 \in \partial G$ close to x_1 with $(x_2 - x_0) \cdot \nu(x_2) < 0$. This is a contradiction to the assumption that G fulfils the geometric condition with respect to x_0.

The other implication was already proved in Section 9.4.2 of [27], which shows the desired equivalence. □

4.3.12 Remark. Notice that not every domain G fulfilling the geometric condition necessarily has to be star-shaped since it is possible that $x_0 \in \mathbb{R}^n$ belongs to the boundary of the domain.

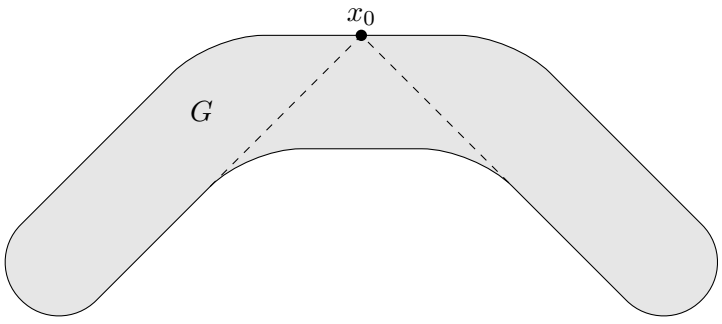

Figure 4.2: The domain G satisfies the geometric condition and is not star-shaped but $\overline{G}$ is star-shaped with respect to $x_0 \in \partial G$.

Before we start the proof of the polynomial stability in the undamped case, we mention the following auxiliary result.

4.3.13 Lemma. *Let $x, y > 0$, $s \in \mathbb{R}$, $k \in \mathbb{N}$ with $k \geqslant 3$ and $0 < \alpha_1 < \ldots < \alpha_k$. Then it holds*

$$\sum_{j=1}^{k} x^{s-\alpha_j} y^{\alpha_j} \leqslant (k-1)\left(x^{s-\alpha_1} y^{\alpha_1} + x^{s-\alpha_k} y^{\alpha_k}\right).$$

Proof. We fix $j \in \{2, \ldots, k-1\}$. Using Young's inequality and $0 < \alpha_1 < \ldots < \alpha_k$, we get

$$\begin{aligned}
x^{s-\alpha_j} y^{\alpha_j} &= x^{\frac{(s-\alpha_1)(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}} y^{\frac{\alpha_1(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}} \cdot x^{s-\alpha_j-\frac{(s-\alpha_1)(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}} y^{\alpha_j-\frac{\alpha_1(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}} \\
&\leqslant \left(x^{\frac{(s-\alpha_1)(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}} y^{\frac{\alpha_1(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}}\right)^{\frac{\alpha_k-\alpha_1}{\alpha_k-\alpha_j}} + \left(x^{s-\alpha_j-\frac{(s-\alpha_1)(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}} y^{\alpha_j-\frac{\alpha_1(\alpha_k-\alpha_j)}{\alpha_k-\alpha_1}}\right)^{\frac{\alpha_k-\alpha_1}{\alpha_j-\alpha_1}} \\
&= x^{s-\alpha_1} y^{\alpha_1} + x^{\frac{(s-\alpha_j)(\alpha_k-\alpha_1)-(s-\alpha_1)(\alpha_k-\alpha_j)}{\alpha_j-\alpha_1}} y^{\frac{\alpha_j(\alpha_k-\alpha_1)-\alpha_1(\alpha_k-\alpha_j)}{\alpha_j-\alpha_1}} \\
&= x^{s-\alpha_1} y^{\alpha_1} + x^{\frac{-\alpha_j\alpha_k-s\alpha_1+\alpha_1\alpha_k+s\alpha_j}{\alpha_j-\alpha_1}} y^{\frac{\alpha_k(\alpha_j-\alpha_1)}{\alpha_j-\alpha_1}} \\
&= x^{s-\alpha_1} y^{\alpha_1} + x^{\frac{s(\alpha_j-\alpha_1)-\alpha_k(\alpha_j-\alpha_1)}{\alpha_j-\alpha_1}} y^{\alpha_k} \\
&= x^{s-\alpha_1} y^{\alpha_1} + x^{s-\alpha_k} y^{\alpha_k}.
\end{aligned}$$

□

For $\gamma \geqslant 0$ and $m = 0$ we can now show the polynomial stability of the semigroup generated by $\mathcal{A}$ in the sense of Definition 2.3.4 under the geometric condition introduced earlier. As we use the Rellich's identity from Lemma 2.4.5 for $n = 2$, we restrict ourselves to space dimension $n = 2$. The following result extends Theorem 5.2 of [11] by the case $\gamma > 0$.

4.3.14 Theorem. *Let $n = 2$, $\gamma \geqslant 0$, $m = 0$ and assume that $\Omega_2 \subset \mathbb{R}^2$ satisfies the geometrical condition with respect to the point $x_0 \in \mathbb{R}^2$. Since $-\nu$ denotes the outward pointing unit normal vector of Ω_2 this means that*

$$q(x) \cdot \nu(x) \leqslant 0 \quad (x \in I),$$

where $q(x) := x - x_0$. Then, the semigroup $(\mathcal{S}(t))_{t \geqslant 0}$ generated by $\mathcal{A}$ decays polynomially.

Proof. By Proposition 2.3.7, the semigroup is polynomially stable if $i\mathbb{R} \subset \rho(\mathcal{A})$ and if there exist $C > 0$, $\lambda_0 > 0$ and $\beta > 0$ with

$$\|(i\lambda - \mathcal{A})^{-1} f\|_{\mathcal{H}_\gamma} \leqslant C|\lambda|^\beta \|\mathcal{A}f\|_{\mathcal{H}_\gamma} \quad (f \in D(\mathcal{A}),\, \lambda \in \mathbb{R},\, |\lambda| \geqslant \lambda_0). \tag{4-43}$$

To prove (4-43) let $\lambda_0 > 0$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geqslant \lambda_0$. Additionally, let $w \in D(\mathcal{A}^2)$ be arbitrary and $f := (i\lambda - \mathcal{A})w$. Then $f \in D(\mathcal{A})$ and

$$\begin{aligned}
i\lambda w_1 - w_2 &= f_1, && (4\text{-}44)\\
i\lambda w_2 - i\lambda\gamma\Delta w_2 + \Delta^2 w_1 + \Delta w_3 &= f_2 - \gamma\Delta f_2, \\
i\lambda w_3 - \Delta w_2 - \Delta w_3 &= f_3, && (4\text{-}45)\\
i\lambda w_4 - w_5 &= f_4, && (4\text{-}46)\\
i\lambda w_5 - \Delta w_4 &= f_5. && (4\text{-}47)
\end{aligned}$$

We show an inequality of the form

$$\|w\|_{\mathcal{H}_\gamma}^2 \leqslant C|\lambda|^{\beta_1}\left(\|w\|_{\mathcal{H}_\gamma}^{2-\beta_2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{\beta_2} + \|w\|_{\mathcal{H}_\gamma}^{2-\beta_3} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{\beta_3}\right)$$

for appropriate $\beta_1, \beta_2, \beta_3 > 0$, which yields (4-43) by Young's inequality.

Replacing (4-44) into (4-45) leads to

$$i\lambda w_3 - i\lambda\Delta w_1 - \Delta w_3 = -\Delta f_1 + f_3. \tag{4-48}$$

Similarly, by replacing (4-46) into (4-47), we get

$$-\lambda^2 w_4 - \Delta w_4 = i\lambda f_4 + f_5. \tag{4-49}$$

First, we notice that

$$\|\nabla w_3\|^2_{L^2(\Omega_1)} + \|w_3\|^2_{L^2(\Gamma)} = \operatorname{Re}\langle f, w\rangle_{\mathcal{H}_\gamma} \leqslant \|w\|_{\mathcal{H}_\gamma}\|f\|_{\mathcal{H}_\gamma}$$

due to (4-11). Poincaré's inequality shows

$$\|w_3\|^2_{H^1(\Omega_1)} \leqslant C\|w\|_{\mathcal{H}_\gamma}\|f\|_{\mathcal{H}_\gamma}. \tag{4-50}$$

As in the proof of Theorem 4.3.1, Corollary 4.2.4 yields

$$\|w\|_{H^3(\Omega_1)\times H^2(\Omega_1)^2\times H^2(\Omega_2)\times H^1(\Omega_2)} \leqslant C(\|w\|_{\mathcal{H}_\gamma} + \|f - i\lambda w\|_{\mathcal{H}_\gamma}) \leqslant C(|\lambda|\|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma}), \tag{4-51}$$

and with Theorem 4.2.3 (ii) we obtain

$$\begin{aligned}\|w_1\|_{H^4(\Omega_1)} &\leqslant C(\|w\|_{\mathcal{H}_\gamma} + \|\mathcal{A}(f - i\lambda w)\|_{\mathcal{H}_\gamma})\\ &\leqslant C(\|w\|_{\mathcal{H}_\gamma} + \|\mathcal{A}f\|_{\mathcal{H}_\gamma} + |\lambda|\|f - i\lambda w\|_{\mathcal{H}_\gamma})\\ &\leqslant C(|\lambda|^2\|w\|_{\mathcal{H}_\gamma} + |\lambda|\|f\|_{\mathcal{H}_\gamma} + \|\mathcal{A}f\|_{\mathcal{H}_\gamma})\\ &\leqslant C(|\lambda|^2\|w\|_{\mathcal{H}_\gamma} + |\lambda|\|\mathcal{A}f\|_{\mathcal{H}_\gamma}).\end{aligned} \tag{4-52}$$

In the last step, we used $0 \in \rho(\mathcal{A})$. Analogously to (4-31) it holds

$$\begin{aligned}&\|w_2\|^2_{L^2(\Omega_1)} + \gamma\|\nabla w_2\|^2_{L^2(\Omega_1)} + \|w_5\|^2_{L^2(\Omega_2)}\\ &= \|(w_1, w_4)\|^2_{H^{2,1}} - \langle w_2, f_1\rangle_{L^2(\Omega_1)} - \gamma\langle\nabla w_2, \nabla f_1\rangle_{L^2(\Omega_1)} - \langle w_5, f_4\rangle_{L^2(\Omega_2)} - \langle f_2, w_1\rangle_{L^2(\Omega_1)}\\ &\quad - \gamma\langle\nabla f_2, \nabla w_1\rangle_{L^2(\Omega_1)} - \langle f_5, w_4\rangle_{L^2(\Omega_2)} + \langle w_3, \Delta w_1\rangle_{L^2(\Omega_1)}.\end{aligned}$$

Therefore, it is enough to find a suitable estimate for $\|(w_1, w_4)\|_{H^{2,1}}$.

Multiplying equation (4-48) by Δw_1 results in

$$\begin{aligned}i\lambda\|\Delta w_1\|^2_{L^2(\Omega_1)} &= \langle i\lambda w_3 - \Delta w_3 + \Delta f_1 - f_3, \Delta w_1\rangle_{L^2(\Omega_1)}\\ &= i\lambda\langle w_3, \Delta w_1\rangle_{L^2(\Omega_1)} + \langle\nabla w_3, \nabla\Delta w_1\rangle_{L^2(\Omega_1)} - \langle\partial_\nu w_3, \Delta w_1\rangle_{L^2(\partial\Omega_1)}\\ &\quad + \langle\Delta f_1 - f_3, \Delta w_1\rangle_{L^2(\Omega_1)}.\end{aligned}$$

Together with (4-50), (4-51), Lemma 2.4.1 and Lemma 4.3.13 it follows

$$\begin{aligned}&\|\Delta w_1\|^2_{L^2(\Omega_1)}\\ &\leqslant C\|w\|^{3/2}_{\mathcal{H}_\gamma}\|f\|^{1/2}_{\mathcal{H}_\gamma} + C|\lambda|^{-1}\|w\|^{1/2}_{\mathcal{H}_\gamma}\|f\|^{1/2}_{\mathcal{H}_\gamma}(|\lambda|\|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma})\\ &\quad + C|\lambda|^{-1}\|w_3\|^{1/2}_{H^1(\Omega_1)}\|w_3\|^{1/2}_{H^2(\Omega_1)}\|\Delta w_1\|^{1/2}_{L^2(\Omega_1)}\|w_1\|^{1/2}_{H^3(\Omega_1)} + C|\lambda|^{-1}\|w\|_{\mathcal{H}_\gamma}\|f\|_{\mathcal{H}_\gamma}\\ &\leqslant C\|w\|^{3/2}_{\mathcal{H}_\gamma}\|f\|^{1/2}_{\mathcal{H}_\gamma} + C|\lambda|^{-1}\|w\|^{1/2}_{\mathcal{H}_\gamma}\|f\|^{1/2}_{\mathcal{H}_\gamma}(|\lambda|\|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma})\\ &\quad + C|\lambda|^{-1}\|w\|^{3/4}_{\mathcal{H}_\gamma}\|f\|^{1/4}_{\mathcal{H}_\gamma}(|\lambda|\|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma}) + C|\lambda|^{-1}\|w\|_{\mathcal{H}_\gamma}\|f\|_{\mathcal{H}_\gamma}\\ &\leqslant C\left(\|w\|^{7/4}_{\mathcal{H}_\gamma}\|f\|^{1/4}_{\mathcal{H}_\gamma} + \|w\|^{3/2}_{\mathcal{H}_\gamma}\|f\|^{1/2}_{\mathcal{H}_\gamma} + \|w\|_{\mathcal{H}_\gamma}\|f\|_{\mathcal{H}_\gamma} + \|w\|^{3/4}_{\mathcal{H}_\gamma}\|f\|^{5/4}_{\mathcal{H}_\gamma} + \|w\|^{1/2}_{\mathcal{H}_\gamma}\|f\|^{3/2}_{\mathcal{H}_\gamma}\right)\\ &\leqslant C\left(\|w\|^{7/4}_{\mathcal{H}_\gamma}\|f\|^{1/4}_{\mathcal{H}_\gamma} + \|w\|^{1/2}_{\mathcal{H}_\gamma}\|f\|^{3/2}_{\mathcal{H}_\gamma}\right).\end{aligned} \tag{4-53}$$

To find an estimate for $\|\nabla w_4\|_{L^2(\Omega_2)}$, we multiply equation (4-49) by w_4 and obtain

$$-\lambda^2\|w_4\|^2_{L^2(\Omega_2)} - \langle \Delta w_4, w_4\rangle_{L^2(\Omega_2)} = \langle i\lambda f_4 + f_5, w_4\rangle_{L^2(\Omega_2)}.$$

Integration by parts leads to

$$\|\nabla w_4\|^2_{L^2(\Omega_2)} = \lambda^2\|w_4\|^2_{L^2(\Omega_2)} + \langle i\lambda f_4 + f_5, w_4\rangle_{L^2(\Omega_2)} - \langle \partial_\nu w_4, w_4\rangle_{L^2(I)}. \tag{4-54}$$

Using the transmission conditions, Lemma 2.4.1, interpolation, (4-51), (4-52) and (4-53), we get

$$\begin{aligned}
&\|\partial_\nu w_4\|_{L^2(I)}\\
&\leqslant \|\partial_\nu \Delta w_1\|_{L^2(I)} + \|\partial_\nu w_3\|_{L^2(I)}\\
&\leqslant C\left(\|\Delta w_1\|^{1/2}_{H^1(\Omega_1)}\|\Delta w_1\|^{1/2}_{H^2(\Omega_1)} + \|w_3\|^{1/2}_{H^1(\Omega_1)}\|w_3\|^{1/2}_{H^2(\Omega_1)}\right)\\
&\leqslant C\left(\|\Delta w_1\|^{1/4}_{L^2(\Omega_1)}\|w_1\|^{3/4}_{H^4(\Omega_1)} + \|w_3\|^{1/2}_{H^1(\Omega_1)}\|w_3\|^{1/2}_{H^2(\Omega_1)}\right)\\
&\leqslant C\left(\|\Delta w_1\|^{1/4}_{L^2(\Omega_1)}\left(|\lambda|^{3/2}\|w\|^{3/4}_{\mathcal{H}_\gamma} + |\lambda|^{3/4}\|\mathcal{A}f\|^{3/4}_{\mathcal{H}_\gamma}\right) + \|w\|^{1/4}_{\mathcal{H}_\gamma}\|f\|^{1/4}_{\mathcal{H}_\gamma}\left(|\lambda|^{1/2}\|w\|^{1/2}_{\mathcal{H}_\gamma} + \|f\|^{1/2}_{\mathcal{H}_\gamma}\right)\right)\\
&\leqslant C|\lambda|^{3/2}\left(\|w\|^{31/32}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{1/32}_{\mathcal{H}_\gamma} + \|w\|^{1/16}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{15/16}_{\mathcal{H}_\gamma}\right),
\end{aligned}$$

where we applied again Lemma 4.3.13 in the last step. Combined with Lemma 2.4.6, identity (4-49), Lemma 2.4.5, the geometric condition and (4-53) it holds

$$\begin{aligned}
\lambda^2\|w_4\|^2_{L^2(\Omega_2)} &= -\lambda^2 \operatorname{Re}\int_{\Omega_2} w_4(q\cdot\overline{\nabla w_4})\,dx - \frac{\lambda^2}{2}\int_I (q\cdot\nu)|w_4|^2\,dS\\
&= \operatorname{Re}\int_{\Omega_2}(\Delta w_4 + i\lambda f_4 + f_5)(q\cdot\overline{\nabla w_4})\,dx - \frac{\lambda^2}{2}\int_I (q\cdot\nu)|w_4|^2\,dS\\
&= \operatorname{Re}\int_{\Omega_2}(i\lambda f_4 + f_5)(q\cdot\overline{\nabla w_4})\,dx - \operatorname{Re}\int_I \partial_\nu w_4(q\cdot\overline{\nabla w_4})\,dS\\
&\quad + \frac{1}{2}\int_I (q\cdot\nu)|\nabla w_4|^2\,dS - \frac{\lambda^2}{2}\int_I (q\cdot\nu)|w_4|^2\,dS\\
&\leqslant C\left(|\lambda|\|f\|_{\mathcal{H}_\gamma}\|w\|_{\mathcal{H}_\gamma} + \|\partial_\nu w_4\|_{L^2(I)}\|\nabla w_4\|_{L^2(I)} + \lambda^2\|w_4\|^2_{L^2(I)}\right)\\
&\leqslant C\left(|\lambda|\|f\|_{\mathcal{H}_\gamma}\|w\|_{\mathcal{H}_\gamma} + \|\partial_\nu w_4\|_{L^2(I)}\|\nabla w_4\|^{1/2}_{L^2(\Omega_2)}\|\nabla w_4\|^{1/2}_{H^1(\Omega_2)} + \lambda^2\|w_1\|^2_{H^2(\Omega_1)}\right)\\
&\leqslant C\left(|\lambda|\|f\|_{\mathcal{H}_\gamma}\|w\|_{\mathcal{H}_\gamma} + \|\partial_\nu w_4\|_{L^2(I)}\|w\|^{1/2}_{\mathcal{H}_\gamma}\left(|\lambda|^{1/2}\|w\|^{1/2}_{\mathcal{H}_\gamma} + \|f\|^{1/2}_{\mathcal{H}_\gamma}\right) + \lambda^2\|\Delta w_1\|^2_{L^2(\Omega_1)}\right)\\
&\leqslant C|\lambda|^2\left(\|w\|^{63/32}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{1/32}_{\mathcal{H}_\gamma} + \|w\|^{1/2}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{3/2}_{\mathcal{H}_\gamma}\right).
\end{aligned}$$

In consequent, (4-54) reduces to

$$\|\nabla w_4\|^2_{L^2(\Omega_2)} \leqslant C|\lambda|^2\left(\|w\|^{63/32}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{1/32}_{\mathcal{H}_\gamma} + \|w\|^{1/2}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{3/2}_{\mathcal{H}_\gamma}\right).$$

Altogether, we obtain with Young's inequality

$$\begin{aligned}
\|w\|^2_{\mathcal{H}_\gamma} &\leqslant C|\lambda|^2\left(\|w\|^{63/32}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{1/32}_{\mathcal{H}_\gamma} + \|w\|^{1/2}_{\mathcal{H}_\gamma}\|\mathcal{A}f\|^{3/2}_{\mathcal{H}_\gamma}\right)\\
&= C\left(\|w\|^{63/32}_{\mathcal{H}_\gamma}(|\lambda|^{64}\|\mathcal{A}f\|_{\mathcal{H}_\gamma})^{1/32} + \|w\|^{1/2}_{\mathcal{H}_\gamma}(|\lambda|^{4/3}\|\mathcal{A}f\|_{\mathcal{H}_\gamma})^{3/2}\right)\\
&\leqslant \frac{1}{2}\|w\|^2_{\mathcal{H}_\gamma} + C|\lambda|^{128}\|\mathcal{A}f\|^2_{\mathcal{H}_\gamma},
\end{aligned}$$

that is,

$$\|w\|_{\mathcal{H}_\gamma} \leqslant C|\lambda|^{64}\|\mathcal{A}f\|_{\mathcal{H}_\gamma}. \tag{4-55}$$

Note that (4-55) implies that $i\lambda - \mathcal{A}$ is injective, and therefore $i\mathbb{R} \subset \rho(\mathcal{A})$. In fact, let $w \in D(\mathcal{A})$ with $f := (i\lambda - \mathcal{A})w = 0$. Then $\mathcal{A}w = i\lambda w \in D(\mathcal{A})$, which shows $w \in D(\mathcal{A}^2)$, and we can apply (4-55) to see $w = 0$.

Now, the claim follows from Proposition 2.3.7. □

4.3.15 Remark. In case $\gamma = 0$, we can use the estimate

$$\|w_1\|_{H^4(\Omega_1)} \leqslant C(|\lambda|\|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma})$$

instead of (4-52) to gain a slightly better decay rate with the same arguments as before. However, the proof gives no information about the optimality of the decay rate.

4.3.16 Remark. Summarised we have seen that in the undamped case $m = 0$, there is a lack of exponential stability but still polynomial stability, at least if $\overline{\Omega_2}$ is star-shaped and the space dimension is 2. In general, it seems to be difficult to predict whether a transmission problem with different damping behaviours on the regions under consideration adapts the long-time asymptotic of the damped part or not. While the plate-membrane system analysed in [12] possesses a similar long-time behaviour as the one studied here, the plate-plate system of [18] has better decay properties in time. Even though in [18] there is only a damping term in the exterior part Ω_1, the whole plate-plate system remains exponentially stable, see Theorem 4.5 of [18]. One explanation for the differences could be the different order structure of the systems. In our proof of Theorem 4.3.5 it seems to be essential that the orders in space of the equations vary in Ω_1 and Ω_2. For instance, to handle the tangential derivatives appearing in the application of Corollary 2.4.4, it is crucial that u already has higher regularity than v in the ground space.

4.4 Excursion to a structurally damped plate-plate system

In Section 2.5 we already saw some resolvent estimates for the structurally damped plate equation with different boundary conditions. Here, we want to illustrate the behaviour of the resolvent of a structurally damped plate-plate transmission problem for large λ with some simulations generated in MATLAB. Since there is no analytical proof for most of the observations given here, this section can be seen as a short outlook and a first guess of the decay rate of the stationary solution with respect to λ in the case $n = 1$ and various Sobolev spaces.

We consider the operator

$$A_\rho = \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta \end{pmatrix}$$

belonging to a plate with structural damping $\rho > 0$ in the case $n = 1$ and take a look at a structurally damped plate-plate transmission problem on the interval $(0, 2\pi)$ with different damping coefficients $\rho_1 > 0$ and $\rho_2 > 0$ on $(0, \pi)$ and $(\pi, 2\pi)$, respectively, as well as canonically transmission conditions.

Let $f^1 \in H^2((0,\pi)) \times L^2((0,\pi))$, $f^2 \in H^2((\pi,2\pi)) \times L^2((\pi,2\pi))$, $\lambda \in \mathbb{R}$ and $\rho_1, \rho_2 > 0$. We rewrite the transmission problem

$$\begin{aligned} (\lambda - A_{\rho_1})u(x) &= f^1(x) && \text{for } x \in (0,\pi), \\ (\lambda - A_{\rho_2})v(x) &= f^2(x) && \text{for } x \in (\pi,2\pi), \\ u_1^{(k)}(\pi) &= v_1^{(k)}(\pi) && \text{for } k = 0,1,2,3 \end{aligned}$$

with canonically transmission conditions as a scalar equation by using $\lambda u_1 = f_1^1 + u_2$ and $\lambda v_1 = f_1^2 + v_2$. This leads to

$$\begin{aligned} \lambda^2 u_1 - \lambda f_1^1 + \Delta^2 u_1 - \lambda\rho_1\Delta u_1 + \rho_1\Delta f_1^1 &= f_2^1 && \text{in } (0,\pi), \\ \lambda^2 v_1 - \lambda f_1^2 + \Delta^2 v_1 - \lambda\rho_2\Delta v_1 + \rho_2\Delta f_1^2 &= f_2^2 && \text{in } (\pi,2\pi), \\ u_1^{(k)}(\pi) &= v_1^{(k)}(\pi) && \text{for } k = 0,1,2,3. \end{aligned}$$

Hence, we would like to solve the ordinary differential equation

$$\begin{aligned} \lambda^2 u_1 + \Delta^2 u_1 - \lambda\rho_1\Delta u_1 &= f_2^1 + \lambda f_1^1 - \rho_1\Delta f_1^1 && \text{in } (0,\pi), \\ \lambda^2 v_1 + \Delta^2 v_1 - \lambda\rho_2\Delta v_1 &= f_2^2 + \lambda f_1^2 - \rho_2\Delta f_1^2 && \text{in } (\pi,2\pi), \\ u_1^{(k)}(\pi) &= v_1^{(k)}(\pi) && \text{for } k = 0,1,2,3 \end{aligned} \tag{4-56}$$

for different boundary conditions. The choice of the spaces for f^1 and f^2 seems to be reasonable in view of the Douglis–Nirenberg structure of the system. Nevertheless, according to the following theorem, we cannot expect that the $H^2 \times L^2$-norm of the solution is bounded by $C|\lambda|^{-1}$ for every choice of f^1 and f^2.

4.4.1 Theorem. [17, Theorem 3.8]. *Let $\Omega \subset \mathbb{R}^n$ be a sufficiently smooth domain and assume that $(\lambda - A, B)$ is parameter-elliptic in some sector $\Lambda \subset \mathbb{C}$ with constant coefficients and orders encoded by $\mathbf{s}, \mathbf{t}$ and $\mathbf{m}$. If $f \in H^{-\mathbf{s}}(\Omega)$ fulfils an estimate of the form*

$$\|w\|_{H^{\mathbf{t}}(\Omega)} + |\lambda|\|w\|_{H^{-\mathbf{s}}(\Omega)} \leqslant C\|f\|_{H^{-\mathbf{s}}(\Omega)}$$

for sufficiently large $\lambda \in \Lambda$, where w is the solution of

$$\begin{aligned} (\lambda - A)w &= f && \text{in } \Omega, \\ Bw &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{4-57}$$

with a constant $C > 0$ independent of λ, then $(Bf)_\ell = 0$ for all $\ell \in \{1,\dots,M\}$ with $-m_\ell > \mu$.

However, if f^1 and f^2 satisfy higher smoothness assumptions, we indeed expect a decay rate of $|\lambda|^{-1}$ due to Proposition 2.5.14. We quickly recall the appropriate part of the proof of Proposition 2.5.14 given in the half-space in the context of boundary value problems.

4.4.2 Reminder. Let $(\lambda - A, B)$ be parameter-elliptic in some sector $\Lambda \subset \mathbb{C}$ with constant coefficients and orders encoded by $\mathbf{s}, \mathbf{t}$ and $\mathbf{m}$. We set $\gamma := \max_{\ell=1,\dots,M}\{-m_\ell - \mu - \frac{1}{2}\}$ and assume $\gamma > 0$. For $f \in \{f \in H^{\mathbf{t}}(\mathbb{R}^n_+) : B(D)f = 0\}$, we obtain

$$\|\lambda w - f\|_{H^{-\mathbf{s}}(\mathbb{R}^n_+)} \leqslant C|\lambda|^{\gamma/\mu - 1}\|Af\|_{H^{-\mathbf{s}}(\mathbb{R}^n_+)}$$

for sufficiently large $\lambda \in \Lambda$, where w is the solution of (4-57) in the case $\Omega = \mathbb{R}^n_+$. For $\gamma < \mu$ we get the convergence $\lambda w \to f$ in $H^{-\mathbf{s}}(\mathbb{R}^n_+)$ for $\lambda \to \infty$.

This gives a motivation to check how the decay rate changes if f^1 and f^2 only satisfy some of the boundary and transmission conditions, which can be formulated in the ground space.

We use the notation $h_j\colon (0,\pi)\cup(\pi,2\pi)\to\mathbb{R}$,

$$h_j(x) := \begin{cases} f_j^1(x), & x\in(0,\pi),\\ f_j^2(x), & x\in(\pi,2\pi)\end{cases}$$

for $j=1,2$. The solution of (4-56), with boundary conditions as described below, is denoted by

$$w_1^\lambda(x) := \begin{cases} u_1^\lambda(x), & x\in(0,\pi),\\ v_1^\lambda(x), & x\in(\pi,2\pi).\end{cases}$$

As the function $(f_1^1, f_1^2)\in H^2((0,\pi))\times H^2((\pi,2\pi))$ already possesses higher regularity than L^2 in the ground space, we can always form the traces up to order 1 and have thus some possibilities to vary between boundary and transmission conditions, which are fulfilled for h_1 or not. Consequently, h_1 seems to be more interesting to modify than h_2. That is why we restrict ourselves to the case $h_2=0$, $\rho_1=10$, $\rho_2=20$, distinguish between various types of functions h_1 and solve (4-56) numerically. We focus on Dirichlet boundary conditions, that is,

$$u_1(0)=u_1'(0)=v_1(2\pi)=v_1'(2\pi)=0. \tag{4-58}$$

In the end, we will give a short comparison to other types of boundary conditions but for the moment we assume (4-58) to hold. We sometimes only write L^2 or H^j for $j=0,1,2$ if the domain is clear from the context.

According to Example 2.5.13, we expect at least a decay of the form $\|w_1^\lambda\|_{H^2}\leqslant C|\lambda|^{-1/4}$ for every $h_1\in H^2$. We want to analyse for which choices of h_1 we can improve the decay rate. In addition, we are interested in a comparison to the H^1- and L^2-norm of w_1^λ.

Besides the decay rate of w_1^λ, we also take a look at $\lambda w_1^\lambda - h_1$ in H^j for $j=0,1,2$ since we expect $\|\lambda w_1^\lambda - h_1\|_{H^2}\leqslant C|\lambda|^{-1/4}$ at least for sufficiently smooth functions h_1, see Reminder 4.4.2.

In a first step, we consider functions $f_1^1\in H^2((0,\pi))$ and $f_1^2\in H^2((\pi,2\pi))$ fulfilling all boundary and transmission conditions, which can be formulated in the H^2-setting, that is,

$$\begin{aligned} f_1^1(0)=f_1^{1\prime}(0)=f_1^2(2\pi)=f_1^{2\prime}(2\pi)&=0,\\ f_1^1(\pi)=f_1^2(\pi),&\\ f_1^{1\prime}(\pi)=f_1^{2\prime}(\pi).&\end{aligned} \tag{4-59}$$

We choose for instance

$$h_1(x) := \begin{cases} 1-\cos(x), & x\in(0,\pi),\\ 1-\cos(x)+(x-\pi)^2(x-2\pi)^2, & x\in(\pi,2\pi).\end{cases} \tag{4-60}$$

Notice that h_1 is differentiable but not twice differentiable in π. To get a better feeling of what the solutions of (4-56) look like, we first want to present a plot of λw_1^λ for various values of λ, including the function h_1.

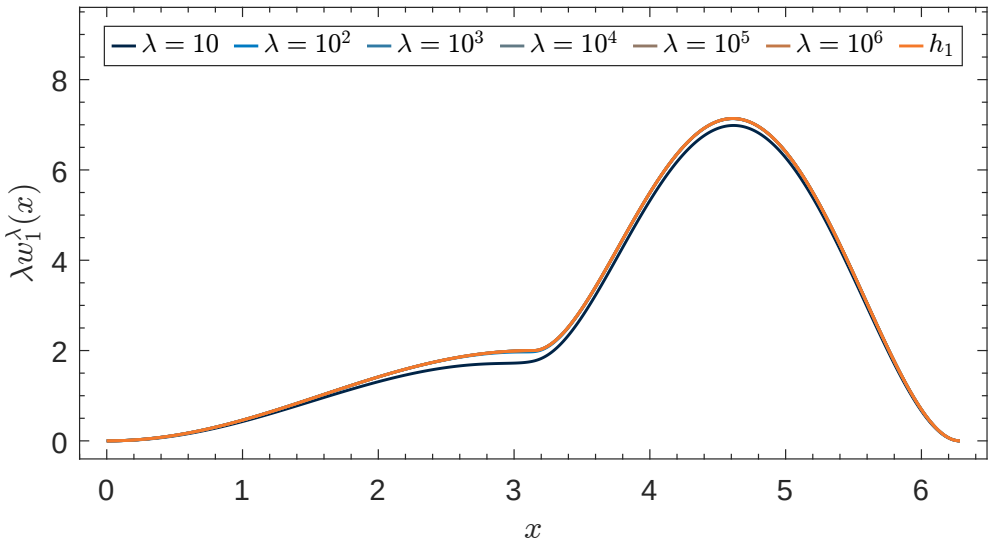

Figure 4.3: Numerical solution w_1^λ of (4-56) with Dirichlet boundary conditions on the interval $(0, 2\pi)$ weighted by λ for different values of λ and h_1 as in (4-60).

As we can see in Figure 4.3, the functions λw_1^λ and h_1 are very close to each other, already for small values of λ. This fact is also reflected in the behaviour of $\lambda w_1^\lambda - h_1$ in H^j for $j = 0, 1, 2$.

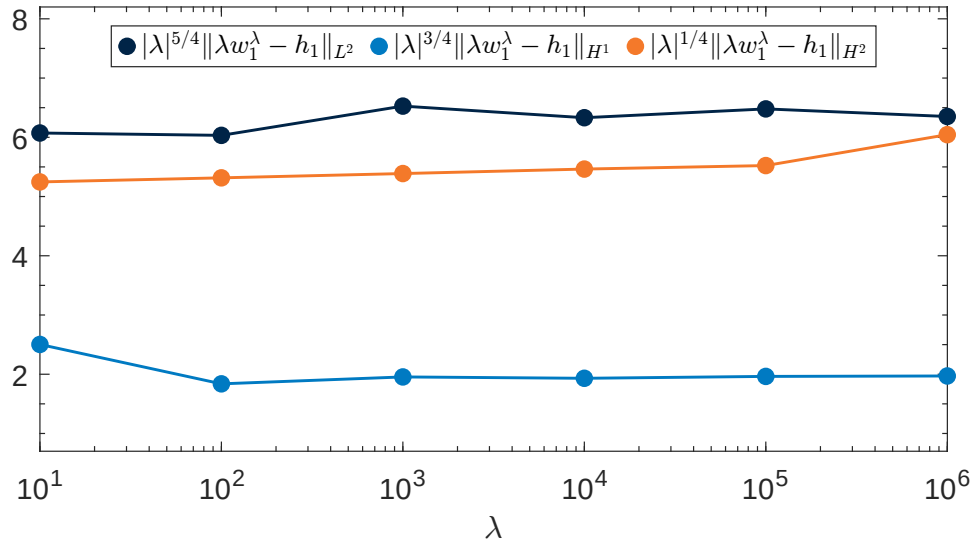

Figure 4.4: Behaviour of the function $\lambda w_1^\lambda - h_1$ with suitable λ-weights in different Sobolev spaces for $\lambda \to \infty$, where h_1 is chosen as in (4-60).

Even though we are not in the situation of Reminder 4.4.2, Figure 4.4 suggests the convergence $\lambda w_1^\lambda \to h_1$ for $\lambda \to \infty$ even in H^2. Comparing the powers of λ, we additionally see that the convergence in H^1 and L^2 is much faster. Therefore, we expect an estimate of the form $|\lambda|\|w_1^\lambda\|_{H^j} \leqslant C$ for $j = 0, 1, 2$. This is numerically confirmed in Figure 4.5.

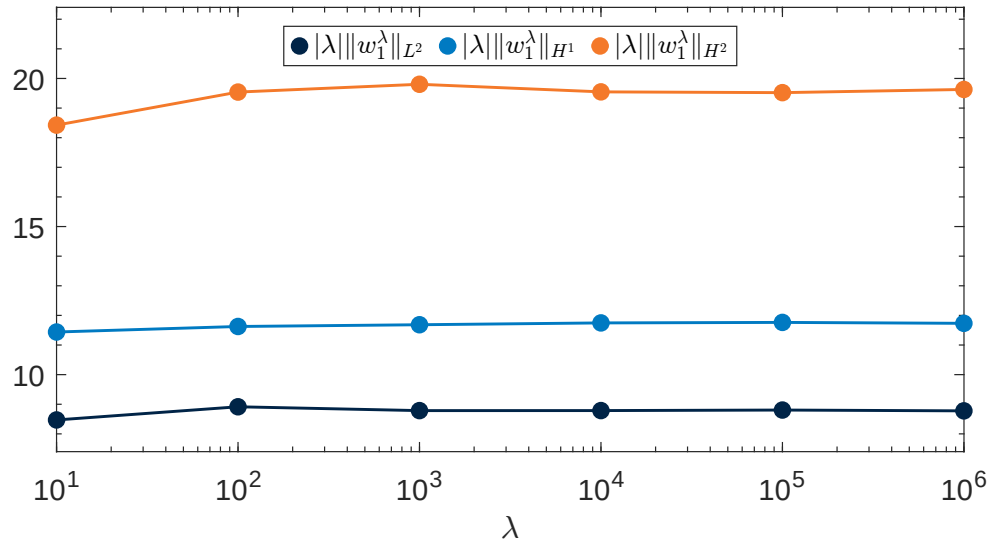

Figure 4.5: Behaviour of λw_1^λ in different Sobolev spaces for $\lambda \to \infty$ with h_1 as in (4-60).

The situation changes if we consider functions $f_1^1 \in H^2((0,\pi))$ and $f_1^2 \in H^2((\pi,2\pi))$ with (4-59) except for the last equality in there, for instance

$$h_1(x) := \begin{cases} 1-\cos(x), & x \in (0,\pi), \\ 1-\cos(x)+(x-\pi)(x-2\pi)^2, & x \in (\pi,2\pi). \end{cases} \tag{4-61}$$

Notice that h_1 is continuous but not differentiable in π. In particular, we have

$$h_1 \in H^1((0,2\pi)) \backslash H^2((0,2\pi)).$$

The functions h_1 and λw_1^λ are illustrated in Figure 4.6 for some values of λ.

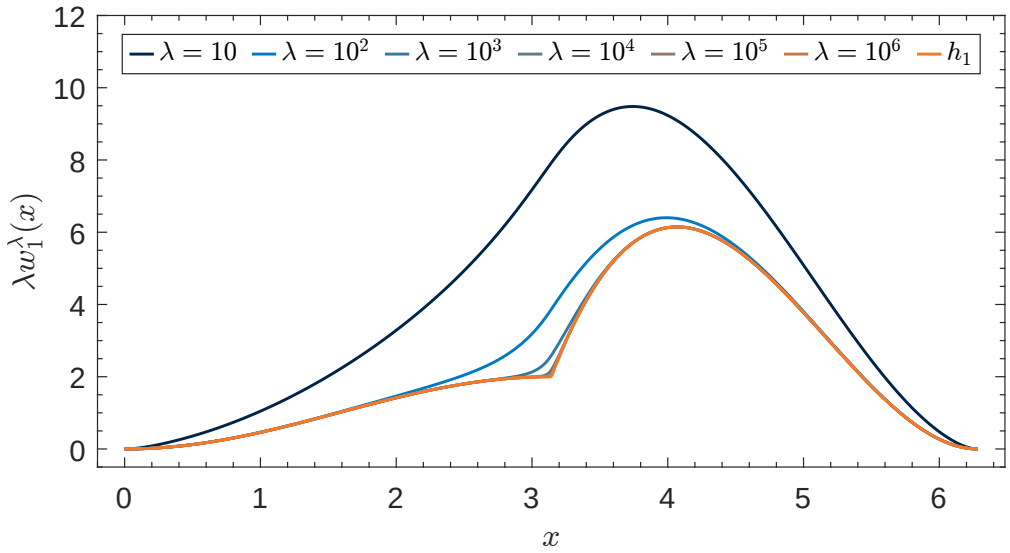

Figure 4.6: Numerical solution w_1^λ of (4-56) with Dirichlet boundary conditions weighted by λ for various values of λ and h_1 as in (4-61).

The pointwise convergence of λw_1^λ in Figure 4.6 is slower than the one in Figure 4.3. A similar slowdown can be observed by studying the convergence of $\lambda w_1^\lambda - h_1$ in H^j for $j = 0, 1, 2$.

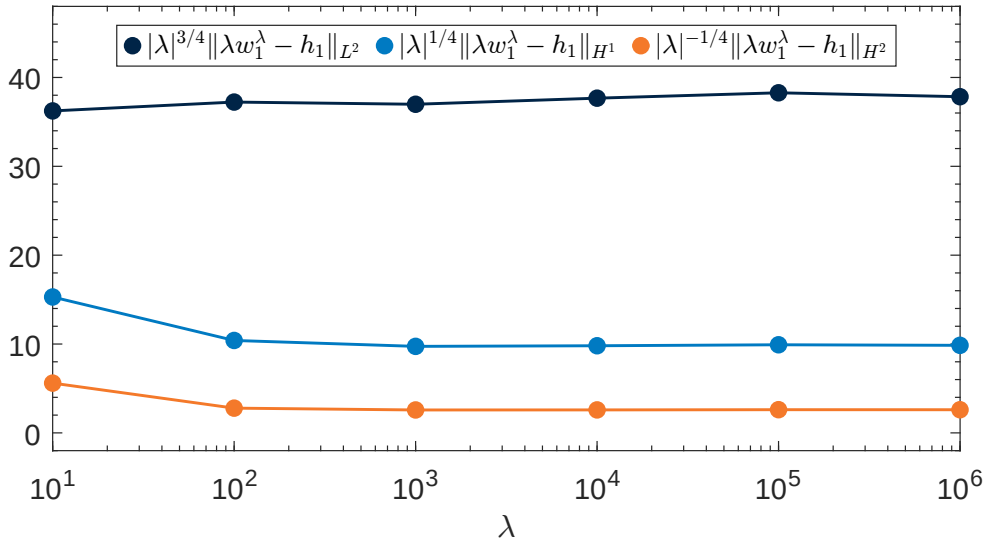

Figure 4.7: Behaviour of the function $\lambda w_1^\lambda - h_1$ multiplied by appropriate powers of λ in different Sobolev spaces for $\lambda \to \infty$, where h_1 is chosen as in (4-61).

As we can see in Figure 4.7, we have no longer $\lambda w_1^\lambda \to h_1$ in H^2 for $\lambda \to \infty$ because of $\|\lambda w_1^\lambda - h_1\| \approx C|\lambda|^{1/4}$. However, the convergence $\lambda w_1^\lambda \to h_1$ still seems to hold in H^1 for $\lambda \to \infty$. We want to take a closer look at the behaviour of $\lambda \mapsto \|w_1^\lambda\|_{H^j}$ for $j = 0, 1, 2$.

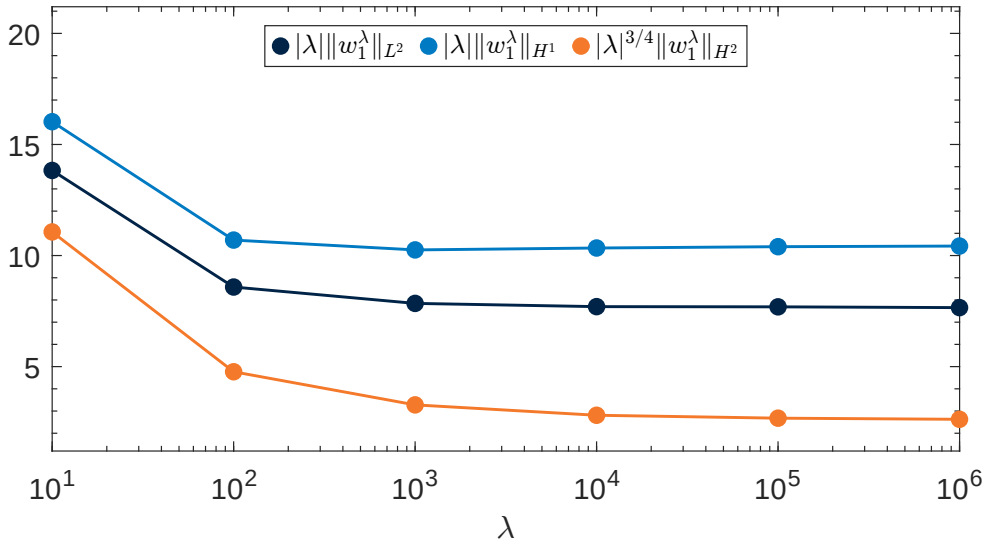

Figure 4.8: Behaviour of the numerical solution w_1^λ with different λ-weights in L^2, H^1 and H^2 for $\lambda \to \infty$, where h_1 is chosen as in (4-61).

Figure 4.8 confirms $|\lambda|\|w_1^\lambda\|_{H^j} \leqslant C$ for $j = 0, 1$, what we already concluded from the convergence $\lambda w_1^\lambda \to h_1$ in H^j for $j = 0, 1$. In addition, we suppose that $|\lambda|^{3/4}\|w_1^\lambda\|_{H^2} \leqslant C$.

In a last example, we analyse the solutions of (4-56) with Dirichlet boundary conditions for the function $h_1 = 1$ on $(0, 2\pi)$. While h_1 satisfies the transmission conditions, the constant one function does not fulfil the Dirichlet boundary conditions. In Figure 4.9 we present a plot of λw_1^λ for various values of λ.

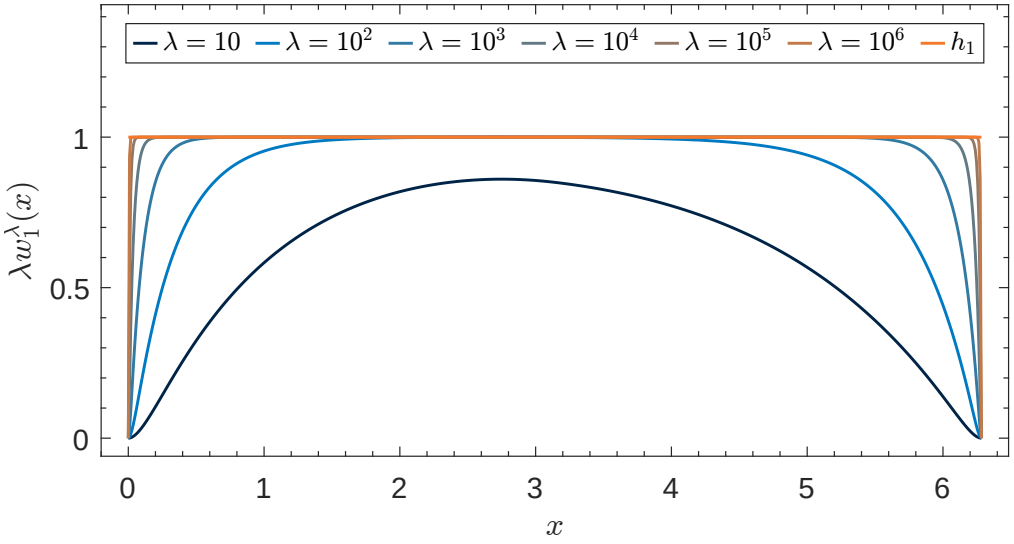

Figure 4.9: Numerical solution w_1^λ of (4-56) with Dirichlet boundary conditions weighted by λ for different values of λ and $h_1 = 1$.

It turns out that the convergence $\lambda w_1^\lambda \to h_1$ only seems to hold in L^2, see Figure 4.10.

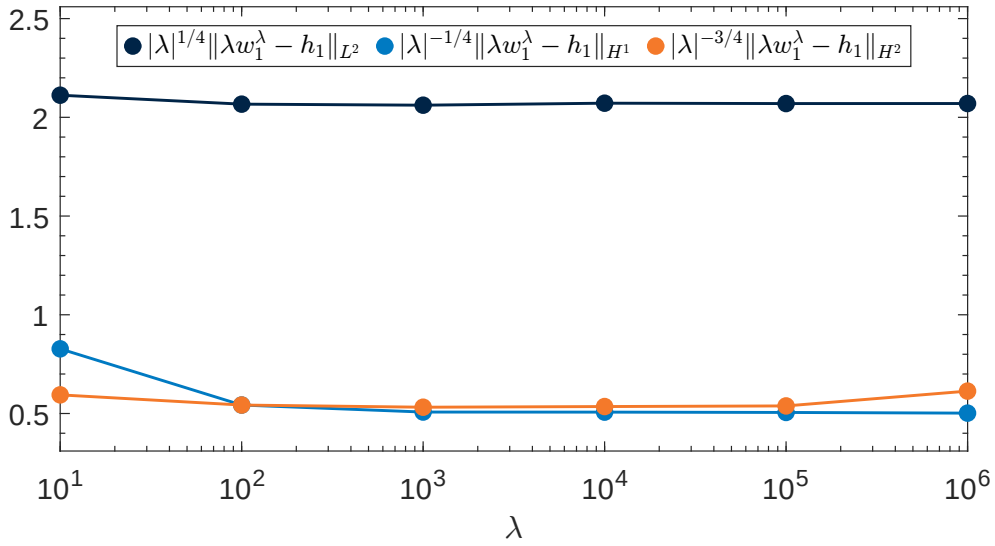

Figure 4.10: Behaviour of the function $\lambda w_1^\lambda - h_1$ multiplied by appropriate powers of λ in different Sobolev spaces for $\lambda \to \infty$, where $h_1 = 1$.

Also the decay rate of $\lambda \mapsto \|w_1^\lambda\|_{H^j}$ downgrades for both $j = 1$ and $j = 2$ with this choice of h_1 compared to the ones in Figure 4.8.

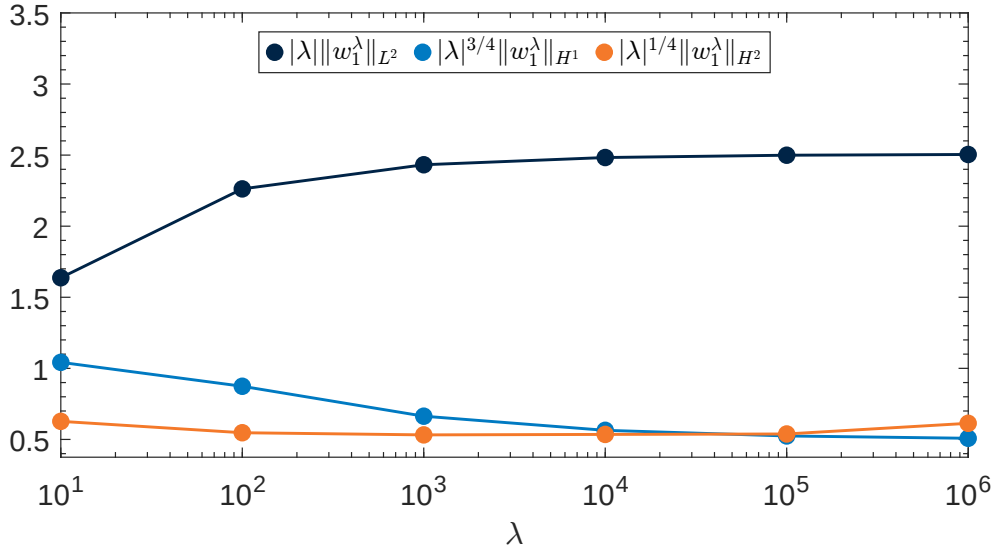

Figure 4.11: Behaviour of the numerical solution w_1^λ with different λ-weights in H^j for $\lambda \to \infty$ and $j = 0, 1, 2$, where $h_1 = 1$.

In Figure 4.11 we find $|\lambda|\|w_1^\lambda\|_{L^2} \leqslant C$, $|\lambda|^{3/4}\|w_1^\lambda\|_{H^1} \leqslant C$ and $|\lambda|^{1/4}\|w_1^\lambda\|_{H^2} \leqslant C$. The last estimate is consistent with Example 2.5.13 and suggests that the decay rate shown there is sharp for $f_1^1 \in H^2((0,\pi))$ and $f_1^2 \in H^2((\pi, 2\pi))$.

However due to the previous examples, the decay rates seem to improve if h_1 belongs to

$$X_0 := \{h_1 \in H^2((0,\pi) \cup (\pi, 2\pi)) : h_1 \text{ fulfils all boundary and transmission conditions of order } 0\}.$$

The best decay rates could have been observed if

$$h_1 \in X_1 := \{h_1 \in X_0 : h_1 \text{ fulfils all boundary and transmission conditions of order } 1\}.$$

In addition to the studies of Dirichlet boundary conditions (4-58), also simulations for Neumann boundary conditions, that is

$$u_1'(0) = u_1''(0) = v_1'(2\pi) = v_1''(2\pi) = 0,$$

as well as for boundary conditions of the form

$$u_1(0) = u_1''(0) = v_1(2\pi) = v_1''(2\pi) = 0$$

were generated.

No matter the boundary conditions, the plots indicate that the decay rates of w_1^λ and the convergence of $\lambda w_1^\lambda - h_1$ in H^j for $j = 0, 1, 2$ only vary for $h_1 \in H^2((0,\pi) \cup (\pi, 2\pi))\backslash X_0$, $h_1 \in X_0\backslash X_1$ and $h_1 \in X_1$. Note that $X_0 \neq H^2((0,\pi) \cup (\pi, 2\pi))$ even in the case of Neumann boundary conditions because of the transmission condition $u_1(\pi) = v_1(\pi)$ of order 0.

We summarise the results in the following table.

Table 4.1: Behaviour of w_1^λ and $\lambda w_1^\lambda - h_1$ in H^j for $j = 0, 1, 2$ and different choices of h_1.

	$h_1 \in H^2((0,\pi) \cup (\pi, 2\pi))\backslash X_0$	$h_1 \in X_0 \backslash X_1$	$h_1 \in X_1$
$\|\lambda w_1^\lambda - h_1\|_{L^2}$	$\|\lambda\|^{-1/4}$	$\|\lambda\|^{-3/4}$	$\|\lambda\|^{-5/4}$
$\|\lambda w_1^\lambda - h_1\|_{H^1}$	$\|\lambda\|^{1/4}$	$\|\lambda\|^{-1/4}$	$\|\lambda\|^{-3/4}$
$\|\lambda w_1^\lambda - h_1\|_{H^2}$	$\|\lambda\|^{3/4}$	$\|\lambda\|^{1/4}$	$\|\lambda\|^{-1/4}$
$\|w_1^\lambda\|_{L^2}$	$\|\lambda\|^{-1}$	$\|\lambda\|^{-1}$	$\|\lambda\|^{-1}$
$\|w_1^\lambda\|_{H^1}$	$\|\lambda\|^{-3/4}$	$\|\lambda\|^{-1}$	$\|\lambda\|^{-1}$
$\|w_1^\lambda\|_{H^2}$	$\|\lambda\|^{-1/4}$	$\|\lambda\|^{-3/4}$	$\|\lambda\|^{-1}$

In the upper part of the table, we perceive that an estimate of the form $\|\lambda w_1^\lambda - h_1\|_{L^2} \leqslant C|\lambda|^\alpha$ implies

$$\|\lambda w_1^\lambda - h_1\|_{H^j} \leqslant C|\lambda|^{\alpha + j/2} \quad (j = 1, 2).$$

Whenever $j \in \{0, 1, 2\}$ with $\|\lambda w_1^\lambda - h_1\|_{H^j} \leqslant C|\lambda|^\alpha$ for a positive value $\alpha > 0$, we expect the estimate $\|w_1^\lambda\|_{H^j} \leqslant C|\lambda|^{\alpha-1}$ for the solution. If $j \in \{0, 1, 2\}$ with $\|\lambda w_1^\lambda - h_1\|_{H^j} \leqslant C|\lambda|^\alpha$ for a negative value $\alpha < 0$, we immediately conclude $\lambda w_1^\lambda \to h_1$ in H^j for $\lambda \to \infty$. In this case, we obtain $\|w_1^\lambda\|_{H^j} \leqslant C|\lambda|^{-1}$. Furthermore, we observe that the estimate $\|\lambda w_1^\lambda - h_1\|_{H^2} \leqslant C|\lambda|^{-1/4}$ suggested in Reminder 4.4.2 seems to be sharp for smooth functions $h_1 \in X_1$.

In the lower part of the table, we see that the necessary condition for a decay rate of $|\lambda|^{-1}$ from Theorem 4.4.1 might also be a sufficient condition in this specific situation.

Deutsche Zusammenfassung

In dieser Arbeit werden L^p-Sobolevräume mit unterschiedlicher Glattheit in Tangential- und Normalenrichtung eingeführt. Anschließend betrachten wir Systeme parameterelliptischer Randwertprobleme gemischter Ordnung der Form

$$\begin{aligned} (\lambda - A)w &= f \quad \text{in } \Omega, \\ Bw &= g \quad \text{auf } \partial\Omega \end{aligned}$$

in diesen Räumen. Hierbei unterscheiden wir zwischen dem Halbraumfall $\Omega = \mathbb{R}^n_+$ und Gebieten $\Omega \subset \mathbb{R}^n$ mit kompaktem Rand hinreichend hoher Glattheit. Während sich im Halbraumfall viele Eigenschaften aus dem klassischen Setting übertragen, ist die Situation im Gebiet schwieriger, da es dort keine kanonische Definition der anisotropen Räume gibt. Aus diesem Grund arbeiten wir im Gebiet mit Einbettungen in die klassischen Sobolevräume, was einige Zusatzforderungen bedingt. Anstelle eines Isomorphismus zwischen dem Raum für die rechte Seite f und g und dem der Lösung w können wir in Gebieten daher nur noch eine injektive Abbildung erwarten. Dennoch lassen sich so a priori Abschätzungen in Sobolevräumen niedrigerer Ordnung zeigen.

Im Anschluss wenden wir die oben genannten Ergebnisse im Hilbertraumfall mit Hilfe der eindimensionalen Fouriertransformation auf zeitabhängige Gleichungen der Form

$$\begin{aligned} (\partial_t - A)w &= f \qquad \text{in } (0,T) \times \Omega, \\ Bw &= g \qquad \text{auf } (0,T) \times \partial\Omega, \\ w(0) &= 0 \end{aligned} \tag{$*$}$$

für $T > 0$ an. Das führt in einigen Spezialfällen für $f = 0$ zu Abschätzungen der Lösung w in passenden Sobolevräumen gegen $\|g\|_{L^2((0,T),L^2(\partial\Omega))}$.

Als Anwendungsbeispiel betrachten wir ein Platte-Membran Transmissionsproblem mit verschiedenen Dämpfungen. Während wir die Platte stets mit einer thermoelastischen Dämpfung versehen, unterscheiden wir zwischen einer gedämpften und einer ungedämpften Membran. Wir zeigen in beiden Fällen die Wohlgestelltheit des Problems sowie höhere Regularität der Lösung.

Abschließend analysieren wir das Langzeitverhalten der Lösung. Im Fall einer gedämpften Wellengleichung können wir exponentielle Stabilität der Lösung nachweisen. Bei einer ungedämpften Membran ist die Lösung nicht mehr exponentiell stabil. Hier geht im Beweis die Theorie für zeitabhängige Systeme $(*)$ mit ein. Dennoch bleibt die Lösung des Platte-Membran Systems auch im Fall einer ungedämpften Membran polynomiell stabil.

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