Backward Stochastic Differential Equations and Stochastic Controls: A New Perspective

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Abstract

It is well known that backward stochastic differential equations (BSDEs) stem from the study on the Pontryagin type maximum principle for optimal stochastic controls. A solution of a BSDE hits a given *terminal* value (which is a random variable) by virtue of an additional martingale term and an indefinite initial state. This paper attempts to view the relation between BSDEs and stochastic controls from s new perspective by interpreting BSDEs as some stochastic optimal control problems. More specifically, associated with a BSDE a new stochastic control problem is introduced with the same dynamics but a *definite* initial state. The martingale term in the original BSDE is regarded as the control and the objective is to minimize the second moment of the difference between the terminal state and the given terminal value. This problem is solved in a closed form by the stochastic linear-quadratic theory developed recently. The general result is then applied to the Black-Scholes model, where an optimal feedback control is obtained explicitly in terms of the option price. Finally, a modified model is investigated where the difference between the state and the expectation of the given terminal value at *any* time is take into account.

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1 Introduction

Backward stochastic differential equation (BSDE for short) theory and applications have remained very active in recent years. Consider the following linear BSDE:

$$\begin{cases} dp(t) = [A(t)p(t) + B(t)q(t) + f(t)]dt + q(t)dW(t), \\ p(T) = \xi, \end{cases} \tag{1.1}$$

where ξ is a random variable that will become certain only at the terminal time T. As is well-known the equation was initially introduced by Bismut [2, 3] when he was studying the adjoint equations associated with the stochastic maximum principle in stochastic optimal controls. Basically, the equation (1.1) tells how to price the *marginal value* of the resource represented by the state variable in a random environment. The solution of (1.1) has two components: p and q, the former being the price while the latter signifies the uncertainty between the present and terminal times. The linear BSDEs were later extended to nonlinear ones by Pardoux and Peng [12] motivated by stochastic control problems, and independently by Duffie and Epstein [6] in their study of recursive utility in finance. The BSDE theory has found wide applications in partial differential equation theory, stochastic controls and, particularly, mathematical finance. For a most updated account of the BSDE theory and applications see the book by Yong and Zhou [14, Chapter 7].

As mentioned a solution of a BSDE consists of two components, p and q. Mathematically q is obtained implicitly by the martingale representation theorem (see [10]). This somehow mysterious second term q is hard to handle both analytically and numerically, and we have not found a clear, tangible explanation for q in the literature. The purpose of this paper is try to view the BSDEs from a control perspective (so it is a reverse of the original "birth process" of BSDEs!) and interpret the term q as a control variable.

To be precise, note that in equation (1.1) the terminal value is specified while the initial value is left open. But if the equation has a solution then the initial value cannot be chosen arbitrarily; rather it is uniquely determined by the solution and is hence *part* of the solution. Therefore, solving (1.1) amounts to the following statement: starting with a *proper* initial condition and choose an *appropriate* diffusion term to hit the given value at the terminal.

Then, it will be very natural to modify the above statement and consider the following stochastic optimal control problem: for the *same* dynamics of (1.1), starting with a *given* initial state x choose a control q so that the terminal state $p(T)$ stays as close to the given terminal value ξ as possible. Note that since now the initial value x is given *a priori*, one in general cannot expect that $p(T)$ will hit ξ exactly by choosing certain q. Hence it is reasonable to require that the difference between the two is minimized. Here, the "difference" may be measured by, say, the second moment of the algebraic difference between the two random variables. More interestingly, if we regard the initial state also as a decision variable then the optimal state-control pair of the problem (p, q) is exactly the solution of the original BSDE!

It turns out that the control problem formulated above is a stochastic optimal linear-quadratic (LQ) problem that can be solved analytically via a stochastic Riccati equation (SRE), employing the similar technique as developed recently in [4, 5].

We then apply the general result obtained to the Black-Scholes model. Taking advantage of the fact that the state (wealth) is a scalar, one can solve the SRE explicitly and hence the stochastic control problem. It turns out that an optimal control consists of the hedging strategy for the claim and the Merton portfolio for a quadratic terminal utility.

Finally we consider a modified model where the difference between the state and and the expected terminal value must be kept small at *any* time. Again explicit optimal control is derived via an SRE which is shown to be always solvable. The result is then applied to the Black-Scholes model which gives rise to a consumption process to correct dynamically any large deviation of the price from the expected value of the claim.

The rest of the paper is organized as follows. In Section 2 we formulate the model and problem. Section 3 presents the optimal solution to the problem. Section 4 is concerned with the solvability of the stochastic Riccati equation necessary for the optimal control derived in Section 3. In Section 5 a special case, namely the Black-Scholes model is considered and an optimal hedging portfolio is derived explicitly based on the results of the previous sections. Section 6 is devoted to a modified model. Finally, Section 7 concludes the paper.

2 Problem Formulation

Throughout this paper $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ is a fixed filtered complete probability space on which defined a standard $\{\mathcal{F}_t\}_{t\geq 0}$-adapted m-dimensional Brownian motion $W(t) \equiv (W^1(t), \cdots, W^m(t))'$ with $W(0) = 0$. It is assumed that $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$. We denote by $L^2_{\mathcal{F}}(0, T; R^d)$ the set of all R^d-valued, measurable stochastic processes $\psi(t)$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, such that $E\int_0^T |\psi(t)|^2 dt < +\infty$.

Notation. We make the following additional notation:

M'	:	the transpose of any vector or matrix M;
$\|M\|$	:	$= \sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix or vector $M = (m_{ij})$;
S^n	:	the space of all $n \times n$ symmetric matrices;
S_+^n	:	the subspace of all nonnegative definite matrices of S^n;
$\hat{S}_+^n$	:	the subspace of all positive definite matrices of S^n;
$C([0,T];X)$	:	the Banach space of X-valued continuous functions on $[0,T]$ endowed with the maximum norm $\|\cdot\|$ for a given Hilbert space X;
$L^2(0,T;X)$	:	the Hilbert space of X-valued integrable functions on $[0,T]$ endowed with the norm $\left(\int_0^T \| f(t) \|_X^2 \, dt\right)^{\frac{1}{2}}$ for a given Hilbert space X;
$L^\infty(0,T;X)$	:	the Banach space of X-valued essentially bounded functions on $[0,T]$ endowed with the norm $\sup_{0\le t\le T} \| f(t) \|_X$ for a given Hilbert space X.

Consider the following controlled system

$$\begin{cases} dx(t) = [A(t)x(t) + \sum_{j=1}^m B_j(t)u_j(t) + f(t)]dt + \sum_{j=1}^m u_j(t)dW^j(t), \ t \in [0,T], \\ x(0) = x, \end{cases} \tag{2.1}$$

where $x(t), x, u_j(t), f(t) \in R^n$ and $A(t), B_j(t) \in R^{n\times n}$. Throughout this paper we assume that $A(t), B_j(t)$ are bounded deterministic functions and $f \in L_{\mathcal{F}}^2(0,T;R^n)$. For a given $\mathcal{F}_T$-measurable square integrable random variable ξ, the problem is to select an ($\mathcal{F}_T$-adapted) control process $u(\cdot) \equiv (u_1(\cdot), \cdots, u_m(\cdot)) \in L_{\mathcal{F}}^2(0,T;R^{mn})$ so as to minimize the cost functional

$$J(x, u(\cdot)) = E\frac{1}{2}|x(T) - \xi|^2. \tag{2.2}$$

To simplify the cost functional, it is natural to define

$$y(t) = x(t) - E(\xi|\mathcal{F}_t). \tag{2.3}$$

Since $E(\xi|\mathcal{F}_t)$ is an $\mathcal{F}_t$-martingale and $\mathcal{F}_t$ is generated by the Brownian motion $W(t)$, by the Martingale Representation Theorem ([10]) there is $z(\cdot) \equiv (z_1(\cdot), \cdots, z_m(\cdot)) \in L_{\mathcal{F}}^2(0,T;R^{mn})$ so that

$$E(\xi|\mathcal{F}_t) = E\xi + \sum_{j=1}^m \int_0^t z_j(s)dW^j(s). \tag{2.4}$$

By (2.1), (2.3) and (2.4), with the new state variable $y(\cdot)$ the controlled system becomes

$$\begin{cases} dy(t) = [A(t)y(t) + \sum_{j=1}^m B_j(t)u_j(t) + g(t)]dt + \sum_{j=1}^m [u_j(t) - z_j(t)]dW^j(t), \ t \in [0,T], \\ y(0) = x - E\xi \equiv y, \end{cases} \tag{2.5}$$

where

$$g(t) = f(t) + A(t)E(\xi|\mathcal{F}_t), \tag{2.6}$$

and the cost functional reduces to

$$J(y,u(\cdot)) = E\frac{1}{2}|y(T)|^2. \tag{2.7}$$

Notice that the above problem is a stochastic linear-quadratic (LQ) control problem with *random* nonhomogeneous terms in both drift and diffusion coefficients.

3 Solutions

In this section we solve the problem (2.1)-(2.2) or (2.5)-(2.7) by LQ techniques. The main idea is simply the *completion of squares.* It should be noted that the problem under consideration is a singular LQ problem in that the running cost is identically zero, therefore can not be solved by the conventional approach as developed by Wonham [13] and others. Indeed, study on the general (possibly singular) stochastic LQ problem is interesting in its own right and has recently been developed extensively, see [4, 5]. For a systematic treatment of stochastic LQ problems, see also [14, Chapter 6].

In the rest of this paper, we may write X for a (deterministic or stochastic) process $X(t)$, omitting the variable t, whenever no confusion arises. Under this convention, when $X \in C([0,T];S^n)$, $X \geq (>)0$ means $X(t) \geq (>)0$, $\forall t \in [0,T]$.

We introduce the following *stochastic Riccati equation* (SRE for short):

$$\begin{cases} \dot{P} + PA + A'P - \sum_{j=1}^m PB_jP^{-1}B_j'P = 0, \\ P(T) = I, \\ P(t) > 0, \ \forall t \in [0,T], \end{cases} \tag{3.1}$$

along with a backward stochastic differential equation (BSDE for short)

$$\begin{cases} d\phi(t) = & -\Big[(A' - \sum_{j=1}^m PB_jP^{-1}B_j')\phi - \sum_{j=1}^m PB_jP^{-1}\beta_j + P(g + \sum_{j=1}^m B_jz_j)\Big](t)dt \\ & + \sum_{j=1}^m \beta_j(t)dW^j(t), \\ \phi(T) = & 0. \end{cases} \tag{3.2}$$

Note that the SRE (3.1) is fundamentally different from the *conventional* Riccati equation[1] in that the equation (3.1) involves the *inverse* of the unknown. In addition, the third constraint of (3.1) must also be satisfied for any solution. In general, such an equation does *not* automatically admit a solution (the sovalbility of this equation is interesting on its own; see Section 4 below). On the other hand, if (3.1) has a solution $P(\cdot)$, then the second equation (3.2) must admit an $\mathcal{F}_t$-adapted solution $(\phi(\cdot), \beta_j(\cdot), j = 1, \cdots, m)$ as (3.2) is a linear BSDE; see [2, 3] or [14, Chapter 7] for more details about BSDEs.

[1]Here by a conventional Riccati equation we mean one associated with the *deterministic* LQ problem, see [1], as opposed to that associated with the *stochastic* LQ problem, see [13, 4].

Theorem 3.1. *If the equations (3.1) and (3.2) admit solutions $P \in C([0,T];\hat{S}_+^n)$ and $(\phi(\cdot), \beta_j(\cdot), j = 1, \cdots, m) \in L^2_{\mathcal{F}}(0,T;R^n) \times L^2_{\mathcal{F}}(0,T;R^{nm})$, respectively, then the problem (2.5)-(2.7) has an optimal feedback control $u^*(\cdot) \equiv (u_1^*(\cdot), \cdots, u_m^*(\cdot))$, where*

$$u_j^*(t) = -P(t)^{-1}B_j(t)'[P(t)y^*(t) + \phi(t)] - P(t)^{-1}\beta_j(t) + z_j(t), \; j = 1, \cdots, m. \tag{3.3}$$

Moreover, the optimal cost value under the above control is

$$\begin{array}{rl} J^*(y) = & \frac{1}{2}y'P(0)y + y'\phi(0) + \frac{1}{2}E\int_0^T \Big[2\phi' g - 2\sum_{j=1}^m \beta_j z_j + \sum_{j=1}^m z_j' P z_j \\ & -\sum_{j=1}^m (P^{-1}B_j'\phi + P^{-1}\beta_j - z_j)'P(P^{-1}B_j'\phi + P^{-1}\beta_j - z_j)\Big](t)dt. \end{array} \tag{3.4}$$

Proof. Applying Ito's formula, we get

$$\begin{array}{rl} \frac{1}{2}d[y(t)'P(t)y(t)] = & \frac{1}{2}\Big[\sum_{j=1}^m (u_j - z_j)'P(u_j - z_j) + 2y'Pg \\ & + \sum_{j=1}^m (y'PB_jP^{-1}B_j'Py + 2u_j'B_j'Py)\Big](t)dt + \frac{1}{2}\{...\}dW(t), \end{array} \tag{3.5}$$

and

$$\begin{array}{rl} d[\phi(t)'y(t)] = & \Big[-\phi'(A - \sum_{j=1}^m B_jP^{-1}B_j'P)y + \sum_{j=1}^m \beta_j'P^{-1}B_j'Py - (g' + \sum_{j=1}^m z_j'B_j')Py \\ & + \phi'(Ay + \sum_{j=1}^m B_ju_j + g) + \sum_{j=1}^m \beta_j'(u_j - z_j)\Big](t)dt + \{...\}dW(t). \end{array} \tag{3.6}$$

Then we integrate both (3.5) and (3.6) from 0 to T, take expectations, and add them together. Trying to complete a square and going through a fairly tedious manipulation, we end up with

$$\begin{array}{rl} & J(y, u(\cdot)) \\ = & \frac{1}{2}E\int_0^T \Big[2\phi' g - 2\sum_{j=1}^m \beta_j z_j + \sum_{j=1}^m z_j'Pz_j \\ & + \sum_{j=1}^m (u_j + P^{-1}B_j'Py + P^{-1}B_j'\phi + P^{-1}\beta_j - z_j)'P(u_j + P^{-1}B_j'Py + P^{-1}B_j'\phi + P^{-1}\beta_j - z_j) \\ & - \sum_{j=1}^m (P^{-1}B_j'\phi + P^{-1}\beta_j - z_j)'P(P^{-1}B_j'\phi + P^{-1}\beta_j - z_j)\Big](t)dt \\ & + \frac{1}{2}y'P(0)y + y'\phi(0). \end{array} \tag{3.7}$$

It follows immediately that the optimal feedback control is given by (3.3) and the optimal value is given by (3.4) provided that the the corresponding equation (2.5) under (3.3) has a solution. But under the linear feedback (3.3), the system (2.5) is a nonhomogeneous linear stochastic differential equation with bounded linear coefficients and square integrable nonhomogeneous terms. Hence it must admit one and only one solution by standard SDE theory. This completes the proof. □

Now we would like to derive the optimal feedback control in terms of the original variable $x(t)$. Interestingly, the optimal control can be obtained via the original BSDE that motivated the optimal control problem (2.1)-(2.2).

Theorem 3.2. *Under the same assumptions of Theorem 3.1, the control problem (2.1)-(2.2) has an optimal feedback control $u^*(\cdot) \equiv (u_1^*(\cdot), \cdots, u_m^*(\cdot))$, where*

$$u_j^*(t) = -P(t)^{-1}B_j(t)'P(t)[x^*(t) - p(t)] + q_j(t), \; j = 1, \cdots, m, \tag{3.8}$$

where $(p(\cdot), q_j(\cdot), j = 1, \cdots, m) \in L^2_{\mathcal{F}}(0, T; R^n) \times L^2_{\mathcal{F}}(0, T; R^{nm})$ *is the unique* $\mathcal{F}_t$*-adapted solution of the following BSDE:*

$$\begin{cases} dp(t) = [A(t)p(t) + \sum_{j=1}^m B_j(t)q_j(t) + f(t)]dt + \sum_{j=1}^m q_j(t)dW^j(t), \ t \in [0, T], \\ p(T) = \xi. \end{cases} \tag{3.9}$$

Proof. First of all, consider the following matrix-valued ordinary differential equation:

$$\begin{cases} \dot{Q} - AQ - QA' + \sum_{j=1}^m B_j Q B_j' = 0, \\ Q(T) = I, \end{cases} \tag{3.10}$$

which must admit a unique solution $Q(\cdot)$ since it is linear with bounded coefficients. Denote $S = PQ$, then by the differential chain rule it is easy to verify that S satisfies

$$\begin{cases} \dot{S} = SA' - A'S + \sum_{j=1}^m PB_jP^{-1}B_j'S - \sum_{j=1}^m PB_jP^{-1}SB_j', \\ S(T) = I. \end{cases} \tag{3.11}$$

This is a linear equation hence it has a unique solution $S \equiv I$. It leads to $Q(t) = P(t)^{-1}$.

Now, noting (2.3), the feedback control (3.3) can be written as

$$u_j^*(t) = -P(t)^{-1}B_j(t)'[P(t)x^*(t) - P(t)E(\xi|\mathcal{F}_t) + \phi(t)] - P(t)^{-1}\beta_j(t) + z_j(t), \ j = 1, \cdots, m. \tag{3.12}$$

Denote

$$\begin{array}{l} p(t) = E(\xi|\mathcal{F}_t) - P(t)^{-1}\phi(t) \equiv E(\xi|\mathcal{F}_t) - Q(t)\phi(t), \\ q_j(t) = z_j(t) - P(t)^{-1}\beta_j(t) \equiv z_j(t) - Q(t)\beta_j(t), \ j = 1, \cdots, m. \end{array} \tag{3.13}$$

Applying Ito's formula to (2.4), (3.10) and (3.2), we can verify that $(p(\cdot), q_j(\cdot), j = 1, \cdots, m)$ satisfies the BSDE (3.9). Therefore the desired results follow by virtue of the uniqueness of solutions to (3.9). □

Remark 3.1. Equation (3.4) also gives the optimal cost functional value as a function of the initial value $y \equiv x - E\xi$. It turns out to be a quadratic function. If the controller has the choice of selecting the initial value y so as to minimize $J^*(y)$, then the "best" initial value would be obtained by setting $\frac{d}{dy}J^*(y)|_{y=y^*} = 0$. This yields $y^* = -P(0)^{-1}\phi(0)$. Returning to the original variable, we get that the best initial value for $x(\cdot)$ will be

$$x^* = y^* + E\xi = -P(0)^{-1}\phi(0) + E\xi = p(0), \tag{3.14}$$

where the last equality is due to (3.13). This makes perfect sense, as it implies that one should choose an initial value $p(0)$ so as to minimize the difference between the terminal state value and the given value ξ (of course, in this case the minimum difference is zero since starting with $p(0)$ one can hit ξ exactly at the end, by the BSDE theory). In this perspective, the solution pair (p, q)

of the BSDE (3.9) may be regarded as the optimal state-control pair of minimizing $J(x, u(\cdot))$ (as given by (2.2)) over $(x, u(\cdot))$ subject to the dynamics (2.1). This gives an interpretation of (p, q) via a stochastic control problem. In this perspective, if a BSDE does not have an adapted solution (e.g., when the underlying filteration is not generated by the Brownian motion involved), we may still define a "pseudo-solution" via the corresponding stochastic control problem.

Remark 3.2. Under the optimal feedback (3.8), the optimal trajectory $x^*(\cdot)$ evolves as

$$\left\{\begin{array}{rl} dx^*(t) = & \Big\{[A - \sum_{j=1}^m B_j P^{-1} B_j' P]x^* + \sum_{j=1}^m B_j(t) P^{-1} B_j' P p + f + \sum_{j=1}^m B_j q_j\Big\}(t)dt \\ & + \sum_{j=1}^m [-P^{-1} B_j' P x^* + P^{-1} B_j' P p + q_j](t) dW^j(t), \\ x^*(0) = & x. \end{array}\right. \tag{3.15}$$

Moreover, the difference $\Delta(t) = x^*(t) - p(t)$ satisfies

$$\left\{\begin{array}{rl} d\Delta(t) = & [A - \sum_{j=1}^m B_j P^{-1} B_j' P](t)\Delta(t)dt - \sum_{j=1}^m [P^{-1} B_j' P](t)\Delta(t) dW^j(t), \\ \Delta(0) = & x - p(0). \end{array}\right. \tag{3.16}$$

Notice that $\Delta(\cdot)$ satisfies a homogeneous linear SDE, and hence must be identically zero if the initial is zero, namely, $x = p(0)$. In this case, by (3.8), the optimal control is $u_j^*(t) = q_j(t)$. This is exactly in line with the observation in Remark 3.1.

Remark 3.3. The results obtained above are based on the LQ approach. LQ models constitute an extremely important class of optimal control problems and their optimal solutions can be obtained explicitly via the Riccati equations, due to the nice underlying structures (see [1, 4, 5, 9, 13, 14]). The general stochastic Riccati equation is introduced in [4] as a BSDE of the Pardoux-Peng type ([12]) for the case where all the coefficients are random. It reduces to (4.1) for the present case. Consequently, the results in this section can be easily extended to the case where the coefficients A, B_j are adapted random processes.

4 Solvability of SRE

In the previous section we derived explicitly an optimal control (in a feedback form) of the problem. However, there is one gap remaining, namely the result depends on the SRE (3.1) being solvable. The solvability of the SRE is by no means trivial, and is interesting in its own right. In [4], a necessary and sufficient condition for the solvability of SREs more general than (3.1) is derived. However the condition there is rather implicit. This section gives an explicit condition which ensures that (3.1) admits a unique solution.

To this end, we first consider a *conventional* Riccati equation

$$\left\{\begin{array}{l} \dot{P} + PA + A'P - \sum_{j=1}^m PB_j K^{-1} B_j' P = 0, \\ P(T) = I, \end{array}\right. \tag{4.1}$$

where $K > 0$ is given *a priori* (compare the equations (4.1) and (3.1)). Note that the above equation is a bit different from the standard Riccati equation arising from deterministic control problems (see [1]) where $m = 1$. But the case $m > 1$ can be treated in the same way without any difficulty. In particular, it is associated with the following deterministic LQ problem:

$$\begin{array}{ll} \text{Minimize} & J(s,y;u(\cdot)) = \int_s^T \frac{1}{2}\sum_{j=1}^m u_j(t)'K(t)u_j(t)dt + \frac{1}{2}|x(T)|^2, \\ \text{Subject to} & \begin{cases} \dot{x}(t) = A(t)x(t) + \sum_{j=1}^m B_j(t)u_j(t), \\ x(s) = y, \end{cases} \end{array}$$

where $(s,y) \in [0,T] \times R^n$. Namely, the value function of the above LQ problem is $\frac{1}{2}y'P(s)y$ where P is the solution to (4.1). Denote $\mathcal{K} = \{K \in L^\infty(0,T;\hat{S}_+^m) \mid K^{-1} \in L^\infty(0,T;\hat{S}_+^m)\}$. It can be checked that $C([0,T];\hat{S}_+^m) \subset \mathcal{K}$. For each $K \in \mathcal{K}$, we know from the classical Riccati theory (as well as the remark above) that (4.1) admits a unique solution $P \in C([0,T];S_+^n)$. Thus we can define a mapping $\Psi : \mathcal{K} \to C([0,T];S^n)$ as $P = \Psi(K)$.

Theorem 4.1. *The SRE (3.1) admits a unique solution if and only if there exist* $K \in C([0,T];\hat{S}_+^m)$ *such that*

$$\Psi(K) \geq K. \tag{4.2}$$

Proof. This is a special case of [4, Theorem 4.2]. □

Theorem 4.2. *If*

$$A(t) + A(t)' \geq \sum_{j=1}^m B_j(t)B_j(t)', \tag{4.3}$$

then the SRE (3.1) admits a unique solution.

Proof. We will show that (4.2) holds for $K = \varepsilon I$ for some $\varepsilon > 0$. To this end, for $\varepsilon > 0$ set $P_\varepsilon = \Psi(\varepsilon I) - \varepsilon I$. Then P_ε satisfies

$$\begin{cases} \dot{P}_\varepsilon + P_\varepsilon(A - \sum_{j=1}^m B_jB_j') + (A - \sum_{j=1}^m B_jB_j')'P_\varepsilon - \varepsilon^{-1}\sum_{j=1}^m P_\varepsilon B_jB_j'P_\varepsilon \\ \quad +\varepsilon(A + A' - \sum_{j=1}^m B_jB_j') = 0, \\ P_\varepsilon(T) = I - \varepsilon I. \end{cases} \tag{4.4}$$

Therefore, under the assumption (4.3) and when $0 < \varepsilon < 1$, the above is a standard conventional Riccati equation which admits a unique solution $P_\varepsilon \geq 0$. This implies that (4.2) holds with $K = \varepsilon I$. The result follows then from Theorem 4.1. □

Remark 4.1. In [4], an algorithm of computing the solution to the SREs is given. For the special SRE (3.1), the algorithm basically stipulates that one start with $K = \varepsilon I$ (with $0 < \varepsilon < 1$) and solve the conventional Riccati equation (4.1) recursively. The resulting sequence of solutions will monotonically converge to the solution of SRE (3.1).

It should be noted that (4.3) only gives an (easily verifiable) *sufficient* condition for the solvability of SRE (3.1). In other special cases (see Section 5 below), solvability of SRE can also be shown without (4.3).

5 Black-Scholes Model

We now apply the general results obtained in the previous sections to a contingent claim problem with the Black-Scholes setup. Suppose there is a market in which $m+1$ assets (or securities) are traded continuously. One of the assets is the *bond* whose price process $P_0(t)$ is subject to the following (deterministic) ordinary differential equation:

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, \quad t \in [0,T], \\ P_0(0) = p_0 > 0, \end{cases} \tag{5.1}$$

where $r(t) > 0$ is the *interest rate* (of the bond). The other m assets are *stocks* whose price processes $P_1(t), \cdots, P_m(t)$ satisfy the following stochastic differential equation:

$$\begin{cases} dP_i(t) = P_i(t)[b_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW^j(t)], \quad t \in [0,T], \\ P_i(0) = p_i > 0, \end{cases} \tag{5.2}$$

where $b_i(t) > 0$ is the *appreciation rate*, and $\sigma_i(t) \equiv (\sigma_{i1}(t), \cdots, \sigma_{im}(t)) : [0,T] \to R^m$ is the *volatility* or the *dispersion* of the stocks. Define the *covariance matrix*

$$\sigma(t) = \begin{pmatrix} \sigma_1(t) \\ \vdots \\ \sigma_m(t) \end{pmatrix} \equiv (\sigma_{ij}(t))_{m\times m}. \tag{5.3}$$

The basic assumption throughout this section is

$$\Sigma(t) \equiv \sigma(t)\sigma(t)' \geq \delta I, \quad \forall t \in [0,T], \tag{5.4}$$

for some $\delta > 0$. This is the so-called *non-degeneracy* condition. We also assume that all the functions are measurable and uniformly bounded in t.

Consider an investor whose total wealth at time $t \geq 0$ is denoted by $x(t)$. Suppose he/she decides to hold $N_i(t)$ shares of i-th asset ($i = 0, 1, \cdots, m$) at time t. Then

$$x(t) = \sum_{i=0}^m N_i(t)P_i(t), \quad t \geq 0. \tag{5.5}$$

Assume that the trading of shares takes place continuously and transaction cost and consumptions are not considered. Then one has

$$\begin{cases} dx(t) &= \sum_{i=0}^m N_i(t)dP_i(t) \\ &= \Big\{r(t)N_0(t)P_0(t) + \sum_{i=1}^m b_i(t)N_i(t)P_i(t)\Big\}dt \\ &\quad + \sum_{i=1}^m N_i(t)P_i(t)\sum_{j=1}^m \sigma_{ij}(t)dW_j(t) \\ &= \Big\{r(t)x(t) + \sum_{i=1}^m [b_i(t)-r(t)]\pi_i(t)\Big\}dt \\ &\quad + \sum_{j=1}^m\sum_{i=1}^m \sigma_{ij}(t)\pi_i(t)dW^j(t), \\ x(0) &= x > 0, \end{cases} \tag{5.6}$$

where

$$\pi_i(t) \equiv N_i(t)P_i(t), \quad i = 0,1,2\cdots,m, \tag{5.7}$$

denotes the total market value of the investor's wealth in the i-th bond/stock. If $\pi_i(t) < 0$ $(i = 1,2,\cdots,m)$, then the investor is *short-selling* i-th stock. If $\pi_0(t) < 0$, then the investor is borrowing the amount $|\pi_0(t)|$ at rate $r(t)$. It is clear that by changing $\pi_i(t)$, the investor changes the "allocation" of his/her wealth in these $m+1$ assets. We call $\pi(t) \equiv (\pi_1(t),\cdots,\pi_m(t))'$ a *portfolio* of the investor. Notice that we exclude the allocation to the bond, $\pi_0(t)$, from the portfolio as it will be determined completely by the allocation to the stocks.

Before going further, let us change the control variable so that the resulting system fits into the one we studied in the previous sections. Set

$$u(t) \equiv (u_1(t),\cdots,u_m(t))' = \sigma(t)'\pi(t), \text{ or } \pi(t) = \Sigma(t)^{-1}\sigma(t)u(t). \tag{5.8}$$

On the other hand, due to (5.4), the model is *arbitrage free*, namely, there exits a *risk premium process* $\theta(\cdot)$ satisfying

$$\theta(t)\sigma(t)' = (b_1(t)-r(t),\cdots,b_m(t)-r(t)). \tag{5.9}$$

In fact, $\theta(\cdot)$ can be constructed as

$$\theta(t) \equiv (\theta_1(t),\cdots,\theta_m(t)) = (b_1(t)-r(t),\cdots,b_m(t)-r(t))\Sigma(t)^{-1}\sigma(t). \tag{5.10}$$

With the above notation the equation (5.6) becomes

$$\begin{cases} dx(t) = [r(t)x(t) + \sum_{j=1}^m \theta_j(t)u_j(t)]dt + \sum_{j=1}^m u_j(t)dW^j(t), \\ x(0) = x. \end{cases} \tag{5.11}$$

The objective is, for each given initial wealth x and a contingent claim ξ (which is an $\mathcal{F}_T$-measurable square integrable random variable), to choose a *hedging portfolio* $\pi(\cdot)$ (or equivalently a control $u(\cdot)$) so as to minimize

$$J(x,u(\cdot)) = \frac{1}{2}E|x(T)-\xi|^2. \tag{5.12}$$

This is a special case of the general model studied in Section 4, so we can apply the results there. Interestingly, in this case the corresponding stochastic Riccati equation is explicitly solvable due to the specific structure that the state variable is a *scalar* (and hence so is the solution to the SRE).

Theorem 5.1. *The optimal portfolio of the hedging problem consisting of (5.6) and (5.11) is*

$$\pi^*(t) = -\Sigma(t)^{-1}(b_1(t)-r(t),\cdots,b_m(t)-r(t))'[x^*(t)-p(t)] + \Sigma(t)^{-1}\sigma(t)q(t), \tag{5.13}$$

where $(p(\cdot),q(\cdot)) \equiv (p(\cdot),q_j(\cdot),j=1,\cdots,m) \in L^2_{\mathcal{F}}(0,T;R^n)\times L^2_{\mathcal{F}}(0,T;R^{nm})$ *is the unique adapted solution of the BSDE*

$$\begin{cases} dp(t) = [r(t)p(t) + \sum_{j=1}^m \theta_j(t)q_j(t)]dt + \sum_{j=1}^m q_j(t)dW^j(t), \\ p(T) = \xi. \end{cases} \tag{5.14}$$

Proof. The SRE (3.1) in the present case reduces to (noting that the unknown $P(t)$ of the equation is a scalar)

$$\begin{cases} \dot{P}(t) + 2r(t)P(t) - \sum_{j=1}^m \theta_j(t)^2 P(t) = 0, \\ P(T) = 1, \\ P(t) > 0,\ t\in[0,T]. \end{cases} \tag{5.15}$$

Denote $\rho(t) = \sum_{j=1}^m \theta_j(t)^2 \equiv (b_1(t)-r(t),\cdots,b_m(t)-r(t))\Sigma(t)^{-1}(b_1(t)-r(t),\cdots,b_m(t)-r(t))'$. Then the above equation has a unique solution $P(t) = e^{-\int_t^T(\rho(s)-2r(s))ds}$. Note the third inequality constraint in (5.15) is automatically satisfied by this solution. On the other hand, the associated equation (3.2) reads

$$\begin{cases} d\phi(t) = & -\Big\{[r(t)-\rho(t)]\phi(t) - \sum_{j=1}^m \theta_j(t)\beta_j(t) + P(t)[g(t) + \sum_{j=1}^m \theta_j(t)z_j(t)]\Big\}dt \\ & + \sum_{j=1}^m \beta_j(t)dW^j(t), \\ \phi(T) = & 0. \end{cases} \tag{5.16}$$

Applying Theorem 3.2 and noticing that $P(t)$ is now a scalar, we obtain

$$u_j^*(t) = -\theta_j(t)[x^*(t)-p(t)] + q_j(t). \tag{5.17}$$

Appealing to (5.8) and writing in a vector form, we obtain the desired result (5.13). □

Remark 5.1. The formula (5.11) has a straightforward interpretation in financial terms. Indeed, it is well-known that the second term on the right hand side of (5.11) is the hedger for the claim ξ when the initial wealth is the initial option price $p(0)$. The other term is exactly the *Merton portfolio* for a terminal utility function $c(x) = x^2$ (Merton [11]). Therefore, our optimal hedging policy (5.11) for our problem is the sum of the hedger for the claim and the Merton portfolio. Consequently, if the initial endowment x is different from the fair initial price $p(0)$ necessary to hedge the contingent claim ξ, then the difference $x - p(0)$ should be invested according to the Merton strategy.

6 A Modified Model

In the previous sections we investigated a model where only the terminal variance is to be minimized. It is more in line with the European option in the context of option theory where only the terminal situation is of interest. Now, in the spirit of the American option, it is natural to consider a modified model where the difference between the state $x(t)$ and the expected value of the claim $E(\xi|\mathcal{F}_t)$ should be kept small at *any* time (rather than just at the terminal time).

Motivated by this, let us consider a modification of the model (2.1)-(2.2). Instead of cost functional (2.2), we consider the following

$$J(x,u(\cdot)) = \frac{1}{2}E[\int_0^T |x(t) - E(\xi|\mathcal{F}_t)|^2 dt + |x(T) - \xi|^2], \tag{6.1}$$

while keeping the same dynamics (2.1). (One can also put different weights on the running cost and the terminal cost, but let us not bother to do it here.)

Employing the same change of variable (2.3), we get the state equation (2.5) with the new cost functional

$$J(y,u(\cdot)) = \frac{1}{2}E[\int_0^T |y(t)|^2 dt + |y(T)|^2]. \tag{6.2}$$

To solve this problem, we only need to slightly modify the argument in Section 3. Specifically, the SRE (3.1) for the present case is changed to

$$\begin{cases} \dot{P} + PA + A'P - \sum_{j=1}^m PB_jP^{-1}B_j'P + I = 0, \\ P(T) = I, \\ P(t) > 0, \ \forall t \in [0,T]. \end{cases} \tag{6.3}$$

The form of the associated equation (3.2) remains unchanged, but with the new $P(\cdot)$ in it as determined by (6.3).

Theorem 6.1. *If the equations (6.3) and (3.2) admit solutions $P \in C([0,T];\hat{S}^n_+)$ and $(\phi(\cdot),\beta_j(\cdot), j = 1,\cdots,m) \in L^2_{\mathcal{F}}(0,T;R^n) \times L^2_{\mathcal{F}}(0,T;R^{nm})$, respectively, then the optimal control problem consisting of (2.1) and (6.1) has an optimal feedback control $u^*(\cdot) \equiv (u_1^*(\cdot),\cdots,u_m^*(\cdot))$, where*

$$u_j^*(t) = -P(t)^{-1}B_j(t)'P(t)[x^*(t) - p(t)] + q_j(t), \ j = 1,\cdots,m, \tag{6.4}$$

where $(p(\cdot), q_j(\cdot), j = 1,\cdots,m) \in L^2_{\mathcal{F}}(0,T;R^n) \times L^2_{\mathcal{F}}(0,T;R^{nm})$ is the unique $\mathcal{F}_t$-adapted solution of the following BSDE:

$$\begin{cases} dp(t) = [A(t)p(t) + \sum_{j=1}^m B_j(t)q_j(t) + f(t) - P(t)^{-2}\phi(t)]dt + \sum_{j=1}^m q_j(t)dW^j(t), \ t \in [0,T], \\ p(T) = \xi, \end{cases} \tag{6.5}$$

or equivalently,

$$\begin{cases} dp(t) = & \Big\{[A(t)+P(t)^{-1}]p(t)+\sum_{j=1}^m B_j(t)q_j(t)+f(t)-P(t)^{-1}E(\xi|\mathcal{F}_t)\Big\}dt \\ & +\sum_{j=1}^m q_j(t)dW^j(t),\ t\in[0,T], \\ p(T) = & \xi. \end{cases} \tag{6.6}$$

Proof. Consider the following matrix-valued ordinary differential equation:

$$\begin{cases} \dot{Q}-AQ-QA'+\sum_{j=1}^m B_jQB_j'-Q^2=0, \\ Q(T)=I, \end{cases} \tag{6.7}$$

which is a conventional Riccati equation. Hence it admits a unique solution $Q(\cdot)$. Denote $S=PQ$, then S satisfies

$$\begin{cases} \dot{S}=SA'-A'S+SQ-Q+\sum_{j=1}^m PB_jP^{-1}B_j'S-\sum_{j=1}^m PB_jP^{-1}SB_j', \\ S(T)=I. \end{cases} \tag{6.8}$$

It has the only solution $S\equiv I$, implying $Q(t)=P(t)^{-1}$. Now do the same change of variable (3.13), we get that $(p(\cdot),q_j(\cdot),j=1,\cdots,m)$ satisfies (6.5). The equation (6.6) is equivalent to (6.5) due to the fact that $P(t)^{-2}\phi(t)=P(t)^{-1}[E(\xi|\mathcal{F}_t)-p(t)]$ (see (3.13)). □

Remark 6.1. We note that in this case $(p(\cdot),q_j(\cdot),j=1,\cdots,m)$ no longer satisfies the *original* BSDE (5.14) which is the starting point of the control problem under consideration in this paper. The reason is that the BSDE (5.14) only concerns the terminal situation, but not any time in between. Therefore a large deviation of $p(t)$ from the expected terminal value, $E(\xi|\mathcal{F}_t)$, is allowed in the setup of (5.14). However, in our modified model, it is required that this deviation cannot be too large (which will be realized by the optimal control), therefore in the optimal feedback control one no longer compares against the original BSDE (5.14).

It is interesting that in this case the SRE (6.3) *automatically* admits a solution.

Theorem 6.2. *The SRE (6.3) admits a unique solution.*

Proof. Employing the same argument as that in the proof of Theorem 4.2, set $P_\varepsilon=\Psi(\varepsilon I)-\varepsilon I$. Then P_ε satisfies

$$\begin{cases} \dot{P}_\varepsilon+P_\varepsilon(A-\sum_{j=1}^m B_jB_j')+(A-\sum_{j=1}^m B_jB_j')'P_\varepsilon-\varepsilon^{-1}\sum_{j=1}^m P_\varepsilon B_jB_j'P_\varepsilon \\ \quad +\varepsilon(A+A'-\sum_{j=1}^m B_jB_j')+I=0, \\ P_\varepsilon(T)=I-\varepsilon I. \end{cases} \tag{6.9}$$

When $\varepsilon>0$ is small enough, $\varepsilon(A+A'-\sum_{j=1}^m B_jB_j')+I>0$, hence the solution of the above equation (which is a conventional Riccati equation) $P_\varepsilon\geq 0$. This implies that (4.2) holds with $K=\varepsilon I$ for sufficiently small $\varepsilon>0$. The result follows then from Theorem 4.1. □

Now let us consider the corresponding Black-Scholes model. The SRE (5.15) is modified to

$$\begin{cases} \dot{P}(t) + 2r(t)P(t) - \sum_{j=1}^m \theta_j(t)^2 P(t) + 1 = 0, \\ P(T) = 1, \\ P(t) > 0, \ t \in [0,T]. \end{cases} \tag{6.10}$$

This equation has an explicit solution $P(t) = e^{-\int_t^T (\rho(s)-2r(s))ds} + \int_t^T e^{-\int_t^s (\rho(\tau)-2r(\tau))d\tau} ds > 0$ (the existence of solutions can also be seen from Theorem 6.2).

Theorem 6.3. *The optimal (feedback) control for the modified Black-Scholes model the following*

$$u_j^*(t) = -\theta_j(t)[x^*(t) - p(t)] + q_j(t), \quad j = 1, \cdots, m. \tag{6.11}$$

where $(p(\cdot), q_j(\cdot), j = 1, \cdots, m) \in L^2_{\mathcal{F}}(0,T;R^n) \times L^2_{\mathcal{F}}(0,T;R^{nm})$ *is the unique* $\mathcal{F}_t$*-adapted solution of the following BSDE:*

$$\begin{cases} dp(t) = [r(t)p(t) + \sum_{j=1}^m \theta_j(t) q_j(t) + \frac{p(t)-E(\xi|\mathcal{F}_t)}{P(t)}]dt + \sum_{j=1}^m q_j(t) dW^j(t), \\ p(T) = \xi. \end{cases} \tag{6.12}$$

Proof. This follows immediately from Theorem 6.1. □

Remark 6.1. Again this result has an interesting interpretation in financial terms. Rewrite (6.12) in the following way

$$\begin{cases} dp(t) = [r(t)p(t) + \sum_{j=1}^m \theta_j(t) q_j(t)]dt + \sum_{j=1}^m q_j(t) dW^j(t) - dC(t), \\ p(T) = \xi, \end{cases} \tag{6.13}$$

where

$$C(t) = \int_t^T \frac{p(s) - E(\xi|\mathcal{F}_s)}{P(s)} ds. \tag{6.14}$$

The process $C(\cdot)$ may be regarded as a cumulative *consumption* process in order to correct any deviation of the price $p(t)$ from the expected value of the claim $E(\xi|\mathcal{F}_t)$ at any time t. Note here the term "consumption" is in a more general sense in our framework; any withdrawal from or injection into the portfolio may be regarded as an action of consumption. It is interesting that an equation similar to (6.14) has been derived for studying American options (e.g., [7]), where (p,q,C) is called a *superhedging* policy and the role of $C(\cdot)$ is to make the price $p(t)$ stay *above* $E(\xi|\mathcal{F}_t)$ at any time t. Moreover, $C(t)$ is the minimum required consumption process in the sense that

$$\int_0^T [p(t) - E(\xi|\mathcal{F}_t)] dC(t) = 0. \tag{6.15}$$

In our case, since the price $p(t)$ is allowed to go either above or under $E(\xi|\mathcal{F}_t)$ (the role of $C(\cdot)$ is to make sure that the variance is minimal), we do not have (6.16). However, we do have the following analogous relations:

$$\int_0^T [p(t) - E(\xi|\mathcal{F}_t)]^+ [dC(t)]^- = 0, \quad \text{and} \quad \int_0^T [p(t) - E(\xi|\mathcal{F}_t)]^- [dC(t)]^+ = 0, \tag{6.16}$$

where $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$. Therefore, the interpretation of the portfolio (6.11) is that one should superhedge/subhedge the claim and invest the rest of the wealth according to the Merton portfolio.

7 Concluding Remarks

In this paper we introduced a stochastic control model associated with a backward stochastic differential equation, and solved the problem in a closed form by virtue of the stochastic linear-quadratic theory developed recently. The results were then applied to solve an optimal hedging problem associated with a Black-Scholes contingent claim problem. Our study suggested that the solution pair of a BSDE can be interpreted as the state-control pair of a stochastic control problem. This finding is expected to lead insights into the nature of the BSDEs as well as their applications in finance problems.

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