Essays in Finance

Doctoral thesis for obtaining the
academic degree
Doctor of Economics (Dr. rer. pol.)

submitted by
Marlene Koch

at the

Universität
Konstanz

Faculty of Politics, Law and Economics
Department of Economics

Konstanz, 2023

Tag der mündlichen Prüfung: 08. September 2023

1. Referent: Prof. Dr. Marcel Fischer
2. Referent: Prof. Dr. Jens Jackwerth
3. Referent: Prof. Bjarne Astrup Jensen

Acknowledgments

First, I would like to thank my first supervisor Marcel Fischer. Marcel, it was a pleasure to be your PhD student and to learn from you by working together on several projects. You taught me how to find a research question, how to write papers, and how to never give up, even if it seems that there is only little progress. I always felt, that you supported me in every way you could and that you always wanted the best for me. On a personal note, I think that the joint breakfasts, game nights, and barbecues you are organizing make this chair a special place to work at and no matter where my academic journey takes me next, I will never forget the “Chair of Finance”.

I would also like to thank my second supervisor Jens Jackwerth for his helpful comments and suggestions in seminars, in which I presented my research, and for agreeing to review my thesis and to serve as an oral examiner. I did not give a single talk in which you did not provide very valuable comments that helped me to improve my manuscripts.

Third, I would like to express my gratitude towards my third supervisor, Bjarne Astrup Jensen. I enormously benefited from your experience when working on joint projects. You also made me realize, how privileged I am to work on topics I am really interested in, and you showed me that academia is fun. Additionally, I am incredibly grateful that you hosted me twice at Copenhagen Business School. Your kindness and warm welcome made my time in Denmark special. Further, these visits did not only help me to improve my research, but I also met colleagues, who became friends with whom I am still in touch today.

This PhD would have been way less fun without the amazing colleagues I had. Thank you, Simon, thank you, Natascha. Simon, I really appreciate how you welcomed me to Konstanz, introduced me to your group of friends and helped me being less homesick when I moved to Konstanz. Further, I learned a lot from you when it comes to research. You showed me how to answer research questions empirically in a joint project that did not make it into this thesis but will hopefully be published in due time. I am also grateful that you shared all the experiences you had, before I even started my PhD with me. Natascha, you only joined our chair about one year ago but from the first day I enjoyed discussing research ideas with you. It is so great to have a colleague who is working on related topics. Thank you for always having an open door and listening to all my research– and non–research–related problems. There are way more colleagues and friends who made my time here in Konstanz unforgettable, but mentioning all of them would make these acknowledgments way too long. You know who you are.

I would also like to thank my friends and family at home who supported me from when I was a little girl until today. I really appreciate that you always had my back even though I had not been home much in the last years due to the pandemic and the rescheduled conference trips afterwards. Without your ongoing support and encouragement, I would clearly not be where I am today.

Finally, thank you Stephan for just being great, for always being there for me, and for being one of the best–educated yellow ducks in the world spending weekends helping me to debug my code and listening to new research ideas of mine. You are truly one of my role models as I have never met a more dedicated researcher than you.

Table of Contents

List of Tables

List of Figures

Summary

This PhD thesis consists of four independent research articles, that are all addressing recent challenges in household and/or real estate finance, namely the redesign of pension systems (Chapter 1 and Chapter 2), the housing affordability crisis (Chapter 3), and the asset allocation problem (Chapter 4).

Chapter 1 is based on joint work with Bjarne Astrup Jensen and Marcel Fischer. We derive closed–form solutions of classical intertemporal consumption–savings problems in a deterministic setting. We assume that individuals can invest in a risk–free asset and that the growth rate of labor income is deterministic. We then solve (i) an unrestricted problem, in which individuals can borrow against future labor income, (ii) a restricted problem with borrowing constraints ensuring that wealth is always non–negative, and (iii) a restricted problem with borrowing constraints ensuring that wealth is always non–negative and with the possibility to save in a tax–deferred retirement account. We show that the non–negativity constraint on wealth can be severely binding; particularly in situations with a high growth rate of labor income. In these situations, it is optimal for individuals to postpone accumulating retirement savings until they are older. Further, we show that with access to a tax–deferred account, it is always optimal to save in this (illiquid) account first, before saving in the (liquid) taxable account. The latter account might be needed to satisfy individuals' needs for consumption smoothing. The derived optimal retirement savings pattern is far from the widespread practice of contributing a fixed fraction of current labor income over the working life to a tax–deferred environment.

After showing in Chapter 1 that constant minimum contribution requirements to tax–deferred accounts are at odds with basic principles of consumption–savings theory, my co–authors (Bjarne Astrup Jensen and Marcel Fischer) and I study the impact of constant minimum contribution requirements on housing decisions in Chapter 2. Therefore, we set up a quantitative life cycle model of optimal consumption, savings, and housing decisions. We find that these constant minimum contributions defer housing market entry and increase loan–to–value ratios. We additionally propose a flexible retirement saving scheme that does not force individuals to build up savings in

a tax–deferred retirement account and only requires them to save in either a taxable account, a tax–deferred retirement account, or through home equity, if they are undersaving. This flexible retirement saving scheme largely alleviates the unintended side effects of mandatory minimum contributions and simultaneously ensures that individuals build up sufficient savings for retirement.

In the single–authored Chapter 3 I also study individuals' optimal housing decisions. More specifically, I explore a novel financing option for real estate—"fractional homeownership"—that enables individuals to become housing market participants despite the prevalent housing affordability crisis. Fractional homeownership means that two parties—an individual and an institutional investor—share full ownership of a property. The individual lives in the property full–time and makes periodic rent payments to the institutional investor who sees the property solely as an investment vehicle. The amount of the rent payments depends on the value of the home and the fractional ownership structure, that can be time–varying. Using (as in Chapter 2) a quantitative life cycle model of optimal consumption, savings, and housing decisions, I investigate the impact of access to fractional homeownership on individuals' decision making. I find that primarily young individuals who are liquidity constrained and old individuals who are in the phase of decumulating their savings make use of the fractional homeownership. Further, this novel financing option leads to earlier housing market entry, later housing market exit, decreases individuals' loan–to–value ratios and reduces their moving activity at old age; all in comparison to a setting in which the individuals' rent–versus–own decisions are binary. However, it does not lead to undersaving as individuals with/without access to fractional homeownership accumulate roughly the same level of total savings over the life cycle.

Chapters 1, 2, and 3 show the potential benefits for individuals when making optimal financial decisions. Yet, solving such complex dynamic optimization problems is too difficult for the average individual. Therefore, Marcel Fischer and I propose a simple heuristic that enables individuals to make close–to–optimal asset allocation decisions in Chapter 4. The individuals' equity exposures are then a linear function of their labor income, total savings, and age. When individuals follow our heuristic, welfare gains when switching to the optimal policy are only 0.06% of lifetime consumption (around USD 1,600). Relative to portfolio strategies used by individuals in the data, welfare gains from using our heuristic are about 1.3% of lifetime consumption (around USD 35,000). Our results are robust to different educational attainments and family structures.

Zusammenfassung

Die vorliegende Dissertation besteht aus vier eigenständigen Forschungsartikeln, welche aktuelle Herausforderungen in den Bereichen “Household Finance” und/oder “Real Estate Finance” thematisieren. Zu diesen Herausforderungen zählen die Reformierung von Pensionssystemen (Kapitel 1 und Kapitel 2), die Bezahlbarkeit von Wohnraum (Kapitel 3) und Anlageentscheidungen am Kapitalmarkt (Kapitel 4).

Kapitel 1 ist in Zusammenarbeit mit Bjarne Astrup Jensen und Marcel Fischer entstanden. In diesem leiten wir durch das Lösen eines deterministischen Modells analytische Lösungen für drei Varianten des klassischen intertemporalen Konsum–Sparproblems her. Wir nehmen an, dass Individuen in ein risikoloses Anlageinstrument investieren können und dass das Wachstum des Arbeitseinkommens deterministisch ist. Anschließend lösen wir (i) ein Problem, in welchem Individuen die Möglichkeit haben, sich Geld zu leihen, (ii) ein beschränktes Problem, in welchem das Vermögen der Individuen nicht–negativ sein muss und (iii) ein beschränktes Problem, in welchem das Vermögen der Individuen nicht–negativ sein muss und in welchem sie zusätzlich die Möglichkeit haben, in einem nachgelagert–besteuerten Pensionskonto zu sparen. Wir zeigen, dass die Restriktion, dass das Vermögen nicht–negativ sein muss, schwerwiegend sein kann; insbesondere, wenn das Arbeitseinkommen im Verlauf der Zeit stark wächst. In diesem Fall ist es optimal, erst in höherem Alter mit dem Pensionssparen zu beginnen. Ferner zeigen wir, dass es—gegeben, dass Individuen Zugriff auf ein nachgelagert–besteuertes Pensionskonto haben—immer optimal ist, zunächst in diesem illiquiden Konto zu sparen und erst anschließend in einem liquiden Bankkonto, um die Konsumkurve über den Lebenszyklus zu glätten. In der Praxis sind Individuen häufig verpflichtet, konstante Mindestbeiträge zu nachgelagert–besteuerten Pensionskonten zu leisten. Unsere Ergebnisse zeigen, dass diese Verpflichtung klar im Widerspruch zu optimalem Ansparverhalten steht.

Nachdem meine Ko–Autoren und ich in Kapitel 1 zeigen, dass konstante Mindestbeiträge zu nachgelagert–besteuerten Pensionskonten im Widerspruch zu fundamentalen Konsum–Spar–Prinzipien stehen, untersuchen wir (Bjarne Astrup Jensen, Marcel Fischer und ich) in Kapitel 2 mithilfe eines quantitativen Lebenszyklusmodells den

Einfluss von eben diesen konstanten Mindestbeiträgen auf Immobilienentscheidungen. Unsere Ergebnisse implizieren, dass konstante Mindestbeiträge zu nachgelagert–besteuerten Pensionskonten zu einer späteren Anschaffung von Immobilien und höheren Beleihungsquoten führen. Ferner schlagen wir in diesem Projekt einen flexiblen Pensionssparplan vor. Wenn Individuen mit Hilfe dieses Plans sparen, sind sie nicht verpflichtet, konstante Mindestbeiträge zu nachgelagert–besteuerten Pensionskonten zu leisten. Stattdessen müssen sie ein altersabhängiges Sparziel erfüllen. Ob sie dieses durch Einzahlungen auf ein konventionelles Bankkonto oder ein nachgelagert–besteuertes Pensionskonto oder durch Immobilieninvestitionen erfüllen, ist ihnen freigestellt. Dieser flexible Pensionssparplan führt dazu, dass Individuen—analog dem Fall, in dem sie konstante Mindestbeiträge zu nachgelagert–besteuerten Pensionskonten leisten müssen—noch immer ausreichend für die Rentenphase sparen. Allerdings sind ihre Beleihungsquoten deutlich niedriger und sie können früher in Immobilien investieren.

In Kapitel 3, welches ich alleine verfasst habe, analysiere ich ebenfalls Immobilienentscheidungen von Individuen. Genauer gesagt befasse ich mich mit einer neuen Form der Immobilienfinanzierung—“fractional homeownership”—welche es Individuen ermöglicht, Immobilien trotz der derzeit schlechten Bezahlbarkeit von Wohnraum zu erwerben. “Fractional homeownership” bedeutet, dass sich eine Immobilie im Besitz zweier Investoren—einem Individuum und einem institutionellen Investor—befindet. Das Individuum lebt permanent in der Immobilie und leistet periodische Mietzahlungen an den institutionellen Investor, für welchen die Immobilie lediglich eine Investitionsmöglichkeit darstellt. Die Höhe der Mietzahlungen hängt von den genauen Eigentumsverhältnissen, welche sich im Laufe der Zeit ändern können, ab. Ich untersuche mithilfe eines quantitativen Lebenszyklusmodells wie die Möglichkeit, eine Immobilie mittels “fractional homeownership” zu erwerben, optimale Konsum–Spar–Entscheidungen von Individuen beeinflusst. Meine Ergebnisse zeigen, dass primär junge Individuen mit beschränkter Liquidität und ältere Individuen, die sich in der Phase des Entsparens befinden, von “fractional homeownership” Gebrauch machen und dadurch früher in den Immobilienmarkt eintreten und ihn später wieder verlassen können. Ferner sind die Beleihungsquoten der Individuen, die über die Möglichkeit verfügen, ihre Immobilie mittels “fractional homeownership” zu finanzieren, deutlich niedriger und eben diese Individuen müssen im hohen Alter seltener umziehen. Dennoch führt “fractional homeownership” nicht dazu, dass Individuen insgesamt zu wenig sparen; deren Gesamtersparnisse sind vergleichbar mit denen von Individuen, die ihren Wohnsitz entweder mieten oder ganz kaufen können.

Kapitel 1, Kapitel 2 und Kapitel 3 zeigen die potenziellen Vorteile für Individuen, wenn sie optimale Finanzmarktentscheidungen treffen. Allerdings ist das Lösen solcher dynamischer Optimierungsprobleme für das durchschnittliche Individuum deutlich zu komplex. Daher entwickeln Marcel Fischer und ich in Kapitel 4 eine Heuristik, welche es Individuen ermöglicht, eine für sie nahezu optimale Aktienquote zu berechnen. Die Aktienquote ist dann eine lineare Funktion, welche vom Arbeitseinkommen, von den Ersparnissen und vom Alter der Individuen abhängt. Wenn Individuen anhand der von uns vorgeschlagenen Heuristik investieren, wären ihre Wohlfahrtsgewinne, wenn sie stattdessen optimal investieren würden, nur 0,06% ihres Konsums über den gesamten Lebenszyklus (zirka USD 1'600). Ein Vergleich von unserer Heuristik mit empirisch–beobachteten Anlagestrategien zeigt, dass Individuen—würden sie unserer Heuristik folgen—Wohlfahrtsgewinne von 1,3% ihres Konsum über den gesamten Lebenszyklus (zirka USD 35'000) erzielen könnten. Unsere Ergebnisse sind robust bezüglich verschiedener Bildungsabschlüsse und Familienzusammensetzungen.

Chapter 1

Optimal Retirement Savings over the Life Cycle: A Deterministic Analysis in Closed Form*

*This chapter is published as Fischer, M., B. A. Jensen, and M. Koch (2023). "Optimal Retirement Savings over the Life Cycle: A Deterministic Analysis in Closed Form." *Insurance: Mathematics and Economics* 112, 48–58, https://doi.org/10.1016/j.insmatheco.2023.05.010. The corresponding "Online Appendix" is placed at the end of this chapter as Appendix 1.A, Appendix 1.B, and Appendix 1.C. We are grateful for helpful comments and suggestions from seminar participants at the University of Amsterdam. This research has been supported by PeRCent, which receives base funding from the Danish pension funds and Copenhagen Business School.

1.1. Introduction

A large literature studies optimal retirement savings and related asset allocation decisions. Particular attention has been devoted to the optimal location of assets to taxable and tax–deferred retirement accounts in both theoretical work providing closed–form solutions to this question (e.g., Black (1980); Tepper (1981); Garlappi and Huang (2006); and Huang (2008)) and life cycle models relying on numerical methods (e.g., Shoven and Sialm (1998); Shoven and Sialm (2003); Amromin (2003); Dammon et al. (2004); and Fischer and Gallmeyer (2017)). Another strand of literature investigates the optimal draw–down of savings during retirement age (e.g., Huang and Milevsky (2016) and Huang et al. (2012)). While these manuscripts have improved our understanding of the optimal location of assets to taxable and tax–deferred environments and the optimal draw–down of savings during retirement age, less attention has been paid to the optimal rate of contributions to tax–deferred retirement accounts over the life cycle. In this manuscript, we shed light on optimal contribution rates to tax–deferred retirement accounts in a setting, in which individuals have access to a taxable and a tax–deferred account, a risk–free investment opportunity and earn deterministic labor income. Our analytical solutions show that contributions to the tax–deferred retirement account are age–dependent, which contrasts with the constant contribution requirements, many countries are imposing on their citizens.

This study is inspired by some institutional features that are common in countries that rely on pension saving in an asset–backed defined contribution system (DC system) such as Denmark and the Netherlands.[1] We assume that the so–called *EET* tax regime is valid. This means that contributions to the retirement account are tax–deductible from labor income (up to certain limits) at the time the money is paid into the system (the first E). During the accumulation phase, the financial return is tax–exempt (the second E). When the pension savers reach retirement age and receive their retirement income by drawing from the accumulated reserves, the payout is taxed as income (the T). The retirement account is thus what is known as a tax–deferred account. The tax rate applicable to withdrawals may be different from the tax rate applicable at time of deduction. If so, it is typically lower.

The *EET* tax regime is commonly found in, e.g., labor market–oriented pension schemes in which individuals are required—either by labor market agreements or by

[1] To be precise in the terminology, we disregard the now widespread occurrence of so–called notionally defined contribution systems (NDC systems), that seemingly operate like a DC system. In an NDC system, the pension liabilities are not asset–backed, but rather rely on the contributions from the (future) working population to finance the future pension payments. For a survey of countries with such an approach, see, e.g., Holzmann and Palmer (2006).

law—to contribute to pension funds throughout their working life. It is also commonly found that the contributions are a constant fraction of labor income throughout the working life. A feature, which may also just be assumed as a convenient modeling device. One example is found in the influential and widely cited paper by Dammon et al. (2004):

> "The treatment of the investor's retirement account is broadly consistent with practice. Prior to retirement, the investor is assumed to contribute a constant fraction k of pre–tax non–financial income to a retirement account each year."
>
> (Dammon et al. (2004), p. 1013)

While the (re–)design of pension systems is an important policy topic in light of the demographic development and the longevity risk inherent in many countries' present arrangements, whether contributing a constant fraction of labor income into a tax–deferred retirement account is optimal has received relatively little attention in the public debate. However, it is a fact of life that liquidity constraints vary over the life cycle and are most severe for younger individuals, who have recently finished their education with a possible debt to be repaid and being in the process of establishing a family with the needs for finding appropriate housing. It is also partly a result of a wage profile that is typically increasing over the working life. Classical references include Deaton (1991), Carroll (1997), Constantinides et al. (2002) and Gourinchas and Parker (2002). More recent work includes Lachance (2012), Blake et al. (2014), Schlafmann et al. (2022), Fischer et al. (2022), and Larsen and Munk (2023).

Our work challenges the idea that contributions to a retirement savings scheme should be a constant fraction of labor income; it also challenges the idea that pension saving should start early in the life cycle. If one searches Google for answers to the question: "When and how to save for retirement?", numerous answers come up. Interestingly, they are pretty much alike. Here are two typical examples:

- The answer is simple: as soon as you can. Ideally, you'd start saving in your 20s, when you first leave school and begin earning paychecks. That's because the sooner you begin saving, the more time your money has to grow.[2]
- Short answer: as soon as you begin working. Too late for that? Remember that saving will be easiest if you start now.[3]

[2]Source: https://money.cnn.com/retirement/guide, last retrieved September 19, 2022.
[3]Source: https://investor.vanguard.com/investor-resources-education, last retrieved September 19, 2022.

These answers are fatally wrong in our view. We ask how a utility optimizing scheme looks like using a deterministic model in order to allow for analytical solutions. We provide a closed–form solution to this problem with both a taxable and a tax–deferred retirement account, a risk–free investment opportunity, and a deterministic labor income stream. Our results show that the widespread institutional feature of contributing a constant share of labor income is typically not optimal.

In the first step, the investor only has the possibility to save in an ordinary way in a taxable account, in which the current return on the investment is taxed in accordance with a mark–to–market principle; we also allow the investor to borrow against future labor income. Saving for retirement is the sole responsibility of the investor, who will behave perfectly rational in this regard.

In the next step, we impose a non–negativity constraint on financial wealth. Depending upon the profile of future labor earnings, this constraint may lead to the investor being liquidity constrained in the initial periods of the working life.

Finally, in our richest model, we allow the investor to save for retirement in a tax–deferred account. The accumulated retirement savings are locked and inaccessible for the investor until attaining retirement age. The compensation for accepting the illiquid nature of the retirement savings is the favorable tax treatment of savings in tax–deferred account. Furthermore, the tax rate applicable to payouts from the pension system may be lower than the one applicable at the time of deduction. Both these features make it optimal for the investor to exploit the tax–advantages offered by accumulating retirement savings in the tax–deferred account instead of accumulating them in the fully liquid taxable account.

During working life, the taxable account is used for intertemporal consumption smoothing and for securing access to funds in cases, in which a stochastic environment triggers bad outcomes, such as low labor income earnings. The latter is well–known as precautionary savings/buffer stock savings motive (see, e.g., Carroll (1997); Gourinchas and Parker (2002); and Amromin (2003)). Both motives imply that—depending on the model parameters—it may be optimal to (also) build up savings in the taxable account. The precautionary savings/buffer stock savings motive is absent in a deterministic setting, whereas intertemporal consumption smoothing via the taxable account may still be needed.

This paper proceeds as follows: In Section 1.2, we solve an unrestricted version of the consumption–savings problem in which individuals can borrow against future labor income. In Section 1.3, we extend the model introduced in Section 1.2 by imposing a non–negativity constraint on wealth and then derive optimal consumption–savings decisions. In Section 1.4, we solve a consumption-savings model that additionally offers individuals the possibility to save in a tax–deferred retirement account. In Section 1.5, we outline some generalizations to our model. Finally, Section 1.6 concludes.

1.2. A simple unconstrained model

The simplest possible scenario is an unconstrained variant of the classical intertemporal consumption–savings problem (see, e.g., Hakansson (1970); Merton (1969); and Merton (1971)), where the consumption stream, C_t, is financed by labor income, and where initial wealth, W_{0-}, is defined to be 0:[4]

$$\max_{\{\{W_t\}_{t=0}^{T-1}\}} \frac{1}{1-\gamma}\left[\sum_{t=0}^{T-1}\rho^t C_t^{1-\gamma} + K\rho^T C_T^{1-\gamma}\right] \tag{1.2.1}$$

subject to

$$C_t = L_t(1-\tau) + W_{t-1}\big(1+r(1-\tau)\big) - W_t, \quad t = 0,1,\ldots,T-1 \tag{1.2.2}$$

$$C_T = W_{T-1}\big(1+r(1-\tau)\big), \tag{1.2.3}$$

in which γ is the individual's risk aversion, $\rho < 1$ its time discount rate, measuring how much the individual prefers present over future consumption, r is the pre–tax interest rate that is earned on savings, τ is a common tax rate applicable to labor income and interest earnings, respectively, and W_t is the wealth invested from time t to $t+1$ in order to finance future consumption. The utility optimizing individual is assumed to have no initial wealth, but receives a deterministic stream of labor income L_t, $t = 0,1,\ldots,T-1$. The weighting factor K is supposed to summarize the discounted utility stream from consumption during the retirement period.[5]

The first–order conditions, also known as Euler conditions, for this utility optimization problem are:

$$\rho^t C_t^{-\gamma} = \rho^{t+1}\big(1+r(1-\tau)\big)C_{t+1}^{-\gamma}, \quad [W_t], \quad t = 0,1,2,\ldots,T-2 \tag{1.2.4}$$

$$\rho^{T-1}C_{T-1}^{-\gamma} = \big(1+r(1-\tau)\big)K\rho^T C_T^{-\gamma}, \quad [W_{T-1}], \tag{1.2.5}$$

[4] We relax the assumption of $W_{0-} = 0$ in Section 1.5.2 by allowing for pre–existing initial wealth.

[5] It is well–known that with a time–additive utility function of the HARA type, this sum has the same functional form as the individual utility functions in the sum. In Appendix 1.C we give examples as to how parameter K could be chosen.

with solutions:

$$C_t = C_0\varepsilon^t, \quad t = 1,2,\ldots,T-1 \tag{1.2.6}$$

$$C_T = C_0\varepsilon^T K^{\frac{1}{\gamma}}, \quad \varepsilon \equiv \left(\rho\left(1+r\left(1-\tau\right)\right)\right)^{\frac{1}{\gamma}}. \tag{1.2.7}$$

In order to derive an analytically tractable solution, we assume a wage process with a uniform gross growth factor g, i.e. $L_t = L_0 g^t$, $t = 1,\ldots,T-1$.[6] The dynamics of the savings account for $t = 0,1,\ldots,T-1$ then become:

$$W_t = (1+r(1-\tau))^{-(T-t)}C_0\varepsilon^T K^{1/\gamma} + \sum_{s=t+1}^{T-1}\left[C_0\varepsilon^s - L_0(1-\tau)g^s\right](1+r(1-\tau))^{-(s-t)}. \tag{1.2.8}$$

The level of C_0 is found by using the initial budget constraint $W_0 = L_0(1-\tau) - C_0$, which implies:

$$0 = (1+r(1-\tau))^{-T}C_0\varepsilon^T K^{1/\gamma} + \sum_{s=0}^{T-1}\left[C_0\varepsilon^s - L_0(1-\tau)g^s\right](1+r(1-\tau))^{-s}. \tag{1.2.9}$$

From here, the consumption–savings policy for $t = 0,1,\ldots,T-1$ is given in closed form by:

$$\begin{aligned} C_t &= C_0\varepsilon^t = L_0(1-\tau)\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}}x^T}\varepsilon^t \\ x &= \frac{\varepsilon}{1+r(1-\tau)}, \quad y = \frac{g}{1+r(1-\tau)}. \end{aligned} \tag{1.2.10}$$

The shape of the consumption pattern and the relative share of current disposable income, which is consumed in every period, is given solely by the relation between g and ε:

$$\frac{C_t}{L_t(1-\tau)} = \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}}x^T}\left(\frac{\varepsilon}{g}\right)^t. \tag{1.2.11}$$

If $g < \varepsilon$ ($\Leftrightarrow \; y < x$), which means that labor income is relatively slowly increasing, we have an increasing profile of consumption out of disposable income and, vice versa, a decreasing profile of the savings rate. For $g < \varepsilon$, it is thus optimal to already build up savings at the beginning of the life cycle in order to smooth consumption in accordance with the preferences for intertemporal consumption.

[6] We allow for time–varying growth rates of labor income in Section 1.5.3.

If $g > \varepsilon$ ($\Leftrightarrow x < y$), labor income is relatively steeply increasing. In that case, we have a decreasing profile of consumption out of disposable income and, vice versa, an increasing profile of the savings rate out of disposable income. The optimal savings rate out of disposable income then has an increasing trend in line with the increasing profile of labor income over the lifetime. Savings may even be negative and allow the individual to borrow against future labor income in the beginning and first turn positive at some later point in time. Saving for the retirement period is optimally postponed as a response to the increasing labor income. To be precise, the borderline between saving and borrowing at time t is determined by whether current income after tax is sufficient to cover current expense, i.e., by the following inequality:

$$L_0(1-\tau)g^t \gtreqqless L_0(1-\tau)\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\varepsilon^t \quad \Leftrightarrow \quad \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\left(\frac{\varepsilon}{g}\right)^t \lesseqqgtr 1. \tag{1.2.12}$$

This set of inequalities determines when and how individuals' budget constraints become binding when we add restrictions to the consumption–investment problems in the following Theorems 1 and 2 and is a key ingredient in the detailed proofs of these theorems.

In both cases ($g < \varepsilon$ and $g > \varepsilon$), a constant savings rate out of disposable income is not part of the optimal solution. Only the borderline situation with $\varepsilon = g$ leads to a constant savings rate out of disposable income.

The value of the utility function, disregarding the factor $1/(1-\gamma)$ in front, has the analytical form:

$$\begin{aligned}
&\left[L_0(1-\tau)\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\right]^{1-\gamma}\left[\sum_{t=0}^{T-1}\left(\varepsilon^{1-\gamma}\right)^t \rho^t + \varepsilon^{(1-\gamma)T}\rho^T K^{\frac{1}{\gamma}}\right] = \\
&\left[L_0(1-\tau)\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\right]^{1-\gamma}\left[\frac{\left(1-\rho^T\varepsilon^{(1-\gamma)T}\right)}{1-\rho\varepsilon^{1-\gamma}} + \rho^T\varepsilon^{(1-\gamma)T}K^{\frac{1}{\gamma}}\right].
\end{aligned} \tag{1.2.13}$$

Once the intertemporal structure of the consumption policy is found, the remaining step is to insert equation (1.2.7) into the intertemporal budget constraint and equate the present value of the labor income stream to the present value of consumption expenditures. x is then the discount factor for a geometrically growing stream of consumption, whereas y is the discount factor for a geometrically growing stream of labor income.

1.3. Non–negative financial wealth

We next account for the fact that, in reality, the individual is not able to finance a consumption level in excess of current financial resources; i.e., the individual is constrained to always be financially solvent and unable to borrow against future labor income. This reflects the usual institutional feature that individuals are unable to collateralize their human capital, e.g., due to moral hazard considerations. The optimization problem and the budget equations are the same as in equations (1.2.2)–(1.2.3), but the solvency constraints $W_t \geq 0$, $t = 0,1,2,\ldots,T-1$, are added.

Variants of this problem appear in many contexts in the economic literature.

The first–order conditions now involve the complementary slackness conditions associated with the non–negativity constraints:[7]

$$\begin{aligned} &\rho^t C_t^{-\gamma} \geq \rho^{t+1}\left(1 + r\left(1-\tau\right)\right) C_{t+1}^{-\gamma} \quad \Leftrightarrow \quad \varepsilon C_t \leq C_{t+1} \\ &\left[C_{t+1} - \varepsilon C_t\right] W_t = 0, \quad t = 0,1,2,\ldots,T-2 \end{aligned} \tag{1.3.1}$$

$$C_{T-1}^{-\gamma} = \left(1 + r\left(1-\tau\right)\right) K\rho C_T^{-\gamma} \quad \Leftrightarrow \quad C_T = K^{1/\gamma}\varepsilon C_{T-1}. \tag{1.3.2}$$

Whereas the discounting mechanism is well defined and exogenously given by the after–tax interest rate in the unrestricted model outlined in Section 1.2, this is no longer true in the current setting with the non–negativity constraint. When the non–negativity constraint is binding, the marginal value of a future unit of wealth is less than the value given by simple discounting as the marginal utility of present consumption calls for the opportunity to convert future consumption into present consumption. The before–tax interest rate (the market rate) is only relevant when the individual is actually saving and has already built up savings in previous periods, i.e., if $W_t > 0$. If $W_t = 0$, the individual might want to borrow against future labor income, but is unable to use the financial market for this purpose.

However, for a labor income process with a constant growth rate, we are still able to derive the optimal consumption–savings profile in closed form:

Theorem 1. *When labor income has a uniform gross growth rate g, there are two possible scenarios, one of which leads to an optimal solution. Which of the two solutions applies depends on how the complementary slackness conditions (1.3.1) and (1.3.2) are fulfilled:*

1. *If $\varepsilon \geq g$, we have $C_0 < L_0(1-\tau)$, and the unconstrained solution is valid throughout.*

[7] For more details, see Appendix 1.A.

2. *If $\varepsilon < g$, we may have a sequence of initial periods in which $C_t = L_t(1-\tau)$, followed by a saving for retirement period, during which the profile follows the unconstrained solution with the reduced remaining time to retirement. The switch point s^*, the last time index at which $C_t = L_t(1-\tau)$, is determined by*

$$\frac{\sum_{s=s^*}^{T-1} y^{(s-s^*)}}{\left[\sum_{s=s^*}^{T-1} x^{s-s^*} + K^{\frac{1}{\gamma}} x^{T-s^*}\right]} > 1 > \frac{\sum_{s=s^*+1}^{T-1} y^{s-(s^*+1)}}{\left[\sum_{s=s^*+1}^{T-1} x^{s-(s^*+1)} + K^{\frac{1}{\gamma}} x^{T-(s^*+1)}\right]}. \tag{1.3.3}$$

Proof. See Appendix 1.A. □

In case of $\varepsilon < g$, i.e., in case 2 of Theorem 1, the unrestricted solution may still be feasible. If the solvency constraint is not binding at any point in time, there is no need to search for a switch point s^*. This happens when

$$\sum_{s=0}^{T-1} y^s < \sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T. \tag{1.3.4}$$

When $\varepsilon = g$ ($\Leftrightarrow$ $x = y$), the inequality in (1.3.4) is clearly fulfilled. The left–hand side of (1.3.4) is increasing in g with no upper limit, whereas the right–hand side is independent of g. Hence, for some value of g the inequality in (1.3.4) is no longer valid. The exact condition on how large the growth in labor income, g, must be for the inequality in (1.3.4) not to hold and the non–negativity constraint to be binding in some initial period(s) depends on the parameters of the model—in particular on K.

The inequalities in (1.3.3) can be rewritten as follows:

$$\frac{\sum_{s=s^*}^{T-1} y^{(s-s^*)}}{\sum_{s=s^*}^{T-1} x^{s-s^*} + K^{\frac{1}{\gamma}} x^{T-s^*}} = \frac{\sum_{s=0}^{T-1-s^*} y^s}{\sum_{s=0}^{T-1-s^*} x^s + K^{1/\gamma} x^{T-s^*}}. \tag{1.3.5}$$

The numerator in the right–hand side of equation (1.3.5) is now the "present" value of labor income starting from $L_0 = 1$, when the clock is reset to start at time s^* (see the standard formula for discounting a geometrically growing series). Similarly, the denominator is the "present" value of consumption starting from $C_0 = 1$, when the clock is reset to start at time s^*.

When $s^* > 0$, the individual is unable to reach the desired unrestricted level of consumption that will fulfill the Euler conditions. The marginal utility is forced to be too high in the beginning, which is captured by a strict inequality in (1.3.1). At some point in time, the labor income is high enough in order for the unrestricted solution to work for the remaining periods (Bellman's optimality principle). So with a sufficiently high growth in labor income, saving is postponed.

We illustrate the impact of the non–negativity condition for wealth graphically in Figure 1.3.1. We choose a time horizon of $T = 10$ periods and compare two different growth rates of labor income: (i) $g = 1.0$ and (ii) $g = 1.05$. Panel A depicts the evolution of the individual's consumption level over the time horizon and Panel B of the consumption–to–labor income ratio. Panel C shows the evolution of total wealth over the time horizon.

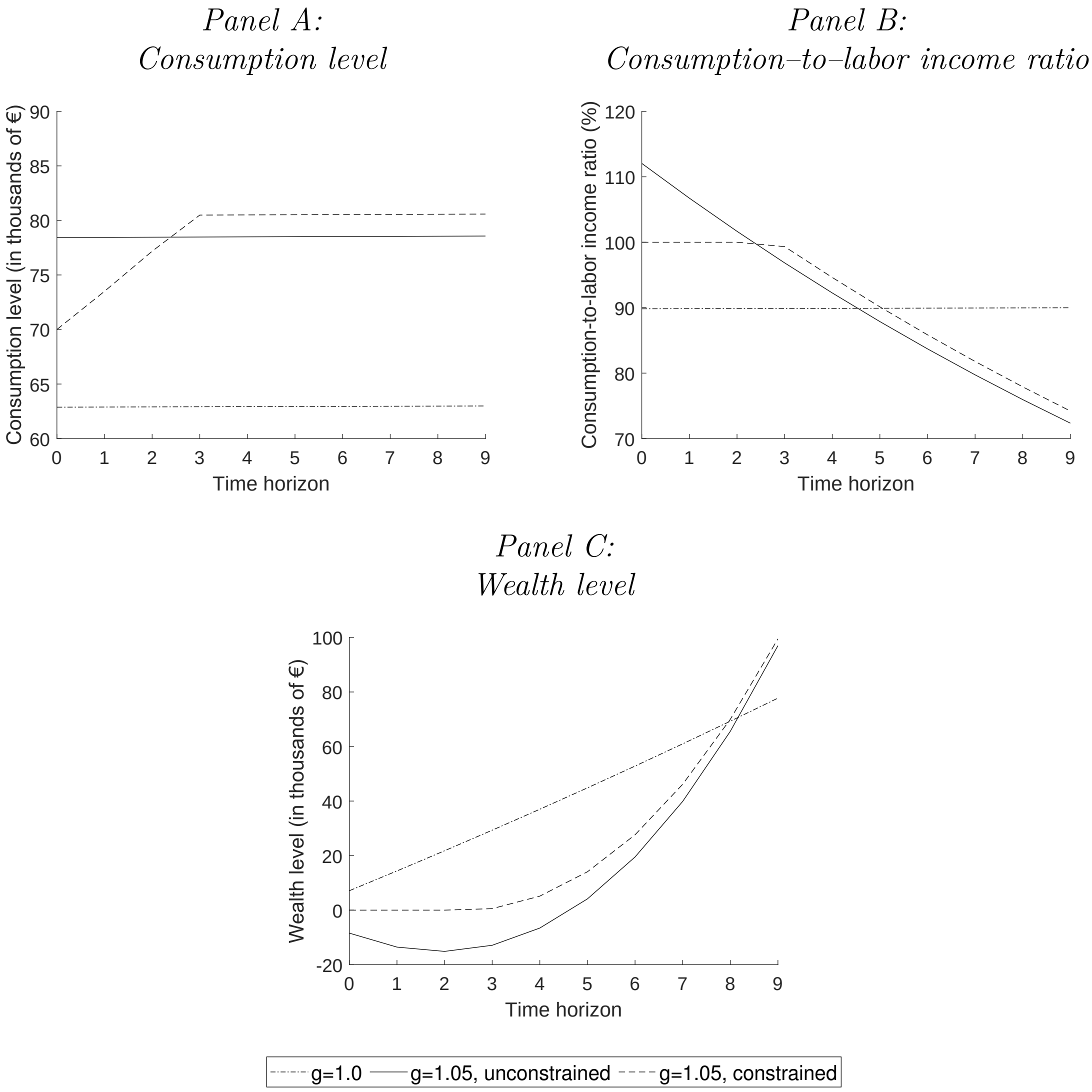
The example illustrates that the financial solvency constraint significantly affects the intertemporal behavior of consumption and saving. Without the constraint and when individuals face growth in labor income, i.e., in the setting depicted by the solid line in Figure 1.3.1, individuals borrow against human capital, i.e., the capitalized value of future labor income, to finance a consumption level well above current earnings in the early periods. Therefore, the consumption–to–labor income ratio, as depicted in Panel B of Figure 1.3.1, is above 100% in early periods and consequently total wealth as depicted in Panel C is negative.

The case, in which individuals face labor income growth and are subject to the non–negativity constraint on financial wealth, is depicted by the dashed line in Figure 1.3.1. In this case, the consumption profile is increasing over the considered time horizon. Panel A shows that the borrowing constraint forces a lower starting level of consumption with a marginal utility well above what it is in the unconstrained case. As a result, the optimal behavior is to wait until the solvency constraint is no longer binding and let the consumption level increase afterwards.[8] Panel B and Panel C of Figure 1.3.1 show that as long as the solvency constraint is binding, individuals consume 100% of their labor income and therefore, postpone accumulating savings until the constraint is no longer binding.

In the case of no labor income growth, i.e., in the setting in which $g = 1.0$, that is depicted by the dash–dotted lines in Figure 1.3.1, the solvency constraint is not binding and the consumption rate is almost constant over the time horizon considered. This can be explained by the fact that there is a discounting effect (in the utility function) which expresses a time preference for early consumption rather than late consumption. But there is also a need for saving for retirement, and with no growth in labor income it is necessary to start early despite the preference for early consumption.

[8] The consumption levels for the unconstrained cases are not constant, although they appear to be. This is due to the fact that the growth rate of consumption is determined by $\varepsilon = 1.00312$, almost equal to one.

Figure 1.3.1: Comparison of consumption–savings patterns in the constrained and unconstrained case



This figure depicts the results for consumption–savings decisions in the constrained and unconstrained case of our model, i.e., we compare the models with and without a non–negativity constraint on wealth, as presented in Section 1.2 and Section 1.3, respectively. We depict results for two different growth rates of labor income, $g = 1.05$ and $g = 1.0$. Panel A depicts the evolution of the consumption level over the time horizon $T = 10$, Panel B of the consumption–to–labor income ratio, and Panel C depicts the evolution of the wealth level over the time horizon. The solid (dashed) lines report results for the unconstrained (constrained) case with $g = 1.05$. The dash–dotted lines report the results for the case with $g = 1.0$ in which the solvency constraint is not binding. The other parameters are chosen as follows: $L_0 = 100$, $K = 2$, $\gamma = 3$, $\rho = 0.98$, $r = 3\%$, $\tau = 30\%$.

Panel C of Figure 1.3.1 shows that for the case with a relatively high growth rate of labor income ($g = 1.05$), the solvency constraint is binding. In order to attain the same level of utility in the constrained and in the unconstrained case, the wage profile has to be lifted by approximately 1% throughout the 10 year period. Although the liquidity constraint skews the time–profile of consumption, the effect is rather small for growth rates of labor income relatively close to the risk–free interest rate r. Increasing this growth rate leads to a markedly larger effect. With a 7% growth rate in labor income, the necessary lift in the labor income profile is approximately 3%, and with a 10% growth rate in labor income it becomes 7.3%.

In all settings, it is clear that the optimal intertemporal consumption–savings decision is far from a policy with a constant contribution rate out of current disposable income.

1.4. A tax–deferred retirement account

Introducing the possibility to save in a tax–deferred retirement account further increases the complexity of the intertemporal consumption–savings problem. The savings decision now additionally involves an asset location decision, i.e., individuals no longer only have to decide how much to save, but also in which account—the taxable or the tax–deferred retirement account. Such a tax–deferred retirement account is a common institutional device for defined–contribution pension systems.

We assume that (i) contributions to such an account are tax–deductible from labor income, (ii) it is not possible to liquidate any of these savings before retirement, and (iii) upon liquidation the withdrawals are subject to the same tax rate, τ, as the one that was applied when the contribution was made.[9] We also impose the solvency constraint introduced in Section 1.3; otherwise, there would be obvious and unlimited arbitrage opportunities by moving borrowed money from the taxable account to the tax–deferred retirement account.

[9] The latter is not a restriction, since any difference between τ and the tax rate applicable to the proceeds from liquidation can be captured by the preference weighting parameter K (see Appendix 1.C).

We denote the contribution to the tax–deferred retirement account by Z_t. The optimization problem can then be written as:

$$\max_{\{\{W_t\geq 0\}_{t=0}^{T-2},\{Z_t\geq 0\}_{t=0}^{T-1}\}} \frac{1}{1-\gamma}\left[\sum_{t=0}^{T-1}\rho^t C_t^{1-\gamma}+\rho^T K C_T^{1-\gamma}\right] \tag{1.4.1}$$

$$C_t = (L_t - Z_t)(1-\tau) + W_{t-1}(1+r(1-\tau)) - W_t, \quad t=0,1,2,\ldots,T-2 \tag{1.4.2}$$

$$C_{T-1} = (L_{T-1} - Z_{T-1})(1-\tau) + W_{T-2}(1+r(1-\tau)) \tag{1.4.3}$$

$$C_T = (1-\tau)\sum_{s=0}^{T-1} Z_s(1+r)^{T-s}. \tag{1.4.4}$$

Assuming again that the wage dynamics are given by a constant gross growth rate g and setting $W_{0-}=0$, we can rewrite the budget dynamics as:

$$C_t = (L_0 g^t - Z_t)(1-\tau) + W_{t-1}(1+r(1-\tau)) - W_t, \quad t=0,1,2,\ldots,T-2 \tag{1.4.5}$$

$$C_{T-1} = (L_0 g^{T-1} - Z_{T-1})(1-\tau) + W_{T-2}(1+r(1-\tau)) \tag{1.4.6}$$

$$C_T = (1-\tau)\sum_{s=0}^{T-1} Z_s(1+r)^{T-s}. \tag{1.4.7}$$

We use the same type of additional variables as in Theorem 1, but we need to distinguish between "before tax" and "after tax" versions. We denote the before tax versions by subscript 1 and the after tax versions by subscript 2:

$$\varepsilon_1 \equiv \big(\rho\,(1+r)\big)^{\frac{1}{\gamma}}, \quad x_1 = \frac{\varepsilon_1}{1+r}, \quad y_1 = \frac{g}{1+r}, \tag{1.4.8}$$

$$\varepsilon_2 \equiv \Big(\rho\,\big(1+r(1-\tau)\big)\Big)^{\frac{1}{\gamma}}, \quad x_2 = \frac{\varepsilon_2}{1+r(1-\tau)}, \quad y_2 = \frac{g}{1+r(1-\tau)}. \tag{1.4.9}$$

Given these definitions, the first-order optimality conditions are:

$$\begin{aligned}&\rho^t C_t^{-\gamma} \geq (1+r(1-\tau))\rho^{t+1}C_{t+1}^{-\gamma}, \quad \left[\rho^t C_t^{-\gamma} - (1+r(1-\tau))\rho^{t+1}C_{t+1}^{-\gamma}\right] W_t = 0 \quad \Leftrightarrow\\ &\varepsilon_2 C_t \leq C_{t+1}, \quad [\varepsilon_2 C_t - C_{t+1}]\, W_t = 0, \quad t=0,1,2,\ldots,T-2\end{aligned} \tag{1.4.10}$$

$$\begin{aligned}&\rho^t C_t^{-\gamma} \geq K\rho^T C_T^{-\gamma}(1+r)^{T-t}, \quad \left[\rho^t C_t^{-\gamma} - K\rho^T C_T^{-\gamma}(1+r)^{T-t}\right] Z_t = 0 \quad \Leftrightarrow\\ &K^{1/\gamma}\varepsilon_1^{(T-t)} C_t \leq C_T, \quad \left[K^{1/\gamma}\varepsilon_1^{(T-t)} C_t - C_T\right] Z_t = 0, \quad t=0,1,2,\ldots,T-1.\end{aligned} \tag{1.4.11}$$

In a model with stochastic outcomes and the possibility of unfavorable shocks, there is a clear "precautionary savings" motive to build a buffer of liquid savings. This result is found in much of the literature (see, e.g., Carroll (1997); Munk (2000); and Amromin (2003)). However, under certainty and with a uniform growth rate of labor income, the savings motive is only driven by two incentives: (i) the fundamental desire to finance consumption in the retirement period and (ii) the desire to smooth consumption over the entire investment horizon in accordance with preferences. The vehicle for the former is the tax–deferred retirement account, and the vehicle for the latter is the taxable account. Theorem 2 shows that in our framework the former motive is the dominating one. The contributions to the tax–deferred retirement account always precede contributions to the taxable account; and for many parameter constellations the taxable account is never used, because the growth in labor income is sufficient in itself to satisfy the desire to smooth consumption over time.

Theorem 2. *When labor income has a uniform gross growth rate g, there are three possible scenarios, one of which leads to an optimal solution. These solutions differ in how the complementary slackness conditions* (1.4.10) *and* (1.4.11) *are fulfilled:*

1. *If $\varepsilon_1 \leq g$, there may be a sequence of initial periods, in which $C_t = L_t(1-\tau)$, followed by a saving for retirement period where the profile follows the unconstrained solution with the reduced remaining time to retirement. The switch point s^*, the last time index for which $C_t = L_t(1-\tau)$, is given by the first value of s^*, for which the following inequality is fulfilled:*

$$\frac{\sum_{s=s^*}^{T-1} y_1^{(s-s^*)}}{\left[\sum_{s=s^*}^{T-1} x_1^{s-s^*} + K^{\frac{1}{\gamma}} x_1^{T-s^*}\right]} > 1 > \frac{\sum_{s=s^*+1}^{T-1} y_1^{(s-s^*-1)}}{\left[\sum_{s=s^*+1}^{T-1} x_1^{s-s^*-1} + K^{\frac{1}{\gamma}} x_1^{(T-s^*-1)}\right]}. \qquad (1.4.12)$$

The contributions to the tax–deferred retirement account develop as

$$Z_t = L_t\left(1 - \frac{\sum_{s=s^*+1}^{T-1} y^{(s-(s^*+1))}}{\sum_{s=s^*+1}^{T-1} x^{s-(s^*+1)} + K^{\frac{1}{\gamma}} x^{(T-(s^*+1))}}\left(\frac{\varepsilon_1}{g}\right)^{t-(s^*+1)}\right), \qquad (1.4.13)$$
$$t = s^* + 1, \ldots, N-1,$$

and the taxable account is not used.

2. *If $\varepsilon_1 > g$, saving for retirement starts immediately and stops after a switch point s^*, given by the first value of s^*, for which the following inequality is fulfilled:*

$$\frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*+1}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^{s^*+1}} \leq 1. \tag{1.4.14}$$

This switch point is the last period, in which contributions are made to the tax–deferred retirement account; at the switch point there are two possibilities:

a) *There are only contributions to the tax–deferred retirement account at time s^*. This is the relevant possibility whenever*

$$\frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^{s^*}} \geq 1. \tag{1.4.15}$$

Then

$$C_t = L_0(1-\tau)\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^t, \quad t = 0,1,\ldots,s^* \tag{1.4.16}$$

$$C_t = L_0(1-\tau)\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^t, \quad t = s^*+1, s^*+2, \ldots, T-1 \tag{1.4.17}$$

$$Z_t = L_t\left[1 - \frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\left(\frac{\varepsilon_1}{g}\right)^t\right], \quad t = 0,1,\ldots,s^*. \tag{1.4.18}$$

b) *There are contributions to both, the taxable and the tax–deferred retirement account at time s^*. This is the relevant possibility whenever*

$$\frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^{s^*}} < 1. \tag{1.4.19}$$

Then

$$W_{s^*} = C_{s^*} \sum_{s=1}^{T-1-s^*} x_2^s - L_{s^*}(1-\tau) \sum_{s=1}^{T-1-s^*} y_2^s \tag{1.4.20}$$

$$Z_{s^*} = L_{s^*} \sum_{s=0}^{T-1-s^*} y_2^s - \frac{C_{s^*}}{1-\tau} \sum_{s=0}^{T-1-s^*} x_2^s \tag{1.4.21}$$

$$C_t = L_0(1-\tau) \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \varepsilon_1^t, \; t = 0,1,\ldots,s^* \tag{1.4.22}$$

$$C_t = L_0(1-\tau) \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \varepsilon_1^{s^*} \varepsilon_2^{t-s^*}, \quad t = s^*+1,\ldots,T-1. \tag{1.4.23}$$

Proof. See Appendix 1.B. □

The nature of s^* as the first time index satisfying the respective inequalities in Theorem 2 resembles the property of a stopping time in a stochastic model.

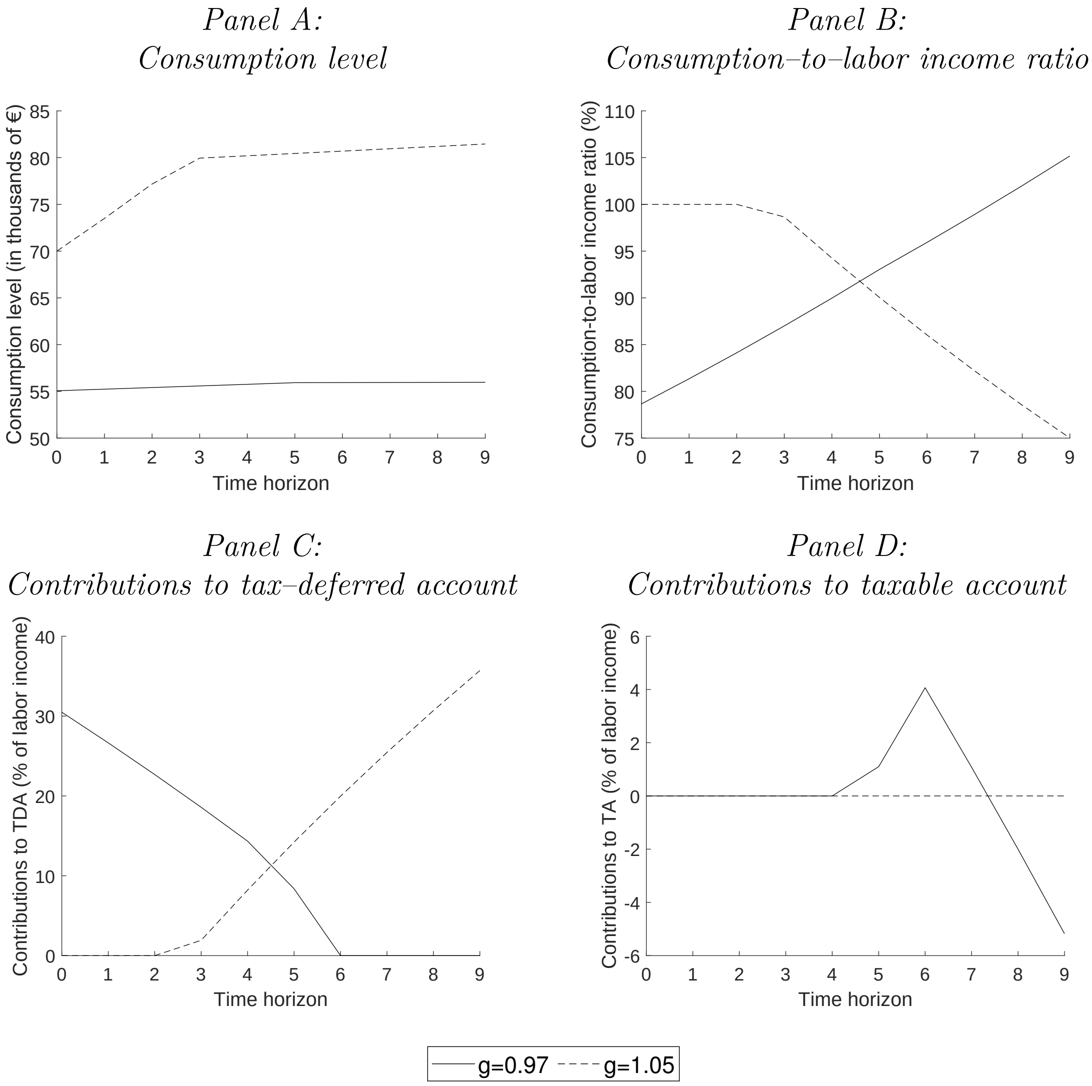
From Theorem 2, the optimal consumption–investment strategy at any given point in time t depends on the model parameters, which determine if and when the complementary slackness conditions are fulfilled with equality. In total, three different cases for the optimal consumption–investment strategy can be considered, corresponding to item 1, item 2a, and item 2b in Theorem 2.

In item 1, the growth rate of labor income is relatively large. More specifically, if $\varepsilon_1 \leq g$, the non–negativity condition on savings becomes binding in earlier periods. In that case, the high growth rate of labor income renders initially deferring saving to a later point in time and consuming all income in the earlier periods optimal.

In the two cases characterized by items 2a and 2b, it is optimal to start saving in the tax–deferred retirement account already in the first period and eventually stop it when the inequality in (1.4.14) is fulfilled. The two cases in item 2 are characterized by a lower growth rate of labor income ($g < \varepsilon_1$) and differ by the optimal location of contributions at the point in time s^*, when the last contribution to the tax–deferred account is made.

In case 2a, contributions are solely made to the tax–deferred retirement account, but not to the taxable account. In case 2b, there are contributions to both the taxable and the tax–deferred retirement account. In this case, the savings in the taxable account do not benefit from the tax–exemption of profits, applicable to savings in the tax–deferred account. Taxable savings may be built up if the growth rate of labor

Figure 1.4.1: Comparison of consumption–savings patterns in the model with a tax–deferred retirement account for various growth rates of labor income



This figure depicts the results for investors having access to a tax–deferred retirement account, i.e., results for the model presented in Section 1.4. Panel A depicts the evolution of the consumption level over the time horizon $T = 10$ and Panel B of the consumption–to–labor income ratio. Panel C depicts the evolution of contributions to the tax–deferred retirement account in percent of labor income received in the period over the time horizon $T = 10$ and Panel D the evolution of contributions to the taxable account in percent of labor income received in the period. In all panels we compare the results for two different growth rates of labor income: (i) $g = 0.97$ (solid line) and (ii) $g = 1.05$ (dashed line). The other parameters are chosen as follows: $L_0 = 100$, $K = 2$, $\gamma = 3$, $\rho = 0.98$, $r = 3\%$, $\tau = 30\%$.

income is so low that parts of the savings are already meant for consumption prior to retirement age.

Figure 1.4.1 illustrates the different cases from Theorem 2 graphically. For that purpose, it compares the evolution of the consumption level (Panel A) and of the consumption–to–labor income ratio (Panel B) as well as the evolution of the contributions to the tax–deferred retirement account in percent of labor income (Panel C) and the taxable account in percent of income (Panel D) for a time horizon of $T = 10$ for two different growth rates of labor income, g, and a tax rate of $\tau = 30\%$.[10] More specifically, the values of g are set to $g = 1.05$ (dashed line) and $g = 0.97$ (solid line), respectively. These two choices are useful to illustrate the two different items in Theorem 2.

In the case with $g = 1.05$ (dashed line), labor income grows over the investment horizon. In that case, the individual is initially consumption–constrained as can be seen from the upward–sloping consumption level in Panel A and the consumption–to–labor income ratio being at 100% from time $t = 0$ to $t = 2$. That is, the individual's situation is characterized by item 1 of Theorem 2. With a growing level of labor income, the consumption–to–labor income ratio required for consumption–smoothing is monotonically decreasing after time $t = 2$. Instead, starting from time $t = 3$, the individual monotonically increases its contributions to the tax–deferred retirement account to build up sufficient savings for retirement age.

In the case with $g = 0.97$ (solid line), labor income drops over the investment horizon. In this case, the individual is able to smooth consumption over the life cycle (Panel A) as the consumption–constraint does not bind (Panel B). That is, for $g = 0.97$, the individual is not ending up in the case characterized by item 1 of Theorem 2. Instead, the case characterized by item 2 of Theorem 2 becomes relevant.

At the beginning of the investment horizon, in period $t = 0$ to $t = 4$, the individual solely contributes to the tax–deferred retirement account. Contributions to the retirement account in percent of labor income are monotonically decreasing, reflecting that due to the decrease in labor income, an increasing share of it is required to uphold the consumption level and smooth consumption over the life cycle. From time $t = 6$, no further contributions to the tax–deferred retirement account are made, i.e., $s^* = 5$. From this point in time, the accumulated savings (plus the past and future interest earned) are already sufficient to cover consumption expenses during retirement age. From time $t = 5$ to $t = 7$, the individual contributes to the taxable account (Panel D) to sustain its consumption level prior to retirement age in periods $t = 8$ to $t = 9$, despite declining labor income. In particular, at time $s^* = 5$, the individual contributes

[10] We also explored other levels of the tax rate. Given that changes to the tax rate only affect our results quantitatively, but not qualitatively, we do not report these different cases here. These results are, however, available from the authors upon request.

to both the taxable and the tax–deferred retirement account. That is, the individual is in the case characterized by item 2b in Theorem 2.[11]

From Panel D, contributions to the taxable account increase monotonically from time $t = 5$ to $t = 6$, reflecting the decline in contributions to the tax–deferred retirement account. After $t = 6$, contributions to the taxable account monotonically decrease as a growing share of the declining labor income is needed for consumption–smoothing. In period $t = 8$ and $t = 9$, labor income has dropped so much that the taxable savings need to be consumed to sustain the desired level of consumption and smooth consumption over the life cycle.

Overall, our illustrative example in Figure 1.4.1 shows that a constant contribution rate to a tax–deferred retirement account can be far from optimal—both for a decent growth rate of labor income ($g = 1.05$) as well as a slightly declining one ($g = 0.97$).

1.5. Generalizations

Throughout we have assumed an identical tax rate τ for labor income and interest earnings. We have also assumed zero initial wealth, i.e., that the individual does not have any pre–existing savings, and a uniform growth rate of labor income. These assumptions can be relaxed without the loss of the existence of a closed–form solution as we outline in this section.

1.5.1. Different tax rates

In this subsection, we ask how allowing for a different tax rate, τ_l, on labor income, affects our closed form solution. A different tax rate on labor income affects the amount of labor income that is available to be consumed and invested after taxes. It does, however, not alter the growth rate of the labor income stream. Hence, by replacing the available labor income after taxes of $L_0(1 - \tau)$ in equation (1.2.2) and subsequently by its adjusted after–tax version

$$L_0(1 - \tau_l), \tag{1.5.1}$$

our closed form solution remains otherwise unaltered. That is, when allowing for a different tax rate on labor income, we can still provide a closed form solution.

[11] The case characterized by item 2a in Theorem 2 is the limiting case, in which it is optimal to still build up tax–deferred savings at time $t = 8$ and to consume the entire income at time $t = 9$, thus avoiding the build–up of taxable savings. It is attained for $g = 0.9815$ (not depicted graphically in Figure 1.4.1, figure is available upon request).

1.5.2. Initial wealth

The case with pre–existing initial wealth, $W_{0-} > 0$ is more challenging to generalize than the case with different tax rates on labor and other income. This reflect that pre–existing initial wealth may affect whether, and if so, when, the budget constraint becomes binding. Yet, we are still able to derive a generalized analytical solution for the optimal consumption–savings profile.

For the simple unconstrained model from Section 1.2 we must alter the initial budget constraint $W_0 = L_0(1-\tau) - C_0 + W_{0-}$ in equation (1.2.9) and have W_{0-} on the left hand side instead of 0. The analytical solution then becomes:

$$\begin{aligned} C_t = C_0\varepsilon^t &= \frac{L_0(1-\tau)\sum_{s=0}^{T-1} y^s + W_{0-}}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\varepsilon^t \\ &= \left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s}\right)\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\varepsilon^t \qquad (1.5.2) \\ &= \left(L_0 + \frac{W_{0-}}{(1-\tau)\sum_{s=0}^{T-1} y^s}\right)(1-\tau)\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\varepsilon^t \\ \frac{C_t}{L_t(1-\tau)} &= \left(1 + \frac{W_{0-}}{L_0(1-\tau)\sum_{s=0}^{T-1} y^s}\right)\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\left(\frac{\varepsilon}{g}\right)^t. \qquad (1.5.3) \end{aligned}$$

The effect of a pre–existing financial wealth is tantamount to lifting the entire profile of before–tax labor income by the factor $\left(1 + \frac{W_{0-}}{L_0(1-\tau)\sum_{s=0}^{T-1} y^s}\right)$. Hence, the optimal solution is identical to the one obtained with the corresponding higher level of labor income. However, it will decrease the savings rate, when consumption is measured against labor income.

A useful interpretation of this is that the initial wealth W_{0-} is converted to a growing annuity with the same present value that is added to the wage earnings:

$$\frac{W_{0-}}{\sum_{s=0}^{T-1} y^s}\left[1, g, g^2, \ldots, g^{T-1}\right]. \qquad (1.5.4)$$

In this way, the property of a constant growth rate is preserved. In this simple case, when the individual does not face any constraints, the consumption–constraint can (by assumption) not become binding, implying that a positive amount of initial wealth is economically equivalent to the payment stream from a growing annuity.

In the presence of possibly binding budget constraints, an amount of initial wealth may relax these constraints. Hence, we should expect initially pre–existing wealth to have a second effect on our closed–form solution through this channel. More technically, it turns out that the optimal solutions for the situations in Theorems 1 and 2 with a positive initial wealth follows the same reasoning as in the unrestricted model. The initial wealth W_{0-} is converted to a growing annuity with the same growth factor as labor income; i.e., instead of $L_t = L_0 g^t$ we use the optimal solution for the situation with labor income

$$L_t(1-\tau) + \frac{W_{0-}g^t}{\sum_{s=0}^{T-1} y^s} = \left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) g^t. \tag{1.5.5}$$

The initial wealth is kept in the taxable account and amortized accordingly, but it must be separated from other savings in this account as it may have an impact on whether the borrowing constraint binds or not. This means that the borrowing constraint must be reinterpreted; it is binding, when the unrestricted solution calls for a level of consumption in excess of $\left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) g^t$. The analogue of the expression in (1.2.12), i.e. the criterion for whether borrowing at time 0 takes place or not, and consequently whether the unrestricted solution is feasible also under the no borrowing constraint, is expressed in the following inequality, similar to the condition expressed in (1.3.4):

$$\left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{1/\gamma} x^T} < \left(L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s} \right) \quad \Leftrightarrow \tag{1.5.6}$$

$$\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{1/\gamma} x^T} < 1. \tag{1.5.7}$$

When the inequality (1.5.7) no longer holds for a sufficiently high growth rate of labor income, we will be in a situation where $L_0(1-\tau) + \frac{W_{0-}}{\sum_{s=0}^{T-1} y^s}$ is consumed at time 0. This is analogous to the analysis in Section 1.3, and the proof follows the exact same line of reasoning as the proof in Appendix 1.A. The same holds for the situation with a tax–deferred account in Theorem 2.

1.5.3. Time varying growth rates of labor income

The assumption of a constant growth rate of labor income can also be generalized to a situation with a time–varying, though still deterministic, growth rate of labor income without affecting the existence of closed–form solution. Assume that the growth rate at time v is g_v instead of just the constant g. The only change to the expression in equation (1.2.10) is found in the following expression with $g_0 = 1$ for notational convenience:

$$C_t = C_0\varepsilon^t = L_0(1-\tau)\frac{1+\sum_{s=1}^{T-1}(1+r(1-\tau))^{-s}\prod_{v=0}^{s}g_v}{\sum_{s=0}^{T-1}x^s + K^{\frac{1}{\gamma}}x^T}\varepsilon^t \tag{1.5.8}$$

$$\frac{C_t}{L_t(1-\tau)} = \frac{C_0}{L_0}\frac{\varepsilon^t}{\prod_{v=0}^{t}g_v}, \tag{1.5.9}$$

and a similar generalization of the growing annuity in equation (1.5.4). This reinterpretation of the constant parameter y to become time–varying also goes for the analogous expressions in Theorems 1 and 2.

However, for our proofs to still hold, we have to impose restrictions on the variability of the time–dependent growth rate of labor income, g_v. For example, if it has to hold that $\left(\varepsilon_1/g\right) > 1$, the variability of g_v must be restricted to situations where $\varepsilon_1^t/\prod_{v=0}^{t}g_v$ is increasing and larger than one. These restrictions on g_v apply whenever the monotonicity of $\varepsilon_1^t/\prod_{v=0}^{t}g_v$ is important for the validity of the proof.

In total, our results in this section shows that our model can be generalized to cases with different tax rates on labor income and interest earnings, positive initial wealth, and time–varying growth rates of labor income.

1.6. Conclusion

We provide analytical solutions to variants of a classical intertemporal consumption–savings problem in a deterministic setting with a time–additive CRRA utility function. We assume a constant growth rate in labor income and solve for

(1) the unrestricted problem, for which it is possible to borrow against future labor income,

(2) the restricted problem with non–negative wealth constraints added, and

(3) the restricted problem with non–negative wealth constraints and the possibility to save in a tax–deferred retirement account.

The benefit of assuming a deterministic setting is that we provide analytical solutions to all cases. Moving from (1) to (2), we show that the non–negativity constraint can be severely binding and that saving for retirement is optimally postponed when the labor income process has a sufficiently high growth rate. Moving from (2) to (3), we show that the tax advantage of saving in a tax–deferred retirement account implies that saving in this account precedes any saving in the taxable account, where the latter is based on a consumption smoothing effect. We also demonstrate that the optimal retirement savings profile is far from the widespread practice of allocating a fixed fraction of labor income to a tax–deferred retirement account during the working life.

References

Amromin, G. (2003). “Household Portfolio Choices in Taxable and Tax-Deferred Accounts: Another Puzzle?” *European Finance Review* 7. (3), 547–582.

Black, F. (1980). “The Tax Consequences of Long-Run Pension Policy.” *Financial Analysts Journal* 36. (4), 21–28.

Blake, D., D. Wright, and Y. Zhang (2014). “Age-Dependent Investing. Optimal Funding and Investment Strategies in Defined Contribution Pension Plans when Members are Rational Life Cycle Financial Planners.” *Journal of Economic Dynamics and Control* 38, 105–124.

Boyd, S. and L. Vandenberghe (2009). *Convex Optimization.* Seventh ed. Cambridge University Press.

Carroll, C. D. (1997). “Buffer-Stock Saving and The Life Cycle/Permanent Income Hypothesis.” *Quarterly Journal of Economics* 112. (1), 1–55.

Constantinides, G. M., J. B. Donaldson, and R. Mehra (2002). “Junior Can’t Borrow: A New Perspective on the Equity Premium Puzzle.” *Quarterly Journal of Economics* 117. (1), 269–296.

Dammon, R. M., C. S. Spatt, and H. H. Zhang (2004). “Optimal Asset Location and Allocation with Taxable and Tax-Deferred Investing.” *Journal of Finance* 59. (3), 999–1037.

Deaton, A. (1991). “Saving and Liquidity Constraints.” *Econometrica* 59. (5), 1221–1248.

Fischer, M. and M. F. Gallmeyer (2017). “Taxable and Tax-Deferred Investing with the Limited Use of Losses.” *Review of Finance* 21. (5), 1847–1873.

Fischer, M., B. A. Jensen, and M. Koch (2022). “Mandatory Pension Saving and Homeownership.” working paper, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4210537.

Garlappi, L. and J. Huang (2006). “Are Stocks Desirable in Tax-Deferred Accounts?” *Journal of Public Economics* 90. (12), 2257–2283.

Gourinchas, P.-O. and J. A. Parker (2002). “Consumption over the Life Cycle.” *Econometrica* 70. (1), 47–89.

Hakansson, N. H. (1970). “Optimal investment and consumption strategies under risk for a class of utility functions.” *Econometrica* 38. (5), 587–607.

Holzmann, R. and E. Palmer (2006). *Pension Reform. Issues and Prospects for Non-Financial Defined Contribution (NDC) Schemes.* The World Bank.

Huang, H. and M. A. Milevsky (2016). "Longevity Risk and Retirement Income Tax Efficiency: A Location Spending Rate Puzzle." *Insurance: Mathematics and Economics* 71. (1), 50–62.

Huang, H., M. A. Milevsky, and T. S. Salisbury (2012). "Optimal Retirement Consumption with a Stochastic Force of Mortality." *Insurance: Mathematics and Economics* 51. (2), 282–291.

Huang, J. (2008). "Taxable and Tax-Deferred Investing: A Tax Arbitrage Approach." *Review of Financial Studies* 21. (5), 2173–2207.

Lachance, M.-E. (2012). "Optimal Onset and Exhaustion of Retirement Savings in a Life-Cycle Model." *Journal of Pension Economics & Finance* 11. (1), 21–52.

Larsen, L. S. and C. Munk (2023). "The Design and Welfare Implications of Mandatory Pension Plans." *Journal of Financial and Quantitative Analysis* online, 1–30.

Merton, R. C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case." *Review of Economics and Statistics* 51. (1), 247–257.

— (1971). "Optimal Consumption and Portfolio Rules in a Continuous-Time Model." *Journal of Economic Theory* 3. (4), 373–413.

Munk, C. (2000). "Optimal Consumption/Investment Policies with Undiversifiable Income Risk and Liquidity Constraints." *Journal of Economic Dynamics and Control* 24. (9), 1315–1343.

Schlafmann, K., O. Setty, and R. Vestman (2022). "Designing Pension Plans According to Consumption-Savings Theory." working paper, available at https://cepr.org/system/files/2022-09/Kathrin%20Schlafmann_Designing%20Pension%20Plans%20According%20to%20Consumption-Savings%20Theory.pdf.

Shoven, J. B. and C. Sialm (1998). "Long Run Asset Allocation for Retirement Savings." *Journal of Private Portfolio Management* 1. (2), 13–26.

— (2003). "Asset Location in Tax-Deferred and Conventional Savings Accounts." *Journal of Public Economics* 88. (1), 23–38.

Tepper, I. (1981). "Taxation and Corporate Pension Policy." *Journal of Finance* 36. (1), 1–13.

Appendix 1.A Proof of Theorem 1

When the requirement of non–negative wealth is added to the optimization problem in equations (1.2.1)–(1.2.3), it looks as follows:

$$\max_{\{\{W_t\geq 0\}_{t=0}^{T-1}\}} \frac{1}{1-\gamma}\left[\sum_{t=0}^{T-1}\rho^t C_t^{1-\gamma} + K\rho^T C_T^{1-\gamma}\right] \tag{1.A.1}$$

subject to

$$C_t = L_t(1-\tau) + W_{t-1}\left(1+r(1-\tau)\right) - W_t,\ t = 0,1,\ldots,T-1 \tag{1.A.2}$$

$$C_T = W_{T-1}\left(1+r(1-\tau)\right). \tag{1.A.3}$$

This optimization problem has a concave objective function and linear constraints. Hence, the Kuhn–Tucker conditions are sufficient conditions for an optimal solution (see, e.g., Boyd and Vandenberghe (2009), p. 244).

Since C_t is given in explicit form in terms of the decision variables W_t, the Kuhn–Tucker conditions, including the complimentary slackness conditions, do not need to be formulated in terms of Lagrangian multipliers. It is sufficient that the derivative of the objective function vanishes in case $W_t > 0$, and is non–positive in case $W_t = 0$. Hence, we have:

$$\rho^{t+1}C_{t+1}^{-\gamma}(1+r(1-\tau)) - \rho^t C_t^{-\gamma} \leq 0 \ \text{ and } = 0 \text{ when } W_t > 0 \tag{1.A.4}$$

$$\rho^T K C_T^{-\gamma}(1+r(1-\tau)) - \rho^{T-1}C_{T-1}^{-\gamma} = 0 \ \Leftrightarrow\ C_T = K^{1/\gamma}\varepsilon C_{T-1}, \tag{1.A.5}$$

After cancellation, the former condition(s) become $\varepsilon C_t \leq C_{t+1}$ with equality whenever $W_t = 0$. This is equation (1.3.1). The latter is guaranteed as an equality because W_{T-1} must necessarily be positive. This is equation (1.3.2).

We restate and rewrite equation (1.2.9), which also provides the structure of the solution to the unrestricted optimization problem:

$$\begin{aligned} 0 &= (1+r(1-\tau))^{-T}C_0\varepsilon^T K^{1/\gamma} + \sum_{s=0}^{T-1}\left[C_0\varepsilon^s - L_0(1-\tau)g^s\right](1+r(1-\tau))^{-s} \\ &= C_0 x^T K^{1/\gamma} + \sum_{s=0}^{T-1}\left[C_0 x^s - L_0(1-\tau)y^s\right]. \end{aligned} \tag{1.A.6}$$

For the unrestricted problem this gives the explicit solution:

$$C_t = \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} L_0(1-\tau)\varepsilon^t = \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} L_t(1-\tau)\left(\frac{\varepsilon}{g}\right)^t. \tag{1.A.7}$$

We now have to distinguish two different cases.

Case 1: $\varepsilon/g > 1$ As $\varepsilon/g = x/y$, we have that $\varepsilon/g > 1$ is tantamount to $x > y$. Hence, in that case we know from equation (1.2.10) that

$$\frac{C_t}{L_t(1-\tau)} = \frac{C_0\varepsilon^t}{L_0(1-\tau)g^t} = \frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T}\left(\frac{\varepsilon}{g}\right)^t. \tag{1.A.8}$$

Since the consumption to labor income (after tax) ratio is increasing throughout, it must be below 1 for some initial period(s), i.e.:

$$\frac{\sum_{s=0}^{T-1} y^s}{\sum_{s=0}^{T-1} x^s + K^{\frac{1}{\gamma}} x^T} < 1. \tag{1.A.9}$$

This also follows directly from the condition $x > y$. Hence, the unrestricted solution shows an initial build up of reserves that either continues to build up or at some point in time may be reversed and drawn down to the level necessary to finance the consumption at time T, see equation (1.2.8).

This covers the first case in Theorem 1. If the optimal solution to the unrestricted problem is feasible for the restricted problem, it is obviously also the optimal solution for the latter.

Case 2: $\varepsilon/g < 1$ When $\varepsilon/g < 1 \Leftrightarrow x < y$, the consumption–to–labor income (after tax) ratio is decreasing throughout for the unrestricted problem. It is still possible that the consumption–to–labor income (after tax) ratio is below unity in all periods for the unrestricted problem. If this is the case ($s^* = 0$), the solution to the unrestricted problem is again also the solution to the restricted problem.

If, alternatively, the solution to the unrestricted problem calls for borrowing in one or more initial periods, some of the solvency constraints $W_t \geq 0$, $t = 1,2,\ldots,T-1$, become active and the consumption follows the wage process. For $t = 0$ this looks like one of the following two cases. The first case assumes that the solvency constraint is active at time $t = 1$:

$$L_0(1-\tau)\varepsilon < L_0(1-\tau)g \;\Leftrightarrow\; \frac{\varepsilon}{g} < 1, \tag{1.A.10}$$

which is satisfied by assumption.

The second case assumes that the solvency constraint is not binding at time $t = 1$. In that case, the remaining periods follow the solution for the unrestricted problem, but with the new starting point at time $t = 1$.[12] When the solvency constraint is not binding we know that

$$\frac{\sum_{s=1}^{T-1} y^{s-1}}{\sum_{s=1}^{T-1} x^{s-1} + K^{\frac{1}{\gamma}} x^{T-1}} < 1. \tag{1.A.11}$$

Consequently:

$$\begin{aligned} L_0(1-\tau)\varepsilon &< \left(\frac{\sum_{s=1}^{T-1} y^{s-1}}{\sum_{s=1}^{T-1} x^{s-1} + K^{\frac{1}{\gamma}} x^{T-1}} L_0(1-\tau)g \right) < L_0(1-\tau)g \quad \Leftrightarrow \\ \frac{\varepsilon}{g} &< \frac{\sum_{s=0}^{T-2} y^{s}}{\sum_{s=0}^{T-2} x^{s} + K^{\frac{1}{\gamma}} x^{T-1}} < 1. \end{aligned} \tag{1.A.12}$$

If the solvency constraint continues to be binding at time $t = 1$, this process continues with $C_t = L_t(1-\tau)$ until we reach a point at which the unrestricted solution for the "tail" of the optimization problem is feasible.

We finally show that s^* is uniquely determined by showing that the sequence:

$$\frac{\sum_{s=s^*}^{T-1} y^{s-s^*}}{\sum_{s=s^*}^{T-1} x^{s-s^*} + K^{\frac{1}{\gamma}} x^{T-s^*}} \tag{1.A.13}$$

is decreasing with increasing values of s^* and that it eventually will be below unity for $s^* \leq T-2$. With $\varepsilon/g < 1 \Leftrightarrow x/y < 1$ we have that

$$\begin{aligned} \frac{\sum_{s=s^*}^{T-1} y^{s-s^*}}{\sum_{s=s^*}^{T-1} x^{s-s^*} + K^{\frac{1}{\gamma}} x^{T-s^*}} &> \frac{\sum_{s=s^*+1}^{T-1} y^{s-(s^*+1)}}{\sum_{s=s^*+1}^{T-1} x^{s-(s^*+1)} + K^{\frac{1}{\gamma}} x^{T-(s^*+1)}} \quad \Leftrightarrow \\ \frac{\sum_{s=0}^{T-1-s^*} y^{s}}{\sum_{p=0}^{T-1-s^*} x^{p} + K^{\frac{1}{\gamma}} x^{T-s^*}} &> \frac{\sum_{s=0}^{T-2-s^*} y^{s}}{\sum_{p=0}^{T-2-s^*} x^{p} + K^{\frac{1}{\gamma}} x^{T-(s^*+1)}}. \end{aligned} \tag{1.A.14}$$

[12] The problem clearly fulfills the conditions for applying the Bellman optimality principle.

Multiplying through by both denominators we get the following inequalities to verify, in which we evaluate the two types of terms separately:

$$\begin{aligned}&\sum_{s=0}^{T-1-s^*}\sum_{p=0}^{T-2-s^*} y^s x^p + K^{\frac{1}{\gamma}} x^{T-(s^*+1)} \sum_{s=0}^{T-1-s^*} y^s > \\ &\sum_{s=0}^{T-2-s^*}\sum_{p=0}^{T-1-s^*} y^s x^p + K^{\frac{1}{\gamma}} x^{T-s^*} \sum_{s=0}^{T-2-s^*} y^s.\end{aligned} \tag{1.A.15}$$

For the terms relating to $K^{1/\gamma}$ we can cancel out $K^{1/\gamma}x^T$ on both sides of the inequality and arrive at:

$$\sum_{s=0}^{T-1-s^*} y^s > x \sum_{s=0}^{T-2-s^*} y^s \Leftrightarrow \sum_{s=0}^{T-2-s^*} y^s(1-x) + y^{T-1-s^*} > 0. \tag{1.A.16}$$

Since $x < y$, this is true if the inequality still holds with x replaced by y:

$$\sum_{s=0}^{T-2-s^*} y^s(1-y) + y^{T-1-s^*} = 1 > 0. \tag{1.A.17}$$

For the other terms we have, after cancellation of identical terms on both sides:

$$y^{T-1-s^*} \sum_{p=0}^{T-2-s^*} x^p > x^{T-1-s^*} \sum_{s=0}^{T-2-s^*} y^s \Leftrightarrow \sum_{p=0}^{T-2-s^*} x^{p+1-T+s^*} > \sum_{s=0}^{T-2-s^*} y^{s+1-T+s^*}. \tag{1.A.18}$$

Since $x < y$ and all powers are negative, also this inequality is obvious.

We need to verify that the lower limit is below one. This is at least true for $s^* = T-2$, since the right–hand side of equation (1.3.3) is clearly less than unity:

$$\frac{1}{1 + K^{\frac{1}{\gamma}} x} < 1. \tag{1.A.19}$$

Whenever the ratio starts above 1 and the borrowing restriction is activated, we then know that there will be a uniquely determined future point in time, s^*, from where the ratio goes below one. From that point in time and onwards, the unrestricted solution works with a "new clock" and with appropriately shortened time to retirement.

This completes the proof of Theorem 1.

Appendix 1.B Proof of Theorem 2

Our optimization problem has a concave objective function and linear constraints. Hence, the Kuhn–Tucker conditions from Equations (1.4.10) and (1.4.11) are not only necessary, but also sufficient conditions for an optimal solution (see, e.g., Boyd and Vandenberghe (2009), p. 244).

1.B.1 Only tax–deferred retirement account

Consider first the case with $Z_t > 0\ \forall t = 0,1,\ldots,T-1$. Given this assumption we know from equation (1.4.11) that $C_{t+1}/C_t = \varepsilon_1$. This implies that $W_t = 0\ \forall t = 0,1,\ldots,T-1$; otherwise there would be conflicting requirements for the growth rate of consumption, because $W_t > 0$ together with equation (1.4.10) would imply that $C_{t+1}/C_t = \varepsilon_2$. We then have the following relations for Z_t:

$$\begin{aligned} C_t &= (L_t - Z_t)(1-\tau) \\ C_{t+1} &= (L_{t+1} - Z_{t+1})(1-\tau) = (L_t g - Z_{t+1})(1-\tau) \\ \frac{C_{t+1}}{C_t} &= \frac{L_t g - Z_{t+1}}{L_t - Z_t} \equiv \varepsilon_1. \end{aligned} \tag{1.B.1}$$

This gives a first–order difference equation for Z_t that also determines the consumption in the horizon term C_T:

$$Z_t = Z_0\varepsilon_1^t + L_0(g^t - \varepsilon_1^t) \iff (1-\tau)Z_t = L_0(1-\tau)g^t - C_0\varepsilon_1^t. \tag{1.B.2}$$

This saving into the tax–deferred retirement account is directly related to the consumption–term for the last period:

$$\sum_{s=0}^{T-1} Z_s(1+r)^{T-s} = (1+r)^T \left[Z_0 \sum_{s=0}^{T-1} x_1^s + L_0 \sum_{s=0}^{T-1} \left(y_1^s - x_1^s \right) \right] \tag{1.B.3}$$

$$\begin{aligned} C_T &= (1-\tau)(1+r)^T \left[Z_0 \sum_{s=0}^{T-1} x_1^s + L_0 \sum_{s=0}^{T-1} \left(y_1^s - x_1^s \right) \right] \\ &= K^{1/\gamma} \varepsilon_1^T C_0 = (1-\tau) K^{1/\gamma} \varepsilon_1^T \left(L_0 - Z_0 \right) \end{aligned} \tag{1.B.4}$$

$$\begin{aligned} K^{1/\gamma} x_1^T \left(L_0 - Z_0 \right) &= \left[Z_0 \sum_{s=0}^{T-1} x_1^s + L_0 \sum_{s=0}^{T-1} \left(y_1^s - x_1^s \right) \right] \quad \Rightarrow \\ Z_0 &= L_0 \left[1 - \frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} \right]. \end{aligned} \tag{1.B.5}$$

This also gives the resulting processes for C_t and Z_t:

$$\begin{aligned} C_t &= (L_t - Z_t)(1-\tau) = L_t(1-\tau) \frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g} \right)^t \\ &= L_0(1-\tau) \frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} \varepsilon_1^t \end{aligned} \tag{1.B.6}$$

$$Z_t = L_t \left(1 - \frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g} \right)^t \right). \tag{1.B.7}$$

For the case in which $\varepsilon_1 \le g$, the only requirement to check in order to ensure that $Z_t \ge 0 \; \forall t = 0,1,\ldots,T-1$ is that

$$\frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} \le 1. \tag{1.B.8}$$

Whenever equation (1.B.8) is fulfilled, the other part of the complementary slackness conditions (see equation (1.4.10)) are fulfilled:

$$
\begin{aligned}
&C_t \leq \varepsilon_2^{-1} C_{t+1} \;\Leftrightarrow\; L_t \frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g}\right)^t \leq \\
&(L_t g)\varepsilon_2^{-1} \frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g}\right)^{t+1} \;\Leftrightarrow \\
&1 \leq \frac{\varepsilon_1}{\varepsilon_2} = \left(\frac{1+r}{1+r(1-\tau)}\right)^{1/\gamma},
\end{aligned}
\tag{1.B.9}
$$

which is true whenever the interest rate is positive which is a basic assumption in all derivations. Clearly, the tax effect of the deferred taxation of the retirement savings is increasing with the interest rate, and the effect vanishes when the interest rate decreases.

If

$$
\frac{\sum_{s=0}^{T-1} y_1^s}{\sum_{s=0}^{T-1} x_1^s + K^{1/\gamma} x_1^T} > 1, \tag{1.B.10}
$$

the savings process must be postponed as in Theorem 1 until

$$
\frac{\sum_{s=s^*}^{T-1} y_1^{s-s^*}}{\sum_{s=s^*}^{T-1} x_1^{s-s^*} + K^{1/\gamma} x_1^{T-s^*}} = \frac{\sum_{s=0}^{T-1-s^*} y_1^s}{\sum_{s=0}^{T-1-s^*} x_1^s + K^{1/\gamma} x_1^{T-s^*}} < 1. \tag{1.B.11}
$$

An economic interpretation of theses cases is that when the taxable account is not used at all, the relevant discount rate is the before tax rate. With a constant (gross) growth rate of wages g, discounting by $y_1 = g/(1+r)$ is the standard textbook formula for finding the present value of a geometrically growing stream of payments. The same argument is applicable to the consumption stream C_t, which displays geometric growth by the factor ε_1 that enters into the discount factor x_1. Hence, the solution portrayed in equations (1.B.6) and (1.B.7) is simply the budget restriction applied to candidates for the consumption process that fulfills the first–order conditions.

When equations (1.B.6) and (1.B.7) do not represent a feasible solution, because they require negative values for Z_t in some initial periods, the optimal solution is to consume everything in these initial periods and wait until the solution represented by equations (1.B.6) and (1.B.7) becomes feasible for the remaining periods. In this case the first–order conditions are clearly satisfied:

$$
\begin{aligned}
C_{t+1} &= gC_t > \varepsilon_1 C_t > \varepsilon_2 C_t,\ t = 0,1,\ldots,s^* - 1 \\
C_{t+1} &= \varepsilon_1 C_t > \varepsilon_2 C_t,\ t = s^*,s^* + 1,\ldots,T - 1 \\
C_T &= K^{1/\gamma}\varepsilon_1 C_{T-1}.
\end{aligned}
\tag{1.B.12}
$$

The reason why the taxable account is not used is that there is no need for further consumption smoothing. The growth in labor income ($g > \varepsilon_1$) is sufficient to satisfy this demand.

1.B.2 Both tax–deferred and taxable account

When $\varepsilon_1 > g$, it is possible that contributions to the tax–deferred retirement account stop at time $s^* < T-1$, and then smoothing of consumption takes place via the taxable account. At time s^*, it is possible to (i) contribute to both accounts or (ii) to solely allocate savings in the tax–deferred retirement account.

These two cases are mutually exclusive and behave differently. That is, either of the two cases will be fulfilled; and taken together they provide a full solution to the optimization problem. We therefore analyze these two cases separately, assuming that we have identified the time index s^*. Later in subsection 1.B.2.3, we show how this time index is determined.

Afterwards, we give the analytical solution to this identification problem and provide criteria for determining which of the two scenarios is the optimal one.

It is clearly never optimal to have $W_{T-1} > 0$; savings at the last point in time where the individual receives labor income must be allocated in the tax–deferred retirement account to earn the tax–free return.

1.B.2.1 No contribution to the taxable account W_t at time s^*

In case there is no contribution to the taxable account at time s^*, the saving into this account starts at time $s^* + 1$:

$$W_t = \sum_{s=s^*+1}^{t} (L_s(1-\tau) - C_s)(1+r(1-\tau))^{t-s} \tag{1.B.13}$$

$$0 = \sum_{s=s^*+1}^{T-1} (L_0(1-\tau)g^s - C_{s^*+1}\varepsilon_2^{s-(s^*+1)})(1+r(1-\tau))^{T-1-s} \quad \Rightarrow \tag{1.B.14}$$

$$0 = \sum_{s=s^*+1}^{T-1} (L_0(1-\tau)g^s - C_{s^*+1}\varepsilon_2^{s-(s^*+1)})(1+r(1-\tau))^{-(s-(s^*+1))} \quad \Rightarrow \tag{1.B.15}$$

$$C_{s^*+1} = L_{s^*+1}(1-\tau)\frac{\sum_{s=0}^{T-2-s^*} y_2^s}{\sum_{s=0}^{T-2-s^*} x_2^s} = L_0(1-\tau)\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*+1}. \tag{1.B.16}$$

The consumption policy for the first period follows, using the dynamics for Z_t derived in equations (1.B.2)–(1.B.3) with upper bound s^* instead of $T-1$:

$$\sum_{s=0}^{s^*} Z_t(1+r)^{T-s} = (1+r)^T \left[Z_0 \sum_{s=0}^{s^*} x_1^s + L_0 \sum_{s=0}^{s^*} \left(y_1^s - x_1^s\right) \right] \tag{1.B.17}$$

$$C_T = K^{1/\gamma}\varepsilon_1^T C_0 = (1-\tau)(1+r)^T \left[Z_0 \sum_{s=0}^{s^*} x_1^s + L_0 \sum_{s=0}^{s^*} \left(y_1^s - x_1^s\right) \right] \tag{1.B.18}$$

$$\begin{aligned} K^{1/\gamma}x_1^T C_0 &= \left[L_0(1-\tau)\sum_{s=0}^{s^*} y_1^s - C_0 \sum_{s=0}^{s^*} x_1^s \right] \quad \Rightarrow \\ \left(\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma}x_1^T \right) C_0 &= L_0(1-\tau)\sum_{s=0}^{s^*} y_1^s. \end{aligned} \tag{1.B.19}$$

This gives the consumption policy for the entire period:

$$C_t = L_0(1-\tau)\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma}x_1^T}\varepsilon_1^t,\ t = 0,1,\ldots,s^* \tag{1.B.20}$$

$$\begin{aligned} C_t &= L_0(1-\tau)\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^t = L_{s^*+1}(1-\tau)\frac{\sum_{s=0}^{T-2-s^*} y_2^s}{\sum_{s=0}^{T-2-s^*} x_2^s}\varepsilon_2^{t-(s^*+1)}, \\ &\quad t = s^*+1, s^*+2, \ldots, T-1, \end{aligned} \tag{1.B.21}$$

$$C_T = K^{1/\gamma}\varepsilon_1^T C_0 \tag{1.B.22}$$

$$\begin{aligned} Z_t &= L_0 \left[g^t - \frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma}x_1^T}\varepsilon_1^t \right] = L_t \left[1 - \frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma}x_1^T} \left(\frac{\varepsilon_1}{g} \right)^t \right] \\ &\quad t = 0,1,\ldots,s^*. \end{aligned} \tag{1.B.23}$$

1.B.2.2 Both accounts (taxable and tax–deferred retirement account) active at time s^*

When there is inflow to both accounts at time s^*, the following relations are valid:

$$C_t = (L_t - Z_t)(1-\tau),\ t = 0,1,\dots,s^*-1 \tag{1.B.24}$$

$$C_{s^*} = (L_{s^*} - Z_{s^*})(1-\tau) - W_{s^*} = (L_0 - Z_0)(1-\tau)\varepsilon_1^{s^*} - W_{s^*} \tag{1.B.25}$$

$$\varepsilon_1 = \frac{(L_0 g^{s^*} - Z_{s^*})(1-\tau) - W_{s^*}}{(L_0 g^{s^*-1} - Z_{s^*-1})(1-\tau)} = \frac{(L_0 g^{s^*} - Z_{s^*}) - \frac{W_{s^*}}{1-\tau}}{(L_0 g^{s^*-1} - Z_{s^*-1})} \tag{1.B.26}$$

$$\begin{aligned} W_t &= \sum_{s=s^*}^{t} ((L_s - Z_s 1_{\{s=s^*\}})(1-\tau)(1-\tau) - C_s)(1+r(1-\tau))^{t-s} \\ &= \sum_{s=s^*}^{t} \left(\left(L_s - Z_s 1_{\{s=s^*\}}\right)(1-\tau) - C_0 \varepsilon_1^{s^*} \varepsilon_2^{s-s^*} \right)(1+r(1-\tau))^{t-s} \\ &\quad (t = s^*,\dots,T-1). \end{aligned} \tag{1.B.27}$$

Since $W_{T-1} = 0$, we have:

$$\begin{aligned} 0 &= ((L_{s^*} - Z_{s^*})(1-\tau) - C_0\varepsilon_1^{s^*})(1+r(1-\tau))^{T-1-s^*} + \\ &\quad \sum_{s=s^*+1}^{T-1} (L_s(1-\tau) - C_0 \varepsilon_1^{s^*} \varepsilon_2^{s-s^*})(1+r(1-\tau))^{T-1-s} \qquad \Rightarrow \end{aligned} \tag{1.B.28}$$

$$\begin{aligned} 0 &= ((L_{s^*} - Z_{s^*})(1-\tau) - C_0\varepsilon_1^{s^*}) + \\ &\quad \sum_{s=s^*+1}^{T-1} (L_s(1-\tau) - C_0 \varepsilon_1^{s^*} \varepsilon_2^{s-s^*})(1+r(1-\tau))^{-(s-s^*)} \qquad \Rightarrow \end{aligned} \tag{1.B.29}$$

$$Z_{s^*} = \sum_{s=0}^{T-1-s^*} (L_0 y_2^s g^{s^*} - \frac{C_0 \varepsilon_1^{s^*}}{1-\tau} x_2^s). \tag{1.B.30}$$

Given these calculations, the following holds:

$$\sum_{s=0}^{s^*} Z_t(1+r)^{T-s} = (1+r)^T \left[\sum_{s=0}^{s^*-1} \left(L_0 y_1^s - \frac{C_0}{1-\tau} x_1^s \right) + \sum_{s=0}^{T-1-s^*} \left(L_0 y_1^{s^*} y_2^s - \frac{C_0}{1-\tau} x_1^{s^*} x_2^s \right) \right] \tag{1.B.31}$$

$$\begin{aligned} C_T = K^{1/\gamma} \varepsilon_1^T C_0 = (1+r)^T & \left[L_0(1-\tau) \left(\sum_{s=0}^{s^*-1} y_1^s + y_1^{s^*} \sum_{s=0}^{T-1-s^*} y_2^s \right) - \right. \\ & \left. C_0 \left(\sum_{s=0}^{s^*-1} x_1^s + x_1^{s^*} \sum_{s=0}^{T-1-s^*} x_2^s \right) \right]. \end{aligned} \tag{1.B.32}$$

This gives the optimal consumption policy:

$$C_t = L_0(1-\tau) \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \varepsilon_1^t,\ t = 0,1,\ldots,s^* \tag{1.B.33}$$

$$C_t = L_0(1-\tau) \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \varepsilon_1^{s^*} \varepsilon_2^{t-s^*},\ t = s^*+1,\ldots,T-1 \tag{1.B.34}$$

$$Z_t = L_t \left[1 - \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g} \right)^t \right],\ t = 0,1,\ldots,s^*-1 \tag{1.B.35}$$

$$Z_{s^*} = L_{s^*} \sum_{s=0}^{T-1-s^*} y_2^s - \frac{C_{s^*}}{1-\tau} \sum_{s=0}^{T-1-s^*} x_2^s. \tag{1.B.36}$$

in accordance with the result in equation (1.B.6).

To collect the results: Having found the critical switch point s^*, we always have that in case of $W_{s^*} > 0$

$$Z_{s^*} = L_{s^*} \sum_{s=0}^{T-1-s^*} y_2^s - \frac{C_{s^*}}{1-\tau} \sum_{s=0}^{T-1-s^*} x_2^s \tag{1.B.37}$$

$$W_{s^*} = C_{s^*} \sum_{s=1}^{T-1-s^*} x_2^s - L_{s^*}(1-\tau) \sum_{s=1}^{T-1-s^*} y_2^s. \tag{1.B.38}$$

The total savings before tax at time s^* is identical in magnitude to the savings into the tax–deferred retirement account in the former case where $W_{s^*} = 0$:

$$Z_{s^*} + \frac{W_{s^*}}{1-\tau} = L_{s^*} - \frac{C_{s^*}}{1-\tau}. \tag{1.B.39}$$

1.B.2.3 Finding s^*

The first–order conditions for the evolution of consumption require the following two inequalities at the switch point, which is the first value of s^* for which the inequality (1.B.40) is fulfilled:

$$\frac{C_{s^*+1}}{C_{s*}} < \varepsilon_1 \quad \Leftrightarrow \quad \frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*+1}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^{s^*+1}} \leq 1 \tag{1.B.40}$$

$$\varepsilon_2 \leq \frac{C_{s^*+1}}{C_{s*}} \quad \Leftrightarrow \quad \frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^{s^*}} \geq 1. \tag{1.B.41}$$

It is tantamount to stopping payments into the tax–deferred retirement account from time $s^* + 1$ and onwards. The inequality (1.B.40) may never be satisfied in which case we have savings into the tax–deferred retirement account at all times and a situation, where the taxable account is never used.

For the rest of the proof we need the following two lemmas.

Lemma 1. *Let A,B,C and D be positive numbers. Then*

$$\frac{A+B}{C+D} > (<) \frac{B}{D} \quad \textit{iff} \quad \frac{A}{C} > (<) \frac{B}{D}. \tag{1.B.42}$$

Proof. Trivial. □

Lemma 2. *The expressions in equations (1.B.40)–(1.B.41) can be rewritten in the following manner.*

$$\frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^{s^*}} = \frac{\frac{\sum_{s=1}^{T-1-s^*} y_2^s\, y_2^{s^*}}{\sum_{s=1}^{T-1-s^*} x_2^s\, x_2^{s^*}}\varepsilon_2^{s^*}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}\varepsilon_1^{s^*}} = \frac{\frac{\sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=1}^{T-1-s^*} x_2^s}\frac{g^{s^*}}{\varepsilon_1^{s^*}}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}} = \frac{\frac{\sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=1}^{T-1-s^*} x_2^s}\frac{y_1^{s^*}}{x_1^{s^*}}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}}. \tag{1.B.43}$$

Proof. The proof relies on straightforward substitutions. □

If both inequalities (1.B.40)–(1.B.41) are fulfilled (which corresponds to item (2a) of Theorem 2), we are in the scenario where $W_{s^*} = 0$, and the two subperiods separate, i.e., there is no point in time where the individual makes contributions to both accounts simultaneously. The entire solution is then as stated in equations (1.4.16)–(1.4.18) and in equations (1.B.20)–(1.B.23). It is straightforward to check the first–order conditions concerning the growth rates of consumption; it is ε_1 ($> \varepsilon_2$) for the first subperiod—as required when $Z_t > 0$ and $W_t = 0$—and ε_2 ($< \varepsilon_1$) for the second subperiod, as required when $W_t > 0$ and $Z_t = 0$. At the transition point, it holds that $C_{s^*+1}/C_{s^*} \in (\varepsilon_2,\varepsilon_1)$. The only thing left to check is that Z_t are positive numbers.

Since Z_t is a decreasing sequence, due to the fact that $\varepsilon_1 > g$, it is enough to check this feasibility for s^*, i.e., to check that Z_t are positive numbers. We need to verify that

$$\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g}\right)^{s^*} < 1. \tag{1.B.44}$$

Making use of the assumption in equation (1.B.41) we have that

$$\begin{aligned} \frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \left(\frac{\varepsilon_1}{g}\right)^{s^*} &\le \frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s} \left(\frac{\varepsilon_2}{g}\right)^{s^*} \\ &= \frac{\sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=1}^{T-1-s^*} x_2^s} \left(\frac{y_2\varepsilon_2}{x_2 g}\right)^{s^*} \\ &= \frac{\sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=1}^{T-1-s^*} x_2^s}. \end{aligned} \tag{1.B.45}$$

The left–hand side of equation (1.B.45) is a decreasing function of g. As only y_1 and g itself depend on g, this property is determined solely by the relation defined in equation (1.B.46):

$$f(g) \equiv g^{-s^*} \sum_{s=0}^{s^*} y_1^s \;\Rightarrow\; f'(g) = g^{-s^*-1} \sum_{s=0}^{s^*} y_1^s \left(\sum_{s=0}^{s^*} s \frac{y_1^s}{\sum_{s=0}^{s^*} y_1^s} - s^* \right) < 0. \tag{1.B.46}$$

As the first term in the parenthesis is a weighted sum of time indices $0 \le s \le s^*$, the whole expression is obviously negative. Hence, as g increases the left–hand side decreases.

The right–hand side of equation (1.B.45) is an increasing function of g, so as g increases the right–hand side increases. For $g = \varepsilon_1$, we have $x_1 = y_1$ and the left–hand side of equation (1.B.45) is clearly less than unity whenever $K > 0$. For $g = \varepsilon_2$, we have $x_2 = y_2$ and the right–hand side is equal to one. As g further decreases we have an increasing left–hand side bounded from above by a decreasing upper limit below unity. Consequently, this relation and the positivity of Z_{s^*} remain valid as long as the condition in equation (1.B.41) is fulfilled.

When the condition in equation (1.B.41) is not fulfilled, we are in the situation in which the following two inequalities are fulfilled (which corresponds to item (2b) of Theorem 2):

$$\frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*+1}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s+K^{1/\gamma}x_1^T}\varepsilon_1^{s^*+1}} \leq \frac{\frac{\sum_{s=s^*+1}^{T-1} y_2^s}{\sum_{s=s^*+1}^{T-1} x_2^s}\varepsilon_2^{s^*}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s+K^{1/\gamma}x_1^T}\varepsilon_1^{s^*}} \leq 1. \tag{1.B.47}$$

There are now contributions into both accounts as time s^* and the solution with $W_{s^*} > 0$ becomes relevant:

$$C_t = L_0(1-\tau)\frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*}\sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*}\sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma}x_1^T}\varepsilon_1^t,\ t = 0,1,\ldots,s^* \tag{1.B.48}$$

$$C_t = L_0(1-\tau)\frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*}\sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*}\sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma}x_1^T}\varepsilon_1^{s^*}\varepsilon_2^{t-s^*},\ t = s^*+1,\ldots,T-1 \tag{1.B.49}$$

$$Z_t = L_0\left[1 - \frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*}\sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*}\sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma}x_1^T}\left(\frac{\varepsilon_1}{g}\right)^t\right],\ t = 0,1,\ldots,s^*-1 \tag{1.B.50}$$

$$Z_{s^*} = L_{s^*}\sum_{s=0}^{T-1-s^*} y_2^s - \frac{C_{s^*}}{1-\tau}\sum_{s=0}^{T-1-s^*} x_2^s \tag{1.B.51}$$

$$W_{s^*} = C_{s^*}\sum_{s=1}^{T-1-s^*} x_2^s - L_{s^*}(1-\tau)\sum_{s=1}^{T-1-s^*} y_2^s. \tag{1.B.52}$$

The first–order conditions concerning the growth rates of consumption are clearly fulfilled. We need to verify that $Z_{s^*} > 0$ as well as $W_{s^*} > 0$. Consider the situation for Z_{s^*}, where we insert the expressions for L_{s^*} and C_{s^*}, respectively, into equation (1.B.53):

$$Z_{s^*} = L_{s^*}\sum_{s=0}^{T-1-s^*} y_2^s - \frac{C_{s^*}}{1-\tau}\sum_{s=0}^{T-1-s^*} x_2^s \tag{1.B.53}$$

$$\begin{aligned} Z_{s^*}(1-\tau) = L_0(1-\tau)(1+r)^{s^*} x_1^{s^*}\sum_{s=0}^{T-1-s^*} x_2^s \\ \left[\frac{y_1^{s^*}\sum_{s=0}^{T-1-s^*} y_2^s}{x_1^{s^*}\sum_{s=0}^{T-1-s^*} x_2^s} - \frac{\sum_{s=0}^{s^*-1} y_1^s + y_1^{s^*}\sum_{s=0}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*}\sum_{s=0}^{T-1-s^*} x_2^s + K^{1/\gamma}x_1^T}\right]. \end{aligned} \tag{1.B.54}$$

This is positive iff

$$\frac{y_1^{s^*} \sum_{s=0}^{T-1-s^*} y_2^s}{x_1^{s^*} \sum_{s=0}^{T-1-s^*} x_2^s} > \frac{\sum_{s=0}^{s^*-1} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \tag{1.B.55}$$

By the nature of s^* as the first time index that satisfies equation (1.B.40), this inequality is fulfilled.

Next we consider the situation for W_{s^*} and insert the consumption policy into equation (1.B.52). After a reorganization of terms we have:

$$\begin{aligned} W_{s^*} &= L_0(1-\tau)\left[\frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} \varepsilon_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s - g^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s\right] \\ &= L_0(1-\tau)\varepsilon_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s \left[\frac{\sum_{s=0}^{s^*} y_1^s + y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{\sum_{s=0}^{s^*} x_1^s + x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s + K^{1/\gamma} x_1^T} - \frac{y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s}\right]. \end{aligned} \tag{1.B.56}$$

This expression is positive, cf. Lemma 1 and Lemma 2, iff

$$\frac{y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s} < \frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T} \Leftrightarrow \frac{\frac{y_1^{s^*} \sum_{s=1}^{T-1-s^*} y_2^s}{x_1^{s^*} \sum_{s=1}^{T-1-s^*} x_2^s}}{\frac{\sum_{s=0}^{s^*} y_1^s}{\sum_{s=0}^{s^*} x_1^s + K^{1/\gamma} x_1^T}} < 1. \tag{1.B.57}$$

This completes the proof of Theorem 2.

Appendix 1.C Different choices of K

In the basic formulation of the objective function, the terminal term was left unspecified except for the parameter K. In this appendix we provide some suggestions as to alternative choices of this parameter.

1.C.1 Identical utility function throughout the entire lifetime

Assume that T marks the time index for the beginning of the retirement period, i.e., the first time index for which the labor market income vanishes. Let T^* be the last time for which the individual is alive. Then the optimization problem in Section 1.2 becomes:

$$\max_{\{\{W_t\}_{t=0}^{T^*-1}\}} \frac{1}{1-\gamma}\left[\sum_{t=0}^{T^*}\rho^t C_t^{1-\gamma}\right] \tag{1.C.1}$$

subject to

$$C_t = L_t(1-\tau) + W_{t-1}\left(1+r(1-\tau)\right) - W_t,\ t = 0, 1, \ldots, T-1 \tag{1.C.2}$$

$$C_s = W_{s-1}\left(1+r(1-\tau)\right) - W_s,\ s = T, T+1, \ \ldots, T^* \tag{1.C.3}$$

$$W_{T^*} = 0. \tag{1.C.4}$$

The restriction in equation (1.C.4) will automatically come out of the optimization and is not really a restriction. The optimal solution will be of the same form as the one in Section 1.2; i.e.,

$$C_t = C_0\varepsilon^t,\ t = 1, 2, \ \ldots, T^*. \tag{1.C.5}$$

The tail of the objective function is then:

$$\sum_{s=T}^{T^*}\rho^s C_0^{1-\gamma}\varepsilon^{(1-\gamma)s} = \rho^T C_T^{1-\gamma}\sum_{s=T}^{T^*}\left(\rho\varepsilon^{1-\gamma}\right)^{s-T} = \rho^T C_T^{1-\gamma}\sum_{s=0}^{T^*-T}\left(\rho\varepsilon^{1-\gamma}\right)^{s}. \tag{1.C.6}$$

Hence, this is represented by setting K to:

$$K = \sum_{s=0}^{T^*-T}\left(\rho\varepsilon^{1-\gamma}\right)^{s} = \frac{1-\left(\rho\varepsilon^{1-\gamma}\right)^{T^*+1-T}}{1-\left(\rho\varepsilon^{1-\gamma}\right)}. \tag{1.C.7}$$

1.C.2 Different tax rates during the savings period and the retirement period

The analysis with a given K can easily be adjusted to incorporate the feature that the tax rate during the retirement period is different—and typically lower—than during the working period. If the latter tax rate is termed τ_r, a simple adjustment of K to $K\frac{1-\tau_r}{1-\tau}$ has the same effect as keeping K constant and redefine the constraint given by equation (1.4.7) as follows:

$$C_T = (1-\tau)\sum_{s=0}^{T-1} Z_s(1+r)^{T-s} \quad \rightarrowtail \quad C_T = (1-\tau_r)\sum_{s=0}^{T-1} Z_s(1+r)^{T-s}. \tag{1.C.8}$$

Chapter 2

Mandatory Pension Saving and Homeownership*

*This chapter is joint work with Marcel Fischer and Bjarne Astrup Jensen. We are grateful for helpful comments and suggestions from Rob Alessie, Rafael González–Val, Jens Jackwerth, Marius Mihai, Lasse Heje Pedersen, Joel Petry, Nic Schaub, Kathrin Schlafmann, and seminar participants at the Universities of Amsterdam, St. Gallen, Konstanz, Regensburg, Strasbourg, Copenhagen Business School, the 2022 American Finance Association Annual Meeting, the 2022 Southwestern Finance Association Annual Meeting, the 29th Annual Global Finance Conference, the 2022 European Actuarial Journal Conference, the 2022 PeRCent conference, the 2022 Annual Meeting of the German Finance Association, the 2023 AREUEA–ASSA Annual Meeting, the 2023 Actuarial and Financial Mathematics Conference, the 2023 Ageing and Sustainable Finance Conference, and the 2023 European Meeting of the Urban Economics Association. This research has been supported by PeRCent, which receives base funding from the Danish pension funds and Copenhagen Business School.

2.1. Introduction

Saving for retirement is among the most important financial challenges individuals face throughout their lives. At present, increasing life expectancies combined with decreasing birth rates and low interest rates impose huge challenges on finding suitable ways to sustain a reasonable level of retirement benefits. Most countries around the world impose some form of mandatory retirement saving scheme on their citizens, often financed via a constant share of labor income used to fund the system, such as under the current U.S. Social Security system, (i) because forced saving can help individuals overcome lack of retirement planning or self–control problems (Samuelson (1975); Feldstein (1985); Moser and Silva (2019)) and (ii) to overcome the Samaritan's dilemma caused by the government's inability to commit to not helping the elderly experiencing poverty.

We set up a realistically calibrated life cycle model which aims to shed light on the side effects of mandatory retirement savings on housing decisions, in particular housing market entries and loan–to–value ratios, and incorporates many important features of models from the literature, including stochastic labor income (e.g., Cocco, Gomes, and Maenhout (2005)), a rent–versus–own decision (e.g., Cocco (2005); Yao and Zhang (2005)), and taxable as well as tax–deferred retirement accounts (e.g., Dammon, Spatt, and Zhang (2004); Fischer and Gallmeyer (2017)). In our model, individuals must make informed decisions about their preferred consumption, their homeownership status, the size of the home they want to live in, the contribution to or, after attaining retirement age, withdrawal from the tax–deferred retirement account, and about the shares of wealth invested into stocks and bonds in the taxable as well as the tax–deferred retirement account.

Existing pension systems of the defined benefit type, in which participants are promised fixed annual benefits for their contributions, face increasing funding problems. As a result, this study focuses on pension systems of the defined contribution type, to which individuals contribute according to a pre–determined schedule, while benefits depend on the amounts of contributions made and their profits earned. Currently, such pension systems already cover large shares of the population's pensions in, e.g., Denmark, the Netherlands, and Sweden.

Retirement accounts are typically set up as tax–deferred accounts. Contributions to these accounts are (within certain limits) deductible from taxable income, and profits earned from them are generally tax–exempt, while withdrawals during retirement age are subject to income taxation.[1]

Whereas existing funded retirement saving schemes typically provide individuals with a degree of freedom in their asset allocation, they usually impose harsh restrictions on minimum contributions. In practice, individuals are often required to contribute a constant minimum share of their labor income to a retirement saving scheme. To explore the implications of such minimum contribution requirements for tax–deferred retirement accounts, we compare a setting with minimum contributions with a setting without mandatory contributions.

Intuitively, mandatory minimum contributions could delay housing market entry, because they render it more challenging for young adults, who are typically severely liquidity constrained, to build up sufficient liquid savings for a down payment. Consistent with this intuition, our model predicts that minimum contributions delay the average age of first home buyers by more than two years. Simultaneously, individuals facing minimum contribution requirements face higher loan–to–value ratios. In essence, these individuals incur the cost of the interest margin on mortgage debt by borrowing their own money.

To counteract the undesirable side effects of mandatory minimum contributions, we propose a flexible retirement saving scheme, which builds on the intuition that it is not important through which channels individuals save for retirement, as long as they build up sufficient total wealth. Intuitively, it may make sense to prioritize savings outside the tax–deferred retirement account at a younger age to facilitate making the down payment required for the acquisition of a home. Our flexible retirement saving scheme builds on this intuition and only forces individuals to save, if they have not built up sufficient savings in either ordinary taxable accounts, as home equity, or through pre–existing tax–deferred retirement savings. More specifically, our scheme only requires individuals to build up savings if they have not built up sufficient wealth in the past, but does not require savings to be held in the tax–deferred retirement account.

Compared to the pension system with mandatory minimum contributions, our flexible retirement saving scheme, which ensures a similar build up of savings, has two key advantages. First, it allows individuals to repay their mortgages before building up savings in the tax–deferred retirement account, thus avoiding paying the interest

[1] Few countries, including Denmark and Sweden, impose taxes on profits earned in tax–deferred accounts. However, the tax rate is low compared to that applicable in a taxable account. Withdrawals prior to retirement age are generally not possible or subject to substantial penalty taxes, thus rendering early withdrawals highly undesirable. Hence, an important difference between taxable accounts and tax–deferred retirement accounts is the illiquidity of the latter.

differential between the borrowing rate and the investment rate. Second, it allows for earlier homeownership. In sum, the flexible retirement saving scheme leads to an increase in welfare compared to the pension system with mandatory minimum contributions while simultaneously ensuring a similar level of savings.

Our work contributes to a growing strand of literature investigating optimal retirement savings decisions over the life cycle. An important early contribution is the work of Carroll (1997), which studies the optimal life cycle profile of savings when individuals do not have access to tax–deferred retirement accounts. Cocco, Gomes, and Maenhout (2005) extends this work by allowing for a stochastic labor income stream, which is calibrated to socio–demographic data. Studies examining optimal savings decisions with taxable accounts as well as tax–deferred retirement accounts include: Amromin (2003); Shoven and Sialm (2003); Dammon, Spatt, and Zhang (2004); Garlappi and Huang (2006); Huang (2008); and Fischer and Gallmeyer (2017). We contribute to this line of literature by highlighting the undesirable side effects of minimum contribution requirements and by proposing a new flexible retirement saving scheme, that largely alleviates the unintended side effects, while simultaneously ensuring that individuals build up sufficient savings.

Our work is also related to a strand of literature exploring optimal pension system design (e.g., Dahlquist, Vestman, and Setty (2018)); Larsen and Munk (2023); and Schlafmann, Setty, and Vestman (2022) also investigate optimal pension plan design in a framework with a taxable account as well as a tax–deferred retirement account, but abstract away from explicitly accounting for housing. The former documents that individuals who undersave or do not invest into stocks can benefit from mandatory pension saving. The latter advocates age– and retirement wealth–to–labor income dependent contribution rates to retirement savings plans, but do not account for wealth held outside retirement plans in determining their contribution rates. Our work has another focus and differs from those two papers in several regards. First, we explicitly model a rent–versus-own decision and mortgages. Our results show that mandatory pension saving delays housing market entry and leads to higher loan–to–value ratios. To the best of our knowledge, our manuscript is the first to investigate the implications of pension system design for homeownership decisions and loan–to–value ratios. Second, our work proposes a flexible retirement saving scheme that (i) only requires saving from individuals that have not built up sufficient wealth, and (ii) does not require mandatory savings to be located in the tax–deferred retirement account, if, e.g., repaying an expensive mortgage is more desirable. This flexible retirement saving scheme leads to earlier housing market entry, lower loan–to–value ratios, and higher welfare than the pension system with mandatory minimum contributions. While other manuscripts have investigated different contribution rules to retirement saving schemes, our manuscript is, to the best of our knowledge, the first to propose a savings rule that does not solely focus on savings in tax–deferred retirement accounts, but also takes other forms of savings into account. Consistent with economic intuition, our results show that savings outside retirement savings accounts are important for determining how much to save for retirement.

This paper proceeds as follows: Section 2.2 outlines our life cycle model and presents its calibration, while Section 2.3 presents the model predictions. Next, Section 2.4 introduces our flexible retirement saving scheme, and Section 2.5 concludes.

2.2. The model

In this section, we outline our life cycle model of optimal consumption, savings, and housing decisions. In our model, individuals work from age 20 to 65 and build up savings for retirement (accumulation phase). At age 66, they are retired and use their accumulated savings during their remaining lifetime. In our model, individuals face a rent–versus–own decision. They can build up savings in both a taxable and a tax–deferred retirement account, and they can invest in a risk–free asset, a risky asset representing a stock market index, and an owner–occupied home. We let the Cocco, Gomes, and Maenhout (2005) labor income process calibrated for high school graduates determine the individual's labor income during working age. That is, we allow for an age–dependent trend as well as stochastic income shocks.

Individuals are subject to mortality risk and make their decisions in order to achieve the objective of maximizing expected lifetime utility. The individual's preference function is given by recursive Epstein–Zin preferences (Epstein and Zin (1989)) to enable a separation between the individual's degree of relative risk aversion, γ, and its elasticity of intertemporal substitution, ψ.

2.2.1. Mandatory contributions

Our goal is to investigate the impact of mandatory minimum contributions to the tax–deferred retirement account, irrespective of whether the account is run by a mandatory public pension system, such as the U.S. Social Security system, a mandatory occupational pension system, such as those widespread in Scandinavian countries, or a mandatory private pension system. We deliberately do not try to replicate a specific retirement savings scheme in a particular country, but aim at contributing to a better understanding of the broader consequences of mandatory minimum contributions.

To investigate the impact of mandatory contributions, we compare two settings with each other. In our benchmark case, individuals are faced with a minimum contribution requirement to a tax–deferred retirement account, corresponding to a constant share, $\xi \in (0,1]$, of the individual's labor income, L_t, at time t during working age:

$$Z_t \geq \xi L_t, \quad t < t_{ret}, \tag{2.2.1}$$

in which Z_t denotes the individual's contribution to the tax–deferred retirement account, and t_{ret} denotes the first time at which the individual is retired. Requiring minimum contributions, which are a constant share of labor income, is a widespread phenomenon in many pension systems around the world. Notable exceptions are mandatory occupational pension systems in Switzerland, for which the minimum contribution rate typically increases with age—from 7% at age 25 to 18% at 65 (see OECD (2021a), in particular the country profile of Switzerland). The age–dependent contribution requirements only apply to the mandatory occupational pension systems. Contribution rates to Switzerland's mandatory state pension insurance are constant, begin at the age of 17 and end at retirement age.

We compare this benchmark case with a setting, in which the minimum contribution during working age is zero, i.e.:

$$Z_t \geq 0, \quad t < t_{ret}. \tag{2.2.2}$$

We deliberately do not allow for early withdrawals from the tax–deferred retirement account, because such withdrawals are either impossible or subject to substantial tax penalties, thus rendering early withdrawals undesirable.

Since contributions to the tax–deferred retirement account during working age are typically tax–deductible, they are usually limited by an upper bound, $\phi \in (0,1]$, of the individual's labor income:

$$Z_t \leq \phi L_t, \quad t < t_{ret}, \tag{2.2.3}$$

in which $\xi \leq \phi$.

2.2.2. Housing

In our model, individuals face a rent–versus–own decision, i.e., they can either live in a rented place or an owner–occupied home. The return on investments in owner–occupied homes, r_t^H, from time t to time $t+1$ is log–normally distributed, with mean μ_H and standard deviation σ_H.

Owner–occupied homes serve a dual role as an asset and a durable consumption good. That is, when owning a home, individuals simultaneously derive utility from living in their home and expose themselves to changes in its value. Following earlier work with owner–occupied housing, such as Cocco (2005), Yao and Zhang (2005), Fischer and Stamos (2013), and Kraft, Munk, and Wagner (2018), we employ a Cobb–Douglas utility function, u, defined over consumption and housing.

Following Kiyotaki, Michaelides, and Nikolov (2011), we account for the preferences of individuals to live in owner–occupied housing, by allowing for a higher utility per home unit when they own it. For that purpose, we multiply the size of the individual's residence, Q, by a factor $1+\zeta h$, in which ζ determines the strength of the preference for living in an owner–occupied home, and h is an indicator variable, taking the value of one if the individual lives in an owner–occupied home, and zero otherwise. Hence:

$$u\left(C,Q\right) = C^{1-\theta}\left(Q(1+\zeta h)\right)^{\theta}, \tag{2.2.4}$$

in which C is the amount consumed, and θ is the relative preference over housing consumption.

Both owners and renters face costs for housing consumption. These costs can be divided into recurring and nonrecurring costs. The former depends on whether the individual lives in an owner–occupied home or a rented place. If H_t denotes the price per housing unit at time t, renters pay an annual rent, $m_r Q_t H_t$, that is assumed to be a constant share, m_r, of the value of their residence, $Q_t H_t$. Owners also face recurring costs, such as property taxes and maintenance costs of $m_o Q_t H_t$, in which m_o is a constant share of the value of their home. The rate of recurring housing costs, m, from time t to $t+1$ can be expressed as

$$m\left(h_t\right) = h_t m_o + \left(1-h_t\right) m_r. \tag{2.2.5}$$

Nonrecurring housing expenses, n, occur when the individual acquires a new home. In that case, the individual faces transaction costs of $f_p Q_t H_t$, in which f_p denotes the percentage cost of acquiring homeownership. This cost includes fees to be paid to a real estate agent and taxes. Individuals face these costs when changing homeownership status from renter to owner $(h_t(1-h_{t-1})=1)$ or when remaining a homeowner, but changing home size, i.e., when moving from one owner–occupied home to another $(h_t h_{t-1}\chi_{\{Q_t\neq Q_{t-1}\}}=1)$, in which $\chi_{\{Q_t\neq Q_{t-1}\}}$ is an indicator variable taking the value one, if $Q_t\neq Q_{t-1}$, and zero otherwise.[2] Nonrecurring transaction costs, n, from time t to $t+1$ can then be summarized as follows:

$$n(Q_t,h_t,h_{t-1})=f_p Q_t H_t\left(h_t(1-h_{t-1})+h_t h_{t-1}\chi_{\{Q_t\neq Q_{t-1}\}}\right). \tag{2.2.6}$$

These transaction costs result in no–trade regions, in which the costs from adjusting the home size outweigh the benefits. That is, trading costs render frequent changes in home size undesirable.

2.2.3. Capital markets

Apart from investing in owner–occupied homes, individuals can invest into a representative stock market index for which the pre–tax return, r_t^S, from time t to $t+1$ is log–normally distributed with mean μ_S and standard deviation σ_S. Returns on investments into the stock market index and owner–occupied homes may be correlated, reflecting that they may depend on common risk factors, such as the evolution of the economy. When τ_g denotes the tax rate applicable to returns on equity investments in the taxable account, the gross returns on an investment in the stock market index in the taxable and the tax–deferred retirement account from time t to $t+1$ are given by $G_{T,t}=1+r_t^S\left(1-\tau_g\right)$ and $G_{R,t}=1+r_t^S$, respectively.

In addition to the stock market, individuals can invest into a risk–free asset paying a constant pre–tax interest rate of r. When τ denotes the income tax rate applicable to labor income, including interest earned in the taxable account, and to withdrawals from the tax–deferred retirement account after attaining retirement age, the gross returns on investments into the risk–free asset in the taxable and the tax–deferred retirement account are given by $R_T=1+r(1-\tau)$ and $R_R=1+r$, respectively. Since profits are taxed on a nominal basis, it is important to explicitly account for inflation. We assume a constant inflation rate, which we denote by i.

[2] In our model, no costs occur when selling a home, i.e., changing homeownership status from owner to renter is not associated with any costs to the seller. In our model, the buyer covers the costs.

Homeowners can use their home as a collateral and borrow up to a share, $\delta \in [0,1]$, of its value, whereas renters are barred from this borrowing opportunity. When B_t denotes the amount invested risk–free ($B_t \geq 0$) or borrowed ($B_t < 0$), it holds that:

$$-B_t \leq \delta h_t Q_t H_t. \tag{2.2.7}$$

To avoid tax arbitrage opportunities by deducting mortgage interest expenses, while simultaneously earning interest in a tax–deferred retirement account tax free, we assume that borrowers face an interest rate margin, i.e., the after–tax interest burden from borrowing exceeds the after–tax interest earned on investments in the risk–free asset. When R_D denotes the gross after–tax borrowing rate after accounting for the tax deductibility of mortgage interest expenses, it thus holds that $R_D > R_R$.

2.2.4. Optimization problem

In our model, in every period, t, individuals must simultaneously make seven interrelated decisions: 1) how much to spend on non–durable consumption; 2) how much to contribute to (or, after attaining retirement age, withdraw from) their tax–deferred retirement account, Z_t; 3) the share of retirement savings invested in stocks, $\alpha_{R,t}$; 4) the size of the home they want to live in, Q_t; 5) their homeownership status, h_t; 6) how much of their wealth in the taxable account to invest in stocks, S_t; and, finally, 7) the amount of wealth to invest in bonds or the size of their mortgage, B_t.

In setting up the individual's optimization problem, $f(t)$ denotes the individual's survival probability from time t to $t+1$ and β denotes its time preference factor. The terminal period in our model is denoted by N. $W_{T,t}$ is the amount of wealth in the taxable account at time t. Retirement savings at time t before (after) contributions/withdrawals are denoted by $W_{R,t-}$ ($W_{R,t+}$). Total wealth, W_t, that effectively belongs to the individual after accounting for the fact that savings in the tax–deferred retirement account are still subject to taxation upon withdrawal can be expressed as:

$$W_t = W_{R,t-}(1-\tau) + W_{T,t} + h_{t-1} Q_{t-1} H_t. \tag{2.2.8}$$

The individual's optimization problem is then:

$$\max_{\{C_t,Z_t,\alpha_{R,t},Q_t,h_t,S_t,B_t\}} V_t\left(X_t\right) = \left((1-\beta)\left(\left(\frac{C_t}{(1+i)^t}\right)^{1-\theta}\left(Q_t\left(1+\zeta h_t\right)\right)^{\theta}\right)^{1-\frac{1}{\psi}} + \beta\left(f\left(t\right)\mathbb{E}_t\left[V_{t+1}\left(X_{t+1}\right)^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right)^{\frac{1}{1-\frac{1}{\psi}}}, \tag{2.2.9}$$

subject to

$$W_{R,t-} = W_{R,(t-1)+}\left(\alpha_{R,t-1}G_{R,t-1} + \left(1-\alpha_{R,t-1}\right)R_R\right) \tag{2.2.10}$$

$$W_{R,t+} = W_{R,t-} + Z_t \tag{2.2.11}$$

$$\begin{aligned} W_{T,t} &= (1-\tau)\,L_t + S_{t-1}G_{T,t-1} + B_{t-1}\left(\chi_{\{B_{t-1}\geq 0\}}R_T + \chi_{\{B_{t-1}<0\}}R_D\right) \\ &= C_t + S_t + B_t + m\left(h_t\right)Q_tH_t + n\left(Q_t,h_t,h_{t-1}\right) + h_tQ_tH_t - h_{t-1}Q_{t-1}H_t + Z_t(1-\tau) \end{aligned} \tag{2.2.12}$$

$$C_t,Q_t > 0,\ S_t \geq 0,\ \alpha_{R,t} \in [0,1],\ W_{R,t+} \geq 0, \tag{2.2.13}$$

and equations (2.2.3) to (2.2.7), in which $X_t = \left[t,W_{R,t-},W_{T,t},L_t,Q_{t-1},h_{t-1}\right]$ is the vector of state variables for this optimization problem. Depending on whether or not we are investigating the optimal consumption–investment strategy with mandatory contributions to the tax–deferred retirement account, the individual's optimization problem is solved subject to the constraint given in equations (2.2.1) or (2.2.2). Appendix 2.A explains the more technical details on how we simplify and solve the optimization problem.

2.2.5. Calibration

This section summarizes the calibration of our model. Individuals enter the life cycle at age 20 ($t = 1$), work until age 65, are retired at age 66 ($t_{ret} = 47$), and die at the latest at age 100. Therefore, the accumulation period is 46 years, and the maximum investment horizon is $N = 81$ years. We let one period correspond to one year. Prior to age 100, we use the life table published by the Centers for Disease Control and Prevention (Arias and Xu (2018)) to determine the survival probabilities for men for another year.

We compare two different settings: In one setting, individuals do not face any mandatory contribution requirements for a tax–deferred retirement account, i.e., $\xi = 0\%$. In the other setting, individuals face minimum contribution requirements. The exact level of mandatory minimum contributions imposed differs between countries. Some countries, such as the U.S. and Germany, also split mandatory contributions between employer and employee contributions. While this difference may be important for employees' net salaries, the exact source of funding is mostly an accounting issue and not of primary importance for exploring the economic implications of minimum contribution requirements for housing decisions. If the employer has to bear parts of retirement contributions, he will account for these costs in the wage, he is willing to offer. That is, the employee's net salary after retirement contributions should not rationally depend on whether retirement contribution are legally borne by the employee, the employer, or shared between them. Total mandatory retirement contributions are 12.4% in the U.S., around 15% in Scandinavian countries and as high as 18.6% in Germany. We deliberately stay at the lower range of these values and set mandatory minimum contributions to $\xi = 12.4\%$.[3] The upper bound for contributions to the tax–deferred retirement account is set to $\phi = 100\%$ of labor income in both settings.

The pre–tax risk–free rate is set to $r = 3.7\%$, the average one–month Treasury Bill rate between 1946 and 2019. The expected annual pre–tax return on the S&P 500 between 1946 and 2019 was 12.4% and its standard deviation $\sigma_S = 16.9\%$. While the historical equity premium was quite high (Mehra and Prescott (1985); Dimson, Marsh, and Staunton (2002)), there are doubts as to whether such a high level will also apply in future periods. As a result, we follow the current consensus, which is around three to four percentage points (Claus and Thomas (2001)) and set the expected return of stocks to $\mu_S = 7.8\%$, implying an equity premium of around four percentage points. Inflation is set to its average value between 1946 and 2019, i.e., $i = 3.7\%$, implying a real risk–free rate of zero.

The tax rate on interest income, labor income, and regular withdrawals from the tax–deferred retirement account is set to $\tau = 35\%$ and the tax rate on capital gains is set to $\tau_g = 20\%$, roughly corresponding to the top tax rates in the U.S.

Following Yao and Zhang (2005), we set the housing preference parameter to $\theta = 0.2$, the maximum borrowing rate to $\delta = 80\%$, the annual maintenance rate to $m_o = 1.5\%$, and the transaction costs when acquiring a new owner–occupied home to $f_p = 6\%$. The annual rent rate is $m_r = 6.7\%$, the empirical estimate of Fischer

[3]We also computed results for $\xi = 18.6\%$, which lead—as to be expected—to qualitatively identical, though quantitatively larger effects. These results are available from the authors upon request.

and Stamos (2013). We set the individual's preference for owning over renting to $\zeta = 10\%$, which is in the order of magnitudes chosen in Kiyotaki, Michaelides, and Nikolov (2011) and Fischer and Khorunzhina (2019).

Using the U.S. S&P's CoreLogic Case–Shiller Home Price Index, we estimated the expected nominal return on housing from 1946 to 2019, μ_H, to be 4.6% and its standard deviation, σ_H, to be 5.2%. Price changes in individual homes are far from perfectly correlated, implying that the House Price Index underestimates the volatility of individual house prices. Case and Shiller (1989) argue that the volatility of individual house prices should be around 15%. Bourassa, Haurin, Haurin, Hoesli, and Sun (2009) provide empirical support. We therefore set the standard deviation of individual homes to $\sigma_H = 15\%$.

To avoid tax arbitrage opportunities, as in Marekwica, Schaefer, and Sebastian (2013), the borrowing rate is set to $r_D = 6.2\%$, corresponding to an after–tax borrowing rate of $(1-\tau)r_D = 4.03\%$, thus exceeding the after–tax risk–free rate in the tax–deferred retirement account of $r = 3.7\%$. Correlation between stock returns and labor income is set to $\rho_{S,L} = 0.047$, correlation between housing return and labor income to $\rho_{H,L} = 0.55$, and correlation between housing and stock returns to $\rho_{H,S} = 0$, which are the empirical estimates from Cocco (2005).

To determine the individual's labor income during working age, we use the Cocco, Gomes, and Maenhout (2005) labor income process estimated for high school graduates. As the goal of our work is to contribute to a better understanding of how contributions to pension systems are best implemented, we require individuals to fully fund their retirement benefits through their own savings. As a result, we do not allow for retirement benefits from other sources and set the replacement ratio to zero, implying that individuals do not receive other income during retirement age besides from their accumulated savings.

Preference parameters are chosen to match empirically observed wealth at retirement age for individuals facing minimum contribution requirements. The degree of risk aversion and the time preference factor are therefore set to $\gamma = 3$ and $\beta = 0.96$, respectively. We set the elasticity of intertemporal substitution to $\psi = 0.15$, which is in the range of estimates in Pakoš (2011), Cashin and Unayama (2016), Gayle and Khorunzhina (2018), and Best, Cloyne, Ilzetzki, and Kleven (2020). Table 2.2.1 summarizes our base case parameter choice.

Table 2.2.1: Base case calibration

Description	Symbol	Value	Source
Preference parameters			
Degree of risk aversion	γ	3	Own choice
Elasticity of intertemporal substitution	ψ	0.15	Own choice
Time preference factor	β	0.96	Own choice
Financial markets			
Expected nominal return on equity	μ_S	7.8%	Historical estimate
Standard deviation of return on equity	σ_S	16.9%	Historical estimate
Inflation	i	3.7%	Historical estimate
Nominal risk–free rate	r	3.7%	Historical estimate
Housing market			
Housing preference parameter	θ	0.2	Yao and Zhang (2005)
Maximum borrowing rate	δ	80.0%	Yao and Zhang (2005)
Risk–free borrowing rate	r_D	6.2%	Yao and Zhang (2005)
Annual rent rate	m_r	6.7%	Fischer and Stamos (2013)
Annual maintenance rate	m_o	1.5%	Yao and Zhang (2005)
Transaction costs for purchasing home	f_p	6.0%	Yao and Zhang (2005)
Expected nominal return on housing	μ_H	4.6%	Historical estimate
Standard deviation of return on housing	σ_H	15.0%	Case and Shiller (1989)
Preference for owning over renting	ζ	10.0%	Kiyotaki et al. (2011)
Taxes			
Tax rate on capital gains	τ_g	20%	Own choice
Tax rate on other income	τ	35%	Own choice
Correlations			
Stock returns and labor income	$\rho_{S,L}$	0.047	Cocco (2005)
Housing return and labor income	$\rho_{H,L}$	0.550	Cocco (2005)
Housing and stock return	$\rho_{H,S}$	0.000	Cocco (2005)

This table summarizes the base case calibration used in our model.

2.3. Results

2.3.1. Model predictions

It is important to ensure that our simulated model predictions are able to match key features in the data, which is why we begin the presentation of our results by comparing wealth in the Panel Study of Income Dynamics (PSID) data with model–predicted wealth. For our data analyses examining the model fit, we use data from all available PSID waves after the financial crisis, i.e., the PSID waves from 2011 to 2019. A comparison between model predictions and the data is challenging, because

Social Security benefits are not part of wealth reported in the data, and there is an extremely high degree of dispersion of wealth in the PSID data. To address these two issues, we proceed as follows.

We add the present value of Social Security benefits to PSID wealth to allow for a meaningful comparison between model predictions and the data. We calculate the present value of Social Security benefits in two steps. First, we calculate the monthly Social Security benefits individuals can expect when retiring at age 65 by following the calculation procedure provided on the U.S. Social Security Administration website.[4] Second, we calculate the present value of the expected benefits with basic actuarial formulas accounting for mortality rates, as provided by the Centers for Disease Control and Prevention (Arias and Xu (2018)) and using our base case interest rate of $r = 3.7\%$. Appendix 2.B outlines the data selection process and the computation of the present value of the Social Security benefits in more detail.

We use median wealth in the data, which reflects that mean wealth is heavily biased upwards by a few outliers. In the data, self–reported median wealth at age 20 is USD 10,000 and grows to USD 110,000 at age 65. The median present value of Social Security benefits at age 65 is USD 250,035.70, implying a total wealth at age 65 of USD 360,035.70. Hence, total wealth grows by a factor of about 36.0 from age 20 to 65 in the PSID data. Preference parameters in our model are chosen to achieve a comparable growth of wealth. Specifically, with the choice of parameters introduced in Section 2.2.5, the growth of average wealth in our model from age 20 to 65 is 35.55, thus matching the growth of wealth in the data very well.

2.3.2. Implications of minimum contribution requirements

In this section, we explore the implications of mandatory minimum contributions for the evolution of the wealth of individuals, W_t, their retirement savings, housing market entry, and loan–to–value ratios. Intuitively, mandatory contributions to the tax–deferred retirement account should affect not only the individuals' accumulation of wealth and the location of their savings between the taxable and the tax–deferred retirement account, but they should also affect the rent–versus–own decision and the timing of housing market entry through two counteracting channels. On the one hand, mandatory contributions reduce the individuals' liquid wealth, thus deferring the age at which individuals have accumulated enough savings for a down payment. Hence, from this liquidity channel, mandatory contributions should defer housing market entry. On the other hand, mandatory contributions should lead to a faster

[4] https://www.ssa.gov/oact/cola/Benefits.html; last accessed November 11, 2021.

overall accumulation of wealth, enabling individuals to bear possible losses on potential housing investments more easily, without running the risk of retirement poverty. Via this wealth channel, mandatory contributions could prepone housing market entry. In this section, we explore, among others, which of these two counteracting channels dominates.

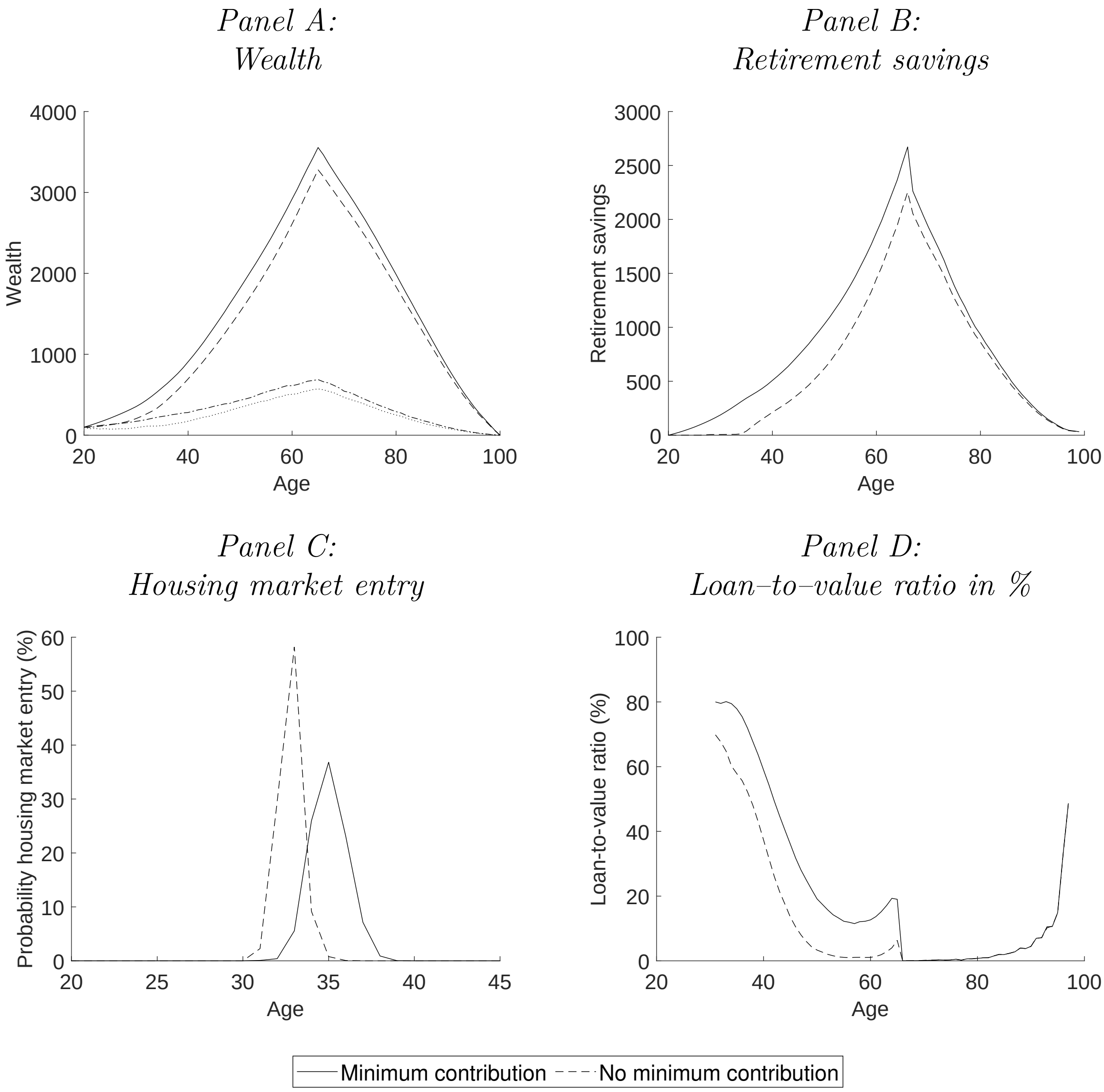
Figure 2.3.1 summarizes our results. Panel A depicts the evolution of wealth over the life cycle and Panel B of retirement savings. Panel C depicts the distribution of first–time housing market entry, while Panel D compares the evolution of loan–to–value ratios over the life cycle. The solid lines report results for individuals facing a minimum contribution requirement, while the dashed lines report no minimum requirement. The dash–dotted and dotted line in Panel A report the results for the first percentile of the wealth distribution from 10,000 simulated paths for minimum and no minimum contributions, respectively. All other results are averages from 10,000 simulated paths.

From Panel A, mandatory contributions have the expected effect of causing individuals to accumulate more wealth. Most importantly, the wealth level of poorer individuals, as measured by the evolution of wealth for the first percentile, also increase. The quality of a pension system, among others, depends crucially on its ability to secure living standard of poorer individuals. In that regards, mandatory contributions are useful. As to be expected, from Panel B, this effect is mainly channeled through retirement savings. Whereas individuals not facing minimum contribution requirements significantly postpone building up retirement savings until their liquidity constraint is less binding, minimum contributions prepone contributions to the tax–deferred retirement account and lead to a higher total amount of retirement savings. While higher retirement savings alone are desirable, as they help in obtaining financial retirement security, they are accompanied by undesirable side effects. Specifically, from Panel C, mandatory minimum contribution requirements lead to deferred housing market entry. With mandatory minimum contributions, the average individual entering the housing market is 2.2 years older than without minimum contributions. That is, our results indicate that the liquidity channel dominates.[5]

Simultaneously, from Panel D, mandatory minimum contributions lead to higher loan–to–value ratios, which is consistent with the empirical evidence in Beshears, Choi, Laibson, Madrian, and Skimmyhorn (2022) that automatic enrollment in a retirement plan increases (second) mortgage debt. Without mandatory minimum contributions, individuals can finance their home with a higher share of equity, thus avoiding the

[5]For individuals with the set of preferences studied in our work, housing market entry are concentrated between the age of 30 to 40. Vestman (2019) documents that heterogeneity in preferences leads to higher dispersion in housing decisions.

Figure 2.3.1: Base case results



This figure depicts the base case results of our model. Panel A depicts the evolution of wealth over the life cycle and Panel B of retirement savings. Panel C depicts the distribution of first–time housing market entry, Panel D compares the evolution of loan–to–value ratios over the life cycle. The solid lines report results for individuals facing a minimum contribution requirement, while the dashed lines report no minimum requirement. The dash–dotted and dotted line in Panel A report the results for the first percentile of the wealth distribution from 10,000 simulated paths for minimum and no minimum contributions, respectively.

relatively high borrowing rate. Individuals facing minimum contribution requirements, on the other hand, are forced to borrow more for a comparable home. That is, these individuals invest at a relatively low rate in their tax–deferred retirement account and simultaneously borrow at a relatively high rate. In essence, these individuals incur the cost of the interest margin by borrowing their own money. From the individual's perspective, it may therefore be more desirable to first repay the relatively expensive mortgage before accumulating savings in the tax–deferred retirement account.[6]

Via deferred housing market entry, constraints on the consumption opportunities of young adults, and by forcing them into higher loan–to–value ratios, mandatory contribution requirements are associated with significant welfare costs. A 20–year old facing mandatory minimum contributions needs to be endowed with a 7.7% higher level of lifetime consumption and housing services to attain the same level of expected presently discounted lifetime utility as an individual not facing minimum contribution requirements.

2.4. A flexible retirement saving scheme

In this section, we propose a flexible retirement saving scheme. This scheme alleviates unintended side effects of mandatory minimum contributions while simultaneously ensuring that individuals save enough for retirement. Our flexible retirement saving scheme builds on the intuition that it does not matter whether individuals build up savings for retirement age in the tax–deferred retirement account, specifically labeled as pension savings, or via other means of saving, as long as they build up sufficient total wealth. Under our flexible retirement saving scheme, individuals are only required to save, if they have not built up sufficient total wealth. Individuals that are required to save do not have to build up savings in their tax–deferred retirement accounts, but can also choose to increase their wealth by (partly) repaying a mortgage or by building up wealth in the taxable account.

From a legal perspective, it is important to ensure that individuals cannot consume their retirement savings before attaining retirement age to prevent the government from having to sustain their living expenses during retirement. In reality, this could be achieved by requiring individuals to label sufficient savings as “for retirement” each

[6]Some pension systems around the world, including the German and the Swiss pension system, allow early withdrawals from tax–deferred retirement accounts, if they are used to support the acquisition of a home. Even though the exact conditions under which such early withdrawals escape penalty taxation vary between national tax laws, the general consensus is that only acquisitions of owner–occupied homes qualify. We also explored a setting, in which individuals are allowed to withdraw from their tax–deferred retirement accounts prior to retirement age when acquiring an owner–occupied home. In line with economic intuition, the opportunity of early withdrawals leads to earlier homeownership. However, unlike a pension system with mandatory contributions, the pension system with early withdrawals may not guarantee that individuals are able to build up sufficient retirement savings without imposing further constraints, because (levered) housing investments are subject to substantial volatility. Not ensuring sufficient retirement savings not only renders a direct comparison difficult, it is also undesirable from a legislative perspective, because it may trigger a need for transfer income. Imposing further constraints adds significantly to the already substantial computational complexity of our modeling framework. We therefore do not further pursue this case.

period, which may not be spent. These labels could, however, be moved between different asset classes. For instance, after an appreciation of a home and/or the reduction of mortgage debt, more home equity could be labeled as "for retirement", while wealth in the taxable account could be freed, thus giving the individual more financial flexibility.

The fundamental idea behind building up savings for retirement is to smooth consumption over the life cycle. To attain this goal, an increase in income should result in an increase in savings, whereas an increase in savings, e.g., due to high realized returns in the past, should decrease the need for saving. Building on this intuition, we advocate an (age–dependent) savings rule that depends on the individual's labor–to–wealth ratio. A high labor–to–wealth ratio indicates that an individual has low savings relative to current income and is more in danger of not being able to sustain the consumption level at retirement. To nevertheless reach the goal of smoothing consumption over the life cycle, the individual has to save more and consume less.

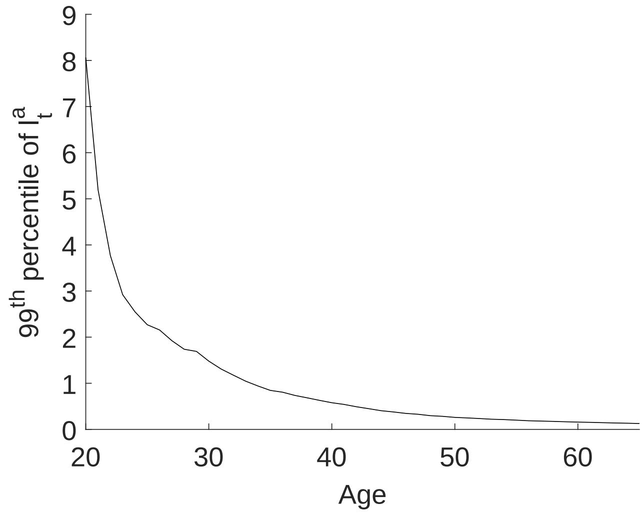
For every given age, we therefore impose a maximum labor–to–wealth ratio after expenses, $l_{t,MAX}$, to be attained. This maximum labor–to–wealth ratio should decrease with age reflecting that more savings should already have been built up. When W_t is the investor's wealth level at the beginning of period t, already including its labor income, L_t, earned in this period, and $E_t = C_t + m\left(h_t\right) Q_t H_t + n\left(Q_t, h_t, h_{t-1}\right)$ are non–durable expenses in period t, i.e., consumption, rents, maintenance, and transaction costs, its labor–to–wealth ratio after expenses in period t is $\frac{L_t}{W_t - E_t}$. Imposing a maximum labor–to–wealth ratio after expenses is then equivalent to imposing a maximum expenses–to–wealth ratio:

$$\frac{L_t}{W_t - E_t} \leq l_{t,MAX} \iff \frac{E_t}{W_t} \leq 1 - \frac{\frac{L_t}{W_t}}{l_{t,MAX}} \tag{2.4.1}$$

From equation (2.4.1), the maximum expenses–to–wealth ratio decreases in the beginning–of–period labor–to–wealth ratio, $\frac{L_t}{W_t}$, prior to expenses. That is, individuals with high initial labor–to–wealth ratios, $\frac{L_t}{W_t}$, face harsher constraints on their maximum expenses–to–wealth ratios, $\frac{E_t}{W_t}$, than individuals with low initial labor–to–wealth ratios, for whom the expenses constraint is more likely not to become binding.[7]. Our flexible retirement saving scheme does, however, not require new savings to be

[7]Technically, the right hand side of equation (2.4.1) can take a negative value—for instance, if an individual faces a dramatic decrease in total wealth. In such (very rare) cases, we set the individual's maximum expenses–to–wealth ratio to 10^{-4}. By restricting the individual's maximum expenses–to–wealth ratio, we are imposing a lower bound on its minimum savings, which may be even negative for individuals with very low labor–to–wealth ratios, i.e., individuals with very low labor–to–wealth ratios may even dissave

Figure 2.4.1: Evolution of the maximum labor–to–wealth ratio after expenses in the flexible retirement saving scheme



This figure depicts the 99^{th} percentile of the labor–to–wealth ratio after consumption, rents, maintenance, and transaction costs, l_t^a, for the case with mandatory minimum contributions, which is used as the age–dependent maximum labor–to–wealth ratio after expenses, $l_{t,MAX}$, in equation (2.4.1) in our flexible retirement scheme.

located in a tax–deferred retirement account. If more desirable, additional savings can, for instance, also be used to repay a relatively expensive mortgage before building up other forms of savings.

An important degree of freedom in our flexible retirement saving scheme is the choice and shape of the age–dependent maximum labor–to–wealth ratio, $l_{t,MAX}$, in equation (2.4.1). To facilitate a direct comparison with the setting with mandatory minimum contributions investigated in Section 2.3, the age–dependent maximum labor–to–wealth ratios, $l_{t,MAX}$, are constructed from the 10,000 simulations on the optimal paths under the minimum contribution requirement. Since ensuring the living standards of the poorest individuals is particularly important in pension system design, we compute the age–dependent maximum labor–to–wealth ratio, $l_{t,MAX}$, as the 99^{th} percentile from our 10,000 simulations on the optimal paths under the minimum contribution requirement.

From Figure 2.4.1, the age-dependent maximum labor–to–wealth ratio, $l_{t,MAX}$, shows the expected decline with age.[8] From equation (2.4.1) higher levels of $l_{t,MAX}$ imply less binding expenses constraints. Hence, in line with economic intuition that

[8]Unlike the labor–to–wealth ratio prior to expenses, $\frac{L_t}{W_t}$, the 99^{th} percentile of the labor–to–wealth ratio after expenses, $\frac{L_t}{W_t - E_t}$, can take values larger than one.

younger individuals can still build up sufficient savings over a longer savings horizon, their higher levels of $l_{t,MAX}$ imply less tightly binding expenses constraints than for older individuals.

The expenses constraint imposed by equation (2.4.1) can unfold its effect in two economically different ways. First, it can be effectively binding, thus increasing savings by force. Second, it can stimulate individuals to build up sufficient savings in advance to avoid ending up facing a tightly binding expenses constraint. Prior to age 30, the expenses constraint binds on more than 90% of the 10,000 simulated paths, indicating that without the expenses constraint individuals would want to choose higher expenses. That is, prior to age 30, the expenses constraint mainly works through the first channel. Beyond the age of 30, the expenses constraint only binds on 1.3% of our simulated paths. That is, beyond the age of 30, the second channel becomes more prevalent.

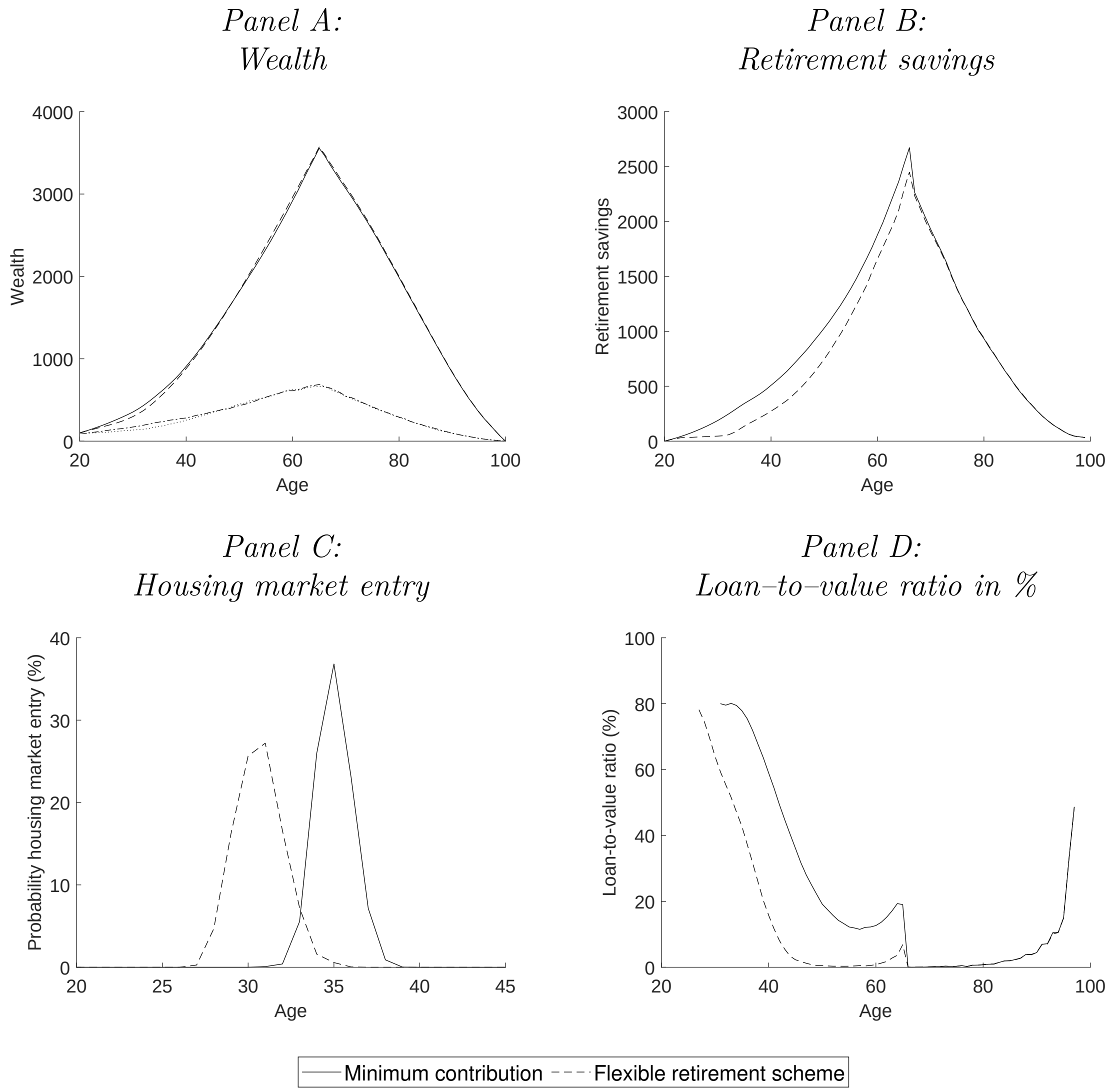
Figure 2.4.2, in a similar fashion as Figure 2.3.1, depicts the implications of our flexible retirement saving scheme relative to the case with mandatory retirement savings on the evolution of wealth (Panel A), retirement savings (Panel B), first–time housing market entry (Panel C), and loan–to–value ratios (Panel D) over the life cycle. More technically, for the flexible retirement saving scheme, we solve the optimization problem presented in Section 2.2.4 subject to equations (2.2.2) and (2.4.1).

Overall, the results in Figure 2.4.2 show that our flexible retirement saving scheme leads to an evolution of wealth and retirement savings very similar to that of the pension system with mandatory contributions. Under our flexible retirement saving scheme, total savings at retirement age are almost exactly as high as under the pension system with mandatory contributions—both, on average and for the first percentile.

Panel C shows that our flexible retirement saving scheme prepones first–time housing market entry relative to the system with mandatory contributions to the tax–deferred retirement account. The average housing market entrant is 4.3 years younger when trading under our flexible scheme, implying that our flexible retirement saving scheme even prepones first–time housing market entry relative to the system with no minimum contribution. Panel D shows that our flexible retirement saving scheme also allows our individuals to repay their relatively expensive mortgages at a faster rate. After all, repaying the relatively expensive mortgage before saving in risk–free assets in the tax–deferred retirement account is an obvious arbitrage opportunity that our flexible retirement saving scheme allows individuals to exploit more efficiently than under the pension system with mandatory contributions.

The opportunities available for more flexible saving requirements are early home-ownership and lower loan–to–value ratios, both of which lead to positive welfare

Figure 2.4.2: Flexible retirement saving scheme results



This figure compares results under our flexible retirement saving scheme (dashed line) with results under a pension system with mandatory minimum contributions (solid line). Panel A depicts the evolution of wealth over the life cycle and Panel B of retirement savings. Panel C depicts the distribution of first–time housing market entry, while Panel D compares the evolution of loan–to–value ratios over the life cycle. The dash–dotted and dotted line in Panel A report the results for the first percentile of the wealth distribution from 10,000 simulated paths for minimum contributions and the flexible scheme, respectively. All other results are averages from 10,000 simulated paths.

effects. Compared to the base case setting with mandatory minimum contributions, welfare increases by 2.3%. That is, a 20–year old obliged to make contributions under the minimum contribution pension scheme needs to be endowed with a 2.3% higher level of lifetime consumption and housing services, to attain the same level of expected presently discounted lifetime utility as an individual trading under our flexible retirement saving scheme.

2.5. Conclusion

In this study, we explore the implications of minimum contribution requirements for tax–deferred retirement accounts found in many countries around the world in a realistically calibrated life cycle model. Our results show that minimum contribution requirements have the unintended side effect of delaying housing market entry and of forcing individuals to sustain higher loan–to–value ratios. That is, individuals are forced to pay the higher borrowing rate on their debt, while simultaneously investing at a lower rate in the tax–deferred retirement account. In other words, individuals essentially borrow their own money and face the cost of the interest margin.

We therefore propose a new flexible retirement saving scheme, which ensures that individuals build up sufficient savings. Intuitively, it should not matter whether individuals build up savings in a tax–deferred retirement account, in a taxable account, or through home equity, as long as they build up sufficient total wealth. Our flexible retirement saving scheme builds on this intuition and only forces individuals to build up savings, if they are undersaving. It does, however, not force individuals to build up (illiquid) tax–deferred retirement savings, if, e.g., repaying an expensive mortgage first is more desirable. Compared to the system with mandatory minimum contributions, the flexible retirement saving scheme results in a similar evolution of wealth and retirement savings, but prepones first–time housing market entry, lowers the loan–to–value ratios, and increases welfare.

In a nutshell, the widespread institutional constraint requiring individuals to contribute a minimum fixed fraction of their income to an illiquid pension savings account throughout their working life induces a suboptimal consumption profile over the life cycle, delayed housing market entry, and forces homeowners into accepting high loan–to–value ratios. Our flexible retirement saving scheme is an attempt to mitigate these issues. Further investigations of the design of optimal pension plans is a fruitful subject for future research.

References

Amromin, G. (2003). "Household Portfolio Choices in Taxable and Tax-Deferred Accounts: Another Puzzle?" *European Finance Review* 7. (3), 547–582.

Arias, E. and J. Xu (2018). "United States Life Tables, 2015. National Vital Statistics Report." *National Center for Health Statistics* 67. (7).

Beshears, J., J. J. Choi, D. Laibson, B. C. Madrian, and W. L. Skimmyhorn (2022). *Journal of Finance* 77. (1), 403–447.

Best, M. C., J. S. Cloyne, E. Ilzetzki, and H. J. Kleven (2020). "Estimating the Elasticity of Intertemporal Substitution Using Mortgage Notches." *Review of Economic Studies* 87. (2), 656–690.

Bourassa, S. C., D. R. Haurin, J. L. Haurin, M. Hoesli, and J. Sun (2009). "House Price Changes and Idiosyncratic Risk: The Impact of Property Characteristics." *Real Estate Economics* 37. (2), 259–278.

Carroll, C. D. (1997). "Buffer-Stock Saving and The Life Cycle/Permanent Income Hypothesis." *Quarterly Journal of Economics* 112. (1), 1–55.

Case, K. E. and R. J. Shiller (1989). "The Efficiency of the Market for Single-Family Homes." *American Economic Review* 79. (1), 125–137.

Cashin, D. and T. Unayama (2016). "Measuring Intertemporal Substitution in Consumption: Evidence from a VAT Increase in Japan." *Review of Economics and Statistics* 98. (2), 285–297.

Claus, J. and J. Thomas (2001). "Equity Premia as Low as Three Percent? Evidence from Analysts' Earnings Forecasts for Domenstic and International Stock Markets." *Journal of Finance* 56. (5), 1629–1666.

Cocco, J. (2005). "Portfolio Choice in the Presence of Housing." *Review of Financial Studies* 18. (2), 535–567.

Cocco, J., F. J. Gomes, and P. J. Maenhout (2005). "Consumption and Portfolio Choice over the Life Cycle." *Review of Financial Studies* 18. (2), 491–533.

Dahlquist, M., R. Vestman, and O. Setty (2018). "On the Asset Allocation of a Default Pension Fund." *Journal of Finance* 73. (4), 1893–1936.

Dammon, R. M., C. S. Spatt, and H. H. Zhang (2004). "Optimal Asset Location and Allocation with Taxable and Tax-Deferred Investing." *Journal of Finance* 59. (3), 999–1037.

Dimson, E., P. Marsh, and M. Staunton (2002). *Triumph of the Optimists: 101 Years of Global Investment Returns*. Princeton University Press.

Epstein, L. G. and S. E. Zin (1989). “Substitution, Risk Aversion and Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework.” *Econometrica* 57. (4), 937–969.

Feldstein, M. (1985). “The Optimal Level of Social Security Benefits.” *Quarterly Journal of Economics* 100. (2), 303–320.

Fischer, M. and M. F. Gallmeyer (2017). “Taxable and Tax-Deferred Investing with the Limited Use of Losses.” *Review of Finance* 21. (5), 1847–1873.

Fischer, M. and N. Khorunzhina (2019). “Housing Decision with Divorce Risk.” *International Economic Review* 60. (3), 1263–1290.

Fischer, M. and M. Stamos (2013). “Optimal Life Cycle Portfolio Choice with Housing Market Cycles.” *Review of Financial Studies* 26. (9), 2311–2352.

Garlappi, L. and J. Huang (2006). “Are Stocks Desirable in Tax-Deferred Accounts?” *Journal of Public Economics* 90. (12), 2257–2283.

Gayle, W. R. and N. Khorunzhina (2018). “Micro-Level Estimation of Optimal Consumption Choice with Intertemporal Nonseparability in Preferences and Measurement Errors.” *Journal of Business & Economic Statistics* 36. (2), 227–238.

Huang, J. (2008). “Taxable and Tax-Deferred Investing: A Tax Arbitrage Approach.” *Review of Financial Studies* 21. (5), 2173–2207.

Kiyotaki, N., A. Michaelides, and K. Nikolov (2011). “Winners and Losers in Housing Markets.” *Journal of Money, Credit and Banking* 43. (2–3), 255–296.

Kraft, H., C. Munk, and S. Wagner (2018). “Housing Habits and Their Implications for Life-Cycle Consumption and Investment.” *Review of Finance* 22. (5), 1737–1762.

Larsen, L. S. and C. Munk (2023). “The Design and Welfare Implications of Mandatory Pension Plans.” *Journal of Financial and Quantitative Analysis* online, 1–30.

Marekwica, M., A. Schaefer, and S. Sebastian (2013). “Life Cycle Asset Allocation in the Presence of Housing and Tax-Deferred Investing.” *Journal of Economic Dynamics and Control* 37. (6), 1110–1125.

Mehra, R. and E. C. Prescott (1985). “The Equity Premium: A Puzzle.” *Journal of Monetary Economics* 15. (2), 145–162.

Moser, C. and P. Olea de Souza e Silva (2019). “Optimal Paternalistic Savings Policies.” working paper, available at https://ssrn.com/abstract=2959844.

OECD (2021a). *Pensions at a Glance 2021: OECD and G20 Indicators*. OECD Publishing, Paris.

Pakoš, M. (2011). “Estimating Intertemporal and Intratemporal Substitutions When Both Income and Substitution Effects are Present: The Role of Durable Goods.” *Journal of Business & Economic Statistics* 29. (3), 439–454.

Samuelson, P. A. (1975). "Optimal Social Security in a Life-Cycle Growth Model." *International Economic Review* 16. (3), 539–544.

Schlafmann, K., O. Setty, and R. Vestman (2022). "Designing Pension Plans According to Consumption-Savings Theory." working paper, available at https://cepr.org/system/files/2022-09/Kathrin%20Schlafmann_Designing%20Pension%20Plans%20According%20to%20Consumption-Savings%20Theory.pdf.

Shoven, J. B. and C. Sialm (2003). "Asset Location in Tax-Deferred and Conventional Savings Accounts." *Journal of Public Economics* 88. (1), 23–38.

Vestman, R. (2019). "Limited Stock Market Participation Among Renters and Homeowners." *Review of Financial Studies* 32. (4), 1494–1535.

Yao, R. and H. H. Zhang (2005). "Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints." *Review of Financial Studies* 18. (1), 197–239.

Appendix 2.A Solution of the optimization problem

We simplify our optimization by normalizing with aggregate wealth, W_t, that effectively belongs to the individual and does not fall to tax authorities upon withdrawal from the tax–deferred retirement account. Hence, in the normalized optimization problem, $c_t = \frac{C_t}{W_t}$ and $q_t = \frac{Q_t H_t}{W_t}$ are the decision variables that affect utility from immediate consumption of both non–durable and durable consumption. Defining $V_t(X_t) = v_t(x_t)\frac{W_t/(1+i)^t}{\left(H_t/(1+i)^t\right)^{\theta}}$, the individual's optimization problem can be rewritten as:

$$\max_{\{c_t,z_t,s_t,b_t,\alpha_{R,t},q_t,h_t\}} v_t(x_t) = \tag{2.A.1}$$

$$\left((1-\beta)\left(c_t^{1-\theta}\left(q_t\left(1+\zeta h_t\right)\right)^{\theta}\right)^{1-\frac{1}{\psi}} + \beta\left(f(t)\,\mathbb{E}_t\left[\left(v_{t+1}(x_{t+1})\frac{\frac{W_{t+1}}{W_t(1+i)}}{\left(\frac{H_{t+1}}{H_t(1+i)}\right)^{\theta}}\right)^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right)^{\frac{1}{1-\frac{1}{\psi}}},$$

subject to

$$w_{R,t-} = w_{R,(t-1)+}\left(\alpha_{R,t-1}G_{R,t-1} + \left(1-\alpha_{R,t-1}\right)R_R\right) \tag{2.A.2}$$

$$w_{R,t+} = w_{R,t-} + z_t \tag{2.A.3}$$

$$\begin{aligned} w_{T,t} &= 1 - w_{R,t-} - (1-\delta)h_{t-1}\frac{Q_{t-1}H_t}{W_t},\\ &= c_t + s_t + b_t + m(h_t)\,q_t + f_p q_t\left(h_t(1-h_t) + h_t h_{t-1}\chi_{\left\{\frac{Q_{t-1}H_t}{W_t}\neq q_t\right\}}\right) + h_t q_t \\ &\quad - h_{t-1}\frac{Q_{t-1}H_t}{W_t} + z_t(1-\tau) \end{aligned} \tag{2.A.4}$$

$$c_t, q_t > 0, s_t \geq 0, \alpha_{R,t-1} \in [0,1], w_{R,t+} \geq 0, \tag{2.A.5}$$

in which $w_{R,t-} = \frac{W_{R,t-}}{W_t}$, $w_{R,t+} = \frac{W_{R,t+}}{W_t}$, $w_{T,t} = \frac{W_{T,t}}{W_t}$, $z_t = \frac{Z_t}{W_t}$, $b_t = \frac{B_t}{W_t}$, and $s_t = \frac{S_t}{W_t}$. The new vector of state variables is

$$x_t = \left[t, w_{R,t-}, \frac{L_t}{W_t}, \frac{Q_{t-1}H_t}{W_t}, h_{t-1}\right]. \tag{2.A.6}$$

We discretize the continuous state variables. The expectation in equation (2.A.1) is computed using Gaussian quadrature. Parallel computing is used to expedite our computations.

Appendix 2.B PSID data and present value of Social Security benefits

This appendix explains in more detail, which PSID data we use in our study and how we calculate the present value of Social Security payments (SSP), individuals can expect to make when working until the age of 65, and being retired at the age of 66.

For our data analyses, we use the biannually available PSID data after the financial crisis from 2011 to 2019, i.e., PSID data from 2011, 2013, 2015, 2017, and 2019. We only consider heads of households (henceforth, individuals) who are high school graduates to ensure that we only use the wealth of the same individuals whose labor income process we consider. Furthermore, we adjust total wealth and labor income for inflation with an annual rate of 3.7% as in our model.

To calculate the growth of median wealth, we proceed as follows: first, we restrict the inflation–adjusted total wealth of individuals to be positive. Second, we construct two subsamples, one containing all individuals aged 20 and one containing all individuals aged 65. Third, we calculate the median wealth of the total wealth of individuals for the respective subsamples. We find that the median wealth of individuals aged 20 is USD 10,000 and for those aged 65 it is USD 110,000.

To calculate the present value of Social Security benefits for individuals working until the age of 65 and being retired at age 66 we proceed as follows: first, we use inflation–adjusted labor income to calculate the average indexed monthly earnings (AIME). We obtain AIME by calculating the median earnings for each age from 26 to 60, taking the mean of the median earnings, and dividing this value by 12. Our result is USD 3,291.45.

Second, we determine the basic Social Security retirement benefit when retiring at age 62, the earliest age at which an individual is generally eligible for Social Security retirement benefits, by calculating the so–called primary insurance account (PIA): for 2019, the PIA is the sum of 90% of the first USD 926 of the AIME, 32% of the AIME exceeding USD 926 but below or equal to USD 5,785, and 15% of the AIME exceeding USD 5,785. Therefore, our PIA is USD 1,590.34.

Third, as the individuals in our model are retired at age 66 and not already at age 62, we consider cost–of–living adjustments (COLA). We assume the annual COLA to be 1.69%, which is the mean of all COLAs between 2011 and 2019. When taking COLA into account, the PIA for individuals retired at age 66 is USD 1,700.61. Based on these values, we compute a replacement ratio in the data of: $\frac{1{,}700.61}{3{,}291.45} = 51.7\%$.

Fourth, to calculate the present value of SSP, PV_{SSP}, we assume an interest rate of 3.7% (the same as in our model) and use the following formula:

$$PV_{SSP} = \text{average labor income} \cdot \text{replacement ratio} \cdot \sum_{i=0}^{\omega-66} v^{i-1} \cdot_i p_{66}, \tag{2.B.1}$$

in which ω denotes the technical maximum life time assumed in the life table (100 years), v the discount factor, and ${}_ip_{66}$ the probability that a person aged 66 survives the next i years. Our result for the present value of the annuity is USD 250,035.70. We do not consider any taxes as Social Security income is (up to certain limits) tax free in the U.S. When adding up the individual's median total wealth right before retirement, i.e., for individuals aged 65, and the present value of SSP, we observe that median total wealth grows by a factor of 36.

Chapter 3

Fractional Homeownership and its Impact on Life Cycle Portfolio Choice*

*This chapter is single-authored. I am grateful for helpful comments and suggestions from Stephan Eckstein, Marcel Fischer, Albert Grafe, Jens Jackwerth, Natascha Jankowski, Bjarne Astrup Jensen, Axel Kind, Torsten Kleinow, Manuel Leininger, Kathrin Schlafmann, Steffen Sebastian, Simon Stehle, and seminar participants at the Universities of Amsterdam, Konstanz, and Regensburg. I would like to thank Ingrid Klarhandler and Christoph Schäfer for excellent research assistance.

3.1. Introduction

Acquiring real estate is among the biggest financial challenges individuals face throughout their lives. In recent years, the housing affordability crisis rendered purchasing properties even more difficult (Anacker (2019))—in 2022, housing affordability was at its lowest level since the 2008 financial crisis due to changes in home prices and interest rates applicable to mortgage debt. Nonetheless, the acquisition of real estate was among the primary economic objectives of 75% of the U.S. population.[1]

Due to the unabated desire for homeownership and the housing affordability crisis, alternative financial products for financing real estate are developed. In our manuscript, we study such an alternative by investigating the implications of access to *fractional homeownership* on individuals' lifetime consumption, savings, and housing decisions. Fractional homeownership means that two parties—an individual and an institutional investor—share full ownership of a property. The individual lives in the property full–time and makes periodic rent payments to the institutional investor who sees the property solely as an investment vehicle. The amount of the rent payments depends on the value of the home and the fractional ownership structure that can be time–varying as the individuals have the right to adjust the fraction owned in every period.

Using a rich and realistically calibrated life cycle model of optimal consumption, savings, and housing decisions, we shed, among others, light on the impact of access to fractional homeownership on individuals' housing market entry and exit as well as their loan–to–value ratios. Our results suggest that primarily young, liquidity constrained individuals and older individuals who are in the phase of decumulating their savings make use of fractional homeownership to finance their property. Further, this novel financing option leads to earlier housing market entry, later housing market exit, lower loan–to–value ratios and to reduced moving activity of elderly individuals; all in comparison to the setting in which the individuals' rent–versus–own decisions are binary, i.e., the setting in which individuals can either be renters or owners.[2]

[1] An evolution of housing affordability can be downloaded from https://www.atlantafed.org/. https://www.cnbc.com/ summarizes surveys on individuals' economic objectives. Both sources last accessed on October 10, 2022.

[2] To be clear, we define the individuals' loan–to–value ratios (LTVs) as fractions of their outstanding mortgage debt and the full value of their property, but we do not allow them to borrow more than 80% of the value of the property they effectively own (see Section 3.3.1). We use this approach for two reasons: (i) The borrowing restriction ensures that individuals do not end up having unsecured loans. (ii) Our calculation of the LTVs highlights that, when living in equally valuable homes, individuals with access to fractional homeownership have lower mortgage debt than those who do not. Yet, we report the fractions of individuals' outstanding mortgage debt and the value of the home they effectively own in Section 3.4; these ratios are also lower than the LTVs of individuals without access to fractional homeownership.

On the real estate market, the concept of fractional homeownership is already well–established for second residences. Several individuals own fractions of a property and the time they are allowed to spend in the property depends on the fraction owned, i.e., individuals owning a larger share of the property are allowed to use it more days per year.

In our manuscript, we study a slight alteration of the concept described above as we study fractional homeownership contracts between individuals who want to live in the property permanently and institutional investors for whom the property is purely an investment vehicle. As compensation for not living in the property the institutional investor receives periodical rent payments from the individual; the exact amount of the rent payments depends on the fraction owned by the institutional investor and the value of the property. The market for this variant of fractional homeownership is recently emerging with the first contract being signed in the U.S. in 2023.[3] Further, this concept is related to cooperative housing which is a common form of housing in, for example, Denmark, Norway, and Sweden, but also in larger cities in the U.S. (see, e.g., Balmer and Gerber (2018); Sørvoll and Bengtsson (2018); Sazama (2000)).

Fractional homeownership contracts have advantages for both parties—for the individual as well as for the institutional investor. It is well–known that young individuals often face harsh liquidity constraints (Gourinchas and Parker (2002)). Consequently, they struggle, for example, with accumulating sufficient liquid savings for the down payment needed to acquire real estate. We find that access to fractional homeownership alleviates these liquidity issues and thereby enables individuals to live in valuable properties without having to take out very high mortgages, i.e., without facing very high loan–to–value ratios. Another rather recently developed product that is worth mentioning in this context as it also aims at increasing housing affordability and alleviating young individuals' liquidity constraints, are so–called interest–only (IO) loans. This type of loans was introduced in several countries such as Denmark, Sweden, and the Netherlands in the 2000's and is studied, for example, by Amromin et al. (2018), Bian et al. (2018), and Bäckman and Lutz (2020). The development of concepts such as fractional homeownership and IO loans, that facilitate the acquisition of real estate matter as, for example, Goodman and Mayer (2018) document that even after the financial crisis real estate is a valuable asset for individuals and that home acquisitions should be stimulated.

For institutional investors, such as pension funds, real estate is an important asset class. Engaging in fractional homeownership contracts has substantial diversification

[3] A press release reporting the first settlement of such a contract can be found on https://www.globenewswire.com/, last accessed April 5, 2023.

benefits in comparison to owning, for example, one large apartment building. Additionally, fractional homeownership alleviates the principal–agent problem present in many real estate transactions (see, e.g., Arnold (1992); Daneshvary and Clauretie (2013); Furst and Evans (2017)). When residents own at least a certain share of their accommodation, they are likely to have stronger incentives to maintain the property's value.

With our manuscript, we contribute to two different strands of literature. First, we contribute to the literature on life cycle portfolio choice. A lot of previous work on life cycle portfolio choice focuses on optimal asset allocation decisions (see, e.g., Cocco et al. (2005); Benzoni et al. (2007)) and optimal asset location decisions (see, e.g., Dammon et al. (2004); Fischer and Gallmeyer (2017)). Life cycle models studying optimal consumption, savings, and housing decisions generally assume that the individuals' rent–versus–own decisions are binary, implying that individuals have to fully fund their owner–occupied homes by taking out a mortgage or by accumulating sufficient savings (see, e.g., Cocco (2005); Kraft and Munk (2011); Fischer and Stamos (2013); Schlafmann (2021)). We are, to the best of our knowledge, the first to soften the assumption of the rent–versus–own decision being binary by studying the impact of access to fractional homeownership on life cycle portfolio choice.

Second, we contribute to the literature on fractional ownership of different asset classes. Whitaker and Kräussl (2020), for example, study fractional ownership of art and introduce a novel fractional equity structure for artists' oeuvres. Bogyrbayeva et al. (2021) focus on fractional ownership of autonomous vehicles and provide a solution for the winners determination problem in autonomous vehicles' auctions. There are several papers studying co–ownership in the energy sector of different countries: Johansen and Emborg (2018) focus, for example, on Denmark and Roth et al. (2018) on Germany.

Three papers studying fractional ownership of real estate are Cauley et al. (2007), who study optimal asset allocation when individuals have/do not have the opportunity to sell a fractional interest in their home, Lowies et al. (2018), who identify the demographic structure of fractional homeowners in Australia, and Rong (2023), who discusses how fractional homeownership could be implemented in practice. More specifically, Cauley et al. (2007) document that homeownership constrains individuals' asset allocation decisions. Giving individuals the opportunity to sell fractions of their homes alleviates the constraint homeownership imposes on asset allocation and thus leads to significant welfare gains. Lowies et al. (2018) conduct an experiment and find that younger individuals are rather willing to become fractional homeowners than older individuals and Rong (2023) highlights that tokenization of properties might facilitate entering the housing market, alleviate liquidity problems, and avoid foreclosures of distressed properties.[4] We contribute to this strand of literature by theoretically studying the impact of access to fractional ownership of real estate on optimal consumption, savings, and housing decisions. We extend the model of Cauley et al. (2007) by, among others, explicitly modeling rent–versus–own decisions and mortgages. Further, our theoretical results support the findings of Lowies et al. (2018) as in our manuscript young individuals often finance their homes via fractional ownership. While Rong (2023) focuses on the implementation of fractional homeownership from a legislative perspective, we focus on the impact of access to fractional homeownership on individuals' decision making over the life cycle.

The remainder of this manuscript is structured as follows: Section 3.2 provides empirical evidence for the housing affordability crisis and discusses the individuals' desire to acquire real estate as well as the details of fractional homeownership contracts. Section 3.3 outlines our model of optimal consumption, savings, and housing decisions as well as its calibration. The results are presented in Section 3.4. Finally, Section 3.5 concludes.

3.2. Housing affordability/desire and fractional homeownership contracts

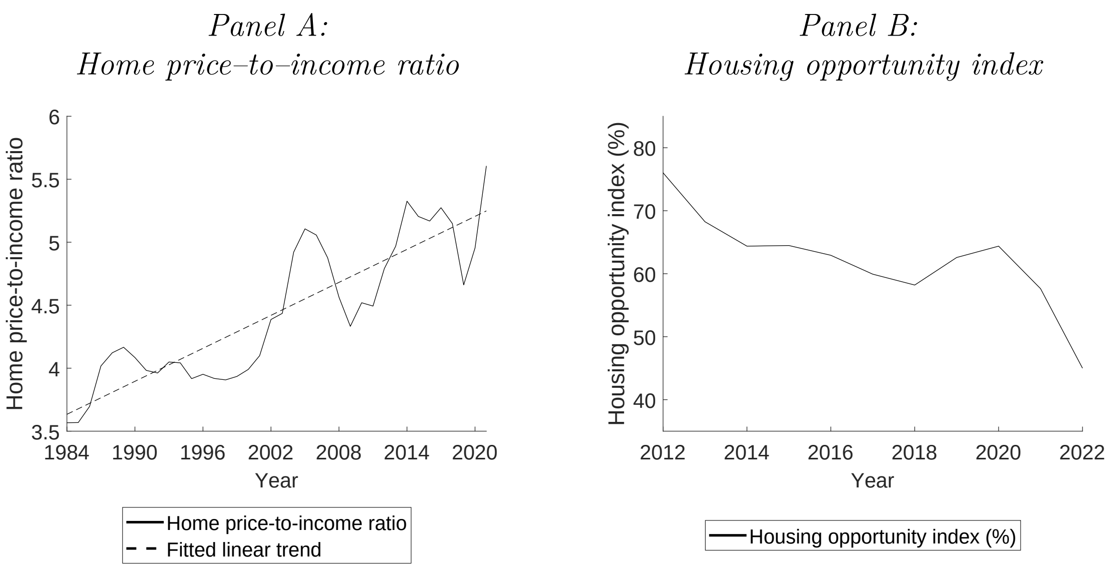
In this section, we (i) motivate the necessity for alternative real estate financing options by analyzing housing affordability and desire in the U.S. and (ii) discuss the design of fractional homeownership contracts.

3.2.1. Housing affordability/desire

In 2021, 49% of the U.S. population saw the availability of affordable housing as a major problem in their community; a 10 percentage point increase in comparison to 2018, when only 39% reported affordable housing to be a major problem (Schaeffer (2022)). The housing affordability crisis also received substantial media attention with, for example, CNBC writing "Millennials and Gen Zers do want to buy homes—they just can't afford it, even as adults" and the New York Times trying to identify the drivers of the housing affordability crisis in an article titled "Is Wall Street Really to

[4] Tokenization of real estate means that a property is split into several parts (the so–called tokens). These tokens are stored on a blockchain and owning a token of a property results in the same rights and obligations as directly owning the property. Gupta et al. (2020) and Konashevych (2020) provide a thorough overview of real estate tokenization.

Figure 3.2.1: Housing affordability in the U.S. over time



This figure provides empirical evidence of the current U.S. housing affordability crisis. Panel A depicts the evolution of the annual median U.S. home price–to–income ratio (solid line) and a fitted linear trend (dashed line) from 1984 to 2021. The ratio is constructed as the fraction of annual median sales prices of houses sold in the U.S. and the corresponding annual U.S. median household income. Panel B depicts the U.S. NAHB/Wells Fargo housing opportunity index from 2012 to 2022. It indicates how many percent of homes sold in a given year could have been afforded by a household earning median income in the same year assuming that standard mortgage underwriting criteria apply. As there have been structural changes in the construction of the housing opportunity index, no longer time series than the one from 2012 to 2022 is available. Data for the annual median sales prices of houses sold in the U.S. and for the annual U.S. median household incomes can be downloaded from https://fred.stlouisfed.org/series/MSPUS, last retrieved May 17, 2023, and https://fred.stlouisfed.org/, last retrieved November 2, 2022, respectively. Data for the housing opportunity index can be downloaded from https://www.nahb.org/, last retrieved March 29, 2023.

Blame for the Affordable Housing Crisis?".[5] The latter article concludes that there are many drivers of the housing affordability crisis, among others, the high inflation rate and the housing shortage, and that institutional investors cannot be held fully responsible for the crisis.

Figure 3.2.1 provides further empirical evidence of the housing affordability crisis by showing the evolution of the median home price–to–income ratio and the evolution of the housing opportunity index, respectively. Panel A depicts the evolution of the annual median U.S. home price–to–income ratio from 1984 to 2021 by the solid line and a fitted linear trend by the dashed line. The ratio is constructed as the fraction of annual median sales prices of houses sold in the U.S. (accessible via https://fred.stlouisfed.org/, last retrieved May 17, 2023) and the annual U.S. median

[5]The articles can be downloaded from https://www.cnbc.com/ and https://www.nytimes.com/, respectively; last accessed November 3, 2022.

household income (accessible via https://fred.stlouisfed.org/, last retrieved November 2, 2022). Economically speaking, a home price–to–income ratio of x indicates that a single family home in the U.S. costs x–times the annual U.S. median household income.

The home price–to–income ratio was particularly high for the first time before the beginning of the financial crisis in 2007 and reached its maximum in the last 35 years at the end of 2021 when an average single family home cost more than 5–times the annual median U.S. household income. From the fitted linear trend, we see that the home price–to–income ratio is increasing over time, i.e., the faster increase in home prices than in income levels renders it financially more challenging for individuals to acquire real estate.

Panel B of Figure 3.2.1 depicts the U.S. housing opportunity index from 2012 to 2022 (accessible via https://www.nahb.org/, last retrieved March 29, 2023). It indicates how many percent of homes sold in a given year could have been afforded by a household earning median income in the same year assuming that standard mortgage underwriting criteria apply. We observe a sharp decrease in the index' value in the last three years; in 2022 only 45% of the homes sold in the U.S. were affordable for households with median income.

Despite these affordability issues, both, individuals and policymakers, see homeownership as a desirable goal. On the individuals' side, there are various surveys providing evidence of individuals still aiming at becoming homeowners. A survey from Ostrowski (2022), for example, finds that for 74% of the U.S. population owning a home has the highest economic priority. On the policymakers' side, the U.S. government, for example, has special programs that support individuals who aim at acquiring real estate such as government–backed home loans and homeownership vouchers.[6] Further, several countries, such as the Netherlands, Sweden, and the United Kingdom, allow for a deduction of mortgage interest rates (OECD (2021b)) and Layton (2022) proposes initiatives that could be implemented by policymakers to increase homeownership rates, such as reducing the labor income tax burden.

Taken together, the housing affordability crisis and the desire for homeownership increase the demand for novel financial products that facilitate the acquisition of real estate. In this manuscript we therefore study such a product, namely fractional homeownership, and outline the details of such a contract in the next section.

[6] Source: https://www.usa.gov/, last accessed January 15, 2023.

3.2.2. Contract details

The concept of fractional homeownership is not entirely new. There is already a well–developed fractional homeownership market for second residences. Several real estate firms, such as "*Pacaso*" and "*Lazazu*", have specialized on setting up co–ownership contracts between different individuals who are interested in acquiring a second residence.[7] While the exact contract details may vary, most contracts share the feature that the time individuals are allowed to spend in their second residence is proportional to the share of the property they own.

In this manuscript we study a slightly different market, that is currently emerging—the fractional homeownership market for owner–occupied residences. In this variant of fractional homeownership, two parties share full ownership of a property: (i) an individual, i.e., a private investor who is the owner–occupier of the property, and (ii) an institutional investor who sees the real estate unit solely as an investment opportunity. More specifically, we study the impact of access to fractional homeownership of owner–occupied residences on individuals' life cycle portfolio choice.

The fractional homeownership contract is structured as follows: The individual owns only h% of the property but lives in it permanently. The institutional investor owns $(100 - h)$% and sees the property solely as an investment vehicle. As the institutional investor does not derive any utility from living in the property, the individual pays (as compensation) a rent rate to the institutional investor. The exact amount of the rent depends on the value of the property and the fractional homeownership structure. Costs associated with the property, such as maintenance costs, are split proportionally to the ownership structure, i.e., the party who owns h% of the property pays h% of the costs, whereas the party who owns $(100 - h)$% pays $(100 - h)$% of the costs.

We assume that individuals can adjust the fraction of the property owned in every period by selling fractions to/buying fractions from the institutional investor. As compensation for these strong rights the individuals gain when entering the contract, we require them to pay a "fractional homeownership premium" that depends on the value of the home to the institutional investor whenever acquiring a new home (not when solely adjusting the fraction owned).

There are several advantages for both parties when entering such a fractional homeownership contract. On the individuals' side, we know that young individuals

[7] A detailed description of the fractional homeownership products these companies offer can be found on https://www.pacaso.com/ and https://www.lazazu.com/, respectively; all online sources last accessed November 16, 2022.

are often liquidity constrained (Gourinchas and Parker (2002)). Hence they struggle, among others, with accumulating savings needed for the down payment of a mortgage. Access to fractional homeownership decreases the level of savings needed for the down payment and also reduces individuals' total mortgage debt. Other advantages for the individuals are, among others, (i) the reduced exposure to house price volatility as the share of total wealth invested in the housing market is smaller and (ii) the possibility of acquiring a larger home without going on compromise with the consumption levels that can be attained.

On the institutional side, we observe that real estate is becoming an increasingly important asset class. Acquiring fractions of homes in different geographic areas leads to an exceedingly well–diversified portfolio than when acquiring, for example, one large apartment building. Additionally, the principle–agent problem inherent in many rental transactions is alleviated (see, e.g., Arnold (1992) and Furst and Evans (2017) for an overview of the principle–agent problem). Individuals who own at least a certain share of the property they live in might have stronger incentives to maintain the property's value than those who are only renting the property. One type of institutional investor, who could be interested in entering such a contract, is pension funds as Wood (2016) documents that pension funds are by far the largest institutional investors in the real estate industry. About 80% of all private and public pension funds have some real estate investments in their portfolio and the capital invested by pension funds is about 35% of the total capital invested into real estate by institutional investors.

Honestly speaking, there are not only advantages but also downsides that must not be neglected. A disadvantage for the institutional investor is, for example, our assumption, that the institutional investor is required to sell/buy shares from the individual whenever the individual considers it to be optimal to adjust the fraction owned. However, we believe (i) that the "fractional homeownership premium", which is introduced above, creates incentives for the institutional investor to nonetheless enter this type of contract as this premium provides the institutional investor with additional liquidity at the time an individual is acquiring a home and leads to the institutional investor selling the property above market value and (ii) that once the institutional investor owns a sufficiently large number of properties and the individuals who own fractions of these properties are heterogeneous enough, the market might converge to an equilibrium in which individuals who want to buy and those who want to sell fractions of their homes balance each other—given no herd behavior of market participants.[8]

[8]A very interesting direction of future research would be to explicitly model the preferences of the institutional investor and then maximize the aggregate utility of the private and institutional investor.

Moreover, there are high information costs for the institutional investor and it might also prefer to be the majority owner to make important property–related decisions on its own. The latter might also be a preference of the private investor. Given the advantages outlined above, we nonetheless consider fractional homeownership to be an interesting new financial product that deserves to receive attention from research and therefore study it in this manuscript.

Now, assuming that both parties (the individual and the institutional investor) are interested in entering a fractional homeownership contract, the precise contract details such as how to deal with different preferences for maintenance work, have to be discussed.

The prevailing solution to protect the parties engaging in fractional homeownership contracts in case of such disagreements is that they found an LLC ("Limited Liability Company"). The LLC then owns the property whereas the parties of the fractional homeownership contract own parts of the LLC but not of the property itself. As already established in fractional homeownership contracts of second homes, independent real estate agents could mediate between the two parties, set up the LLC, and negotiate the contract details (e.g., the splitting of costs, the rights to sell parts of the property) with both parties in advance. The independent real estate agents should then also have the final say in case of disagreements.[9]

The costs for setting up such an LLC could then either be covered by a part of the "fractional homeownership premium" or be splitted between both parties and be incorporated in the transaction costs when acquiring a new home. Founding an LLC for fractional ownership is not only common in the real estate sector but also, for example, in the energy sector (Lowitzsch (2019)). This procedure is also similar to the management procedures of housing cooperatives in, for example, Sweden.

3.3. The model

In this chapter, we introduce our life cycle model of optimal consumption, savings, and housing decisions. With this model, we aim at comparing optimal decision making for individuals who have access to fractional homeownership and for individuals who do not have access to fractional homeownership, i.e., for individuals whose rent–versus–own decision is binary with each other. Individuals can invest into two financial markets,

In case of aggregate utility maximization, it might even be possible to derive a Pareto–optimal cost structure for both parties in fractional homeownership contracts.

[9]A lot of real estate companies who focus on co–ownership of second homes already offer these services to its customers, see, e.g., https://www.pacaso.com/, last accessed November 7, 2022.

the housing market as well as the capital market, which are introduced in Sections 3.3.1 and 3.3.2, respectively, and make six interrelated decisions at every point in time simultaneously to achieve their goal of maximizing their present discounted lifetime utility as outlined in Section 3.3.3. The model's calibration is given in Section 3.3.4.

3.3.1. Housing market

In life cycle literature, it is common to model rent–versus–own decisions as binary decisions, i.e., individuals can either live in a rented place or in an owner–occupied home (see, e.g., Cocco (2005); Fischer and Stamos (2013); Schlafmann (2021)).

In this manuscript, we extend existing life cycle models of optimal consumption, savings, and housing decisions by allowing individuals to partly own the property they live in and to pay a rent rate for the other part, i.e., we allow individuals to be fractional homeowners.

Let Δ_t denote the individual's homeownership status from time t to $t+1$. $\Delta_t \in \{0,1\}$ takes value zero if the individual is a renter and value one if the individual owns (at least partly) the residence she lives in. In case of fractional homeownership, individuals additionally have to decide upon the fraction of the property owned from time t to $t+1$, $h_t \in (0,1]$ in case $\Delta_t = 1$. Consequently, they then rent the remaining share of the property, $(1-h_t)$, from the institutional investor. In the case of a binary rent–versus–own decision, h_t is not needed as decision variable as it follows directly from Δ_t, i.e., in the case of binary homeownership it holds that $\Delta_t = h_t \in \{0,1\}$.

Owner–occupied homes serve a dual role as (i) a durable consumption good and (ii) an asset. That means, that (i) individuals derive utility from living in the owner–occupied home but (ii) that they also expose themselves to changes in the home's value. In line with, for example, Cocco (2005) and Kraft et al. (2018), we employ a Cobb–Douglas utility function, u, over consumption and housing. As Kiyotaki et al. (2011), we account for individual's preferences of living in an owner–occupied home instead of a rented place by assigning a $\zeta\%$ higher utility per home unit when the individual owns at least a part of the property they live in, i.e., when $\Delta = 1$. When C

denotes the amount consumed, Q the home size, and θ the relative preference over housing consumption, the utility function can be written as[10]

$$u\left(C,Q\right) = C^{1-\theta}\left(Q\left(1+\zeta\mathbb{1}_{\{\Delta=1\}}\right)\right)^{\theta}. \tag{3.3.1}$$

We use a log–normal distribution to model returns on investments into the housing market and denote the distribution's mean by μ_H, its standard deviation by σ_H, and its gross return from time t to $t+1$ by G_t^H.

In our model, there are recurring and non–recurring housing market costs. Recurring costs cover the rent rate and/or maintenance costs and have to be paid in every period, whereas non–recurring costs only occur when the individual acquires a new owner–occupied home or adjusts the fraction of the home already owned. These non–recurring costs cover, for examples, fees that have to be paid to a real estate agent and taxes.

As discussed in Section 3.2.2, we further assume that individuals who have access to fractional homeownership have to pay a "fractional homeownership premium" when they are buying a new property. This premium depends on the value of the home acquired and has to be paid to the institutional investor as compensation for the right to adjust the fraction of the property effectively owned, h_t, at any later point in time.

Let H_t denote the price per home unit at date t. If m_r denotes the rent rate applicable to the home units rented and m_o the maintenance costs applicable to the home units owned, then the recurring costs, $m(\Delta_t,Q_t,H_t,h_t)$, depend on the individual's homeownership status, Δ_t, the value of the home, Q_tH_t, and in case of access to fractional homeownership the fraction of the home owned, h_t. Let $\chi \in \{0,1\}$ be an indicator function taking value zero if individuals do not have access to fractional homeownership and value one if they do. Then the recurring housing costs are given by

$$m(\Delta_t,Q_t,H_t,h_t) = \begin{cases} \left(m_r\mathbb{1}_{\{\Delta_t=0\}} + m_o\mathbb{1}_{\{\Delta_t=1\}}\right)Q_tH_t & \text{if} \quad \chi = 0, \\ \mathbb{1}_{\{\Delta_t=0\}}m_rQ_tH_t + \mathbb{1}_{\{\Delta_t=1\}}\left(m_r\left(1-h_t\right)+m_oh_t\right)Q_tH_t & \text{if} \quad \chi = 1. \end{cases} \tag{3.3.2}$$

[10] It would also be valid to assume that the utility gains are proportional to the fraction effectively owned, i.e., to assume that $u\left(C,Q\right) = C^{1-\theta}\left(Q\left(1+\zeta h_t\right)\right)^{\theta}$. Our assumption that individuals derive a higher utility per home unit if they own at least a part of the property is based on the endowment effect studied, for example, by Reb and Connolly (2007). They argue that the individuals' feelings of ownership towards a property are mainly mediated by subjective feelings of possession rather than by factual ownership. We believe that owning at least a part of a property already leads to such a strong feeling of possession.

Let f_p denote the transaction costs, that occur when individuals change their homeownership status from renter to owner or when moving from one owner–occupied home to another one, and f_f the transaction costs that occur when individuals change the fraction of the owner–occupied home owned. We assume $f_f < f_p$, reflecting that acquiring a new home is more expensive than only adjusting the fraction owned. If f_a denotes the "fractional homeownership premium", that individuals, who have access to fractional homeownership, have to pay whenever acquiring a new property, the non–recurring costs applicable to the investment in the real estate market, n, can be summarized as follows:

$$n(h_t,h_{t-1},Q_t,Q_{t-1},\Delta_t,\Delta_{t-1}) = \begin{cases} f_p Q_t H_t \left(\Delta_t \left(1-\Delta_{t-1}\right) + \Delta_t \Delta_{t-1} \mathbb{1}_{\{Q_{t-1} \neq Q_t\}}\right) & \text{if} \quad \chi = 0, \\ f_p h_t Q_t H_t \left(\Delta_t \left(1-\Delta_{t-1}\right) + \Delta_t \Delta_{t-1} \mathbb{1}_{\{Q_{t-1} \neq Q_t\}}\right) & \\ +f_f h_t Q_t H_t \left(\Delta_t \Delta_{t-1} \mathbb{1}_{\{Q_{t-1} = Q_t\}} \mathbb{1}_{\{h_{t-1} \neq h_t\}}\right) & \\ +f_a Q_t H_t \left(\Delta_t \left(1-\Delta_{t-1}\right) + \Delta_t \Delta_{t-1} \mathbb{1}_{\{Q_{t-1} \neq Q_t\}}\right) & \text{if} \quad \chi = 1. \end{cases} \tag{3.3.3}$$

In case of $\chi = 1$ in equation (3.3.3), the transaction costs (line 1 and line 2 in the second case of the respective equation) are split proportionally between the individual and the institutional investor, i.e., if individuals own a larger fraction, h_t, of the property, they also have to cover a larger share of the transaction costs.

3.3.2. Capital market

On the capital market, individuals can either invest in a risky asset representing a stock market index or in a risk–free asset paying a constant return r. The risky asset's return from time t to $t+1$ is log–normally distributed with mean μ_S and standard deviation σ_S, and its random gross return from time t to $t+1$ is denoted by G_t^S. We impose a short–selling constraint, i.e., for the individual's investment in the risky asset, S_t, it has to hold that $S_t \geq 0$.

In case of $\Delta_t = 1$, i.e., when individuals are (at least fractional) owners, they can use their home as a collateral and borrow up to a certain share, $\delta \in [0,1]$, of the fraction of the home's value they own. If B_t denotes the amount invested risk–free (in case of $B_t \geq 0$) or the size of the mortgage (in case of $B_t < 0$), it has to hold that

$$-B_t \leq \begin{cases} \delta Q_t H_t & \text{if} \quad \chi = 0, \\ \delta h_t Q_t H_t & \text{if} \quad \chi = 1. \end{cases} \tag{3.3.4}$$

We assume that (as in reality) the interest rate applicable to mortgage debt exceeds the risk–free interest rate. Let $\lambda \in [0,1]$ denote the interest rate margin on mortgage debt and $R_f = 1 + r$ the gross return on investments in the risk–free asset, then $R_D = R_f + \lambda$ denotes the gross interest rate applicable to mortgage debt.

3.3.3. Optimization problem

In our model, individuals have to make six interrelated decisions at every point in time, $t = \{1,\ldots,T-1\}$, simultaneously in case they have access to fractional homeownership and five interrelated decisions if their rent–versus–own decision is restricted to be binary. They have to decide upon (i) their consumption level, C_t; (ii) their investment into stocks, S_t; (iii) their homeownership status, Δ_t; (iv) the size of their home, Q_t; (v) the investment in the risk–free asset or the size of their mortgage, B_t, and (vi) upon the fraction of the property owned in case of fractional homeownership, h_t.

The total wealth, W_t, that belongs to the individual at time t (before decisions are made) can be expressed as

$$W_t = \begin{cases} X_t + \mathbb{1}_{\{\Delta_{t-1}=1\}} Q_{t-1} H_t & \text{if} \quad \chi = 0, \\ X_t + \mathbb{1}_{\{\Delta_{t-1}=1\}} h_{t-1} Q_{t-1} H_t & \text{if} \quad \chi = 1, \end{cases} \tag{3.3.5}$$

where X_t denotes cash–on–hand, i.e., the sum of (i) labor income received in period t, L_t, (ii) the bond investment from time $t-1$ to t times the gross risk–free rate or the mortgage times the interest rate applicable on mortgage debt, i.e., $B_{t-1}\left(\mathbb{1}_{\{B_{t-1}\geq 0\}} R_f + \mathbb{1}_{\{B_{t-1}<0\}} R_D\right)$, and (iii) the stock investment from time $t-1$ to t times the return earned, i.e., $S_t G^S_{t-1}$.

We use Epstein–Zin preferences (see Epstein and Zin (1989)) to allow for a separation between elasticity of intertemporal substitution, ψ, and risk aversion, γ. When β denotes the individuals time preference, $f(t)$ its probability of survival from period t to $t+1$, and i the inflation rate, the individual's optimization problem is given by[11]

[11]Note that for individuals whose homeownership decision is binary, h_t is not needed as decision variable as it directly follows from Δ_t. To avoid having to state two optimization problems—one for the case with binary homeownership decisions and one for the case with fractional homeownership decisions—we state the optimization problem using both variables. The reader can just apply $\Delta_t = h_t \in \{0,1\}$ when thinking about the case with a binary rent–versus–own decision.

$$\max_{\{C_t,S_t,\Delta_t,Q_t,B_t,h_t\}} V_t\left(\mathfrak{X}_t\right) = \left((1-\beta)\left(\left(\frac{C_t}{(1+i)^t}\right)^{1-\theta}\left(Q_t\left(1+\zeta\mathbb{1}_{\{\Delta_t=1\}}\right)\right)^{\theta}\right)^{1-\frac{1}{\psi}} \right. \tag{3.3.6}$$
$$\left. +\beta\left(f(t)\,\mathbb{E}_t\left[V_{t+1}\left(\mathfrak{X}_{t+1}\right)^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right)^{\frac{1}{1-\frac{1}{\psi}}},$$

subject to

$$\begin{aligned} X_t &= L_t + B_{t-1}\left(\mathbb{1}_{\{B_{t-1}\geq 0\}}R_f + \mathbb{1}_{\{B_{t-1}<0\}}R_D\right) + S_{t-1}G_{t-1}^S \\ &= B_t + S_t + C_t + h_tQ_tH_t - h_{t-1}Q_{t-1}H_t + m(\Delta_t,Q_t,H_t,h_t) + n(h_t,h_{t-1},Q_t,Q_{t-1},\Delta_t,\Delta_{t-1}), \end{aligned} \tag{3.3.7}$$

$$C_t,Q_t > 0,\ S_t \geq 0, \tag{3.3.8}$$

and subject to the borrowing constraint introduced in equation (3.3.4) and the budget constraint in equation (3.3.5). $\mathfrak{X}_t = [t,\Delta_t,L_t,W_t,h_{t-1},Q_{t-1}]$ is the vector of state variables required to solve this optimization problem.[12] We explain in greater detail how we solve the optimization problem in Appendix 3.A.

3.3.4. Calibration

This section outlines the model's base case calibration which is also summarized in Table 3.3.1. We assume that individuals enter the life cycle at age $t_0 = 20$, are retired at age $t_{ret} = 66$ and die at the latest at age $t_\Omega = 100$. Consequently, their maximum investment horizon is $T = 81$ years. Prior to t_Ω, we use the empirical estimates of Arias and Xu (2018) to determine the individuals' probabilities of surviving from period t to $t+1$.

We use the estimates of Fischer and Koch (2022), who update the estimates for the labor income process by Cocco et al. (2005) using more recent data, to calibrate the individuals' labor income process. That means that individuals' labor income prior to retirement age is stochastic whereas it is a constant fraction of the last working period's labor income after attaining retirement age. To parameterize the labor income

[12]As h_t is not a decision variable in the case of binary homeownership decisions, h_{t-1} is not needed as a state variable in this case either.

process, we assume—following Fischer and Koch (2022)—that the individuals are married and have attained high school degree.[13]

We set the financial market parameters to historical estimates. The annual risk–free rate of return on investments into the risk–free asset is set to $r = 3.9\%$, corresponding to the average U.S. 3–month Treasury Bill rate between 1946 and 2021 and the inflation rate is set to $i = 3.7\%$, the average U.S. inflation rate from 1946 to 2021. We set the interest rate applicable to mortgage debt to $r_D = 7.9\%$, the annual 30–year fixed rate mortgage average from 1971 to 2021.[14] That implies an interest rate margin on mortgage debt of $\lambda = 3.9\%$.

We estimate the average annual return on the S&P500 from 1946 to 2021 to be 12.6% and its standard deviation to be $\sigma_S = 16.6\%$. Following, among others, Claus and Thomas (2001), who doubt that such high equity premia will occur in future periods, we set the equity premium to 3%, implying an expected annual return on the investment into the risky asset of $\mu_S = 6.9\%$.

Using the U.S. S&P's CoreLogic Case–Shiller Home Price Index, we estimate the average annual return on housing from 1946 to 2021 to be $\mu_H = 5\%$ and its standard deviation to be 5.8%. Case and Shiller (1989) and Chinloy et al. (2022) argue that the volatility of individual's home's prices is substantially higher. We follow their consensus and set the standard deviation of the annual expected return on housing to $\sigma_H = 8.7\%$, the empirical estimate of Chinloy et al. (2022) for the U.S. housing market for 1990 to 2020. this is to the best of our knowledge the most recent estimate available.

We mostly use empirical estimates from the literature for parameterizing the housing market. Following Yao and Zhang (2005), we set the individual's relative preference over housing consumption to $\theta = 0.2$, the annual maintenance costs, m_o, to 1.5%, the rent rate to $m_r = 6\%$, and the transaction cost when acquiring a new home, f_p, to 6%. We set the transaction costs that occur when solely adjusting the fraction owned to $f_f = 1\%$ and the "fractional homeownership premium" to $f_a = 2.5\%$.[15] Individuals are allowed to borrow up to 80% of the value of the home they effectively own, i.e., $\delta = 80\%$ (see, e.g., Yao and Zhang (2005); Bacher (2021); Paz-Pardo (2021)).

[13]Our results are robust to different parameterizations of the labor income process, i.e., our results are robust to individuals' different educational and family statuses.

[14]We use a shorter time horizon for estimating the historical mortgage rate than for estimating the other financial market parameters as the Federal Reserve Bank only provides data from 1971 onwards (source: https://fred.stlouisfed.org/series/MORTGAGE30US; last accessed April 27, 2023).

[15]As we are, to the best of our knowledge, the first to study fractional homeownership using a quantitative life cycle model, there are no empirical estimates for f_f and f_a available yet. Our results are qualitatively similar for different values of $f_f \in [0.5\%,5.0\%]$ and $f_a \in [0\%,5.0\%]$ and are available upon request.

Following Kiyotaki et al. (2011) and Fischer and Khorunzhina (2019), we set the individual's preference for living in an owner–occupied home instead of a rented place to $\zeta = 10\%$, i.e., we assign a 10% higher utility per home unit if individuals own at least a fraction of the home they live in.

Following Cocco (2005), we set the correlation between stock return and labor income to $\rho_{S,L} = 0.047$, the correlation between housing return and labor income to $\rho_{H,L} = 0.550$, and we assume that housing and stock returns are uncorrelated, i.e., that $\rho_{H,S} = 0$.

We aim at choosing preference parameters such that we match empirically–observed growth of wealth for individuals not having access to fractional homeownership. Therefore, we set the risk aversion to $\gamma = 5$, the elasticity of intertemporal substitution to $\psi = 0.1$, and the time preference to $\beta = 0.96$. All these values are common in life cycle literature.

3.4. Results

In this section, we present the results obtained when solving the quantitative life cycle model outlined in Section 3.3. First, it is important to show that key model predictions match key features observed in the data.

Figure 3.4.1 compares the model–predicted evolution of total savings for individuals whose rent–versus–own decision is restricted to be binary (solid line) with the empirically–observed evolution of total savings (dashed line). The model–predicted evolution of wealth is the average of 10,000 simulated paths on the policy functions obtained when solving the model outlined in Section 3.3 with the restriction that the rent–versus–own decision is binary. We use the mean net worth of individuals as reported in the 2019 Survey of Consumer Finances (the most recent survey published) as a proxy for empirically–observed growth of wealth. The data can be downloaded from https://www.federalreserve.gov/ (last retrieved April 8, 2023). Our model's predictions match the empirically–observed evolution of total wealth well as the solid line is intersecting with the dashed line for all age intervals the Survey of Consumer Finances provides data for.

Figure 3.4.2 compares the model's predictions for the case of binary rent–versus–own decisions (solid line) with the case of fractional homeownership (dashed line). All results are averages of 10,000 simulated paths on the optimal policy functions obtained when solving the optimization problem outlined in Section 3.3. Panel A compares the evolution of total wealth. Panel B compares the distribution of first–time housing market entry (depicted by the two spikes on the left around age 30) and the

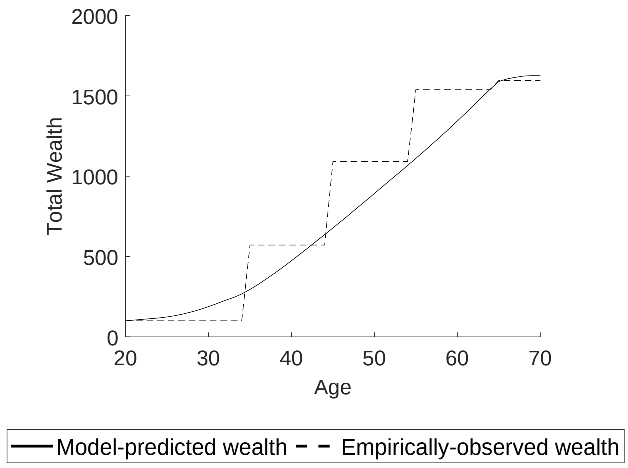
Table 3.3.1: Base case calibration

Description	Symbol	Value	Source
	Preference parameters		
Degree of risk aversion	γ	5.00	Own choice
Elasticity of intertemporal substitution	ψ	0.10	Own choice
Time preference factor	β	0.96	Own choice
	Financial markets		
Expected nominal return on equity	μ_S	6.9%	Claus and Thomas (2001)
Standard deviation of return on equity	σ_S	16.6%	Historical estimate
Nominal risk–free rate	r	3.9%	Historical estimate
Inflation	i	3.7%	Historical estimate
	Housing market		
Housing preference parameter	θ	0.2	Yao and Zhang (2005)
Maximum borrowing rate	δ	80.0%	Yao and Zhang (2005)
Risk–free borrowing rate	r_D	7.9%	Historical estimate
Annual rent rate	m_r	6.0%	Yao and Zhang (2005)
Annual maintenance rate	m_o	1.5%	Yao and Zhang (2005)
Transaction costs: home purchase	f_p	6.0%	Yao and Zhang (2005)
Transaction costs: adjustment of fraction	f_f	1.5%	Own choice
Fractional homeownership premium	f_a	2.5%	Own choice
Expected nominal return on housing	μ_H	5.0%	Historical estimate
Standard deviation of return on housing	σ_H	8.7%	Chinloy et al. (2022)
Preference for owning over renting	ζ	10.0%	Kiyotaki et al. (2011)
	Correlations		
Stock return and labor income	$\rho_{S,L}$	0.047	Cocco (2005)
Housing return and labor income	$\rho_{H,L}$	0.550	Cocco (2005)
Housing return and stock return	$\rho_{H,S}$	0.000	Cocco (2005)

This table summarizes the base case calibration used in our model.

distribution of last–time housing market exit (depicted by the two spikes on the right around age 95). The evolution of the loan–to–value ratios is depicted in Panel C. The solid line depicts the loan–to–value ratio for individuals whose rent–versus–own decision is binary. Following up on Footnote 2, the dashed line depicts the loan–to–value ratio that is calculated as the fraction of the outstanding mortgage debt and the value of the home, i.e., the loan–to–value ratio that is calculated as $\frac{-B_t}{Q_t H_t}$, whereas the dotted line depicts the loan–to–value ratio that is calculated as the fraction of the outstanding mortgage debt and the value of the home individuals effectively own, i.e., the loan–to–value ratio that is calculated as $\frac{-B_t}{h_t Q_t H_t}$. Panel D depicts the fraction owned conditional on the individuals being owners, i.e., conditional on $\Delta_t = 1$. Note that the fraction owned is by definition 100% for individuals whose rent–versus–own decision is binary as they are not allowed to own shares of the property they live in.

Figure 3.4.1: Evolution of model–predicted and empirically–observed growth of wealth

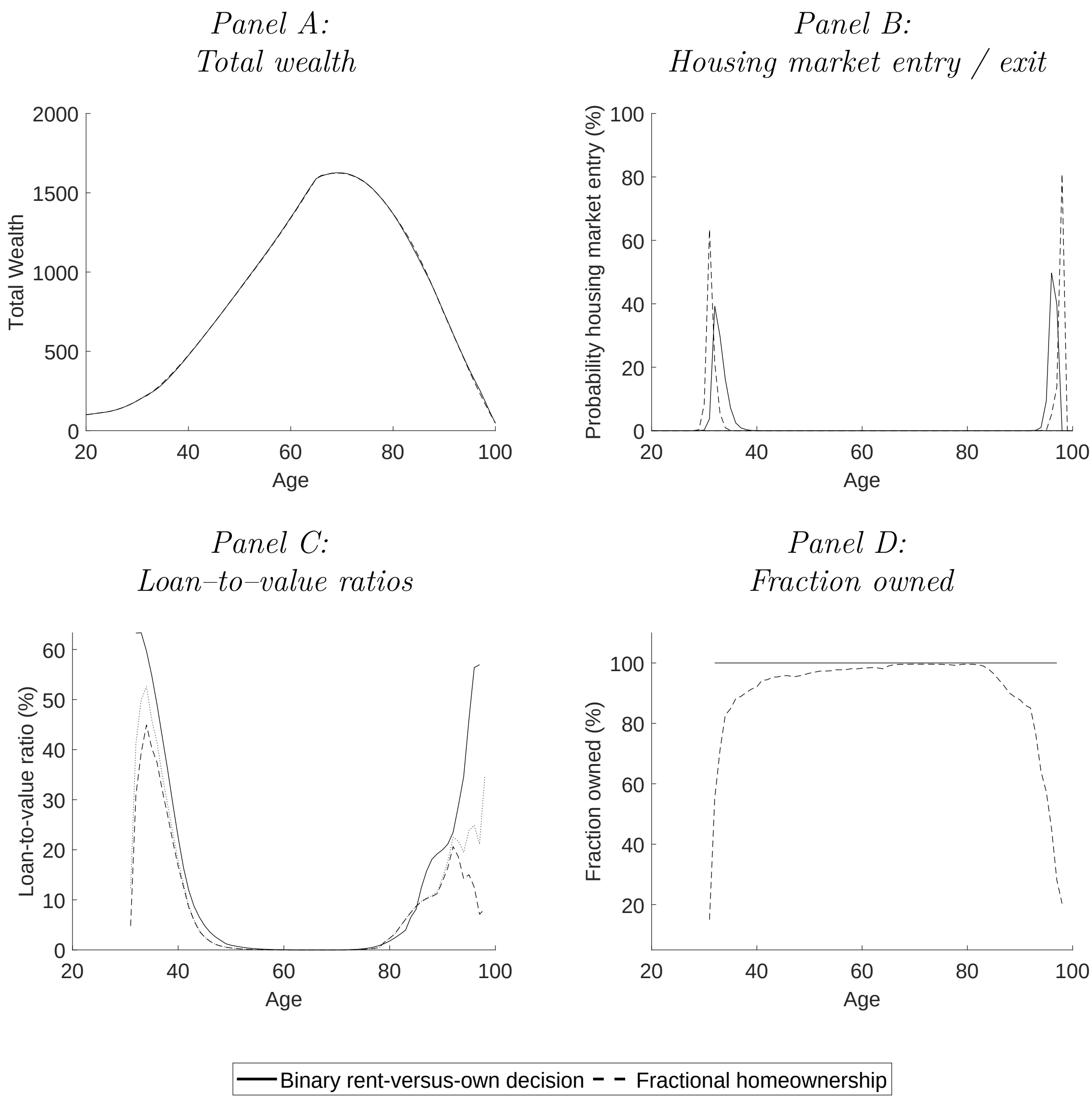


This figure compares the evolution of model–predicted wealth for individuals not having access to fractional homeownership (solid line) with the evolution of empirically–observed growth of wealth (dashed line) for the accumulation phase of savings from age 20 to age 70. The solid line is the average of 10,000 simulated paths on the policy functions obtained when solving the model outlined in Section 3.3 but when restricting the homeownership decision to be binary. We use the mean net worth of individuals as reported in the 2019 Survey of Consumer Finances (the most recent survey published) as a proxy for the empirically–observed growth of wealth. The data can be downloaded from https://www.federalreserve.gov/; last retrieved April 8, 2023.

Panel A of Figure 3.4.2 indicates that individuals who have/do not have access to fractional homeownership accumulate roughly the same level of total savings. This is an important finding from the policymakers' perspective as it implies that individuals do not procrastinate on their savings even when they are provided with easier access to the housing market, i.e., this finding provides evidence that fractional homeownership could also be stimulated by policymakers. Instead of leading to procrastination on savings, it is rather the case that individuals with access to fractional homeownership have lower loan–to–value ratios (see Panel C of Figure 3.4.2) implying that (i) they can avoid paying the interest rate margin on mortgage debt and (ii) have more liquid wealth available to invest and to consequently earn financial income on.

From Panel B of Figure 3.4.2, we see that individuals with access to fractional homeownership enter the housing market earlier and exit it later. The average first–time housing market entrant with access to fractional homeownership is almost two years younger than the average first–time housing market entrant who does not have access to fractional homeownership. Further, individuals with access to fractional

Figure 3.4.2: Results for binary and fractional homeownership decisions



This figure compares the model's predictions for individuals whose rent–versus–own decision is binary (solid line) with those for individuals who have access to fractional homeownership (dashed line). Panel A compares the evolution of total wealth and Panel B the distribution of the first–time housing market entry (depicted by the two spikes on the left around age 30) and of the last–time housing market exit (depicted by the two spikes on the right around age 95). Panel C compares the individuals' loan–to–value ratios conditional on the individuals being owners, i.e., conditional on $\Delta_t = 1$. Following up on Footnote 2, the dashed line depicts the loan–to–value ratio that is calculated as the fraction of the outstanding mortgage debt and the value of the home, i.e., as $\frac{-B_t}{Q_t H_t}$, whereas the dotted line depicts the loan–to–value ratio that is calculated as the fraction of the outstanding mortgage debt and the value of the home individuals effectively own, i.e., as $\frac{-B_t}{h_t Q_t H_t}$. Panel D compares the evolution of the fraction owned conditional on the individuals being owners. Note that the fraction owned is not a decision variable when the individuals' homeownership decision is binary but is by definition 100% for all individuals living in owner–occupied homes. All lines are averages of 10,000 simulated paths on the optimal policy functions obtained when solving the optimization problem outlined in Section 3.3.

homeownership exit the housing market for the last time when they are almost one and a half years older than individuals who do not have access to fractional homeownership. Taken these two results together, fractional homeownership extends individuals' time of housing market participation on average by more than three years.[16]

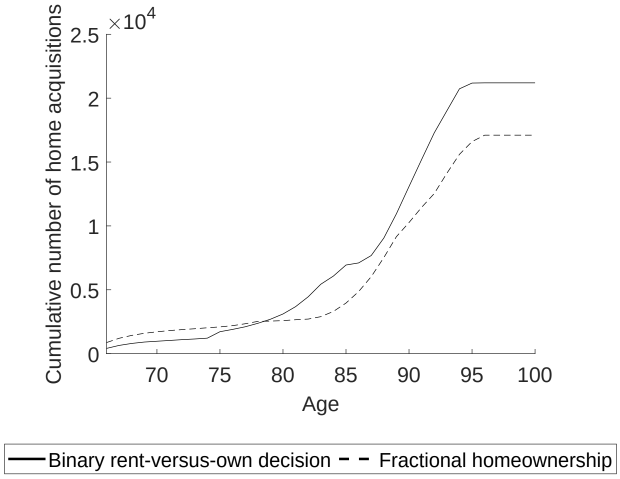
Panel D of Figure 3.4.2 provides evidence that the extended housing market participation is really driven by the access to fractional homeownership as particularly young and very old housing market participants make use of the fractional homeownership concept. Younger individuals use the fractional homeownership structure to finance valuable properties while simultaneously keeping their mortgage debt low by just buying small shares of the property they wish to live in. In contrast, old housing market participants reduce the fraction of the home owned instead of borrowing against their housing wealth (see Panel C of Figure 3.4.2) or selling the home completely. To put it differently, the elderly with access to fractional homeownership sell fractions of their home to have more liquid wealth available to finance, for example, consumption in their last years instead of fully selling the property or moving to a smaller property.

Panel C of Figure 3.4.2 indicates that the loan–to–value ratios are significantly lower for individuals having access to fractional homeownership, irrespective of whether the ratio is calculated as the fraction of outstanding mortgage debt and the full value of the home or as the fraction of outstanding mortgage debt and the value of the home that effectively belongs to the individual. This finding can be explained by the individuals' preferences to only own fractions of a home and borrow less instead of owning the full property, having more outstanding mortgage debt, and facing the interest rate margin on mortgage debt. We find that young individuals have very low loan–to–value ratios when they first enter the housing market but then take out larger mortgages to finance an increase in the fraction of the property effectively owned (see Panel D of 3.4.2). However, these mortgages are then repaid at a very fast rate.

In practice, the elderly frequently borrow against the value of their home by entering reverse mortgages contracts (see, e.g., Van Ooijen et al. (2015)). In contrast to reverse mortgages/the case with binary rent–versus–own decisions, the loan–to–value ratios of individuals with access to fractional homeownership are still relatively low and it is not the case that the house belongs to the bank after the death of the resident (which would be the case if the individual took out a reverse mortgage).

[16] In our manuscript, we assume that individuals are homogeneous, i.e., that all individuals have the same set of preferences. Therefore, the individuals' housing market entry and exit is concentrated around the ages 30 and 95, respectively. Vestman (2019) shows that the assuming that individuals are heterogeneous, i.e., that individuals have different sets of preferences, would lead to more variation in housing choices.

Figure 3.4.3: Cumulative number of home acquisitions during retirement period



This figure depicts the cumulative number of home acquisitions during retirement age, i.e., from age 66 to age 100. The results for the case in which the individual's rent–versus–own decision is binary are depicted by the solid line and the results for the case in which individuals have access to fractional homeownership are depicted by the dashed line. The lines are constructed using 10,000 simulated paths on the optimal policy functions obtained when solving the optimization problem outlined in Section 3.3.

Moving can be stressful, particularly for older individuals, and can even cause adverse health effects such as relocation stress syndrome. Various papers studying moving of the elderly include (i) Yawney and Slover (1973), who document that it is generally more difficult for older individuals to adapt to new living situations, (ii) Venti and Wise (2004), who find—using U.S. panel data—that the majority of individuals does not move even though it would help them to smooth consumption over their life cycle, and (iii) Weeks et al. (2012), who identify drivers (among others gender, age, and household income) of individuals' moving decisions. In this manuscript, we solely study home–to–home relocations for financial reasons but we consider incorporating non–voluntary moving to, for example, nursing homes into life–cycle models of optimal consumption, savings, and housing decisions to be an interesting avenue for further research.

Given that individuals tend to refrain from moving even though it would be optimal for them from a financial perspective, we believe that it would be advantageous if a novel financing option for real estate would require individuals to move less frequently. To shed light on this issue, Figure 3.4.3 depicts the cumulative number of home acquisitions during the retirement period, i.e., from age 66 to age 100 by the solid line for the case of binary rent–versus–own decisions and by the dashed line for the case of fractional homeownership. Until the age of 80, individuals acquire homes to a roughly equal extent. After the age of 80, the two lines start to diverge indicating that individuals whose rent–versus–own decision is binary have to move more frequently to make optimal decisions. In contrast, individuals with access to fractional homeownership can just sell fractions of homes they currently live in as indicated by Panel D of Figure 3.4.2. Fractional homeownership is thus not only beneficial for younger individuals as it facilitates their housing market entry but also for older individuals as they are not required to move as frequently as individuals whose rent–versus–own decision is binary to still being able to make optimal financial decisions during their retirement phase.

3.5. Conclusion

The market of fractional homeownership of owner–occupied homes is an emerging market in the U.S., that has up to now received very little attention from researchers. We contribute to closing this gap in the literature by using a realistically calibrated quantitative life cycle model to study the impact of fractional homeownership on optimal consumption, savings, and housing decisions. We define a fractional homeownership contract to be a contract between two parties who share full ownership of a property: (i) an individual who lives in the property permanently and (ii) an institutional investor who sees the property solely as an investment vehicle. As compensation for the right to live in the property full–time, the individual makes periodic rent payments to the institutional investor. The exact amount of these rent payments depends on the value of the home and the fractional ownership structure that can change over time as individuals have the right to adjust the fraction owned in every period.

Our results suggest that, in particular, young individuals facing liquidity constraints and older individuals who are in the phase of decumulating their savings make use of fractional homeownership to finance real estate. Further, we find that individuals who have access to fractional homeownership enter the housing market earlier, exit it later, have lower loan–to–value ratios and have to move less frequently at old age; all in comparison to a setting in which the individuals' rent–versus–own decisions are binary. Nonetheless, individuals in both settings accumulate roughly the same level of total savings over the life cycle.

References

Amromin, G., J. Huang, C. Sialm, and E. Zhong (2018). "Complex Mortgages." *Review of Finance* 22. (6), 1975–2007.

Anacker, K. B. (2019). "Introduction: Housing Affordability and Affordable Housing." *International Journal of Housing Policy* 19. (1), 1–16.

Arias, E. and J. Xu (2018). "United States Life Tables, 2015. National Vital Statistics Report." *National Center for Health Statistics* 67. (7).

Arnold, M. A. (1992). "The Principal-Agent Relationship in Real Estate Brokerage Services." *Real Estate Economics* 20. (1), 89–106.

Bacher, A. (2021). "Housing and Savings Behavior across Family Types." working paper, available at https://annikabacher.github.io/Bacher_JMP.pdf.

Bäckman, C. and C. Lutz (2020). "The Impact of Interest-Only Loans on Affordability." *Regional Science and Urban Economics* 80. Special Issue on Housing Affordability, article no. 103376.

Balmer, I. and J.-D. Gerber (2018). "Why are Housing Cooperatives Successful? Insights from Swiss Affordable Housing Policy." *Housing Studies* 33. (3), 361–385.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein (2007). "Portfolio Choice over the Life-Cycle When the Stock and Labor Markets Are Cointegrated." *Journal of Finance* 62. (5), 2123–2168.

Bian, X., Z. Lin, and Y. Liu (2018). "House Price, Loan-to-Value Ratio and Credit Risk." *Journal of Banking & Finance* 92, 1–12.

Bogyrbayeva, A., M. Takalloo, H. Charkhgard, and C. Kwon (2021). "An Iterative Combinatorial Auction Design for Fractional Ownership of Autonomous Vehicles." *International Transactions in Operational Research* 28. (4), 1681–1705.

Case, K. E. and R. J. Shiller (1989). "The Efficiency of the Market for Single-Family Homes." *American Economic Review* 79. (1), 125–137.

Cauley, S. D., A. D. Pavlov, and E. S. Schwartz (2007). "Homeownership as a Constraint on Asset Allocation." *Journal of Real Estate Finance and Economics* 34. (3), 283–311.

Chinloy, P., C. Jiang, and K. John (2022). "Spreads and Volatility in House Returns." *Journal of Risk and Financial Management* 15. (8), 369.

Claus, J. and J. Thomas (2001). "Equity Premia as Low as Three Percent? Evidence from Analysts' Earnings Forecasts for Domenstic and International Stock Markets." *Journal of Finance* 56. (5), 1629–1666.

Cocco, J. (2005). "Portfolio Choice in the Presence of Housing." *Review of Financial Studies* 18. (2), 535–567.

Cocco, J., F. J. Gomes, and P. J. Maenhout (2005). “Consumption and Portfolio Choice over the Life Cycle.” *Review of Financial Studies* 18. (2), 491–533.

Dammon, R. M., C. S. Spatt, and H. H. Zhang (2004). “Optimal Asset Location and Allocation with Taxable and Tax-Deferred Investing.” *Journal of Finance* 59. (3), 999–1037.

Daneshvary, N. and T. M. Clauretie (2013). “Agent Change and Seller Bargaining Power: A Case of Principal Agent Problem in the Housing Market.” *Journal of Real Estate Finance and Economics* 47. (3), 416–433.

Epstein, L. G. and S. E. Zin (1989). “Substitution, Risk Aversion and Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework.” *Econometrica* 57. (4), 937–969.

Fischer, M. and M. F. Gallmeyer (2017). “Taxable and Tax-Deferred Investing with the Limited Use of Losses.” *Review of Finance* 21. (5), 1847–1873.

Fischer, M. and N. Khorunzhina (2019). “Housing Decision with Divorce Risk.” *International Economic Review* 60. (3), 1263–1290.

Fischer, M. and M. Koch (2022). “Heuristic Portfolio Rules with Labor Income.” working paper, University of Konstanz.

Fischer, M. and M. Stamos (2013). “Optimal Life Cycle Portfolio Choice with Housing Market Cycles.” *Review of Financial Studies* 26. (9), 2311–2352.

Furst, R. T. and D. N. Evans (2017). “Renting Apartments to Felons: Variations in Real Estate Agent Decisions Due to Stigma.” *Deviant Behavior* 38. (6), 698–708.

Goodman, L. S. and C. Mayer (2018). “Homeownership and the American Dream.” *Journal of Economic Perspectives* 32. (1), 31–58.

Gourinchas, P.-O. and J. A. Parker (2002). “Consumption over the Life Cycle.” *Econometrica* 70. (1), 47–89.

Gupta, A., J. Rathod, D. Patel, J. Bothra, S. Shanbhag, and T. Bhalerao (2020). “Tokenization of Real Estate Using Blockchain Technology.” In: *Applied Cryptography and Network Security Workshops: Proceedings 18*. Springer, 77–90.

Johansen, K. and J. Emborg (2018). “Wind Farm Acceptance For Sale? Evidence from the Danish Wind Farm Co-Ownership Scheme.” *Energy Policy* 117, 413–422.

Kiyotaki, N., A. Michaelides, and K. Nikolov (2011). “Winners and Losers in Housing Markets.” *Journal of Money, Credit and Banking* 43. (2–3), 255–296.

Konashevych, O. (2020). “General Concept of Real Estate Tokenization on Blockchain.” *European Property Law Journal* 9. (1), 21–66.

Kraft, H. and C. Munk (2011). “Optimal Housing, Consumption, and Investment Decisions over the Life Cycle.” *Management Science* 57. (6), 1025–1041.

Kraft, H., C. Munk, and S. Wagner (2018). “Housing Habits and Their Implications for Life-Cycle Consumption and Investment.” *Review of Finance* 22. (5), 1737–1762.

Layton, D. (2022). “The Homeownership Rate and Housing Finance Policy – Part 2: The Very Heavy Lift to Maintain and Then Break the 65 Percent Barrier.” Joint Center for Housing Studies of Harvard University, available at https://www.jchs.harvard.edu/sites/default/files/research/files/harvard_jchs_homeownership_rate_ii_layton_2022.pdf.

Lowies, B., R. B. Whait, C. Viljoen, and S. McGreal (2018). “Fractional Ownership – An Alternative Residential Property Investment Vehicle.” *Journal of Property Investment & Finance* 36. (6), 513–522.

Lowitzsch, J., ed. (2019). *Energy Transition: Financing Consumer Co-Ownership in Renewables*. Springer.

OECD (2021b). “Tax Relief for Home Ownership.” technical report, available at https://www.oecd.org/els/family/PH2-2-Tax-relief-for-home-ownership.pdf.

Ostrowski, J. (2022). “Nearly Two-Thirds of Non-Homeowners Polled Say Affordability Woes Block Homeownership.” Bankrate, available at https://www.bankrate.com/mortgages/homeownership-remains-centerpiece-of-american-dream/.

Paz-Pardo, G. (2021). “Homeownership and Portfolio Choice over the Generations.” working paper, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3797142.

Reb, J. and T. Connolly (2007). “Possession, Feelings of Ownership and the Endowment Effect.” *Judgment and Decision Making* 2. (2), 107–114.

Rong, H. (2023). “A Proposal for Fractional Property Ownership and Collective Governance in Local Development.” working paper, available at https://ssrn.com/abstract=4353724.

Roth, L., J. Lowitzsch, Ö. Yildiz, and A. Hashani (2018). “Does (Co-) Ownership in Renewables Matter for an Electricity Consumer’s Demand Flexibility? Empirical Evidence from Germany.” *Energy Research & Social Science* 46, 169–182.

Sazama, G. W. (2000). “Lessons from the History of Affordable Housing Cooperatives in the United States: A Case Study in American Affordable Housing Policy.” *American Journal of Economics and Sociology* 59. (4), 573–608.

Schaeffer, K. (2022). “A Growing Share of Americans Say Affordable Housing is a Major Problem Where They Live.” Pew Research Center, available at https://www.pewresearch.org/fact-tank/2022/01/18/a-growing-share-of-americans-say-affordable-housing-is-a-major-problem-where-they-live/.

Schlafmann, K. (2021). “Housing, Mortgages, and Self-Control.” *Review of Financial Studies* 34. (5), 2648–2687.

Sørvoll, J. and B. Bengtsson (2018). “The Pyrrhic Victory of Civil Society Housing? Co-operative Housing in Sweden and Norway.” *International Journal of Housing Policy* 18. (1), 124–142.

Van Ooijen, R., R. Alessie, and A. Kalwij (2015). “Saving Behavior and Portfolio Choice After Retirement.” *De Economist* 163, 353–404.

Venti, S. F. and D. A. Wise (2004). “Aging and Housing Equity: Another Look.” In: *Perspectives on the Economics of Aging*. University of Chicago Press, 127–180.

Vestman, R. (2019). “Limited Stock Market Participation Among Renters and Homeowners.” *Review of Financial Studies* 32. (4), 1494–1535.

Weeks, L. E., J. Keefe, and D. J. Macdonald (2012). “Factors Predicting Relocation Among Older Adults.” *Journal of Housing for the Elderly* 26. (4), 355–371.

Whitaker, A. and R. Kräussl (2020). “Blockchain, Fractional Ownership, and the Future of Creative Work.” *Management Science* 66. (10), 4594–4611.

Wood, C. (2016). “Pension Funds Investing in Real Estate.” Preqin, available at https://docs.preqin.com/newsletters/re/Preqin-RESL-September-16-Pension-Funds-Investing-in-Real-Estate.pdf.

Yao, R. and H. H. Zhang (2005). “Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints.” *Review of Financial Studies* 18. (1), 197–239.

Yawney, B. A. and D. L. Slover (1973). “Relocation of the Elderly.” *Social Work* 18. (3), 86–95.

Appendix 3.A Solution of the optimization problem

To simplify the optimization problem, we normalize with total wealth, W_t, that belongs to the individual at time t. Consequently, $c_t = \frac{C_t}{W_t}$ and $q_t = \frac{Q_t H_t}{W_t}$ are the two decision variables that directly affect the individual's utility. If $V_t\left(\mathfrak{X}_t\right) = v_t\left(\mathfrak{x}_t\right)\frac{W_t/(1+i)^t}{\left(H_t/(1+i)^t\right)^{\theta}}$, the simplified optimization problem is:

$$\max_{\{c_t,s_t,\Delta_t,q_t,b_t,h_t\}} v_t\left(\mathfrak{x}_t\right) = \tag{3.A.1}$$

$$\left(\left(1-\beta\right)\left(c_t^{1-\theta}\left(q_t\left(1+\zeta\mathbb{1}_{\{\Delta_t=1\}}\right)\right)^{\theta}\right)^{1-\frac{1}{\psi}} + \beta\left(f\left(t\right)\mathbb{E}_t\left[\left(v_{t+1}\left(\mathfrak{x}_{t+1}\right)\frac{\frac{W_{t+1}}{W_t(1+i)}}{\left(\frac{H_{t+1}}{H_t(1+i)}\right)^{\theta}}\right)^{1-\gamma}\right]\right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right)^{\frac{1}{1-\frac{1}{\psi}}}$$

subject to

$$\begin{aligned} x_t &= \frac{L_t}{W_t} + b_{t-1}\left(\mathbb{1}_{\{b_{t-1}\geq 0\}}R_f + \mathbb{1}_{\{b_{t-1}<0\}}R_D\right) + s_{t-1}G^S_{t-1} \\ &= b_t + s_t + c_t + h_t q_t - h_{t-1}\frac{Q_{t-1}H_t}{W_t} + m\left(.\right) + n\left(.\right), \end{aligned} \tag{3.A.2}$$

$$-\,b_t \leq \mathbb{1}_{\{\chi=0\}}\delta q_t + \mathbb{1}_{\{\chi=1\}}\delta h_t q_t, \tag{3.A.3}$$

$$x_t + \mathbb{1}_{\{\Delta_{t-1}=1\}}\left(\mathbb{1}_{\{\chi=0\}}\frac{Q_{t-1}H_t}{W_t} + \mathbb{1}_{\{\chi=1\}}h_{t-1}\frac{Q_{t-1}H_t}{W_t}\right) = 1, \tag{3.A.4}$$

$$c_t, q_t > 0,\ s_t \geq 0, \tag{3.A.5}$$

in which $s_t = \frac{S_t}{W_t}$, $b_t = \frac{B_t}{W_t}$, $x_t = \frac{X_t}{W_t}$, and $m(.)$ and $n(.)$ are the recurring and non–recurring costs as described in equations (3.3.2) and (3.3.3), respectively. $\mathfrak{x}_t = \left[t, \Delta_t, \frac{L_t}{W_t}, h_{t-1}, \frac{Q_{t-1}H_t}{W_t}\right]$ is the new vector of state variables needed to solve this optimization problem. For solving the model numerically using backward induction, the continuous state variables h_{t-1}, $\frac{L_t}{W_t}$, and $\frac{Q_{t-1}H_t}{W_t}$ are discretized and the expectation in equation (3.A.1) is calculated using Gaussian quadrature. We use parallel computing to expedite our calculations.

Chapter 4

Heuristic Portfolio Rules with Labor Income*

*This chapter is joint work with Marcel Fischer. We are grateful for helpful comments and suggestions from Aras Canipek, Günter Franke, Jens Jackwerth, Simon Stehle, and seminar participants at the University of Konstanz and Copenhagen Business School. This research has been supported by PeRCent, which receives base funding from the Danish pension funds and Copenhagen Business School.

4.1. Introduction

Determining an optimal investment strategy over the life cycle is a challenging dynamic optimization problem. Solving this problem extends beyond what can reasonably be expected from a private investor. In fact, a substantial share of individuals fails at much easier tasks, such as being aware of the importance of diversification (Van Rooij et al. (2011); Gaudecker (2015)), understanding the concept of compound interest rates (Song (2020)), or determining how much to save for retirement (Lusardi and Mitchell (2007)). As a result, individuals often base their financial decisions on suggestions received from personal financial advisors. Choi (2022) identifies the simplicity of the rules provided as their biggest advantage, but stresses that individuals might benefit from rules of thumb developed by economists who have a more thorough understanding of the drivers of financial decision making than personal financial advisors.

In our manuscript, we propose such a heuristic portfolio rule that enables individuals to easily determine a close to optimal equity exposure. Consistent with economic intuition, under our heuristic, the optimal equity exposure depends on the individual's age and its labor–to–wealth ratio. Accounting for age is important as it proxies for the expected length of the remaining investment horizon, while the labor–to–wealth ratio accounts for how much the evolution of total wealth depends on investments, i.e., how large risks can reasonably be taken.

To derive our heuristic, we first determine the individuals' optimal consumption–savings decisions by solving a life cycle model similar to the one proposed by Cocco et al. (2005). In a second step, we then estimate a linear regression of the optimal equity exposure on age and the labor–to–wealth ratio using the simulated paths on optimal decisions and state variables from our life cycle model as data. In other words, our regression aims at determining a heuristic relationship between the optimal equity exposure and age as well as the labor–to–wealth ratio, that is simple enough to be applied by a private investor or implemented by a financial advisor.

Welfare gains when investing optimally instead of according to our heuristic are very small. A 20–year old individual investing according to our heuristic only has to be endowed with a 0.06% higher level of lifetime consumption to achieve the same level of present discounted lifetime utility as a 20–year old individual investing optimally. In other words, it costs less than 0.1% of lifetime consumption to rely on our heuristic investment strategy compared to following the optimal, though computationally more demanding, investment strategy. Given an average lifetime consumption level of around USD 2,800,000, the empirical estimate from the 2019 Consumer Expenditures

Survey, only a one–time amount of around USD 1,600 could be gained when investing optimally instead of according to our heuristic.

Non–participation in the stock market, however, which is a widespread problem (Guiso et al. (2003); Guiso and Sodini (2013)), has substantially larger welfare implications. Welfare gains from following our heuristic instead of not investing into stocks at all are 1.9% of lifetime consumption, which corresponds to a financial gain of around USD 52,000, roughly corresponding to the price of a new Tesla Model 3. Similarly, using the average equity exposures in the PSID data reveals that U.S. individuals' equity exposures' are far from optimal. In fact, using our heuristic portfolio strategy would improve lifetime consumption opportunities by 1.3% or USD 35,000, roughly corresponding to the price of a new Chevrolet Silverado. That is, loosely speaking, by following our heuristic portfolio strategy, a typical private investor on average gains a Chevrolet Silverado, whereas a typical individual not investing into the stock market at all, on average gains a Tesla Model 3.

Our welfare effects are robust to different educational attainments and family compositions, i.e., our heuristic approximates the optimal equity exposure over the life cycle well for single females, single males, and married couples with different educational attainments. That is, simple rules of thumb can be applied to various types of households without going on compromise with welfare effects. Likewise, our portfolio strategy could also be easily implemented at low costs by financial advisors.

With our paper, we contribute to a constantly growing strand of literature about heuristics for financial decision making. An early work is the paper of Clarkson and Meltzer (1960), which provides a heuristic for portfolio selection. More recent papers include Cocco et al. (2005), who, among others, propose an age–dependent rule, Fischer et al. (2013), who show that ignoring the realization–based feature of capital gains taxation only leads to very modest welfare costs, and Fischer and Gallmeyer (2016), who propose a simple tax–efficient overlay to existing trading strategy to improve after–tax portfolios. Winter et al. (2012) show that following heuristics for consumption–savings decisions has only very small welfare costs for the decision about the share of wealth to save. Our work complements their work by documenting that simple heuristics can also be applied to asset allocation strategies with very small welfare effects. In a related work, Love (2013) thoroughly studies various rules of thumb for individuals' consumption–savings decisions. He identifies a rule of thumb dependent on the individuals' age and their financial wealth–to–total lifetime resources ratio as the best-performing one. In contrast to Love (2013), we aim at developing an even simpler rule which does not require individuals to estimate their total lifetime resources, i.e., their human capital. Instead, our heuristic only requires them to know their current labor income and total wealth, but still leads to comparable welfare losses as the rule in Love (2013).

This manuscript proceeds as follows: Section 4.2 outlines our life cycle model of optimal consumption–savings decisions and introduces our heuristic. Section 4.3 presents the base case results and assesses the welfare implications of following our heuristic strategy and of under–investing in the stock market. Section 4.4 shows that our results are robust to different educational attainments and family compositions. Finally, Section 4.5 concludes.

4.2. Theory

In our manuscript, we propose a heuristic portfolio rule that (i) enables individuals to determine their equity exposures using a simple rule of thumb and (ii) ensures that welfare gains when instead investing optimally are small. In this section, we first introduce a model of optimal consumption and savings decisions, which is closely related to the one proposed by Cocco et al. (2005). Second, we derive our rule of thumb by running linear regressions on the simulated paths of optimal decisions and state variables. Our heuristic is then a linear function only dependent on the individual's labor–to–wealth ratio and age and should therefore be easy enough to be employed by private investors but could also be offered at low costs by financial advisors.

4.2.1. Optimal portfolio rule

For determining the optimal consumption–savings decision of individuals at each age, we follow closely the procedure outlined by Cocco et al. (2005). In our model, individuals have two investment opportunities. First, they can invest into a risk–free asset paying a constant return r. Second, they can invest into a risky asset representing a stock market index. Its return from time t to $t+1$, r_t^S, is log–normally distributed with mean μ_S and standard deviation σ_S. Individuals are short–selling constrained.

After having received period t's labor income, L_t, individuals have to take two interrelated decisions simultaneously. First, they have to decide how much to consume at time t, C_t. Second, they have to decide how much of their savings to invest into stocks at time t, $\omega_{t,opt}$.

The structure of the labor income process is as in Cocco et al. (2005), i.e., labor income prior to attaining retirement age is stochastic, whereas labor income after attaining retirement age is a constant share of the labor income received in the last working period. We provide more details on the calibration of the labor income process, which we updated relative to the work of Cocco et al. (2005) using more recent empirical data, in Section 4.2.3.

We employ a constant relative risk aversion (CRRA) utility function with risk aversion parameter γ, denote the individual's time preference parameter by β, and include mortality risk in our model. $f(t)$ denotes the individual's probability of survival from time t to $t+1$.

When $R = 1 + r$ is the gross return on the risk–free asset, $G_t = 1 + r_t^S$ the gross return on the risky asset from time t to $t+1$, and W_t the total wealth (already including period t's labor income L_t) that belongs to the individual at time t, the individual's optimization problem can be written as

$$\max_{\{C_t,\omega_{t,opt}\}} V_t\left(X_t\right) = \left(C_t^{1-\gamma} + \beta f(t)\mathbb{E}_t\left[V_{t+1}\left(X_{t+1}\right)^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \tag{4.2.1}$$

such that

$$W_{t+1} = \left(W_t - C_t\right)\left(\omega_{t,opt}G_t + \left(1-\omega_{t,opt}\right)R\right) + L_{t+1}, \tag{4.2.2}$$

with $C_t > 0$ and $\omega_{t,opt} \in [0,1]$. $X_t = [t,L_t,W_t]$ denotes the vector of state variables for this optimization problem.

It is not possible to solve this optimization problem analytically. We therefore solve it numerically using backward induction and reduce the number of state variables by normalizing with total wealth. The more technical details are outlined in Appendix 4.B.

4.2.2. Heuristic portfolio rule

The complexity of the optimization problem outlined in Section 4.2.1 renders solving for optimal investment decisions very difficult to basically impossible for most individuals. Yet, a careful choice of ones investment strategy is also important for these individuals—among others to sustain ones living standards during retirement age. In this manuscript, we propose a heuristic portfolio rule according to which the individual's equity exposure is a linear function of its labor–to–wealth ratio and age. Plugging ones age and ones personal labor–to–wealth ratio into the formula then also allows individuals without expert knowledge in portfolio choice to determine a close to optimal equity exposure. More precisely, to obtain the coefficients for our linear function we run linear regressions for the working and retirement periods separately using 10,000 simulated paths on optimal decisions and state variables as data. Separating between working and retirement age is important as labor income is typically more volatile than retirement benefits suggesting that retirees could bear higher financial risks.

We let $\omega_{i,t,opt} \in [0,1]$ denote the optimal equity exposure for simulated individual i at time t, $\frac{L_{i,t}}{W_{i,t}}$ its simulated labor–to–wealth ratio, and $age_{i,t}$ its age.[1] When t_{ret} denotes the first period in which the individual is retired, our regression equations for working and retirement period are given by

$$\omega_{i,t,opt} = \begin{cases} \alpha_w + \beta_{1,w} \cdot \frac{L_{i,t}}{W_{i,t}} + \beta_{2,w} \cdot age_{i,t} + \epsilon_{i,t,w}, & t < t_{ret} \\ \alpha_r + \beta_{1,r} \cdot \frac{L_{i,t}}{W_{i,t}} + \beta_{2,r} \cdot age_{i,t} + \epsilon_{i,t,r}, & t \geq t_{ret}, \end{cases} \tag{4.2.3}$$

in which α_w (α_r) is the intercept, $\beta_{1,w}$ ($\beta_{1,r}$) and $\beta_{2,w}$ ($\beta_{2,r}$) are the regression coefficients, and $\epsilon_{i,t,w}$ ($\epsilon_{i,t,r}$) is the error term of the regression equation for the working (retirement) period, respectively.

When $\omega_{i,t,heur} \in [0,1]$ denotes individual i's heuristic equation–implied equity exposure at time t, our **heuristic portfolio rule** is given by

$$\omega_{i,t,heur} = \begin{cases} \hat{\alpha}_w + \hat{\beta}_{1,w} \cdot \frac{L_{i,t}}{W_{i,t}} + \hat{\beta}_{2,w} \cdot age_{i,t}, & t < t_{ret} \\ \hat{\alpha}_r + \hat{\beta}_{1,r} \cdot \frac{L_{i,t}}{W_{i,t}} + \hat{\beta}_{2,r} \cdot age_{i,t}, & t \geq t_{ret}, \end{cases} \tag{4.2.4}$$

in which $\hat{\alpha}_w,\hat{\alpha}_r,\hat{\beta}_{1,w},\hat{\beta}_{1,r},\hat{\beta}_{2,w},\hat{\beta}_{2,r}$ are the coefficients estimated from equation (4.2.3) for working and retirement period, respectively, using OLS.

4.2.3. Calibration

In this section, we outline the calibration of our model. We set the financial market parameters to historical estimates. Individuals can invest into two different assets: (i) a risk–free asset paying a constant annual return of $r = 3.9\%$, the average 3–month Treasury Bill rate between 1946 and 2021 and (ii) a risky asset representing a stock market index.

The average annual return of the S&P500 from 1946 to 2021 was 12.6% and its standard deviation, σ_S, 16.6%. As there are doubts whether such high returns can be achieved in the future (Mehra and Prescott (1985); Dimson et al. (2002)), we follow

[1]We exclude all observations with $\omega_{i,t,opt} = 1$ to avoid using observations, in which individuals might want to violate their short–selling constraint.

Claus and Thomas (2001) and set the equity premium to 3%, implying an expected annual return, μ_S, of 6.9%.

We choose the preference parameters to match the empirically–observed growth of wealth in the Survey of Consumer Finances data. More specifically, we set the individual's time preference factor, β, to 0.95, its risk aversion, γ, to 10, and assume that the individuals do not have a bequest motive.

We assume that individuals enter the life cycle at age 20, after having graduated from high school and work until the age of 65.[2] During the working period, the individual's labor income process is subject to a stochastic growth rate and correlated to the return on the risky asset. We set the correlation between stock return and labor income, $\rho_{S,L}$, to 0.047, the empirical estimate of Cocco (2005). At age 66, individuals are retired, i.e., $t_{ret} = 66$, and receive a deterministic income determined by the labor income received in the last working period. Individuals die at latest at the age of 100. Prior to age 100, we use the unisex life table of the Center of Disease Control and Prevention to determine the probabilities of survival from period t to $t + 1$ (Arias and Xu (2018)).[3]

Using PSID data from all waves between 1980 and 2019 and closely following the methodology outlined, for example, in Cocco et al. (2005), Love (2010), and Fischer and Khorunzhina (2019), we estimate labor income processes for married households, single females, and single males, respectively. For each of the three groups, we distinguish three different educational attainments: (i) no high school degree, (ii) high school degree, and (iii) college degree. For married households, we let the educational attainment of the head of household determine the couples educational attainment. Table 4.2.1 summarizes our estimated coefficients. We provide more details on the estimation of the labor income process in Appendix 4.A.

In our base case analysis, whose calibration is summarized in Table 4.2.2, we focus on married households as more than the half of the U.S. population aged 25 to 54 is married (Fry and Parker (2021)).[4] When taking a look at the total U.S. population, even about two thirds are married, and we aim at covering a large part of the U.S. population with our analysis.[5]

[2]We document the robustness of our results to other educational attainments in Section 4.4.

[3]We document the robustness of our results to different mortality tables for males and females in Section 4.4.

[4]https://www.statista.com/, last retrieved September 6, 2022, provides an overview of the marital status of the total U.S. population.

[5]We document the robustness of our results to other marital statuses in Section 4.4.

Table 4.2.1: Coefficient estimates for labor income processes

Description	No high school	High school	College
	Panel A: Married		
Constant	-1.6552	-1.3672	-3.3179
Age	0.1389	0.1046	0.2550
$Age^2/100$	-0.2917	-0.1905	-0.4717
$Age^3/10{,}000$	0.1960	0.1165	0.2905
Replacement rate	0.8231	0.8493	0.8888
Standard deviation permanent shock	0.0778	0.0850	0.1086
	Panel B: Single female		
Constant	-3.9171	-1.9646	-5.8781
Age	0.3226	0.1560	0.4805
$Age^2/100$	-0.7358	-0.3242	-1.0068
$Age^3/10{,}000$	0.5368	0.2201	0.6875
Replacement rate	0.9219	0.9269	0.9290
Standard deviation permanent shock	0.1037	0.0825	0.0873
	Panel C: Single male		
Constant	-1.7660	-2.5223	-6.3490
Age	0.1392	0.2046	0.5000
$Age^2/100$	-0.3087	-0.4429	-1.0779
$Age^3/10{,}000$	0.2170	0.3108	0.7615
Replacement rate	0.8076	0.8751	0.8609
Standard deviation permanent shock	0.0801	0.1016	0.1587

This table summarizes the parameter estimates for labor income processes for married households (Panel A), single females (Panel B), and single males (Panel C), respectively. For each of the three groups, we distinguish three different educational attainments: (i) no high school degree (first column), (ii) high school degree (second column), and (iii) college degree (third column). For married households, we let the educational attainment of the head of household determine the couples' educational attainment. The reported coefficients are results from fixed effect regressions using PSID data from all waves between 1980 and 2019. We provide more details on the estimation process, which closely follows the one proposed by Cocco et al. (2005), in Appendix 4.A.

4.3. Results

In this section, we first present our base case results and show that the evolution of model–predicted wealth matches the empirically–observed evolution of wealth. Second, we show that welfare gains when investing optimally instead of according to our heuristic are very low whereas welfare gains from investing according to our heuristic compared to following the average investment strategy of U.S. individuals in the PSID data are high. In other words, our heuristic leads to very close to optimal investment decisions, such that individuals following our strategy are almost as well off as individuals following the computationally very demanding to determine optimal

Table 4.2.2: Base case calibration

Description	Symbol	Value	Source
	Individuals' specifics		
Degree of risk aversion	γ	10	Cocco et al. (2005)
Time preference factor	β	0.95	Own calibration
Educational attainment	–	high school	Cocco et al. (2005)
Marital status	–	married	Own calibration
Entry age life cycle	–	20	Cocco et al. (2005)
Retirement age	t_{ret}	66	Cocco et al. (2005)
	Financial market		
Expected nominal return on equity	μ_S	6.9%	Claus and Thomas (2001)
Standard deviation of return on equity	σ_S	16.6%	Historical estimate
Risk–free rate	r	3.9%	Historical estimate
	Correlation		
Correlation stock return/labor income	$\rho_{S,L}$	0.047	Cocco (2005)

This table summarizes the base case calibration of our model.

investment strategy. Simultaneously, individuals following our heuristic strategy are substantially better off than individuals investing as in the data. That is, our model predicts that by investing according to our simple heuristic, most individuals could be better off.

4.3.1. Base case results

Table 4.3.1 summarizes the estimated regression coefficients for our base case for the working and retirement period, respectively. For both, working and retirement period, we estimate small negative coefficients for age but large positive coefficients for the labor–to–wealth ratio. The positive coefficients on the labor–to–wealth ratio indicate that individuals with relatively high labor income relative to their savings can take a few more chances in their investment strategies as returns on stocks and labor income are only weakly positively correlated and possible losses on investments can be easier made up for with future labor income or retirement benefits. The negative coefficients on age are substantially smaller than, for example, those under the “100–age rule”, which is a stock allocation rule frequently provided by personal financial advisors and which states that individuals should invest (100–age)% in stocks at every point in life (Choi (2022)). Unfortunately, this rule does not take into account how much an individual has been saving, thus potentially leading to too risky investment strategies

Table 4.3.1: Regression coefficients for base case

	Work	Retirement
Intercept	0.2952	0.8017
Coefficient $\frac{L}{W}$	1.1571	1.1631
Coefficient age	-0.0023	-0.0072
R^2	0.9114	0.7336

This table summarizes the regression coefficients for our base case calibration for working period (first column) and retirement period (second column), respectively. Intercept corresponds to $\hat{\alpha}_w$ ($\hat{\alpha}_r$), coefficient $\frac{L}{W}$ to $\hat{\beta}_{1,w}$ ($\hat{\beta}_{1,r}$), and coefficient age to $\hat{\beta}_{2,w}$ ($\hat{\beta}_{2,w}$) in equation (4.2.4) for the working (retirement) period.

for individuals with very large savings and too few upside potential for individuals with low savings.

It is important to ensure that the model predictions match key features in the data. We therefore compare the average model–predicted net worth with the average empirically–observed net worth. Panel A of Figure 4.3.1 depicts the evolution of the average empirically–observed net worth (dashed line) and of the average model–predicted net worth (solid line), respectively. We determine the average empirically–observed net worth by using data from the 2019 Survey of Consumer Finances, the most recent survey available.[6] The average model–predicted net worth is the average of 10,000 simulations on the optimal paths of the optimization problem outlined in Section 4.2.1.

We find that the evolution of our model–predicted net worth matches the one of the empirically–observed net worth very well as the solid and dashed line in Panel A of Figure 4.3.1 intersect for every age bracket.

Panel B of Figure 4.3.1 depicts the evolution of the equity exposure over the life cycle for individuals investing optimally (solid line) and according to our heuristic (dashed line). We find that our heuristic matches the evolution of the equity exposure over the life cycle well.

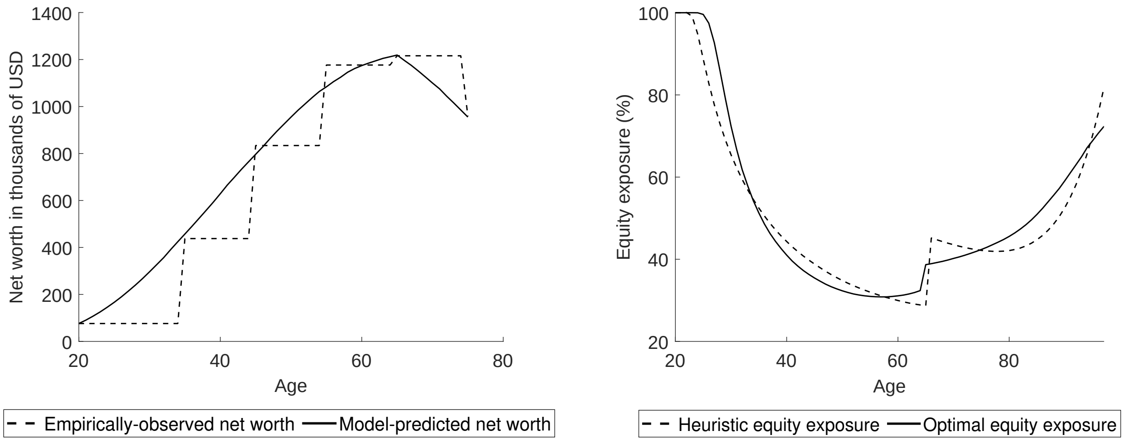
The welfare gains individuals can obtain when investing optimally instead of according to our heuristic are close to zero. 20–year old individuals investing according to our heuristic only have to be endowed with a 0.06% higher level of lifetime consumption than those investing optimally to achieve the same level of of present discounted lifetime utility. In other words, by giving up less than 0.1% of lifetime consumption, individuals can refrain from going through the complicated dynamic optimizations

[6]The data can be downloaded from https://www.federalreserve.gov, last retrieved August 2, 2022.

Figure 4.3.1: Base case results

Panel A:
Net worth (in thousands of USD)

Panel B:
Equity exposure (in %)



Panel A compares the evolution of model–predicted net worth (solid line) with empirically–observed net worth (dashed line). The average model–predicted net worth is the average of 10,000 simulations on the optimal paths of the optimization problem outlined in Section 4.2.1. The average empirically–observed net worth is obtained using data from the 2019 Survey of Consumer Finances. Panel B compares the optimal (solid line) and heuristic (dashed line) equity exposure over the life cycle. The optimal and heuristic equity exposure are averages of 10,000 simulated paths, respectively.

required to determine optimal equity exposures. Given a lifetime consumption level of about USD 2,800,000 (the empirical estimate of the 2019 Consumer Expenditures Survey) individuals would only gain a one–time amount of around USD 1,600 when investing optimally.[7] In other words, until age 60, individuals gain less than USD 40 annually when following the optimal investment strategy.

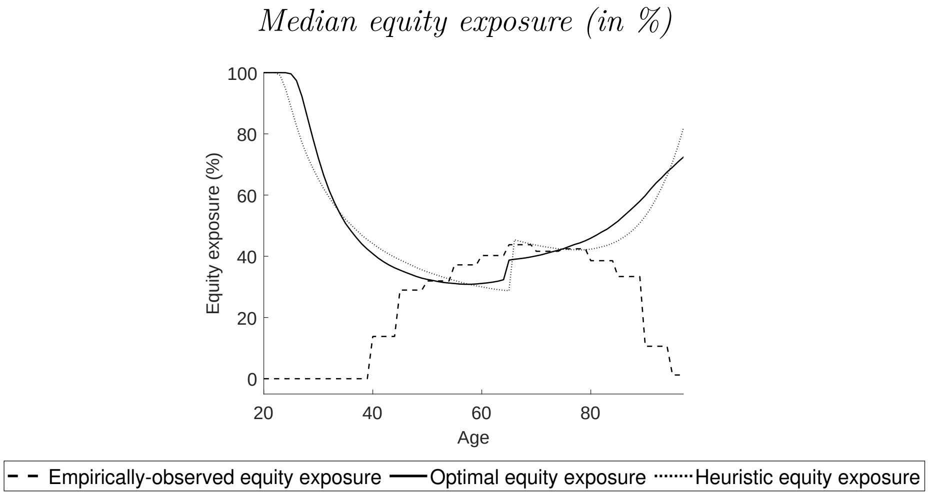
4.3.2. Welfare implications of sub–optimal asset allocation

The average U.S. individual does either only invest a limited share of its net worth into stocks (see Figure 4.3.2) or does not participate in the stock market at all (Guiso and Sodini (2013)). This holds especially true for younger individuals and individuals at older age. In this section, we quantify the welfare implications of under–investment in equity and of stock market non–participation, respectively.

To determine the equity exposure of U.S. individuals, we use PSID data from all waves available between 2001 and 2019, the years for which detailed data on investments is available in the Wealth Supplements. We restrict our sample to high

[7] We provide details on how we calculate the lifetime consumption level in Appendix 4.D.

Figure 4.3.2: Optimal, heuristic as well as empirically–observed equity exposures



This figure depicts (i) the optimal equity exposure for high school graduates obtained when solving the optimization problem introduced in Section 4.2.1 by the solid line, (ii) the heuristic equity exposure for high school graduates obtained when following the methodology outlined in Section 4.2.2 by the dotted line, and (iii) the empirically–observed median equity exposure of U.S. high school graduates by the dashed line. To calculate the empirically–observed equity exposure we use data from all PSID waves between 2001 and 2019. For a better readability of the graph, we depict the average empirically–observed equity exposures for five–year age brackets between age 20 and 100. We outline how we calculate the empirically-observed equity exposure in greater detail in Appendix 4.C. The optimal and heuristic equity exposure are averages of 10,000 simulated paths, respectively.

school graduates that are aged 20 to 100. We provide more details on the data selection process in Appendix 4.C.

Figure 4.3.2 compares the optimal equity exposure (solid line), which is the average of 10,000 simulated paths on the optimal decisions obtained by solving the optimization problem outlined in Section 4.2.1 and the heuristic equity exposure (dotted line) estimated as outlined in Section 4.2.2, with the empirically–observed median equity exposure (dashed line) of the U.S. population. For a better readability of the graph, we depict the average empirically–observed equity exposures for five–year age brackets between age 20 and 100.

Figure 4.3.2 provides descriptive evidence that the average U.S. individual's asset allocation strategy is far from optimal: Equity exposures in younger ages and very old ages are too low relative to our model–implied rates, implying that individuals forego the equity premium. This sub–optimal investment pattern further highlights the importance of simple rules of thumbs, such as the one we are proposing.

We use the empirically–observed median equity exposure to calculate the utility from following this asset allocation strategy and subsequently the welfare gains when investing according to our heuristic instead of according to the investment strategy observed in the data.[8]

As a consequence of the sub–optimal investment pattern depicted in Figure 4.3.2, we observe significant welfare implications. A 20–year old individual, whose equity exposure corresponds to the median equity exposure of U.S. individuals, has to be endowed with a 1.3% higher level of lifetime consumption to achieve the same present discounted lifetime utility as the individual investing according to our heuristic. In other words, individuals would gain about USD 35,000 by following our heuristic portfolio strategy, roughly corresponding to the price of a new Chevrolet Silverado.

44% of the U.S. population do not invest into stocks at all.[9] When not investing into the stock market at all, the welfare gains from following our heuristic are even higher: A 20–year old individual never investing into stocks over the life cycle has to be endowed with a 1.9% higher level of lifetime consumption to achieve the same present discounted lifetime utility as the individual investing according to our heuristic. Following our heuristic would lead to financial gains of around USD 52,000, which is more than the 2019 average after–tax U.S. household income for individuals whose educational attainment is a high school degree.[10] That is, by following our heuristic portfolio strategy, these individuals would on average gain more than the value of a year's salary or, roughly equivalently, the value of a new Tesla Model 3.

4.4. Robustness tests

In this section, we show that our results are robust to different educational attainments, i.e., that our results also hold for married couples whose heads do not have a high school degree and for married couples whose heads are college graduates. Furthermore, we show that our heuristic is a suitable rule of thumb for single males and single females with different educational attainments.

[8]To calculate the utility from the empirically–observed strategy we plug in the median equity exposure into the utility function and use the same returns on labor income and stocks as when calculating the utility from the heuristic strategy to ensure comparability of our results. That is, our welfare effects are rather conservatively measured and should be substantially higher for many individuals.

[9]https://www.statista.com/, last retrieved August 2, 2022, provides an overview of stock market participation rates of U.S. individuals over time.

[10]An overview of U.S. post–tax household income for different educational attainments can be found in Table C–1 on https://www.census.gov, last retrieved September 9, 2022.

Table 4.4.1: Regression coefficients for married couples with different educational attainments

	Work	**Retirement**
Panel A: No high school degree		
Intercept	0.3007	0.7526
Coefficient $\frac{L}{W}$	1.1819	1.1526
Coefficient age	-0.0025	-0.0068
R^2	0.9159	0.7224
Panel B: College degree		
Intercept	0.1999	0.8384
Coefficient $\frac{L}{W}$	0.9973	1.1869
Coefficient age	-0.0009	-0.0077
R^2	0.9134	0.7369

This table summarizes the regression coefficients for married couples whose heads of household do not have a high school degree (Panel A) and whose heads of household have a college degree (Panel B), respectively. The first column (Work) reports the coefficients for the working period and the second column (Retirement) the coefficients for the retirement period. Intercept corresponds to $\hat{\alpha}_w$ ($\hat{\alpha}_r$), coefficient $\frac{L}{W}$ to $\hat{\beta}_{1,w}$ ($\hat{\beta}_{1,r}$), and coefficient age to $\hat{\beta}_{2,w}$ ($\hat{\beta}_{2,w}$) in equation (4.2.4) for the working (retirement) period.

More specifically, relative to our base case setting, we consecutively vary two assumptions: (i) the educational attainment and thereby the labor income process and (ii) when analyzing the case of single females and single males also the probabilities of survival by using the respective estimates of Arias and Xu (2018).

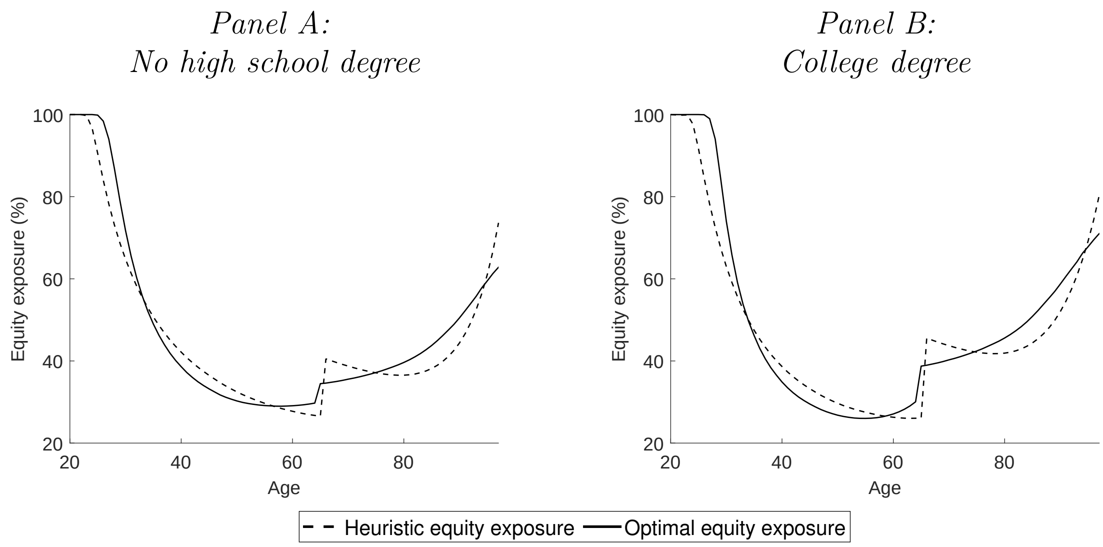
4.4.1. Educational attainment

Our base case analysis outlined in Section 4.3.1 focuses on married couples whose heads of household have a high school degree. Table 4.4.1 summarizes the regression coefficients for working and retirement period for married couples whose heads do not have a high school degree and whose heads have a college degree, respectively.

Similarly to Table 4.3.1, we find coefficients close to zero for age but large positive coefficients for the labor–to–wealth ratio, once more shedding light on the importance of taking labor income into account when making asset allocation decisions.

Panel A of Figure 4.4.1 compares the evolution of the optimal equity exposure (solid line) with the heuristic equity exposure (dashed line) over the life cycle for married couples whose heads do not have a high school degree. In a similar fashion as Panel A, Panel B of Figure 4.4.1 compares the evolution of the optimal equity exposure (solid line) with the heuristic equity exposure (dashed line) over the life

Figure 4.4.1: Optimal and heuristic equity exposures for married couples with different educational attainments



Panel A and Panel B compare the evolution of the optimal and heuristic equity exposure over the life cycle for married couples whose heads do not have a high school degree and for married couples whose heads have a college degree, respectively. The optimal equity exposure is depicted by the solid line and the heuristic equity exposure by the dashed line. All results are averages of 10,000 simulated paths.

cycle for married couples whose heads do have a college degree. Overall, Figure 4.4.1 indicates that our heuristic captures the shape of the equity exposure well for different educational attainments. Yet, our heuristic strategies remain able to closely match optimal equity exposures.

Consequently, the welfare gains when investing optimally instead of according to our heuristic are close to zero: 0.05% of lifetime consumption for married couples whose heads do not have a college degree and 0.07% for married couples whose heads have a college degree.

4.4.2. Single individuals

Even though the largest share of the U.S. population is married, the share of individuals being single has been increasing over the last 30 years. In 1990, only 29% of the U.S. population aged 25 to 54 have been unpartnered whereas in 2019 already 38% have been unpartnered (Fry and Parker (2021)). Therefore, we show that our results remain robust for single females and single males with three different educational attainments: (i) no high school degree, (ii) high school degree, and (iii) college degree.

Table 4.4.2: Regression coefficients for single individuals with different educational attainments

	Single females		Single males	
	Work	**Retirement**	**Work**	**Retirement**
	Panel A: No high school degree			
Intercept	-0.0653	0.9250	0.2460	0.6823
Coefficient $\frac{L}{W}$	1.4073	1.2257	1.1975	1.1243
Coefficient age	0.0034	-0.0088	-0.0017	-0.0059
R^2	0.9500	0.7351	0.9224	0.7530
	Panel B: High school degree			
Intercept	0.1913	0.9023	0.0187	0.7460
Coefficient $\frac{L}{W}$	1.2747	1.1677	1.1387	1.1514
Coefficient age	-0.0004	-0.0082	0.0018	-0.0066
R^2	0.9108	0.7224	0.9534	0.7604
	Panel C: College degree			
Intercept	0.2811	0.9696	-0.1391	0.8202
Coefficient $\frac{L}{W}$	0.8007	1.0485	1.1068	1.1831
Coefficient age	-0.0003	-0.0081	0.0043	-0.0077
R^2	0.8903	0.6978	0.9344	0.7745

This table summarizes the regression coefficients for single females and single males with three different educational attainments: (i) no high school degree (Panel A), (ii) high school degree (Panel B), and (iii) college degree (Panel C). The first and third column (Work) report the results for the working period and the second and fourth column (Retirement) the results for the retirement period, respectively. Intercept corresponds to $\hat{\alpha}_w$ ($\hat{\alpha}_r$), coefficient $\frac{L}{W}$ to $\hat{\beta}_{1,w}$ ($\hat{\beta}_{1,r}$), and coefficient age to $\hat{\beta}_{2,w}$ ($\hat{\beta}_{2,w}$) in equation (4.2.4) for the working (retirement) period.

Table 4.4.2 provides the regression coefficients for the heuristic portfolio rules, Figure 4.4.2 compares the evolution of the optimal and heuristic equity exposure over the life cycle, and Table 4.4.3 summarizes the welfare gains when investing optimally instead of according to our heuristic.

Table 4.4.2 provides further evidence, that the labor–to–wealth ratio has a stronger impact on the asset allocation strategy than age, as all coefficient estimates for the labor–to–wealth ratio are positive and significantly different from zero, whereas all coefficient estimates for age are very close to zero.

Figure 4.4.2 shows that the optimal equity exposures (depicted by the solid lines) and the heuristic equity exposures (depicted by the dashed lines) are qualitatively similar for single females and single males. Quantitatively, the equity exposure of single females is higher (lower) for those who have a high school and college degree (for those who do not have a high school degree), which can be explained by the

Table 4.4.3: Welfare gains for single individuals with different educational attainments when investing optimally instead of according to our heuristic

	Single females	**Single males**
No high school degree	0.07%	0.05%
High school degree	0.06%	0.05%
College degree	0.01%	0.19%

This table summarizes welfare gains in percent of lifetime consumption for single females (first column) and single males (second column) with three different educational attainments: (i) no high school degree (first row), (ii) high school degree (second row), and (iii) college degree (third row) when investing optimally instead of according to our heuristic. The reported welfare gains can be interpreted as follows: A 20–year old individual investing according to our heuristic only has to be endowed with the reported percent higher level of lifetime consumption to achieve the same level of present discounted lifetime utility as a 20–year old individual investing optimally. For example, it costs less than 0.1% of lifetime consumption for a single female with no high school degree to rely on our heuristic investment strategy compared to following the optimal one.

lower (higher) volatility of the labor income stream (see Table 4.2.1). Overall, our heuristic captures the evolution of the optimal equity exposure over the life cycle very well—irrespective of the specific educational attainment.

That our rule of thumb works very well for single males and single females is further supported by the welfare implications reported in Table 4.4.3. Single males who have a college degree face the highest welfare gains when switching from the heuristic to the optimal portfolio policy. Yet, even a 20–year old male college graduate following our heuristic only has to be endowed with a 0.19% higher level of lifetime consumption to achieve the same level of present discounted lifetime utility as a 20–year college graduate investing optimally.

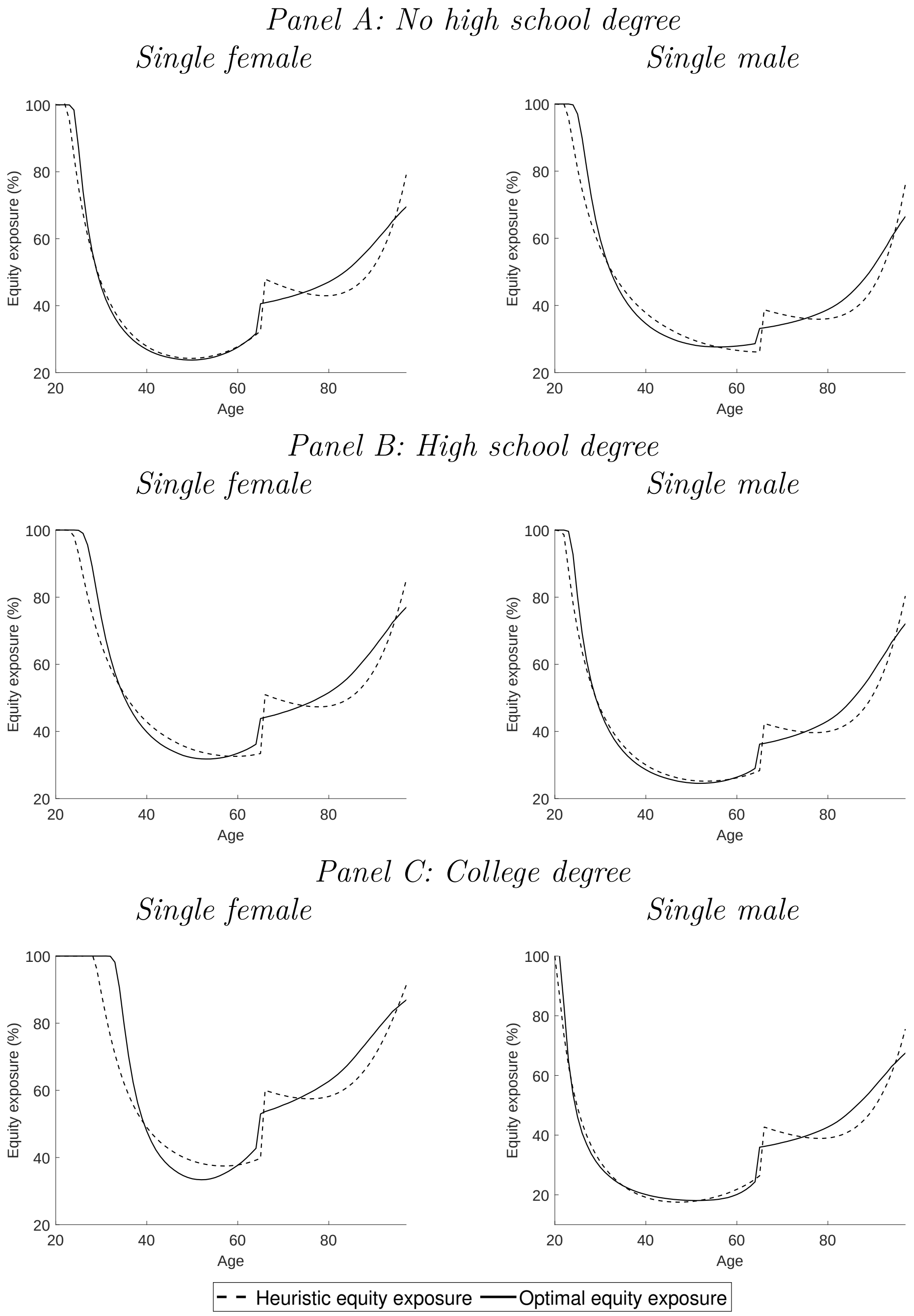
4.5. Conclusion

In our manuscript, we propose a heuristic portfolio rule, which enables individuals to easily determine their equity exposure as a linear function of their age and labor–to–wealth ratio. We find that welfare gains when investing optimally instead of according to our heuristic are extremely small. That is, our heuristic portfolio rule is both, easily implementable for private investors and not very costly. Likewise, it could also easily be implemented at low costs by private investors' financial advisors.

Furthermore, we shed light on the welfare implications of sub–optimal asset allocation strategies. Following our heuristic instead of not participating in the stock market over the life cycle leads to welfare gains of 1.9% (around USD 52,000), roughly corresponding to the value of a new Tesla Model 3; and the average U.S. individual with its current equity exposure faces welfare gains of 1.3% (around USD 35,000), roughly corresponding to the value of a new Chevrolet Silverado, when investing according to our heuristic. That is, when following our simple heuristic, individuals are typically one car better off.

In extensive robustness checks, we demonstrate our heuristic's ability to also work for different educational attainments and family compositions. In all these scenarios, our heuristic portfolio strategies remain close to their optimal counterparts and consequently investing optimally only leads to very small welfare gains.

Figure 4.4.2: Optimal and heuristic equity exposures for single individuals with different educational attainments



This figure compares the optimal (solid line) and heuristic (dashed line) equity exposure for single males and single females over the life cycle for three different educational attainments: (i) no high school degree (Panel A), (ii) high school degree (Panel B), and (iii) college degree (Panel C). All results are averages of 10,000 simulated paths.

References

Arias, E. and J. Xu (2018). "United States Life Tables, 2015. National Vital Statistics Report." *National Center for Health Statistics* 67. (7).

Baker, S. R., R. A. Farrokhnia, S. Meyer, M. Pagel, and C. Yannelis (2020). "How Does Household Spending Respond to an Epidemic? Consumption During the 2020 Covid-19 Pandemic." *Review of Asset Pricing Studies* 10. (4), 834–862.

Carroll, C. D. and A. A. Samwick (1997). "The Nature of Precautionary Wealth." *Journal of Monetary Economics* 40. (1), 41–71.

Choi, J. J. (2022). "Popular Personal Financial Advice versus the Professors." *Journal of Economic Perspectives* 36. (4), 167–92.

Clarkson, G. P. and A. H. Meltzer (1960). "Portfolio Selection: A Heuristic Approach." *Journal of Finance* 15. (4), 465–480.

Claus, J. and J. Thomas (2001). "Equity Premia as Low as Three Percent? Evidence from Analysts' Earnings Forecasts for Domenstic and International Stock Markets." *Journal of Finance* 56. (5), 1629–1666.

Cocco, J. (2005). "Portfolio Choice in the Presence of Housing." *Review of Financial Studies* 18. (2), 535–567.

Cocco, J., F. J. Gomes, and P. J. Maenhout (2005). "Consumption and Portfolio Choice over the Life Cycle." *Review of Financial Studies* 18. (2), 491–533.

Dimson, E., P. Marsh, and M. Staunton (2002). *Triumph of the Optimists: 101 Years of Global Investment Returns*. Princeton University Press.

Fischer, M. and M. F. Gallmeyer (2016). "Heuristic Portfolio Trading Rules with Capital Gain Taxes." *Journal of Financial Economics* 119. (3), 611–625.

Fischer, M. and N. Khorunzhina (2019). "Housing Decision with Divorce Risk." *International Economic Review* 60. (3), 1263–1290.

Fischer, M., H. Kraft, and C. Munk (2013). "Asset Allocation over the Life Cycle: How Much Do Taxes Matter?" *Journal of Economic Dynamics and Control* 37. (11), 2217–2240.

Foster, A. C. (2015). "Consumer Expenditures Vary by Age." *Beyond the Numbers: Prices & Spending* 4. (14).

Fry, R. and K. Parker (2021). "Rising Share of U.S. Adults are Living Without a Spouse or Partner." Pew Research Center, available at https://www.pewresearch.org/social-trends/2021/10/05/rising-share-of-u-s-adults-are-living-without-a-spouse-or-partner/.

Gaudecker, H.-M. v. (2015). "How Does Household Portfolio Diversification Vary with Financial Literacy and Financial Advice?" *Journal of Finance* 70. (2), 489–507.

Guiso, L., M. Haliassos, and T. Japelli (2003). "Household Stockholding in Europe: Where Do We Stand and Where Do We Go?" *Economic Policy* 18. (1), 123–170.

Guiso, L. and P. Sodini (2013). "Household Finance: An Emerging Field." In: *Handbook of the Economics of Finance*. Vol. 2. Elsevier, 1397–1532.

Love, D. A. (2010). "The Effect of Marital Status and Children on Savings and Portfolio Choice." *Review of Financial Studies* 23. (1), 385–432.

— (2013). "Optimal Rules of Thumb for Consumption and Portfolio Choice." *The Economic Journal* 123. (571), 932–961.

Lusardi, A. and O. S. Mitchell (2007). "Baby Boomer Retirement Security: The Roles of Planning, Financial Literacy, and Housing Wealth." *Journal of Monetary Economics* 54. (1), 205–224.

Mehra, R. and E. C. Prescott (1985). "The Equity Premium: A Puzzle." *Journal of Monetary Economics* 15. (2), 145–162.

Song, C. (2020). "Financial Illiteracy and Pension Contributions: A Field Experiment on Compound Interest in China." *Review of Financial Studies* 33. (2), 916–949.

Van Rooij, M., A. Lusardi, and R. Alessie (2011). "Financial Literacy and Stock Market Participation." *Journal of Financial Economics* 101. (2), 449–472.

Winter, J. K., K. Schlafmann, and R. Rodepeter (2012). "Rules of Thumb in Life-Cycle Saving Decisions." *The Economic Journal* 122. (560), 479–501.

Appendix 4.A Estimation of the labor income process

In this section, we outline in greater detail how we obtain the coefficient estimates for the labor income process reported in Table 4.2.1. Our estimation procedure closely follows the one proposed by Cocco et al. (2005) and subsequently used, for example, by Love (2010) and Fischer and Khorunzhina (2019).

Using PSID data from the waves from 1980 to 2019 and excluding individuals from the SRC, Immigrants, SEO, and Latino sample, we estimate labor income processes for married households, single females, and single males, separately. For each of the three groups, we distinguish three different educational attainments: (i) no high school degree, (ii) high school degree, and (iii) college degree. For married households, we let the educational attainment of the head of household determine the couples' educational attainment.

First, we run a fixed effect regression of the natural logarithm of after–tax labor income on a full set of age dummies for ages 20 to 65 for each of the above defined nine subgroups. In line with Cocco et al. (2005), we allow for a broad definition of labor income by adding potential unemployment compensation, workers compensation, (supplemental) social security, other welfare, child support, and total transfers to the reported labor income. As PSID data reports before–tax labor income, we use the *TAXSIM* software to calculate income tax liabilities and subsequently the after–tax labor income.

Second, we fit a third order polynomial to the coefficient estimates for the age dummies obtained by running a fixed effect regression.

Third, we calculate the replacement rate, by running two fixed effect regressions on a full set of age dummies, one for working individuals aged between 55 and 62 and one for retired individuals aged between 67 to 80. Then, to obtain the replacement rate we divide the implied income from the coefficient estimates for ages 67 to 80 by the implied income from the coefficient estimates for ages 55 to 62.

Finally, to obtain the standard deviation of the permanent shock to the labor income, we follow—in line with Cocco et al. (2005)—the variance decomposition procedure introduced by Carroll and Samwick (1997).

Appendix 4.B Solution of the optimization problem

It is impossible to solve the optimization problem presented in Section 4.2.1 analytically. We therefore have to solve it numerically. To expedite the computations, we normalize the optimization problem with total wealth at time t, W_t.

Defining $c_t = \frac{C_t}{W_t}$, $v_t(x_t) = \frac{V_t(X_t)}{W_t}$, and denoting the gross return on labor income from period t to $t+1$ by R_t^L, the normalized optimization problem is

$$\max_{\{c_t,\omega_{t,opt}\}} v_t(x_t) = \left(c_t^{1-\gamma} + \beta f(t)\,\mathbb{E}_t\left[v_{t+1}(x_{t+1})^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}, \tag{4.B.1}$$

such that

$$\frac{W_{t+1}}{W_t} = (1-c_t)\left(\omega_{t,opt}G_t + \left(1-\omega_{t,opt}\right)R\right) + \frac{L_t}{W_t}R_t^L, \tag{4.B.2}$$

$$c_t > 0,\ \omega_{t,opt} \in [0,1]\,. \tag{4.B.3}$$

$x_t = \left[t, \frac{L_t}{W_t}\right]$ is the new vector of state variables for this optimization problem. We discretize the continuous state variable $\frac{L_t}{W_t}$ and compute the expectation in equation (4.B.1) using Gaussian quadrature. The resulting simplified optimization problem is then solved using backward induction.

Appendix 4.C Empirically–observed equity exposure

We use data from all PSID waves between 2001 and 2019, i.e., from in total 10 waves, to determine the average U.S. individual's equity exposure.

As there is no variable directly reporting equity exposure, we calculate the equity exposure for each individual i aged t, $\omega_{i,t,emp}$, using the following formula:

$$\omega_{i,t,emp} = \frac{\text{wealth with equity}_{i,t} - \text{wealth without equity}_{i,t}}{\text{wealth with equity}_{i,t}} \tag{4.C.1}$$

.

and exclude, in a first data cleaning step, all observations for which $\omega_{i,t,emp} \notin [0,1]$, which is about 15% of all observations. This rather high number of dismissed observations can be explained by the fact that PSID data is self–reported and therefore subject to substantial inaccuracies. After this cleaning step, our data set consists of 72,736 observations.

In a second cleaning step, we dismiss all observations that do not report their educational attainment to be a high school degree to be in line with the calibration of our theoretical model. After this cleaning step, the sample used for our analysis consists of 21,076 observations.

Finally, we are able to calculate the median equity exposure for high school graduates for each age between 20 and 100, which is reported in Figure 4.3.2.

Appendix 4.D Consumer expenditures

We use data from the 2019 Consumer Expenditures Survey to calculate U.S. individuals' lifetime consumption levels.[11] We focus on the year 2019 in our analysis as 2020 and 2021 U.S. individuals' consumption patterns were strongly affected by the Covid–19 pandemic (Baker et al. (2020)).

First, we dismiss all observations whose marital status is not married and whose reference person's educational attainment is not a high school degree to be in line with our base case calibration. After this cleaning step our sample consists of 11,133 observations.

Second, we sort the individuals—in line with Foster (2015)—into seven different age brackets: 1) age 20 to 24; 2) age 25 to 34; 3) age 35 to 44; 4) age 45 to 54; 5) age 55 to 64; 6) age 65 to 74; and 7) age 75 to 100.

Then, we calculate the median quarterly expenditures for individuals in each of the these seven age brackets separately.

Finally, to obtain the average lifetime consumption of U.S. married couples whose reference person's educational attainment is a high school degree, we (i) multiply the median quarterly expenditures for individuals by four to get the median annual expenditures, (ii) multiply the median annual expenditures by the length of each age bracket, and (iii) add the numbers for all age brackets up. We estimate an average lifetime consumption of USD 2,711,151.

[11]The data can be downloaded from https://www.bls.gov/cex/pumd_data.htm, last retrieved September 9, 2022.

Complete References

Amromin, G. (2003). "Household Portfolio Choices in Taxable and Tax-Deferred Accounts: Another Puzzle?" *European Finance Review* 7. (3), 547–582.

Amromin, G., J. Huang, C. Sialm, and E. Zhong (2018). "Complex Mortgages." *Review of Finance* 22. (6), 1975–2007.

Anacker, K. B. (2019). "Introduction: Housing Affordability and Affordable Housing." *International Journal of Housing Policy* 19. (1), 1–16.

Arias, E. and J. Xu (2018). "United States Life Tables, 2015. National Vital Statistics Report." *National Center for Health Statistics* 67. (7).

Arnold, M. A. (1992). "The Principal-Agent Relationship in Real Estate Brokerage Services." *Real Estate Economics* 20. (1), 89–106.

Bacher, A. (2021). "Housing and Savings Behavior across Family Types." working paper, available at https://annikabacher.github.io/Bacher_JMP.pdf.

Bäckman, C. and C. Lutz (2020). "The Impact of Interest-Only Loans on Affordability." *Regional Science and Urban Economics* 80. Special Issue on Housing Affordability, article no. 103376.

Baker, S. R., R. A. Farrokhnia, S. Meyer, M. Pagel, and C. Yannelis (2020). "How Does Household Spending Respond to an Epidemic? Consumption During the 2020 Covid-19 Pandemic." *Review of Asset Pricing Studies* 10. (4), 834–862.

Balmer, I. and J.-D. Gerber (2018). "Why are Housing Cooperatives Successful? Insights from Swiss Affordable Housing Policy." *Housing Studies* 33. (3), 361–385.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein (2007). "Portfolio Choice over the Life-Cycle When the Stock and Labor Markets Are Cointegrated." *Journal of Finance* 62. (5), 2123–2168.

Beshears, J., J. J. Choi, D. Laibson, B. C. Madrian, and W. L. Skimmyhorn (2022). *Journal of Finance* 77. (1), 403–447.

Best, M. C., J. S. Cloyne, E. Ilzetzki, and H. J. Kleven (2020). "Estimating the Elasticity of Intertemporal Substitution Using Mortgage Notches." *Review of Economic Studies* 87. (2), 656–690.

Bian, X., Z. Lin, and Y. Liu (2018). "House Price, Loan-to-Value Ratio and Credit Risk." *Journal of Banking & Finance* 92, 1–12.

Black, F. (1980). "The Tax Consequences of Long-Run Pension Policy." *Financial Analysts Journal* 36. (4), 21–28.

Blake, D., D. Wright, and Y. Zhang (2014). "Age-Dependent Investing. Optimal Funding and Investment Strategies in Defined Contribution Pension Plans when Members are Rational Life Cycle Financial Planners." *Journal of Economic Dynamics and Control* 38, 105–124.

Bogyrbayeva, A., M. Takalloo, H. Charkhgard, and C. Kwon (2021). "An Iterative Combinatorial Auction Design for Fractional Ownership of Autonomous Vehicles." *International Transactions in Operational Research* 28. (4), 1681–1705.

Bourassa, S. C., D. R. Haurin, J. L. Haurin, M. Hoesli, and J. Sun (2009). "House Price Changes and Idiosyncratic Risk: The Impact of Property Characteristics." *Real Estate Economics* 37. (2), 259–278.

Boyd, S. and L. Vandenberghe (2009). *Convex Optimization*. Seventh ed. Cambridge University Press.

Carroll, C. D. (1997). "Buffer-Stock Saving and The Life Cycle/Permanent Income Hypothesis." *Quarterly Journal of Economics* 112. (1), 1–55.

Carroll, C. D. and A. A. Samwick (1997). "The Nature of Precautionary Wealth." *Journal of Monetary Economics* 40. (1), 41–71.

Case, K. E. and R. J. Shiller (1989). "The Efficiency of the Market for Single-Family Homes." *American Economic Review* 79. (1), 125–137.

Cashin, D. and T. Unayama (2016). "Measuring Intertemporal Substitution in Consumption: Evidence from a VAT Increase in Japan." *Review of Economics and Statistics* 98. (2), 285–297.

Cauley, S. D., A. D. Pavlov, and E. S. Schwartz (2007). "Homeownership as a Constraint on Asset Allocation." *Journal of Real Estate Finance and Economics* 34. (3), 283–311.

Chinloy, P., C. Jiang, and K. John (2022). "Spreads and Volatility in House Returns." *Journal of Risk and Financial Management* 15. (8), 369.

Choi, J. J. (2022). "Popular Personal Financial Advice versus the Professors." *Journal of Economic Perspectives* 36. (4), 167–92.

Clarkson, G. P. and A. H. Meltzer (1960). "Portfolio Selection: A Heuristic Approach." *Journal of Finance* 15. (4), 465–480.

Claus, J. and J. Thomas (2001). "Equity Premia as Low as Three Percent? Evidence from Analysts' Earnings Forecasts for Domenstic and International Stock Markets." *Journal of Finance* 56. (5), 1629–1666.

Cocco, J. (2005). "Portfolio Choice in the Presence of Housing." *Review of Financial Studies* 18. (2), 535–567.

Cocco, J., F. J. Gomes, and P. J. Maenhout (2005). "Consumption and Portfolio Choice over the Life Cycle." *Review of Financial Studies* 18. (2), 491–533.

Constantinides, G. M., J. B. Donaldson, and R. Mehra (2002). "Junior Can't Borrow: A New Perspective on the Equity Premium Puzzle." *Quarterly Journal of Economics* 117. (1), 269–296.

Dahlquist, M., R. Vestman, and O. Setty (2018). "On the Asset Allocation of a Default Pension Fund." *Journal of Finance* 73. (4), 1893–1936.

Dammon, R. M., C. S. Spatt, and H. H. Zhang (2004). "Optimal Asset Location and Allocation with Taxable and Tax-Deferred Investing." *Journal of Finance* 59. (3), 999–1037.

Daneshvary, N. and T. M. Clauretie (2013). "Agent Change and Seller Bargaining Power: A Case of Principal Agent Problem in the Housing Market." *Journal of Real Estate Finance and Economics* 47. (3), 416–433.

Deaton, A. (1991). "Saving and Liquidity Constraints." *Econometrica* 59. (5), 1221–1248.

Dimson, E., P. Marsh, and M. Staunton (2002). *Triumph of the Optimists: 101 Years of Global Investment Returns*. Princeton University Press.

Epstein, L. G. and S. E. Zin (1989). "Substitution, Risk Aversion and Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework." *Econometrica* 57. (4), 937–969.

Feldstein, M. (1985). "The Optimal Level of Social Security Benefits." *Quarterly Journal of Economics* 100. (2), 303–320.

Fischer, M. and M. F. Gallmeyer (2016). "Heuristic Portfolio Trading Rules with Capital Gain Taxes." *Journal of Financial Economics* 119. (3), 611–625.

— (2017). "Taxable and Tax-Deferred Investing with the Limited Use of Losses." *Review of Finance* 21. (5), 1847–1873.

Fischer, M., B. A. Jensen, and M. Koch (2022). "Mandatory Pension Saving and Homeownership." working paper, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4210537.

Fischer, M. and N. Khorunzhina (2019). "Housing Decision with Divorce Risk." *International Economic Review* 60. (3), 1263–1290.

Fischer, M. and M. Koch (2022). "Heuristic Portfolio Rules with Labor Income." working paper, University of Konstanz.

Fischer, M., H. Kraft, and C. Munk (2013). "Asset Allocation over the Life Cycle: How Much Do Taxes Matter?" *Journal of Economic Dynamics and Control* 37. (11), 2217–2240.

Fischer, M. and M. Stamos (2013). "Optimal Life Cycle Portfolio Choice with Housing Market Cycles." *Review of Financial Studies* 26. (9), 2311–2352.

Foster, A. C. (2015). "Consumer Expenditures Vary by Age." *Beyond the Numbers: Prices & Spending* 4. (14).

Fry, R. and K. Parker (2021). "Rising Share of U.S. Adults are Living Without a Spouse or Partner." Pew Research Center, available at https://www.pewresearch.org/social-trends/2021/10/05/rising-share-of-u-s-adults-are-living-without-a-spouse-or-partner/.

Furst, R. T. and D. N. Evans (2017). "Renting Apartments to Felons: Variations in Real Estate Agent Decisions Due to Stigma." *Deviant Behavior* 38. (6), 698–708.

Garlappi, L. and J. Huang (2006). "Are Stocks Desirable in Tax-Deferred Accounts?" *Journal of Public Economics* 90. (12), 2257–2283.

Gaudecker, H.-M. v. (2015). "How Does Household Portfolio Diversification Vary with Financial Literacy and Financial Advice?" *Journal of Finance* 70. (2), 489–507.

Gayle, W. R. and N. Khorunzhina (2018). "Micro-Level Estimation of Optimal Consumption Choice with Intertemporal Nonseparability in Preferences and Measurement Errors." *Journal of Business & Economic Statistics* 36. (2), 227–238.

Goodman, L. S. and C. Mayer (2018). "Homeownership and the American Dream." *Journal of Economic Perspectives* 32. (1), 31–58.

Gourinchas, P.-O. and J. A. Parker (2002). "Consumption over the Life Cycle." *Econometrica* 70. (1), 47–89.

Guiso, L., M. Haliassos, and T. Japelli (2003). "Household Stockholding in Europe: Where Do We Stand and Where Do We Go?" *Economic Policy* 18. (1), 123–170.

Guiso, L. and P. Sodini (2013). "Household Finance: An Emerging Field." In: *Handbook of the Economics of Finance*. Vol. 2. Elsevier, 1397–1532.

Gupta, A., J. Rathod, D. Patel, J. Bothra, S. Shanbhag, and T. Bhalerao (2020). "Tokenization of Real Estate Using Blockchain Technology." In: *Applied Cryptography and Network Security Workshops: Proceedings 18*. Springer, 77–90.

Hakansson, N. H. (1970). "Optimal investment and consumption strategies under risk for a class of utility functions." *Econometrica* 38. (5), 587–607.

Holzmann, R. and E. Palmer (2006). *Pension Reform. Issues and Prospects for Non-Financial Defined Contribution (NDC) Schemes*. The World Bank.

Huang, H. and M. A. Milevsky (2016). "Longevity Risk and Retirement Income Tax Efficiency: A Location Spending Rate Puzzle." *Insurance: Mathematics and Economics* 71. (1), 50–62.

Huang, H., M. A. Milevsky, and T. S. Salisbury (2012). "Optimal Retirement Consumption with a Stochastic Force of Mortality." *Insurance: Mathematics and Economics* 51. (2), 282–291.

Huang, J. (2008). "Taxable and Tax-Deferred Investing: A Tax Arbitrage Approach." *Review of Financial Studies* 21. (5), 2173–2207.

Johansen, K. and J. Emborg (2018). "Wind Farm Acceptance For Sale? Evidence from the Danish Wind Farm Co-Ownership Scheme." *Energy Policy* 117, 413–422.

Kiyotaki, N., A. Michaelides, and K. Nikolov (2011). "Winners and Losers in Housing Markets." *Journal of Money, Credit and Banking* 43. (2–3), 255–296.

Konashevych, O. (2020). "General Concept of Real Estate Tokenization on Blockchain." *European Property Law Journal* 9. (1), 21–66.

Kraft, H. and C. Munk (2011). "Optimal Housing, Consumption, and Investment Decisions over the Life Cycle." *Management Science* 57. (6), 1025–1041.

Kraft, H., C. Munk, and S. Wagner (2018). "Housing Habits and Their Implications for Life-Cycle Consumption and Investment." *Review of Finance* 22. (5), 1737–1762.

Lachance, M.-E. (2012). "Optimal Onset and Exhaustion of Retirement Savings in a Life-Cycle Model." *Journal of Pension Economics & Finance* 11. (1), 21–52.

Larsen, L. S. and C. Munk (2023). "The Design and Welfare Implications of Mandatory Pension Plans." *Journal of Financial and Quantitative Analysis* online, 1–30.

Layton, D. (2022). "The Homeownership Rate and Housing Finance Policy – Part 2: The Very Heavy Lift to Maintain and Then Break the 65 Percent Barrier." Joint Center for Housing Studies of Harvard University, available at https://www.jchs.harvard.edu/sites/default/files/research/files/harvard_jchs_homeownership_rate_ii_layton_2022.pdf.

Love, D. A. (2010). "The Effect of Marital Status and Children on Savings and Portfolio Choice." *Review of Financial Studies* 23. (1), 385–432.

— (2013). "Optimal Rules of Thumb for Consumption and Portfolio Choice." *The Economic Journal* 123. (571), 932–961.

Lowies, B., R. B. Whait, C. Viljoen, and S. McGreal (2018). "Fractional Ownership – An Alternative Residential Property Investment Vehicle." *Journal of Property Investment & Finance* 36. (6), 513–522.

Lowitzsch, J., ed. (2019). *Energy Transition: Financing Consumer Co-Ownership in Renewables*. Springer.

Lusardi, A. and O. S. Mitchell (2007). "Baby Boomer Retirement Security: The Roles of Planning, Financial Literacy, and Housing Wealth." *Journal of Monetary Economics* 54. (1), 205–224.

Marekwica, M., A. Schaefer, and S. Sebastian (2013). "Life Cycle Asset Allocation in the Presence of Housing and Tax-Deferred Investing." *Journal of Economic Dynamics and Control* 37. (6), 1110–1125.

Mehra, R. and E. C. Prescott (1985). "The Equity Premium: A Puzzle." *Journal of Monetary Economics* 15. (2), 145–162.

Merton, R. C. (1969). "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case." *Review of Economics and Statistics* 51. (1), 247–257.

— (1971). "Optimal Consumption and Portfolio Rules in a Continuous-Time Model." *Journal of Economic Theory* 3. (4), 373–413.

Moser, C. and P. Olea de Souza e Silva (2019). "Optimal Paternalistic Savings Policies." working paper, available at https://ssrn.com/abstract=2959844.

Munk, C. (2000). "Optimal Consumption/Investment Policies with Undiversifiable Income Risk and Liquidity Constraints." *Journal of Economic Dynamics and Control* 24. (9), 1315–1343.

OECD (2021a). *Pensions at a Glance 2021: OECD and G20 Indicators*. OECD Publishing, Paris.

— (2021b). "Tax Relief for Home Ownership." technical report, available at https://www.oecd.org/els/family/PH2-2-Tax-relief-for-home-ownership.pdf.

Ostrowski, J. (2022). "Nearly Two-Thirds of Non-Homeowners Polled Say Affordability Woes Block Homeownership." Bankrate, available at https://www.bankrate.com/mortgages/homeownership-remains-centerpiece-of-american-dream/.

Pakoš, M. (2011). "Estimating Intertemporal and Intratemporal Substitutions When Both Income and Substitution Effects are Present: The Role of Durable Goods." *Journal of Business & Economic Statistics* 29. (3), 439–454.

Paz-Pardo, G. (2021). "Homeownership and Portfolio Choice over the Generations." working paper, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3797142.

Reb, J. and T. Connolly (2007). "Possession, Feelings of Ownership and the Endowment Effect." *Judgment and Decision Making* 2. (2), 107–114.

Rong, H. (2023). "A Proposal for Fractional Property Ownership and Collective Governance in Local Development." working paper, available at https://ssrn.com/abstract=4353724.

Roth, L., J. Lowitzsch, Ö. Yildiz, and A. Hashani (2018). "Does (Co-) Ownership in Renewables Matter for an Electricity Consumer's Demand Flexibility? Empirical Evidence from Germany." *Energy Research & Social Science* 46, 169–182.

Samuelson, P. A. (1975). "Optimal Social Security in a Life-Cycle Growth Model." *International Economic Review* 16. (3), 539–544.

Sazama, G. W. (2000). "Lessons from the History of Affordable Housing Cooperatives in the United States: A Case Study in American Affordable Housing Policy." *American Journal of Economics and Sociology* 59. (4), 573–608.

Schaeffer, K. (2022). "A Growing Share of Americans Say Affordable Housing is a Major Problem Where They Live." Pew Research Center, available at https://www.pewresearch.org/fact-tank/2022/01/18/a-growing-share-of-americans-say-affordable-housing-is-a-major-problem-where-they-live/.

Schlafmann, K. (2021). "Housing, Mortgages, and Self-Control." *Review of Financial Studies* 34. (5), 2648–2687.

Schlafmann, K., O. Setty, and R. Vestman (2022). "Designing Pension Plans According to Consumption-Savings Theory." working paper, available at https://cepr.org/system/files/2022-09/Kathrin%20Schlafmann_Designing%20Pension%20Plans%20According%20to%20Consumption-Savings%20Theory.pdf.

Shoven, J. B. and C. Sialm (1998). "Long Run Asset Allocation for Retirement Savings." *Journal of Private Portfolio Management* 1. (2), 13–26.

— (2003). "Asset Location in Tax-Deferred and Conventional Savings Accounts." *Journal of Public Economics* 88. (1), 23–38.

Song, C. (2020). "Financial Illiteracy and Pension Contributions: A Field Experiment on Compound Interest in China." *Review of Financial Studies* 33. (2), 916–949.

Sørvoll, J. and B. Bengtsson (2018). "The Pyrrhic Victory of Civil Society Housing? Co-operative Housing in Sweden and Norway." *International Journal of Housing Policy* 18. (1), 124–142.

Tepper, I. (1981). "Taxation and Corporate Pension Policy." *Journal of Finance* 36. (1), 1–13.

Van Ooijen, R., R. Alessie, and A. Kalwij (2015). "Saving Behavior and Portfolio Choice After Retirement." *De Economist* 163, 353–404.

Van Rooij, M., A. Lusardi, and R. Alessie (2011). “Financial Literacy and Stock Market Participation.” *Journal of Financial Economics* 101. (2), 449–472.

Venti, S. F. and D. A. Wise (2004). “Aging and Housing Equity: Another Look.” In: *Perspectives on the Economics of Aging*. University of Chicago Press, 127–180.

Vestman, R. (2019). “Limited Stock Market Participation Among Renters and Homeowners.” *Review of Financial Studies* 32. (4), 1494–1535.

Weeks, L. E., J. Keefe, and D. J. Macdonald (2012). “Factors Predicting Relocation Among Older Adults.” *Journal of Housing for the Elderly* 26. (4), 355–371.

Whitaker, A. and R. Kräussl (2020). “Blockchain, Fractional Ownership, and the Future of Creative Work.” *Management Science* 66. (10), 4594–4611.

Winter, J. K., K. Schlafmann, and R. Rodepeter (2012). “Rules of Thumb in Life-Cycle Saving Decisions.” *The Economic Journal* 122. (560), 479–501.

Wood, C. (2016). “Pension Funds Investing in Real Estate.” Preqin, available at https://docs.preqin.com/newsletters/re/Preqin-RESL-September-16-Pension-Funds-Investing-in-Real-Estate.pdf.

Yao, R. and H. H. Zhang (2005). “Optimal Consumption and Portfolio Choices with Risky Housing and Borrowing Constraints.” *Review of Financial Studies* 18. (1), 197–239.

Yawney, B. A. and D. L. Slover (1973). “Relocation of the Elderly.” *Social Work* 18. (3), 86–95.

Eigenabgrenzung

Das erste Kapitel mit dem Titel „*Optimal Retirement Savings over the Life Cycle: A Deterministic Analysis in Closed Form*“ ist in Zusammenarbeit mit Marcel Fischer, Professor für Finance an der Universität Konstanz, und Bjarne Astrup Jensen, Professor für Finance an der Copenhagen Business School, entstanden. Mein individueller Anteil an diesem Projekt beträgt 33%.

Das zweite Kapitel mit dem Titel „*Mandatory Pension Saving and Homeownership*“ ist ebenso wie das erste Kapitel in Zusammenarbeit mit Marcel Fischer und Bjarne Astrup Jensen entstanden. Mein individueller Anteil an diesem Projekt beträgt 50%.

Das dritte Kapitel mit dem Titel „*Fractional Homeownership and its Impact on Life Cycle Portfolio Choice*“ habe ich in Eigenarbeit erstellt. Mein individueller Anteil beträgt somit 100%.

Das vierte Kapitel mit dem Titel „*Heuristic Portfolio Rules with Labor Income*“ ist in Zusammenarbeit mit Marcel Fischer entstanden. Wir haben zu gleichen Teilen zu dem Projekt beigetragen; mein individueller Anteil beträgt somit 50%.