Uniform Decay for the Linear Version of Ruggeri's Relaxed Navier-Stokes-Fourier Equations

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According to [1], the linearization of Ruggeri's model [4] at a homogeneous equilibrium state $(\bar\theta, \bar\psi, \bar u, \bar\Sigma, \bar\sigma, \bar q) = (\bar\theta, \bar\psi, 0, 0, 0, 0)$ is given by

$$A^0 U_t + \sum_{j=1}^{3} A^j U_{x_j} = BU, \quad U = (V, W) = ((\check\theta, \check\psi, u), (\Sigma, \sigma, q)) \tag{1}$$

with

$$A^0_\epsilon = \begin{pmatrix} \theta^{-1}\hat p_{\psi\psi} & 0 & -\hat p_\psi + \theta \hat p_{\theta\psi} & 0 & 0 & 0 \\ 0 & \hat p_\psi & 0 & 0 & 0 & 0 \\ -\hat p_\psi + \theta\hat p_{\theta\psi} & 0 & \theta^3 \hat p_{\theta\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^2\tau_1\rho\delta^{(kl)(rs)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon^2\tau_2\rho & 0 \\ 0 & 0 & 0 & 0 & 0 & \epsilon^2\tau_3\rho\delta^{mn} \end{pmatrix}$$

$$A^j = \begin{pmatrix} 0 & \hat p_\psi\delta^{nj} & 0 & 0 & 0 & 0 \\ \hat p_\psi\delta^{mj} & 0 & \theta^2\hat p_\theta\delta^{mj} & \theta C^{mjrs} & \theta\delta^{mj} & 0 \\ 0 & \theta^2\hat p_\theta\delta^{nj} & 0 & 0 & 0 & \theta^2\delta^{nj} \\ 0 & \theta C^{njkl} & 0 & 0 & 0 & 0 \\ 0 & \theta\delta^{nj} & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta^2\delta^{mj} & 0 & 0 & 0 \end{pmatrix}, \quad j = 1, 2, 3,$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1/\eta)\delta^{(kl)(rs)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1/\zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1/\chi)\delta^{mn} \end{pmatrix},$$

where $p = \hat p(\theta, \psi)$, its derivatives, and $\rho = \hat p_\psi(\theta, \psi)/\theta$ are evaluated at the reference state $(\bar\theta, \bar\psi)$. The goal of this note is to show the following L^2 estimate, which is the starting point for its obvious extensions to L^2-based Sobolev spaces.

Theorem 1. *For any $\tau_1, \tau_2, \tau_3 > 0$ there exists a constant $C > 0$ such that for any $\epsilon \in (0, 1]$ and any data*

$$U_0 = (V_0, W_0) = ((\check{\theta}_0, \check{\psi}_0, u_0), (\Sigma_0, \sigma_0, q_0)) \in L^2 \cap L^1,$$

the solution $t \mapsto (V(t), W(t))$ of system (1) with these data obeys the estimate

$$\|(V, \epsilon W)\|_{L^2} \le C(1+t)^{-1/4}(\|(V_0, \epsilon W_0)\|_{L^2} + \|(V_0, \epsilon W_0)\|_{L^1}) \quad \text{for all } t > 0.$$

We write

$$A^0_\epsilon = \begin{pmatrix} A^0_1 & 0 \\ 0 & \epsilon^2 A^0_2 \end{pmatrix}, \quad \sum_{j=1}^{3} \xi_j A^j = \begin{pmatrix} A_{11}(\boldsymbol{\xi}) & A_{12}(\boldsymbol{\xi}) \\ A_{21}(\boldsymbol{\xi}) & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -\mathcal{J} \end{pmatrix}$$

and consider, with $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) = \xi \boldsymbol{\omega}, \boldsymbol{\omega} \in S^2, \xi \ge 0$,

$$M(\xi, \epsilon, \boldsymbol{\omega}) = (A^0_\epsilon)^{-1/2} \left(i \sum_{j=1}^{3} \xi_j A^j - B \right) (A^0_\epsilon)^{-1/2}$$

i.e.,

$$M(\xi, \epsilon, \boldsymbol{\omega}) = \begin{pmatrix} i\xi (A^0_1)^{-1/2} A_{11}(\boldsymbol{\omega}) (A^0_1)^{-1/2} & i\epsilon^{-1}\xi (A^0_1)^{-1/2} A_{12}(\boldsymbol{\omega}) (A^0_2)^{-1/2} \\ i\epsilon^{-1}\xi (A^0_2)^{-1/2} A_{21}(\boldsymbol{\omega}) (A^0_1)^{-1/2} & \epsilon^{-2} (A^0_2)^{-1/2} \quad \mathcal{J} \quad (A^0_2)^{-1/2} \end{pmatrix}$$

or, briefly,

$$= \begin{pmatrix} i\xi \tilde{A}_{11}(\boldsymbol{\omega}) & i\epsilon^{-1}\xi \tilde{A}_{12}(\boldsymbol{\omega}) \\ i\epsilon^{-1}\xi \tilde{A}_{21}(\boldsymbol{\omega}) & \epsilon^{-2} \tilde{\mathcal{J}} \end{pmatrix},$$

with

$$\tilde{A}_{11}(\boldsymbol{\omega}) = (A^0_1)^{-1/2} A_{11}(\boldsymbol{\omega}) (A^0_1)^{-1/2}, \quad \tilde{A}_{12}(\boldsymbol{\omega}) = (A^0_1)^{-1/2} A_{12}(\boldsymbol{\omega}) (A^0_2)^{-1/2},$$
$$\tilde{A}_{11}(\boldsymbol{\omega}) = (A^0_2)^{-1/2} A_{21}(\boldsymbol{\omega}) (A^0_1)^{-1/2}, \quad \tilde{\mathcal{J}} = (A^0_2)^{-1/2} \quad \mathcal{J} \quad (A^0_2)^{-1/2}.$$

From arguments given in [3, 5], it is obvious that Theorem 1 is implied by the following assertion.

Proposition 1. *There exists $c > 0$ and uniform transformations $R(\xi, \epsilon, \boldsymbol{\omega})$ such that for all $\boldsymbol{\omega} \in S^{d-1}, \epsilon \in (0, 1], \xi \ge 0$, the transformed version*

$$\tilde{M}(\xi, \epsilon, \boldsymbol{\omega}) = R(\xi, \epsilon, \boldsymbol{\omega})^{-1} M(\xi, \epsilon, \boldsymbol{\omega}) R(\xi, \epsilon, \boldsymbol{\omega})$$

satisfies

$$(*) \quad \tilde{M}(\xi, \epsilon, \boldsymbol{\omega}) + \tilde{M}(\xi, \epsilon, \boldsymbol{\omega})^* \ge c \frac{|\boldsymbol{\xi}|^2}{1 + |\boldsymbol{\xi}|^2}.$$

The rest of the note serves to prove this assertion. We do that separately in three regimes.

(i) Small ξ

Look at

$$\epsilon^2 M(\xi,\epsilon,\boldsymbol{\omega}) = \begin{pmatrix} i\epsilon^2\xi\tilde{A}_{11}(\boldsymbol{\omega}) & i\epsilon\xi\tilde{A}_{12}(\boldsymbol{\omega}) \\ i\epsilon\xi\tilde{A}_{21}(\boldsymbol{\omega}) & \tilde{\mathcal{J}} \end{pmatrix} = \begin{pmatrix} i\epsilon\eta\tilde{A}_{11}(\boldsymbol{\omega}) & i\eta\tilde{A}_{12}(\boldsymbol{\omega}) \\ i\eta\tilde{A}_{21}(\boldsymbol{\omega}) & \tilde{\mathcal{J}} \end{pmatrix} =: C(\eta) = \begin{pmatrix} C_{11}(\eta) & C_{12}(\eta) \\ C_{21}(\eta) & C_{22}(\eta) \end{pmatrix}$$

where $\eta = \epsilon\xi \geq 0$. To identify a base transform $T(\eta)$ that makes $C(\eta)$ block diagonal,

$$T(\eta)^{-1}C(\eta)T(\eta) = \begin{pmatrix} V(\eta) & 0 \\ 0 & W(\eta) \end{pmatrix},$$

we make the ansatz

$$T(\eta) = \begin{pmatrix} I & X(\eta) \\ Y(\eta) & I \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} C_{11}(\eta) & C_{12}(\eta) \\ C_{21}(\eta) & C_{22}(\eta) \end{pmatrix} \begin{pmatrix} I & X(\eta) \\ Y(\eta) & I \end{pmatrix} = \begin{pmatrix} I & X(\eta) \\ Y(\eta) & I \end{pmatrix} \begin{pmatrix} V(\eta) & 0 \\ 0 & W(\eta) \end{pmatrix}.$$

This is equivalent to the four matrix equations

$$\begin{aligned} C_{11} + C_{12}Y &= V && (2) \\ C_{11}X + C_{12} &= XW && (3) \\ C_{21} + C_{22}Y &= YV && (4) \\ C_{21}X + C_{22} &= W, && (5) \end{aligned}$$

which combine to

$$\begin{aligned} C_{11}X + C_{12} &= X(C_{21}X + C_{22}) && (6) \\ Y(C_{11} + C_{12}Y) &= C_{21} + C_{22}Y, && (7) \end{aligned}$$

equations that, according to the Implicit Function Theorem, uniquely determine $X(\eta)$ and $Y(\eta)$, as unique smooth functions for small values of η, with

$$X(0) = 0, \quad Y(0) = 0.$$

From these one obtains $V(\eta), W(\eta)$ through (2) and (5). As

$$W(0) = C_{22}(0) = \tilde{\mathcal{J}} > 0,$$

the W part causes no problem with respect to our goal $(*)$. However, in view of

$$V(0) = C_{11}(0) = 0,$$

we must inspect $V(\eta)$. Differentiating equation (2) yields

$$V' = C'_{11} + C'_{12}Y + C_{12}Y', \tag{8}$$

whence

$$V'(0) = C'_{11}(0) = i\epsilon\tilde{A}_{11}(\boldsymbol{\omega}). \tag{9}$$

Differentiating equations (7) and (8) gives the relations

$$\begin{aligned} Y'(C_{11} + C_{12}Y) + Y(C_{11} + C_{12}Y)' &= C'_{21} + C'_{22}Y + C_{22}Y', \\ V'' &= C''_{11} + 2C'_{12}Y' + C_{12}Y'', \end{aligned}$$

which show that

$$Y'(0) = -(C_{22}(0))^{-1}C'_{21}(0) = -i\tilde{\mathcal{J}}^{-1}\tilde{A}_{12}(\boldsymbol{\omega})$$

and thus

$$V''(0) = 2C'_{12}(0)Y'(0) = 2A_{12}(\boldsymbol{\omega})\mathcal{J}^{-1}A_{21}(\boldsymbol{\omega}) > 0. \tag{10}$$

This gives $C(\eta) \geq c\eta^2 I$ for $0 \leq \eta \leq \underline{k}$ with some $\underline{k} > 0$. We have proved $(*)$ for $0 \leq \xi \leq \underline{k}/\epsilon$.

(ii) Large $|\xi|$

Look at

$$\epsilon^2 M(\Xi/\epsilon, \epsilon, \boldsymbol{\omega}) = \begin{pmatrix} i\epsilon\Xi\tilde{A}_{11}(\boldsymbol{\omega}) & i\Xi\tilde{A}_{12}(\boldsymbol{\omega}) \\ i\Xi\tilde{A}_{21}(\boldsymbol{\omega}) & \tilde{\mathcal{J}} \end{pmatrix} = \Xi \begin{pmatrix} i\epsilon\tilde{A}_{11}(\boldsymbol{\omega}) & i\tilde{A}_{12}(\boldsymbol{\omega}) \\ i\tilde{A}_{21}(\boldsymbol{\omega}) & \Xi^{-1}\tilde{\mathcal{J}} \end{pmatrix} =: \Xi N(\Xi^{-1}, \epsilon)$$

with

$$N(x, \epsilon, \boldsymbol{\omega}) = \begin{pmatrix} i\epsilon\tilde{A}_{11}(\boldsymbol{\omega}) & i\tilde{A}_{12}(\boldsymbol{\omega}) \\ i\tilde{A}_{21}(\boldsymbol{\omega}) & x\tilde{\mathcal{J}}. \end{pmatrix}$$

With $S(0, \epsilon, \boldsymbol{\omega})$ a diagonalizer for $N(0, \epsilon, \boldsymbol{\omega})$, using a lemma from [2], we find $S(x, \epsilon, \boldsymbol{\omega})$ such that

$$\tilde{N}(x, \epsilon, \boldsymbol{\omega}) = S(x, \epsilon, \boldsymbol{\omega})^{-1}N(x, \epsilon, \boldsymbol{\omega})S(x, \epsilon, \boldsymbol{\omega})$$

satisfies

$$\tilde{N}(x, \epsilon, \boldsymbol{\omega}) + \tilde{N}(x, \epsilon, \boldsymbol{\omega})^* \geq cxI \quad 0 \leq x \leq \overline{x}$$

with certain $\overline{x}, c > 0$. For

$$\tilde{M}(\xi, \epsilon, \boldsymbol{\omega}) = S(x, \epsilon, \boldsymbol{\omega})^{-1}(\epsilon^{-1}\xi N((\epsilon\xi)^{-1}, \epsilon, \boldsymbol{\omega})S(x, \epsilon, \boldsymbol{\omega})$$

this means

$$\tilde{M}(\xi, \epsilon, \boldsymbol{\omega}) + \tilde{M}(\xi, \epsilon, \boldsymbol{\omega}) \geq \epsilon^{-2}\epsilon\xi(N((\epsilon\xi)^{-1}, \epsilon) + N((\epsilon\xi)^{-1}, \epsilon)^*) \geq c\epsilon^{-2}I, \quad \xi > \overline{k}/\epsilon$$

with $\overline{k} = 1/\overline{x}$. We have proved $(*)$ for $\xi \geq \overline{k}/\epsilon$.

(iii) Intermediate $|\xi|$

For $|\epsilon\xi|$ in the compact interval $[\underline{k},\overline{k}]$, it suffices to invoke the Kawashima condition. We use

Lemma 1. *Let $P \subset \mathbb{R}^N$ be compact and $K : P \to \mathbb{C}^{n\times n}$ be continuous such that all eigenvalues of all $K(p)$, $p \in P$, have positive real part. Then there exist a constant $c > 0$ and a map $T : P \to GL(n,\mathbb{C})$ such that $\{T(p) : p \in P\}$ and $\{T(p)^{-1} : p \in P\}$ are bounded and*

$$\tilde{K}(p) = T(p)^{-1}K(p)T(p)$$

satisfies

$$\tilde{K}(p) + \tilde{K}(p)^* \geq cI \quad \textit{for all } p \in P.$$

Due to the Kawashima condition, no matrix

$$\epsilon^2 M(\eta/\epsilon,\epsilon,\boldsymbol{\omega}) = \begin{pmatrix} i\epsilon\eta\tilde{A}_{11}(\boldsymbol{\omega}) & i\eta\tilde{A}_{12}(\boldsymbol{\omega}) \\ i\eta\tilde{A}_{21}(\boldsymbol{\omega}) & \tilde{\mathcal{J}} \end{pmatrix}, \quad 0 \leq \epsilon \leq 1,\ \underline{k} \leq \eta \leq \overline{k},$$

has a purely imaginary eigenvalue. By continuity for η tending to small or large values, all eigenvalues have positive real part. The lemma thus implies that there exist uniform transformations R such that

$$\tilde{M}(\xi,\epsilon,\boldsymbol{\omega}) = R(\xi,\epsilon,\boldsymbol{\omega})^{-1}M(\xi,\epsilon,\boldsymbol{\omega})R(\xi,\epsilon,\boldsymbol{\omega})$$

satisfies

$$\tilde{M}(\xi,\epsilon,\boldsymbol{\omega}) + \tilde{M}(\xi,\epsilon,\boldsymbol{\omega})^* \geq \epsilon^{-2}cI \quad \text{for all } 0 < \epsilon \leq 1,\ \underline{k}/\epsilon \leq \xi \leq \overline{k}/\epsilon.$$

This implies $(*)$ for all ξ with $\underline{k}/\epsilon \leq \xi \leq \overline{k}/\epsilon$.

References

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