BSDES WITH STOCHASTIC LIPSCHITZ CONDITION

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Abstract. We prove an existence and uniqueness theorem for backward stochastic differential equations driven by a Brownian motion, where the uniform Lipschitz continuity is replaced by a stochastic one.

1. Introduction

In this paper we study Backward Stochastic Differential Equations (BSDEs for short) of the form

$$\begin{aligned} -dY(t) &= f(t, Y(t), Z(t))dt - Z(t)dW(t) \\ Y(\tau) &= \xi \end{aligned}$$

A wellposedness result was obtained by Pardoux and Peng [P-P] in the case that the stopping time τ is deterministic and bounded and the driver f is uniformly Lipschitz continuous. This last condition is very restrictive and cannot be assumed in many interesting applications. Let us have a look at the pricing problem of a European claim for example. This problem is equivalent to solving the linear BSDE

$$\begin{aligned} -dY(t) &= -[r(t)Y(t) + \theta(t)Z(t)]dt - Z(t)dW(t) \\ Y(T) &= \xi \end{aligned}$$

Here $r(t)$ is the interest rate and $\theta(t)$ is the risk premium vector. Both will not be bounded in general. So Pardoux-Peng's theorem cannot be applied.

Consequently, one is interested in relaxing the Lipschitz condition. But some examples show (see e.g. [ElK]) that this condition is necessary in the standard setting. Hence, one must strengthen other conditions while relaxing the Lipschitz continuity. Indeed we will have stronger integrability conditions on the driver as well as on the solutions. These integrability conditions make it possible to replace the uniform Lipschitz condition by a stochastic one, which was introduced

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in [E-H], and allow an infinite and random time horizon, too. Due to the fact that the BSDE is driven by a Brownian motion the third part of the solution which was necessary in [E-H] can be proved to vanish. In this way related results in [E-H], [M-Y], and [Y-Z] are generalized.

The paper is organized as follows. In chapter 2 we introduce some notation including some spaces, which are different from the standard spaces used in BSDE-theory. In chapter 3 we set the problem and state the main results on existence, uniqueness and continuous dependence. A priori estimates are given in chapter 4. An existence and uniqueness theorem for a class of "easy" BSDEs is obtained in chapter 5, while chapter 6 contains the proofs of the main results.

2. Some Notations

Let $(\Omega, \mathbb{F}, \mathbb{F}(t), P)$ be a filtered probability space. Let $W(t)$ be a n-dimensional Brownian motion. We assume, that $\mathbb{F}(t)$ is the standard filtration generated by the Brownian motion and augmented by all P-null-sets. It follows, that $\mathbb{F}(t)$ is complete and continuous (see [K-S] for a proof).

The standard inner product of the $\mathbb{R}^d$ is denoted by $\langle ., . \rangle$, the Euklidean norm by $|.|$. A norm on $\mathbb{R}^{d\times n}$ is defined by $tr(ZZ^*)$. We will denote this norm by $|.|$, too.

For any nonnegative $\mathbb{F}(t)$-adapted process a we define the increasing continuous process

$A(t) = \int_0^t a^2(s)ds$

We can introduce the appropriate spaces now:

Let $\beta \geq 0$ and a be a nonnegative $\mathbb{F}(t)$-adapted process. We set

$$\begin{aligned}
L^2(\beta, a, \tau, \mathbb{R}^d) &= \{\xi; \mathbb{R}^d\text{-valued and } \mathbb{F}(\tau)\text{-measurable such that} \\
&\quad ||\xi||_\beta^2 = E[e^{\beta A(\tau)}|\xi|^2] < \infty\} \\
L^2(\beta, a, [0,\tau], \mathbb{R}^d) &= \{Y; \mathbb{R}^d\text{-valued and } \mathbb{F}(t) - \text{adapted such that} \\
&\quad ||Y||_\beta^2 = E\int_0^\tau e^{\beta A(s)}|Y(s)|^2 ds < \infty\} \\
L^{2,a}(\beta, a, [0,\tau], \mathbb{R}^d) &= \{Y; \mathbb{R}^d\text{-valued and } \mathbb{F}(t) - \text{adapted} \\
&\quad \text{such that } ||aY||_\beta^2 < \infty\} \\
L^{2,c}(\beta, a, [0,\tau], \mathbb{R}^d) &= \{Y; \mathbb{R}^d\text{-valued, } \mathbb{F}(t)\text{-adapted and continuous} \\
&\quad \text{such that } ||Y||_{\beta,c}^2 = E\sup_{0\leq t\leq\tau} e^{\beta A(t)}|Y(t)|^2 < \infty\}
\end{aligned}$$

We notice that $L^2(\beta, a, [0,\tau], \mathbb{R}^d)$ is a Banach space with the norm $||Y||_\beta$.

Consequently,

$$M(\beta, a, \tau) = L^{2,a}(\beta, a, [0,\tau], \mathbb{R}^d) \times L^2(\beta, a, [0,\tau], \mathbb{R}^{d\times n})$$

is a Banach space with the norm

$$||(Y,Z)||_\beta^2 = ||aY||_\beta^2 + ||Z||_\beta^2$$

Our main interest is in a subspace of $M(\beta, a, \tau)$, namely

$$\begin{aligned} & M^c(\beta,a,\tau) \\ = \quad & (L^{2,a}(\beta,a,[0,\tau],\mathbb{R}^d) \cap L^{2,c}(\beta,a,[0,\tau],\mathbb{R}^d)) \times L^2(\beta,a,[0,\tau],\mathbb{R}^{d\times n}) \end{aligned}$$

We define a norm on $M^c(\beta, a, \tau)$ by

$$||(Y,Z)||^2_{\beta,c} = ||Y||^2_{\beta,c} + ||aY||^2_\beta + ||Z||^2_\beta$$

3. Statement of the main results

We assume $(\Omega, \mathbb{F}, \mathbb{F}(t), P)$ to be a filtered probability space as in the previous chapter.

Let $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^{n\times d} \to \mathbb{R}^d$ such that for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{n\times d}$ $f(.,.,y,z)$ is $\mathbb{F}(t)$-adapted. Let further τ be a stopping time which may take values in $[0,\infty]$ and let ξ be a $\mathbb{F}(\tau)$-measurable random variable.

We consider the following BSDE (suppressing ω):

$$\begin{aligned} -dY(t) &= f(t,Y(t),Z(t))dt - Z(t)dW(t) \\ Y(\tau) &= \xi \end{aligned} \tag{3.1}$$

Definition 1. Let $\beta > 0$ and a a nonnegative $\mathbb{F}(t)$-adapted process. A pair $(Y,Z) \in M^c(\beta,a,\tau)$ is called a *solution* of BSDE (3.1), if

$$Y(t\wedge\tau) = \xi + \int_{t\wedge\tau}^{\tau} f(s,Y(s),Z(s))ds - \int_{t\wedge\tau}^{\tau} Z(s)dW(s) \tag{3.2}$$

Equation (3.1) is said to be *uniquely solvable*, if for any two solutions (Y,Z) and (Y',Z') the following holds

(3.3)

$$Y(t) = Y'(t)\ \ P-a.s.\ \forall t\in[0,\tau];\ \ Z(t) = Z'(t)\ \ P-a.s.\ \ a.e.\ t\in[0,\tau]$$

Definition 2. Let $\beta > 0$. We call a triple (τ,ξ,f) *standard data*, if the following holds:

(H1) τ is a stopping time of the filtration $\mathbb{F}(t)$.

(H2) There are two nonnegative $\mathbb{F}(t) -$ *adapted* processes $r(t)$and $u(t)$ such that $\forall (y,z,y',z') \in \mathbb{R}^d \times \mathbb{R}^{d\times n} \times \mathbb{R}^d \times \mathbb{R}^{d\times n}$

$$|f(t,y,z) - f(t,y',z')| \le r(t)|y-y'| + u(t)|z-z'| \tag{3.4}$$

We refer to (H2) as the stochastic Lipschitz condition.

(H3) $\exists \varepsilon > 0 \quad a^2(t) := r(t) + u^2(t) \ge \varepsilon$

(H4) $\xi \in L^2(\beta,a,\tau,\mathbb{R}^d)$

(H5) $\frac{f(.,.,0,0)}{a} \in L^2(\beta,a,[0,\tau],\mathbb{R}^d)$

We can now state the main results

Theorem 3. *Let (τ, ξ, f) be standard data for a sufficiently large β. Then the BSDE (3.1) has a unique solution.*

We will further obtain the following result concerning the continuous dependence:

Theorem 4. *Assume (τ, ξ, f) and (τ, ξ', f') to be standard data with associated solutions (Y, Z) and (Y', Z'). For β large enough there is a constant $K > 0$ independent of τ such that*

$$\begin{aligned} & ||(Y,Z)-(Y',Z')||_{\beta,c}^2 \\ \leq \;& K||\xi-\xi'||_\beta^2 + K||\frac{f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))}{a}||_\beta^2 \end{aligned} \tag{3.5}$$

We split the proofs in several steps. First we give some a-priori-estimates. Then we show a wellposedness result for some "easy" BSDEs. Finally, we use the contraction mapping theorem to prove theorem 3. Theorem 4 turns out to be a corollary of the a priori estimates.

4. A Priori Estimates

Lemma 5. *(A-priori-estimates)*
Let (τ, ξ, f) be a triple of data satisfying (H1) and (H4) for some $\beta > 0$ and an $\mathbb{F}(t)$-adapted process $a \geq \varepsilon > 0$. Let (Y, Z) be a solution of the BSDE associated with the data and assume

$$\frac{f(t,Y(t),Z(t))}{a} \in L^2(\beta, a, [0,\tau], \mathbb{R}^d)$$

Then

$$||aY||_\beta^2 \leq \frac{2}{\beta}||\xi||_\beta^2 + \frac{4}{\beta^2}||\frac{f(t,Y(t),Z(t))}{a}||_\beta^2 \tag{4.1}$$

$$||Y||_{\beta,c}^2 \leq C||\xi||_\beta^2 + \frac{C'}{\beta}||\frac{f(t,Y(t),Z(t))}{a}||_\beta^2 \tag{4.2}$$

$$||Z||_\beta^2 \leq ||\xi||_\beta^2 + \frac{2}{\beta}||\frac{f(t,Y(t),Z(t))}{a}||_\beta^2 \tag{4.3}$$

for positive constants C and C' independent of β and τ.

Proof. Since (Y, Z) is a solution associated with the given data, we obtain from (3.2) for $T \geq t \geq 0$

$$Y(t\wedge\tau) = Y(T\wedge\tau) + \int_{t\wedge\tau}^{T\wedge\tau} f(s,Y(s),Z(s))ds - \int_{t\wedge\tau}^{T\wedge\tau} Z(s)dW(s)$$

Let us recall that

$$A(t) = \int_0^t a^2(s)ds.$$

Applying Itô's formula to

$$|Y(t\wedge\tau)|^2 e^{\beta A(t\wedge\tau)}$$

yields

$$\begin{aligned}
&|Y(t\wedge\tau)|^2 e^{\beta A(t\wedge\tau)} + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds \\
= \;& |Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} - \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}[\beta a^2(s)|Y(s)|^2 \\
& +2\langle f(s,Y(s),Z(s)),Y(s)\rangle]ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle Y(s),Z(s)dW(s)\rangle \\
\leq \;& |Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}[-\beta a^2(s)|Y(s)|^2 \\
& +2|f(s,Y(s),Z(s))||Y(s)|]ds + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle Y(s),Z(s)dW(s)\rangle \\
& \leq |Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}[-\frac{\beta a^2(s)}{2}|Y(s)|^2 \\
& +\frac{2}{\beta a^2(s)}|f(s,Y(s),Z(s))|^2]ds \\
& + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle Y(s),Z(s)dW(s)\rangle
\end{aligned} \tag{4.4}$$

Here the last estimate is due to Young's inequality.

Noting that $\int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}\langle Y(s),Z(s)dW(s)\rangle$ is a martingale and taking expectation we have

$$\begin{aligned}
&2E\int_0^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds + E\int_0^{T\wedge\tau} e^{\beta A(s)}\beta a^2(s)|Y(s)|^2 ds \\
\leq \;& 2E|Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} + E\int_0^{T\wedge\tau} e^{\beta A(s)}\frac{4}{\beta a^2(s)}|f(s,Y(s),Z(s))|^2 ds
\end{aligned}$$

Now $E\sup_{0\leq t\leq\tau} e^{\beta A(t)}|Y(t)|^2 < \infty$ because $(Y,Z)\in M^c(\beta,a,\tau)$. Letting $T\to\infty$ we obtain by the dominated convergence theorem

$$\begin{aligned}
&2E\int_0^{\tau} e^{\beta A(s)}|Z(s)|^2 ds + \beta E\int_0^{\tau} e^{\beta A(s)} a^2(s)|Y(s)|^2 ds \\
\leq \;& 2E|\xi|^2 e^{\beta A(\tau)} + \frac{4}{\beta}E\int_0^{\tau} e^{\beta A(s)}\frac{1}{a^2(s)}|f(s,Y(s),Z(s))|^2 ds
\end{aligned}$$

(4.1) and (4.3) follow easily.

By the Burkholder-Davis-Gundy's inequalities

$$\begin{aligned}
& E \sup_{0\le t\le T\wedge\tau} \left| \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)} \langle Y(s), Z(s)dW(s)\rangle \right| \\
\le\ & E\left| \int_0^{T\wedge\tau} e^{\beta A(s)} \langle Y(s), Z(s)dW(s)\rangle \right| \\
& + E \sup_{0\le t\le T\wedge\tau} \left| \int_0^{t\wedge\tau} e^{\beta A(s)} \langle Y(s), Z(s)dW(s)\rangle \right| \\
\le\ & 2KE[\int_0^{T\wedge\tau} e^{2\beta A(s)}|Y(s)|^2|Z(s)|^2 ds]^{\frac{1}{2}} \\
\le\ & 2KE[(\sup_{0\le t\le \tau\wedge T} |Y(t)|^2 e^{\beta A(t)})^{\frac{1}{2}} (\int_0^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds)^{\frac{1}{2}}] \\
\le\ & \frac{1}{2} E \sup_{0\le t\le \tau\wedge T} |Y(t)|^2 e^{\beta A(t)} + 2K^2 E\int_0^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds \qquad (4.5)
\end{aligned}$$

Combining this with (4.4) we have

$$\begin{aligned}
& E \sup_{0\le t\le T\wedge\tau} |Y(t)|^2 e^{\beta A(t)} \\
\le\ & E|Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} + E\int_0^{T\wedge\tau} e^{\beta A(s)} \frac{2}{\beta a^2(s)} |f(s,Y(s),Z(s))|^2 ds \\
& + E \sup_{0\le t\le T\wedge\tau} \left| \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)} \langle Y(s), Z(s)dW(s)\rangle \right| \\
\le\ & E|Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} + E\int_0^{T\wedge\tau} e^{\beta A(s)} \frac{2}{\beta a^2(s)} |f(s,Y(s),Z(s))|^2 ds \\
& + \frac{1}{2} E \sup_{0\le t\le \tau\wedge T} |Y(t)|^2 e^{\beta A(t)} + 2K^2 E\int_0^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds
\end{aligned}$$

Thus,

$$\begin{aligned}
& E \sup_{0\le t\le T\wedge\tau} |Y(t)|^2 e^{\beta A(t\wedge\tau)} \\
\le\ & 2E|Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} + 4E\int_0^{T\wedge\tau} e^{\beta A(s)} \frac{1}{\beta a^2(s)} |f(s,Y(s),Z(s))|^2 ds \\
& + 4K^2 E\int_0^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds
\end{aligned}$$

Letting $T \to \infty$ and using Fatou's Lemma, dominated convergence theorem and (4.3)

$$\begin{aligned} & E \sup_{0\le t\le\tau} |Y(t)|^2 e^{\beta A(t\wedge\tau)} \\ \le & \lim_{T\to\infty} E \sup_{0\le t\le T\wedge\tau} |Y(t\wedge\tau)|^2 e^{\beta A(t\wedge\tau)} \\ \le & (2+4K^2)E|\xi|^2 e^{\beta A(\tau)} + \frac{4+8K^2}{\beta} E\int_0^\tau e^{\beta A(s)} \frac{1}{a^2(s)} |f(s,Y(s),Z(s))|^2 ds \end{aligned}$$

So we obtain (4.2) and finish the proof.

Lemma 6. *Assume (τ,ξ,f) and (τ,ξ',f') to be standard data with associated solutions (Y,Z) and (Y',Z') respectively. Then*

$$\frac{f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))}{a} \in L^2(\beta,a,[0,\tau],\mathbb{R}^d) \tag{4.6}$$

$$\frac{f(t,Y(t),Z(t)) - f'(t,Y'(t),Z'(t))}{a} \in L^2(\beta,a,[0,\tau],\mathbb{R}^d) \tag{4.7}$$

Moreover,

$$\begin{aligned} & ||\frac{f(t,Y(t),Z(t)) - f'(t,Y'(t),Z'(t))}{a}||_\beta^2 \\ \le & 3(||a(Y(t)-Y'(t))||_\beta^2 + ||(Z(t)-Z'(t))||_\beta^2 \\ & +||\frac{f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))}{a}||_\beta^2) \end{aligned} \tag{4.8}$$

Proof. By the stochastic Lipschitz condition we have

$$\begin{aligned} & |f(t,Y(t),Z(t)) - f'(t,Y'(t),Z'(t))| \\ \le & |f(t,Y(t),Z(t)) - f(t,Y'(t),Z'(t))| \\ & +|f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))| \\ \le & r(t)|Y(t)-Y'(t)| + u(t)|Z(t)-Z'(t)| \\ & +|f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))| \end{aligned}$$

Now by Young's inequality and the definition of a^2

$$\begin{aligned} & |f(t,Y(t),Z(t)) - f'(t,Y'(t),Z'(t))|^2 \\ \le & 3(r^2(t)|Y(t)-Y'(t)|^2 + u^2(t)|Z(t)-Z'(t)|^2 \\ & +|f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))|^2) \\ \le & 3a^2(t)[a^2(t)|Y(t)-Y'(t)|^2 + |Z(t)-Z'(t)|^2 \\ & +\frac{|f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))|^2}{a^2(t)}] \end{aligned}$$

Consequently,

$$\begin{aligned} & \frac{|f(t,Y(t),Z(t)) - f'(t,Y'(t),Z'(t))|^2}{a^2(t)} e^{\beta A(t)} \\ \leq \; & 3e^{\beta A(t)}[a^2(t)|Y(t)-Y'(t)|^2 + |Z(t)-Z'(t)|^2 \\ & + \frac{|f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))|^2}{a^2(t)}] \end{aligned}$$

Integrating from 0 to τ and taking expectation yield (4.8).

Furthermore we have

$$\begin{aligned} & ||\frac{f(t,Y'(t),Z'(t)) - f'(t,Y'(t),Z'(t))}{a}||_\beta^2 \\ \leq \; & ||\frac{f(t,Y'(t),Z'(t))}{a}||_\beta^2 + ||\frac{f'(t,Y'(t),Z'(t))}{a}||_\beta^2 \end{aligned}$$

By (H2)

$$|f(t,Y'(t),Z'(t))| \leq r(t)|Y'(t)| + u(t)|Z'(t)| + |f(t,0,0)|$$

By Young's inequality and the definition of a^2 we obtain

$$\frac{|f(t,Y'(t),Z'(t))|^2}{a^2(t)} \leq 3(a^2(t)|Y'(t)|^2 + |Z'(t)|^2 + \frac{|f(t,0,0)|^2}{a^2(t)})$$

Using (H5) one can easily check that $||\frac{f(t,Y'(t),Z'(t))}{a}||_\beta^2 < \infty$. One obtains $||\frac{f'(t,Y'(t),Z'(t))}{a}||_\beta^2 < \infty$ in the same way. Hence, (4.6) holds and (4.7) follows directly from (4.8) and (4.6).

5. Wellposedness for "easy" BSDEs

In this chapter we consider a class of "easy" BSDEs. The meaning of "easy" is, that the driver f of the BSDE is independent of Y and Z. We are going to obtain the following result.

Proposition 7. *Let $\beta > 0$ and let a be a nonnegative $\mathbb{F}(t)$-adapted process bounded away from 0 by an $\varepsilon > 0$. Assume*

$$\tfrac{f}{a} \in L^2(\beta, a, [0,\tau], \mathbb{R}^d)$$

and $\xi \in L^2(\beta, a, \tau, \mathbb{R}^d)$. Then the BSDE

$$\begin{aligned} -dY(t) &= f(t)dt - Z(t)dW(t) \\ Y(\tau) &= \xi \end{aligned} \tag{5.1}$$

has a unique solution.

The idea of the proof is the same as in [P-P], but the proof becomes by far more technical because of the more general situation. Let us give a heuristic argument first.

Let (Y, Z) be a solution of the BSDE. We have

$$Y(t \wedge \tau) = \xi + \int_{t\wedge\tau}^{\tau} f(s)ds - \int_{t\wedge\tau}^{\tau} Z(s)dW(s)$$

Taking conditional expectation yields

$$\begin{aligned} Y(t \wedge \tau) &= E[Y(t \wedge \tau)|\mathbb{F}(t \wedge \tau)] \\ &= E[\xi + \int_0^{\tau} f(s)ds|\mathbb{F}(t \wedge \tau)] - \int_0^{t\wedge\tau} f(s)ds \end{aligned}$$

So we have a candidate for one part of the solution. Define

$$\begin{aligned} M(t \wedge \tau) &= E[\xi + \int_0^{\tau} f(s)ds|\mathbb{F}(t \wedge \tau)] \\ Y(t \wedge \tau) &= M(t \wedge \tau) - \int_0^{t\wedge\tau} f(s)ds \end{aligned}$$

We obtain the following estimates for M and Y.

Lemma 8.

$$E[|M(t \wedge \tau)|^2] \leq 2E|\xi|^2 + \frac{2}{\beta} E \int_0^{\tau} \frac{|f(s)|^2}{a^2(s)} ds \tag{5.2}$$

$$E \sup_{0\leq t\leq\tau} [e^{\beta A(t)}|Y(t)|^2] \leq 8E|\xi|^2 e^{\beta A(\tau)} + \frac{8}{\beta} E \int_0^{\tau} \frac{|f(s)|^2}{a^2(s)} e^{\beta A(s)} ds \tag{5.3}$$

Consequently, $M(t)$ $(t \in [0, \tau))$ is a square integrable martingale.

Proof. By the definition of Y we obtain using Jensen's, Young's and Hölder's inequality respectively

$$
\begin{aligned}
e^{\beta/2A(t\wedge\tau)}|Y(t\wedge\tau)| &= |E[\xi+\int_{t\wedge\tau}^{\tau} f(s)ds|\mathbb{F}(t\wedge\tau)]|e^{\beta/2A(t\wedge\tau)} \\
&\leq E[\sqrt{|\xi+\int_{t\wedge\tau}^{\tau} f(s)ds|^2 e^{\beta A(t\wedge\tau)}}|\mathbb{F}(t\wedge\tau)] \\
&\leq \sqrt{2}E[\sqrt{|\xi|^2 e^{\beta A(\tau)}+|\int_{t\wedge\tau}^{\tau} f(s)ds|^2 e^{\beta A(t\wedge\tau)}}|\mathbb{F}(t\wedge\tau)] \\
&\leq \sqrt{2}E[\{|\xi|^2 e^{\beta A(\tau)}+ \\
&\quad (\int_{t\wedge\tau}^{\tau} a^2(s)e^{-\beta A(s)}ds) \\
&\quad (\int_{t\wedge\tau}^{\tau} \frac{|f(s)|^2}{a^2(s)}e^{\beta A(s)}ds)e^{\beta A(t\wedge\tau)}\}^{\frac{1}{2}}|\mathbb{F}(t\wedge\tau)] \\
&\leq \sqrt{2}E[\sqrt{|\xi|^2 e^{\beta A(\tau)}+\frac{1}{\beta}\int_0^{\tau}\frac{|f(s)|^2}{a^2(s)}e^{\beta A(s)}ds}|\mathbb{F}(t\wedge\tau)]
\end{aligned}
$$

Thus, $e^{\beta/2A(t\wedge\tau)}|Y(t\wedge\tau)|$ is dominated by a martingale. By Doob's inequality and Jensen's inequality one has the estimate for Y. The estimate for M follows in a similiar way.

Let us now construct the second part of the solution. Because M is a square integrable martingale we can make use of the martingale representation theorem (see e.g. [I-W]). We have:

There is an $\mathbb{F}(t)$-adapted process $Z:\Omega\times\mathbb{R}_+\to\mathbb{R}^{d\times n}$ such that

$$P(\int_0^{\infty}|Z(s)|ds<\infty)=1$$

and

$$M(t\wedge\tau)=M(0)+\int_0^{t\wedge\tau} Z(s)dW(s)$$

One can easily show (using the definitions above) that for any fixed $T<\infty$ and $t\leq T$

$$Y(t\wedge\tau)=Y(T\wedge\tau)+\int_{t\wedge\tau}^{T\wedge\tau} f(s)ds-\int_{t\wedge\tau}^{T\wedge\tau} Z(s)dW(s) \tag{5.4}$$

Letting $T\to\infty$ we see that Z is the natural candidate for the second part of the solution. For this purpose let us prove that $Y(T\wedge\tau)\to\xi$ $(P-a.s.)$. We briefly discuss two cases.

(i) Let $\{\tau \le T\}$for a $T < \infty$. By Problem 1.2.17 in [K-S] we have

$$E[\xi + \int_0^\tau f(s)ds|\mathbb{F}(T \wedge \tau)] = E[\xi + \int_0^\tau f(s)ds|\mathbb{F}(\tau)] = \xi + \int_0^\tau f(s)ds$$

Hence, $Y(T \wedge \tau) \to \xi$ on $\{\tau < \infty\}$.

(ii) $\{\tau = \infty\}$. Since $\frac{f}{a} \in L^2(\beta, a, [0,\tau], \mathbb{R}^d)$ and $\xi \in L^2(\beta, a, \tau, \mathbb{R}^d)$, we have $f = 0$ and $\xi = 0$ on $\{\tau = \infty\}$. Thus $Y(T \wedge \tau) \to \xi$ holds trivially. Notice that we need a bounded away from 0 to conclude that f and ξ equal 0.

Consequently,

$$Y(t \wedge \tau) = \xi + \int_{t\wedge\tau}^\tau f(s)ds - \int_{t\wedge\tau}^\tau Z(s)dW(s)$$

We have to prove that $(Y, Z) \in M^c(\beta, a, \tau)$.

Lemma 9. $(Y, Z) \in M^c(\beta, a, \tau)$

Proof. From the previous lemma we know that

$$Y \in L^{2,c}(\beta, a, [0,\tau], \mathbb{R}^d).$$

Applying Itô's formula to $e^{\beta A(t\wedge\tau)}|Y(t\wedge\tau)|^2$ noting (5.4) yields

$$\begin{aligned} & |Y(t\wedge\tau)|^2 e^{\beta A(t\wedge\tau)} + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds \\ = \; & |Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} - \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)}[\beta a^2(s)|Y(s)|^2 + 2\langle f(s), Y(s)\rangle]ds \\ & + \int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)} \langle Y(s), Z(s)dW(s)\rangle \end{aligned}$$

We have to pay attention to the fact that

$$\textstyle\int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)} \langle Y(s), Z(s)dW(s)\rangle$$

could be a real local martingale. Hence, we cannot assume that

$$E \textstyle\int_{t\wedge\tau}^{T\wedge\tau} e^{\beta A(s)} \langle Y(s), Z(s)dW(s)\rangle = 0.$$

But similiar to the a priori estimates in lemma 5 one has (using the Burkholder-Davis-Gundy's inequalities)

$$\begin{aligned} & E\int_0^{T\wedge\tau} e^{\beta A(s)}|Z(s)|^2 ds + E\int_0^{T\wedge\tau} e^{\beta A(s)}|a(s)Y(s)|^2 ds \\ \le \; & C[E|Y(T\wedge\tau)|^2 e^{\beta A(T\wedge\tau)} + E\int_0^\tau |f(s)|^2 e^{\beta A(s)} ds + E\sup_{0\le t\le\tau}(e^{\beta A(t)}|Y(t)|^2)] \end{aligned}$$

with a constant C only dependent of β but not of T. The details are left to the reader. Then we can make use of (5.3) and pass to the limit to obtain the desired result.

It remains to prove the uniqueness:
Let $\xi = 0$ and $f = 0$. Applying Itô's formula to $|Y(t \wedge \tau)|^2$ we obtain

$$E|Y(t \wedge \tau)|^2 = -E \int_{t \wedge \tau}^{\tau} |Z(s)|^2 ds$$

Hence,

$$\begin{aligned} Y(t \wedge \tau) &= 0 \quad P-a.s. \\ Z(t) &= 0 \quad P-a.s. \quad a.e.\ t \in [0, \tau] \end{aligned}$$

Uniqueness follows from the linearity of the equation.
The proof of proposition 7 is complete now.

6. Proofs of the main results

After these preparations we are able to prove the main results. We make use of the contraction mapping theorem. So recall that

$$(M(\beta, a, \tau), ||(.,.)||_\beta)$$

is a Banach space.

Proof of Theorem 3.
For fixed $(y, z) \in M(\beta, a, \tau)$ consider the BSDE

$$\begin{aligned} -dY(t) &= f(t, y(t), z(t))dt - Z(t)dW(t) \\ Y(\tau) &= \xi \end{aligned} \tag{6.1}$$

By (H2)

$$|f(t, y(t), z(t))| \leq r(t)|y(t)| + u(t)|z(t)| + |f(t, 0, 0)|$$

By Young's inequality and the definition of a^2

$$\begin{aligned} \frac{|f(t, y(t), z(t))|^2}{a^2(t)} \leq\ & 3(a^2(t)|y(t)|^2 + |z(t)|^2 \\ & + \frac{|f(t, 0, 0)|^2}{a^2(t)}) \end{aligned}$$

Thus, using (H5) we have

$$\frac{f(t, y(t), z(t))}{a(t)} \in L^2(\beta, a, [0, \tau], \mathbb{R}^d).$$

Hence the BSDE (6.1) has a unique solution by proposition 7.

So we can define the operator

$$\Pi : M(\beta, a, \tau) \to M^c(\beta, a, \tau) \subset M(\beta, a, \tau)$$

such that $\Pi(y, z)$ is the solution of the correspondig BSDE (6.1).

We will show, that for a β large enough Π is a contraction mapping. Using the contraction mapping theorem we find a unique fixed point, which is the unique solution of BSDE (3.1).

Assume (y, z), $(y', z') \in M(\beta, a, \tau)$. By lemma 6

$$||\frac{f(t, y(t), z(t)) - f(t, y'(t), z'(t))}{a}||_\beta^2 \leq 3(||a(y(t) - y'(t))||_\beta^2 + ||z(t) - z'(t))||_\beta^2)$$

Combining this with the results of lemma 5 - using the obvious fact that $\Pi(y, z) - \Pi(y', z')$ is the solution of the BSDE given by the data $(\tau, 0, f(t, y(t), z(t)) - f(t, y'(t), z'(t)))$ - one has

$$\begin{aligned} ||\Pi(y, z) - \Pi(y', z')||_\beta^2 &\leq (\frac{12}{\beta^2} + \frac{6}{\beta})(||a(y(t) - y'(t))||_\beta^2 + ||z(t) - z'(t))||_\beta^2) \\ &\leq (\frac{12}{\beta^2} + \frac{6}{\beta})||(y, z) - (y', z')||_\beta^2 \end{aligned}$$

Hence, for β large enough Π is a contraction mapping. Thus the BSDE (3.1) has a unique solution. The proof is complete now.

It remains to prove the continuous dependence property

Proof of Theorem 4.

Let

$$g(t, y, z) = f(t, y + Y'(t), z + Z'(t)) - f'(t, -y - Y(t), -z - Z(t))$$

Then $(Y - Y', Z - Z')$ is a solution of the BSDE induced by $(\tau, \xi - \xi', g)$. By lemma 6

$$\frac{g(t, Y(t) - Y'(t), Z(t) - Z'(t))}{a} \in L^2(\beta, a, [0, \tau], \mathbb{R}^d)$$

Hence, lemma 5 can be applied and we obtain

$$\begin{aligned}
& ||(Y-Y', Z-Z')||^2_{\beta,c} \\
\leq\ & (C+1+\frac{2}{\beta})||\xi-\xi'||^2_\beta \\
& +(\frac{C'+2}{\beta}+\frac{4}{\beta^2})||\frac{f(t,Y(t),Z(t))-f'(t,Y'(t),Z'(t))}{a}||^2_\beta \\
\leq\ & (C+1+\frac{2}{\beta})||\xi-\xi'||^2_\beta \\
& +3(\frac{C'+2}{\beta}+\frac{4}{\beta^2})\{||a(Y(t)-Y'(t))||^2_\beta+||(Z(t)-Z'(t))||^2_\beta \\
& +||\frac{f(t,Y'(t),Z'(t))-f'(t,Y'(t),Z'(t))}{a}||^2_\beta\}
\end{aligned}$$

The second inequality follows from lemma 6. Choosing β large enough the proof is finished.

Finally, we compare the results with Pardoux-Peng's theorem in the case of the standard setting. So let the stopping time τ be bounded by some $T_0 < \infty$ and let a uniform Lipschitz condition hold, i.e.

$$|f(t,y,z)-f(t,y',z')| \leq K(|y-y'|+|z-z'|)$$

Obviously, under these assumptions (H4) and (H5) are equivalent to
(H4') $\xi \in L^2(0,0,\tau,\mathbb{R}^d)$
(H5') $f(.,.,0,0) \in L^2(0,0,[0,\tau],\mathbb{R}^d)$

Hence, the conditions for the data are perfectly the same as in [P-P].

Consequently, theorem 3 covers Pardoux-Peng's theorem in the case of the standard setting.

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