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The length and other invariants of a real field

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Abstract The length of a field is the smallest integer m such that any totally positive quadratic form of dimension m represents all sums of squares. We investigate this field invariant and compare it to others such as the u-invariant, the Pythagoras number, the Hasse number, and the Mordell function related to sums of squares of linear forms.

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1 Introduction

Let K be a field of characteristic different from 2. The set of nonzero sums of squares in K is denoted by $\sum K^2$. For $n \in \mathbb{N}$ we write $D_K(n)$ for the set of nonzero elements of K that can be expressed as sums of n squares in K. By the Artin-Schreier Theorem, K admits a field ordering if and only if $-1 \notin \sum K^2$. If this is the case one says that the field K is *real* (or *formally real*), otherwise *nonreal*. We write $K(i) = K(\sqrt{-1})$.

We assume that the reader is familiar with the basic theory of quadratic forms over fields, for which we refer to [17] and [20]. By a 'form' or a 'quadratic form' we mean a regular (that is, non-degenerate) quadratic form. Since we identify quadratic forms up to isometry, we use the equality sign to indicate that two quadratic forms are isometric. Given a quadratic form φ over K, we denote by $D_K(\varphi)$ the set of nonzero elements of K represented by φ and, for $n \in \mathbb{N}$, by $n \times \varphi$ the *n-fold sum* $\varphi \perp \ldots \perp \varphi$. If $D_K(\varphi) \subseteq \sum K^2$, then φ is said to be *totally positive*, and if $D_K(\varphi) = \sum K^2$, then we say that φ is *positive-universal* over K.

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We say that φ is *torsion* if $n \times \varphi$ is hyperbolic for some integer $n \geq 1$. We say that φ is *totally indefinite* if it is indefinite with respect to every ordering of K.

In this article we study certain field invariants taking values in $\mathbb{N} \cup \{\infty\}$, namely the *Pythagoras number* $p(K)$, the *length* $\ell(K)$, the *u-invariant* $u(K)$, and the Hasse number $\widetilde{u}(K)$. They are defined by

$$\begin{aligned} p(K) &= \inf\left\{n \in \mathbb{N} \mid D_K(n) = \textstyle\sum K^2\right\}, \\ \ell(K) &= \inf\left\{n \in \mathbb{N} \,\middle|\, \begin{array}{l}\text{any totally positive form over } K \\ \text{of dimension } n \text{ is positive-universal}\end{array}\right\}, \\ u(K) &= \sup\{\dim(\varphi) \mid \varphi \text{ anisotropic torsion form over } K\}, \\ \widetilde{u}(K) &= \sup\{\dim(\varphi) \mid \varphi \text{ anisotropic totally indefinite form over } K\}. \end{aligned}$$

Here, the infimum and the supremum are taken in $\mathbb{N} \cup \{\infty\}$, so in particular, $\inf \emptyset = \infty$ and $\sup \emptyset = 0$. Each of these invariants is related to the possible dimensions of a specific type of anisotropic quadratic form. Since any torsion form is totally indefinite, we have $u(K) \leq \widetilde{u}(K)$. In [10, (2.5)] it is shown that $\widetilde{u}(K) < \infty$ holds if and only if $u(K) < \infty$ and K is an *ED*-field, in the terminology of [22]. If K is nonreal, then every quadratic form over K is totally positive and torsion, so that $\ell(K) = u(K) = \widetilde{u}(K)$. One of the aims of this article is to investigate the relations between these invariants for real fields with a special focus on the length. This invariant was introduced in [2] and studied together with a certain function $g_K : \mathbb{N} \longrightarrow \mathbb{N} \cup \{\infty\}$ related to sums of squares of linear forms over K.

In Sect. 2 of this article we study the powers of the ideal $I_t K$ of even dimensional torsion forms in the Witt ring WK of quadratic forms over K and obtain some other preliminary results. In Sect. 3, we study the function g_K and obtain upper bounds for its values under conditions on the vanishing of a certain power of $I_t K$. In Sect. 4 we revisit and partially improve results from [2] on the length of a field. In Sect. 5 we obtain upper and lower bounds on the length in terms of the other field invariants introduced above. In particular, we show the estimate

$$\ell(K) \leq g_K\left(\frac{u(K)}{2}\right) + 1 \leq \tfrac{1}{2} p(K) u(K) + 1 \tag{1.1}$$

for an arbitrary real field K with $u(K) < \infty$. For *ED*-fields, we obtain results relating $\widetilde{u}(K)$ and $\ell(K)$. We further show that $\ell(K) \leq |\sum K^2 / K^{\times 2}|$, which is a generalization of Kneser's bound $u(K) \leq |K^\times / K^{\times 2}|$ for nonreal fields K. In Sect. 6 we give bounds on $\ell(K)$ in terms of $u(K(i))$. In particular, we obtain a bound for the length of an extension of $\mathbb{R}$ in terms of the transcendence degree. Section 7 is devoted to the construction of examples that show many of our results to be best possible. Following a suggestion by D. Hoffmann, we show that, for any integer $n \geq 1$ there is a uniquely ordered field K with $\ell(K) = p(K) = n$. Moreover, we give examples realizing all possible values for $\ell(K)$ for fields K with $(I_t K)^3 = 0$. In (7.13) we give an example where the bounds in (1.1) are sharp.

2 Powers of the torsion ideal in the Witt ring

We denote by IK the fundamental ideal in the Witt ring WK of K, which consists of the classes of even dimensional quadratic forms over K, and by $I_t K$ the torsion part of IK. Let n be a positive integer. Given $a_1, \dots, a_n \in K^\times$ we write $\langle\langle a_1, \dots, a_n \rangle\rangle = \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle$ and call this an *n-fold Pfister form*. We further write $I^n K = (IK)^n$ and $I_t^n K = I^n K \cap I_t K$.

It is a basic fact that $I^n K$ is generated (even as a group) by the n-fold Pfister forms over K. It was shown in [1] that $I_t^n K = 0$ holds if and only if every $(n-1)$-fold Pfister form over K represents all totally positive elements of K, which is condition (A_n) in the terminology of [9]. Whereas this relies on a deep result in [19], (2.3) below gives a simple argument that $(I_t K)^n = 0$ is equivalent to condition (B_n) in [9], namely that every totally positive $(n-1)$-fold Pfister form over K represents all elements of $\sum K^2$.

Remark 2.1 Note that $\langle\langle a, b\rangle\rangle = \langle\langle a, -ab\rangle\rangle$ for any $a, b \in K^\times$. If $a \in \sum K^2$, then b and $-ab$ have opposite signs with respect to any ordering of K. Using this fact, given an arbitrary Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ with $a_1 \in \sum K^2$ and $a_2, \dots, a_n \in K^\times$, we may change signs as above in the last $n-1$ slots as convenient. In particular, any Pfister form $\langle\langle a_1, \dots, a_n\rangle\rangle$ with $a_1, \dots, a_n \in \sum K^2$ can be rewritten as a product of a torsion onefold Pfister form and a totally positive $(n-1)$-fold Pfister form, and conversely.

Proposition 2.2 *Let $n \in \mathbb{N}$. The ideal $(I_t K)^{n+1}$ is generated by the differences of totally positive $(n+1)$-fold Pfister forms over K. It is also generated by the forms $\langle 1, -a\rangle \otimes \pi$ where $a \in \sum K^2$ and π is a totally positive n-fold Pfister form over K.*

Proof Both statements follow by induction from the case $n = 0$.

Corollary 2.3 *For $n \in \mathbb{N}$, the following are equivalent:*

(*i*) $(I_t K)^{n+1} = 0$.
(*ii*) $2^{n+1} \times \langle 1\rangle$ *is the only totally positive $(n+1)$-fold Pfister form over K.*
(*iii*) $D_K(\pi) = \sum K^2$ *for every totally positive n-fold Pfister form π over K.*

Proof The first set of generators provided by (2.2) yields the equivalence $(i) \Leftrightarrow (ii)$, the second one the equivalence $(i) \Leftrightarrow (iii)$.

Lemma 2.4 *Let $\varphi \in I^2 K$ and assume that φ has a diagonalization with entries in $\pm \sum K^2$. Then φ is Witt equivalent to a sum of 4-dimensional forms over K of trivial determinant and having a diagonalization with entries in $\pm \sum K^2$.*

Proof If $\dim(\varphi) \leq 2$, the statement is trivial. Assume that $\dim(\varphi) > 2$ and write $\varphi = \psi \perp \langle x, y, z\rangle$ where $x, y, z \in \pm \sum K^2$ and where ψ is diagonalizable with entries in $\pm \sum K^2$. Then φ is Witt equivalent to $\psi \perp \langle -xyz\rangle \perp \langle x, y, z, xyz\rangle$. Thus the statement follows by induction on the dimension of φ.

For a form φ over K we denote by $G_K(\varphi)$ the group of similarity factors, i.e. $G_K(\varphi) = \{a \in K^\times \mid a\varphi = \varphi\}$. Note that $G_K(\varphi) \cdot D_K(\varphi) \subseteq D_K(\varphi)$.

Lemma 2.5 *If φ is a nontrivial form over K such that $D_K(\varphi) \subseteq G_K(\varphi)$, then $D_K(\varphi) = G_K(\varphi)$.*

Proof Let $a \in D_K(\varphi)$ be given and assume that $D_K(\varphi) \subseteq G_K(\varphi)$. Then one has $G_K(\varphi) = G_K(\varphi) \cdot a \subseteq G_K(\varphi) \cdot D_K(\varphi) \subseteq D_K(\varphi)$, showing the equality.

Proposition 2.6 *Assume that $(I_t K)^3 = 0$. Let φ be a quadratic form of even dimension over K with $\det(\varphi) = \pm 1$ and having a diagonalization with entries in $\pm \sum K^2$. Then $D_K(2) \subseteq G_K(\varphi)$. Moreover, if $\varphi \in I^2 K$, then $\sum K^2 \subseteq G_K(\varphi)$.*

Proof If $\dim(\varphi) = 4$ and $\det(\varphi) = 1$, then $\langle 1, -a\rangle \otimes \varphi \in (I_t K)^3 = 0$ for any $a \in \sum K^2$ by (2.1) and thus $\sum K^2 \subseteq G_K(\varphi)$. Using (2.4) the same then follows for any $\varphi \in I^2 K$ having a diagonalization with entries in $\pm \sum K^2$. Assume now that $\varphi \notin I^2 K$. Then $\varphi \perp \langle 1, 1\rangle \in I^2 K$ so that we obtain $\sum K^2 \subseteq G_K(\varphi \perp \langle 1, 1\rangle)$ and therefore $D_K(2) = G_K(\langle 1, 1\rangle) \subseteq G_K(\varphi)$.

Proposition 2.7 *Let $n \in \mathbb{N}$.*

(*a*) *If $u(K) < 2^{n+1}$, then $I_t^{n+1} K = 0$.*
(*b*) *If $(I_t K)^{n+1} = 0$, then $p(K) \leq 2^n$.*

Proof To show (*a*), assume that $I_t^{n+1} K \neq 0$. With a nontrivial anisotropic torsion form $\varphi \in I^{n+1} K$ we obtain $u(K) \geq \dim(\varphi) \geq 2^{n+1}$ by the Arason–Pfister–Hauptsatz [17, Chap. X, (5.1)]. Part (*b*) is clear from (2.3).

Corollary 2.8 *One has $p(K) \leq u(K)$.*

Proof If $n \in \mathbb{N}$ is such that $2^n \leq u(K) < 2^{n+1}$, then $(I_t K)^{n+1} \subseteq I_t^{n+1} K = 0$ and $p(K) \leq 2^n \leq u(K)$, by (2.7).

3 The Mordell function

For a field K and $n \in \mathbb{N}$, we denote by $g_K(n)$ the infimum in $\mathbb{N} \cup \{\infty\}$ of the set of natural numbers m such that $m \times \langle 1\rangle$ contains every totally positive form of dimension n over K. This defines a function $g_K : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, which we call the *Mordell function* of K. By [2, (2.4)] g_K coincides with the function introduced in [7] in terms of sums of squares of linear forms, based on work by Mordell in [18] on positive definite integral quadratic forms.

Proposition 3.1 *For $m, n, r \in \mathbb{N}$, the following hold:*

(*a*) *$g_K(0) = 0$ and $g_K(1) = p(K)$.*
(*b*) *$g_K(n) \geq n$ and $g_K(n+1) \geq g_K(n) + 1$.*
(*c*) *$g_K(m+n) \leq g_K(m) + g_K(n)$ and $g_K(mn) \leq m\, g_K(n)$.*
(*d*) *If $p(K) \leq 2^r$ and $g_K(n) < m\, 2^r$, then $g_K(n+1) \leq m\, 2^r$.*

Proof Parts (*a*)–(*c*) are obvious, and (*d*) is [2, (2.19)].

Corollary 3.2 *If $p(K) < \infty$, then $g_K(n) < \infty$ for every $n \in \mathbb{N}$, otherwise $g_K(n) = \infty$ for all $n \geq 1$.*

Proof For $n \geq 1$ one has $p(K) = g_K(1) \leq g_K(n) \leq n g_K(1) = n\, p(K)$.

Corollary 3.3 *Assume that $p(K) \leq 3$. Then $g_K(m) \leq 2m + 1$ for any $m \in \mathbb{N}$. Moreover, $g_K(m) \leq 2m$ if m is even or if $p(K) \leq 2$.*

Proof Using part (*d*) of (3.1) for $m = n = 1$ and $r = 2$ yields $g_K(2) \leq 4$. The statement follows using part (*c*) of (3.1).

Theorem 3.4 *Assume that $(I_t K)^{n+1} = 0$. Then the following hold:*

(*a*) *If $n \geq 1$, then $g_K(m) \leq 2^{n-1}(m+1)$ for any $m \in \mathbb{N}$.*
(*b*) *$g_K(m) \leq 2^{n-m+1}(2^m - 1)$ for $1 \leq m \leq n+1$.*
(*c*) *$g_K(n+2) \leq 2^{n+1}$.*

Proof (*a*) By the hypothesis and (2.3) we have $2^{n-1} \times \langle a, b\rangle = 2^{n-1} \times \langle 1, ab\rangle$ for any $a, b \in \sum K^2$. Hence, if φ is a totally positive quadratic form of dimension m over K, then $2^{n-1} \times (\varphi \perp \langle d\rangle) = 2^{n-1}(m+1) \times \langle 1\rangle$ with $d = \det(\varphi)$, so φ is a subform of $2^{n-1}(m+1) \times \langle 1\rangle$. Therefore $g_K(m) \leq 2^{n-1}(m+1)$.

(*b*) Let ρ be a totally positive form of dimension $m \leq n+1$ over K. Write $\rho = \langle a_1, \ldots, a_m\rangle$ with $a_1, \ldots, a_m \in K^\times$ and put $\pi = \langle 1, a_1\rangle \otimes \cdots \otimes \langle 1, a_m\rangle$. Since $(I_t K)^{n+1} = 0$, we have $2^{n-m+1} \times \pi = 2^{n+1} \times \langle 1\rangle$ by (2.3). We write $\pi = \langle 1\rangle \perp \pi'$ and obtain that $2^{n-m+1} \times \pi' = 2^{n-m+1}(2^m - 1) \times \langle 1\rangle$. Then ρ is a subform of π' and thus of $2^{n-m+1}(2^m - 1) \times \langle 1\rangle$.

(*c*) By (2.7) we have $p(K) \leq 2^n$. Hence, $g_K(n+1) \leq 2^{n+1} - 1$ by (*b*) and therefore $g_K(n+2) \leq 2^{n+1}$ by part (*d*) of (3.1).

Corollary 3.5 *Assume that* $(I_t K)^{n+1} = 0$. *Then* $g_K(m) \leq \lceil \frac{m}{n+2} \rceil \cdot 2^{n+1}$ *for any* $m \in \mathbb{N}$. *More precisely, writing* $m = k(n+2) - r$ *with* $k, r \in \mathbb{N}$ *such that* $r \leq n+1$, *one has*

$$g_K(m) \leq k \cdot 2^{n+1} - \begin{cases} 2^{r-1} & \text{if } 1 \leq r \leq n+1, \\ 0 & \text{if } r = 0. \end{cases}$$

Proof With k and r as above, we have $k = \lceil \frac{m}{n+2} \rceil$ and $m \leq k(n+2)$. Then $g_K(m) \leq g_K(k(n+2)) \leq k \cdot g_K(n+2) \leq \lceil \frac{m}{n+2} \rceil \cdot 2^{n+1}$, showing the first estimate, and also the second estimate for $r = 0$. Assume now that $1 \leq r \leq n+1$. By part (*b*) of (3.4) one has $g_K(n+2-r) \leq 2^{n+1} - 2^{r-1}$. As $m = (k-1)(n+2) + n + 2 - r$, we get $g_K(m)$ $g_K((k-1)(n+2)) + g_K(n+2-r)$, and the second estimate follows.

We state explicitly the special case where $n = 2$ of the estimate in (3.5) and give a separate proof, for simplicity.

Corollary 3.6 *Assume that* $(I_t K)^3 = 0$. *Then, for any* $m \geq 1$, *one has*

$$g_K(m) \leq \begin{cases} 2m & \text{for } m \equiv 0 \mod 4, \\ 2m+2 & \text{for } m \equiv 1, 2 \mod 4, \\ 2m+1 & \text{for } m \equiv 3 \mod 4. \end{cases}$$

Proof We write $m = 4k + m'$ with $k, m' \in \mathbb{N}$ such that $1 \leq m' \leq 4$. Then $g_K(m) \leq k \cdot g_K(4) + g_K(m')$. Therefore, it suffices to check the claimed inequality for $1 \leq m \leq 4$. In this case the inequality is exactly what part (*c*) of (3.4) yields for $n = 2$.

Question 3.7 Is $g_K(n) \leq p(K) + 2n$ for any field K and any $n \in \mathbb{N}$?

Recall that $p(K) = g_K(1) < g_K(2)$. On the level of natural examples, it seems to happen often that $g_K(2) = p(K) + 1$. In (7.16) we will give an example of a uniquely ordered field K such that $g_K(2) = p(K) + 2$.

4 The length

A quadratic form φ over K is called *positive-universal*, if $D_K(\varphi) = \sum K^2$. By definition, the *length* $\ell(K)$, if finite, is the smallest number $m \geq 1$ such that any totally positive form of dimension m over K is positive-universal.

Proposition 4.1 *One has* $p(K) \leq \ell(K)$. *Moreover,* $p(K) = 1$ *if and only if* $\ell(K) = 1$, *if and only if* $I_t K = 0$.

Proof This is obvious.

Proposition 4.2 *Let $m \in \mathbb{N}$.*

(*a*) $g_K(m) \leq m + \ell(K) - 1$.
(*b*) *If $m < \ell(K)$, then $g_K(m) \geq 2m$, otherwise $g_K(m) < 2m$.*
(*c*) *If $m \geq \ell(K) - 1$ then $g_K(m) = m + \ell(K) - 1$.*

Proof See [2, (2.10), (2.13), and (2.15)].

By (4.2) the length determines the final behavior of the Mordell function. To determine $g_K(m)$ for $m < \ell(K) - 1$ remains a problem, which is only solved in special cases, such as the following (see [2, (2.17)]):

Corollary 4.3 *If $p(K) = 2$ then $g_K(m) = 2m$ for all $m \in \mathbb{N}$ with $m < \ell(K)$.*

Proof If $p(K) = 2$, then $g_K(m) \leq m \cdot g_K(1) = 2m$, and by part (*b*) of (4.2), then the equality follows for $m < \ell(K)$.

Corollary 4.4 *One has $p(K) = \ell(K)$ if and only if $g_K(m) = m + \ell(K) - 1$ holds for all integers $m \geq 1$.*

Proof Since g_K is strictly increasing and by part (*a*) of (4.2), we have $p(K) - 1 = g_K(1) - 1 \leq g_K(m) - m \leq \ell(K) - 1$ for any $m \geq 1$. This shows one direction, and the other direction follows by setting $m = 1$.

Corollary 4.5 *If $\ell(K) \leq 3$, then $p(K) = \ell(K)$. Moreover, for $1 \leq r \leq 3$, the following are equivalent:*

(*i*) $\ell(K) = r$.
(*ii*) $g_K(m) = m + r - 1$ *for all integers $m \geq 1$.*
(*iii*) $g_K(m) = m + r - 1$ *for some integer $m \geq r$.*

Proof The first statement is clear from (4.1) for $\ell(K) \leq 2$. Assume that $\ell(K) = 3$. There exist $a, b \in \sum K^2$ with $1 \in D_K(\langle a, b, ab\rangle) \backslash D_K(\langle a, b\rangle)$. Then $\langle a, b, ab\rangle = \langle 1, c, c\rangle$ for some $c \in \sum K^2$. Since $\langle a, b\rangle$ does not represent 1, we have $\langle 1, c, c\rangle = \langle a, b, ab\rangle \neq \langle 1, ab, ab\rangle$. Therefore $\langle c, c\rangle \neq \langle ab, ab\rangle$ and thus $abc \in \sum K^2 \backslash D_K(2)$. In particular, $3 \leq p(K) \leq \ell(K) \leq 3$, so $p(K) = 3$. This shows that, if $\ell(K) \leq 3$, then $p(K) = \ell(K)$.

Using this fact, the implication $(i) \Rightarrow (ii)$ follows from (4.4). Since $(ii) \Rightarrow (iii)$ is obvious, it remains to show that $(iii) \Rightarrow (i)$. Assume now that (iii) holds. Let $m \geq r$ be such that $g_K(m) = m + r - 1$. Since g_K is strictly increasing, it follows that $g_K(r) - r \leq g_K(m) - m = r - 1$, thus $g_K(r) < 2r$ and therefore $\ell(K) \leq r$ by part (*b*) of (4.2). But $r - 1 = g_K(m) - m \leq \ell(K) - 1$ by part (*a*) of (4.2), so $r \leq \ell(K)$. This shows that $\ell(K) = r$.

Only the proof of $(i) \Rightarrow (ii)$ required the hypothesis that $1 \leq r \leq 3$. An example of a real field K with $\ell(K) = 3$ will be given in (7.3).

We next want to relate the length of a field to conditions on the vanishing of powers of the torsion ideal in the Witt ring.

Proposition 4.6 *One has $\ell(K) \leq 2$ if and only if $(I_t K)^2 = 0$.*

Proof With (2.3), both say that $\langle a, b\rangle = \langle 1, ab\rangle$ for any $a, b \in \sum K^2$.

Proposition 4.7 *Let $n \in \mathbb{N}$.*

(*a*) *If $\ell(K) < 2^{n+1}$, then $(I_t K)^{n+1} = 2 \cdot (I_t K)^n$.*
(*b*) *If $\ell(K) \leq 2^n$, then $(I_t K)^{n+1} = 0$.*

Proof (*a*) Assume that $\ell(K) < 2^{n+1}$. Then any totally positive $(n+1)$-fold Pfister form contains $\langle 1, 1\rangle$ and therefore is of the shape $2 \times \tau$ for an n-fold Pfister form τ, by [17, Chap. X, (1.5)]. Note that here τ is also totally positive. Using (2.2), it follows that $(I_t K)^{n+1} = 2(I_t K)^n$.

(*b*) Assume $\ell(K) \leq 2^n$. Let π be an arbitrary totally positive $(n+1)$-fold Pfister form over K. As $\dim(\pi) \geq \ell(K) + 2^n$ then π contains $2^n \times \langle 1\rangle$ and thus $\pi = 2^n \times \langle 1, a\rangle$ for some $a \in \sum K^2$. As $p(K) \leq \ell(K) \leq 2^n$, we have $a \in D_K(2^n)$, so $\pi \cong 2^{n+1} \times \langle 1\rangle$. By (2.3) this shows that $(I_t K)^{n+1} = 0$.

We now improve [2, (4.11)]. There $I_t^3 K = 0$ is assumed, a strictly stronger hypothesis than $(I_t K)^3 = 0$ as one can see from either of the examples $K = \mathbb{R}(X, Y)((t))$ and $K = \mathbb{Q}((t))$.

Theorem 4.8 *If K is not pythagorean and $(I_t K)^3 = 0$, then $\ell(K) \not\equiv 1 \bmod 4$.*

Proof Suppose that $\ell(K) \leq 4k + 1$ with $k \in \mathbb{N}$. Assuming that K is not pythagorean, we have $k > 0$. Assuming in addition that $(I_t K)^3 = 0$, we will show that any totally positive form of dimension $4k$ over K represents 1, hence proving that $\ell(K) \leq 4k$.

Let φ be an arbitrary totally positive quadratic form of dimension $4k$ over K. Let $d = d_\pm(\varphi) = \det(\varphi) \in \sum K^2$. Then $\varphi \perp \langle d\rangle$ is a totally positive form over K of dimension $4k + 1 = \ell(K)$, and it therefore represents 1. We thus have $\varphi \perp \langle d\rangle = \psi \perp \langle 1\rangle$ for some totally positive form ψ of dimension $4k$ over K. Comparing determinants, we obtain that $d_\pm(\psi) = 1$, whence $\psi \in I^2 K$. Now, since ψ is totally positive and $(I_t K)^3 = 0$, we obtain using (2.6) that $D_K(\psi) \subseteq \sum K^2 \subseteq G_K(\psi)$ and therefore $D_K(\psi) = \sum K^2$, by (2.5). In particular $\psi = \psi' \perp \langle d\rangle$ for some form ψ' over K. Then Witt cancellation yields $\varphi = \psi' \perp \langle 1\rangle$, whence $1 \in D_K(\varphi)$, as claimed.

Examples (7.9) and (7.11) will show that all integers $m \geq 1$ except those excluded by (4.8) are possible values for $\ell(K)$ for a field K with $(I_t K)^3 = 0$.

The following characterization of the length will be used in (5.6) below.

Lemma 4.9 *One has $\ell(K) = \inf(S)$ where S denotes the set of natural numbers m with the property that every totally positive form φ over K with $\dim(\varphi) \in \{m, m+1\}$ and $\det(\varphi) = 1$ is positive-universal.*

Proof It is clear that $\inf S \leq \ell(K)$. Assume now that $m = \inf S < \infty$. Let m' denote the odd integer among m and $m + 1$. We claim that any totally positive form φ of dimension m' over K is positive-universal. In fact, since $m' \in \{m, m+1\}$ and $m \in S$, this holds at least whenever $\det(\varphi) = 1$. Using now that m' is odd and scaling φ by its determinant, it follows that the claim holds in general. In other terms, we have shown that $\ell(K) \leq m'$. If m is odd, then this yields the statement, because $m' = m$ in this case.

Assume now that m is even. Let ψ be a totally positive form of dimension m over K. Denoting $d = \det(\psi)$ and $\varphi = \psi \perp \langle d\rangle$, we have $\dim(\varphi) = m + 1 = m' \geq \ell(K)$ and therefore $1 \in D_K(\varphi)$. We write $\varphi = \langle 1\rangle \perp \varphi'$ with a form φ' of dimension m over K. Then φ' is totally positive and $\det(\varphi') = \det(\varphi) = 1$, so that φ' is positive-universal over K and thus $\varphi' = \langle d\rangle \perp \psi'$ for some form ψ' of dimension $m - 1$ over K. Then $\psi \perp \langle d\rangle = \varphi = \psi' \perp \langle 1, d\rangle$, and Witt Cancellation shows that $1 \in D_K(\psi)$. This shows that $\ell(K) \leq m$.

5 Comparison of field invariants

We will compare the length of a real field with other invariants such as the u-invariant, the Hasse number, and the cardinality of the square class group.

Proposition 5.1 *If K is not pythagorean, then $\ell(K) \leq \widetilde{u}(K)$.*

Proof Let n be any positive integer with $n < \ell(K)$. Let φ be a totally positive form over K of dimension n that is not positive-universal. Choose $a \in \sum K^2 \backslash D_K(\varphi)$. Then $\varphi \perp \langle -a \rangle$ is anisotropic and totally indefinite, so $n + 1 = \dim(\varphi \perp \langle -a \rangle) \leq \widetilde{u}(K)$. This shows the claim.

Theorem 5.2 *For any real field K with $u(K) < \infty$, one has*

$$\ell(K) \quad \leq \quad g_K\left(\frac{u(K)}{2}\right) + 1 \quad \leq \quad \tfrac{1}{2}p(K)u(K) + 1.$$

Proof Since K is real and $u(K) < \infty$, we have $u(K) = 2m$ for some $m \in \mathbb{N}$. As $g_K(m) \leq m \cdot p(K)$ by (3.1), the second inequality is clear. Note that $u(K) < \infty$ implies that $p(K) < \infty$, whence $g_K(m) < \infty$. Let $n = g_K(m) + 1$. By part (b) of (4.2), to prove the first inequality it is enough to show that $g_K(n) < 2n$.

Let φ be a totally positive form of dimension n over K. Then $\varphi \perp -(n \times \langle 1 \rangle)$ is a torsion form of dimension $2n$ whose anisotropic part has dimension at most $u(K) = 2m$. This means that φ and $n \times \langle 1 \rangle$ have a common subform ψ of dimension $n - m$. We write $\varphi = \varphi' \perp \psi$ and $n \times \langle 1 \rangle = \psi \perp \psi'$ with quadratic forms φ' and ψ', both of dimension m. Then $\varphi \perp \psi' = \varphi' \perp n \times \langle 1 \rangle$. Since φ' is a totally positive form of dimension m, it is contained in the form $g_K(m) \times \langle 1 \rangle$, whence φ is contained in $(n + g_K(m)) \times \langle 1 \rangle$. This argument shows that $g_K(n) \leq n + g_K(m) < 2n$.

The following estimate was obtained independently by D. Hoffmann.

Corollary 5.3 *For any field K, one has $\ell(K) \leq \frac{1}{2}p(K)u(K) + 1$.*

Proof Assume that $u(K) < \infty$. If K is real then the claim follows from (5.2). If K is nonreal then $\ell(K) = u(K)$ and the bound is still correct.

In particular, (5.3) yields that, if $p(K) \leq 2$, then $\ell(K) \leq u(K) + 1$. In (7.13) we shall give an example where the estimates in (5.2) and (5.3) are sharp. We now give bounds for the length in terms of the u-invariant.

Proposition 5.4 *If K is not pythagorean and $\ell(K) \leq 5$ then $\ell(K) \leq u(K)$.*

Proof If $u(K) > 4$ there is nothing to show. If $u(K) = 4$, then $(I_t K)^3 = 0$ and $\ell(K) \neq 5$ by (4.8). If $u(K) = 2$, then $(I_t K)^2 \subseteq I_t^2 K = 0$, thus $\ell(K) \leq 2$. Finally, $u(K) \neq 3$, and $u(K) \leq 1$ holds if and only if K is pythagorean.

In (7.16) below we give an example of a uniquely ordered field K with $\ell(K) = 6$ and $u(K) = 4$. The next result improves and extends [2, (4.6)].

Proposition 5.5 *If $(I_t K)^{n+1} = 0$, then $\ell(K) \leq 2^{n-2}(u(K) + 2) + 1$.*

Proof For $n \leq 1$, this follows from (4.1) and (4.6). Assume now that $n \geq 2$ and $(I_t K)^{n+1} = 0$. If K is nonreal, then $\ell(K) = u(K) \leq 2^{n-2}(u(K)+2)+1$. So, we may assume that K is real and, moreover, that $u(K) < \infty$, whence $u(K) = 2m$ for some $m \in \mathbb{N}$. Then (5.2) and part (*a*) of (3.4) yield that

$$\ell(K) \leq g_K(m) + 1 \leq 2^{n-1}(m+1) + 1 = 2^{n-2}(u(K)+2) + 1.$$

Corollary 5.6 *Assume that* $(I_t K)^3 = 0$. *Then* $\ell(K) \leq u(K) + 2$. *Moreover, if* K *is not pythagorean and* $u(K) \equiv 0 \bmod 8$, *then* $\ell(K) \leq u(K)$.

Proof As before, we may assume that K is real with $u(K) < \infty$, thus $u(K) = 2m$ for some $m \in \mathbb{N}$. By (5.5) we have $\ell(K) \leq u(K)+3 = 2m+3$. If m is odd, then $2m+3 \equiv 1 \bmod 4$, thus $\ell(K) \neq 2m+3$, by (4.8). Assume now that m is even. By (4.9), as we know that $\ell(K) \leq 2m+3$, in order to show that $\ell(K) \leq 2m+2$, it suffices to prove that any totally positive form φ over K with $\dim(\varphi) = 2m+2$ and $\det(\varphi) = 1$ represents 1. Given such a form φ, [2, (4.1)] yields that $\varphi = \psi \perp \langle a, a\rangle$ with a totally positive form ψ over K and $a \in \sum K^2$. Then $\psi \in I^2 K$ and $\dim(\psi) = 2m$ is divisible by 4, whence (2.6) and (2.5) yield that $\sum K^2 = D_K(\psi) = D_K(\varphi)$. So we have shown that $\ell(K) \leq 2m+2 = u(K)+2$ in either case. Suppose finally that K is not pythagorean and $u(K) \equiv 0 \bmod 8$. Then $m \equiv 0 \bmod 4$, and (5.2) and (3.6) together yield that $\ell(K) \leq g_K(m)+1 \leq 2m+1$. Since $\ell(K) \not\equiv 1 \bmod 4$ by (4.8), we then obtain $\ell(K) \leq 2m = u(K)$.

With the sole hypothesis that $(I_t K)^3 = 0$, the bound $\ell(K) \leq u(K) + 2$ is best possible, as (7.16) will show.

Corollary 5.7 *If* $(I_t K)^3 = 0$ *and* $p(K) = 2$, *then* $\ell(K) \leq u(K)$. *Moreover, neither of the two invariants is an odd number.*

Proof If K is real, then $u(K)$ is either even or infinite, and if K is nonreal, then $\ell(K) = u(K)$. Observe now that, with $p(K) = 2$, (5.3) yields the estimate $\ell(K) \leq u(K) + 1$. It therefore remains only to show that the assumptions imply that also $\ell(K)$ is either infinite or an even number. To this aim, we suppose that $\ell(K) \leq 2m+1$ for some $m \in \mathbb{N}$ and want to deduce that $\ell(K) \leq 2m$. Let φ be a totally positive form of dimension $2m$ over K, and let $d = \det(\varphi)$. Then $\varphi \perp \langle d\rangle$ is a totally positive form of dimension $2m+1 \geq \ell(K)$ and of determinant 1. Hence $\varphi \perp \langle d\rangle = \psi \perp \langle 1\rangle$ for some totally positive form ψ over K of dimension $2m$ and of determinant 1. Since $p(K) = 2$ and $(I_t K)^3 = 0$, we obtain $D_K(\psi) \subseteq \sum K^2 = D_K(2) \subseteq G_K(\psi)$ using (2.6), and thus $D_K(\psi) = G_K(\psi) = \sum K^2$ by (2.5). Hence, $d \in D_K(\psi)$ and $\psi = \psi' \perp \langle d\rangle$ for some form ψ' over K. Then $\varphi \perp \langle d\rangle = \psi' \perp \langle 1, d\rangle$ and cancellation yields $1 \in D_K(\varphi)$. Therefore $\ell(K) \leq 2m$.

Proposition 5.8 *Assume that* K *is not quadratically closed and that any torsion form over* K *has a diagonalization with entries in* $\pm \sum K^2$. *Then* $u(K) \leq 2\ell(K) - 2$.

Proof If K is nonreal, then $\ell(K) > 1$ as $K^\times \neq K^{\times 2}$, so $u(K) = \ell(K) \leq 2\ell(K) - 2$. Assume now that K is real. Let φ be an anisotropic torsion form over K. By the hypothesis, there exist $a_1, b_1, \ldots, a_m, b_m \in \sum K^2$ such that $\varphi = \langle a_1, \ldots, a_m, -b_1, \ldots, -b_m\rangle$. Then the totally positive form $\langle a_1, \ldots, a_m\rangle$ over K does not represent b_1, whence $\ell(K) > m = \frac{1}{2}\dim(\varphi)$. This shows that $\ell(K) \geq \frac{1}{2}u(K) + 1$.

In (7.14) below we give an example where the bound in (5.8) is sharp.

The next statements are concerned with *ED*-fields. We refer to [22] for the definitions of effective diagonalization of a quadratic form and of the *ED*-property for fields. Note that *ED*-fields satisfy the second condition in (5.8). The following is a simplified presentation of [13, (2.16)].

Theorem 5.9 *Assume that K is an ED-field and not pythagorean. Then*

$$\tfrac{1}{2}\widetilde{u}(K)+1 \le \ell(K) \le \widetilde{u}(K).$$

Proof By (5.1) the second inequality is clear. Let $m \geq 2$ be an integer such that there exists an anisotropic totally indefinite form φ of dimension m over K. Since K is an *ED*-field it follows that there exists $a \in K^\times$ such that the signature of $a\varphi$ with respect to any ordering of K is an integer between 0 and $m-2$. (If $\langle a_1,\dots,a_m\rangle$ is an effective diagonalization of φ, one may choose $a = a_r$ for $r = \lceil \frac{m}{2} \rceil$.) Replacing φ by $a\varphi$, we may as well assume that $a = 1$. Effectively diagonalizing φ over K leads to a decomposition $\varphi = \psi \perp \vartheta$ where ψ and ϑ are forms over K such that ψ is totally positive, $\dim(\psi) \geq \dim(\vartheta)$, and ϑ represents an element $-s$ with $s \in \sum K^2$. Then $\psi \perp \langle -s\rangle$ is a subform of φ and thus anisotropic, and therefore $\ell(K) \geq \dim(\psi)+1 \geq \frac{m}{2}+1$. This argument and the definition of $\widetilde{u}(K)$ yield that $\ell(K) \geq \frac{1}{2}\widetilde{u}(K)+1$.

Examples (7.14) and (7.16) will show that equality can occur in either of the two bounds in (5.9). We now retrieve the characterizations of fields with finite Hasse number from [10, (2.5)] and [13, (2.17)].

Corollary 5.10 *For the field K the following are equivalent:*

- (i) *K is an ED-field with $u(K) < \infty$.*
- (ii) *K is an ED-field with $\ell(K) < \infty$.*
- (iii) $\widetilde{u}(K) < \infty$.

Proof If $\widetilde{u}(K) < \infty$, then K is an *ED*-field by [22, (2.5)], and $u(K) \leq \widetilde{u}(K) < \infty$. This shows $(iii \Rightarrow i)$. The implication $(i \Rightarrow ii)$ follows readily from (5.3) and (2.8). If K is not pythagorean, then (5.9) yields that $(ii \Rightarrow iii)$. Finally, assume that K is pythagorean and that (ii) holds. Then the *ED*-property implies that any totally indefinite form of dimension at least two over K contains a subform $\langle a, -b\rangle$ with $a, b \in \sum K^2$, and as K is pythagorean this form is isotropic. This shows that $\widetilde{u}(K) \leq 1$. Hence, $(ii \Rightarrow iii)$ also holds in the case when K is pythagorean.

Corollary 5.11 *Assume that $\widetilde{u}(K) = 6$. Then $u(K), \ell(K) \in \{4, 6\}$, and if K is uniquely ordered, then* $\max\{u(K), \ell(K)\} = 6$.

Proof Obviously, $\max\{u(K), \ell(K)\} \leq \widetilde{u}(K) = 6$. Note that $u(K) \neq 5$. Moreover, since $u(K) < 8$ we have $(I_tK)^3 = I_t^3K = 0$ by (2.7) and thus $\ell(K) \neq 5$ by (4.8). From (5.9) we obtain that $\ell(K) \geq 4$. If $\ell(K) = 6$, then $u(K) \geq 4$ by (5.6). If $\ell(K) \leq 5$, then $\ell(K) \leq u(K)$ by (5.4). This altogether shows that $u(K), \ell(K) \in \{4, 6\}$.

Assume that K is uniquely ordered. Let φ be a 6-dimensional anisotropic indefinite form over K. Then $\varphi = \varphi_1 \perp -\varphi_2$ for two nontrivial positive forms φ_1 and φ_2 over K with $D_K(\varphi_1) \cap D_K(\varphi_2) = \emptyset$. Replacing φ by $-\varphi$ if necessary, we may assume that $\dim(\varphi_1) \geq \dim(\varphi_2)$. If φ is torsion we have $u(K) = \dim(\varphi) = 6$. If φ is not torsion, then $\dim(\varphi_1) \geq 4$, so $\ell(K) > 4$ and therefore $\ell(K) = 6$.

In (7.14) and (7.16) we give examples of uniquely ordered fields K, one with $u(K) = \widetilde{u}(K) = 6$ and $\ell(K) = 4$, the other one with $u(K) = 4$ and $\ell(K) = \widetilde{u}(K) = 6$. We now generalize an observation due to M. Kneser [17, Chap. XI, (6.5) and (6.6)].

Lemma 5.12 *Let φ be a totally positive quadratic form over K. If φ is not positive-universal, then $D_K(\varphi) \subsetneq D_K(\varphi \perp \langle a \rangle)$ for any $a \in \sum K^2$.*

Proof The proof is straightforward and can be found in [22, (4.1)].

Theorem 5.13 *One has $\ell(K) \leq |\sum K^2/K^{\times 2}|$.*

Proof Let $n < \ell(K)$. Let φ be a totally positive form of dimension n that is not positive-universal. Choose a chain of subforms $\varphi_1 \subsetneq \varphi_2 \subsetneq \cdots \subsetneq \varphi_n = \varphi$ where $\dim(\varphi_i) = i$ for $1 \leq i \leq n$. As $\varphi_1, \ldots, \varphi_n$ are totally positive and not positive-universal, we obtain $D_K(\varphi_1) \subsetneq D_K(\varphi_2) \subsetneq \cdots \subsetneq D_K(\varphi_n) \subsetneq \sum K^2$ from (5.12), whence $|\sum K^2/K^{\times 2}| > n$.

If K is a nonreal field, then $\sum K^2 = K^\times$ and $\ell(K) = u(K)$, so in this case we retrieve Kneser's statement. Since (5.12) only holds for totally positive forms, there is no adequate version of Kneser's statement on the u-invariant that would cover real fields in general. However, it was shown in [22, (4.2)] that, if K is an *ED*-field, then $u(K) \leq |\sum K^2/K^{\times 2}|$.

6 Quadratic extensions

There are upper bounds on the u-invariant and the length of K in terms of $u(K(i))$. Note that, if K is an extension of transcendence degree n of a real closed field, then $u(K(i)) \leq 2^n$ by Tsen-Lang Theory. In [6] it is shown that, if $u(K(i)) \leq 2^n$ with $n \geq 2$, then $u(K) \leq 2^{n+2} - 2n - 6$, improving for $n \geq 3$ a bound from [9]. The following improvement of a similar bound in [2] on the length was pointed out to us by C. Scheiderer.

Theorem 6.1 *Let n be a positive integer such that $u(K(i)) \leq 2^n$. Then*

$$\ell(K) \leq 2^n(2^{n-1} - 1) + 2.$$

Proof The hypothesis implies that any totally positive form φ over K with $\dim(\varphi) > 2^n$ has a subform $\langle b, b \rangle$ where $b \in \sum K^2$. Applying [2, (4.3)] with $m = 2^n$, $t = n - 1$, and $a = 1$ yields that any totally positive form φ over K with $\dim(\varphi) > 2^{n-1}(2^n - 1)$ has a subform $2^{n-1} \times \langle b, bc \rangle$ with $b, c \in \sum K^2$. As $u(K(i)) \leq 2^n$, it follows with [17, Chap. XI, (4.9)] that $2^{n-1} \times \langle b, bc \rangle = 2^{n-1} \times \langle 1, c \rangle$ for any $b, c \in \sum K^2$. Hence, every totally positive form φ over K with $\dim(\varphi) \geq 2^{n-1}(2^n - 1) + 1$ contains $2^{n-1} \times \langle 1 \rangle$. It follows that every totally positive form ψ over K with $\dim(\psi) \geq 2^{n-1}(2^n - 1) - 2^{n-1} + 2$ represents 1, whence $\ell(K) \leq 2^{n-1}(2^n - 1) - 2^{n-1} + 2 = 2^n(2^{n-1} - 1) + 2$.

Remark 6.2 For $n \geq 3$, (6.1) improves the bound obtained from [2, (4.9)]. However, if K is nonreal and $n \geq 2$, then $\ell(K) = u(K) \leq 2^{n+2} - 2n - 6$ by [6], which for $n \geq 3$ is better than the bound in (6.1).

Proposition 6.3 *Assume that $p(K) \leq 2$. Then $\ell(K) \leq u(K(i))$.*

Proof If K is nonreal, then $\ell(K) = u(K)$, and with $p(K) \leq 2$ one has $u(K) \leq u(K(i))$ by [4, (3.5)]. Assume now that K is real and $u(K(i)) < \infty$. Set $m = u(K(i))$ and let φ be a totally positive form of dimension $m + 1$ over K. Then φ is anisotropic over K, whereas $\varphi_{K(i)}$ is isotropic. So φ contains a subform $\langle a, a\rangle$ with $a \in K^\times$. Now, $a \in D_K(\varphi) \subseteq \sum K^2 = D_K(2)$ yields that $\langle a, a\rangle = \langle 1, 1\rangle$. This shows that every $(m+1)$-dimensional totally positive form over K contains $\langle 1, 1\rangle$, so any m-dimensional totally positive form over K represents 1. Therefore $\ell(K) \le m = u(K(i))$.

With the assumption that $p(K) = 2$, we have $u(K) \le u(K(i))$ by [4, (3.5)], $\ell(K) \le u(K(i))$ by (6.3), and $\ell(K) \le u(K) + 1$ by (5.3).

In [5, (4.4)] it was shown that, if $I_t^3 K = 0$, then $u(K) \le u(K(i)) + 2$. We shall now obtain a similar bound for the length. Note that the result in [5, (4.4)] combined with (5.6) gives $\ell(K) \le u(K) + 2 \le u(K(i)) + 4$, while the bound we obtain now is actually better.

Proposition 6.4 *Assume that $(I_t K)^3 = 0$. Then $\ell(K) \le u(K(i)) + 2$.*

Proof Let $m = u(K(i))$ and assume that m is finite. We may assume that $m \ge 2$, since otherwise K is pythagorean and thus $\ell(K) = 1$. Let φ be a totally positive form of dimension $m + 2$ over K. Note that $(\varphi \perp \langle 1\rangle)_{K(i)}$ is isotropic of Witt index at least 2.

We claim that the form $\varphi \perp \langle 1\rangle$ over K admits a subform $\langle a, a, b, b\rangle$ with $a, b \in K^\times$. In fact, this is a standard argument if $\varphi \perp \langle 1\rangle$ is anisotropic. Assume that $\varphi \perp \langle 1\rangle$ is isotropic. Then $\varphi \perp \langle 1\rangle = \psi \perp \langle 1, -1\rangle$ with a form ψ over K with $\dim(\psi) = m + 1 \ge 3$. If ψ is isotropic, then it contains $\langle 1, -1\rangle$, so we may choose $a = 1$ and $b = -1$. If ψ is anisotropic then, as $\psi_{K(i)}$ is isotropic, it follows that ψ contains a form $\langle a, a, b\rangle$ with $a, b \in K^\times$, and as $\langle 1, -1\rangle$ represents b, then $\varphi \perp \langle 1\rangle = \psi \perp \langle 1, -1\rangle$ contains $\langle a, a, b, b\rangle$.

Now, we have $a, b \in D_K(\varphi \perp \langle 1\rangle) \subseteq \sum K^2$, and since $(I_t K)^3 = 0$, it follows from (2.3) that $\langle a, a, b, b\rangle = \langle 1, 1, ab, ab\rangle$. Hence, φ contains $\langle 1, ab, ab\rangle$ and thus represents 1. Therefore $\ell(K) \le m + 2$.

By (7.15) and (7.16), the bound given in (6.4) is best possible.

Example 6.5 Let K be an extension of transcendence degree 2 of a real closed field. Then $I_t^3 K = 0$ and $u(K(i)) \le 4$, because $K(i)$ is a C_2-field. Therefore $\ell(K) \le 6$ by (6.4) and $\ell(K) \ne 5$ by (4.8). In fact, this is the same bound as obtained for $n = 2$ in (6.1). Note that $u(K) \le 6$ by [8, (4.11)].

We now show that $\ell(\mathbb{R}((X, Y))) = 4$, correcting a statement in [2, Sect. 3].

Example 6.6 Let $K = \mathbb{R}((X, Y))$. Then $K(i) = \mathbb{C}((X, Y))$. We have $p(K) = 2$ and $u(K(i)) = 4$ by [7, (5.14) and (5.16)]. Using (6.3) this allows to conclude that $\ell(K) \le 4$. Note that $K(i)$ is the quotient field of the unique factorization domain $\mathbb{C}[[X, Y]]$. Let v denote the $\mathbb{Z}$-valuation on $K(i)$ associated to the irreducible element $X - iY$ in $\mathbb{C}[[X, Y]]$. The residue field of v is $\mathbb{C}((\overline{X}))$. Let $a = X^2 + Y^2$ and $b = X^2(X+1)^2 + Y^2(X-1)^2 = (X^2 + Y^2)(X^2 + 1) + 2X(X^2 - Y^2)$. Then b is a unit with respect to v and its residue is the element $4\overline{X}^3$ in $\mathbb{C}((\overline{X}))$. Both residue forms of $\langle 1, a, b, ab\rangle$ over $K(i)$ with respect to the valuation v and the uniformizing element a are equal to $\langle 1, 4\overline{X}^3\rangle$. Since $-4\overline{X}^3$ is not a square in $\mathbb{C}((\overline{X}))$, it follows that the form $\langle 1, a, b, ab\rangle = \langle -1, a, b, ab\rangle$ is anisotropic over $K(i)$. Hence, 1 is not represented by the totally positive form $\langle a, b, ab\rangle$ over K, in particular $\ell(K) \ge 4$. This shows that $\ell(K) = 4$.

Furthermore, for the subfield $L = \mathbb{R}((X))(Y)$ of K, we also obtain that $\ell(L) = 4$. In fact, $p(L) = 2$ and $u(L(i)) = 4$, thus $\ell(L) \le 4$ by (6.3), and equality follows as $a, b \in \sum L^2$ and $1 \notin D_K(\langle a, b, ab\rangle) \supseteq D_L(\langle a, b, ab\rangle)$.

7 Construction of examples

In order to obtain examples of fields where the invariants discussed in this article take certain prescribed values, we use a classical technique going back to Merkurjev's u-invariant construction. We begin by recalling from [3] a reformulation of this into a general construction principle.

Theorem 7.1 *Let k be a field of characteristic different from 2 and C a class of field extensions of k such that the following conditions are satisfied:*

(i) C is closed under direct limits,
(ii) if K'/k is a subextension of an extension $K/k \in C$, then $K'/k \in C$,
(iii) $k/k \in C$.

Then there exists a field extension $K/k \in C$ such that $K(\varphi)/k \notin C$ for any anisotropic quadratic form φ over K of dimension at least 2. Moreover, if k is infinite, then K may be chosen to have the same cardinality as k.

Proof As in Merkurjev's original construction the idea is to start with $K_0 = k$ and then recursively define K_{i+1} as a composite of certain function fields of quadratic forms over K_i, for each $i \in \mathbb{N}$, and finally, to choose $K = \bigcup_{i\in\mathbb{N}} K_i$. For the details see [3, (6.1)].

Using this principle to produce examples of fields with prescribed invariants requires, on the one hand, a good choice for a starting field k along with a class C of field extensions of k, and on the other hand, special results to derive conclusions about the extension K/k thus obtained by (7.1). The results on function fields of quadratic forms that we will use in the sequel are classical and contained in [17, Chap. X, Sect. 4].

We say that a quadratic form φ over K is *partially indefinite* if it is indefinite with respect to some ordering P of K. Note that this is equivalent to saying that $\dim(\varphi) \geq 2$ and the function field $K(\varphi)$ is real. Recall that φ is *totally indefinite* if it is indefinite with respect to all orderings of K; if K is uniquely ordered we omit 'partially' or 'totally' and just say 'indefinite'.

Remark 7.2 Let K be a real field. If there is an upper bound on the dimensions of anisotropic partially indefinite forms over K, then K is uniquely ordered. In fact, if K is not uniquely ordered, then there is $a \in K^\times \setminus \pm \sum K^2$ and this gives rise to many anisotropic partially indefinite forms over K such as $n \times \langle 1\rangle \perp \langle a\rangle$ or $2^n \times \langle 1, a\rangle$ for arbitrary $n \in \mathbb{N}$.

We now give examples of fields of length 3 and 5.

Example 7.3 Let k be a real field with $p(k) \geq 3$ and fix $a \in D_k(3)\setminus D_k(2)$. Consider the class of real field extensions K/k such that $a \notin D_K(2)$. Using (7.1) it follows that there exists a real extension K/k with $a \notin D_K(2)$ and such that, for any anisotropic partially indefinite form φ over K, we have $a \in D_{K(\varphi)}(2)$, so $\langle 1, 1, -a, -a\rangle$ becomes hyperbolic over $K(\varphi)$. With [17, Chap. X, (4.5)] we conclude that any anisotropic partially indefinite form over K is similar to a subform of $\langle 1, 1, -a, -a\rangle$.

Therefore $p(K) \leq \ell(K) \leq 3, u(K) \leq \widetilde{u}(K) \leq 4, I_t^3 K = 0$, and K is uniquely ordered by (7.2). As $a \in D_K(3)\setminus D_K(2)$ we get $p(K) = \ell(K) = 3$ and $u(K) = \widetilde{u}(K) = 4$. Moreover, we have that $\sum K^2 = D_K(2) \cup aD_K(2)$; in fact, $\langle 1, 1, -a, -a\rangle$ is the only anisotropic indefinite twofold Pfister form over K, but for $b \in K^\times\setminus D_K(2) \cup aD_K(2)$ the form $\langle 1, 1, -b, -b\rangle$ is anisotropic and different from $\langle 1, 1, -a, -a\rangle$, so it is totally definite, whence $b \in -\sum K^2$.

Example 7.4 Using (7.1) in a similar way as in (7.3) above, we find a real field K with an element $a \in \sum K^2 \backslash D_K(4)$ such that $a \in D_{K(\varphi)}(4)$ for every anisotropic partially indefinite form φ over K. Adapting the arguments from (7.3) we obtain the following: The Pfister form $4 \times \langle 1, -a\rangle$ over K is anisotropic and contains up to similarity every anisotropic partially indefinite form over K. Furthermore, K is uniquely ordered, $\sum K^2 = D_K(4) \cup aD_K(4)$, $u(K) = \tilde{u}(K) = 8$ and $I_t^4 K = 0$. Now, every positive definite 5-dimensional form ψ over K represents 1, because $\psi \perp \langle -1\rangle$ is indefinite and not similar to a subform of $4 \times \langle 1, -a\rangle$, by a signature argument. Hence $p(K) \leq \ell(K) \leq 5$. As $a \in D_K(5)\backslash D_K(4)$ we get $p(K) = \ell(K) = 5$.

We now prove the existence of fields with an arbitrary prescribed length, equal to the Pythagoras number, enhancing the result in [11]. The idea is due to D. Hoffmann.

Lemma 7.5 *Let $m \in \mathbb{N}$ and let φ be a form over K with $\dim(\varphi) = m+2$ and $\operatorname{sign}_P(\varphi) = m$ for some ordering P on K. Then for any anisotropic form ϑ over K with $\dim(\vartheta) \leq m + 1$, the form $\vartheta_{K(\varphi)}$ is anisotropic.*

Proof For an anisotropic form ψ over K, let $i(\psi)$ denote the first Witt index of ψ, i.e. the Witt index of $\psi_{K(\psi)}$. The hypothesis implies that φ is indefinite with respect to P, so P extends to an ordering P' on $K(\varphi)$. Since $\operatorname{sign}_{P'}(\varphi_{K(\varphi)}) = \operatorname{sign}_P(\varphi) = m$, it is clear that $i(\varphi) = 1$. Let ϑ be any anisotropic form over K such that $\vartheta_{K(\varphi)}$ is isotropic. By [15, (4.1)], then $\dim(\vartheta) - i(\vartheta) \geq \dim(\varphi) - i(\varphi) = m + 1$ and therefore $\dim(\vartheta) \geq m + 2$.

Proposition 7.6 *Let $m \in \mathbb{N}$. Assume that K is a real field and there is an element $s \in \sum K^2 \backslash D_K(m)$ such that $s \in D_{K(\varphi)}(m)$ for any anisotropic partially indefinite form φ over K with $\dim(\varphi) = m + 2$. Then K is uniquely ordered and $\ell(K) = p(K) = m + 1$.*

Proof We obviously have $m+1 \leq p(K) \leq \ell(K)$. Using (7.5) with $\vartheta = m \times \langle 1\rangle \perp \langle -s\rangle$, we obtain from the hypothesis that any form φ over K with $\dim(\varphi) = m+2$ and $\operatorname{sign}_P(\varphi) = m$ for some ordering P of K is isotropic. Thus K is uniquely ordered by (7.2), and every totally positive form over K of dimension $m + 1$ represents 1, so that $\ell(K) = m + 1$.

Example 7.7 Let $m \in \mathbb{N}$. Let k_0 be a real field, $k = k_0(X_1, \ldots, X_m)$, and $s = 1 + X_1^2 + \cdots + X_m^2$. Then $s \in \sum K^2 \backslash D_k(m)$. The class of real field extensions K of k such that $s \notin D_K(m)$ satisfies the hypotheses in (7.1). Therefore there exists a field extension K/k as in (7.6), and then K is uniquely ordered with $\ell(K) = p(K) = m + 1$.

For the following examples we need the Clifford algebra of a quadratic form; we refer to [17, Chap. V] and [20, Chap. 9]. Given a form φ over K we denote by $C(\varphi)$ its Clifford algebra and by $C_0(\varphi)$ the even part of $C(\varphi)$. If φ has even dimension then $C(\varphi)$ is a central simple algebra over K. For a central simple algebra A we denote by $\operatorname{ind}(A)$ its Schur index. If φ has even dimension $2m > 0$, then $\operatorname{ind}(C(\varphi)) \leq 2^{m-1}$, and equality is only possible in case φ is anisotropic; moreover, if $\varphi \in I^3 K$, then $\operatorname{ind}(C(\varphi)) = 1$.

We now construct fields K with $I_t^3 K = 0$ showing that all values except those excluded by (4.8) are possible for $\ell(K)$. In [12, Sect. 3], Hornix has constructed similar examples, but our examples satisfy additional properties.

Proposition 7.8 *Let m be a positive integer. Assume that K is real and φ is a totally positive form over K such that $\dim(\varphi) = 2m+2$ and $\det(\varphi) = 1$, and such that $\operatorname{ind}(C(\varphi_{K(i)})) = 2^m$, whereas $\operatorname{ind}(C(\varphi_{K(i)(\psi)})) < 2^m$ for any anisotropic partially indefinite form ψ over K. Then K is uniquely ordered, $I_t^3 K = 0$, $p(K) = 2$, and $\ell(K) = u(K) = \tilde{u}(K) = 2m + 2$.*

Proof It follows from Merkurjev's index reduction criterion [21, Théorème 1] that, if ψ is a form over K with $\mathrm{ind}(C(\varphi_{K(i)(\psi)})) < 2^m = \mathrm{ind}(C(\varphi_{K(i)}))$, then $\dim(\psi) \leq 2m+2$ and ψ is not contained in I^3K. Thus by the hypothesis we have $\widetilde{u}(K) \leq 2m+2$ and there is no anisotropic partially indefinite form in I^3K. In particular, $I_t^3K = 0$ and by (7.2) K is uniquely ordered.

Given $s \in \sum K^2$, the indefinite form $\psi = \langle 1, 1, -s\rangle$ over K becomes isotropic over $K(i)$, in particular $\mathrm{ind}(C(\varphi_{K(i)(\psi)})) = \mathrm{ind}(C(\varphi_{K(i)}))$, so the hypothesis implies that ψ is isotropic over K. Therefore $p(K) \leq 2$.

We write $\varphi = \varphi' \perp \langle a\rangle$ with some $a \in \sum K^2$. Since $\mathrm{ind}(C(\varphi_{K(i)})) = 2^m$, the form $\varphi_{K(i)}$ is anisotropic, so that φ does not contain the form $\langle a, a\rangle$ over K. It follows that φ' is a totally positive form that does not represent a, whence $\ell(K) > \dim(\varphi') = 2m+1$ and therefore $\ell(K) \geq 2m+2$. As $\ell(K) > 1$, K is not pythagorean, so we have $p(K) = 2$. As $\ell(K) \leq \widetilde{u}(K)$ by (5.1), it follows that $\ell(K) = \widetilde{u}(K) = 2m+2$. Now (5.3) yields that $2m+2 = \ell(K) \leq u(K)+1$. Hence $2m+1 \leq u(K) \leq \widetilde{u}(K) = 2m+2$. As $u(K)$ is even we conclude that $u(K) = \widetilde{u}(K) = 2m+2$.

Example 7.9 Let $m \in \mathbb{N}$, k_0 a real field, and $k = k_0(X_1, \ldots, X_{2m})$. Let φ' be the $(2m+1)$-dimensional form over k with $\det(\varphi') = 1$ whose first $2m$ entries are $1+X_i^2$ $(1 \leq i \leq 2m)$. Then $\varphi = \varphi' \perp \langle 1\rangle$ is totally positive over k and $\mathrm{ind}(C(\varphi_{k(i)})) = 2^m$. By (7.1), there exists a real field extension K/k such that $\mathrm{ind}(C(\varphi_{K(i)})) = 2^m$, whereas $\mathrm{ind}((C(\varphi_{K(i)(\psi)})) < 2^m$ for any anisotropic partially indefinite form ψ over K. By (7.8) then K is uniquely ordered, $I_t^3K = 0$, $p(K) = 2$, and $\widetilde{u}(K) = \ell(K) = u(K) = 2m+2$.

Proposition 7.10 *Let $m \in \mathbb{N}$. Let K be real, φ' a totally positive form over K with* $\dim(\varphi') = 4m+2$ *and* $\det(\varphi') = 1$, *and let* $\varphi = \varphi' \perp \langle -1, -1\rangle$. *Assume that* $\mathrm{ind}(C(\varphi)) = 2^{2m+1}$, *whereas* $\mathrm{ind}(C(\varphi_{K(\psi)})) < 2^{2m+1}$ *for any anisotropic partially indefinite form ψ over K. Then K is uniquely ordered, $I_t^3K = 0$, $\ell(K) = 4m+3$, and $\widetilde{u}(K) = 4m+4$. Furthermore, $u(K)$ equals either $4m+2$ or $4m+4$, and the latter value is only possible if m is even.*

Proof Since $\mathrm{ind}(C(\varphi)) = 2^{2m+1}$, there is a central division algebra D over K of degree 2^{2m+1} such that $C(\varphi) \cong M_2(D)$. By Merkurjev's index reduction criterion [21, Théorème 1], given a form ψ over K, $D \otimes_K K(\psi)$ has nontrivial zero-divisors if and only if D contains a homomorphic image of $C_0(\psi)$. If ψ is a threefold Pfister form, then $C_0(\psi)$ contains zero-divisors and thus cannot be contained in D. Therefore every anisotropic threefold Pfister form over K is totally positive. Using (7.2) it follows that K is uniquely ordered and $I_t^3K = 0$.

Let ψ be an arbitrary anisotropic indefinite form over K. By the hypothesis, D contains a homomorphic image of $C_0(\psi)$. This implies in particular that $\dim(\psi) \leq 4m+4$. Since $\dim(\varphi) = 4m+4$ and $\mathrm{ind}(C(\varphi)) = 2^{2m+1}$, the indefinite form φ is anisotropic, so $\widetilde{u}(K) = \dim(\varphi) = 4m+4$.

Moreover, if $\dim(\psi) = 4m+4$, then the homomorphism $C_0(\psi) \longrightarrow D$ cannot be injective and thus $C_0(\psi)$ cannot be simple, hence ψ has trivial discriminant. In particular, ψ cannot have signature $4m+2$. Therefore every totally positive form of dimension $4m+3$ over K represents 1, so that $\ell(K) \leq 4m+3$. But as φ is anisotropic, φ' does not represent any sum of two squares, which shows that $\ell(K) > \dim(\varphi') = 4m+2$. Hence $\ell(K) = 4m+3$.

Using (5.6) we obtain that $4m+1 = \ell(K) - 2 \leq u(K) \leq \widetilde{u}(K) \leq 4m+4$. Since $u(K)$ is even it is either $4m+2$ or $4m+4$. Suppose that $u(K) = 4m+4$. Then there exists an anisotropic torsion form τ of dimension $4m+4$ over K. The above argument shows that τ has trivial discriminant and D contains a homomorphic image of $C_0(\tau)$. Then $\tau \in I^2K$ and therefore $C(\tau) \cong M_2(D) \cong C(\varphi)$. As τ becomes hyperbolic over the real closure of K, it follows that $4m = \mathrm{sign}(\varphi)$ is divisible by 8, so m is even.

Example 7.11 Let $m \in \mathbb{N}$, k_0 a real field, and $k = k_0(X_1, \dots, X_{4m+1})$. Let φ' be the $(4m+2)$-dimensional form over k with $\det(\varphi') = 1$ and whose first $4m+1$ diagonal entries are $1+X_j^2$ $(1 \le j \le 4m+1)$. Put $\varphi = \varphi' \perp \langle -1, -1 \rangle$. Note that $\operatorname{ind}(C(\varphi_{k(i)})) = 2^{2m+1}$. Using (7.1), there exists a real field extension K/k such that $\operatorname{ind}(C(\varphi_K)) = 2^{2m+1}$, whereas for any anisotropic partially indefinite form ψ over K one has $\operatorname{ind}((C(\varphi_{K(\psi)})) < 2^{2m+1}$. By (7.10) then K is uniquely ordered $I_t^3 K = 0$, $\widetilde{u}(K) = 4m+4$, and $\ell(K) = 4m+3$; moreover $u(K) = 4m+2$ if m is odd, otherwise $u(K) = 4m+2$ or $u(K) = 4m+4$.

We now give an example where the bounds in (5.2) and (5.3) are sharp, elaborating an idea suggested to us by D. Hoffmann.

Following [14], a 9-dimensional form φ over K is said to be *essential* if it is anisotropic and not a Pfister neighbor and if $\operatorname{ind}(C_0(\varphi)) \ge 4$.

Proposition 7.12 *Let K be a real field and ψ a totally positive form over K with* $\dim(\psi) = 8$ *and such that $(\psi \perp \langle -1 \rangle)_{K(i)}$ is essential. Assume further that, for any anisotropic and partially indefinite form φ over K, $(\psi \perp \langle -1 \rangle)_{K(i)(\varphi)}$ is not essential. Then K is uniquely ordered, $p(K) = 2$, $u(K) = 8$, and $\ell(K) = \widetilde{u}(K) = 9$.*

Proof Let $s \in \sum K^2$. We put $\varphi = \langle 1, 1, -s \rangle$. Then $\varphi_{K(i)}$ is isotropic, so $K(i)(\varphi)$ is a rational function field over $K(i)$, and this allows to conclude that the extended form $(\psi \perp \langle -1 \rangle)_{K(i)(\varphi)}$ is still essential. Since φ is totally indefinite over K it must be isotropic, so $s \in D_K(2)$. Therefore $p(K) \le 2$.

By [14, (0.4)] the form $(\psi \perp \langle -1 \rangle)_{K(i)(\varphi)}$ is essential for any 10-dimensional form φ over K. Therefore any 10-dimensional partially indefinite form over K is isotropic. Hence, $\widetilde{u}(K) \le 9$ and K is uniquely ordered. As $(\psi \perp \langle -1 \rangle)_{K(i)}$ is essential, $\psi \perp \langle -1 \rangle$ is anisotropic over $K(i)$ and thus also over K. Thus ψ does not represent 1 over K. In particular, K is not pythagorean and thus $p(K) = 2$. Moreover, with (5.1) we obtain that $8 = \dim(\psi) < \ell(K) \le \widetilde{u}(K) \le 9$ and conclude that $\ell(K) = \widetilde{u}(K) = 9$. Now $u(K) \le \widetilde{u}(K) = 9$, and since K is real, $u(K)$ is even. Since $\ell(K) = 9$, (4.8) yields that $(I_t K)^3 \ne 0$, whence $u(K) \ge 8$. This shows that $u(K) = 8$.

Example 7.13 Let k_0 be a real field and $k = k_0(X_1, \dots, X_8)$. Consider the form $\psi = \langle 1 + X_1^2, \dots, 1 + X_8^2 \rangle$ over k. Note that $(\psi \perp \langle -1 \rangle)_{k(i)}$ is essential. The class of real field extensions K of k such that $(\psi \perp \langle -1 \rangle)_{k(i)}$ is essential satisfies the hypotheses in (7.1). Therefore there exists a real field extension K/k as in (7.12), hence with $p(K) = 2$, $u(K) = 8$, and $\ell(K) = \widetilde{u}(K) = 9$. It follows then that $g_K(4) = 8$ and thus $\ell(K) = g_K(\frac{u(K)}{2}) + 1 = \frac{1}{2} p(K) u(K) + 1$.

We now give two examples of uniquely ordered fields K with $\widetilde{u}(K) = 6$, illustrating (5.11).

Example 7.14 Let α be an anisotropic 6-dimensional torsion form of trivial discriminant over some real field k – for example $\alpha = \langle 1, 1, -3, -21, X, -7X \rangle$ over $k = \mathbb{Q}(X)$. By (7.1), there exists a real field extension K/k such that α_K is anisotropic, whereas $\alpha_{K(\psi)}$ is isotropic over the function field $K(\psi)$ of any anisotropic partially indefinite form ψ over K. Using [16, (1.1)], it follows that every anisotropic partially indefinite form over K is similar either to a twofold Pfister form or to a subform of α_K. Therefore $\widetilde{u}(K) = u(K) = \dim(\alpha) = 6$, and $\ell(K) \le 4$. Furthermore, K is uniquely ordered by (7.2), and thus $\ell(K) = 4$ by (5.11).

Proposition 7.15 *Let $a, b \in \sum K^2$. Assume that $\alpha = \langle 1, 1, 1, a, -b, ab \rangle$ is anisotropic and that $\alpha_{K(\psi)}$ is isotropic for any anisotropic partially indefinite form ψ over K. Then K is* *uniquely ordered,* $p(K) = u(K) = 4$, $\ell(K) = \widetilde{u}(K) = 6$, $g_K(m) = m + 4$ *for* $2 \leq m \leq 4$, *and* $g_K(m) = m + 5$ *for* $m \geq 5$. *Furthermore* $p(K(X)) \geq 6$.

Proof By [16, (1.1)], it follows from the properties of α that any anisotropic partially indefinite form over K is similar to a twofold Pfister form or to a subform of α. In particular, $\widetilde{u}(K) = \dim(\alpha) = 6$ and K is uniquely ordered. By a signature argument, α does not contain any torsion form of dimension larger than 2, so it follows from the above that $u(K) \leq 4$.

Since α is anisotropic, the totally positive form $\langle 1, 1, 1, a, ab\rangle$ does not represent the element $b \in \sum K^2$, showing that $\ell(K) \geq 6$ and $p(K) \geq 4$. As $p(K) \leq u(K)$ by (2.8), we obtain that $p(K) = u(K) = 4$. With (5.1) we get that $\ell(K) \leq \widetilde{u}(K) = 6$ and conclude that $\ell(K) = 6$.

Since $u(K) = 4$, we have $(I_t K)^3 = I_t^3 K = 0$. Thus part (*c*) of (3.4) yields that $g_K(4) \leq 8$. Moreover, $\langle 1, b\rangle$ is not a subform of $\langle 1, 1, 1, 1, a\rangle = 5 \times \langle a\rangle$ and thus $\langle a, ab\rangle$ is not a subform of $5 \times \langle 1\rangle$, showing that $g_K(2) \geq 6$. As g_K is strictly increasing, we obtain that $g_K(m) = m+4$ for $2 \leq m \leq 4$. Since $\ell(K) = 6$, part (*c*) of (4.2) yields that $g_K(m) = m + 5$ for any $m \geq 5$.

Finally, since $b \notin D_K(\langle 1, 1, 1, a\rangle)$, the element $X^2 + b$ is not represented by the form $\langle 1, 1, 1, 1, a\rangle = 5 \times \langle a\rangle$ over $K(X)$ by [17, Chap. IX, (2.1)], so $a(X^2 + b)$ is not a sum of 5 squares, whence $p(K(X)) \geq 6$.

Example 7.16 We start with constructing a real field k with elements $a, b \in \sum k^2$ such that the quadratic form $\langle 1, 1, 1, a, -b, ab\rangle$ over k is anisotropic. To this end we take $a, c \in \mathbb{N}$ such that $a \equiv 1 \bmod 8$, $c \equiv 7 \bmod 8$, and a is not a square number (e.g., $a = 17$ and $c = 7$). We consider the form $\alpha = \langle 1, 1, 1, a, -(T^2+c), a(T^2+c)\rangle$ over $\mathbb{Q}(T)$ and its residue forms with respect to $T^2 + c$. Note that the residue field $\mathbb{Q}(\sqrt{-c})$ is contained in $\mathbb{Q}_2$ and that a is a square in $\mathbb{Q}_2$ but not in $\mathbb{Q}(\sqrt{-c})$. The first residue form is $\langle 1, 1, 1, a\rangle$, which is anisotropic over $\mathbb{Q}(\sqrt{-c})$ because it becomes isometric over $\mathbb{Q}_2$ to the anisotropic form $4 \times \langle 1\rangle$. The second residue form is $\langle -1, a\rangle$ over $\mathbb{Q}(\sqrt{-c})$, which is also anisotropic, as a is not a square in $\mathbb{Q}(\sqrt{-c})$. This shows that α is anisotropic over $k = \mathbb{Q}(T)$, and we take $b = T^2 + c$.

By (7.1) there is a real extension K/k such that α_K is anisotropic while $\alpha_{K(\psi)}$ is isotropic for any anisotropic partially indefinite form ψ over K. By (7.15) then K is uniquely ordered, $p(K) = u(K) = 4$, $\ell(K) = \widetilde{u}(K) = 6$, $p(K(X) \geq 6$, $g_K(m) = m + 4$ for $2 \leq m \leq 4$, and $g_K(m) = m + 5$ for $m \geq 5$.

To our knowledge, this is the first example of a field K with $p(K) = 4$ and $p(K(X)) \geq 6$.

Remark 7.17 The hypotheses in (7.15) imply also that $u(K(i)) = 4$ and $I^3 K(i) = 0$. In particular, this holds for the field K obtained in (7.16). Using a stronger version of (7.1) in (7.16), one can achieve that K also has no proper finite extensions of odd degree. Then it follows for any finite extension L/K that $I^3 L(i) = 0$, in particular $I_t^3 L = 0$ and $p(L) \leq 4$, and Milnor's Exact Sequence for the Witt ring of $K(X)$ [17, Chap. IX, Sect. 3] then yields that $6 \leq p(K(X)) \leq 8$.

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References

1. Arason, J.K., Elman, R.: Powers of the fundamental ideal in the Witt ring. J. Algebra **239**, 150–160 (2001)

2. Baeza, R., Leep, D., O'Ryan, M., Prieto, J.P.: Sums of squares of linear forms. Math. Z. **193**, 297–306 (1986)
3. Becher, K.J.: Supreme Pfister forms. Comm. Algebra **32**, 217–241 (2004)
4. Becher, K.J.: On fields of u-invariant 4. Arch. Math. **86**, 31–35 (2006)
5. Becher, K.J.: Decomposability for division algebras of exponent two and associated forms. Math. Z. **258**, 691–709 (2008)
6. Becher, K.J.: On the u-invariant of a real function field. Math. Ann. **346**, 245–249 (2010)
7. Choi, M.D., Dai, Z.D., Lam, T.Y., Reznick, B.: The Pythagoras number of some affine algebras and local algebras. J. Reine Angew. Math. **336**, 45–82 (1982)
8. Elman, R., Lam, T.Y.: Quadratic forms and the u-invariant, I. Math. Z. **131**, 283–304 (1973)
9. Elman, R., Lam, T.Y.: Quadratic forms under algebraic extensions. Math. Ann. **219**, 21–42 (1976)
10. Elman, R., Prestel, A.: Reduced stability of the Witt ring of a field and its Pythagorean closure. Am. J. Math. **106**, 1237–1260 (1984)
11. Hoffmann, D.W.: Pythagoras numbers of fields. J. Am. Math. Soc. **12**, 839–848 (1999)
12. Hornix, E.A.M.: Formally real fields with prescribed invariants in the theory of quadratic forms. Indag. Math. (N.S.) **2**, 65–78 (1991)
13. Hornix, E.A.M.: Totally indefinite quadratic forms over formally real fields. Nederl. Akad. Wetensch. Indag. Math. **47**, 305–312 (1985)
14. Izhboldin, O.: Fields of u-invariant 9. Ann. Math. **154**, 529–587 (2001)
15. Karpenko, N., Merkurjev, A.S.: Essential dimension of quadrics. Inventiones Math. **153**, 361–372 (2003)
16. Laghribi, A.: Formes quadratiques de dimension 6. Math. Nachr. **204**, 125–135 (1999)
17. Lam, T.Y.: Introduction to quadratic forms over fields. Graduate Studies in Mathematics, **67**, American Mathematical Society, Providence (2005)
18. Mordell, L.J.: On the representation of a binary quadratic form as a sum of squares of linear forms. Math. Z. **35**, 1–15 (1932)
19. Orlov, D., Vishik, A., Voevodsky, V.: An exact sequence for $K_*^M/2$ with applications to quadratic forms. Ann. Math. **165**, 1–13 (2007)
20. Scharlau, W.: Quadratic and Hermitian forms. Grundlehren der Mathematischen Wissenschaften, **270**. Springer, Berlin (1985)
21. Tignol, J.-P.: Réduction de l'indice d'une algèbre simple centrale sur le corps des fonctions d'une quadrique. Bull. Soc. Math. Belgique Sér. A **42**, 735–745 (1990)
22. Ware, R.: Hasse principles and the u-invariant over formally real fields. Nagoya Math. J. **61**, 117–125 (1976)