Journal of Algebra 660 (2024) 882–907

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Research Paper

On positivity preservers with constant coefficients and their generators

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ARTICLE INFO

Article history:
Received 14 May 2024
Available online 14 August 2024
Communicated by Franz-Viktor Kuhlmann

MSC:
primary 44A60, 47A57, 15A04
secondary 12D15, 45P05, 47B38

Keywords:
Positivity preserver
Generator
Constant coefficient
Moments

ABSTRACT

In this work we study positivity preservers $T : \mathbb{R}[x_1, \ldots, x_n] \to \mathbb{R}[x_1, \ldots, x_n]$ with constant coefficients and define their generators A if they exist, i.e., $\exp(A) = T$. We use the theory of regular Fréchet Lie groups to show the first main result. A positivity preserver with constant coefficients has a generator if and only if it is represented by an infinitely divisible measure (Main Theorem 4.7). In the second main result (Main Theorem 4.11) we use the Lévy–Khinchin formula to fully characterize the generators of positivity preservers with constant coefficients.

1. Introduction

Non-negative polynomials are widely applied and studied. A first step to investigate non-negative polynomials is to study the convex cone

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https://doi.org/10.1016/j.jalgebra.2024.07.050

$$\operatorname{Pos}(K) := \{p \in \mathbb{R}[x_1,\dots,x_n] \,|\, p(x) \geq 0 \text{ for all } x \in K\}$$

of non-negative polynomials on some $K \subseteq \mathbb{R}^n$.

The second logical step is to study (linear) maps

$$T : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}[x_1,\dots,x_n]$$

between polynomials, especially when they preserve non-negativity, i.e.,

$$T\operatorname{Pos}(K) \subseteq \operatorname{Pos}(K),$$

see e.g. [9,16,4]. With the multi-index notation $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ for all $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}_0^n$ with $n \in \mathbb{N}$ the following is long known. An explicit proof can e.g. be found in [16].

Lemma 1.1 *(folklore, see e.g. [16, Lem. 2.3]). Let $n \in \mathbb{N}_0$ and let*

$$T : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}[x_1,\dots,x_n]$$

be linear. Then for all $\alpha \in \mathbb{N}_0^n$ there exist unique $q_\alpha \in \mathbb{R}[x_1,\dots,x_n]$ such that

$$T = \sum_{\alpha\in\mathbb{N}_0^n} q_\alpha \cdot \partial^\alpha.$$

The map $T : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}[x_1,\dots,x_n]$ is called to have *constant coefficients* if $q_\alpha \in \mathbb{R}$ for all $\alpha \in \mathbb{N}_0^n$. The subset with $q_0 = 1$ is denoted by

$$\mathfrak{D} := \left\{ \sum_{\alpha\in\mathbb{N}_0^n} q_\alpha \cdot \partial^\alpha \,\middle|\, q_\alpha \in \mathbb{R},\ q_0 = 1 \right\} \subsetneq \mathbb{R}[[\partial_1,\dots,\partial_n]].$$

Positivity preservers are fully described by their polynomial coefficients. A sequence $s = (s_\alpha)_{\alpha\in\mathbb{N}_0^n}$ is called a *moment sequence* if there exists a measure μ such that $s_\alpha = \int x^\alpha \,\mathrm{d}\mu(x)$ holds for all $\alpha \in \mathbb{N}_0^n$. We have the following.

Theorem 1.2 *(see e.g. [4, Thm. 3.1]). Let $n \in \mathbb{N}$ and let $T : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}[x_1,\dots,x_n]$ be linear. Then the following are equivalent:*

(i) $T\operatorname{Pos}(\mathbb{R}^n) \subseteq \operatorname{Pos}(\mathbb{R}^n)$.
(ii) $(\alpha! \cdot q_\alpha(y))_{\alpha\in\mathbb{N}_0^n}$ *is a moment sequence for all $y \in \mathbb{R}^n$.*

Example 1.3. Let $n = 1$. Then

$$\exp(t \cdot \partial_x^2) = \sum_{j\in\mathbb{N}_0} \frac{t^j \cdot \partial_x^{2j}}{j!}$$

is a positivity preserver for all $t \geq 0$. ◦

The following is a linear operator T which is not a positivity preserver.

Example 1.4. Let $k \geq 3$ and $a \in \mathbb{R} \setminus \{0\}$. Then

$$\exp(a\partial_x^k) := \sum_{j\in\mathbb{N}_0} \frac{a^j \cdot \partial_x^{j\cdot k}}{j!}$$

is not a positivity preserver since $q_{2k} = \frac{a^2}{2} \neq 0$ but $q_{2k+2} = 0$, i.e., $(j! \cdot q_j)_{j\in\mathbb{N}_0}$ is not a moment sequence. ◦

From elementary calculations and measure theory we get the following additional properties of positivity preservers. Let T be a positivity preserver with constant coefficients and μ be a representing measure of the corresponding moment sequence $s = (s_\alpha)_{\alpha\in\mathbb{N}_0^n} = (\alpha! \cdot q_\alpha)_{\alpha\in\mathbb{N}_0^n}$, then

$$(Tf)(x) = \int_{\mathbb{R}^n} f(x+y)\ \mathrm{d}\mu(y)$$

for all $f \in \mathbb{R}[x_1, \ldots, x_n]$. For short we call μ also a *representing measure of* T. It follows for two such operators T and T' with representing measures μ and μ' that

$$(TT'f)(x) = \int f(x+y)\ \mathrm{d}(\mu * \mu')(y) \tag{1}$$

where $\mu * \mu'$ is the *convolution of the two measures* μ and μ':

$$\begin{aligned}(\mu * \mu')(A) :=& \int_{\mathbb{R}^n\times\mathbb{R}^n} \chi_A(x+y)\ \mathrm{d}\mu(x)\ \mathrm{d}\mu'(y)\\ &= \int_{\mathbb{R}^n} \mu(A-y)\ \mathrm{d}\mu'(y) = \int_{\mathbb{R}^n} \mu'(A-x)\ \mathrm{d}\mu(x)\end{aligned} \tag{2}$$

for any $A \in \mathfrak{B}(\mathbb{R}^n)$, see e.g. [3, Sect. 3.9]. Here, χ_A is the characteristic function of the Borel set A. For the supports we have

$$\mathrm{supp}\,(\mu * \mu') = \mathrm{supp}\,\mu + \mathrm{supp}\,\mu'. \tag{3}$$

This can easily be proved from (2) or found in the literature, see e.g. [7, Prop. 14.5 (ii)] for a special case. We abbreviate

$$\mu^{*k} := \underbrace{\mu * \cdots * \mu}_{k\text{-times}}$$

for all $k \in \mathbb{N}$ and set $\mu^{*0} := \delta_0$ with δ_0 the Dirac measure centered at 0.

Previously we investigated the heat semi-group (see Example 1.3 for $n = 1$) and its action on (non-negative) polynomials, see [5,6]. For $n \in \mathbb{N}$ we have that $\exp(t\Delta)p_0$ is the unique solution of the polynomial valued heat equation

$$\partial_t p = \Delta p$$

with initial values $p_0 \in \mathbb{R}[x_1, \ldots, x_n]$. In [6] we observed the very strange behavior that several non-negative polynomials which are not sums of squares (e.g. the Motzkin and the Choi–Lam polynomial) become a sum of squares in finite time under the heat equation. In [6, Thm. 3.20] we showed that every non-negative $p \in \mathbb{R}[x, y, z]_{\leq 4}$ becomes a sum of squares in finite time. All these observations hold for the heat equation, i.e., the family $(\exp(t\Delta))_{t\geq 0}$ of positivity preservers with constant coefficients with the generator $\Delta = \partial_1^2 + \cdots + \partial_n^2$. So a natural question to further study these effects is to understand any family $(\exp(tA))_{t\geq 0}$ of positivity preservers and hence to determine all possible generators A. This is the main question of the current work for the constant coefficient case. It is answered in Main Theorem 4.11.

The paper is structured as follows. To define and work with $\exp(A)$ we repeat in the preliminaries for the readers convenience the notion of Lie groups, Lie algebras, and especially their lesser known infinite dimensional versions of regular Fréchet Lie groups. We will see that $\mathfrak{D}$ is a regular Fréchet Lie group with a Lie algebra $\mathfrak{d}$. By Theorem 1.2 we transport this property to sequences in Section 3. Section 4 contains the two main results. In the first Main Theorem 4.7 we show that a positivity preserver with constant coefficients has a generator if and only if it is represented by an infinitely divisible measure. Using the Lévy–Khinchin formula (which is also given in the preliminaries, see Theorem 2.18) we give in the second Main Theorem 4.11 the full description of all generators of positivity preservers with constant coefficients. In Section 5 we discuss a strange action on non-negative polynomials on $[0, \infty)$ caused by the heat equation with Dirichlet boundary conditions. We end this paper with a summary and an open question.

2. Preliminaries

Lie groups and Lie algebras are standard concepts in mathematics [28]. However, this only applies to the finite dimensional cases. For the readers convenience we give here the infinite dimensional definitions and examples we need to make the paper as self-contained as possible. For the sake of completeness we also include the explicit statement of the Lévy–Khinchin formula in Theorem 2.18.

2.1. Lie groups and their Lie algebras

Definition 2.1. A group $(G, \cdot)$ is called a *Lie group* if G is also a n-dimensional smooth manifold, $n \in \mathbb{N}$, such that $G \times G \to G : (A, B) \mapsto AB^{-1}$ is smooth. The *Lie algebra* $\mathfrak{g}$ *of* G is the tangent space $T_e G$ at the identity $e \in G$.

The connection between the Lie algebra $\mathfrak{g}$ and the Lie group G is given by the *exponential map* $\exp : \mathfrak{g} \to G$. For the special case $G = \mathrm{Gl}(m, \mathbb{C})$, $m \in \mathbb{N}$, the exponential mapping fulfills the following.

Lemma 2.2 *(see e.g. [28, p. 134, Ex. 15]). Let $m \in \mathbb{N}$. The following hold:*

(i) The exponential map

$$\exp : \mathbb{C}^{m\times m} \to \mathrm{Gl}(m, \mathbb{C}), \quad A \mapsto \exp A := \sum_{k\in\mathbb{N}_0} \frac{A^k}{k!}$$

is surjective.

(ii) Let $\mathrm{id} + N \in \mathrm{Gl}(m, \mathbb{C})$ be such that N is nilpotent. Then

$$\log(\mathrm{id} + N) := -\sum_{k\in\mathbb{N}} \frac{(-N)^k}{k}$$

is well-defined and

$$\exp(\log(\mathrm{id} + N)) = \mathrm{id} + N$$

holds.

The proof of Lemma 2.2 (ii) follows from formal power series calculations. For more on Lie groups and Lie algebras see e.g. [28].

The charts $\varphi : U \subseteq \mathbb{R}^n \to G$ of the n-dimensional smooth manifold G induce the Euclidean topology on the group G and $(A, B) \mapsto AB^{-1}$ is smooth with respect to this topology. When extending this to infinite dimensions more than one topology is possible and the choice of topology on G is important.

2.2. Fréchet spaces

Definition 2.3. A topological vector space $\mathcal{V}$ is called a *Fréchet space* if the following three conditions are fulfilled:

(i) $\mathcal{V}$ is metrizable (i.e., $\mathcal{V}$ is Hausdorff),
(ii) $\mathcal{V}$ is complete, and
(iii) $\mathcal{V}$ is locally convex.

It is clear that $\mathbb{R}[x_1, \dots, x_n]_{\leq d}$ is a Fréchet space for all $d \in \mathbb{N}_0$ since they are finite dimensional. Their Fréchet topology is unique.

Example 2.4 *(see e.g. [27, pp. 91–92, Ex. III]).* Let $n \in \mathbb{N}$. The vector space $\mathbb{R}[[x_1, \ldots, x_n]]$ of formal power series

$$p = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \cdot x^\alpha \qquad \text{with } c_\alpha \in \mathbb{R}$$

equipped with the topology induced by the semi-norms

$$|p|_d := \sup_{|\alpha| \leq d} |c_\alpha| \qquad \text{with } d \in \mathbb{N}_0$$

is a Fréchet space. In other words we have the convergence

$$p_i = \sum_{\alpha \in \mathbb{N}_0^n} c_{i,\alpha} \cdot x^\alpha \quad \to \quad p = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \cdot x^\alpha \qquad \text{for } i \to \infty$$

if and only if $c_{i,\alpha} \xrightarrow{i \to \infty} c_\alpha$ for all $\alpha \in \mathbb{N}_0^n$. $\circ$

For more on topological vector spaces see e.g. [27,26].

2.3. Regular Fréchet Lie groups and their Lie algebras

Already Hideki Omori stated the following, see [17, pp. III-IV]:

[G]eneral Fréchet manifolds are very difficult to treat. For instance, there are some difficulties in the definition of tangent bundles, hence in the definition of the concept of C^∞-mappings. Of course, there is neither an implicit function theorem nor a Frobenius theorem in general. Thus, it is difficult to give a theory of general Fréchet Lie groups.

A more detailed study is given by Omori in [18] and the theory successfully evolved since then, see e.g. [17,12,18,25,29,23] and references therein. We will give here only the basic definitions which will be needed for our study.

For the definition of $C^1(G, \mathfrak{g})$ see e.g. [18, pp. 9–10]. Since $\mathbb{R}[[\partial_1, \ldots, \partial_n]]$ has the Fréchet topology in Example 2.4, i.e., the coordinate-wise convergence, we have that for every $m \in \mathbb{N}_0$ a function $F : \mathbb{R} \to \mathbb{R}[[\partial_1, \ldots, \partial_n]]$, $F(t) = \sum_{\alpha \in \mathbb{N}_0^n} F_\alpha(t) \cdot \partial^\alpha$ is C^m if and only if every coordinate F_α is C^m.

Definition 2.5 *(see e.g. [18, p. 63, Dfn. 1.1]).* We call $(G, \cdot\,)$ a (*regular*) *Fréchet Lie group* if the following conditions are fulfilled:

(i) G is an infinite dimensional smooth Fréchet manifold.
(ii) $(G, \cdot\,)$ is a group.
(iii) The map $G \times G \to G$, $(A, B) \mapsto A \cdot B^{-1}$ is smooth.

(iv) The *Fréchet Lie algebra* $\mathfrak{g}$ of G is isomorphic to the tangent space T_eG of G at the unit element $e \in G$.

(v) $\exp : \mathfrak{g} \to G$ is a smooth mapping such that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tu) = u$$

holds for all $u \in \mathfrak{g}$.

(vi) The space $C^1(G, \mathfrak{g})$ of C^1-curves in G coincides with the set of all C^1-curves in G under the Fréchet topology.

For more on infinite dimensional manifolds, differential calculus, Lie groups, and Lie algebras see e.g. [15,18,23].

2.4. The Lie group $\mathfrak{D}_d$

Definition 2.6. Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. We define $\mathfrak{D}_d := \mathfrak{D}|_{\mathbb{R}[x_1,\dots,x_n]_{\leq d}}$.

Since

$$A\mathbb{R}[x_1, \dots, x_n]_{\leq d} \subseteq \mathbb{R}[x_1, \dots, x_n]_{\leq d}$$

holds for all $d \in \mathbb{N}_0$ and $A \in \mathfrak{D}$ the $\mathfrak{D}_d$ are well-defined. From Definition 2.6 we see that $\mathfrak{D}_d$ consists only of operators of the form

$$\sum_{\alpha \in \mathbb{N}_0^n : |\alpha| \leq d} c_\alpha \cdot \partial^\alpha$$

with $c_\alpha \in \mathbb{R}$ and $c_0 = 1$ since on $\mathbb{R}[x_1, \dots, x_n]_{\leq d}$ every operator ∂^β with $|\beta| > d$ fulfills $\partial^\beta p = 0$ for all $p \in \mathbb{R}[x_1, \dots, x_n]_{\leq d}$, i.e., $\partial^\beta = 0$ on $\mathbb{R}[x_1, \dots, x_n]_{\leq d}$.

Remark 2.7. We can also define $\mathfrak{D}_d$ by $\mathfrak{D}/\langle \partial^\alpha \,|\, |\alpha| = d + 1\rangle$. Both definitions are almost identical. However, Definition 2.6 has the following advantage. In $\mathfrak{D}/\langle \partial^\alpha \,|\, |\alpha| = d + 1\rangle$ we have the problem that we are working with equivalence classes and hence we can not calculate $A + B$ for $A \in \mathfrak{D}_d$ and $B \in \mathfrak{D}_e$ for $d \neq e$. With Definition 2.6 we can calculate $A + B$ for $A \in \mathfrak{D}_d$ and $B \in \mathfrak{D}_e$ with $d \neq e$ since $A + B$ is defined on

$$\mathrm{dom}\,(A + B) = \mathrm{dom}\,A \cap \mathrm{dom}\,B = \mathbb{R}[x_1, \dots, x_n]_{\leq \min\,\{d,e\}}$$

as usual for (unbounded) operators [21]. Definition 2.6 can then even be used to calculate $A + B$ for A on $\mathbb{R}[x_1, \dots, x_n]_{\leq d}$ and B on $\mathbb{R}[x_1, \dots, x_m]_{\leq e}$ for $n \neq m$ and $d \neq e$ on

$$\mathrm{dom}\,(A + B) = \mathbb{R}[x_1, \dots, x_{\min\{n,m\}}]_{\leq \min\{d,e\}}. \qquad \circ$$

Example 2.8. Let $n = 1$ and $d = 3$. Then

$$\mathfrak{D}_3 = \{1 + c_1\partial_x + c_2\partial_x^2 + c_3\partial_x^3 \mid c_1, c_2, c_3 \in \mathbb{R}\} \quad \text{on} \quad \mathbb{R}[x]_{\leq 3}.$$

Let

$$A = 1 + a_1\partial_x + a_2\partial_x^2 + a_3\partial_x^3 \quad \text{and} \quad B = 1 + b_1\partial_x + b_2\partial_x^2 + b_3\partial_x^3$$

be in $\mathfrak{D}_3$. Then

$$\begin{aligned} AB = BA &= (1 + a_1\partial_x + a_2\partial_x^2 + a_3\partial_x^3) \cdot (1 + b_1\partial_x + b_2\partial_x^2 + b_3\partial_x^3) \\ &= 1 + (a_1 + b_1)\partial_x + (a_2 + a_1b_1 + b_2)\partial_x^2 + (a_3 + a_2b_1 + a_1b_2 + b_3)\partial_x^3 \end{aligned} \tag{4}$$

since derivatives ∂^i with $i \geq 4$ are the zero operators on $\mathbb{R}[x]_{\leq 3}$. Hence, $(\mathfrak{D}_3, \cdot)$ is a commutative semi-group with neutral element $\mathbb{1} = 1$.

We will now see that $\mathfrak{D}_3$ is even a commutative group. For that it is sufficient to find for any $A \in \mathfrak{D}_3$ a $B \in \mathfrak{D}_3$ with $AB = \mathbb{1}$. By (4) $AB = \mathbb{1}$ is equivalent to

$$\begin{aligned} 0 &= a_1 + b_1 & &\Rightarrow & b_1 &= -a_1 \\ 0 &= a_2 + a_1b_1 + b_2 & &\Rightarrow & b_2 &= -a_2 + a_1^2 \\ 0 &= a_3 + a_2b_1 + a_1b_2 + b_3 & &\Rightarrow & b_3 &= -a_3 + 2a_2a_1 - a_1^3, \end{aligned}$$

i.e., every $A \in \mathfrak{D}_3$ has the unique inverse

$$A^{-1} = 1 - a_1\partial_x + (-a_2 + a_1^2)\partial_x^2 + (-a_3 + 2a_2a_1 - a_1^3)\partial_x^3 \quad \in \mathfrak{D}_3.$$

Hence, $(\mathfrak{D}_3, \cdot)$ is a commutative group. $\circ$

We have seen in the previous example that $(\mathfrak{D}_d, \cdot)$ for $n = 1$ and $d = 3$ is a commutative group. This holds for all $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$.

Lemma 2.9. *Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. Then $(\mathfrak{D}_d, \cdot)$ is a commutative group.*

Proof. Let $A = \sum_{\alpha:|\alpha|\leq d} a_\alpha\partial^\alpha, B = \sum_{\beta:|\beta|\leq d} b_\beta\partial^\beta \in \mathfrak{D}_d$, i.e., $a_0 = b_0 = 1$. Then

$$AB = C = \sum_{\gamma:|\gamma|\leq d} c_\gamma \cdot \partial^\gamma$$

holds with

$$c_\gamma = \sum_{\alpha,\beta\in\mathbb{N}_0^n:\alpha+\beta=\gamma} a_\alpha b_\beta. \tag{5}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \succeq \beta = (\beta_1, \ldots, \beta_n)$ on $\mathbb{N}_0^n$ if and only if $\alpha_i \geq \beta_i$ for all $i = 1, \ldots, n$. Then (5) can be solved by induction on $|\gamma|$. For $|\gamma| = 0$ we have $c_0 = a_0 \cdot b_0$ and $a_0 = c_0 = 1$, i.e., $b_0 = 1$. So assume (5) is solved for all c_γ with $|\gamma| \leq d-1$. Then for any $\gamma \in \mathbb{N}_0$ with $|\gamma| = d$ we have

$$b_\gamma = a_0 b_\gamma = -\sum_{\alpha\in\mathbb{N}_0^n\setminus\{0\}:\gamma\succeq\alpha} a_\alpha \cdot b_{\gamma-\alpha}, \tag{6}$$

i.e., the system (5) of equations has a unique solution gained by induction. Hence, for every $A \in \mathfrak{D}_d$ there exists a unique $B \in \mathfrak{D}_d$ with $AB = BA = \mathbb{1}$. $\square$

From Lemma 2.9 we have seen that $(\mathfrak{D}_d, \cdot\,)$ for any $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$ is a commutative group. Let

$$\iota_d : \{1\} \times \mathbb{R}^{\binom{n+d}{n}-1} \to \mathfrak{D}_d, \quad (a_\alpha)_{\alpha\in\mathbb{N}_0^n:|\alpha|\leq d} \mapsto \sum_{\alpha\in\mathbb{N}_0^n:|\alpha|\leq d} a_\alpha \cdot \partial^\alpha \tag{7}$$

be an affine linear map. Then ι_d in (7) is a diffeomorphism and it is a coordinate map for $\mathfrak{D}_d$. The smooth manifold $\{1\} \times \mathbb{R}^{\binom{n+d}{n}-1}$ inherits the group structure of $\mathfrak{D}_d$ through ι_d, i.e.,

$$(\mathfrak{D}_d, \cdot\,) \overset{\iota_d}{\cong} \left(\{1\} \times \mathbb{R}^{\binom{n+d}{n}-1}, \odot\right) \tag{8}$$

Hence, the map ι_d shows the following.

Theorem 2.10. *Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. Then $(\mathfrak{D}_d, \cdot\,)$ is a Lie group.*

Proof. The map ι_d in (7) is a diffeomorphism between $\mathfrak{D}_d$ and $\{1\} \times \mathbb{R}^{\binom{n+d}{n}-1}$. Hence, $\mathfrak{D}_d$ is a differentiable manifold which possesses the group structure $(\mathfrak{D}_d, \cdot\,)$. By (5) and (6) we have that the map

$$\mathfrak{D}_d \times \mathfrak{D}_d \to \mathfrak{D}_d, \quad (A, B) \mapsto AB^{-1}$$

is smooth. Hence, $(\mathfrak{D}_d, \cdot\,)$ is a commutative Lie group. $\square$

2.5. The Lie algebra $\mathfrak{d}_d$ of $\mathfrak{D}_d$

Since every $A \in \mathfrak{D}_d$ is a linear map

$$A : \mathbb{R}[x_1, \ldots, x_n]_{\leq d} \to \mathbb{R}[x_1, \ldots, x_n]_{\leq d}$$

between finite-dimensional vector spaces we can choose a basis of $\mathbb{R}[x_1, \ldots, x_n]_{\leq d}$ and get a matrix representation $\tilde{A}$ of A. Take the monomial basis of $\mathbb{R}[x_1, \ldots, x_n]_{\leq d}$. Then $\tilde{A}$ is an upper triangular matrix with diagonal entries 1.

Example 2.11 *(Example 2.8 continued).* Let $n = 1$ and $d = 3$. Then every $A = 1 + a_1\partial_x + a_2\partial_x^2 + a_3\partial_x^3 \in \mathfrak{D}_3$ has with the monomial basis $\{1, x, x^2, x^3\}$ of $\mathbb{R}[x]_{\leq 3}$ the matrix representation

$$\tilde{A} = \begin{pmatrix} 1 & a_1 & 2a_2 & 6a_3 \\ 0 & 1 & 2a_1 & 6a_2 \\ 0 & 0 & 1 & 3a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and we therefore set

$$\tilde{\mathfrak{D}}_3 := \left\{ \begin{pmatrix} 1 & a_1 & 2a_2 & 6a_3 \\ 0 & 1 & 2a_1 & 6a_2 \\ 0 & 0 & 1 & 3a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \,\middle|\, a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

Hence, $(\tilde{A} - \mathrm{id})^4 = 0$ as a matrix and also $(A - \mathbb{1})^4 = 0$ as an operator on $\mathbb{R}[x]_{\leq 3}$. From Lemma 2.2 we find that the matrix valued exponential map

$$\exp : \mathrm{gl}(4, \mathbb{C}) \to \mathrm{Gl}(4, \mathbb{C}), \quad \tilde{A} \mapsto \exp(\tilde{A}) := \sum_{k \in \mathbb{N}_0} \frac{\tilde{A}^k}{k!}$$

is surjective and the logarithm

$$\tilde{A} \mapsto \log \tilde{A} := -\sum_{k \in \mathbb{N}} \frac{(\mathrm{id} - \tilde{A})^k}{k}$$

is well-defined for all $\mathrm{id} + N \in \mathrm{Gl}(4, \mathbb{C})$ with N nilpotent. Since $\tilde{\mathfrak{D}}_3 \subseteq \mathrm{Gl}(4, \mathbb{C})$ with $(\mathrm{id} - \tilde{A})^4 = 0$ for all $\tilde{A} \in \tilde{\mathfrak{D}}_3$ we have

$$\log : \tilde{\mathfrak{D}}_3 \to \mathrm{gl}(4, \mathbb{C}), \quad \tilde{A} \mapsto \log \tilde{A} = -\sum_{k=1}^{3} \frac{(\mathrm{id} - \tilde{A})^k}{k}. \tag{9}$$

Since also $(A - \mathbb{1})^4 = 0$ for all $A \in \mathfrak{D}_3$ we can use (9) also for the differential operators in $\mathfrak{D}_3$:

$$\log : \mathfrak{D}_3 \to \left\{ d_0 + d_1\partial_x + d_2\partial_x^2 + d_3\partial_x^3 \,\middle|\, d_0, \ldots, d_3 \in \mathbb{R} \right\}, \quad A \mapsto -\sum_{k=1}^{3} \frac{(\mathbb{1} - A)^k}{k}.$$

To determine the image $\log \mathfrak{D}_3$ recall that also log is an injective map by Lemma 2.2 and hence $\log \mathfrak{D}_3$ is 3-dimensional with $d_0 = 0$, i.e., we have

$$\log \mathfrak{D}_3 = \left\{ d_1\partial_x + d_2\partial_x^2 + d_3\partial_x^3 \,\middle|\, d_1, d_2, d_3 \in \mathbb{R} \right\} =: \mathfrak{d}_3.$$

In summary, since $A^4 = 0$ for all $A \in \mathfrak{d}_3$ we have that

$$\exp : \mathfrak{d}_3 \to \mathfrak{D}_3, \quad A \mapsto \sum_{k=0}^{3} \frac{A^k}{k!} \tag{10}$$

is surjective with inverse

$$\log : \mathfrak{D}_3 \to \mathfrak{d}_3, \quad A \mapsto -\sum_{k=1}^{3} \frac{(\mathbb{1} - A)^k}{k}. \tag{11}$$

Therefore, $\mathfrak{d}_3$ is the Lie algebra of $\mathfrak{D}_3$ and exp in (10) is the exponential map between the Lie algebra $\mathfrak{d}_3$ and its Lie group $\mathfrak{D}_3$ with inverse log in (11). ∘

The previous example of the Lie algebra $\mathfrak{d}_3$ of the Lie group $\mathfrak{D}_3$ holds for all $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. We define the following.

Definition 2.12. Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. We define

$$\mathfrak{d}_d := \left\{ \sum_{\alpha\in\mathbb{N}_0^n\setminus\{0\}:|\alpha|\leq d} d_\alpha \cdot \partial^\alpha \,\middle|\, d_\alpha \in \mathbb{R} \text{ for all } \alpha \in \mathbb{N}_0^n \setminus \{0\} \text{ with } |\alpha| \leq d \right\}.$$

It is clear that $(\mathfrak{d}_d, \cdot, +)$ is an algebra on $\mathbb{R}[x_1, \dots, x_n]_{\leq d}$ and we have the following.

Theorem 2.13. *Let $n \in \mathbb{N}$ and $d \in \mathbb{N}_0$. Then $(\mathfrak{d}_d, \cdot, +)$ is the Lie algebra of the Lie group $(\mathfrak{D}_d, \cdot)$ with exponential map*

$$\exp : \mathfrak{d}_d \to \mathfrak{D}_d, \quad A \mapsto \sum_{k=0}^{d} \frac{A^k}{k!}$$

with inverse

$$\log : \mathfrak{D}_d \to \mathfrak{d}_d, \quad A \mapsto -\sum_{k=1}^{d} \frac{(\mathbb{1} - A)^k}{k}.$$

Proof. Follows from Lemma 2.2 similar to Example 2.11. □

2.6. The regular Fréchet Lie group $\mathfrak{D}$ and its Lie algebra $\mathfrak{d}$

In Section 2.4 and 2.5 we have seen that $(\mathfrak{D}_d, \cdot)$ is a Lie group with Lie algebra $(\mathfrak{d}_d, \cdot, +)$ for all $d \in \mathbb{N}_0$. Hence, similar to Definition 2.12 we define the following.

Definition 2.14. Let $n \in \mathbb{N}$. We define

$$\mathfrak{d} := \left\{ \sum_{\alpha\in\mathbb{N}_0^n\setminus\{0\}} d_\alpha \cdot \partial^\alpha \,\middle|\, d_\alpha \in \mathbb{R} \text{ for all } \alpha \in \mathbb{N}_0^n \setminus \{0\} \right\}.$$

For $\mathfrak{D}$ and $\mathfrak{d}$ we have the following.

Theorem 2.15. *Let $n \in \mathbb{N}$. Then the following hold:*

(i) $(\mathfrak{d}, \cdot, +)$ *is a commutative algebra.*
(ii) $(\mathfrak{D}, \cdot)$ *is a commutative group.*
(iii) The map

$$\exp : \mathfrak{d} \to \mathfrak{D}, \quad A \mapsto \sum_{k \in \mathbb{N}_0} \frac{A^k}{k!}$$

is bijective.
(iv) The map

$$\log : \mathfrak{D} \to \mathfrak{d}, \quad A \mapsto -\sum_{k \in \mathbb{N}} \frac{(\mathbb{1} - A)^k}{k}$$

is bijective.
(v) The maps $\exp : \mathfrak{d} \to \mathfrak{D}$ *and* $\log : \mathfrak{D} \to \mathfrak{d}$ *are inverse to each other.*

Proof. (i): That is clear.

(ii): That $A \cdot B = B \cdot A$ holds for all $A, B \in \mathfrak{D}$ is clear. The inverse of $A \in \mathfrak{D}$ is uniquely determined by solving (5) to get (6) for all $\gamma \in \mathbb{N}_0^n$. This is a formal power series argument with coordinate-wise convergence (as in the Fréchet topology, Example 2.4).

(iii): At first we show that $\exp : \mathfrak{d} \to \mathfrak{D}$ is well-defined. To see this note that for any $A \in \mathfrak{d}$ we have

$$A^k = \sum_{\alpha \in \mathbb{N}_0^n : |\alpha| \geq k} c_\alpha \cdot \partial^\alpha,$$

i.e., A^k contains no differential operators of order $\leq k - 1$. Hence, the sum

$$\sum_{k=0}^{K} \frac{A^k}{k!} = \sum_{\alpha \in \mathbb{N}_0^n} c_{K,\alpha} \cdot \partial^\alpha$$

converges coefficient-wise to

$$\exp A = \sum_{k \in \mathbb{N}_0} \frac{A^k}{k!} = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \cdot \partial^\alpha,$$

i.e., in the Fréchet topology of $\mathfrak{D} \subsetneq \mathbb{R}[[\partial_1, \ldots, \partial_n]] \cong \mathbb{R}[[x_1, \ldots, x_n]]$, see Example 2.4. In other words, the coefficients c_α depend only on A^k for $k = 0, \ldots, |\alpha|$, we therefore have $c_{K,\alpha} = c_\alpha$ for all $K > |\alpha|$, and hence $\exp A \in \mathfrak{D}$ is well-defined. With that we have $\exp \mathfrak{d} \subseteq \mathfrak{D}$. For equality we give the inverse map in (v).

(iv): To show that $\log : \mathfrak{D} \to \mathfrak{d}$ is well-defined the same argument as in (iii) holds for $(\mathbb{1} - A)^k$ with $A \in \mathfrak{D}$. It shows that $\log A \in \mathfrak{d}$ for all $A \in \mathfrak{D}$ is well-defined and we have $\log \mathfrak{D} \subseteq \mathfrak{d}$.

(v): To prove that exp and log are inverse to each other we remark

$$\mathbb{R}[x_1, \dots, x_n] = \bigcup_{d \in \mathbb{N}_0} \mathbb{R}[x_1, \dots, x_n]_{\leq d}.$$

For $d \in \mathbb{N}_0$ define

$$\exp_d A := \sum_{k=0}^{d} \frac{A^k}{k!} \quad \text{and} \quad \log_d A := -\sum_{k=1}^{d} \frac{(\mathbb{1} - A)^k}{k}.$$

Then for every $p \in \mathbb{R}[x_1, \dots, x_n]$ with $d = \deg p$ we have

$$\exp(\log A)p = \exp_d(\log_d A)p = Ap$$

for all $A \in \mathfrak{d}$ by Theorem 2.13, i.e., $\exp(\log A) = A$ for all $A \in \mathfrak{D}$. Similarly, we have $\log(\exp A)p = Ap$ for all $A \in \mathfrak{d}$. This also shows the remaining assertions $\exp(\mathfrak{d}) = \mathfrak{D}$ and $\log \mathfrak{D} = \mathfrak{d}$ from (iii) and (iv). $\square$

For $\mathbb{R}[[\partial_1, \dots, \partial_n]] \supseteq \mathfrak{d}, \mathfrak{D}$ being a Fréchet space with the before mentioned topology (of coordinate-wise convergence, Example 2.4) we have the following.

Corollary 2.16. *Let $n \in \mathbb{N}$ and let $\mathfrak{d}, \mathfrak{D} \subseteq \mathbb{R}[[\partial_1, \dots, \partial_n]]$ be Fréchet spaces (equipped with the coordinate-wise convergence). Then the following hold:*

(i) $\mathfrak{D} \times \mathfrak{D} \to \mathfrak{D}$, $(A, B) \mapsto AB^{-1}$ *is smooth.*
(ii) $\exp : \mathfrak{d} \to \mathfrak{D}$ *is smooth and*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tu) = u$$

holds for all $u \in \mathfrak{g}$.
(iii) $\log : \mathfrak{d} \to \mathfrak{D}$ *is smooth.*

Proof. (i): Let $A = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \partial^\alpha$ and $B = \sum_{\alpha \in \mathbb{N}_0^n} b_\alpha \partial^\alpha$ with $a_0 = b_0 = 1$. From (5) we see that the multiplication is smooth since every coordinate c_γ of the product $AB = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \partial^\alpha$ is a polynomial in a_α and b_α with $|\alpha| \leq |\gamma|$. The inverse $B^{-1} = \sum_{\alpha \in \mathbb{N}_0^n} d_\alpha \partial^\alpha$ is smooth because of (6), i.e., also the coefficients d_γ of the inverse depend polynomially on the coefficients b_α with $|\alpha| \leq |\gamma|$.

(ii): In the proof of Theorem 2.15 (iii) we have already seen that the coefficients c_γ of $\exp A = \sum_{\alpha\in\mathbb{N}_0^n} c_\alpha\partial^\alpha$ depend polynomially on the coefficients a_α of $A = \sum_{\alpha\in\mathbb{N}_0^n\setminus\{0\}} a_\alpha\partial^\alpha$ with $|\alpha|\leq|\gamma|$.

The condition

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tu) = u$$

then follows by direct calculations.

(iii): Follows like (ii) from Theorem 2.15 (iv). □

It is easy to see that $\mathfrak{d}$ and $\mathfrak{D}$ are both infinite dimensional smooth (Fréchet) manifolds. Hence, summing everything up we have the following.

Theorem 2.17. *Let $n\in\mathbb{N}$. Then $(\mathfrak{D},\cdot)$ as a Fréchet space is a commutative regular Fréchet Lie group with the commutative Fréchet Lie algebra $(\mathfrak{d},\cdot,+)$. The exponential map*

$$\exp:\mathfrak{d}\to\mathfrak{D},\quad A\mapsto\sum_{k\in\mathbb{N}_0}\frac{A^k}{k!}$$

is smooth and bijective with the smooth and bijective inverse

$$\log:\mathfrak{D}\to\mathfrak{d},\quad A\mapsto-\sum_{k\in\mathbb{N}}\frac{(\mathbb{1}-A)^k}{k}.$$

Proof. We have that $\mathfrak{D}$ is an infinite dimensional smooth manifold, $\mathfrak{D}$ is a Fréchet space (with the coefficient-wise convergence topology, Example 2.4) and by Theorem 2.15 (ii) we also have that $(\mathfrak{D},\cdot)$ is a commutative group. By Corollary 2.16 (i) we have that $(A,B)\mapsto AB^{-1}$ is continuous in the Fréchet topology. Hence, $(\mathfrak{D},\cdot)$ is an infinite dimensional commutative Fréchet Lie group.

The properties about exp and log are Theorem 2.15 (iii)–(v).

We now prove the regularity condition (vi) in Definition 2.5. Let $F:\mathbb{R}\to\mathfrak{D}$ be a C^1-differentiable function, i.e.,

$$\lim_{n\to\infty}\left(F\left(t+\frac{s}{n}\right)\cdot F(t)^{-1}\right) \tag{12}$$

converges uniformly on each compact interval to a one-parameter subgroup

$$\exp(sf(t))$$

where $f:\mathbb{R}\to\mathfrak{d}$ is the derivative $\dot{F}(t)$ of $F(t)$ at $t\in\mathbb{R}$, see [18, p. 10]. But we can take the logarithm of F

$$\tilde{f}(t) := \log F(t)$$

for all $t \in \mathbb{R}$ to see that $\tilde{f}$ is C^1 since log is smooth by Corollary 2.16 (iii). By Corollary 2.16 (ii) we have $f = \tilde{f}$. Hence, $C^1(\mathfrak{D}, \mathfrak{d})$ coincides with the set of all C^1-curves in $\mathfrak{D}$ under the Fréchet topology of $\mathfrak{D}$. □

In the previous proof we can also replace (12) by the fact that a function $F : \mathbb{R} \to \mathbb{R}[[\partial_1, \ldots, \partial_n]]$, $t \mapsto F(t) = \sum_{\alpha\in\mathbb{N}_0^n} F_\alpha(t)\cdot\partial^\alpha$ is C^m for some $m \in \mathbb{N}_0$ if and only if every coordinate function F_α is C^m.

2.7. Lévy–Khinchin formula

A measure μ on $\mathbb{R}^n$ is called *divisible by* $k \in \mathbb{N}$ if there exists a measure ν such that $\mu = \nu^{*k}$. A measure μ on $\mathbb{R}^n$ is called *infinitely divisible* if it is divisible by any $k \in \mathbb{N}$. Infinitely divisible measures are fully characterized by the *Lévy–Khinchin formula*.

Theorem 2.18 *(Lévy–Khinchin, see e.g. [13, Cor. 15.8] or [14, Satz 16.17]). Let $n \in \mathbb{N}$ and μ be a measure on $\mathbb{R}^n$. The following are equivalent:*

- *(i) μ is infinitely divisible.*
- *(ii) There exist a vector $b \in \mathbb{R}^n$, a symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$ with $\Sigma \succeq 0$, and a measure ν on $\mathbb{R}^n$ such that*

$$\log \int e^{itx}\, \mathrm{d}\mu(x) = itb - \frac{1}{2}t^T\Sigma t + \int (e^{itx} - 1 - itx\cdot\chi_{\|x\|_2<1})\, \mathrm{d}\mu(x)$$

holds for the characteristic function of μ.

3. The regular Fréchet Lie group structure of sequences

We define appropriate sets $\mathfrak{S}, \mathfrak{s} \subsetneq \mathbb{R}^{\mathbb{N}_0^n}$ of sequences with convolution $*$. From these definitions we will see that the set of sequences $\mathbb{R}^{\mathbb{N}_0^n}$, and therefore also $\mathfrak{S}$ and $\mathfrak{s}$, inherit the Fréchet topology (Example 2.4) from $\mathbb{R}[[x_1, \ldots, x_n]]$.

Definition 3.1. Let $n \in \mathbb{N}$. We define

$$\mathfrak{S} := \left\{ (s_\alpha)_{\alpha\in\mathbb{N}_0^n} \in \mathbb{R}^{\mathbb{N}_0^n} \,\middle|\, s_0 = 1 \right\} \quad \text{and} \quad \mathfrak{s} := \left\{ (s_\alpha)_{\alpha\in\mathbb{N}_0^n} \in \mathbb{R}^{\mathbb{N}_0^n} \,\middle|\, s_0 = 0 \right\}.$$

Define the linear and bijective map

$$D : \mathbb{R}^{\mathbb{N}_0^n} \to \mathbb{R}[[\partial_1, \ldots, \partial_n]], \quad s = (s_\alpha)_{\alpha\in\mathbb{N}_0^n} \mapsto D(s) := \sum_{\alpha\in\mathbb{N}_0^n} \alpha!^{-1}\cdot s_\alpha\cdot\partial^\alpha \tag{13}$$

and on $\mathbb{R}^{\mathbb{N}_0^n}$ the convolution $*$ as

$$*: \mathbb{R}^{\mathbb{N}_0^n} \times \mathbb{R}^{\mathbb{N}_0^n} \to \mathbb{R}^{\mathbb{N}_0^n}, \quad (s,t) \mapsto s * t := D^{-1}(D(s)D(t)).$$

We abbreviate

$$s^0 = \mathbb{1} := (\delta_{0,\alpha})_{\alpha\in\mathbb{N}_0^n} \qquad \text{and} \qquad s^{*k} := \underbrace{s * \cdots * s}_{k\text{-times}}$$

for all $k \in \mathbb{N}$ and $s \in \mathbb{R}^{\mathbb{N}_0^n}$.

The map D in (13) is of course a Fréchet space isomorphism. The map D also equips $\mathfrak{S}$ and $\mathfrak{s}$ with the Fréchet topology of $\mathfrak{D}$ and $\mathfrak{d}$, respectively. In summary, we have the following.

Corollary 3.2. *Let $n \in \mathbb{N}$. Then the following hold:*

(i) $s * t \in \mathfrak{S}$ *holds for all* $s, t \in \mathfrak{S}$.
(ii) $D|_{\mathfrak{S}} : (\mathfrak{S}, *) \to (\mathfrak{D}, \,\cdot\,)$ *is a Fréchet group isomorphism.*
(iii) $s * t \in \mathfrak{s}$ *and* $s + t \in \mathfrak{s}$ *hold for all* $s, t \in \mathfrak{s}$.
(iv) $D|_{\mathfrak{s}} : (\mathfrak{s}, *, +) \to (\mathfrak{d}, \,\cdot\,, +)$ *is a Fréchet algebra isomorphism.*

Theorem 3.3. *Let $n \in \mathbb{N}$. Then $(\mathfrak{S}, *)$ is a commutative regular Fréchet Lie group with the commutative Lie algebra $(\mathfrak{s}, *, +)$. The exponential map*

$$\exp : \mathfrak{s} \to \mathfrak{S}, \quad s \mapsto \exp(s) := \sum_{k\in\mathbb{N}_0} \frac{s^{*k}}{k!}$$

is smooth and bijective with the smooth and bijective inverse

$$\log : \mathfrak{S} \to \mathfrak{s}, \quad s \mapsto \log(s) := -\sum_{k\in\mathbb{N}} \frac{(\mathbb{1} - s)^{*k}}{k}.$$

Proof. From Corollary 3.2 we see that the map $D : \mathbb{R}^{\mathbb{N}_0^n} \to \mathbb{R}^{\mathbb{N}_0^n}$ induces the Fréchet Lie group isomorphism $D|_{\mathfrak{S}} : (\mathfrak{S}, *) \to (\mathfrak{D}, \,\cdot\,)$ and the Fréchet Lie algebra isomorphism $D|_{\mathfrak{s}} : (\mathfrak{s}, *, +) \to (\mathfrak{d}, \,\cdot\,, +)$. Apply both isomorphisms to Theorem 2.17 and the assertion is proved. □

We want to point out that Hirschman and Widder [11] extensively investigated the inversion theory of convolution of measures and functions. But from Theorem 3.3 we see that the convolution of their moments is trivial when we allow signed moment sequences.

For $*$ on $\mathfrak{S}$ we find from (2) the following.

Corollary 3.4. *Let $n \in \mathbb{N}$. If $s \in \mathfrak{S}$ is represented by the signed measure μ and $t \in \mathfrak{S}$ is represented by the signed representing measure ν then $s * t$ is represented by the measure $\mu * \nu$.*

Note, that every sequence $s \in \mathbb{R}^{\mathbb{N}_0^n}$ is represented by a signed measure, see e.g. [19,2,24,20]. It is even possible to restrict the support to $[0,\infty)^n$, see e.g. [2,24] or to use only linear combinations of Dirac measures [1].

Because of the connection of positivity preservers to moment sequences by Theorem 1.2 we define the following.

Definition 3.5. Let $n \in \mathbb{N}$. We define

$$\mathfrak{S}_+ := \{s \in \mathfrak{S} \,|\, s \text{ is a moment sequence}\}.$$

Then $\mathfrak{S}_+$ is a base of the moment cone.

Remark 3.6. For a general moment sequence $s = (s_\alpha)_{\alpha\in\mathbb{N}_0^n}$ we only have $s_0 > 0$. But, scaling $\tilde{s} = s_0^{-1} \cdot s \in \mathfrak{S}$ gives $s = \exp(\ln s_0 + \log \tilde{s})$. ∘

The Fréchet topology on $\mathfrak{S}$ and Corollary 3.4 imply the following.

Corollary 3.7. *Let $n \in \mathbb{N}$. The following hold:*

(i) The set $\mathfrak{S}_+$ is convex and closed.
(ii) For all $s, t \in \mathfrak{S}_+$ we have $s * t \in \mathfrak{S}_+$.

Proof. (i): Convexity is clear. It suffices to prove that $\mathfrak{S}_+$ is closed in the Fréchet topology.

Let $s^{(n)} = (s_\alpha^{(n)})_{\alpha\in\mathbb{N}_0^n} \in \mathfrak{S}_+$ for all $n \in \mathbb{N}_0$ be such that $s^{(n)} \to s$ in the Fréchet topology (Example 2.4), i.e., $s_\alpha^{(n)} \to s_\alpha$ for all $\alpha \in \mathbb{N}_0^n$. Define $L_s : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}$ by $L_s(x^\alpha) := s_\alpha$ for all $\alpha \in \mathbb{N}_0^n$ and extend it linearly to $\mathbb{R}[x_1,\dots,x_n]$. Let $p \in \operatorname{Pos}(\mathbb{R}^n)$ then $0 \le L_{s^{(n)}}(p) \to L_s(p) \ge 0$ shows that L_s is $\operatorname{Pos}(\mathbb{R}^n)$-positive, i.e., by Haviland's Theorem [10] L_s is a moment functional and $s \in \mathfrak{S}_+$.

(ii): Since $s, t \in \mathfrak{S}_+$ we have that s is represented by μ and t is represented by ν. Therefore, $s * t$ is represented by $\mu * \nu$ by Corollary 3.4. □

The following is another trivial consequence of (2). A moment sequence is called *indeterminate* if it has two and therefore infinitely many representing measures.

Corollary 3.8. *Let $n \in \mathbb{N}$ and $s, t \in \mathfrak{S}_+$. If s or t is indeterminate then $s * t \in \mathfrak{S}_+$ is indeterminate.*

4. Generators of positivity preservers with constant coefficients

In Section 2.6 we showed that $\mathfrak{D}$ is a Fréchet Lie group with Fréchet Lie algebra $\mathfrak{d}$. With the smooth and bijective $\exp : \mathfrak{d} \to \mathfrak{D}$ with the inverse $\log : \mathfrak{D} \to \mathfrak{d}$ we can now easily go from positivity preservers with constant coefficients to their generators.

Definition 4.1. Let $n \in \mathbb{N}$. We define the set

$$\mathfrak{D}_+ := \{A \in \mathfrak{D} \mid A \text{ is a positivity preserver}\}$$

of all *positivity preservers with constant coefficients* and we define the set

$$\mathfrak{d}_+ := \{A \in \mathfrak{d} \mid \exp(tA) \in \mathfrak{D}_+ \text{ for all } t \geq 0\}$$

of all *generators of positivity preservers with constant coefficients.*

From Theorem 1.2 we see that $D|_{\mathfrak{S}_+} : \mathfrak{S}_+ \to \mathfrak{D}_+$ in (13) is an isomorphism.

Examples 4.2. Let $n \in \mathbb{N}$. Then we have the following:

(a) $c\cdot\Delta = c\cdot(\partial_1^2+\dots+\partial_n^2) \in \mathfrak{d}_+$ for all $c \geq 0$ since Δ generates the heat equation/kernel.
(b) $c_1 \cdot \partial_1 + \dots + c_n \cdot \partial_n \in \mathfrak{d}_+$ for all $c_1, \dots, c_n \in \mathbb{R}$ since ∂_i generates the translation group in the direction $(c_1, \dots, c_n) \in \mathbb{R}^n$. ○

Example 4.3. Let $k \in \mathbb{N}$ with $k \geq 3$. Then $\partial_i^k \notin \mathfrak{d}_+$, see Example 1.4. ○

The following result shows that the cases in Examples 4.2 are the only generators of positivity preservers of finite rank.

Lemma 4.4. *Let* $A = \sum_{j=1}^k a_j \partial_x^j \in \mathfrak{d}_+$. *Then* $k \leq 2$.

Proof. Let $k \geq 3$ and $a_k = 1$. By Example 1.4 (and 4.3) we have that $\exp(\partial_x^k) \notin \mathfrak{D}_+$, i.e., it is not a positivity preserver. Hence, there exists a $f_0 \in \mathbb{R}[x]$ with $f_0 \geq 0$ and $x_0 \in \mathbb{R}$ such that $[\exp(\partial_x^k) f_0](x_0) = -1$.

Assume to the contrary that $A \in \mathfrak{d}_+$. By scaling x and A we have that

$$A_\lambda := \sum_{j=1}^k \lambda^{k-j} a_j \partial_x^j \quad \in \mathfrak{d}_+ \tag{14}$$

holds for all $\lambda > 0$. By Theorem 2.17 we have that $[\exp(A_\lambda) f_0](x_0)$ is continuous in λ. Since $(\exp(A_0) f_0)(x_0) = -1$ there exists a $\lambda_0 > 0$ such that $[\exp(A_{\lambda_0}) f_0](x_0) < 0$, i.e., $A \notin \mathfrak{d}_+$ and therefore we have $k \leq 2$. □

It is easy to see that the previous result also holds for $n \geq 2$. To see this let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 3$. Then from Theorem 1.2 it follows that $\exp(\partial^\alpha) \notin \mathfrak{D}_+$. Choosing the same scaling argument (14) we find $\partial^\alpha \notin \mathfrak{d}_+$.

For $\mathfrak{D}_+$ and $\mathfrak{d}_+$ the following holds.

Corollary 4.5. *Let* $n \in \mathbb{N}$. *Then the following hold:*

(i) $\mathfrak{D}_+$ *is a closed and convex set.*
(ii) $\mathfrak{d}_+$ *is a non-trivial, closed, and convex cone.*

Proof. (i): Follows from Corollary 3.7 (i) with the bijective and linear map D from Definition 3.1 eq. (13).

(ii): For the non-triviality we have the non-trivial Examples 4.2.

For the convexity let $A, B \in \mathfrak{d}_+$. Since $(\mathfrak{d}, \cdot, +)$ is a commutative algebra we have that $\exp(t(A+B)) = \exp(tA)\exp(tB)$ and since the product of two positivity preservers is again a positivity preserver we have $A + B \in \mathfrak{d}_+$.

For the closeness let $A_n \in \mathfrak{d}_+$ for all $n \in \mathbb{N}_0$ with $A_n \to A$ in the Fréchet topology of $\mathbb{R}[[\partial_1, \dots, \partial_n]]$, see Example 2.4. By Theorem 3.3 we have that $\exp : \mathfrak{d} \to \mathfrak{D}$ is smooth, i.e., especially continuous. Hence,

$$\mathfrak{D}_+ \ni \exp(tA_n) \;\to\; \exp(tA) \in \mathfrak{D}_+$$

for all $t \geq 0$ since $\mathfrak{D}_+$ is closed by (i). Hence, $A \in \mathfrak{d}_+$.

For the cone property let $A \in \mathfrak{d}_+$ then also $cA \in \mathfrak{d}_+$ for all $c \geq 0$. $\square$

Corollary 4.6. *Let* $n \in \mathbb{N}$ *and* $A \in \mathfrak{d}$. *Then the following are equivalent:*

(i) $A \in \mathfrak{d}_+$.
(ii) The unique solution p_t *of*

$$\partial_t p = Ap \tag{15}$$

for any initial value $p_0 \in \mathrm{Pos}(\mathbb{R}^n)$ *fulfills* $p_t \in \mathrm{Pos}(\mathbb{R}^n)$ *for all* $t \geq 0$.

Proof. Since $p_t = \exp(tA)p_0$ is the unique solution of the time evolution (15) we have that (i) $\Leftrightarrow$ $\exp(tA)$ is a positivity preserver for all $t \geq 0 \Leftrightarrow$ (ii). $\square$

While we have $\mathfrak{d}_+ \subseteq \log \mathfrak{D}_+$ equality does not hold as we will see in Corollary 4.10. The existence of a positivity preserver is equivalent to the existence of an infinitely divisible representing measure as the following result shows.

Main Theorem 4.7. *Let* $n \in \mathbb{N}$. *Then the following are equivalent:*

(i) $A \in \mathfrak{d}_+$.
(ii) $\exp A$ *has an infinitely divisible representing measure.*
(iii) $\exp(tA)$ *has an infinitely divisible representing measure for some* $t > 0$.
(iv) $\exp(tA)$ *has an infinitely divisible representing measure for all* $t > 0$.

Proof. (i) $\Rightarrow$ (ii): Let $A \in \mathfrak{d}_+$, i.e., $\exp(tA) \in \mathfrak{D}_+$ has a representing measure μ_t for all $t \in [0, \infty)$. Set $\nu_k := (\mu_{1/k!})^{*k!}$. Then ν_k is a representing measure of $\exp A$ for all

$k \in \mathbb{N}$. Since $\mathbb{R}[x_1, \ldots, x_n]$ is an adapted space $(\nu_k)_{k\in\mathbb{N}}$ is vaguely compact by [22, Thm. 1.19] and there exists a subsequence $(k_i)_{i\in\mathbb{N}}$ such that $\nu_{k_i} \to \nu$ and ν is a representing measure of $\exp A$.

It remains to show that ν is infinitely divisible, i.e., for every $l \in \mathbb{N}$ there exists a measure ω_l with $\omega_l^{*l} = \nu$.

Let $l \in \mathbb{N}$. For $i \geq l$ we define

$$\omega_{l,i} := (\mu_{1/k_i!})^{*k_i!/l}$$

i.e., $\omega_{l,i}$ is a representing measure of $\exp(A/l)$. Again, since $\mathbb{R}[x_1, \ldots, x_n]$ is an adapted space by [22, Thm. 1.19] there exists a subsequence $(i_j)_{j\in\mathbb{N}}$ such that ω_{l,i_j} converges to some ω_l, i.e., $\omega_{l,i_j} \xrightarrow{j\to\infty} \omega_l$. Hence,

$$(\omega_l)^{*l} = \lim_{j\to\infty} (\omega_{l,i_j})^{*l} = \lim_{j\to\infty} \nu_{k_{i_j}} = \nu,$$

i.e., ν is divisible by all $l \in \mathbb{N}$ and hence ν is an infinitely divisible representing measure of $\exp A$.

(ii) $\Rightarrow$ (i): Let μ_1 be an infinitely divisible representing measure of $\exp A$. Then

$$\mu_q := \mu_1^{*q}$$

exists for all $q \in \mathbb{Q}\cap[0, \infty)$ and it is a representing measure of $\exp(qA)$, i.e., $\exp(qA) \in \mathfrak{D}_+$ for all $q \in [0, \infty) \cap \mathbb{Q}$. Since by Theorem 2.17 $\exp : \mathfrak{d} \to \mathfrak{D}$ is continuous and by Corollary 4.5 (i) $\mathfrak{D}_+$ is closed we have that $\exp(qA) \in \mathfrak{D}_+$ for all $q \geq 0$. Hence, we have $A \in \mathfrak{d}_+$.

(iv) $\Rightarrow$ (iii): Clear.

(iii) $\Rightarrow$ (i): By "(ii) $\Leftrightarrow$ (i)" we have that $qtA \in \mathfrak{d}_+$ for $t > 0$ and all $q \in [0, \infty) \cap \mathbb{Q}$. Since $\mathfrak{d}_+$ is closed by Corollary 4.5 (ii) we have $q_i tA \to A \in \mathfrak{d}_+$ for $q_i \in \mathbb{Q}$ with $q_i \to t^{-1}$ as $i \to \infty$.

(i) $\Rightarrow$ (iv): Since $A \in \mathfrak{d}_+$ and $\mathfrak{d}_+$ is a closed convex cone by Corollary 4.5 (ii) we have that $tA \in \mathfrak{d}_+$ for all $t > 0$ and hence by "(i) $\Leftrightarrow$ (ii)" we have that $\exp(tA)$ has an infinitely divisible representing measure for all $t > 0$. $\square$

Example 4.8 *(Example 4.2 (b) continued).* Let $n = 1$ and $a \in \mathbb{R}$. Then

$$\exp(a\partial_x) = \sum_{k\in\mathbb{N}_0} \frac{a^k}{k!} \cdot \partial_x^k$$

is represented by $\mu = \delta_a$ since δ_a is the representing measure of the moment sequence $(a^k)_{k\in\mathbb{N}_0}$. For any $r > 0$ we have $\delta_{a/r}^{*r} = \delta_a$, i.e., δ_a is infinitely divisible. In fact, δ_a are the only compactly supported infinitely divisible measures, see e.g. [14, p. 316]. Hence, by Main Theorem 4.7 we have $\partial_x \in \mathfrak{d}_+$. $\circ$

Example 4.9. Let $A \in \mathfrak{D}_+$ be the positivity preserver represented by the measure

$$\mathrm{d}\mu = \chi_{[0,1]^n}\ \mathrm{d}\lambda$$

where λ is the n-dimensional Lebesgue measure and $\chi_{[0,1]^n}$ is the characteristic function of $[0,1]^n$. Since $\operatorname{supp}\mu$ is compact μ is unique.

It is known that the only infinitely divisible measures with compact support are δ_x for $x \in \mathbb{R}^n$, see e.g. [14, p. 316]. Therefore, we have that μ is not infinitely divisible and hence $\log A \notin \mathfrak{d}_+$. ∘

The previous example implies that the inclusion $\mathfrak{d}_+ \subseteq \log \mathfrak{D}_+$ is proper.

Corollary 4.10. *Let $n \in \mathbb{N}$. Then $\mathfrak{d}_+ \subsetneq \log \mathfrak{D}_+$.*

Proof. We have $\log \mathfrak{D}_+ \setminus \mathfrak{d}_+ \neq \emptyset$ by Example 4.9. □

We have seen in Main Theorem 4.7 the one-to-one correspondence between a positivity preserver $A \in \mathfrak{D}_+$ having an infinitely divisible representing measure and $A \in \mathfrak{D}_+$ having a generator. The infinitely divisible measures are fully characterized by the Lévy–Khinchin formula, see Theorem 2.18. The Lévi–Khinchin formula is used in the following result to fully characterize the generators $\mathfrak{d}_+$ of the positivity preservers $\mathfrak{D}_+$.

Main Theorem 4.11. *Let $n \in \mathbb{N}$. The following are equivalent:*

(i) $A = \sum_{\alpha \in \mathbb{N}_0^n \setminus \{0\}} \frac{a_\alpha}{\alpha!} \cdot \partial^\alpha \in \mathfrak{d}_+$.

(ii) There exists a symmetric matrix $\Sigma = (\sigma_{i,j})_{i,j=1}^n \in \mathbb{R}^n$ with $\Sigma \succeq 0$, a vector $b = (b_1, \dots, b_n)^T \in \mathbb{R}^n$, and a measure ν on $\mathbb{R}^n$ with

$$\int_{\|x\|_2 \geq 1} |x_i|\ \mathrm{d}\nu(x) < \infty \qquad \text{and} \qquad \int_{\mathbb{R}^n} |x^\alpha|\ \mathrm{d}\nu(x) < \infty$$

for all $i = 1, \dots, n$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \geq 2$ such that

$$a_{e_i} = b_i + \int_{\|x\|_2 \geq 1} x_i\ \mathrm{d}\nu(x) \qquad \text{for all } i = 1, \dots, n,$$

$$a_{e_i+e_j} = \sigma_{i,j} + \int_{\mathbb{R}^n} x^{e_i+e_j}\ \mathrm{d}\nu(x) \qquad \text{for all } i,j = 1, \dots, n,$$

and

$$a_\alpha = \int_{\mathbb{R}^n} x^\alpha\ \mathrm{d}\nu(x) \qquad \text{for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \geq 3.$$

Proof. By Main Theorem 4.7 "(i) ⇔ (ii)" we have that (i) $A \in \mathfrak{d}_+$ if and only if $\exp A$ has an infinitely divisible representing measure μ, i.e., by Theorem 1.2 we have

$$\exp A = \sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} \cdot \int_{\mathbb{R}^n} x^\alpha \,\mathrm{d}\mu(x) \cdot \partial^\alpha. \tag{16}$$

By Theorem 2.17 we can take the logarithm and hence (16) is equivalent to

$$A = \sum_{\alpha\in\mathbb{N}_0^n\setminus\{0\}} \frac{a_\alpha}{\alpha!} \cdot \partial^\alpha = \log\left(\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} \cdot \int_{\mathbb{R}^n} x^\alpha \,\mathrm{d}\mu(x) \cdot \partial^\alpha \right). \tag{17}$$

With the isomorphism

$$\mathbb{C}[[\partial_1,\dots,\partial_n]] \to \mathbb{C}[[t_1,\dots,t_n]], \quad \partial_1 \mapsto it_1,\ \dots,\ \partial_n \mapsto it_n$$

we have that (17) is equivalent to

$$\sum_{\alpha\in\mathbb{N}_0^n\setminus\{0\}} \frac{a_\alpha}{\alpha!} \cdot (it)^\alpha = \log\left(\sum_{\alpha\in\mathbb{N}_0^n} \frac{1}{\alpha!} \cdot \int_{\mathbb{R}^n} x^\alpha \,\mathrm{d}\mu(x) \cdot (it)^\alpha \right). \tag{18}$$

But the right hand side of (18) is now the characteristic function

$$= \log \int e^{itx} \,\mathrm{d}\mu(x)$$

of μ. Hence, by the Lévy–Khinchin formula (see Theorem 2.18) we have

$$= ibt - \frac{1}{2} t^T \Sigma t + \int (e^{itx} - 1 - itx \cdot \chi_{\|x\|_2<1}) \,\mathrm{d}\nu(x). \tag{19}$$

After a formal power series expansion of e^{itx} in the Fréchet topology of $\mathbb{C}[[x_1,\dots,x_n]]$, see Example 2.4, and a comparison of coefficients we have that (19) is equivalent to (ii) which ends the proof. □

From the previous result we see that the difference between $\mathfrak{d}_+$ and a moment sequence is that the representing (Lévy) measure ν in (19) can have a singularity of order ≤ 2 at the origin.

5. A strange action on Pos$([0,\infty))$

Before we end this work we want to discuss an example of a strange positivity action on $[0,\infty)$. We take the following example.

Example 5.1. Let $\Delta = \partial_x^2$ on $L^2([0,\infty),\mathbb{R})$ with Dirichlet boundary conditions

$$\begin{aligned} \partial_t u &= \Delta u \\ u(0,t) &= 0 \quad \text{for all } t \geq 0 \\ u(\,\cdot\,,0) &= u_0 \in L^2([0,\infty),\mathbb{R}). \end{aligned}$$

Then

$$u(x,t) = \int_0^\infty k_t(x,y) u_0(y)\ \mathrm{d}y$$

with

$$k_t(x,y) = \frac{1}{\sqrt{4\pi t}} \cdot e^{-(x-y)^2/4} \cdot \left[1 - e^{xy/t}\right]$$

for all $x, y \geq 0$ and $t > 0$, see [8, Exm. 4.1.1].

For $u_0 \in \mathcal{S}([0,\infty),\mathbb{R})$ a Schwartz function on $[0,\infty)$ we have that $k_t * u_0 \in \mathcal{S}([0,\infty),\mathbb{R})$ for all $t > 0$ and hence we look at the time-dependent moments

$$s_j(t) = \int_K x^j \cdot u(x,t)\ \mathrm{d}x = \int_{[0,\infty)^2} x^j \cdot k_t(x,y) \cdot u_0(y)\ \mathrm{d}x\ \mathrm{d}y$$

for all $j \in \mathbb{N}_0$ and $t > 0$.

The action on $\mathbb{R}[x]$ is then given by $(T_t p)(x) = \int_0^\infty p(y) \cdot k_t(x,y)\ \mathrm{d}y$. We find for $p = 1$ that

$$(T_t 1)(x) = \int_0^\infty k_t(x,y)\ \mathrm{d}y$$

is a continuous, non-decreasing function in $x \in [0,\infty)$ with $(T_t 1)(0) = 0$ and $\lim_{x\to\infty} (T_t 1)(x) = 1$, i.e., $T_t 1 \notin \mathbb{R}[x]$ for any $t \in (0,\infty)$. While for $p = x^j$ with $j \in \mathbb{N}$ we get for the time-dependent moments by partial integration

$$\begin{aligned} \partial_t s_j(t) &= \partial_t \int_0^\infty x^j \cdot u(x,t)\ \mathrm{d}x \\ &= \int_0^\infty x^j \cdot u(x,t)''\ \mathrm{d}x \\ &= \underbrace{x^j \cdot u(x,t)' \Big|_{x=0}^{\infty}}_{=0} - \int_0^\infty j \cdot x^{j-1} \cdot u(x,t)'\ \mathrm{d}x \end{aligned}$$

$$= \underbrace{-j \cdot x^{j-1} \cdot u(x,t) \Big|_{x=0}^{\infty}}_{=0} + \int_0^\infty j \cdot (j-1) \cdot x^{j-2} \cdot u(x,t) \, \mathrm{d}x$$
$$= j \cdot (j-1) \cdot s_{j-2}(t).$$

These are the recursive relations we already encountered with the heat equation on $\mathbb{R}$, see [5,6]. Therefore, for *odd* polynomials $p \in \mathbb{R}[x]$ we have

$$(T_t p)(x) = \int_0^\infty p(y) \cdot k_t(x,y) \, \mathrm{d}y = (e^{t\partial_x^2} p)(x) \in \mathbb{R}[x]$$

with $T_t x^{2d+1} = \mathfrak{p}_{2d+1}(x,t)$ for all $d \in \mathbb{N}_0$ in [6, Dfn. 3.1] and if additionally $p \geq 0$ on $[0,\infty)$, then $T_t p \geq 0$ on $[0,\infty)$. On the other hand for *even* polynomials $p \in \mathbb{R}[x]$ we find that $T_t p \in C^\infty([0,\infty)) \setminus \mathbb{R}[x]$ but at least $T_t p \geq 0$ on $[0,\infty)$. The reason for this strange behavior is of course the Dirichlet boundary condition and the resulting reflection principle. This effect needs further investigations. ∘

This example shall also serve as a warning. Just because the operator (symbol) is ∂_x^2 does not mean that the positivity preservers are $\exp(t\partial_x^2)$. The domain and the corresponding boundary conditions have to be taken into account as is done and is well-known in the partial differential equation literature and semi-group theory. Besides the Open Problem 6.1 below this is another direction where further studies have to be done.

6. Summary and open question

If

$$A = \sum_{\alpha \in \mathbb{N}_0^n} q_\alpha \cdot \partial^\alpha \quad \text{with} \quad q_\alpha \in \mathbb{R}[x_1,\dots,x_n]_{\leq |\alpha|} \tag{20}$$

then

$$A\mathbb{R}[x_1,\dots,x_n]_{\leq d} \subseteq \mathbb{R}[x_1,\dots,x_n]_{\leq d} \quad \text{for all} \quad d \in \mathbb{N}_0, \tag{21}$$

i.e., A is called *degree preserving*. In fact for any linear $A : \mathbb{R}[x_1,\dots,x_n] \to \mathbb{R}[x_1,\dots,x_n]$ we have (20) ⇔ (21). (20) ⇒ (21) is clear. For (21) ⇒ (20) take in (20) the smallest α with respect to the lex-order such that $\deg q_\alpha > |\alpha|$. Then $\deg Ax^\alpha > |\alpha|$ which is a contradiction to (21).

Similar to Corollary 4.6 we have that the partial differential equation

$$\partial_t p(x,t) = Ap(x,t) \quad \text{with} \quad p(x,0) = p_0(x) \in \mathbb{R}[x_1,\dots,x_n]$$

and A as in (20) is equivalent to a linear system of ordinary differential equations in the coefficients of p. Hence, it has a unique solution $p(\,\cdot\,,t) = \exp(tA)p_0$ for all $t \in \mathbb{R}$, i.e., $\exp(tA)$ is well-defined for any A as in (20) and $t \in \mathbb{R}$. If additionally A is a positivity preserver then $\exp(tA)$ is a degree preserving positivity preserver with polynomial coefficients for all $t \geq 0$. In Main Theorem 4.11 we have already seen for the positivity preservers with constant coefficients that additional A appear. The restriction that A is degree preserving is a natural restriction since otherwise $\exp(tA)\mathbb{R}[x_1,\dots,x_n] \not\subseteq \mathbb{R}[x_1,\dots,x_n]$ for any $t > 0$. So we arrive at the following open problem.

Open Problem 6.1. What is the description of the set

$$\left\{ A = \sum_{\alpha\in\mathbb{N}_0^n} q_\alpha \cdot \partial^\alpha \;\middle|\; \exp(tA) \text{ is a positivity preserver for all } t \geq 0 \right\}$$

of all positivity generators, i.e., not necessarily with constant coefficients?

The constant coefficient case is completely solved by Main Theorem 4.11. Part of the non-constant coefficient cases are covered by the previous case that A is a degree preserving positivity preserver. For the non-constant coefficient cases note that we are then in the framework of a non-commutative infinite dimensional Lie group and its Lie algebra. Here the theory is much richer, see e.g. [17,12,18,25,29,23] and references therein.

Funding

The author and this project are supported by the Deutsche Forschungsgemeinschaft DFG with the grant DI-2780/2-1 and his research fellowship at the Zukunfskolleg of the University of Konstanz, funded as part of the Excellence Strategy of the German Federal and State Government.

Data availability

No data was used for the research described in the article.

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