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BOUNDARY VALUE PROBLEMS FOR ELLIPTIC MIXED ORDER SYSTEMS WITH PARAMETER

R. Denk
Universität Regensburg, Regensburg, Germany

L. R. Volevich[1]
Keldysh Institute of Applied Mathematics, Moscow, Russia

This paper gives a survey on the concept of parameter-ellipticity and parabolicity for mixed order systems depending on a complex parameter. We consider both systems of operators acting on closed manifolds and on manifolds with boundary. The main results include unique solvability of the corresponding equations or boundary value problems and uniform estimates on the solutions in terms of parameter-dependent norms.

Keywords: Mixed order systems, Newton polygon, parameter-dependent Sobolev spaces

1. Elliptic systems of constant and mixed order

Consider a polynomial matrix $A(\xi) = (A_{ij}(\xi))_{i,j=1,\dots,N}$ where $\operatorname{ord} A_{ij} \le r$. Denote by $A^0_{ij}(\xi)$ the homogeneous part of order r (note that this part is identically zero if the order is less than r), and define the principal part $A^0(\xi) := (A^0_{ij}(\xi))_{i,j=1,\dots,N}$. The matrix $A(\xi)$ is called elliptic if

$$\det A^0(\xi) \neq 0, \quad |\xi| \neq 0\,. \tag{1}$$

In the case of matrices elliptic in the sense of Douglis-Nirenberg (mixed order systems) it is supposed that there exist $2N$ integers $s_1,\dots,s_N,\ t_1,\dots,t_N$ such that $\operatorname{ord} A_{ij} \le s_i + t_j$. Denote by $A^0_{ij}(\xi)$ the homogeneous part of order $s_i + t_j$. Then the principal part can be defined in the same way as before, and the ellipticity conditions is again given by (1).

2. Parameter-elliptic constant order systems

Consider a matrix $A(\xi,\lambda)$ depending polynomially on $\xi \in \mathbb{R}^n$ and $\lambda \in \mathcal{L}$, where $\mathcal{L}$ is a ray in the complex plane starting at the origin. We assign the weight p to the variable λ (where p is a natural number) and suppose that – taking into account this weight – the orders of the polynomials $A_{ij}(\xi,\lambda)$ are not greater than r. In this way we define the parameter-dependent principal parts $A^0_{ij}(\xi,\lambda)$ and the principal part $A^0(\xi,\lambda)$. The matrix $A(\xi,\lambda)$ is called parameter-elliptic along the ray $\mathcal{L}$ if

$$\det A^0(\xi,\lambda) \neq 0, \quad |\xi| + |\lambda| > 0, \quad \lambda \in \mathcal{L}\,. \tag{2}$$

This is the well-known Agranovich-Vishik condition which can be found in [4]. Agmon [1] formulated it in the following equivalent way: let the ray $\mathcal{L}$ be of the form $\{\arg\lambda = \theta, |\lambda| \ge 0\}$. Then we replace λ by $e^{i\theta}\mu^p$ and consider the matrix $A_\theta(\xi,\mu) := A(\xi, e^{i\theta}\mu^p)$. Agmon supposed that this matrix considered as a polynomial matrix of the variables ξ and μ is elliptic in the sense above. These equivalent conditions will be called AAV condition.

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3. Parameter-elliptic mixed order systems (preliminary remarks)

The AAV condition can be trivially extended to mixed order systems. Indeed, we assign the (integer) weight p to the parameter λ and suppose that the matrix $A(\xi, e^{i\theta}\mu^p)$, considered as a polynomial matrix of the variables ξ and μ is elliptic in the sense of Douglis–Nirenberg. This definition is due to Solonnikov [18] (cf. also Roitberg [17]).

But there is a principal difference between the definitions of parameter-ellipticity for constant and mixed order systems. In the first case it is possible to formulate the parameter-ellipticity condition for the symbol of the resolvent, i.e. the matrix $A(\xi) - \lambda I$. In this case the AAV condition means that

$$\det(A^0(\xi) - \lambda I) \neq 0, \quad |\xi| + |\lambda| > 0, \quad \lambda \in \mathcal{L}.$$

In the second case the matrix $A(\xi) - \lambda I$, in principle, cannot satisfy this definition. To explain this fact we consider the simplest example where $A(\xi)$ is a diagonal matrix: $A(\xi) := \mathrm{diag}\{A_{jj}(\xi), j = 1, \ldots, N\}$. In this case it is natural to expect that parameter-ellipticity condition for $A(\xi) - \lambda I$ is equivalent to the AAV condition for each element $A_{jj}(\xi) - \lambda, j = 1, \ldots, N$:

$$A^0_{jj}(\xi) - \lambda \neq 0, \quad |\xi| + |\lambda| > 0, \quad \lambda \in \mathcal{L}\,.$$

Here A_{jj} are polynomials of order r_j, and we successively adjust variable λ the weights $r_1, \ldots, r_N$. If all these numbers are pairly different, we obtain N conditions of parameter-ellipticity. In the case there are $N' < N$ groups of different numbers r_j we obtain N' conditions.

Now we return to the general case. We have a polynomial matrix and the orders of the elements of the principal part of this matrix are defined by the integers $s_1, \ldots, s_N, t_1, \ldots, t_N$. We suppose, in addition, that these numbers are nonnegative. This assumptions excludes exotic systems of the form

$$\begin{pmatrix} \Delta & M \\ 0 & \Delta \end{pmatrix},$$

where M is an operator of arbitrary high order.

Let us pose

$$r_j = s_j + t_j, \quad j = 1, \ldots, N.$$

By changing (if necessary) the indexing of lines (equations) and columns (unknown functions) we can suppose that

$$r_1 \geq r_2 \geq \cdots \geq r_N \geq 0$$

(see also Section 7 below). In [3], M. Agranovich made the following observation: in the case

$$r_1 = r_2 = \cdots = r_N = r$$

we can adjust the weight r to the variable λ and include $A(\xi) - \lambda I$ in the class of matrices considered by Roitberg. In this case the parameter-ellipticity condition is of the form (2).

4. Parameter-elliptic mixed order systems (definition)

For simplicity of notation we suppose in what follows that

$$r_1 > r_2 > \cdots > r_N > 0. \tag{3}$$

For each $\kappa = 1, \ldots, N$ we consider the submatrix

$$A(\kappa)(\xi) = \begin{pmatrix} A_{11} \ldots A_{1\kappa} \vdots\vdots A_{\kappa 1} \ldots A_{\kappa\kappa} \end{pmatrix}.$$

We denote by E_κ the $\kappa \times \kappa$ matrix which elements except the element in the right lower corner are zeros and this last element is equal to 1. Kozhevnikov [14], [15] gave the following

Definition 1. Under condition (3) the matrix $A(\xi) - \lambda I$ is called parameter-elliptic if there exists a ray $\mathcal{L}$ in the complex plane such that for each $\kappa = 1, \dots, N$

$$\det(A^0(\kappa)(\xi) - \lambda E_\kappa) \neq 0, \quad |\xi| \neq 0, \quad \lambda \in \mathcal{L}. \tag{4}$$

Here, as in the case of diagonal matrices, we have N separate conditions. At the first glance, these conditions seem very similar to the AAV condition, but in reality they differ very much. Let us analyze in detail conditions (4). Setting $\lambda = 0$ we obtain

$$\det A^0(\kappa)(\xi) \neq 0, \quad |\xi| \neq 0, \quad \kappa = 1, \dots, N \tag{5}$$

i.e. all subsystems $A(\kappa)$ are elliptic in the sense of Douglis–Nirenberg. Moreover, it follows from the definition of the determinant that

$$\det(A^0(\kappa)(\xi) - \lambda E_\kappa) = \det A^0(\kappa)(\xi) - \lambda \det A^0(\kappa - 1)(\xi)$$

It follows from (4) and this relation that

$$|\det(A^0(\kappa)(\xi) - \lambda E_\kappa)| > \mathrm{const}|\xi|^{r_1 + \dots + r_{\kappa-1}}(|\xi| + |\lambda|^{\frac{1}{r_\kappa}})^{r_\kappa}, \quad \lambda \in \mathcal{L}. \tag{6}$$

Polynomials satisfying an inequality of type (6) were treated in detail in [7], [6]. Such polynomials will be called weakly parameter-elliptic. Correspondingly, the matrices $A^0(\kappa)(\xi) - \lambda E_\kappa$ will be called weakly parameter-elliptic. Now we can reformulate the Kozhevnikov condition in the following form:

Lemma 1. *The matrix $A - \lambda$ is parameter-elliptic if and only if there exists a ray $\mathcal{L}$ such that all the submatrices $A(\kappa) - \lambda E_\kappa$, $\kappa = 1, \dots, N$ are weakly parameter-elliptic.*

Let us make a remark about the matrices $A(\kappa) - \lambda E_\kappa$. We give λ the weight r, where $r_{\kappa-1} > r \geq r_\kappa$ and introduce the numbers

$$s'_j = s_j, \quad t'_j = t_j, \quad j = 1, \dots, \kappa,$$

$$s'_j = s_j + \frac{r - r_j}{2}, \quad t'_j = t_j + \frac{r - r_j}{2}, \quad j = \kappa + 1, \dots, N.$$

Now we define the principal part of $A - \lambda I$ with respect to s'_j, t'_j. We obtain the following block-diagonal matrices:

$$(A(\kappa) 0 0 \lambda I_{N-\kappa}),$$

when $r > r_\kappa$ and

$$(A(\kappa) - \lambda E_\kappa 0 0 \lambda I_{N-\kappa}),$$

when $r = r_\kappa$. For $r > r_1$ we obtain $-\lambda I_N$, and for $r < r_N$ we obtain A^0. Due to the remarks above, we can call the matrices $A(\kappa)$ and $A(\kappa) - \lambda E_\kappa$ the r-principal parts of the mixed order polynomial matrix A. Now the definition of Kozhevnikov can be reformulated as follows:

Lemma 2. *The matrix $A - \lambda I$ is parameter-elliptic along the ray $\mathcal{L}$ iff for all $r > 1$ the r-principal part is weakly parameter-elliptic along this ray.*

We end this section with a remark about weakly-parameter elliptic and parabolic symbols. Consider a polynomial symbol of the variables $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ of the form

$$P(\xi, \lambda) = P_m(\xi, \lambda) + \lambda P_{m-p}(\xi) + \dots + \lambda^k \tag{7}$$

where $P_j(\xi)$ denote polynomials of ξ of order j. This polynomial is called weakly parameter-elliptic along the ray $\mathcal{L}$ if the inequality

$$|P(\xi, \lambda)| > \mathrm{const}|\xi|^{m-pk}(|\xi| + |\lambda|^{\frac{1}{p}})^{pk}, \quad \lambda \in \mathcal{L} \tag{8}$$

holds. Such polynomials already appeared before as the determinants of the r-principal parts of the matrix $A(\xi) - \lambda I$. We mention other examples.

Suppose $p=1$, m and $\mu:=m-k$ are even. If we replace λ by $1/\varepsilon$, multiply our polynomial by $\varepsilon^{m-\mu}$ and consider the ray $\mathcal{L}=\mathbb{R}_+$ we obtain the polynomial

$$P_\varepsilon=\varepsilon^{m-\mu}P_m(\xi)+\varepsilon^{m-\mu-1}P_{m-1}(\xi)+\cdots+P_\mu(\xi)$$

satisfying for $\varepsilon\geq 0$

$$|P_\varepsilon(\xi)|>\mathrm{const}|\xi|^\mu(1+\varepsilon|\xi|)^{m-\mu}.$$

Such symbols were introduced in the Lyusternik-Vishik theory of small singular perturbations of PDO, see [19] and also [12] and [16].

Consider now polynomial (7) with even $p=2b$. We call such polynomial weakly parabolic, or – more exactly – weakly $2b$-parabolic, if inequality (8) holds in the lower half-plane $\operatorname{Im}\tau\leq 0$. Obviously weakly $2b$-parabolic polynomials are symbols of $2b$-parabolic differential operators, which are not resolved with respect to the highest time derivative.

In the same way we can define weakly parabolic matrices of the form $A(\xi)-\lambda E_N$. As above, the matrix $A(\xi)-\lambda I$ is called parabolic, if all its r-principal parts $A(\kappa)(\xi)-\lambda E_\kappa$ are weakly parabolic.

5. Newton's polygon and parameter-elliptic mixed order systems

In the case of parameter-dependent constant order elliptic systems the AAV condition permits to prove in the parameter-dependent norms two-sided estimates for the operator with variable coefficients

$$A(x,D)-\lambda I, \tag{9}$$

say, on a manifold without boundary. Moreover, the norms of the remainder terms in the asymptotic series for the parametrix decrease as large negative powers of the parameter. Due to this fact it is possible to prove the existence of the inverse of (9) for large enough λ.

On the other side, the above formulated Kozhevnikov's condition of parameter-ellipticity of mixed order systems for the first sight is not connected with the conditions of the invertibility of (9). In connection with this M. S. Agranovich (1996) asked whether under the above parameter-ellipticity condition it is possible to obtain the analog of Agranovich-Vishik theory [4]. The answer was given in [5], where the crucial role was played by the notion of the Newton polygon of the determinant

$$P(\xi,\lambda):=\det(A(\xi)-\lambda I) \tag{10}$$

and the two-sided estimates of polynomials connected with the Newton polygon, see [13].

Let us write polynomial (10) in the form

$$P(\xi,\lambda)=\sum_{\alpha,k}p_{\alpha k}\xi^\alpha\lambda^k$$

and denote by $N(P)$ the convex hull of the points (i,k), where $\{p_{\alpha k}\neq 0, |\alpha|=i\}$, their projections on the coordinate axes $(i,0),(0,k)$ and the origin $(0,0)$. If condition (3) is satisfied, the Newton polygon of (10) has vertices

$$(0,0),(0,N),(r_1,N-1),(r_1+r_2,N-2),\ldots,(r_1+\cdots+r_N,0).$$

Denote

$$W_P(\xi,\lambda):=\sum_{(i,k)\in N(P)}|\xi|^i|\lambda|^k.$$

Obviously

$$|P(\xi,\lambda)|\leq\mathrm{const}\,W_P(\xi,\lambda)$$

with a constant independent of λ.

In [5] the following result was proved

Theorem 1. *For the matrix $A(\xi)-\lambda I$ following conditions are equivalent.*

(I) *There exists $\lambda_0>0$ such that*

$$|P(\xi,\lambda)| \geq \mathrm{const}\, W_P(\xi,\lambda), \quad \lambda\in\mathcal{L}, |\lambda|>|\lambda_0|. \tag{11}$$

(II) *Denote*

$$G(\xi,\lambda) := (A(\xi)-\lambda I)^{-1} := (G_{ij}(\xi,\lambda))_{i,j=1,\dots,N}.$$

Then the estimates

$$|G_{ij}(\xi,\lambda)| \leq const(|\xi+|\lambda|^{\frac{1}{r_i}})^{-t_i}(|\xi+|\lambda|^{\frac{1}{r_j}})^{-s_j}$$

hold.

(III) *The conditions of Definition* 1 *are satisfied.*

In particular, condition (II) permits us for parameter-elliptic systems to repeat the approach developed for systems satisfying the AAV condition.

Let us make some remarks on the proof of Theorem 1. It can be shown (see [5]) that

$$W_P(\xi,\lambda) \approx \prod_{j=1}^{N}(|\xi|+|\lambda|^{\frac{1}{r_j}})^{r_j}.$$

and inequality (11) can be replaced by

$$|P(\xi,\lambda)| \geq \mathrm{const}\prod_{j=1}^{N}(|\xi|+|\lambda|^{\frac{1}{r_j}})^{r_j}. \quad \lambda\in\mathcal{L}, |\lambda|>|\lambda_0|. \tag{12}$$

The equivalence of (I) and (II) is based on (12) and the formula for the elements of the inverse matrix:

$$G_{ij} = \frac{\det(A-\lambda I)^{ij}}{\det(A-\lambda I)},$$

where $(A-\lambda I)^{ij}$ is the complementary submatrix of the element A_{ji}.

The most meaningful part of the theorem is equivalence of (I) and (III). It is easy to check that r-principal parts of polynomial (10) are the determinants of submatrices $A(\kappa)$ and $A(\kappa)-\lambda E_\kappa$. In [13], in fact, it is proved that estimate (11) is equivalent to weak parameter-ellipticity of all r-principal parts of the polynomial P.

Remark 1. Replacing in the above theorem the ray $\mathcal{L}$ by the lower half-plane $\{\mathrm{Im}\,\lambda \leq \lambda_0\}$ we obtain the description of parabolic systems.

6. Parameter-elliptic mixed order systems on manifolds without boundary

Now we shall consider the system with variable coefficients

$$A(x,D)u(x)-\lambda u(x) = f(x). \tag{13}$$

For simplicity, we begin from the case of equation in whole $\mathbb{R}^n$. The first question is the choice of functional spaces with λ-dependent norms. In the traditional theory of systems, elliptic in the sense of Douglis–Nirenberg, the operator $A(x,D)$ is realized as a bounded operator in the pair of spaces

$$\prod_{j=1}^{N} H^{(k+t_j)}(\mathbb{R}^n) \to \prod_{j=1}^{N} H^{(k-s_j)}(\mathbb{R}^n).$$

In the case of systems with parameter (constant order systems, Roitberg's theory) the traditional norm in the Sobolev spaces $H^{(k)}$ is replaced by the norms corresponding to pseudodifferential operators with symbols $(|\xi|^2+|\lambda|^{\frac{2}{p}})^{k/2} \approx (|\xi|+|\lambda|^{\frac{1}{p}})^k$. To introduce norms in our more complicated case we start from inequality (12), which, in fact, is two-sided.

For arbitrary $\sigma = (\sigma_1, \dots, \sigma_N) \in \mathbb{R}^N$ we introduce the function

$$\Phi_\sigma(\xi, \lambda) = (|\xi|^2 + |\lambda|^{2/r_1})^{\sigma_1/2} \cdot \dots \cdot (|\xi|^2 + |\lambda|^{2/r_N})^{\sigma_N/2}$$

and consider the corresponding space $H^{(\sigma)}(\mathbb{R}^n)$ with the norm

$$||F^{-1}\Phi_\sigma(\xi, \lambda)Fu||_{L^2(\mathbb{R}^n)},$$

where we denote by F the Fourier transform.

Using the standard localization technique we can extend the spaces $H^{(\sigma)}(\mathbb{R}^n)$ to the case of a bounded manifold without boundary; the corresponding space will be denoted by $H^{(\sigma)}(M)$.

Now we correspond to the equation (13) an operator

$$A(x, D) - \lambda I : \prod_{j=1}^{N} H^{(\sigma_1,\dots,\sigma_j+t_j,\dots,\sigma_N)}(M) \to \prod_{j=1}^{N} H^{(\sigma_1,\dots,\sigma_j-s_j,\dots,\sigma_N)}(M) \qquad (14)$$

Obviously this operator is continuous and the norm is uniformly bounded with respect to λ. The main result is

Theorem 2. *For a mixed order elliptic system and a ray $\mathcal{L} \in \mathbb{C}$ following conditions are equivalent.*

(A) *For each σ there exists $\lambda_0 = \lambda_0(\sigma)$ such that for $\lambda \in \mathcal{L}, \quad |\lambda| \geq \lambda_0$ operator* (14) *has a bounded inverse and it's norm is uniformly bounded with respect to λ.*

(B) *For each $x^0 \in M$ the polynomial matrix $A(x^0, \xi) - \lambda I$ is parameter-elliptic along $\mathcal{L}$* (*i. e. it satisfies the equivalent conditions of Theorem* 1).

In the particular case $\sigma = (0, \dots, 0)$ this theorem is proved in [5]. The extension of this theorem to the case of arbitrary σ does not demand new ideas.

In addition, turn our attention to parabolic systems. We shall consider the homogeneous Cauchy problem. To treat it we introduce functional spaces in $\mathbb{R}^{n+1} = \{(x, t), x \in \mathbb{R}^n, t \in \mathbb{R}\}$. Denote by $(\xi, \tau), \tau = \xi_{n+1} + i\gamma$ the dual variables and by $H^{(\sigma)}_{[\gamma]}(\mathbb{R}^{n+1})$ the space of functions in $\mathbb{R}^{n+1}$ with finite norm

$$||F^{-1}_{[\gamma]}\Psi_\sigma(\xi_1, \dots, \xi_n, \xi_{n+1} + i\gamma)F_{[\gamma]}u||_{L^2(\mathbb{R}^{n+1})},$$

where $F_{[\gamma]}u$ is the Fourier transform of $e^{\gamma t}u$ and

$$\Psi_\sigma(\xi, \tau) := \prod_{j=1}^{N} (|\xi|^{r_j} + i\tau)^{\sigma_j/r_j}.$$

Denote by $H^{(\sigma)}_{[\gamma]+}(\mathbb{R}^{n+1})$ the subspace of $H^{(\sigma)}_{[\gamma]}(\mathbb{R}^{n+1})$ consisting of elements equal to zero for $t > 0$. The homogeneous Cauchy problem is realized as the operator

$$A(x, D) - D_t I : \prod_{j=1}^{N} H^{(\sigma_1,\dots,\sigma_j+t_j,\dots,\sigma_N)}_{[\gamma]+}(\mathbb{R}^{n+1}) \to \prod_{j=1}^{N} H^{(\sigma_1,\dots,\sigma_j-s_j,\dots,\sigma_N)}_{[\gamma]+}(\mathbb{R}^{n+1}) \qquad (15)$$

Theorem 2 can be reformulated in following form.

Theorem 3. *For a mixed order system following conditions are equivalent.*

(A) *For each σ there exists $\gamma_0 = \gamma_0(\sigma)$ such that for $\gamma \leq \gamma_0$ operator* (15) *has a bounded inverse.*

(B) *For each $x^0 \in M$ the polynomial matrix $A(x^0, \xi) - \lambda I$ is parabolic.*

7. Generalizations and remarks

Let us now consider the case where (3) does not hold. In this case we have to modify the definition of parameter-ellipticity in the following way: the matrix $A(\kappa)(\xi)$ now consists of blocks A_{ij} where the dimension of the block A_{11}, for instance, is given by the index $k \geq 1$ for which we have

$$r_1 = \cdots = r_k > r_{k+1} \geq \cdots \geq r_N \geq 0\,.$$

Similarly, the matrix E_κ is now a block matrix of the corresponding dimensions whose right lower block equals the identity matrix. With these modifications, we obtain analog results as in the case (3) for parameter-elliptic operators (see [5]).

In the case of parabolic time-dependent operators the same analysis works if we have

$$r_1 \geq \cdots \geq r_N > 0$$

with the modifications indicated above. If $r_N = 0$, the Newton polygon corresponding to the polynomial $\det(A(\xi) - \lambda I)$ has a vertical edge. Such operators were studied in [13] where they were called stable correct. Solvability results for such operators can be proved in the same way as for parabolic operators; note, however, that stable operators in general are not hypoelliptic.

Such systems naturally arise in mathematical physics. As an example consider equations for small oscillations of viscous, barotropic, compressible fluid. Indeed, the viscous compressible flow is described by the system

$$\rho\,\frac{\partial u}{\partial t} - \nu\,\Delta u + \rho\,(u\nabla)u + \operatorname{grad} p = 0,$$
$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0.$$

Here $u = (u_1, u_2, u_3)$ is the velocity, p is the pressure and ρ is the density, and the parameter ν is positive. In the case of barotropic fluid $p = p(\rho)$ and the derivative (the velocity of the sound) $p'(\rho)$ is positive. If we linearize the above system near the solution $(0, 0, 0, 1)$, we obtain a linear system of the form

$$\frac{\partial v}{\partial t} - \nu\,\Delta v + c\;\operatorname{grad}\rho = 0,$$
$$\frac{\partial \rho}{\partial t} + \operatorname{div} v = 0.$$

Here $c := p'(1) > 0$. Now we obtained a 4×4 system with $s_1 = s_2 = s_3 = 1, s_4 = 0$ and $t_1 = t_2 = t_3 = 1, t_4 = 0$. From this we get

$$r_1 = r_2 = r_3 = 2, \quad r_4 = 0.$$

The symbol of this system is $A(\xi) - \lambda I_4$, where

$$A(\xi) = \begin{pmatrix} i\nu|\xi|^2 & 0 & 0 & -c\xi_1 \\ 0 & i\nu|\xi|^2 & 0 & -c\xi_2 \\ 0 & 0 & i\nu|\xi|^2 & -c\xi_3 \\ -\xi_1 & -\xi_2 & -\xi_3 & 0 \end{pmatrix}.$$

This matrix is of block structure in the sense indicated at the beginning of this section where the left upper block has dimension 3. As we have $r_4 = 0$, the Newton polygon corresponding to this problem has a vertical edge, and the operator is not parabolic (but stable correct in the sense of [13]). The principal part of $\det(A - \lambda)$ equals

$$\lambda(\lambda - i\nu|\xi|^2)^3\,,$$

i.e. it is the product of a 2-parabolic operator and the symbol of the time derivative $\partial/\partial t$, which is the essential property of stable correct operators. See [13] for details.

8. Position of boundary value problems

We consider a smooth compact manifold M with smooth boundary ∂M and the problem

$$A(x,D)u(x) - \lambda u(x) = f(x), \quad x \in M, \tag{16}$$
$$B(x',D)u(x') = g(x'), \quad x' \in \partial M. \tag{17}$$

As above A is a mixed order system with $\operatorname{ord} A_{ij} \leq s_i + t_j, \quad r_j := s_j + t_j$,

$$r_1 > r_2 > \cdots > r_N > 0.$$

From the definitions below it will follow that the numbers r_j are even. So we can pose

$$R_k := \frac{r_1 + \cdots + r_k}{2}, \quad k = 1, \ldots, N, \quad R := R_N.$$

Boundary conditions (17) are defined as a rectangular $R \times N$ matrix with matrix symbol $(B_{jk}(x', \xi))$, where $\operatorname{ord} B_{jk} \leq m_j + t_k$.

Contrary to traditional elliptic theory the mean of indexing of the boundary conditions plays an important role. We suppose that

$$m_1 \leq \cdots \leq m_R.$$

Moreover, we additionally suppose that

$$m_{R_k} < m_{R_k+1}, \quad k = 1, \ldots, N-1. \tag{18}$$

As above we shall denote by A^0_{ij} and B^0_{jk} the principal homogeneous parts of order $s_i + t_j$ and $m_j + t_k$, respectively, of the symbols A_{ij} and B_{jk}, and let A^0, B^0 be the corresponding matrices of principal parts.

Our main goal is to formulate the analog of the Theorem 2 for the problem (16), (17). The reformulation of the condition (A) of this theorem is rather technical task. The main step is the definition of the spaces $H^{(\sigma)}(M)$ in the case when the manifold M has a boundary. The standard localization technique reduces this problem to the case when $M = \mathbb{R}^n_+ := \{x \in \mathbb{R}^n \colon x_n \geq 0\}$. Denote by ξ' the dual variables to $x' = (x_1, \ldots, x_{n-1})$ and consider the symbol

$$\tilde{\Phi}_\sigma(\xi, \lambda) = \prod_{j=1}^{N} (-i\xi_n + \sqrt{|\xi'|^2 + |\lambda|^{2/r_j}})^{\sigma_j}.$$

Obviously $\Phi_\sigma(\xi, \lambda) \approx \tilde{\Phi}_\sigma(\xi, \lambda)$. We can define the space $H^{(\sigma)}(\mathbb{R}^n_+)$ by means of the norm

$$||F^{-1}\tilde{\Phi}_\sigma(\xi, \lambda) F L u||_{L^2(\mathbb{R}^n_+)},$$

where $Lu(x)$ is an extension of the function $u(x)$ on the whole space. Since the symbol $\tilde{\Phi}_\sigma(\xi, \lambda)$ is holomorphic for $\operatorname{Im} \xi_n < 0$ the norm is independent of the continuation operator L.

Now we can consider the operator

$$\begin{aligned}(A(x,D) - \lambda I, B) : \prod_{j=1}^{N} & H^{(\sigma_1, \ldots, \sigma_j + t_j, \ldots, \sigma_N)}(M) \to \\ & \prod_{j=1}^{N} H^{(\sigma_1, \ldots, \sigma_j - s_j, \ldots, \sigma_N)}(M) \times \prod_{k=1}^{R} H^{(\sigma)(-m_k - 1/2)}(\partial M).\end{aligned} \tag{19}$$

Here $H^{(\sigma)(-m_k-1/2)}(\partial M)$ is the image on ∂M of the space $H^{(\sigma_1, \ldots, \sigma_j + t_j, \ldots, \sigma_N)}(M)$ under the action of the operator $B_{jk}(x', D)$. This space is $H^{\sigma_1 + \cdots + \sigma_n - m_k - 1/2}(\partial M)$ with special parameter-dependent norm. Norms of such type are described in [6].

9. Parameter-ellipticity conditions for boundary value problems

As customary in elliptic theory, the conditions on the system and the boundary operator are called conditions on the inner and boundary symbols. The first is condition (III) of Theorem 1.

Condition on the inner symbol. For each $x^0 \in M \setminus \partial M$ the matrix $A^0(x^0, \xi) - \lambda I$ satisfies equivalent conditions of Theorem 1.

To formulate the conditions on the boundary symbol, we fix a point $x'^0 \in \partial M$ and choose such coordinate system, that x' are tangential variables and the direction of x_n is the normal to ∂M at the point x'^0. The boundary symbol is the problem on the half-line $x_n > 0$ for the matrix ordinary differential operator $A(\xi, D_n) - \lambda I$, where $A(\xi, D_n) := A^0(x'^0, 0, \xi', D_n)$ and the boundary operator $B(\xi', D_n) := B^0(x'^0, \xi', D_n)$.

The condition on the inner symbol is equivalent to weak parameter- ellipticity conditions on matrices $A(\kappa)(\xi) - \lambda I, \quad \kappa = 1, \dots, N$. Now we consider the boundary operators

$$B(\kappa)(\xi' D_n) = (B_{jk}(\xi', D_n))_{j=1,\dots,R_\kappa,\, k=1,\dots,\kappa}$$

and following [6] formulate the weak parameter-ellipticity condition for the boundary problem

$$A(\kappa)(\xi', D_n) - \lambda I, \quad B(\kappa)(\xi', D_n) \tag{20}$$

on the half-line $x_n > 0$.

First of all it contains a natural

Condition (i). For each $\kappa = 1, \dots, N, \quad \lambda \in \mathcal{L}$ and $|\xi'| \neq 0$ the boundary problem

$$(A(\kappa)(\xi', D_n) - \lambda I_\kappa) w^\kappa(x_n) = 0, \quad x_n > 0; \tag{21}$$

$$B(\kappa)(\xi', D_n) w^\kappa(0) = g \in \mathbb{C}^{R_\kappa}; \tag{22}$$

$$|w^\kappa(x_n)| \to 0, \quad x_n \to +\infty \tag{23}$$

has a unique solution.

Here we posed $w^\kappa = (w_1, \dots, w_\kappa)$. Setting $\lambda = 0$ in (20) we obtain, as a corollary,

Condition (i$'$). For each $\kappa = 1, \dots, N$ and $|\xi'| \neq 0$ the boundary problem

$$A(\kappa)(\xi', D_n) w^\kappa(x_n) = 0, \quad x_n > 0;$$
$$B(\kappa)(\xi', D_n) w^\kappa(0) = g \in \mathbb{C}^{R_\kappa};$$
$$|w^\kappa(x_n)| \to 0, \quad x_n \to +\infty$$

has a unique solution.

Condition (i$'$) means that the problem $A(\kappa)(x, D), B(\kappa)(x', D)$ satisfies the standard Shapiro-Lopatinskii condition.

Note that condition (i) cannot be fulfilled for $\xi' = 0$. Indeed, for $\xi' \neq 0$ and $\lambda \in \mathcal{L}$ the equation

$$\det(A(\kappa)(\xi', z) - \lambda I_\kappa) = 0$$

has R_κ zeros in the upper half-plane of the complex plane. In the case $\xi' = 0$ this equation takes the form

$$a_\kappa z^{r_1 + \dots + r_\kappa} - a_{\kappa-1} \lambda z^{r_1 + \dots + r_{\kappa-1}} = 0$$

and has only $r_\kappa/2$ zeros with positive imaginary part.

Now we formulate

Condition (ii). For each $\kappa = 1, \dots, N, \quad \lambda \in \mathcal{L}, \quad |\lambda| = 1$ the boundary problem

$$(A(\kappa)(0, D_n) - \lambda I) w^\kappa(x_n) = 0, \quad x_n > 0; \tag{24}$$

$$\sum_{k=1}^{\kappa} B_{jk}(0, D_n) w_k^{\kappa}(0) = g_j, \quad j = R_{\kappa-1}+1, \dots, R_\kappa; \tag{25}$$

$$|w^\kappa(x_n)| \to 0, \quad x_n \to +\infty \tag{26}$$

has a unique solution.

Definition 2. The boundary value problem (20) is called weakly parameter-elliptic if it satisfies conditions (i) and (ii).

A deeper understanding of this notion is connected with the formal asymptotic solution of the above problem with respect to the small parameter $\varepsilon = 1/\lambda$ and will be given in the next section.

Condition on the boundary symbol. For each $\kappa = 1, \dots, N$ the boundary symbols (20) are weak parameter-elliptic.

The main theorem for the boundary value problem (16), (17) can be formulated in the same form as Theorem 2.

Theorem 4. *For a boundary value problem* (16), (17) *for a mixed order elliptic system and a ray* $\mathcal{L} \in \mathbb{C}$ *following conditions are equivalent.*

(A) *For each* σ *there exists* $\lambda_0 = \lambda_0(\sigma)$ *such that for* $\lambda \in \mathcal{L}$, $|\lambda| \geq \lambda_0$ *operator* (19) *has a bounded inverse and its norm is uniformly bounded with respect to* λ.

(B) *The above formulated conditions on the inner and boundary symbols are satisfied.*

10. Formal asymptotic solutions for weakly parameter-elliptic systems

Let us consider a model weakly parameter-elliptic problem

$$(A(D) - \lambda E_N) u(x', x_n) = 0, \quad x_n > 0, \tag{27}$$

$$B_j(D) u(x', 0) = g_j, \quad j = 1, \dots, R_N, \tag{28}$$

where A_{ij} and B_{jk} do not contain lower order terms. In (28) we used the notation $B_j := (B_{j1}, \dots, B_{jN})^\top$.

We pose $\lambda = \varepsilon^{-r_N}$ and multiply the last equation in the system (27) by ε^{r_N}. Then the system can be rewritten in the form

$$A_\varepsilon(D) u(x', x_n) = 0, \quad x_n > 0, \tag{29}$$

where

$$A_\varepsilon(D) = \operatorname{diag}\{1, \dots, 1, \varepsilon^{r_N}\} A(D) - E_N. \tag{30}$$

Our goal is to find the formal asymptotic solution (FAS)

$$\sum \varepsilon^l u^{(l)}(x, \varepsilon).$$

The partial sums of this formal power series satisfy (29), (28) up to an arbitrary power of ε. Following the Lyusternik-Vishik method we search the FAS as the sum of the so-called exterior expansion

$$u(x, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^l u^{(l)}(x) \tag{31}$$

and the so-called interior expansion, or boundary layer

$$v(x', x_n/\varepsilon, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^{l_0 + l} \operatorname{diag}\{\varepsilon^{t_1}, \dots, \varepsilon^{t_N}\} v^{(l)}(x', x_n/\varepsilon.) \tag{32}$$

The number l_0 will be chosen later.

Differential equations for exterior expansion. Substituting (31) in (29) and posing $u'^{(l)} = (u_1^{(l)}, \dots, u_{N-1}^{(l)})$ we obtain

$$\sum_{l=0}^{\infty} \varepsilon^l (A(N-1)(D)u'^{(l)} + \mathrm{colon}(A_{1N}(D), \dots, A_{N-1,N}(D))u_N^{(l)}) = 0,$$

$$\sum_{l=0}^{\infty} \varepsilon^l (\varepsilon^{r_N} \sum_{j=1}^{N} A_{Nj}(D)u_j^{(l)} - u_N^{(l)}) = 0.$$

Equating to zero the terms corresponding to the same power of ε we obtain relations

$$u_N^{(l)} = -\sum A_{Nj}(D)u_j^{(l-r_N)}, \tag{33}$$

$$A(N-1)(D)u'^{(l)} = -\,\mathrm{col}(A_{1N}(D), \dots, A_{N-1,N}(D))u_N^{(l)}) = \mathcal{F}(u^{(0)}, \dots, u^{(l-r_N)}). \tag{34}$$

Differential equations for interior expansion. Pose $t = x_n/\varepsilon$. Then

$$\begin{aligned} A_\varepsilon(D)v(x', x_n/\varepsilon, \varepsilon) &= A_\varepsilon(D', \frac{1}{\varepsilon}D_t)v(x', t, \varepsilon) \\ &= [\mathrm{diag}(\varepsilon^{-s_1}, \dots, \varepsilon^{-s_{N-1}}, \varepsilon^{t_N})A(\varepsilon D', D_t)\,\mathrm{diag}(\varepsilon^{-t_1}, \dots, \varepsilon^{-t_N}) - E_N]v(x', t, \varepsilon). \end{aligned}$$

Replacing $v(x, t, \varepsilon)$ by expansion (32) we obtain

$$\mathrm{diag}(\varepsilon^{-s_1}, \dots, \varepsilon^{-s_{N-1}}, \varepsilon^{t_N})(A(\varepsilon D', D_t) - E_N) \sum_{l=0}^{\infty} \varepsilon^{l_0+l} v^{(l)}.$$

After multiplication by $\mathrm{diag}(\varepsilon^{-s_1}, \dots, \varepsilon^{-s_{N-1}}, \varepsilon^{t_N})$ we obtain equation

$$\sum_{l=0}^{\infty} \varepsilon^{l_0+l} v^{(l)}(A(\varepsilon D', D_t) - E_N)v^{(l)} = 0.$$

Expanding $A(\varepsilon D', D_t)$ with respect to $\varepsilon D'$ we obtain

$$A(\varepsilon D', D_t) = A(0, D_t) + \sum_{|\alpha|\geq 1} \varepsilon^{|\alpha|} A^{(\alpha)}(0, D_t)D'^{\alpha}/\alpha!$$

Substituting this relation we obtain recurrent relations

$$A(0, D_t)v^{(l)}(x', t) + \sum_{k=1} \sum_{|\alpha|=k} \frac{1}{\alpha!} A^{(\alpha)}(0, D_t)D'^{\alpha} v^{(l-k)} = 0. \tag{35}$$

Systems (34) and (35) and conditions of weak parameter-ellipticity show, that we will be able to define successively vector functions $u'^{(l)}, u^{(l)}, v^{(l)}$ if we know

$$g'_{lj} := B_j(D)u'^{(l)}(x', 0), \quad j = 1, \dots, R_{N-1}, \quad l = 0, 1, \dots$$

and

$$g''_{lj} := B_j(0, D_t)v^{(l)}(x', 0), \quad j = R_{N-1}+1, \dots, R_N, \quad l = 0, 1, \dots.$$

First of all note that

$$B_j(D)u(x', 0, \varepsilon) = \sum_{l=0}^{\infty} \varepsilon^l B_j(D)u^{(l)}(x', 0). \tag{36}$$

In the case of inner expansion arguing as above we obtain

$$\begin{aligned} B_j(D)v(x', 0, \varepsilon) &= \sum_{l=0}^{\infty} \varepsilon^{l_0+l} B_j(D', \frac{1}{\varepsilon}D_t)\,\mathrm{diag}(\varepsilon^{t_1}, \dots, \varepsilon^{t_N})v^{(l)}(x', 0) \\ &= \sum_{l=0}^{\infty} \varepsilon^{l+l_0-m_j} B_j(\varepsilon D', D_t)v^{(l)}(x', 0). \end{aligned}$$

Replacing $B_j(\varepsilon D', D_t)$ by

$$B_j(0, D_t) + \sum_{k=1} \varepsilon^k C_k(D)$$

and gathering the terms with the same power of ε we finally obtain

$$B_j(D)v(x', 0, \varepsilon) = \sum_{l=l_0-m_j}^{\infty} \varepsilon^l (B_j(0, D_t)v^{(l-l_0+m_j)}(x', 0) + C_1(D)v^{(l-l_0+m_j-1)} + \dots). \qquad (37)$$

Now we pose $l_0 = m_{R_{N-1}+1}$. According to our assumption $l_0 > m_j, \quad j = 1, \dots, R_{N-1}$ and the first R_{N-1} boundary conditions take form

$$B_j(D)u^{(l)}(x', 0) = \delta_0^l g_j(x') + B_j(0, D_t)v^{(l-l_0+m_j)}(x', 0) + C_1(D)v^{(l-l_0+m_j-1)} + \dots. \qquad (38)$$

If we already know $u^{(k)}, v^{(k)}, \quad k = 1, \dots, l-1$, we can define

$$B_j(D){u'}^{(l)}(x', 0), \quad j = 1, \dots, R_{N-1}.$$

Using system (34) and these boundary conditions we can define ${u'}^{(l)}$ and, consequently $u^{(l)}$.

For $j = R_{N-1} + 1$ we obtain relation

$$B_j(0, D_t)v^{(l)} = \delta_0^l g_l - B_j u^{(l)} - \sum_{k \geq 1} C_k(D)v^{(l-k)}.$$

To obtain other conditions for $j > R_{N-1} + 1$ we must take the result of application of the operator $B_j(D)$ to the term obtained from equating to zero the coefficient before $\varepsilon^{l+R_{N-1}+1-j}$. Then we obtain for $j = R_{N-1} + 2, \dots, R_N$

$$B_j(0, D_t)v^{(l)} = \delta_0^{l+R_{N-1}+1-j} g_l - B_j(D)u^{(l+R_{N-1}+1-j)} - \sum_{k \geq 1} C_k(D)v^{(l-k)}.$$

Now we can find $v^{(l)}$ and continue our process.

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Robert Denk, NWF I-Mathematik, Universität Regensburg, D-93040 Regensburg, Germany
E-mail address: robert.denk@mathematik.uni-regensburg.de

Leonid Volevich, Keldysh Institute of Applied Mathematics, Russian Acad. Sci., Miusskaya sqr. 4, 125047 Moscow, Russia
E-mail address: volevich@spp.keldysh.ru