CONVERGENCE OF ARBITRAGE-FREE DISCRETE TIME MARKOVIAN MARKET MODELS

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Abstract. We consider two sequences of Markov chains inducing equivalent measures on the discrete path space. We establish conditions under which these two measures converge weakly to measures induced on the Wiener space by weak solutions of two SDEs, which are unique in the sense of probability law. We are going to look at the relation between these two limits and at the convergence and limits of a wide class of bounded functionals of the Markov chains. The limit measures turn out not to be equivalent in general. The results are applied to a sequence of discrete time market models given by an objective probability measure, describing the stochastic dynamics of the state of the market, and an equivalent martingale measure determining prices of contingent claims. The relation between equivalent martingale measure, state prices, market price of risk and the term structure of interest rates is examined. The results lead to a modification of the Black-Scholes formula and an explanation for the surprising fact that continuous-time arbitrage-free markets are complete under weak technical conditions.

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Introduction

We consider a discrete time markovian market model. Instead of modeling the market by defining price processes of a certain generating system of stocks and a bond, we work with an abstract state process, given as a Markov chain. We define only the risk-free bond maturing after one period of time and focus on the state prices. The underlying process describes the market state and the risk-free spot interest rate is assumed to be a function of this state. It is well known that in an arbitrage free discrete time market there exists a *risk free probability measure* or *equivalent martingale measure* P such that prices of attainable contingent claims are expectations of discounted payoffs with respect to this measure and discounted price processes are martingales. In a continuous time setting the change of measure from objective probabilities to risk-free probabilities is expressed by the Girsanov functional. We are going to consider a sequence of discrete time market models together with objective and risk-free probabilities. We want to establish conditions such that the market models converge (weakly with respect to the objective probabilities Q) to a continuous time state process, given as the weak solution of a stochastic differential equation, which is unique in probability law, and such that, at the same time, the risk-free probability measures weakly converge too. We will derive a result about the convergence of prices of a wide class of bounded contingent claims. In the case that the market is modeled as a multinomial branching process we explicitly calculate the drift and diffusion coefficients of the continuous state process with respect to the limit of the equivalent martingale measures. We calculate the Arrow-Debreu state prices and show how to fit a market model to a given initial term structure of interest rates. This will give some insight into the relation between equivalent martingale measure, state prices, market price of risk and zero bond prices.

Several papers address the problem of convergence of discrete time models to continuous time models. It seems not yet to be clear which type of convergence (weak, almost sure, D^2, uniformly tight) is appropriate, see [5], [11], [12], [21], [4], [18] and [20] for an overview. However, all these approaches start with an arbitrage-free continuous time model and approximate the continuous time price processes by discrete time price processes. Therefore the limit of the discrete time equivalent martingale measures is assured to be an *equivalent* martingale measure. The approach followed here is more general since we assume only weak convergence of the objective probability measure describing the stochastic dynamics of the state process (some or even all of its components could represent prices) and weak convergence of the discrete time equivalent martingale measures to a measure which turns out not to be equivalent to the objective probability measure in general. If we assume that the market has found in its equilibrium state the prices for a sufficiently large number of contingent claims such that the market becomes complete, then there is a unique corresponding equivalent martingale measure. There is a one-to-one correspondence between such arbitrage-free price systems and equivalent martingale measures. Therefore we are working with a sequence of incomplete discrete markets made complete by choosing equivalent martingale measures which allow then to price any contingent claim. By this means we avoid the problems of an equilibrium approach for the market model. Choosing these equivalent martingale measures is done in such a way that the weak limit of the measures exists.

The paper is organized as follows: Section 1 contains preliminary material. We introduce the *martingale problem* and cite a theorem concerning the convergence of a sequence of Markov chains to a solution of a stochastic differential equation. Section 2 contains the main result about the convergence of a certain class of bounded functionals of a Markov chain representing price processes of contingent claims. Section 3 focuses on market models where the state process is driven by a random walk respectively a normally distributed random variable. In order to explicitly calculate the limit of the equivalent martingale measures we classify the probability measures on $\{-1,1\}^m$ describing the underlying random walk driving the Markov chains. In Subsection 3.5 we consider Arrow-Debreu state prices and give an explanation why continuous time markets using continuous square integrable hedging are complete under weak technical assumptions. In Section 3.6 a modification of the Black-Scholes European option valuation formula is derived. Section 4 contains some remarks on Zero Bonds.

1. Preliminary Material

We are going to consider Markov processes in discrete time as well as in continuous time. We first define the spaces on which we will model these processes. For $H \subseteq [0,\infty)$ let $\Omega_H := C\left(H;\mathbb{R}^d\right)$ be the space of continuous functions from H into $\mathbb{R}^d$ and let $\epsilon_t^H : \Omega_H \to \mathbb{R}^d$, $\omega \mapsto \omega(t)$ be the evaluation at $t \in H$. Let $\mathcal{B}_d$ be the Borel σ-algebra of $\mathbb{R}^d$ and set $H_t := [0,t] \cap H$. A metric on Ω_H is given by $D_H : \Omega_H \times \Omega_H \to \mathbb{R}$

$$D_H(\omega,\tilde{\omega}) := \sum_{i=1}^{\infty} 2^{-i} \frac{\sup_{t\in H_i} |\,\omega(t)-\tilde{\omega}(t)|}{1+\sup_{t\in H_i}|\,\omega(t)-\tilde{\omega}(t)|}\,.$$

Let $\mathcal{M}_t^H := \sigma\left[\epsilon_s^H | s \in H_t\right]$ and $\mathcal{M}_H := \sigma\left[\bigcup_{s\in H} \mathcal{M}_s^H\right]$. $\{\mathcal{M}_t^H |\, t \in H\}$ is a non-decreasing family of sub σ-algebras of $\mathcal{M}_H$ which equals the Borel σ-field of subsets of the metric space (Ω_H, D_H). $(\Omega_H, \mathcal{M}_H)$ is a subspace of the Skorokhod space, see [16]. The set of all probability measures on $(\Omega_H, \mathcal{M}_H)$ is denoted by $M(\Omega_H)$. We will model Markov chains on $(\tilde{\Omega}, \tilde{\mathcal{M}}) := (\Omega_{\mathbb{N}}, \mathcal{M}_{\mathbb{N}})$ and continuous time Markov processes on $(\Omega, \mathcal{M}) := (\Omega_{[0,\infty)}, \mathcal{M}_{[0,\infty)})$. Set $\epsilon_t := \epsilon_t^{[0,\infty)}$ for $t \in [0,\infty)$ and $\tilde{\epsilon}_i := \epsilon_i^{\mathbb{N}}$ for $i \in \mathbb{N}$.

1.1. **Markov Chains.** By [9], Theorem 2.4.3, p. 81, any stochastic kernel Π on $(\mathbb{R}^d, \mathcal{B}_d)$ defines a unique measure $P_x^\Pi \in M(\tilde{\Omega})$ for all $x \in \mathbb{R}^d$ such that

$$P_x^\Pi(\tilde{\epsilon}_0 = x) = 1, \tag{1.1}$$

and P_x^Π-almost sure for all $i \in \mathbb{N}$

$$P_x^\Pi(\tilde{\epsilon}_{i+1} \in A |\, \tilde{\mathcal{M}}_i) = \Pi(\tilde{\epsilon}_i, A), \quad \forall\, A \in \mathcal{B}_d. \tag{1.2}$$

The triple $(\tilde{\epsilon}_i, \tilde{\mathcal{M}}_i, P_x^\Pi)$ is called a *time-homogeneous Markov chain* on $(\tilde{\Omega}, \tilde{\mathcal{M}})$.

We embed $\tilde{\Omega}$ into Ω by mapping a discrete path to a piecewise linear path by interpolation. For $h > 0$ define $\Phi_h : \tilde{\Omega} \to \Omega$ by

$$\omega \longmapsto \left(t \mapsto \omega\left(\left[\tfrac{t}{h}\right]\right)\left(1 - \left(\tfrac{t}{h} - \left[\tfrac{t}{h}\right]\right)\right) + \omega\left(\left[\tfrac{t}{h}\right] + 1\right)\left(\tfrac{t}{h} - \left[\tfrac{t}{h}\right]\right),\ t \in \mathbb{R}^+\right).$$

Since $D(\Phi_h(\omega), \Phi_h(\tilde{\omega})) = \tilde{D}(\omega, \tilde{\omega}),\ \ \forall\, \omega, \tilde{\omega} \in \tilde{\Omega}$ it follows that Φ_h is continuous. Φ_h induces a map from $M(\tilde{\Omega})$ to $M(\Omega)$ by $P \mapsto P \circ \Phi_h^{-1}$. Since $\epsilon_{ih}(\Phi_h(\tilde{\omega})) = \tilde{\epsilon}_i(\tilde{\omega})$, $\forall\, \tilde{\omega} \in \tilde{\Omega}$ it follows for all $A \in \mathcal{B}_d$ that

$$\begin{aligned} \Phi_h^{-1}\Big(\{\omega \in \Omega |\, \epsilon_{ih}(\omega) \in A\}\Big) &= \{\tilde{\omega} \in \tilde{\Omega} |\, \epsilon_{ih}(\Phi_h(\tilde{\omega})) \in A\} \\ &= \{\tilde{\omega} \in \tilde{\Omega} |\, \tilde{\epsilon}_i(\tilde{\omega}) \in A\}. \end{aligned}$$

Therefore the following lemma holds:

Lemma 1.1. *The evaluation map on $\tilde{\Omega}$ at time $i \in \mathbb{N}$ has the same distribution under $P \in M(\tilde{\Omega})$ as the evaluation map on Ω at time ih under $P \circ \Phi_h^{-1}$.*

For a stochastic kernel Π we define

$$\mathbf{P}_x^h(\Pi) := P_x^\Pi \circ \Phi_h^{-1}. \tag{1.3}$$

This definition will allow us to work on one single space, namely $(\Omega, \mathcal{M})$.

For the special case where the measures $\Pi(y, \cdot)$ are concentrated on a finite discrete subset $Z_y = Z_y(\Pi) \subseteq \mathbb{R}^d$ for all $y \in \mathbb{R}^d$ we introduce some more notation. Set $\tilde{\mathcal{M}}_0^* = \tilde{\mathcal{M}}_0^*(P_x^\Pi) := \Big\{\{\omega \in \tilde{\Omega} | \omega(0) = x\}\Big\}$ and for $i > 0$ define recursively the set of *states at time i of the process $\tilde{\epsilon}$.* by $\tilde{\mathcal{M}}_i^* = \tilde{\mathcal{M}}_i^*(P_x^\Pi) := \Big\{\{\omega \in w | \omega(i) = z\}\big| w \in \tilde{\mathcal{M}}_{i-1}^*, z \in Z_{w(i-1)}\Big\}$, where $w(j) := \omega(j)$ for $\omega \in w \in \tilde{\mathcal{M}}_i^*$ for $0 \leq j \leq i$. $\mathcal{M}_i^*$ can be interpreted as the set of paths of length i with positive probability under P_x^Π since for $\omega, \tilde{\omega} \in w \in \tilde{\mathcal{M}}_i^*$ we have $\omega(j) = \tilde{\omega}(j), 0 \leq j \leq i$. Observe that for $w_1, w_2 \in \tilde{\mathcal{M}}_i^*$, $w_1 \cap w_2 = \emptyset$ iff $w_1 \neq w_2$ and $\sum_{w \in \tilde{\mathcal{M}}_i^*} P_x^\Pi(w) = 1$ and $\tilde{\mathcal{M}}_i^* \subseteq \tilde{\mathcal{M}}_i$ for all $i \in \mathbf{N}$. For $w \in \tilde{\mathcal{M}}_i^*, i > 0$ set $w^{-^j} := \{\omega \in \tilde{\Omega} | \omega(k) = w(k), 0 \leq k \leq i - j\} \in \tilde{\mathcal{M}}_{i-1}^*$ for $0 \leq j \leq i$ and $w^- := w^{-^1}$. For $w \in \tilde{\mathcal{M}}_i^*, i \geq 0$ and $z \in Z_{w(i)}$ set $[w, z] := \{\omega \in w | \omega(i+1) = z\} \in \tilde{\mathcal{M}}_{i+1}^*$, set $w^+ := \{[w, z] | z \in Z_{w(i)}\} \subseteq \tilde{\mathcal{M}}_{i+1}^*$ and define $z(w) \in Z_{w(i-1)}$ for $i > 0$ implicitly by $w = [w^-, z(w)]$. Note that $\cup_{z \in Z_{w(i)}} [w, z] = w$ holds for $w \in \tilde{\mathcal{M}}_i^*$ for all $i \in \mathbf{N}$.

1.2. **Continuous Markov Processes.** The σ-algebra $\mathcal{M}$ together with its filtration $\{\mathcal{M}_t | t \geq 0\}$ is rich enough to support continuous Markov processes. To cite some results we need the notion of a *transition probability function* or *stochastic kernel.*

Definition 1.2. A function $\Pi(s, x; t, A)$, $0 \leq s < t$, $x \in \mathbb{R}^d$, and $A \in \mathcal{B}_d$ is called a *transition probability function* if

1. $\Pi(s, x; t, \cdot), 0 \leq s < t, x \in \mathbb{R}^d$, is a probability measure on $(\mathbb{R}^d, \mathcal{B}_d)$,
2. $\Pi(s, \cdot\,; t, A)$, $0 \leq s < t$, $A \in \mathcal{B}_d$, is $\mathcal{B}_d$-measurable,
3. if $0 \leq s < t < u$, $x \in \mathbb{R}^d$, and $A \in \mathcal{B}_d$, then the Chapman-Kolmogorov equation holds:

$$\Pi(s, x; u, A) = \int_{\mathbb{R}^d} \Pi(s, y; u, A) \Pi(s, x; t, dy). \tag{1.4}$$

Definition 1.3. Let Π be a transition probability function and μ a probability measure on $(\mathbb{R}^d, \mathcal{B}_d)$. A triple $(\epsilon_t, \mathcal{M}_t, P)$, with $P \in M(\Omega)$ is called a *continuous Markov process* (on $(\Omega, \mathcal{M})$) with *transition probability function* Π and *initial distribution* μ if for all $A \in \mathcal{B}_\lceil$ and $0 \leq s < t$

$$P(x_0 \in A) = \mu(A), \tag{1.5}$$

and P-almost sure

$$P(\epsilon_t \in A | \, \mathcal{M}_s) = \Pi(s, \epsilon_s; t, A). \tag{1.6}$$

There exists a measure $W_x \in M(\Omega)$ (the Wiener measure) such that $(x_t, \mathcal{M}_t, W_x)$ is a Brownian Motion starting at $x \in \mathbb{R}^d$.

Denote the set of symmetric non-negative definite $d \times d$ real matrices by S_d.

Definition 1.4. Given locally bounded measurable functions $a = (a_{i,j})_{1\leq\, i,j \leq\, d} : \mathbb{R}^+ \times \mathbb{R}^d \to S_d$ and $\mu = (\mu_i)_{1\leq i \leq\, d} : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}^d$, with

$$L_t := \frac{1}{2} \sum_{i,\, j\, =1}^{d} a_{i,j}(t, \cdot) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{d} \mu_i(t, \cdot) \frac{\partial}{\partial x_i}. \tag{1.7}$$

A *solution to the martingale problem for* (a, μ) *starting from* $(s, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ is a probability measure $P \in M(\Omega)$ such that

$$P(\epsilon_t = x, \quad 0 \leq\; t \leq\; s) = 1 \tag{1.8}$$

and

$$f(\epsilon_t) - \int_s^{s \vee t} L_u f(\epsilon_u) du, \quad t \geq 0 \tag{1.9}$$

is an $\mathcal{M}_t$-adapted P-martingale for all $f \in C_0^\infty(\mathbb{R}^d)$.

1.3. **Weak Convergence of Markov Chains.** We are now going to establish conditions under which a sequence of Markov chains converges weakly to a continuous Markov process. More precisely, given a set $\{\Pi^h,\ h > 0\}$ of stochastic kernels on $(\mathbb{R}^d, \mathcal{B}_d)$, we seek conditions such that $P_x^h := \mathbf{P}_x^h(\Pi^h)$ converges weakly to a measure $P_x \in M(\Omega)$ for $h \searrow 0$.

Denote the set of symmetric non-negative definite $d \times d$ real matrices by S_d. We define two functions $\mu_h^{\Pi^h} : \mathbb{R}^d \to \mathbb{R}^d$ and $a_h^{\Pi^h} : \mathbb{R}^d \to S_d \subseteq \mathbb{R}^{d \times d}$, approximating the *drift*- and *diffusion*-coefficients of a time-homogeneous Markov chain for small h:

$$\mu_h^{\Pi^h}(x) := \left(\frac{1}{h} \int_{|x-y| \leq 1} (y_i - x_i) \Pi^h(x, dy) \right)_{1 \leq i \leq d}, \tag{1.10}$$

and

$$a_h^{\Pi^h}(x) := \left(\frac{1}{h} \int_{|x-y| \leq 1} (y_i - x_i)(y_j - x_j) \Pi^h(x, dy) \right)_{1 \leq i,j \leq d}. \tag{1.11}$$

We assume that the following conditions hold: There exist continuous functions $\mu : \mathbb{R}^d \to \mathbb{R}^d$ and $a : \mathbb{R}^d \to S_d$ such that for all $R > 0$:

$$\lim_{h \searrow 0} \sup_{|x| \leq R} |\mu_h^{\Pi^h}(x) - \mu(x)| = 0, \tag{1.12}$$

$$\lim_{h \searrow 0} \sup_{|x| \leq R} \|a_h^{\Pi^h}(x) - a(x)\| = 0, \tag{1.13}$$

where $\|\cdot\|$ denotes the operator norm, and

$$\lim_{h\searrow 0}\sup_{|x|\leq R}\frac{1}{h}\Pi^h(x,\mathbb{R}^d\setminus B(x,\epsilon))=0,\quad \forall\epsilon>0. \tag{1.14}$$

Conditions (1.12)-(1.14) are quite plausible conditions which we will need in order to establish the convergence result. However, we need one more condition being not intuitive. Define the differential operator L by

$$L:=\frac{1}{2}\sum_{i,\,j\,=1}^{d}a_{i,j}\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_{i=1}^{d}\mu_i\frac{\partial}{\partial x_i}. \tag{1.15}$$

Definition 1.5. A *solution to the martingale problem for* (a,μ) *starting from* $x\in\mathbb{R}^d$ is a probability measure $P_x\in M(\Omega)$ such that

$$P(\epsilon_0=x)=1 \tag{1.16}$$

and

$$f(\epsilon_t)-\int_0^t Lf(\epsilon_u)du,\quad t\geq 0 \tag{1.17}$$

is an $\mathcal{M}_t$-adapted P_x-martingale for all $f\in C_0^\infty(\mathbb{R}^d)$.

We can now cite the main theorem of this section which is a generalization of the Donsker invariance principle, see [19], Chapter 11.2., Theorem 11.2.3, p. 272.

Theorem 1.6. *Assume conditions (1.12)-(1.14) to hold. If there exists for each* $x\in\mathbb{R}^d$ *a unique solution* P_x *to the martingale problem for* (a,μ) *starting from* x, *then* $\lim_{h\searrow 0}P_x^h=P_x$.

1.4. **Stochastic Differential Equations.** The martingale problem is related to the problem of solving a corresponding SDE. This relation will lead to useful conditions on μ and σ allowing to apply Theorem 1.6. We will only consider time-homogeneous SDEs.

Let an m-dimensional Brownian Motion $(W_t,\mathcal{F}_t,Q)$ be given on the probability space $(E,\mathcal{F},Q)$, where $\mathcal{F}_t$, $t\geq 0$, is assumed to satisfy the usual conditions.

We consider the SDE:

$$dX(t)=\mu(X(t))dt+\sigma(X(t))dW_t,\quad t\in[0,\infty), \tag{1.18}$$

$$X(0)=x, \tag{1.19}$$

where $\mu:\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times m}$ are measurable.

There are two different main notions of a solution for an SDE:

Definition 1.7.

1. An $\mathcal{F}_t$-adapted continuous process X_t, $t \geq 0$ on $(E, \mathcal{F}, Q)$ is called a *strong solution* of (1.18),(1.19) with respect to W if $Q(X(0) = x) = 1$,

(1.20)
$$Q\left(\int_0^t |\mu(X(u))| + \operatorname{Tr}\left[\sigma(X(u))\sigma(X(u))^*\right] du < \infty\right) = 1, \quad \forall\, t \in [0, \infty),$$

and

$$X(t) = x + \int_0^t \mu(X(u))du + \int_0^t \sigma(X(u))dW_u, \quad \forall\, t \in [0, \infty), \tag{1.21}$$

holds.

2. A triple $(X, \tilde{W}), (\tilde{E}, \mathcal{G}, P), \{\mathcal{G}_t\}$, where $(\tilde{E}, \mathcal{G}, P)$ is a probability space, $\{\mathcal{G}_t\}$ is right-continuous, augmented filtration of $\mathcal{G}$ such that $\{\tilde{W}_t,\, \mathcal{G}_t,\, 0 \leq t < \infty\}$ is an m-dimensional Brownian motion and X satisfies (1.20) and (1.21) (where P replaces Q and $\tilde{W}$ replaces W), is called a *weak solution.*

Theorem 1.8. *The martingale problem and SDEs are related in the following way:*

1. *The existence of a solution P_x for the martingale problem for $(\sigma\sigma^*, \mu)$ starting from $x \in \mathbb{R}^d$ is equivalent to the existence of a weak solution $(\tilde{X}_x, \tilde{W}), (\tilde{E}, \mathcal{G}, \tilde{P}), \{\mathcal{G}_t\}$, to (1.18), (1.19). The two solutions are related by $P_x = \tilde{P}\tilde{X}_x^{-1}$.*
2. *The uniqueness of the solution P to the martingale problem is equivalent to the uniqueness in the sense of probability law of a weak solution.*

Proof. See [17], Corollaries 5.4.8 and 5.4.9, p. 317 together with Proposition 5.4.11. □

If we model a market by a process being a solution of a SDE and study properties of the market depending only on the law of that process then existence and uniqueness in the sense of probability law of a weak solution to this SDE is a kind of minimum requirement we need to get a well-defined model by specifying the drift- and diffusion-coefficients μ, σ of the SDE. In this case, by Theorem 1.6 and Theorem 1.8 the conditions (1.12)-(1.14) are sufficient for the weak convergence of P_x^h to P_x. In general it is difficult to prove weak existence *and* uniqueness in law of solutions for a given SDE. Since strong existence and pathwise uniqueness implies weak existence and uniqueness in law we cite the following result in order to have a handy criterion:

Theorem 1.9. *Suppose μ and σ satisfy the following growth condition:*

$$|\mu(x)| \leq K(1+|x|), \quad |\sigma(x)| \leq K(1+|x|), \tag{1.22}$$

and the local Lipschitz condition for $|x|, |\tilde{x}| < N$:

$$|\mu(x)-\mu(\tilde{x})| \leq K_N|x-\tilde{x}|, \quad |\sigma(x)-\sigma(\tilde{x})| \leq K_N|x-\tilde{x}|, \tag{1.23}$$

with K, $K_N \in \mathbb{R}$. Then there is a unique strong solution for (1.18), (1.19).

Proof. See [17]. □

Remark 1.10. If for instance a weak solution for a SDE exists, then uniqueness in probability law follows from the existence of solutions to a corresponding Cauchy problem, see [17], Theorem 5.4.28.

1.5. **Convergence of Functionals of Markov Chains.** In order to price contingent claims we have to evaluate functionals of Markov chains and calculate their limits. We present some auxiliary results. Let a set $\{\Pi^h,\ h>0\}$ of stochastic kernels on $(\mathbb{R}^d, \mathcal{B}_d)$ and a set of uniformly bounded random variables $\{\tilde{f}_h | h>0,\ |\tilde{f}_h| \leq K\}$ on $\tilde{\Omega}$ such that for some $0 \leq T < \infty$ all $\tilde{f}_h$ are measurable with respect to $\tilde{\mathcal{M}}_{[\frac{T}{h}]}$ be given. Set $\tilde{P}^h_x := P^{\Pi^h}_x$ and $P^h_x := \mathbf{P}^h_x(\Pi^h)$. We consider functionals $F^h_x := E_{\tilde{P}^h_x}\big[\tilde{f}_h\big]$. Since $\Phi_h(\tilde{\mathcal{M}}_i) \subseteq \mathcal{M}_{ih}$ for all $i \in \mathbb{N}$, we find $\mathcal{M}_{[\frac{T}{h}]h}$-measurable random variables f_h on Ω such that $f_h = \tilde{f}_h \Phi_h^{-1}$ on $\Phi_h(\tilde{\Omega})$. By Lemma 1.1 we have that $F^h_x = E_{P^h_x}\big[f_h\big]$. Assume that f_h to converges uniformly on compact subsets of Ω to an $\mathcal{M}_T$-measurable random variable f. If P^h_x converges weakly to P_x, then $\lim_{h\searrow 0} F^h_x = E_{P_x}\big[f\big]$, since $\{P^h_x,\ h>0\}$ is tight. We are going to apply a version of the Arzelà-Ascoli theorem. Define the modulus of continuity on $[0,T]$:

$$m^T_\delta := \max_{\substack{|s-t|\leq\delta \\ 0\leq s,t\leq T}} |\epsilon_s - \epsilon_t|. \tag{1.24}$$

Theorem 1.11. *A subset A of Ω has compact closure if and only if the following conditions hold:*

$$\sup_{\omega\in A} |\omega(0)| < \infty, \tag{1.25}$$

$$\lim_{\delta\searrow 0} \sup_{\omega\in A} m^T_\delta(\omega) = 0, \quad \forall\, T>0. \tag{1.26}$$

Proof. See [17], Theorem 2.4.9 and Remark 2.3.13. □

We first show locally uniform convergence for some sequences of functions on Ω. Let $Y_h, r_h,\ h>0$, be two families of measurable functions on $\mathbb{R}^d$ converging locally uniform to continuous functions Y and r respectively. Define $f_h, f:\Omega\to\mathbb{R}$ for $0\le S<T<\infty$ by

$$f_h^{S,T} := \sum_{i=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} h r_h(\epsilon_{ih}), \quad f^{S,T} := \int_S^T r(\epsilon_t)dt.$$

$f_h^{S,T}$ and $f^{S,T}$ are $\mathcal{M}_T$-measurable and we find

Lemma 1.12. *$f_h^{S,T}$ converges locally uniform to $f^{S,T}$, and if g_h converges locally uniform on Ω to a continuous function g then $Y_h(g_h)$ converges locally uniform to $Y(g)$.*

Proof. We only show the first assertion. Let $A\subset\Omega$ be compact. Since the function $\sup_T := \sup_{0\le t\le T}|\epsilon_t|$ is continuous we find $K := \sup_{\omega\in A}\sup_T(\omega)<\infty$ and for $\gamma>0$ there exists a $\bar{h}$ such that for all $0<h\le\bar{h}$ the following three conditions hold: $\sup_{|y|\le K}|r_h(y)-r(y)|<\gamma,\ h\sup_{|y|\le K}|r(y)|<\gamma$ and $|r(y_1)-r(y_2)|<\gamma$ if $|y_1-y_2|<\bar{h}$ and $|y_1|,|y_2|\le K$. By Theorem 1.11 we find a $\bar{\delta}>0$ such that $\sup_{\omega\in A} m_\delta^T(\omega)<\bar{h}$ for all $0<\delta\le\bar{\delta}$. On A we have for $h\le\min(\bar{h},\bar{\delta})$

$$\begin{aligned}
&\left|f_h^{S,T}-f^{S,T}\right| = \\
&\left|\left(\sum_{i=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} h r_h(\epsilon_{ih}) - \int_{ih}^{(i+1)h} r(\epsilon_t)dt\right) + \int_{\left[\frac{S}{h}\right]h}^{S} r(\epsilon_t)dt - \int_{\left[\frac{T}{h}\right]h}^{T} r(\epsilon_t)dt\right| \\
&\le \left(\sum_{i=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} \int_{ih}^{(i+1)h} |r_h(\epsilon_{ih})-r(\epsilon_t)|\,dt\right) + \int_{\left[\frac{S}{h}\right]h}^{S} |r(\epsilon_t)|\,dt + \int_{\left[\frac{T}{h}\right]h}^{T} |r(\epsilon_t)|\,dt \\
&\le \left(\sum_{i=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} \int_{ih}^{(i+1)h} |r_h(\epsilon_{ih})-r(\epsilon_{ih})| + |r(\epsilon_{ih})-r(\epsilon_t)|\,dt\right) + 2\gamma \\
&\le \left(\sum_{i=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} \int_{ih}^{(i+1)h} 2\gamma dt\right) + 2\gamma \le 2(T+1)\gamma,
\end{aligned}$$

hence the assertion follows. ∎

With this lemma we immediately find

Proposition 1.13. *Let the family r_h be uniformly bounded from below and let $Y_h:\mathbb{R}^d\to\mathbb{R}$ be uniformly bounded respectively let $Z_h:\Omega\to\mathbb{R}$ be* *a family of* $\mathcal{M}_{\left[\frac{T}{h}\right]h}$*-measurable uniformly bounded functions converging locally uniform to a* $\mathcal{M}_T$ *-measurable function* $Z : \Omega \to \mathbb{R}$*. Then*

(1.27)
$$\lim_{h \searrow 0} E_{P_x^h}\left[\exp\left(-\sum_{i=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} h r_h(\epsilon_{ih})\right) Y_h\left(\epsilon_{\left[\frac{T}{h}\right]h}\right)\right] = E_{P_x}\left[e^{-\int\limits_S^T r(\epsilon_t)dt} Y(\epsilon_T)\right]$$

respectively

$$\lim_{h \searrow 0} E_{P_x^h}\left[\exp\left(-\sum_{i=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} h r_h(\epsilon_{ih})\right) Z_h\right] = E_{P_x}\left[e^{-\int_S^T r(\epsilon_t)dt} Z\right]. \tag{1.28}$$

A similar result holds for the limits of the conditional expectations if $r_h, h > 0$ is uniformly bounded from below. For fixed $0 \leq T < \infty$ and a compact $A \subseteq \Omega$ we find with Lemma 1.5 for $\gamma > 0$ a $\bar{h}_A^T > 0$ such that

$$\left\| \exp\left(-f_h^{S,T}\right) Z_h - \exp\left(-f^{S,T}\right) Z \right\|_A \leq \gamma,$$

for all $0 < h \leq \bar{h}_A^T$ *and* all $0 \leq S \leq T$. Thus for all $P \in M(\Omega)$ and all $0 \leq s < \infty$

$$\left\| E_P\left[\exp\left(-f_h^{S,T}\right) Z_h - \exp\left(-f^{S,T}\right) Z \middle| \mathcal{M}_s\right] \right\|_A \leq \gamma. \tag{1.29}$$

Define measurable maps $F_h^{0,T}, \tilde{F}_h^T : \Omega \to \Omega$ by

$$F_h^{0,T}(\omega) := \Phi_h\left(\left(i \mapsto E_{P_x^h}\left[\exp\left(-f_h^{0,T}\right) Z_h \middle| \mathcal{M}_{ih}\right](\omega)\right)\right), \tag{1.30}$$

and

(1.31)
$$\tilde{F}_h^T(\omega) := \Phi_h\left(\left(i \mapsto E_{P_x^h}\left[\exp\left(-f_h^{ih \wedge T, T}\right) Y_h\left(\epsilon_{\left[\frac{T}{h}\right]h}\right) \middle| \mathcal{M}_{ih}\right](\omega)\right)\right).$$

By (1.29) we have for any sequence $h_n > 0, \lim_{n\to\infty} h_n = 0$ and $\omega_n \in \Omega$ with $\lim_{n\to\infty} \omega_n = \omega$

(1.32)
$$\lim_{n\to\infty} F_{h_n}^{0,T}(\omega_n) = \left(s \mapsto E_{P_x}\left[\exp\left(-f^{0,T}\right) Z \middle| \mathcal{M}_s\right](\omega)\right) =: F^{0,T}(\omega),$$

and

$$\lim_{n\to\infty} \tilde{F}_{h_n}^T(\omega_n) = \left(s \mapsto E_{P_x}\left[\exp\left(-f^{s,T}\right) Z \middle| \mathcal{M}_s\right](\omega)\right) =: \tilde{F}^T(\omega), \tag{1.33}$$

By [1], Theorem 1.5.5, p. 34, we have

Proposition 1.14. *Under the assumptions of Proposition 1.13 and if* P_x^h *converges weakly to* P_x *then* $P_x^h\left(F_h^{0,T}\right)^{-1}$ *converges weakly to* $P_x\left(F^{0,T}\right)^{-1}$ *and* $P_x^h\left(\tilde{F}_h^T\right)^{-1}$ *converges weakly to* $P_x\left(\tilde{F}^T\right)^{-1}$.

With the last two propositions we will prove weak convergence of the discounted discrete time price processes respectively discrete time price processes.

We have now reached a quite general framework in which we can model a sequence of approximating markets weakly converging to a continuous time market model and have found conditions guaranteeing the convergence of a wide class of bounded functionals. The advantage of a discrete time model is that the calculations necessary to price a contingent claim can in principle be done exactly, but with decreasing time period h the number of operations increases dramatically. The convergence of functionals allows to use results about continuous time models as approximations for the discrete time models. For example, differentiability, boundedness and growth conditions on μ, σ, Y allow to find the limit of such functionals by solving a partial differential equation with boundary condition, see [7].

2. Valuation of Contingent Claims

We assume a sequence of markovian discrete time *complete* market models to be given. The state process with respect to the equivalent martingale measure $P_x^{\Pi^h}$, see [10], is then given by a Markov chain described by stochastic kernels Π^h. The assumption we make here is that the Π^h are *time homogeneous* stochastic kernels. In Section 3.5 we will argue from an economical point of view why this assumption is reasonable in our time homogeneous setting, see Remark 3.20. Assuming conditions (1.12) -(1.14) to hold for some μ and $a := \sigma\sigma^*$ such that the martingale problem for (a, μ) starting from $x \in \mathbb{R}^d$ has a unique solution for all x, the measures $P_x^{\Pi^h}$ converge weakly to a measure P_x induced by a process X being a weak solution to the SDE (1.18), (1.19) which is unique in law.

We assume the existence of a bank account without default risk. The risk-free spot interest paid on this account over a time interval of length $h > 0$ is given by a measurable function $R_h : \mathbb{R}^d \to (-1+\delta, \infty)$ of the state variable X_t for some $\delta > 0$. Define $r_h := \frac{\log(1+R_h)}{h}$. r_h is the equivalent interest *rate* being constant over a time interval of length h with $\frac{1}{1+R_h} = exp(-r_h h)$ as the 1-period discount factor. r_h is uniformly bounded from below since $\delta > 0$ is independent of h. We can think of r_h being given by either an interest rate policy of a central bank that sets the rate $r_h(x)$ if the market is in state x, or as the equilibrium interest rate in the market. Note that this definition allows for $R_h(\tilde{\epsilon}_i)$ or $r_h(\tilde{\epsilon}_i)$ to be a component of $\tilde{\epsilon}_i$. The interest rate is then part of the market state and the market dynamic explicitly depends on the interest rate.

A *contingent claim* Y_h^T with maturity $\left[\frac{T}{h}\right] h$ is defined as a $\tilde{\mathcal{M}}_{\left[\frac{T}{h}\right]}$-measurable random variable on $\tilde{\Omega}$. All contingent claims can be priced using the equivalent martingale measure $P_x^{\Pi^h}$:

Proposition 2.1 (Contingent claim pricing formula)**.** *For $0 \leq S \leq T < \infty$ the price $V_{\left[\frac{S}{h}\right]h}(Y_h^T)$ of the contingent claim Y_h^T at time $\left[\frac{S}{h}\right] h$ is given by the expectation of the discounted payoff at maturity:*

$$V_{\left[\frac{S}{h}\right]h}(Y_h^T) = E_{P_x^{\Pi^h}}\left[\prod_{j=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} \frac{1}{1+R_h(\tilde{\epsilon}_j)} Y_h^T \middle| \tilde{\mathcal{M}}_{\left[\frac{S}{h}\right]}\right] \tag{2.1}$$

$$= E_{P_x^{\Pi^h}}\left[\exp\left(-\sum_{j=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} h r_h(\tilde{\epsilon}_j)\right) Y_h^T \middle| \tilde{\mathcal{M}}_{\left[\frac{S}{h}\right]}\right]. \tag{2.2}$$

For a path-independent contingent claim $\tilde{Y}_h^T$ with measurable payoff $\tilde{Y}_h\left(\tilde{\epsilon}_{\left[\frac{T}{h}\right]}\right)$ we have

$$V_{\left[\frac{S}{h}\right]h}(\tilde{Y}_h^T) = E_{P^{\Pi^h}_{\tilde{\epsilon}_{\left[\frac{S}{h}\right]}}}\left[\exp\left(-\sum_{j=0}^{\left[\frac{T}{h}\right]-\left[\frac{S}{h}\right]-1} h r_h(\tilde{\epsilon}_j)\right) \tilde{Y}_h\left(\tilde{\epsilon}_{\left[\frac{T}{h}\right]-\left[\frac{S}{h}\right]}\right)\right]. \tag{2.3}$$

Proof. See [10]. □

Define a map $V^h_{Y_h^T} : \tilde{\Omega} \to \tilde{\Omega}$ by

$$V^h_{Y_h^T}(\tilde{\omega}) := \left(s \mapsto V_{\left(s \wedge \left[\frac{T}{h}\right]\right)h}\left(Y_h^T\right)(\tilde{\omega})\right). \tag{2.4}$$

By (2.3) we have for $s \in \mathbb{N}$

$$\begin{aligned} \tilde{\epsilon}_s\left(V^h_{\tilde{Y}_h^T}\right) &= V_{\left(s \wedge \left[\frac{T}{h}\right]\right)h}(\tilde{Y}_h^T) \\ &= E_{P^{\Pi^h}_{\tilde{\epsilon}_{s \wedge \left[\frac{T}{h}\right]}}}\left[\exp\left(-\sum_{j=0}^{\left[\frac{T}{h}\right]-s-1} h r_h(\tilde{\epsilon}_j)\right) \tilde{Y}_h\left(\tilde{\epsilon}_{\left(\left[\frac{T}{h}\right]-s\right)^+}\right)\right]. \end{aligned}$$

$V^h_{Y^T_h}$ is the discrete time price process of the contingent claim Y^T_h up to its maturity time $\left[\frac{T}{h}\right] h$.

2.1. **Convergence of Prices of Contingent Claims.** Set $P^h_x := \mathbf{P}^h_x(\Pi^h)$ and for simplicity consider contingent claims $\tilde{Y}^T_h = \tilde{Y}_h\left(\tilde{\epsilon}_{\left[\frac{T}{h}\right]}\right)$ for some measurable $\tilde{Y}_h : \mathbb{R}^d \to \mathbb{R}$, i.e. we consider contingent claims depending only on the terminal state and not on the whole path of the state process up to time T. We obtain by Lemma 1.1 and Proposition 2.1 for $0 \leq S \leq T < \infty$:

$$V_{\left[\frac{S}{h}\right]h}(\tilde{Y}^T_h) = E_{P^h_x}\left[\exp\left(-\sum_{j=\left[\frac{S}{h}\right]}^{\left[\frac{T}{h}\right]-1} h r_h(\epsilon_{jh})\right)\tilde{Y}_h\left(\epsilon_{\left[\frac{T}{h}\right]h}\right)\right]. \tag{2.5}$$

Assume that

$$\lim_{h\searrow 0}\sup_{|x|\leq R}|r_h(x) - r(x)| = 0, \tag{2.6}$$

for $R > 0$ and a continuous function r bounded from below by $-L$ for some $L \in \mathbb{R}$ and

$$\lim_{h\searrow 0}\sup_{|x|\leq R}|\tilde{Y}_h(x) - \tilde{Y}(x)| = 0, \tag{2.7}$$

for $R > 0$ and a continuous bounded function $\tilde{Y}$. We also assume $|\tilde{Y}_h| \leq \tilde{L}$ for all $h > 0$. By Proposition 1.13 we immediately find:

Proposition 2.2. *If P^h_x converges weakly to P_x, then the prices $V_{\left[\frac{S}{h}\right]h}(\tilde{Y}^T_h)$ of the contingent claims $\tilde{Y}^T_h$ at time $\left[\frac{S}{h}\right] h$ in the approximating discrete time markets converge to a value $V_{\tilde{Y}^T}(S)$ given by:*

$$V_{\tilde{Y}^T}(S) := \lim_{h\searrow 0} V_{\left[\frac{S}{h}\right]h}(\tilde{Y}^T_h) = E_{P_x}\left[e^{-\int_S^T r(\epsilon_t)dt}\tilde{Y}(\epsilon_T)\right], \quad \forall\, 0 \leq S \leq T. \tag{2.8}$$

By Proposition 2.2 and Proposition 1.14 we find:

Theorem 2.3. *If P^h_x converges weakly to P_x then the discrete time price processes $V^h_{\tilde{Y}^T_h}$ of the contingent claims $\tilde{Y}^T_h$ in the approximating discrete time markets converge weakly to the continuous time process $V_{\tilde{Y}^T}(\cdot \wedge T)$:*

$$V_{\tilde{Y}^T}(s \wedge T) = E_{P_x}\left[e^{-\int_{s\wedge T}^T r(\epsilon_t)dt}\tilde{Y}(\epsilon_T)\,\middle|\,\mathcal{M}_s\right], \quad \forall\, 0 \leq s < \infty. \tag{2.9}$$

Remark 2.4. Theorem 2.3 is easily extended to more general bounded contingent claims Z of the form

$$Z = \tilde{Z}\left(\{\epsilon_{T_i}, i \le n_1\}, \left\{\sup_{S_j \le t \le \tilde{S}_j} g_j(\epsilon_t), j \le n_2\right\}, \left\{\int_{R_k}^{\tilde{R}_k} h_k(\epsilon_t)dt, k \le n_3\right\}\right)$$

where for $1 \le i \le n_1 < \infty$, $1 \le j \le n_2 < \infty$ and $1 \le k \le n_3 < \infty$, $T_i, S_j, \tilde{S}_j, R_k, \tilde{R}_k \le T$ and $\tilde{Z} : \mathbb{R}^{n_1+n_2+n_3} \to \mathbb{R}$ is bounded and continuous, and g_j, h_k are continuous. This allows to price path dependent contingent claims like Barrier Options or Asian Options, see [15].

By the Markov property of weak solutions of SDEs we have

Corollary 2.5. *Under the assumptions of Theorem 2.3 there exists a function* $F_{\tilde{Y}} : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ *such that for all* $0 \le s < \infty$

$$V_{\tilde{Y}^T}(s) = E_{P_{\epsilon_{s\wedge T}}}\left[e^{-\int_0^{(T-s)^+} r(\epsilon_t)dt}\tilde{Y}\left(\epsilon_{(T-s)^+}\right)\right] = F_{\tilde{Y}}((T-s)^+, \epsilon_{s\wedge T}), \tag{2.10}$$

and $F_{\tilde{Y}}(0,x) = \tilde{Y}(x)$ *for all* $x \in \mathbb{R}^d$.

Under differentiability, boundedness and growth conditions on σ, μ, r, $\tilde{Y}$ the function $F_{\tilde{Y}}$ is $C^{1,2}$ and solves the following PDE for $t \ge 0$:

$$\left(-\frac{\partial}{\partial t} + \sum_{i=0}^{d} \mu_i(x)\frac{\partial}{\partial x_i} + \sum_{i,j=0}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} - r(x)\right)F(t,x) = 0, \tag{2.11}$$

with boundary condition $F(0,x) = \tilde{Y}(x)$, see [7].

In the next section we are going to look at the relation between the limit processes under the martingale probabilities respectively under the objective probabilities.

3. Markov Chains driven by a Random Walk

For $m > 0$ set $Z_m := \{-1,1\}^m$ and denote the power set of Z_m by $\mathcal{P}(Z_m)$. For $h > 0$ let π_h be a stochastic kernel on $(\mathbb{R}^d, \mathcal{P}(Z_m))$. Given measurable functions $\mu_h : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma_h : \mathbb{R}^d \to \mathbb{R}^{d\times m}$, define a function $\Pi_h = \Pi_h^{\mu_h,\sigma_h,\pi_h}$ on $\mathbb{R}^d \times \mathcal{B}_d$ by

$$\Pi_h(x,B) := \sum_{z\in Z_m} \mathbf{1}_B\Big(x + \mu_h(x)h + \sqrt{h}\sigma_h(x)z\Big)\pi_h(x,\{z\}) \tag{3.1}$$

Lemma 3.1. Π_h *is a stochastic kernel on* $(\mathbb{R}^d, \mathcal{B}_d)$.

Proof. It is easy to see that $\Pi_h(x,\cdot)$ is a probability measure on $(\mathbb{R}^d,\mathcal{B}_d)$. Since $\mathbf{1}_B$ is measurable for $B\in\mathcal{B}_d$ and $\pi_h(\cdot,\{z\})$ is a measurable function for $z\in Z$ by definition of a stochastic kernel, $\Pi_h(\cdot,B)$ is measurable as a sum of products of measurable functions. □

Π_h defines a Markov chain $(\tilde\epsilon_i,\tilde\Omega,P_x^{\Pi_h})$ such that the states reachable from any state $y\in\mathbb{R}^d$ in the next step are a subset of $\{y+\mu_h(y)h+\sqrt{h}\sigma_h(y)z|z\in Z_m\}$. If $\sigma_h(y)$ is invertible, then $P_x^{\Pi_h}\big(\tilde\epsilon_{i+1}=y+\mu_h(y)h+\sqrt{h}\sigma_h(y)z|\tilde\epsilon_i=y\big)=\pi_h(y,\{z\})$ for all $z\in Z_m$. The stochastic kernel $\Pi_h^{0,I,\hat\pi_h}$, where $\hat\pi_h(y,\{z\})=2^{-m}$, $\forall\, z\in Z_m$, $\forall\, y\in\mathbb{R}^d$, describes an m-dimensional independent random walk in some sense 'driving' the Markov chain $(\tilde\epsilon_t,\tilde\Omega,P_x^{\Pi_h})$.

In order to get a better understanding of Π_h we first take a look at the possible measures π_h. The results will allow us to establish conditions on π_h sufficient for the weak convergence of $\mathbf{P}_x^h(\Pi_h)$.

3.1. **Probability Measures on $\{-1,1\}^m$.** We want to characterize the set $M(Z_m)$ of probability measures on $(Z_m,\mathcal{P}(Z_m))$ for $m\geq 1$. There is an one-to-one correspondence between $M(Z_m)$ and the set $M_m^+:=\{f\in\mathbb{R}^{Z_m}|f\geq 0,\ \sum_{z\in Z_m}f(z)=1\}$. We will construct a basis $\mathcal{B}=\{b_1,\cdots b_{2^m}\}$ for the linear space $\mathbb{R}^{Z_m}$, such that the coefficients of $P\in M(Z_m)\cong M_m^+$ with respect to $\mathcal{B}$ have a nice probability theoretical interpretation.

$\mathbb{R}^{Z_m}$ is an euclidean linear space with the scalar product $<\cdot,\cdot>$: $\mathbb{R}^{Z_m}\times\mathbb{R}^{Z_m}\to\mathbb{R}$, $(f,g)\mapsto\sum_{z\in Z_m}f(z)g(z)$. Let $N_m:=\{1,\cdots,m\}$ and for a subset $U\subseteq N_m$ denote the elements of U by u_i, $i=1,\cdots,|U|$. For $z=(z_1,\cdots,z_m)\in Z_m$ define $z^U:=\prod_{i=1}^{|U|}z_{u_i}$, with $\prod_{i=1}^{0}:=1$. Set $\mathcal{B}=\{(z\mapsto z^U)\in\mathbb{R}^{Z_m}|U\subseteq N_m\}$. Note that $|\mathcal{B}|=2^m=\dim(\mathbb{R}^{Z_m})$. For $z\in Z_m$ and $U,V\subseteq N_m$ we have $z^Uz^V=z^{U\cup V\setminus U\cap V}$ and $\sum_{z\in Z_m}z^W=0$ for all non-empty $W\subseteq N_m$, hence $\mathcal{B}$ is an ortho*gonal* basis of $\mathbb{R}^{Z_m}$.

Lemma 3.2. *A function $f\in\mathbb{R}^{Z_m}$ admits the unique representation*

$$f(z)=\sum_{U\subseteq N_m}2^{-m}\tilde\lambda_U z^U, \tag{3.2}$$

with $\tilde\lambda_U=\tilde\lambda_U^f$ given by

$$\tilde\lambda_U=<f(\cdot),(z\mapsto z^U)>. \tag{3.3}$$

Proof. Observe $<(z\mapsto z^U),(z\mapsto z^U)>=\sum_{z\in Z_m}z^\emptyset=2^m$. □

Consider the identity $I_m=(I_{m,1},\cdots,I_{m,m})$ on Z_m as a $\mathbb{R}^d$-valued random variable on $(Z_m,\mathcal{P}(Z_m))$. Given a measure $P\in M(Z_m)$, we apply this representation to the function $p : Z_m \to [0,1], z \mapsto P(\{z\}) = P(I_m = z)$.

Proposition 3.3. *p admits the unique representation*

$$p(z) = \prod_{i=1}^{m} \frac{1+\lambda_i z_i}{2} + \sum_{1\le i<j\le m} \frac{z_i z_j \lambda_{i,j}}{2^m} + \sum_{U\subseteq N_m, |U|>2} \frac{\lambda_U z^U}{2^m}, \tag{3.4}$$

with

1.
$$\lambda_i = \lambda_i^P := \tilde{\lambda}_i^P = E_P[I_{m,i}] \in [-1,1], \quad 1 \le i \le m, \tag{3.5}$$

2.
$$\lambda_{i,j} = \lambda_{i,j}^P := \tilde{\lambda}_{\{i,j\}}^P - \tilde{\lambda}_{\{i\}}^P \tilde{\lambda}_{\{j\}}^P = Cov_P[I_{m,i}, I_{m,j}], \quad 1 \le i < j \le m, \tag{3.6}$$

3.
$$\lambda_U = \lambda_U^P := \tilde{\lambda}_U^P - \prod_{i=1}^{|U|} \tilde{\lambda}_{u_i}^P, \quad U \subseteq N_m, \ |U| > 2. \tag{3.7}$$

Furthermore we have $P(I_{m,i} = \pm 1) = \frac{1\pm\lambda_i}{2}$ for $1 \le i \le m$ and for the positive semi-definite covariance matrix of I_m

$$\Lambda = \Lambda^P := Cov_P[I_m] = (\lambda_{i,j})_{1\le i,j\le m}, \tag{3.8}$$

holds with $\lambda_{j,i} := \lambda_{i,j}$ for $1 \le i < j \le m$ and $\lambda_{i,i} := 1 - \lambda_i^2$ for $1 \le i \le m$.

Proof. By expanding the product in (3.4) and comparing with the unique representation given in Lemma 3.2 the first part of the Proposition is clear. Furthermore for $1 \le i \le m$ we have

$$P(I_{m,i} = 1) = \sum_{z\in Z_m,\, z_i=1} p(z) = \sum_{z\in Z_m,\, z_i=1} \prod_{j=1}^{m} \frac{1+\lambda_j z_j}{2} = \frac{1+\lambda_i}{2},$$

and

$$V_P[I_{m,i}^2] = 1 - \lambda_i^2.$$

□

Definition 3.4. The elements of the tuple (λ^P, Λ^P) are called the *first and second order data* of P.

It is now interesting to ask, if we can find a probability measure P for any tuple (λ, Λ) such that $(\lambda, \Lambda) = (\lambda^P, \Lambda^P)$, as long as the necessary conditions $|\lambda_i| \le 1$, Λ non-negative definite and $\lambda_{i,i} = 1 - \lambda_i^2$ are satisfied. As it will become clear in the next section, the answer to this question is related to the problem of determining the coefficients of the SDE satisfied by the continuous time Markov process described by the weak limit of the measures $\mathbf{P}_x^h(\Pi_h)$. Denote the set of $m \times m$-matrices with all diagonal elements in the interval $[0,1]$ by D_m and consider the map $Cov^m : M(Z_m) \to S_m \cap D_m,\ P \mapsto Cov_P(I_m)$. It is easy to see that Cov^m is onto for $m = 1, 2$. However, we have

Proposition 3.5. *For $m > 2$, Cov^m is not onto.*

Proof. (Sketch of a proof by contradiction)

$M(Z_m)$ is a compact convex subset of the set of all functions from Z_m to $\mathbb{R}$ with a finite set of extremal points. The same is true for the Image of Cov^m. By considering appropriate projections and intersections with linear subspaces the problem can be reduced to the case $m = 3$, where one finds that $S_3 \cap D_3$ has an infinite set of extremal points, which contradicts the assumption Cov^m being surjective. □

We are especially interested in the image of Cov^m restricted to the set of measures P with first order data $\lambda^P = 0$, i.e. $Cov^m(P)$ is a non-negative definite matrix with all diagonal entries equal to 1. In this case we give an explicit correlation matrix which is not in the image of Cov^m. In the case $m = 3$, given a probability measure P with first and second order data $(0, \Lambda^P)$, we can define a new probability measure P_+ by $P_+(\{z\}) := P(\{z\}) + P(\{-z\})$. Since $(-z)^{N_3} = (-z_1)(-z_2)(-z_3) = -z^{N_3}$ and $(-z)^U = z^U$ for $U \subseteq N_3$ with $|U| = 2$, we find $(\lambda^{P_+}, \Lambda^{P_+}) = (0, \Lambda^P)$ and $\lambda_{N_3}^{P_+} = 0$, hence

$$P_+(\{z\}) := \frac{1}{2^3} + \sum_{1 \le i < j \le 3} \frac{z_i z_j \Lambda_{i,j}^P}{2^3}. \tag{3.9}$$

Define

$$A(\lambda) := \begin{pmatrix} 1 & \lambda & \sqrt{2}/2 \\ \lambda & 1 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 & 1 \end{pmatrix} \tag{3.10}$$

$A(\lambda)$ is non-negative definite for $\lambda \in [0,1]$. It is now easy to check, that the function

$$p(z) := \frac{1}{2^3} + \sum_{1 \le i < j \le 3} \frac{z_i z_j A(\lambda)_{i,j}}{2^3}, \tag{3.11}$$

is non-negative iff $\lambda \in [\sqrt{2} - 1, 1]$ and for general m it is easy to see, that a correlation matrix having $A(\lambda), \lambda \in [0, \sqrt{2} - 1)$ as sub-matrix is not in the image of Cov^m.

3.2. **Weak Convergence of Markov Chains driven by a Random Walk.** The representation of measures on $\{-1,1\}^m$ developed in the last section allows us to calculate the *drift*- and the *diffusion*-coefficients for the Markov chain given by the stochastic kernel $\Pi_h := \Pi_h^{\mu_h,\sigma_h,\pi_h}$ defined by (3.1). Define

$$\tilde{\mu}_h^{\Pi_h}(x) := \left(\frac{1}{h}\int (y_i - x_i)\Pi_h(x, dy)\right)_{1\le i\le d}, \tag{3.12}$$

and

$$\tilde{a}_h^{\Pi_h}(x) := \left(\frac{1}{h}\int (y_i - x_i)(y_j - x_j)\Pi_h(x, dy)\right)_{1\le i,j\le d}. \tag{3.13}$$

For $x, y \in \mathbb{R}^m$ define the matrix $x \otimes y \in \mathbb{R}^{m\times m}$ by $x \otimes y := (x_i y_j)_{1\le i,j\le m}$. Given $A, B \in \mathbb{R}^{d\times m}$, $A = (a_{i,j}), B = (b_{i,j}), B^* = (b^*_{i,j})$ we have

$$\begin{aligned}(Ax)\otimes(By) &= \Big(\sum_{k=1}^m a_{i,k}x_k \sum_{l=1}^m b_{j,l}y_l\Big)_{1\le i,j\le d} \\ &= \Big(\sum_{k,l=1}^m a_{i,k}x_k\, y_l b^*_{l,j}\Big)_{1\le i,j\le d} = A(x\otimes y)B^*.\end{aligned}$$

Now (3.6) reads as

$$\Lambda = \Big(E[z_i z_j]\Big)_{1\le i,j\le m} - \lambda\otimes\lambda. \tag{3.14}$$

Proposition 3.6. *Let $(\lambda_h(x), \Lambda_h(x))$ be the first and second order data of the measure $\pi_h(x,\cdot) \in M(Z_m)$. Then*

$$\tilde{\mu}_h^{\Pi_h}(x) = \mu_h(x) + \sigma_h(x)\frac{\lambda_h(x)}{\sqrt{h}}, \tag{3.15}$$

and

$$\begin{aligned}\tilde{a}_h^{\Pi_h}(x) &= h\mu_h(x)\otimes\mu_h(x) + \sqrt{h}\Big(\mu_h(x)\otimes\big(\sigma_h(x)\lambda_h(x)\big) \\ &\quad + \big(\sigma_h(x)\lambda_h(x)\big)\otimes\mu_h(x)\Big) + \sigma_h(x)\Big(\Lambda_h(x) + \lambda_h(x)\otimes\lambda_h(x)\Big)\sigma_h^*(x).\end{aligned} \tag{3.16}$$

Proof. By the discrete nature of Π_h we have

$$\frac{1}{h}\int (y-x)\Pi_h(x,dy) = \frac{1}{h}\sum_{z\in Z_m}\Big(\mu_h(x)h + \sqrt{h}\sigma_h(x)z\Big)\ \pi_h(x,\{z\}) =$$

$$\mu_h(x) + \frac{1}{\sqrt{h}}\,\sigma_h(x)\sum_{z\in Z_m} z\pi_h(x,\{z\}) = \mu_h(x) + \frac{1}{\sqrt{h}}\,\sigma_h(x)\lambda_h(x),$$

and

$$\frac{1}{h}\int (y-x)\otimes(y-x)\Pi_h(x,dy) =$$

$$\sum_{z\in Z_m}\Big(\mu_h(x)\sqrt{h}+\sigma_h(x)z\Big)\otimes\Big(\mu_h(x)\sqrt{h}+\sigma_h(x)z\Big)\,\pi_h(x,\{z\}) =$$

$$\mu_h(x)\otimes\mu_h(x)h+\mu_h(x)\otimes\big(\sqrt{h}\sigma_h(x)\lambda_h(x)\big)+$$

$$\big(\sqrt{h}\sigma_h(x)\lambda_h(x)\big)\otimes\mu_h(x)+\sum_{z\in Z_m}\big(\sigma_h(x)z\big)\otimes\big(\sigma_h(x)z\big)\pi_h(x,\{z\}).$$

Since

$$\begin{aligned}&\sum_{z\in Z_m}\big(\sigma_h(x)z\big)\otimes\big(\sigma_h(x)z\big)\,\pi_h(x,\{z\})\\ &= \sigma_h(x)\left(\sum_{z\in Z_m} z\otimes z\,\pi_h(x,\{z\})\right)\sigma_h^*(x)\\ &= \sigma_h(x)\big(\Lambda_h(x)+\lambda_h(x)\otimes\lambda_h(x)\big)\sigma_h^*(x)\end{aligned}$$

the assertion follows. □

In the last section the market model was given with respect to the equivalent martingale measure. Now the market model is given with respect to the objective probabilities, i.e. the market state process is given as a time homogeneous Markov chain $(\tilde{\epsilon}_t,\tilde{\Omega},P_x^{\hat{\Pi}_h})$ for a stochastic kernel $\hat{\Pi}_h := \Pi_h^{\mu_h,\sigma_h,\hat{\pi}_h}$ on $\mathbb{R}^d\times\mathcal{B}_d$, with $\hat{\pi}_h(y,\{z\}) = 2^{-m}$, $\forall\, z\in Z_m$, $\forall\, y\in\mathbb{R}^d$. The market state process is driven by m independent random walks. First and second order data of the measure $\hat{\pi}_h(y,\cdot)$ are $(0,I)$ for all $y\in\mathbb{R}^d$ and $h>0$. We assume that the following conditions hold: There exist continuous functions $\mu:\mathbb{R}^d\to\mathbb{R}^d$ and $\sigma:\mathbb{R}^d\to\mathbb{R}^{d\times m}$ such that for all $R>0$:

$$\lim_{h\searrow 0}\sup_{|x|\le R}|\mu_h(x)-\mu(x)| = 0, \tag{3.17}$$

and

$$\lim_{h\searrow 0}\sup_{|x|\le R}\|\sigma_h(x)-\sigma(x)\| = 0. \tag{3.18}$$

Since μ and σ are continuous, they are locally bounded and the same is true for μ_h and σ_h if h is small enough. Therefore for arbitrary $\epsilon>0$, $\hat{\Pi}_h(x,B(x,\epsilon)) = 1$ for all $x\in B(0,R)$ if $h<\hat{h}$, where $\hat{h}$ depends on R and ϵ. This means that condition (1.14) holds and $\tilde{\mu}_h^{\hat{\Pi}_h} = \mu_h^{\hat{\Pi}_h}$ and

$\tilde{a}_h^{\hat{\Pi}_h} = a_h^{\hat{\Pi}_h}$ on $B(0,R)$ for $h < \hat{h}$. By Proposition 3.6 conditions (1.12) and (1.13) hold for μ and $a := \sigma\sigma^*$. Denote the unique solution to the martingale problem for (a,μ) starting from $(0,x)$ by $\hat{P}_x$ if it exists and set $\hat{P}_x^h := \mathbf{P}_x^h(\hat{\Pi}_h)$. We can now apply Theorem 1.6.

Proposition 3.7. *If μ and σ satisfy the growth and Lipschitz conditions (1.22) and (1.23), then* $\lim_{h\searrow 0} \hat{P}_x^h = \hat{P}_x$.

3.3. **Convergence of Equivalent Measures.** We will now apply Theorem 1.6 to a second set of Markov chains given by stochastic kernels on $(\mathbb{R}^d, \mathcal{P}(Z_m))$, defined as $\Pi^h := \Pi_h^{\mu_h,\sigma_h,\pi_h}$, with measurable μ_h, σ_h and a stochastic kernel π_h on $(\mathbb{R}^d, \mathcal{P}(Z_m))$ for $h > 0$. Set $P_x^h := \mathbf{P}_x^h(\Pi_h^{\mu_h,\sigma_h,\pi_h})$ and denote the first and second order data of π_h as (λ_h, Λ_h).

Definition 3.8. The family of stochastic kernels π_h is called *equivalent* to $\hat{\pi}_h$ if $\pi_h(x,\cdot)$ and $\hat{\pi}_h(x,\cdot)$ are equivalent measures, i.e. $\pi_h(x,\{z\}) > 0$, $\forall\, z \in Z_m$, for all $x \in \mathbb{R}^d$. The family π_h is called (λ,Λ)*-converging* if $\lambda : \mathbb{R}^d \to \mathbb{R}^m$ and $\Lambda : \mathbb{R}^d \to \mathbb{R}^{m\times m} \cap S_m$ are continuous functions such that the following conditions hold

1.

$$\lim_{h\searrow 0} \sup_{|x|\le R} \left| \frac{\lambda_h(x)}{\sqrt{h}} - \lambda(x) \right| = 0, \tag{3.19}$$

2.

$$\lim_{h\searrow 0} \sup_{|x|\le R} \|\Lambda_h(x) - \Lambda(x)\| = 0. \tag{3.20}$$

Remark 3.9. Observe that $\Lambda(x) \in \big\{(d_{i,j}) \in \mathbb{R}^{m\times m} | d_{i,i} = 1, d_{i,j} \in [-2,2]\big\}, \forall\, x \in \mathbb{R}^d$.

If π_h and $\hat{\pi}_h$ are equivalent, then P_x^h and $\hat{P}_x^h$ are equivalent on $\mathcal{M}_{ih}$, $\forall\, i \in \mathbf{N}$, since any path of finite length has positive probability with respect to both measures. If π_h is (λ,Λ)-converging, then by Proposition 3.6 we find that condition (1.12) and (1.13) hold with μ replaced by $\mu+\sigma\lambda$ and $a := \sigma\Lambda\sigma^*$. It is well known that a non-negative definite matrix has a unique non-negative definite square root. If we denote this unique square root of Λ by $\sqrt{\Lambda}$, we can again apply Theorem 1.6.

Proposition 3.10. *If $\mu+\sigma\lambda$ and $\sigma\sqrt{\Lambda}$ satisfy the growth and Lipschitz conditions (1.22) and (1.23), then* $\lim_{h\searrow 0} P_x^h = P_x$.

Remark 3.11. By Theorem 1.2 in [8], Chapter 6, p. 129, we have that $\sigma\sqrt{\Lambda}$ is Lipschitz continuous on compact sets if σ is Lipschitz continuous on compact sets and Λ is $C^2(\mathbb{R}^m)$.

Remark 3.12. The results of Section 3.1 show that the possible limit functions Λ are only a proper subset of the set of all correlation matrices. This has its reason in the discrete nature of π. If $\hat{\pi}$ and π are gaussian measures this restriction does not apply.

Remark 3.13. By considering time-space processes these results can be extended to the time-inhomogeneous case.

We now turn to the first main result of this paper. We assume the necessary conditions on $\mu, \sigma, \lambda, \Lambda, r$ guaranteeing $F_{\tilde{Y}}$ to solve (2.11) with μ replaced by $\tilde{\mu}_i := \mu_i + (\sigma\lambda)_i$ and $a_{i,j}$ replaced by $\tilde{a}_{i,j} := (\sigma\sqrt{\Lambda})_{i,j}$, for all $\tilde{Y} \in C_0^2(\mathbb{R}^d)$, see [7], Appendix E. In addition we assume μ, λ, Λ and σ to be bounded. Consider a weak solution $(X, W), (\tilde{E}, \mathcal{G}, P), \{\mathcal{G}_t\}$ of the SDE (1.18), (1.19). We have proved that the discrete time price processes $V^h_{\tilde{Y}^T_h}$ weakly converge to the continuous processes $F_{\tilde{Y}}\big((T - s)^+, X_{s\wedge T}\big)$ (Corollary 2.5). We seek a measure Q equivalent to P such that the discounted processes

(3.21)

$$A_{\tilde{\mu},\tilde{a},r} := \left\{ e^{-\int_0^{\cdot\wedge T} r(X_t)dt} F_{\tilde{Y}}\big((T - \cdot)^+, X_{\cdot\wedge T}\big) \Big| T \in \mathbb{R}^+, \tilde{Y} \in C_0^2(\mathbb{R}^d) \right\},$$

become martingales with respect to Q.

Proposition 3.14. *If $Q \sim P$ and all processes $Z \in A_{\tilde{\mu},\tilde{a},r}$ are local martingales with respect to Q then $\sigma\sigma^* = \sigma\Lambda\sigma^*$.*

Proof. Under a measure Q equivalent to P, X remains a continuous semimartingale with respect to Q. There exists a decomposition $X = L + A$, where L is a continuous local martingale and A is a continuous process with finite variation, see [16], Lemma 4.24, p. 44. For $0 \le t \le T$ we apply Itô's formula to $Z \in A_{\tilde{\mu},\tilde{a},r}$ and make use of (2.11):

$$\begin{aligned} Z_t = e^{-\int_0^t r(X_s)ds} F_{\tilde{Y}}\big(T - t, X_t\big) &= F_{\tilde{Y}}\big(T, X_0\big) \\ &+ \int_0^t e^{-\int_0^s r(X_u)du} \left(-r(X_s) F_{\tilde{Y}}\big(T - s, X_s\big) - \frac{\partial F_{\tilde{Y}}}{\partial T}\big(T - s, X_s\big) \right) ds \\ &+ \sum_{i=0}^d \int_0^t e^{-\int_0^s r(X_u)du} \frac{\partial F_{\tilde{Y}}}{\partial x_i}\big(T - s, X_s\big) dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=0}^d \int_0^t e^{-\int_0^s r(X_u)du} \frac{\partial^2 F_{\tilde{Y}}}{\partial x_i \partial x_j}\big(T - s, X_s\big) d[X^i, X^j]_s, \end{aligned}$$

hence we have

$$Z_t = \sum_{i=0}^{d} \int_0^t e^{-\int_0^s r(X_u)du} \frac{\partial F_{\tilde{Y}}}{\partial x_i}\big(T-s, X_s\big)(dA_s^i - \tilde{\mu}_i(X_s)ds)$$
$$+\frac{1}{2}\sum_{i,j=0}^{d} \int_0^t e^{-\int_0^s r(X_u)du} \frac{\partial^2 F_{\tilde{Y}}\big(T-s, X_s\big)}{\partial x_i \partial x_j}(d[X^i, X^j]_s - \tilde{a}_{i,j}(X_s)ds)$$
$$+\sum_{i=0}^{d} \int_0^t e^{-\int_0^s r(X_u)du} \frac{\partial F_{\tilde{Y}}}{\partial x_i}\big(T-s, X_s\big)dL_s^i.$$

The finite variation part of Z must vanish Q-almost surely by assumption. Since $d[X^i, X^j]_s = a_{i,j}(X_s)ds$ holds with respect to Q, we have Q-almost surely

$$\int_0^t e^{-\int_0^s r(X_u)du} \tilde{L}F_{\tilde{Y}}\big(T-s, X_s\big)ds = \int_0^t e^{-\int_0^s r(X_u)du} \sum_{i=0}^d \frac{\partial F_{\tilde{Y}}\big(T-s, X_s\big)}{\partial x_i} dA_s^i, \tag{3.22}$$

where $\tilde{L} := \sum_{i=0}^d \tilde{\mu}_i \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=0}^d (\tilde{a}_{i,j} - a_{i,j})\frac{\partial^2}{\partial x_i \partial x_j}$. Differentiating the left side of (3.22) at $t = T$ from the left, we obtain

$$\lim_{\delta \searrow 0} \frac{1}{\delta} \int_{T-\delta}^T e^{-\int_0^s r(X_u)du} \tilde{L}F_{\tilde{Y}}\big(T-s, X_s\big)ds =$$
$$e^{-\int_0^T r(X_u)du} \tilde{L}F_{\tilde{Y}}\big(0, X_T\big) = e^{-\int_0^T r(X_u)du} \tilde{L}\tilde{Y}(X_T).$$

Differentiating the right side of (3.22) at $t = T$ from the left, we obtain

$$\lim_{\delta \searrow 0} \frac{1}{\delta} \int_{T-\delta}^T e^{-\int_0^s r(X_u)du} \sum_{i=0}^d \frac{\partial F_{\tilde{Y}}\big(T-s, X_s\big)}{\partial x_i} dA_s^i =$$
$$e^{-\int_0^T r(X_u)du} \lim_{\delta \searrow 0} \frac{1}{\delta} \int_{T-\delta}^T e^{\int_s^T r(X_u)du} \sum_{i=0}^d \frac{\partial F_{\tilde{Y}}\big(T-s, X_T\big)}{\partial x_i} dA_s^i.$$

Multiplying both sides with $e^{\int_0^T r(X_u)du}$ we have

$$\tilde{L}\tilde{Y}(X_T) = \lim_{\delta \searrow 0} \frac{1}{\delta} \int_{T-\delta}^T e^{\int_s^T r(X_u)du} \sum_{i=0}^d \frac{\partial F_{\tilde{Y}}\big(T-s, X_T\big)}{\partial x_i} dA_s^i$$
$$= \lim_{\delta \searrow 0} \frac{1}{\delta} \int_{T-\delta}^T e^{\int_s^T r(X_u)du} \sum_{i=0}^d \frac{\partial \tilde{Y}\big(X_T\big)}{\partial x_i} dA_s^i +$$
$$\lim_{\delta \searrow 0} \frac{1}{\delta} \int_{T-\delta}^T e^{\int_s^T r(X_u)du} \sum_{i=0}^d \frac{\partial (F_{\tilde{Y}}\big(T-s, X_T\big) - \tilde{Y}\big(X_T\big))}{\partial x_i} dA_s^i,$$

hence we get

$$\begin{aligned}\tilde{L}\tilde{Y}(X_T) &= \lim_{\delta\searrow 0}\frac{1}{\delta}\int_{T-\delta}^{T}\sum_{i=0}^{d}\frac{\partial\tilde{Y}\big(X_T\big)}{\partial x_i}dA_s^i = \lim_{\delta\searrow 0}\sum_{i=0}^{d}\frac{\partial\tilde{Y}\big(X_T\big)}{\partial x_i}\frac{1}{\delta}\int\limits_{T-\delta}^{T}dA_s^i \\ &= \lim_{\delta\searrow 0}\sum_{i=0}^{d}\frac{\partial\tilde{Y}\big(X_T\big)}{\partial x_i}\frac{A_T^i - A_{T-\delta}^i}{\delta},\end{aligned}$$

for all $\tilde{Y} \in C_0^2(\mathbb{R}^d)$ and therefore, choosing $\tilde{Y}_i$ such that $\frac{\partial}{\partial x_i}\tilde{Y}_i(y) \neq 0$ and $\frac{\partial}{\partial x_j}\tilde{Y}_i(y) = 0$ for $i \neq j$ and $\frac{\partial^2}{\partial x_k \partial x_l}\tilde{Y}_i(y) = 0, \forall\, 1 \leq k, l \leq d$, for all $|y| < R$, we find $\frac{dA_t^i}{dt} = \tilde{\mu}_i(X_t), \forall 1 \leq i \leq d$, Q-almost sure for all $0 \leq t < \infty$. Choosing $\tilde{Y}_{i,j}$ such that $\frac{\partial^2}{\partial x_i \partial x_j}\tilde{Y}_{i,j}(y) \neq 0$ and $\frac{\partial^2}{\partial x_k \partial x_l}\tilde{Y}_{i,j}(y) = 0$ for all $\{k,l\} \neq \{i,j\}$, for all $|y| < R$, we find and $a_{i,j} = \tilde{a}_{i,j}$, $\forall\, 1 \leq i, j \leq d$. □

Remark 3.15. Even if π_h and $\hat{\pi}_h$ are equivalent and π_h is assumed to be (λ, Λ)-*converging*, we can *not* conclude in general that P_x and $\hat{P}_x$ are equivalent as well. If μ and σ satisfy the growth and Lipschitz conditions (1.22), (1.23) and if $\sigma\Lambda\sigma^* = \sigma\sigma^*$ and the Girsanov-functional $\frac{1}{2}\exp(-\int_0^t \lambda(X_u)^2 du + \int_0^t \lambda(X_u) dW_u)$ is uniformly integrable up to time $T \leq \infty$, then P_x and $\hat{P}_x$ are equivalent on $\mathcal{M}_T$. λ is usually called the *market price of risk*. If $\sigma\Lambda\sigma^* \neq \sigma\sigma^*$ then there does not exist a locally equivalent martingale measure for the continuous time limit market. By a result of [6] this is in the case of a *finite* number of stocks equivalent to the existence of a *free lunch with vanishing risk* (FLVR). This case is considered in Subsection 3.4.

In the current literature on continuous finance only measures equivalent to $\hat{P}_x$ are considered. If we see the continuous time market as an approximation for a discrete market working with a high frequency then we have to take a wider class of measures, induced by the weak solutions of SDEs with drift- and diffusion-coefficients $\mu + \sigma\lambda$ and $\sigma\sqrt{\Lambda}$, into account.

3.4. **An Example.** In this subsection we consider the market described by $\Pi_h := \Pi_h^{\mu_h, \sigma_h, \pi_h}$ and $\hat{\Pi}_h := \Pi_h^{\mu_h, \sigma_h, \hat{\pi}_h}$ under the additional condition that the underlying d-dimensional state process $\tilde{\epsilon}_i, i \in \mathbf{N}$, describes the prices of d stocks or contingent claims. As a consequence the discounted price processes

$$S_i := \tilde{\epsilon}_i \prod_{j=0}^{i-1} \frac{1}{1 + R_h(\tilde{\epsilon}_j)}$$

must be martingales under $P_x^{\Pi_h}$. Calculating for $i \geq 1$

$$\begin{aligned} & E_{P_x^{\Pi_h}}\left[S_i \middle| \tilde{M}_{i-1}\right] \\ = \; & E_{P_x^{\Pi_h}}\left[\tilde{\epsilon}_i \middle| \tilde{M}_{i-1}\right] \prod_{j=0}^{i-1} \frac{1}{1+R_h(\tilde{\epsilon}_j)} \\ = \; & \left(\tilde{\epsilon}_{i-1} + \tilde{\mu}_h^{\Pi_h}(\tilde{\epsilon}_{i-1})h\right) \prod_{j=0}^{i-1} \frac{1}{1+R_h(\tilde{\epsilon}_j)} \\ = \; & \tilde{\epsilon}_{i-1} \prod_{j=0}^{i-2} \frac{1}{1+R_h(\tilde{\epsilon}_j)} + \left(\frac{\tilde{\epsilon}_{i-1} + \tilde{\mu}_h^{\Pi_h}(\tilde{\epsilon}_{i-1})h}{1+R_h(\tilde{\epsilon}_{i-1})} - \tilde{\epsilon}_{i-1}\right) \prod_{j=0}^{i-2} \frac{1}{1+R_h(\tilde{\epsilon}_j)} \\ = \; & S_{i-1} + \left(\tilde{\mu}_h^{\Pi_h}(\tilde{\epsilon}_{i-1})h - \tilde{\epsilon}_{i-1} R_h(\tilde{\epsilon}_{i-1})\right) \prod_{j=0}^{i-1} \frac{1}{1+R_h(\tilde{\epsilon}_j)}, \end{aligned}$$

we find by Proposition 3.6 and Proposition 3.10:

Proposition 3.16. *The discounted price processes S_i are martingales with respect to $P_x^{\Pi_h}$ if and only if*

$$\mu_h h + \sigma_h \lambda_h \sqrt{h} = Id\, R_h, \tag{3.23}$$

*on $\left\{y \in \mathbb{R}^d | P_x^{\hat{\Pi}_h}(\{\exists i : \tilde{\epsilon}_i = y\}) > 0\right\}$, the range of the process $\tilde{\epsilon}_i$, $i \in \mathbf{N}$. ($Id(y) := y$, $\forall\, y \in \mathbb{R}^d$). Assume (2.6), (3.18) and that π_h is (λ, Λ)-*converging *(see Definition 3.8). For $\mu_h := Id\,\frac{R_h}{h} - \sigma_h \frac{\lambda_h}{\sqrt{h}}$ then (3.17) holds with $\mu := Id\, r - \sigma\lambda$. Furthermore assume that μ and σ as well as $Id r$ and $\sigma\sqrt{\Lambda}$ satisfy the growth and Lipschitz conditions (1.22) and (1.23), then the weak limit of the price processes with respect to the martingale measure $P_x^{\Pi_h}$ is a solution to the SDE*

$$dS_t = r(S_t) S_t dt + \sigma(S_t)\sqrt{\Lambda}(S_t) dW_t. \tag{3.24}$$

Remark 3.17. Observe that a market modeled as above with invertible σ_h is only complete for $d = 1$ and $m = 1$, i.e. one Stock and the risk-free bond are traded and there is only one source of uncertainty. All $\pi_h > 0$ with first order data λ^{π_h} such that $\mu_h + \sigma_h \lambda^{\pi_h}\sqrt{h} = \text{Id}\, R_h$ lead to a martingale measure $P_x^{\Pi_h}$, being locally equivalent, i.e. equivalent on $\tilde{\mathcal{M}}_i, \forall\, i \in \mathbf{N}$, which is markovian in the sense that the price process is a time-homogeneous Markov chain with respect to it and any such martingale measure is of that form.

3.5. **The Market Price for Risk and State Prices.** In this subsection we want to reach a better understanding of the market price for risk. We want to clarify the relation between the equivalent martingale measure, market price for risk and the Arrow-Debreu state prices, see [7].

We are still considering the Markov chain $(\tilde{\epsilon}_i, \tilde{\Omega}, P_x^{\hat{\Pi}_h})$ modeling the discrete time approximation of the market with respect to the objective probabilities together with an locally equivalent martingale measure $P_x^{\Pi_h}$ given by $\Pi_h = \Pi_h^{\mu_h,\sigma_h,\pi_h}$. Observe that $Z_y = Z_y(\Pi_h) = \{y + \mu_h(y)h + \sqrt{h}\sigma_h(y)z | z \in Z_m\}$ and $|Z_y| \leq 2^m$. For $w \in \tilde{\mathcal{M}}_i^*, i > 0$ define $\tilde{z}_j(w) := \{z \in Z_m | w(j) = w(j-1) + \mu_h(w(j-1))h + \sqrt{h}\sigma_h(w(j-1))z\} \in \mathcal{P}(Z_m)$ for $0 < j \leq i$. We denote the only element of $\tilde{\mathcal{M}}_i^*$ by x and set for $z \in Z_m$ and $w \in \tilde{\mathcal{M}}_i^*, i \geq 0$, $[w, z] := [w, w(i) + \mu_h(w(i))h + \sqrt{h}\sigma_h(w(i))z]$.

For $w \in \tilde{\mathcal{M}}_i^*, i > 0$ let $Y(w)$ be the contingent claim that pays 1 unit at time ih if the process is in state w at time i and becomes worthless otherwise. By the contingent claim pricing formula we have

(3.25)
$$V_0\left(Y(w)\right) = E_{P_x^{\Pi_h}}\left[\frac{\mathbf{1}_w}{\prod_{j=0}^{i-1} 1 + R_h(\tilde{\epsilon}_j)}\right] = \prod_{j=0}^{i-1} \frac{\pi_h(w(j), \tilde{z}_{j+1}(w))}{1 + R_h(w(j))}.$$

Remark 3.18. $V_0(Y([x, z]))$ is the $1-$period Arrow-Debreu state-price for the state $[x, x + \mu_h(x)h + \sqrt{h}\sigma_h(x)z]$. Furthermore we introduce contingent claims $Y_i(z|w)$ traded at time $ih, i \in \mathbb{N}$ if the process is in state $w \in \tilde{\mathcal{M}}_i^*$ at time i, which pay 1 unit at time $(i+1)h$ if the process moves from state w to state $[w, z]$ and become worthless otherwise. $Y_i(z|w)$ can be interpreted as a bet on the development of the Markov chain in the next period and $Y([x, z]) = Y_0(z|x)$. We find for the price of $Y_i(z|w)$ at time ih if the process is in state w at time i

(3.26)
$$V_{ih}\left(Y_i(z|w)\right) = E_{P_x^{\Pi_h}}\left[\frac{\mathbf{1}_{[w,z]}}{1 + R_h(w(i))}\Big|\tilde{\mathcal{M}}_i\right](w) = \frac{\pi_h(w(i), \tilde{z}_{i+1}([w, z]))}{1 + R_h(w(i))}.$$

For $w \in \tilde{\mathcal{M}}_i^*$ we have by (3.25)

$$V_0\left(Y(w)\right) \;=\; \prod_{j=0}^{i-1} V_{jh}\left(Y_j(z_{j+1}|w^{-(i-j)})\right),$$

where $z_j \in \tilde{z}_j(w)$ for $1 \leq j \leq i$. We find immediately a hedging strategy for the contingent claim $Y(w)$:

Buy $\prod_{j=1}^{i-1} V_{jh}\left(Y_j(z_{j+1}|w^{-(i-j)})\right)$ units of $Y_0(z_1|x)$ at time 0. After one period this portfolio either becomes worthless, in which case it equals the value of $Y(w)$ at time h, or it pays off

$$\prod_{j=1}^{i-1} V_{jh}\left(Y_j(z_{j+1}|w^{-(i-j)})\right)$$

units of money which can be reinvested, buying

$$\prod_{j=2}^{i-1} V_{jh}\left(Y_j(z_{j+1}|w^{-(i-j)})\right)$$

units of $Y_1(z_2|w^{-(i-1)})$. Iterating this we replicate the payoff of $Y(w)$. This procedure also reveals the dynamics of the state prices.

The contingent claim pricing formula now reads as

Proposition 3.19. *The price $V_0(Y)$ at time 0 of a contingent claim Y maturing at time Nh is given by*

$$V_0(Y) = \sum_{w\in\tilde{\mathcal{M}}_N^*} Y(w)V_0\left(Y(w)\right) \tag{3.27}$$

Proof.

$$\begin{aligned}
V_0(Y) &= E_{P_x^{\Pi h}}\left[\prod_{j=0}^{N-1}\frac{1}{1+R_h(\tilde{\epsilon}_j)}Y\right] \\
&= E_{P_x^{\Pi h}}\left[\sum_{w\in\tilde{\mathcal{M}}_N^*}\mathbf{1}_w\prod_{j=0}^{N-1}\frac{1}{1+R_h(\tilde{\epsilon}_j)}Y\right] \\
&= \sum_{w\in\tilde{\mathcal{M}}_N^*} E_{P_x^{\Pi h}}\left[\mathbf{1}_w Y\prod_{j=0}^{N-1}\frac{1}{1+R_h(w(j))}\right] \\
&= \sum_{w\in\tilde{\mathcal{M}}_N^*} Y(w)E_{P_x^{\Pi h}}\left[\mathbf{1}_w\prod_{j=0}^{N-1}\frac{1}{1+R_h(w(j))}\right] \\
&= \sum_{w\in\tilde{\mathcal{M}}_N^*} Y(w)V_0\left(Y(w)\right),
\end{aligned}$$

since Y is constant on w for all $w\in\tilde{\mathcal{M}}_N^*$. □

Hence the market becomes dynamically complete by introducing the contingent claims $Y_i(z|w)$, $z\in Z_m$, $w\in\tilde{\mathcal{M}}_i^*$, $i\in\mathbb{N}$. This means that a system of 1−period Arrow-Debreu state-prices for each state of the market constant in time leads to a complete market.

Proposition 3.19 allows to price future streams of cash flows. Consider a contract C entitling to future payments generated by a portfolio of contingent claims C_i with maturity ih, $i \in \mathbb{N}$. Then the value $V_0(C)$ of C at time 0 is $\sum_{i=0}^{\infty} V_0(C_i)$. If we assume a dividend paying stock S in this market where the dividend depends on the state of the market only, then we can price the stream of dividend payments generated by the stock S. The close relation between capital markets and stock markets becomes clear.

Remark 3.20. The economical reason why we consider locally equivalent martingale measures of the form $P_x^{\Pi_h}$ is that in a market model being time-homogeneous with respect to the objective probabilities we expect prices of contingent claims to depend on the state of the market and the remaining time to maturity only. Assuming completeness of the market there is a one-to-one correspondence between time-homogeneous system of positive prices for the claims $Y_i(z|w)$, $z \in Z_m$, $w \in \tilde{\mathcal{M}}_i^*$ and the stochastic kernels $\Pi_h^{\mu_h,\sigma_h,\pi_h} \sim \Pi_h^{\mu_h,\sigma_h,\hat{\pi}_h}$ which uniquely determine the measures $\pi_h(w(j),\cdot)$ restricted to the σ-algebra generated by the sets $([w,\cdot](j+1))^{-1}$ for $w \in \tilde{\mathcal{M}}_j*$, $j \in \mathbf{N}$. For invertible σ_h, π_h is uniquely determined. Asymptotically only the first and second order data $(\lambda^{\pi_h}, \Lambda^{\pi_h})$ of π_h matter as we have seen in Section 3.3.

The close relation between state-prices and the first and second order data $(\lambda^{\pi_h}, \Lambda^{\pi_h})$ of π_h is explained by Proposition 3.3. For invertible σ_h, $w \in \tilde{\mathcal{M}}_i^*$, $i \geq 0$, and $1 \leq j \leq m$ we obtain for the contingent claims $Y_\pm^j(w) := \sum_{\substack{z\in Z_m \\ z_j=\pm 1}} Y_i(z|w)$

$$V_{ih}\left(Y_\pm^j(w)\right) = \frac{\sum_{\substack{z\in Z_m \\ z_j=\pm 1}} \pi_h(w(i),\{z\})}{1+R_h(w(i))} = \frac{\frac{1\pm\lambda_j^{\pi_h}(w(i))}{2}}{1+R_h(w(i))}, \tag{3.28}$$

and for $1 \leq j,k \leq m$ we obtain for the $Y_{\pm\pm}^{j,k}(w) := \sum_{\substack{z\in Z_m \\ z_j=\pm 1, z_k=\pm 1}} Y_i(z|w)$

$$\begin{aligned} V_{ih}\left(Y_{\pm\pm}^{j,k}(w)\right) &= \frac{\sum\limits_{\substack{z\in Z_m \\ z_j=\pm 1, z_k=\pm 1}} \pi_h(w(i),\{z\})}{1+R_h(w(i))} \\ &= \frac{\frac{\left(1\pm\lambda_j^{\pi_h}(w(i))\right)\left(1\pm\lambda_k^{\pi_h}(w(i))\right)\pm(\pm\lambda_{j,k}^{\pi_h}(w(i)))}{4}}{1+R_h(w(i))}. \end{aligned} \tag{3.29}$$

Remark 3.21. In the situation of Subsection 3.4, where we have d stocks and if σ_h is invertible, $\lambda_h := -\frac{\sigma_h^{-1}}{\sqrt{h}}(\mu_h h - IdR_h) \in (-1,1)$ is a necessary and sufficient condition for the existence of a locally equivalent martingale measure, namely any π_h with first order data λ_h leads to a locally equivalent martingale measure $P_x^{\Pi_h^{\mu_h,\sigma_h,\pi_h}}$. Assuming (3.17), (3.18), (2.6) and (3.19) for μ_h, σ_h, r_h and λ_h the objective probabilities converge weakly but there is no guarantee that the martingale measures will converge too. If we choose Λ_h, such that (λ_h, Λ_h) are the first and second order data of some stochastic kernel π_h and assume π_h to be (λ, Λ)-converging for some continuous Λ, then we fix asymptotically the prices of portfolios of the form $Y_{\pm\pm}^{j,k}(w), 1 \leq j, k \leq m, w \in \tilde{\mathcal{M}}_i^*$. The discrete time markets do in general not become complete if we choose Λ_h, since the higher order data of π_h are not unique. Assuming (3.20), we get weak convergence of the martingale measures. From the theory of Backward Stochastic Differential Equations, see e.g. [22], the limit market is known to be arbitrage-free and complete under weak conditions on μ, σ, r, λ, $\Lambda = I$ and allowing continuous-time square-integrable hedging. The continuous-time market together with an *equivalent* martingale measure can only be approximated by discrete time markets if we assume the introduction of asymptotic prices for the contingent claims $Y_{\pm\pm}^{j,k}(w)$ in such a way that Λ_h converges locally uniform to I. Introducing these asymptotic prices does not cause the discrete time markets to be complete, but there exist now asymptotically enough traded contingent claims to make the limit market complete. This explains the surprising fact that continuous time markets are complete under such weak technical conditions. In a more realistic situation we can not assume $\Lambda = I$.

3.6. **Markov Chains driven by a normally distributed random variable.** We consider again a discrete time market described by stochastic kernels $\Pi_h := \Pi_h^{\mu_h,\sigma_h,\pi_h}$ and $\hat{\Pi}_h := \Pi_h^{\mu_h,\sigma_h,\hat{\pi}_h}$ defined similar to (3.1) with the difference that we now allow π_h and $\hat{\pi}_h$ to be stochastic kernels on $(\mathbb{R}^d, \mathcal{B}_m)$:

$$\Pi_h(x, B) := \int_{\mathbb{R}^m} \mathbf{1}_B\Big(x + \mu_h(x)h + \sqrt{h}\sigma_h(x)z\Big) d\pi_h(x, z) \tag{3.30}$$

For simplicity we set $\hat{\pi}_h(x) := \mathcal{N}(0, I)$ and assume $\pi_h(x)$ to be of the form $\pi_h(x) = \mathcal{N}(\lambda_h(x), \Lambda_h(x))$ for some continuous functions $\lambda_h : \mathbb{R}^d \to \mathbb{R}^m$ and $\Lambda_h : \mathbb{R}^d \to S_m \subseteq \mathbb{R}^{m\times m}$. It is easy to see that the resulting measures $P_x^{\Pi_h}$ and $P_x^{\hat{\Pi}_h}$ describing the state process with respect to objective probabilities respectively the risk-free probabilities are equivalent on $\tilde{\mathcal{M}}_i$, $\forall\, i \in \mathbf{N}$ since $\pi_h(x) \sim \hat{\pi}_h(x)$, $\forall\, x \in \mathbb{R}^d$. Calling $(\lambda_h(x), \Lambda_h(x))$ the first and second order data of $\mathcal{N}(\lambda_h(x), \Lambda_h(x))$, Proposition 3.6 holds word by word in this new context. Assume that

conditions (1.12) and (1.13) hold. If $\pi_h(x)$ is normally distributed, then condition (1.14) holds for Π_h. Therefore Proposition 3.7 and Proposition 3.10 hold in the present situation too if we assume (3.19) and (3.20) to hold. The main difference to the situation considered so far is that Remark 3.12 and Remark 3.9 does not apply anymore. The set of measures now possibly appearing as a weak limit of discrete time equivalent martingale measures $\mathbf{P}_x^h(\Pi_h^{\mu_h,\sigma_h,\pi_h})$ is enormous. In general such a limit measure is not equivalent to the limit of the measures $\mathbf{P}_x^h(\Pi_h^{\mu_h,\sigma_h,\hat{\pi}_h})$ describing the objective probabilities. If the continuous time market is only seen as an approximation for a discrete time market working at a high frequency this is not a problem since from non-equivalentness of the objective measure and the martingale measure there follows only the existence of an FLVR which is a weaker property than the existence of an arbitrage and an FLVR involving infinitely fast trading might not be realizable in a real market.
We restrict the form of π_h further by assuming π_h to be normally distributed: $\pi_h(x,\cdot) = \mathcal{N}(\lambda_h(x), \tilde{\Lambda}_h(x)I)$, where $\tilde{\Lambda}_h : \mathbb{R}^d \to (0,\infty)$ is a continuous function. (If the state process represents d stocks, $\pi_h(x)$ can be modified such that $P_x^{\Pi_h}(\tilde{\epsilon}_i < 0) = 0, \ \forall\, i \in \mathbf{N}$.)

In the case $d = 1$, $r(x) \equiv r_0 > 0, \ \forall\, x \in \mathbb{R}$, $\sigma(x) = \sigma_0 x > 0, \ \forall\, x \in \mathbb{R}$ the price $E_K^T(t,x)$ of an European option with exercise price K and maturity T can be calculated explicitly by the famous Black-Scholes formula. Allowing continuous hedging, it is well known that E_K^T is a solution to the Cauchy problem

$$-\frac{\partial f}{\partial t}(t,x) + r_0 f(t,x) = \frac{1}{2}\sigma_0^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x) + r_0 x \frac{\partial f}{\partial x}(t,x), \tag{3.31}$$

on $[0,T) \times (0,\infty)$ with the boundary condition $f(T,x) = (x-K)^+$ for $x \geq 0$. Observing the interest rate r_0, the price x of the underlying stock and the price of the option $E_K^T(t,x)$ it is possible to calculate the implied relative volatility σ_0. Since it is difficult to estimate σ_0 from observing the stock price directly practitioners proceed in this way, see [15]. Assuming $\lim_{h\searrow 0} \tilde{\Lambda}_h(x) \equiv \tilde{\Lambda} > 0$ uniformly on $(0,\infty)$ our results show that the implied relative volatility turns out to be $\tilde{\Lambda}\sigma_0$. E_K^T now solves (3.31) with σ_0 replaced by $\tilde{\Lambda}\sigma_0$:

$$-\frac{\partial f}{\partial t}(t,x) + r_0 f(t,x) = \frac{1}{2}(\tilde{\Lambda}\sigma_0)^2 x^2 \frac{\partial^2 f}{\partial x^2}(t,x) + r_0 x \frac{\partial f}{\partial x}(t,x). \tag{3.32}$$

The factor $\tilde{\Lambda}$ can be interpreted as a demanded *compensation for risk due to discrete hedging* which can not be derived from the model but has to be estimated from historical data. This model does not explain the smile-shaped graph of the implied relative volatility for different strike prices, see [15], Figure 19.3, p. 503, since $\tilde{\Lambda}$ is independent of K, but one might argue that the compensation demanded for risk due to discrete hedging is different for options right at-the-money and options deep-out-of-the-money or deep-in-the-money. Using Δ-hedging, the sensitivity $\Gamma := \frac{\partial \Delta}{\partial x}$ of $\Delta := \frac{\partial E_K^T}{\partial x}$ with respect to changes in the price of the underlying stock is high at-the-money and low out-of-the-money resp. in-the-money, see [15], Figure 14.10, p. 326. Since Δ is the amount of stock in the hedging portfolio replicating the option, this means that the portfolio has to be rebalanced with higher frequency at-the-money than out-of-the-money resp. in-the-money. Assuming the writer of the option to demand a compensation for this, option prices at-the-money should be increased relatively to prices out-of-the-money resp. in-the-money. This can be achieved by using an increased volatility at-the-money, since the *vega* $:= \frac{\partial E_K^T}{\partial \sigma_0}$, see [15], Figure 14.11, p. 329, is always positive. This leads to a reverted smile for the graph of the implied relative volatilities, an effect which is also observed in the real markets for some options.

4. Zero Bonds

A zero bond at time 0 with maturity Nh is a contingent claim B_x^{Nh} that pays 1 unit at time Nh. We consider only default free zero bonds. This class of contingent claims is especially suited for studying the properties of the market model, first for their simplicity and second for the fact that there is a infinite number of zero bonds, one for each time of maturity. Furthermore, zero bonds are traded in real markets, allowing to analyze data with statistical methods. They reflect the markets attitude towards the uncertainty of the future in a very pure way since their payoff is deterministic.

By the contingent claim pricing formula we find

(4.1)
$$V_0\left(B_x^{Nh}\right) = E_{P_x^{\Pi_h}}\left[\prod_{j=0}^{N-1} \frac{1}{1+R_h(\tilde{\epsilon}_j)}\right] = E_{P_x^{\Pi h}}\left[\exp\left(-\sum_{j=0}^{N-1} h r_h(\tilde{\epsilon}_j)\right)\right].$$

For $\Pi_h = \Pi_h^{\mu_h,\sigma_h,\pi_h}$ set $B_h^{\pi_h}(N) := V_0(B_x^{Nh})$. Assume now that the objective probability measure $P_x^{\hat{\Pi}_h}$ describes a real market where we observe a term structure of interest rates, i.e there is a function $\tilde{B}_h : \mathbb{N} \to \mathbb{R}^+$ such that $\tilde{B}_h(N)$ equals the observed price of the zero bond B_x^{Nh} at time 0 in the market. We can try to fit our model to the given initial data $\tilde{B}_h$ by choosing a π_h equivalent to $\hat{\pi}_h$ such that the resulting function $B_h^{\pi_h}$ fits $\tilde{B}_h$ best with respect to some optimality criterion.

Define $\tilde{R}_h^F : \mathbb{N} \to (-1, \infty)$ by $\frac{1}{1+\tilde{R}_h^F(i)} := \frac{\tilde{B}_h(i+1)}{\tilde{B}_h(i)}$ and the corresponding rate $\tilde{r}_h^F(t) := \frac{\ln(1+\tilde{R}_h^F(i))}{h}$ for $t \in [i, i+1)$. Since

$$\tilde{B}_h(i) = \prod_{j=0}^{i-1} \frac{1}{1+\tilde{R}_h^F(j)} = \exp\left(-\int_0^i \tilde{r}_h^F(t)dt\right),$$

$\tilde{R}_h^F(i)$ can be interpreted as the *forward interest* for time i implied by the initial term structure. $\tilde{r}_h^F(i)$ is the implied *forward interest rate* for the time interval $[i, i+1)$. Similarly define $R_{\pi_h}^F$ by $\frac{1}{1+R_{\pi_h}^F(i)} := \frac{B_h^{\pi_h}(i+1)}{B_h^{\pi_h}(i)}$ and the corresponding rate $r_{\pi_h}^F(t) := \frac{\ln(1+R_{\pi_h}^F(i))}{h}$ for $t \in [i, i+1)$. In case that there does not exist a measure π_h equivalent to $\hat{\pi}_h$ such that $\tilde{r}_h^F = r_{\pi_h}^F$ we can try to find a measure π_h such that $\lim_{i\to\infty} \left|\tilde{r}_h^F(i) - r_{\pi_h}^F(i)\right|$ exists and is minimal, preferably zero.

Since $V_0\left(B_x^{Nh}\right) = \exp\left(-\sum_{j=0}^{N-1} hr_{\pi_h}^F(j)\right)$, it is always possible to fit the model to an initial term structure by changing the time independent function r_h to the time dependent function $\tilde{r}_h(i,\cdot) := r_h + \tilde{r}_h^F(i) - r_{\pi_h}^F(i)$. For such a model it is desirable to have $\lim_{i\to\infty} \tilde{r}_h^F(i) - r_{\pi_h}^F(i) = 0$, since in that case the model is at least asymptotically time homogeneous. Using this condition we can try to estimate π_h from historical data.

An alternative approach which leads to term structure models are forward rate models, see Ho-Lee [14], Heath, Jarrow, Morton [13] and [3]. These models take the whole term structure curve r_h^F as the state of the market and model the stochastic dynamics of this curve in such a way that the resulting market is arbitrage-free. Naturally any initial term structure can just be taken as the initial state of an infinite dimensional SDE, see [2]. This high flexibility makes it easy to fit the model to an observed initial term structure, but very difficult to find the equivalent martingale measure. It is easy to translate our model into the Ho-Lee model in the discrete case and into the Heath, Jarrow, Morton model in the continuous time case. But it then becomes difficult to identify initial term structures which lead to time homogeneous models and it is not clear how to find the market price for risk.

Concluding remarks

We have approximated a continuous time market model by a sequence of discrete models. In a real market the situation is rather the other way round. Such a market is discrete if trading and updating of information about the market state takes place in discrete time steps, but if the frequency of transactions is high, then the continuous time model, where prices can be calculated by solving partial differential equations with boundary conditions, can serve as an approximation to the discrete time market model. We have seen that the limit of the equivalent martingale measures is in general not equivalent to the limit of the objective probability measures (Remark 3.15).

Weak limit theorems using the existence of a unique solution to a martingale problem hold in much greater generality for semimartingales. In the case that the continuous time market and its discrete time approximations are described by a converging sequence of semimartingales (with respect to the objective probabilities) it becomes difficult to identify reasonable equivalent martingale measures for the approximating sequence of discrete time markets, the set of all possible equivalent martingale measures is just too big. In this paper we have therefore confined ourselves to a very special type of approximating semimartingales. Working on the Skorokhod space instead of $(\Omega, \mathcal{M})$ our results can be generalized to a market described by a jump-diffusion, see [16], Theorem 4.8, p. 515 and Theorem 2.32, p. 145. Now the set of possible equivalent martingale measures is quite big since we can choose in addition the intensities of the jumps with respect to the equivalent martingale measures quite arbitrarily. A reasonable choice would be to multiply all intensities by a constant factor.

Given the objective probabilities for the market, prices of contingent claims are determined by the equivalent martingale measure. This measure allows to derive simple pricing formulas for contingent claims (Proposition 2.1). However, the complexity of the market, investors preferences, risk aversion and other constraints that led to an equilibrium described by the model are hidden behind the equivalent martingale measure. The martingale transition probabilities can be calculated from the state prices which are closely related to the market price of risk (Remark 3.18). We have argued why a time-homogeneous price of risk is reasonable. This allows to describe the equivalent martingale measures by generating kernels π_h. Assuming π_h to be (λ, Λ)-converging, in the limit, the appropriate measure for pricing contingent claims can be specified by calculating the limit (λ, Λ) of the first and second order data $\left(\frac{\lambda_h^{\pi_h}}{\sqrt{h}}, \Lambda_h^{\pi_h}\right)$ of the kernels π_h. Fixing the limit $\lim_{h \searrow 0} \Lambda_h = \Lambda$ leads to an asymptotically complete market, explaining the completeness of continuous time markets (Remark 3.21). Now the complexity of the market is just hidden behind the two functions (λ, Λ) which we can try to estimate from historical zero bond prices (Section 4).

If for example there is a strong interest to minimize a weighted average Z of zero bond or option prices and their volatilities and if investors today expect investors in the future to have the same preferences and risk aversions like today, then this can be achieved by choosing the probability measure π_h such that $\Pi_h^{\mu_h,\sigma_h,\pi_h}$ minimizes Z. This is a kind of self-fulfilling expectation or prophecy, consistent with arbitrage-free pricing, which leads to a complete market if the minimizing π_h is unique. In some sense we can not expect more from a theory depending on future, non-rational and psychological factors like investors risk aversion, preferences and expectations.

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