Universität Konstanz

Out-of-Sample Performance of Norm-Constrained Portfolios

Bachelor Thesis

submitted by

Marlon Haudek

Supervisor: Prof. Dr. Fischer

Konstanz 2021

Abstract

This paper extends the academic research around the Norm-Constrained Portfolio Strategies introduced by DeMiguel et al. in 2009 in terms of out-of-sample performance evaluation on the German Stock Market (DAX) for the years 2003 – 2020. Therefore, four different variations of the Norm-Constrained Portfolios are used that are: 1) 1-Norm Constrained Portfolio calibrated for low variance, 2) 1-Norm Constrained Portfolio calibrated for maximum return, 3) 2-Norm Constrained Portfolio calibrated for low variance, and 4) 2-Norm Constrained Portfolio calibrated for maximum portfolio return. A rolling window approach is applied to evaluate out-of-sample performance. Thereby Return, Variance, and Sharpe Ratio are taken as performance measurements. Empirical results show that the 1-Norm Constrained Portfolio calibrated for low variance is able to provide a significantly lower variance than the DAX given same return. However, this result may be viewed with a grain of salt since the corresponding Sharpe Ratio is not superior as it should be compared to the DAX' one. Hence the result may suffer from a calculation error which I was not able to locate.

Table of Contents

List of Figures

List of Abbreviations

MVP = minimum variance portfolio

DAX = German major market index

NC1V = 1-Norm Constrained Portfolio calibrated for low variance

NC1R = 1-Norm Constrained Portfolio calibrated for maximum portfolio return

NC2V = 2-Norm Constrained Portfolio calibrated for low variance

NC2R = 2-Norm Constrained Portfolio calibrated for maximum portfolio return

1. Introduction

Stock markets throughout history have shown periods of greed with high returns on stock investments and periods of fear where investors often lost all the profits they made and more. If you look at the chart of the major German stock market index, the DAX, you can easily see the periods of greed and fear. For example, in the years 1999 - 2003, there was initially a period of greed, which realized an increase in the stock market of about 57%. After the euphoria turned into fear, the DAX lost about 65%. Today, this event is known as the dot-com bubble. Other examples of such events are the financial crisis, which resulted in a painful drop of about 45%, or the Corona crisis, which led to a 30% drop in the German stock index. From the point of view of a long-term investor, such periods are very volatile and associated with the loss of a decent part of the hardly gained wealth.

To minimize the painful effects of a stock market decline and to achieve more constant returns, investor demand for less volatile portfolio strategies is increasing. This is where the norm-constraint portfolio strategy invented by DeMiguel et al. in 2009 comes in. This strategy extends the minimum variance portfolio (MVP) invented by Markowitz in 1952. While the MVP aims to construct a portfolio that has the lowest variance, the norm-constraint strategy adds an additional constraint to the minimization problem (DeMiguel, Garlappi, Nogales, & Uppal, 2009a). Now the optimization of the Norm-Constraint Portfolio consists of two constraints. The first constraint ensures that all weights sum up to one. The second additional constraint ensures that the norm of the portfolio vector is smaller than a given threshold (DeMiguel et al., 2009a).

Since there is no academic research on this topic done for the German stock market, I want to close this gap by offering some academic insights for the out-of-sample performance of the Norm-Constraint Portfolio Strategy for the major German stock market index DAX. In the following paper I want to research around the topic: “**Out-of-Sample Performance of Norm-Constrained Portfolios**” by testing the following hypotheses:

Hypothesis I: Norm-Constraint Portfolio Strategies composed only of stocks from the DAX had a higher return than the DAX itself as a benchmark index from 2003 – 2020.

Hypothesis II: Norm-Constraint Portfolio Strategies composed only of stocks from the DAX had a lower variance than the DAX itself as a benchmark index from 2003 – 2020.

Hypothesis III: Norm-Constraint Portfolio Strategies composed only of stocks from the DAX had a higher Sharpe Ratio than the DAX itself as a benchmark index from 2003 – 2020.

The empirical results show that the Norm-Constraint Portfolio Strategies mostly do not outperform the value-weighted German market index DAX in the years 2003 – 2020. Only one out of four different Norm-Constraint Portfolios has a lower variance than the DAX given the same return.

The following paper is structured as follows. In Section 2 I provide some background information and the framework of the Norm-Constraint Strategy. Furthermore, I summarize the current empirical findings of the literature. Section 3 describes the methodology and data I use and explains the setup of the model. Section 4 presents the empirical findings and compares them to other existing empirical findings. The paper ends with a conclusion in section 5.

2. Review of literature

Now I would like to give some background information on the optimization problem used in this paper. Furthermore, this section gives the reader an overview of the academic findings for the minimum variance problem and its extension to norm-constrained portfolio selection since its introduction by DeMiguel et al. in 2009.

2.1 Framework of Norm-Constrained Portfolio Strategy

The Norm-Constraint Portfolio Strategy consists of two parts. Which are marked in the following optimization problem:

Part 1

$$\min_{w} w^T \hat{\sum} w \quad (1)$$

$$s.t. \ \ w^T e = 1 \quad (2)$$

Part 2

$$||w|| \leq \delta \quad (3)$$

The first part was actually not invented by DeMiguel et al. themselves. It was invented by Nobel Prize winner Harry Markowitz in 1952 and is called the minimum variance portfolio (MVP). Harry Markowitz was the founding father of modern portfolio theory, in which the MVP can be classified. The second part is the extension of DeMiguel et al. in 2009 to the norm-constraint portfolio problem. To understand the idea of the optimization problem, I give some important background on the MVP and the norm-constraint extension.

2.1.1 Minimum-Variance Portfolio

The actual approach from Markowitz in his 1952 paper "Portfolio Selection" was the mean-variance approach. Its focus is on an investor who should construct a portfolio that delivers a maximum expected return for a given minimum variance (Markowitz, 1952). In this context, it is important to note that the portfolio with the maximum expected return is not necessarily the portfolio with the lowest variance (Markowitz, 1952). Furthermore, Markowitz stated that one cannot completely eliminate the variance of a portfolio through diversification. Another important statement made by Markowitz was that to minimize variance it is not enough to simply invest in many stocks, one must invest in different sectors of the economy. For example, the variance of a portfolio that invests only in oil companies has a higher variance than a portfolio with the same amount of stocks but consisting of stocks from different sectors such as oil, technology, pharmaceuticals and consumer goods.

Given these thoughts, one can construct a line of portfolios with different expected returns for a given possible minimum variance. This line is called the efficient frontier (figure 1).

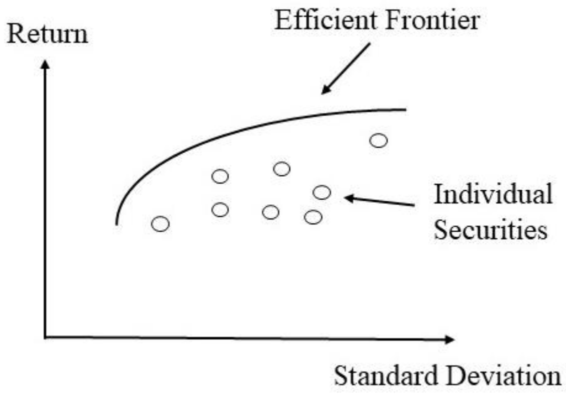

Figure 1: Framework Efficient Frontier (composed by author)

The efficiency frontier specifies all optimal portfolios with a maximum expected return for a given minimum variance. All portfolios below the efficient frontier can be substituted with a portfolio that has a higher expected return for the same minimum variance. One can see by looking at the efficient frontier that an investor can choose a portfolio that has a higher expected return, but at the cost of a higher variance or vice versa.

To calculate these portfolios there are two parameters necessary. First, the expected future returns to calculate the expected means. Second, the expected variances of the stocks to calculate the covariance matrix. Since future expected terms have to be estimated there are always estimation errors. It turns out that it is more difficult to estimate futures means than covariances and the estimation error of the expected mean is much larger and has a greater impact on the outcome of the optimization problem than the covariances (Merton, 1980). Further academy research shows, that the estimation error of expected means is so large that it is not very different when ignoring them (Jagannathan & Ma, 2003). This is the reason why most of the literature today only focuses on the MVP, which is also located on the efficient frontier (figure 2), but it does not rely on expected means.

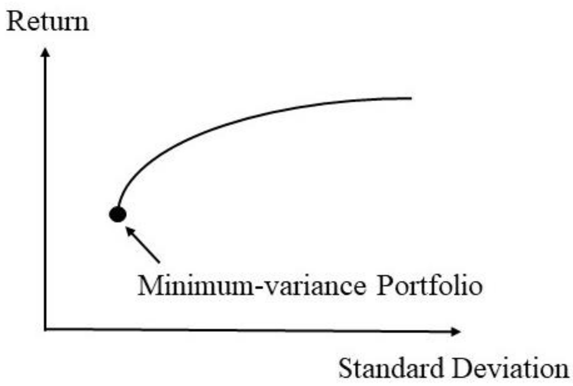

Figure 2: Minimum-variance Portfolio at the Efficient Frontier (composed by author)

In formal terms, the mean-variance approach is reduced to the MVP and is the solution to the following optimization problem approach:

$$\min_{w} \mathrm{w}^{\mathrm{T}} \hat{\sum} w \tag{4}$$

$$s.t. \;\; w^{T}\mathrm{e} = 1 \tag{5}$$

vector w stands for portfolio weights, $\hat{\Sigma}$ is the estimated covariance matrix of all stocks in the portfolio, vector e consists of ones. (4) is the minimization problem, in which optimal portfolio weights are assigned to the covariance matrix such that the variance of the portfolio is minimized. This optimization is bounded through the constraint (5) such that in sum all portfolio weights must be one and thus all capital is invested into the portfolio.

2.1.2 Extension to the Norm-Constrained Portfolio Strategy

Since there is only one parameter left to estimate in the MVP and the estimation error for this parameter is smaller than in the mean-variance approach, the literature is searching for a way to deal with the remaining estimation error. One approach to deal with the estimation error from the covariance is the Norm-Constrained Portfolio approach (DeMiguel et al., 2009a). This general approach allows calibrating the model only by using historical data, which results in an improvement of out-of-sample performance (DeMiguel et al., 2009a). Despite this advantage compared to the mean-variance approach from Markowitz this approach has the disadvantage that the estimation error and complexity grow by the equities in the portfolio (DeMiguel et al., 2009a). Other portfolio optimization approaches like the parametric portfolio policy by Brandt et al. in 2004 do not suffer from this problem. Their complexity only grows with the number of firm parameters used for the calibration of the model and not with the number of assets in the portfolio (Brandt, Santa-Clara, & Valkanov, 2004). To my best knowledge this disadvantage of the Norm-Constraint Portfolio cannot be overcome until today.

The name of the Norm-Constrained Portfolio Strategy comes from the method used in the optimization problem. The optimization problem optimizes the weights of each equity and "norms" the equity-weights, such that they are smaller than a given threshold. This method is introduced by the second constraint which is added to the MVP. (6) states the second constraint:

$$||w|| \leq \delta \tag{6}$$

in which δ is the threshold value for which ||w||, the norm of the portfolio-weight vector must be equal or smaller. The vector is defined for the 1-norm and 2-norm as:

$$||w||_p = \left(\sum_{i=1}^{N} |w_i|^p\right)^{1/p} \tag{7}$$

for p = 1 and 2, respectively.

For the rest of the paper the 1-Norm and 2-Norm Constrained Portfolios are the focus of interest. The two portfolios are now explained to the reader more in detail.

2.1.2.1 1-Norm Constrained Portfolio Strategy

To modify the general Norm-Constrained Portfolio Optimization Approach to the 1-Norm Constrained Portfolio the second constraint is reformulated to (10) resulting in the 1-Norm Constrained Portfolio Problem:

$$\min_{w} \mathrm{w}^T \hat{\sum} w \tag{8}$$

$$s.t. \;\; w^T \mathrm{e} = 1 \tag{9}$$

$$\sum_{i=1}^{N} |w_i| \leq \delta \tag{10}$$

The 1-Norm Constrained Portfolio allows the investor to limit the total amount of short-selling in the portfolio by optimizing the portfolio-weights vector to be strictly smaller than one (DeMiguel et al., 2009a). Equation (10) can be reformulated to equation (11):

$$-\sum_{i \epsilon N(W)} w_i < \frac{\delta - 1}{2} \tag{11}$$

which shows the short-sale budget $\frac{\delta-1}{2}$ and the total proportion of wealth that is sold short $-\sum_{i \epsilon N(W)} w_i$. Properties of the 1-Norm Constraint Portfolio are, that the short-sale budget can be freely distributed among all equities and in some cases, it assigns zero weights to certain equities (DeMiguel et al., 2009a). These kinds of portfolios have become quite popular over time and are the best choice for investors who want to limit their short-selling budget (DeMiguel et al., 2009a).

2.1.2.2 2-Norm Constrained Portfolio Strategy

To modify the general Norm-Constrained Portfolio Optimization Approach to the 2-Norm Constrained Portfolio the second constraint is reformulated to (14) resulting in the 2-Norm Constrained Portfolio Problem:

$$\min_{w} \mathrm{w}^T \hat{\sum} w \tag{12}$$

$$s.t.\ \ w^T \mathrm{e} = 1 \tag{13}$$

$$\sum_{i=1}^{N} w_i^2 \leq \delta \tag{14}$$

Compared to the 1-Norm Constrained Portfolio the 2-Norm Constrained Portfolio invests into all equities of the portfolio (DeMiguel et al., 2009a). It either assigns to every asset positive weights, which stands for long positions or negative weights, which stands for short positions. Further the 2-Norm Constrained Portfolio is very close to the 1/N Portfolio (DeMiguel et al., 2009a). In conclusion, the 2-Norm Constrained Portfolio is the best choice for an investor who prefers a well-diversified portfolio.

2.2 State of research on MVP and Norm-Constrained Portfolio Strategy

Since the publishment of the Norm-Constrained Portfolio Strategy of DeMiguel et al. in 2009 a decent amount of time passed by and many empirical investigations were done to improve the general framework.

In 2011 there was an introduction of flexible upper and lower bounds to each portfolio-weight to manage the tradeoff between the reduction of the sampling error and the loss of sample information (Behr, Guettler, & Miebs, 2011). To bound the estimation risk in a large L1 Norm Constrained Portfolio a gross-exposure constraint on the portfolio allocation vector was invented (Fan, Zhang, & Yu, 2012). After investigation of reducing estimation risk and estimation error was done, further research was contributed regarding the absolute weight in the Norm Constrained Portfolios. Since large absolute, especially negative, values are very unfavorable in a portfolio an additional L1 norm constraint has been added to the minimum variance problem such that it allows to control the largest absolute value a portfolio-weight can take (Xing, Hu, & Yang, 2014). Besides modifying the MVP by additional constraints another approach has been shown up which implements an L-regularization term directly into the objective function. This is a reformulation of the mean-variance problem of Markowitz to a constrained least-squares regression problem which regularizes the optimization problem, encourages for sparse portfolios (portfolio with only a few stocks), and allows to take transaction costs into account (Brodie, Daubechies, Mol, Giannone, & Loris, 2009). In the next years empirical research found that the Norm Constraint Portfolio selection method tries to obtain stable portfolios, whereas the objective function regularization method tries to obtain sparse portfolios (Dai & Wen, 2018). One of the most current literature refers to the mean-variance portfolio and how to improve the poor out-of-sample performance caused by the estimation errors of the mean return and the covariance matrix. It was considered a new efficient mean-variance portfolio by using the L1-regularization in the objective function, the shrinkage method of Ledoit and Wolf from the Journal of Economics, 2003, 10, 603-621, and a robust optimization method (Dai & Kang, 2021). The results of the proposed strategies showed a better out-of-sample performance (Dai & Kang, 2021).

To sum up, the amount of empirical research and new findings to the MVP and the Norm-Constraint Portfolio show a strongly positive result for using such a portfolio strategy for investments nowadays. With the background information about the Norm-Constraint Portfolio Strategy and its current state of research the paper now turns to the application.

3. Methodology & Data

The focus of this section is to provide the reader an insight into the methodology and structure which is used to answer the hypotheses of this paper. Further, the section answers the question of why the DAX is chosen as a benchmark index and which data source is used to feed the model.

3.1 Methodology

In general, the concept of the methodology leads to different distributions of each performance measures for the 1-Norm, 2-Norm Constrained Portfolio Strategy and the DAX as a benchmark index. For evaluation of the performance return, variance and Sharpe Ratio are used. Figure 3 shows the structure in form of steps of the methodology.

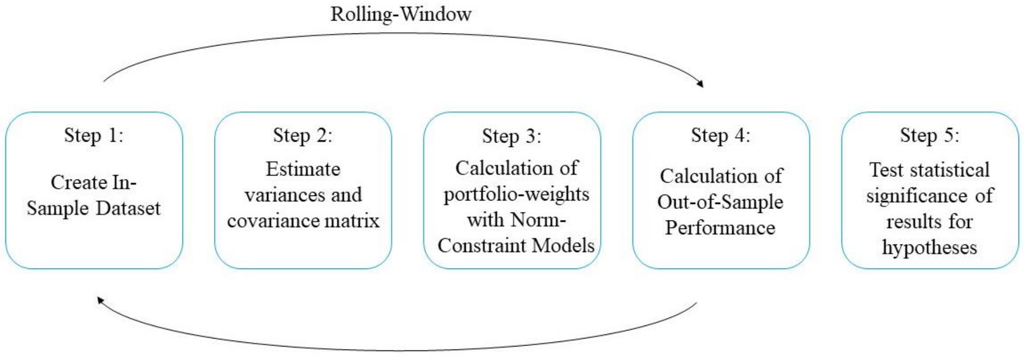

Figure 3: Application of model (composed by author)

In the following for each step further information for a better understanding are provided.

Step 1: Create In-Sample Dataset

The in-sample Dataset consists of 60 Periods to estimate the covariance matrix. This in-sample period length is used by DeMiguel et al. for a conceptual similar optimization problem (DeMiguel, Garlappi, & Uppal, 2009b). Further, the dataset uses monthly data instead of daily data. The reason therefore is nested in the less idiosyncratic risk in monthly data compared to daily data. This is supported by a statement of Eugene Fama:

"The usefulness of the portfolio model depends not on whether the normality assumption which underlies it is an exact description of the world (we know it is not), but on whether the model yields useful insights into the essential ingredients of a rational portfolio decision. Likewise, the usefulness of the model for securities prices depends on how well it describes observed relationships between average returns and risk. If the model does well on this score, we can live with the small observed departures from normality in monthly returns, at least until better models come along." (Fama, 1976)

The dataset contains all stocks of the DAX at the current state in the last month of the in-sample period. From this point in time, it uses the last 60 periods (months) for the calibration of the model.

Step 2: Estimate expected variances to calculate the covariance matrix

In order to calculate the covariance matrix several computational steps have to be done before. First, the average monthly adjusted close prices of each stock are used to calculate the returns. Next, the returns are used to calculate the expected variances, which then are used to construct the covariance matrix. The advantage of using average monthly close prices is the reduction of idiosyncratic risk and they provide more information on the overall market behavior from the month. The reason for using adjusted close prices instead of close prices is because they reflect a stock's value after accounting for any corporate actions like dividend payments (Norton, 2011).

Step 3: Calculate the portfolio-weights of each stock by using 1-Norm and 2-Norm Constraint Portfolio model

Since the formula of the 1-Norm and 2-Norm Constrained Portfolio is explained in sections 2.1.2.1 and 2.1.2.2 there is no more explanation provided here. If necessary, the reader can look up the corresponding sections. On the other hand, a point that needs more explanation is the optimization problem itself. Namely, to calculate the optimal portfolio-weights the optimization problem requires for program-technically reasons an initial guess of the weights. I calibrate the initial guess of the weights such that all stocks of the portfolio are equally weighted by 1/N, where N is the number of stocks that are in the portfolio. Second, the threshold values used in the optimization of both strategies are the same as in the paper of DeMiguel et al. in 2009. More precisely, I use the same threshold values for the 25FF dataset from their paper, since there are almost the same number of stocks (25 stocks) in this dataset as in my DAX dataset (30 stocks). Moreover, in the paper they list different threshold values which on the one hand calibrate the models for a maximum return or on the other hand for a low variance (DeMiguel et al., 2009a). The corresponding threshold values are listed in table 1 in the appendix. The resulting optimized portfolio-weights are rounded to four digits for overview purposes.

Step 4: Calculation of Out-of-Sample Performance

After receiving the optimal portfolio-weights I apply them to the portfolio in an out-of-sample period. Whereby out-of-sample period is the next month following after the 60 periods in-sample periods. For the performance evaluation three different performance measures are used: Return, Variance, and Sharpe Ratio. The return is the percentage change of the optimized portfolio from the last in-sample period to the out-of-sample period. To calculate the variance an out-of-sample covariance-matrix is constructed for which the optimized portfolio-weights are applied. The Sharpe Ratio is calculated by the following formula:

$$Sharpe\ Ratio = \frac{Portfolio\ Return\ - Risk\ Free\ Rate}{Portfolio\ Volatility} \tag{15}$$

As a risk-free rate, a 1-month risk-free rate of 0.1307% is used. This is the calculated mean of the Germany 10-Years Bond Yield for the sample period. For the calculation, the data is downloaded from the website investing.com. All performance measures for each portfolio strategy are saved in different datasets.

After this step is done, to obtain more out-of-sample performances the 60 in-sample periods are shifted one period forward as a rolling window and steps 1-4 are repeated. One important remark for step 1: By every forward shifting of the in-sample it is checked whether a company leaves the DAX for a company that joins it such that the in-sample always consists of the current DAX composition on a monthly basis. The overall sample in which the rolling-window is applied is from the beginning of 2003 until the end of 2020. Through that enough out-of-sample performance data points can be generated to provide a statistical power of the results. Since older historical data for the in-sample window is missing, it is not possible to start further backwards in the history.

Step 5: Test the results for statistical significance and check hypotheses

To check the hypotheses stated in section 1 the results are tested for statistical significance. Therefore, a T-test is used to compare whether the distributions of the resulting performance measures of the 1-Norm and 2-Norm Constrained Portfolio Strategies and the DAX are different. As a choice of significance level at which the Null Hypotheses are rejected, the common values of 0.01, 0.05, and 0.1 are used.

3.2 Data

I want to close this section by providing insights into the data used for this paper. More precisely, I want to answer the questions of why the DAX is chosen as a database for this paper and which data source is chosen. Let us start with the question of why the DAX is chosen as a database. To my best knowledge there is no academic research done on the Norm-Constrained Portfolio Strategy which uses the DAX as a database until today. For example, the paper of DeMiguel et al. uses randomly selected stocks as a database to calculate the performance measures. For this reason, I aim to fill this gap in academic Norm-Constrained Strategy research and provide insights for investors into the out-of-sample performance of the DAX. Now let us take a closer look at which data source is chosen. I use Yahoo Finance as a data source for two reasons. First, the data is easily available for all stocks that are used in the model to calculate the performance measures. Second, there is a long history of available data, so there is no restriction on what data can be used.

4. Empirical Findings

This section is divided into two subsections. The first subsection is about the stated hypotheses in section 1 regarding the performance of the Norm-Constrained Portfolio Strategies compared to the DAX as a benchmark index. The second subsection is a comparison between my empirical findings and other empirical findings. All empirical results from this paper are listed in appendix 2.

4.1 Hypotheses

Hypothesis I:

Let us start by looking at the results regarding the 1-Norm Constrained Strategies (NC1V, NC1R) and the DAX. The Null Hypothesis for monthly equal return distributions of the NC1V and the DAX cannot be rejected even at a significance level of 10%. The same rejection appears for the NC1R and the DAX. This means that neither the average monthly return of the NC1V and the DAX nor the average monthly return of the NC1R and the DAX are significantly different from each other. Whereas both Null Hypotheses for monthly equal return distributions of NC2V and DAX, and NC2R and DAX can be rejected at a significance level of 10%. This means that both 2-Norm Constrained Strategies have different monthly returns compared to the DAX. The average monthly returns of both 2-Norm Constrained Strategies are lower than the monthly return of the DAX.

According to these results *Hypothesis I is fully rejected*. None of the Norm-Constrained Strategies manages to get a significantly higher return than DAX from 2003 - 2020.

Hypothesis II:

By looking at the Null Hypotheses for monthly equal variance distributions between the 1-Norm Constrained, 2- Norm Constrained Portfolio, and the DAX only the NC1V can be rejected at a significance level of 10%. This means NC1V has a significantly different variance than the DAX, which is also lower than the variance of the DAX. All other strategies, namely NC1R, NC2V, and NC2R failed to reject the Null Hypothesis for equal variance distributions even at a significance level of 10%.

From this point of view *Hypothesis II is mostly rejected*. As already mentioned only one out of four Norm-Constrained Strategies has a significantly lower variance than the DAX from 2003 - 2020.

Hypothesis III:

The Null Hypotheses regarding equality of the Sharpe Ratio distributions of the Norm-Constrained Strategies and the DAX can be rejected at a significance level of at least 5% for NC1V, NC1R, and NC2V. Further, NC1V rejected the Null Hypothesis even at a 1% significance level. Only NC2R cannot reject the Null Hypothesis even at a significance level of 10%. More precisely NC1V, NC1R, and NC2V have different Sharpe Ratios compared to the DAX as a benchmark index. Further the results show a lower Sharpe Ratio of all Norm-Constrained Portfolio Strategies compared to DAX, in general.

Therefore, *Hypothesis III can be completely rejected*. No Norm-Constrained Strategy achieves a higher Sharpe Ratio than the DAX from 2003 – 2020.

After the empirical results of the Norm-Constrained Strategies are now presented to the reader I would like to compare those results with other empirical findings of the literature.

4.2 Comparison with other empirical findings

Since the Norm-Constrained Portfolio Strategy was invented and tested by DeMiguel et al. I would start to compare my empirical results with their empirical results. For the comparison I use the results for the 25FF dataset from their paper because this is the dataset from which the threshold values for the calibration of the Norm-Constrained Strategies are used. Secondly, the 25FF dataset contains nearly the same amount of stocks, namely 25 stocks, such that the Norm-Constrained Portfolio Strategies had the same starting point in terms of stocks for their optimizations.

I want to start by determining that all strategies in my empirical results do not show a completely better performance compared to the results of DeMiguel et al. On the contrary, most of my empirical results show a worse performance except the NC1V shows a lower variance compared to the results in the paper of DeMiguel. Reasons for the poorer results may different stocks that are used in the datasets and the different time horizons for the sample. Nevertheless, the empirical results show mostly the same numerical behavior in the performance measurements. To be more precise, both findings show a lower variance of the NC1V compared to the NC2R, a higher return of the NC2R compared to the NC2V and a lower variance of the NC2R compared to the NC2V (DeMiguel et al., 2009a). The only numerical behavior in performance measurements that do not behave similarly is that the NC1V has a higher return than the NC1R in my empirical findings. This is reverse to their findings (DeMiguel et al., 2009a).

Since DeMiguel et al. are not the only ones who have done some research around the Norm-Constrained Portfolio Strategies I would like to compare my empirical findings with a paper by Behr et al. from 2011. Their findings show a lower portfolio variance of the 1-Norm Constrained Strategy compared to the 2-Norm Constrained strategy (Behr et al., 2011). This does not perfectly mirror my results. In my findings, NC1R has a higher variance than NC2V and NC2R. Furthermore, their findings show a higher portfolio return of the 2-Norm Constrained Strategy compared to the 1-Norm Constraint Portfolio Strategy (Behr et al., 2011). This result also does not match my result. In general, my result shows a higher return of the 1-Norm Constrained Portfolio Strategy compared to the 2-Norm Constrained Strategy. Reasons for the difference could be other threshold values or the properties of the dataset they used.

5. Conclusion

The paper expands existing studies in the literature of the Norm-Constrained Portfolio Strategies for the German Market. For the sample period of January 2003 until the end of December in 2020, at least one of the Norm-Constrained Portfolios shows a significantly lower variance given the same return than the DAX. On the other hand one has to conclude that most of the Norm-Constrained Portfolios are outperformed by the DAX in terms of return, variance, and Sharpe Ratio. Although, the paper shows that it is possible to develop an investment strategy based on the Norm-Constrained optimization that offers a lower variance than the German benchmark index in the long run.

At this point I would like to provide some critical points to my empirical results. First, I recognize that the calculation of the Sharpe Ratio may suffer from an error. To be more precise, from a logical point of view the Sharpe Ratio of a strategy with a lower variance compared to another strategy should be higher given the same amount of return. This is not the case for the NC1V portfolio which has a significantly lower variance compared to the DAX given the same amount of return. The Sharpe Ratio of NC1V is lower. This indicates an error, but it was not possible to locate it. Additionally, for all calculations of the Norm-Constrained Portfolios no transaction costs are considered. By adding them to the calculations the results may be drastically different.

Overall Norm-Constrained Portfolio Strategies may be interesting for institutional investors, like pension funds, who prefer a defensive investment strategy with low variances and a guarantee for steady yields for their clients. Moreover, since the application of the Norm-Constrained Strategy is very simple, it may implement among private investors as well. Further, Norm-Constrained Strategies could be combined with other investment approaches. Technical analysis may be used as a timing factor to locate better entries and exits. Moreover, to select stocks for the Norm-Constraint Strategy fundamental stock selection techniques may be used.

6. References

Behr, P., Guettler, A., & Miebs, F. (2011). On Portfolio Optimization: Imposing the Right Constraints. *SSRN Electronic Journal*.

Brandt, M., Santa-Clara, P., & Valkanov, R. (2004). Parametric Portfolio Policies: Exploiting Characteristics in the Cross Section of Equity Returns. *National Bureau of Economic Research*, from https://www.nber.org/papers/w10996.

Brodie, J., Daubechies, I., Mol, C. de, Giannone, D., & Loris, I. (2009). Sparse and stable Markowitz portfolios. *Proceedings of the National Academy of Sciences of the United States of America, 106*(30), 12267–12272.

Dai, Z., & Kang, J. (2021). Some new efficient mean–variance portfolio selection models. *International Journal of Finance & Economics*.

Dai, Z., & Wen, F. (2018). A generalized approach to sparse and stable portfolio optimization problem. *Journal of Industrial & Management Optimization, 14*(4), 1651–1666.

DeMiguel, V., Garlappi, L., Nogales, F. J., & Uppal, R. (2009a). A Generalized Approach to Portfolio Optimization: Improving Performance by Constraining Portfolio Norms. *Management Science, 55*(5), 798–812, from http://www.jstor.org/stable/40539189.

DeMiguel, V., Garlappi, L., & Uppal, R. (2009b). Optimal versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy? *The Review of Financial Studies, 22*(5), 1915–1953, from http://www.jstor.org/stable/30226017.

Fama, E. F. (1976). *Foundations of finance: Portfolio decisions and securities prices.* New York, NY: Basic Books.

Fan, J., Zhang, J., & Yu, K. (2012). Vast Portfolio Selection with Gross-exposure Constraints. *Journal of the American Statistical Association, 107*(498), 592–606.

Jagannathan, R., & Ma, T. (2003). Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. *The Journal of Finance, 58*(4), 1651–1683.

Markowitz, H. (1952). PORTFOLIO SELECTION*. *The Journal of Finance, 7*(1), 77–91.

Merton, R. C. (1980). On estimating the expected return on the market. *Journal of Financial Economics*, *8*(4), 323–361, from https://www.sciencedirect.com/science/article/pii/0304405X80900070.

Norton, V. (2011). *Adjusted Closing Prices*, from https://arxiv.org/pdf/1105.2956.

Xing, X., Hu, J., & Yang, Y. (2014). Robust minimum variance portfolio with L-infinity constraints. *Journal of Banking & Finance, 46*, 107–117.

7. Appendix

7.1 Appendix 1

Table 1: Average threshold values

This table contains the average threshold values from the paper of DeMiguel et al. for the 25FF dataset. The "Strategy" column lists the respective strategies used in this paper. The column "25FF" contains the corresponding average threshold values.

Strategy	25FF
NC1V (δ)	2.39926
NC1R (δ)	1.80242
NC2V (δ)	1.23985
NC2R (δ)	0.96842

7.2 Appendix 2

Table 2: Portfolio Variances

All quantities reported in this table are monthly out-of-sample variances for the Norm-Constrained Portfolios and the DAX as a benchmark index. The table also contains the corresponding P-value that the portfolio variance is different from that for the benchmark index DAX.

Strategy	Variance
NC1V	0.00059
	(0.09)
NC1R	0.00394
	(0.19)
NC2V	0.00193
	(0.44)
NC2R	0.00165
	(0.96)
DAX	0.00163

Table 3: Portfolio Returns

All quantities reported in this table are monthly out-of-sample returns for the Norm-Constrained Portfolios and the DAX as a benchmark index. The table also contains the corresponding P-value that the portfolio return is different from that for the benchmark index DAX.

Strategy	Return
NC1V	0.00879
	(0.26)
NC1R	0.00331
	(0.28)
NC2V	-0.00050
	(0.07)
NC2R	0.00015
	(0.08)
DAX	0.00944

Table 4: Portfolio Sharpe Ratios

All quantities reported in this table are monthly out-of-sample Sharpe Ratios for the Norm-Constraint Portfolios and the DAX as a benchmark index. The table also contains the corresponding P-value that the portfolio Sharpe Ratio is different from that for the benchmark index DAX.

Strategy	Sharpe Ratio
NC1V	-0.24750
	(0.00)
NC1R	-0.82997
	(0.02)
NC2V	-0.31874
	(0.02)
NC2R	-4.42096
	(0.18)
DAX	1.00478