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A $\mathbb{C}^*$–Action without Categorical Quotient

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Let X be a complex algebraic variety endowed with a regular action of an algebraic group G. A categorical quotient (in the category of algebraic varieties) for the action of G on X is a G–invariant regular map $p: X \to Y$ such that for any G–invariant regular map $f: X \to Z$ there is a unique regular map $\widetilde{f}: Y \to Z$ with $f = \widetilde{f} \circ p$.
Categorical quotients can be defined analogously with respect to other categories. It is known that in the category of algebraic varieties categorical quotients need not exist in general (see e.g. [Po;Vi]). In certain cases, e.g. actions of finite groups, one can obtain quotients by passing to the category of algebraic spaces (see also [Ke;Mo]).
The purpose of this note is to provide an explicit elementary example of a $\mathbb{C}^*$–action on a smooth variety that does not admit a categorical quotient, not even in the category of algebraic spaces. In [AC;Ha], we discuss counterexamples of this type more systematically in the framework of subtorus actions on toric varieties. Here we consider the open subvariety

$$X := \mathbb{C}^2 \times (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2 \times \mathbb{C}^2$$

of $\mathbb{C}^4$ and the regular $\mathbb{C}^*$–action on X given by

$$t*(x_1, x_2, x_3, x_4) := (tx_1, tx_2, x_3, t^{-1}x_4).$$

Proposition. *The above $\mathbb{C}^*$–action admits no categorical quotient, not even in the categories of algebraic and analytic spaces.*

Proof. Assume that there exists a categorical quotient $p: X \to Y$ in one of the categories in question. Then it follows from the universal property of categorical quotients that p is surjective and that the canonical action of the torus $T := (\mathbb{C}^*)^4$ on X (denoted by $t \cdot x$) induces a (set theoretical) action of T on Y such that p is equivariant. Let

$$f: X \to \mathbb{C}^3, \qquad (x_1, x_2, x_3, x_4) \mapsto (x_1x_4, x_2x_4, x_3).$$

Then $f|_T: T \to (\mathbb{C}^*)^3$ is a surjective homomorphism of tori and induces a T–action on $\mathbb{C}^3$ making f equivariant. Moreover, f is constant on the orbits of the above–defined $\mathbb{C}^*$–action and an explicit calculation yields

$$f(X) = \mathbb{C}^3 \setminus (T \cdot (0,1,0) \cup T \cdot (1,0,0)).$$

In particular, $f(X)$ is not open in $\mathbb{C}^3$ with respect to the complex topology. Now, by the universal property of p, there is a holomorphic map $\widetilde{f}: Y \to \mathbb{C}^3$ such that $f = \widetilde{f} \circ p$. We will show that $\widetilde{f}$ must be injective. Since Y is necessarily irreducible, this implies that $\widetilde{f}$ is an open map, hence $\widetilde{f}(Y) = f(X)$ is open in $\mathbb{C}^3$ and we arrive at a contradiction.

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To obtain injectivity of $\widetilde{f}$, since p is surjective, it suffices to show that p is constant on the fibres of f. First consider the open T–stable subset

$$U_1 \ := \ \{x \in X;\ x_4 = 0\} \ = \ X \setminus (T\cdot(1,1,1,0) \cup T\cdot(1,1,0,0)) \, .$$

Then U_1 is also $\mathbb{C}^*$–stable and the restriction $f|_{U_1}$ separates $\mathbb{C}^*$–orbits. Hence p is constant on the fibres of $f|_{U_1}$. Next we consider the set

$$U \ := \ U_1 \cup T\cdot(1,1,1,0) \ = \ X \setminus T\cdot(1,1,0,0).$$

Note that

$$f(T\cdot(1,1,1,0)) \ = \ T\cdot(0,0,1) \ = \ f(T\cdot(0,0,1,1)) \in f(U_1).$$

Hence, in order to prove that p is constant on the fibres of the restriction $f|_U$, it suffices by T–equivariance to show that p maps the set

$$f^{-1}(0,0,1) \cap T\cdot(1,1,1,0) \ = \ T_{(0,0,1)}\cdot(1,1,1,0)$$

to the point $p(0,0,1,1)$. So, let $t \in T_{(0,0,1)}$. Then t is of the form $(t_1,t_2,1,t_4)$. Since p is T–equivariant and $\mathbb{C}^*$–invariant, we see

$$\begin{aligned} p(t\cdot(1,1,1,0)) \ &= \ t\cdot p(1,1,1,0) \ = \ t\cdot\lim_{s\to 0} p(1,1,1,s) \ = \ t\cdot \lim_{s \to 0} p(s*(1,1,1,s)) \\ &= \ t\cdot\lim_{s\to 0} p(s,s,1,1) \ = \ t\cdot p(0,0,1,1) \ = \ p(0,0,1,t_4) \\ &= \ p(t_4^{-1}*(0,0,1,1)) \ = \ p(0,0,1,1). \end{aligned}$$

Thus p is even constant on the fibres of $f|_U$. Finally, we have to treat the set $T\cdot(1,1,0,0)$. Since

$$f(T\cdot(1,1,0,0)) = (0,0,0) \notin f(U),$$

we only need to prove that p is constant on $T\cdot(1,1,0,0)$. So far we know

$$\widetilde{f}^{-1}(\overline{T\cdot(0,0,1)}) \ = \ p(f^{-1}(T\cdot(0,0,1))) \cup p(f^{-1}(0,0,0)) \ = \ p(T\cdot(1,1,1,0)) \cup p(T\cdot(1,1,0,0)),$$

where the closure is taken with respect to the complex topology. In particular, the above set is analytic and contains $p(T\cdot(1,1,1,0))$ as an open subset. Since $T\cdot(1,1,0,0)$ lies in the closure of $T\cdot(1,1,1,0)$ and these sets are separated by f, we obtain in addition

$$p(T\cdot(1,1,0,0)) \ = \ \overline{p(T\cdot(1,1,1,0))} \setminus p(T\cdot(1,1,1,0)).$$

Now, $p(T\cdot(1,1,1,0)) = p(T\cdot(0,0,1,1))$ is of dimension one. Thus it follows that $p(T\cdot(1,1,0,0))$ is of dimension zero and hence it is a point. □

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