

Branch Dependence in the “Consistent Histories” Approach to Quantum Mechanics

Thomas Müller¹

In the consistent histories formalism one specifies a family of histories as an exhaustive set of pairwise exclusive descriptions of the dynamics of a quantum system. We define branching families of histories, which strike a middle ground between the two available mathematically precise definitions of families of histories, viz., product families and Isham’s history projector operator formalism. The former are too narrow for applications, and the latter’s generality comes at a certain cost, barring an intuitive reading of the “histories”. Branching families retain the intuitiveness of product families, they allow for the interpretation of a history’s weight as a probability, and they allow one to distinguish two kinds of coarse-graining, leading to reconsidering the motivation for the consistency condition.

KEY WORDS: quantum histories; branching time; probability; coarse graining.

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1. INTRODUCTION

The consistent histories approach to quantum mechanics^(5–8,10,14,15,19) studies the dynamics of closed quantum systems as a stochastic process within a framework of alternative possible histories. Such a framework, or *family of histories*, must consist of pairwise exclusive and jointly exhaustive descriptions of the system’s dynamics.

This intuitive characterization does not yet state what a family of histories is mathematically. There are two formal definitions in the literature: So-called *product families* are straightforward generalizations of one-time descriptions of a system’s properties in terms of projectors to the

¹ Institut für Philosophie, Lennéstr. 39, 53113 Bonn, Germany; e-mail: Thomas.Mueller@uni-bonn.de

case of multiple times.⁽⁸⁾ The so-called *history projector operator* formalism, introduced by Isham,⁽¹¹⁾ is vastly more general. However, histories in that formalism do not necessarily have an intuitive interpretation in terms of temporal sequences of one-time descriptions.

In our paper, we make formally precise the notion of a *branching* family of histories, which is meant to balance generality and intuitiveness: histories in such a family do correspond to temporal sequences of one-time descriptions, yet branching families are much more general than product families. In the context of quantum histories, the notion of branching, or branch dependence, was originally proposed by Gell-Mann and Hartle.⁽⁶⁾ It is invoked informally in many publications, but a formal definition is so far lacking. Our definition of *branching families of histories* is based on the theory of branching temporal logic. Apart from providing a precise reading of a useful concept, our approach allows us to comment on the relation between the consistency condition for families of histories and probability measures on such families. It turns out that the consistency condition is best viewed not as a precondition for introducing probabilities, as some authors suggest, but as the requirement that interference effects be absent from the description of a system's dynamics. Our definition allows for consistent as well as inconsistent families of histories. It is therefore neutral with respect to the discussion about the pros and cons of consistency,^(5,9,14,15) and we refrain from taking a stance in that discussion.

Our paper is structured as follows: In Sec. 2, we introduce some basic facts about consistent histories and probabilities and review the definition of product families of histories and the history projection operator approach. We also sketch the intuitive motivation for a notion of branch-dependent families. In our central Sec. 3, we give our formal definition of branch-dependent families of histories and prove some relevant properties of such families. In the final Sec. 4, we discuss the relation between our new definition and the two mentioned definitions of families of histories, and we comment on the consistency condition.

2. CONSISTENT HISTORIES

2.1. Histories, Chain Operators, and Weights

In the consistent histories approach, a *history* is specified via properties of the system in question at a finite number of times.² The system's prop-

² Continuous extensions of the theory have also been studied,⁽¹³⁾ but these will not be considered in this paper.—This section closely follows the notational conventions of Ref. 10, which book provides a detailed and readable introduction to consistent histories.

erties are expressed through orthogonal projectors³ on (closed) subspaces of the system’s Hilbert space \mathcal{H} , i.e., operators P for which

$$P \cdot P = P^\dagger = P. \quad (1)$$

Thus, a single history Y^α consists of a number of projectors P_α^i at given times t_i , $i = 1, \dots, n$:

$$Y^\alpha = P_\alpha^1 \odot P_\alpha^2 \odot \dots \odot P_\alpha^n. \quad (2)$$

So far, the symbol “ \odot ” should be read as “and then”; in Isham’s *history projection operator* version of the history formalism, the symbol can be read as a tensor product (cf. Sec. 2.5).

A *family of histories* \mathfrak{F} (sometimes also called a *framework*) is an exhaustive set of alternative histories. In line with most of the literature on consistent histories, we will only consider finite families in this paper.

Associated with a history Y^α is a *chain operator* $K(Y^\alpha)$, which is formed by multiplying together the projectors P_α^i associated with the times t_i ($i = 1, \dots, n$), interleaved with the respective unitary time development operators $T(t_i, t_{i+1})$. Employing the convention of Ref. 10, we define

$$K^\dagger(Y^\alpha) = P_\alpha^1 \cdot T(t_1, t_2) \cdot P_\alpha^2 \cdot \dots \cdot T(t_{n-1}, t_n) \cdot P_\alpha^n. \quad (3)$$

These chain operators are often taken to be *representations* of the respective histories. This is appropriate in that $K(Y^\alpha)$ correctly describes the successive action of the projectors forming Y^α on the system. However, the representation relation is in general many-one, which may be seen as a disadvantage in that one cannot recover a history from the associated chain operator uniquely.

The system’s dynamics explicitly enters the definition of the chain operators through the time development operators. Thus, a history Y^α can have a zero chain operator even though the history involves no zero projectors; such histories are *dynamically impossible*, i.e., ruled out by the system’s dynamics.

We assume that the initial state of the system is described by a density matrix ρ , which might be proportional to unity if no information is given.⁴ The inner product of two operators, $\langle K_1, K_2 \rangle_\rho$, given ρ , is defined via

³ More generally, one can specify POVMs or completely positive maps; cf. Ref. 20. We will only consider projectors in this paper.

⁴ Many definitions implicitly take $\rho = I$ in the case of lacking information, which will lead to wrong scaling of inner products and thus, of weights, by a factor of $\dim(\mathcal{H})$.

$$\langle K_1, K_2 \rangle_\rho = \text{Tr}[\rho \cdot K_1^\dagger \cdot K_2]. \quad (4)$$

The chain operators allow us to associate with any history Y^α a *weight* $W(Y^\alpha)$, which is the inner product of the history's chain operator with itself:

$$W(Y^\alpha) = \langle K(Y^\alpha), K(Y^\alpha) \rangle_\rho. \quad (5)$$

In general, one would hope that these weights correspond to probabilities for histories from a given family. We will comment on that issue below, but first we review a few notions from probability theory.

2.2. Probabilities

A probability space is a triple $\mathfrak{P} = \langle S, A, \mu \rangle$, where S is the sample space (the set of alternatives), A is a Boolean σ -algebra on S , and μ is a normalized, countably additive measure on A , i.e., a function

$$\mu: A \rightarrow [0, 1] \quad \text{s.t.} \quad \mu(S) = 1 \quad (6)$$

and such that for any countable family $(a_j)_{j \in J}$ of disjoint elements of A , μ is additive:

$$\mu\left(\bigcup_{j \in J} a_j\right) = \sum_{j \in J} \mu(a_j). \quad (7)$$

For a finite set S , the algebra A is isomorphic to the so-called power set algebra, i.e., the Boolean algebra of subsets of S , with minimal element \emptyset and maximal element S ; the operations of join, meet, and complement are set-theoretic union, intersection, and set-theoretic complement, respectively. In the finite case, μ is uniquely specified by its value on the singletons (atoms), and normalization is expressed by the condition

$$\sum_{s \in S} \mu(\{s\}) = 1. \quad (8)$$

For our treatment of quantum histories, which follows the literature in assuming finite families, these latter, simplified conditions are sufficient: A finite probability space is completely specified by giving a finite set S of alternatives and an assignment μ of non-negative numbers fulfilling (8). The algebra A is then given as the power set algebra of S , and μ is extended to all of A via (7).

2.3. Families of Histories: Product Families

Intuitively, a family \mathfrak{F} of histories should consist of an exhaustive set of exclusive alternatives. Thus, any one history $h \in \mathfrak{F}$ should rule out all other histories from \mathfrak{F} , and \mathfrak{F} should have available enough histories to describe any possible dynamic evolution of the system in question. Exclusiveness must in some way be linked to orthogonality of projectors. This rules out taking \mathfrak{F} to be the set of all time-ordered sets of projectors of the form (2)—such a family would be far too large. The simplest way to ensure the requirements of exclusiveness and exhaustiveness is to fix a sequence of time points

$$t_1 < t_2 < \cdots < t_n \quad (9)$$

and to specify, for each time t_i , a decomposition of the identity operator I on the system’s Hilbert space \mathcal{H} into n_i orthogonal projectors $\{P_1^i, \dots, P_{n_i}^i\}$, so that

$$P_j^i \cdot P_{j'}^i = \delta_{jj'} P_j^i, \quad \sum_{j=1}^{n_i} P_j^i = I. \quad (10)$$

In this case, the index α specifying a history Y^α can be taken to be the list of numbers $(\alpha_1, \dots, \alpha_n)$, where $1 \leq \alpha_i \leq n_i$. The size $|\mathfrak{F}|$ of the family \mathfrak{F} is given by

$$|\mathfrak{F}| = n_1 \times n_2 \times \cdots \times n_n. \quad (11)$$

Histories in \mathfrak{F} are pairwise exclusive, since any two different histories use different, orthogonal projectors at some time t_i . Such a family is also exhaustive, as at every time, the decomposition of the identity specifies an exhaustive set of alternatives. Formally, \mathfrak{F} corresponds to a cartesian product of decompositions of the identity at different times. Product families are thus the obvious generalization of one-time descriptions of a system’s properties in terms of projectors to the case of multiple times.

2.4. Branch-dependent Families of Histories

While the construction of a product family of histories is the simplest way to ensure exclusiveness and exhaustiveness, that construction is by no means the only possibility. Many authors have noted that in applications,

it will often be necessary to choose a time point t_{i+1} , or the decomposition of I at t_{i+1} , dependent on the projector P_α^i employed at time t_i . Thus, e.g., in order to describe a “delayed choice” quantum correlation experiment,⁽¹⁾ one chooses the direction of spin projection at time t_2 depending on the outcome of a previous selection event at time t_1 .

It is intuitively quite clear what such “branch dependence” would mean; Eqs. (22)–(25) below give an example of a branch-dependent family of histories. However, no formally rigorous description of such families of histories is available so far. Before we go on to give such a description in Sec. 3, we introduce Isham’s⁽¹¹⁾ generalized definition of families of histories in terms of history projection operators (HPOs), with which our definition of branch dependence will be compared below.

2.5. Isham’s History Projection Operators (HPO)

The guiding idea of the HPO framework is to single out, as for product families, n times t_1, \dots, t_n at which the system’s properties will be described. One then forms the n -fold tensor product of the system’s Hilbert space:

$$\tilde{\mathcal{H}} = \mathcal{H} \otimes \dots \otimes \mathcal{H} \quad (n \text{ times}). \quad (12)$$

In that large history Hilbert space $\tilde{\mathcal{H}}$, a history Y^α is read as a tensor product of projectors:

$$Y^\alpha = P_\alpha^1 \odot P_\alpha^2 \odot \dots \odot P_\alpha^n = P_\alpha^1 \otimes P_\alpha^2 \otimes \dots \otimes P_\alpha^n. \quad (13)$$

That tensor product operator Y^α is itself a projection operator on $\tilde{\mathcal{H}}$, a so-called *history projection operator*, fulfilling

$$Y^\alpha \cdot Y^\alpha = (Y^\alpha)^\dagger = Y^\alpha. \quad (14)$$

Along these lines one can give an abstract definition of a family of histories $\mathfrak{F} = \{Y^1, \dots, Y^n\}$ as a decomposition of the history Hilbert space identity \tilde{I} :

$$Y^\alpha Y^\beta = \delta_{\alpha\beta} Y^\alpha, \quad \sum_{Y^\alpha \in \mathfrak{F}} Y^\alpha = \tilde{I}. \quad (15)$$

This generalization is formally rigorous, and it allows for further (e.g., continuous) extensions. However, the generality comes at a certain cost,

since there is no condition that would ensure that a history projector Y^α should factor into a product of n projectors on the system’s Hilbert space at the n given times, as in (2). History projectors that do factor in this way are called *homogeneous histories*. As the main motivation for introducing histories is given in terms of homogeneous histories, some authors have expressed doubts as to whether the full generality of HPO is really appropriate.^(10, p. 118)

One possibility for constructing a narrower framework is to restrict the HPO formalism to the homogeneous case by requiring that all the $Y^\alpha \in \mathfrak{F}$ be products of projectors of the form (2). Such a restriction may be implicitly at work in Ref. 10. However, such a restriction is to some extent alien to the HPO formalism. Nor does it single out a useful class of families of histories: as we will show below (Eqs. (54)–(57)), not all homogeneous families are branch-dependent families, and some homogeneous families do not admit an interpretation of weights in terms of probabilities.

We now describe an alternative approach in which branch dependence is the natural result of an inductive definition. Furthermore, the framework allows one to distinguish two different types of coarse-graining for families of histories.

3. A FORMAL FRAMEWORK FOR BRANCH-DEPENDENT HISTORIES

The idea of a branching family of histories is based on the theory of branching temporal logic that originated in the work of Prior.^{(21)⁵}

3.1. Branching Structures

For the purpose of constructing finite families of branching histories, branching temporal logic boils down to the following inductive definition of a *branching structure*, which is a set M of *moments* m_i partially ordered by \preceq :⁶

⁵ Branching temporal logic has already found many applications in computer science, linguistics, and philosophy. The interested reader is referred to Ref. 4 for an overview.

⁶ One should think of \preceq in analogy to “less than or equal” (\leq), so there is a companion strict order (excluding equality), which is denoted by $<$. However, note that \preceq is only a partial order, i.e., some elements of M may be incomparable.

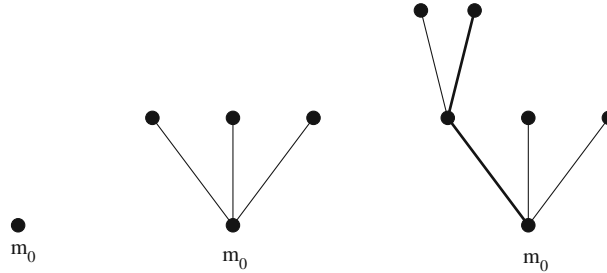


Fig. 1. Three stages in the construction of a branching structure. The thick line in the structure on the right indicates one of the four histories in that structure.

Definition 1. (*Branching structure*)

- (1) A singleton set $M = \{m_0\}$ together with the relation $m_0 \preceq m_0$ is a branching structure.
- (2) Let $\langle M, \preceq \rangle$ be a branching structure and $m \in M$ a maximal element, and let m_1^*, \dots, m_n^* be new elements. Let \preceq^* be the reflexive and transitive closure of the relation \preceq together with the new relations $m \preceq^* m_1^*, \dots, m \preceq^* m_n^*$. Then the set $M \cup \{m_1^*, \dots, m_n^*\}$ together with the relation \preceq^* is again a branching structure.

By taking a finite number of steps along this definition, one constructs a finite branching tree in the form of a partially ordered set with the unique root element m_0 .⁷ The maximal nodes in the tree are called “leaves”.

Figure 1 shows the inductive process. Except for the root element m_0 , each node has a unique direct predecessor, and except for the leaves, each node m has one or more direct successors, which correspond to branching at m . Paths in the tree, i.e., maximal linearly ordered subsets, extend from the root to one of the leaves and are thus in one-to-one correspondence with the leaves. These paths are often called *histories* by logicians, and they can indeed be given an interpretation in terms of quantum histories, as we will show in the next section.

⁷Note that the construction ensures the following formal features: The ordering \preceq is transitive (if $x \preceq y$ and $y \preceq z$, then $x \preceq z$), reflexive ($x \preceq x$) and antisymmetric (if $x \preceq y$ and $y \preceq x$, then $x = y$). Furthermore, the ordering fulfills the axioms of “no backward branching” (if $x \preceq z$ and $y \preceq z$, then either $x \preceq y$ or $y \preceq x$) and of “historical connection”, meaning the existence of a common lower bound for any two elements (for any $x, y \in M$, there is $z \in M$ s.t. $z \preceq x$ and $z \preceq y$). In a more general approach to branching temporal logic, these formal features are taken as axioms of a logical framework.

3.2. Branching Families of Histories

A branching family of histories can be viewed as a quantum-mechanical interpretation of a branching structure $\langle M, \preceq \rangle$. We assume that a system with Hilbert space \mathcal{H} (identity operator I) is given. The interpretation is given by two functions τ and P that associate times and projectors with the elements of M , respectively. Formally, we define:

Definition 2. (*Branching family of histories*)
A *branching family of histories* is a quadruple

$$\mathfrak{F} = \langle M, \preceq, \tau, P \rangle, \quad (16)$$

where $\langle M, \preceq \rangle$ is a finite branching structure and τ is a function from M to the real numbers respecting the partial ordering \preceq :

$$\text{if } m < m', \quad \text{then } \tau(m) < \tau(m'). \quad (17)$$

P is a function from M to the set of projectors on \mathcal{H} that assigns projection operators to the elements of M in the following way: If $m \in M$ is not a maximal element, and m_1, \dots, m_{n_m} are the n_m immediate successors of m , then a set of orthogonal projectors $P_1^m, \dots, P_{n_m}^m$ forming a decomposition of the identity,

$$P_i^m P_j^m = \delta_{ij} P_i^m, \quad \sum_{i=1}^{n_m} P_i^m = I \quad (18)$$

is assigned to the m_1, \dots, m_{n_m} via $P(m_i) = P_i^m$.

The number $\tau(m) \in \mathbb{R}$ is the time associated with $m \in M$. While the same time may be assigned to moments in different histories, we require that τ respect the partial ordering, as expressed via (17).⁸ As regards the assignment of projection operators, note that $P(m_i) = P_i^m$ means that at

⁸The time parameter will only be needed in specifying the time development operators in the definition of the chain operators later on. Working in the Heisenberg picture, the function τ would not be needed; the condition on τ would be replaced by requiring an appropriate temporal ordering of the Heisenberg operators specified through P . This already points toward a relativistic generalization of the current approach to quantum histories, which is currently under preparation.⁽¹⁷⁾

m (the unique predecessor of m_i , not m_i itself), the system had the property expressed by P_i^m .⁹ The function P thus associates decompositions of the Hilbert space identity with instances of branching. In this way, each maximal path α of length $n_\alpha + 1$ in $\langle M, \leq \rangle$,

$$m_0^\alpha < m_1^\alpha < \dots < m_{n_\alpha}^\alpha \quad (19)$$

corresponds to the n_α elements long chain of projection operators

$$P(m_1^\alpha) \odot P(m_2^\alpha) \odot \dots \odot P(m_{n_\alpha}^\alpha). \quad (20)$$

Here, $m_0^\alpha = m_0$ is the root node, and $m_{n_\alpha}^\alpha$ is one of the maximal elements. The projectors give information for the n_α many times

$$\tau(m_0^\alpha) < \tau(m_1^\alpha) < \dots < \tau(m_{n_\alpha-1}^\alpha). \quad (21)$$

Our Definition 2 captures the intuitive notion of branch dependence described in Sec. 2.4 in a formally exact manner, as illustrated by Fig. 2. In that example of a branching family of histories, the root node m_0 has two successors, m_1 and m_2 , corresponding to the system's having the properties corresponding to projection operators P_0^1 and P_0^2 at $\tau(m_0)$, respectively. The vertical position of m_1 versus m_2 indicates that $\tau(m_1) \neq \tau(m_2)$, which is one aspect of branch dependence that is not available in product families: the time for which the system's property is described after $\tau(m_0)$ depends on the system's property at $\tau(m_0)$. Furthermore, the decompositions of the Hilbert space identity at m_1 ($\{P_1^1, P_1^2, P_1^3\}$) and at m_2 ($\{P_2^1, P_2^2\}$) are different, thus exhibiting the second form of branch dependence that is not available in product families.

3.3. Properties of Quantum Branching Histories

We first note that every product family of histories (cf. Sec. 2.3) is a branching family of a very symmetric kind.

⁹The slightly awkward reference to the previous node in (18) can be avoided if one assigns projectors more properly not to nodes, but to *elementary transitions* in the branching structure; cf. Refs. 22,3 for details. We stick to our simplified exposition in order not to clutter this paper with technicalities—which will, however, be relevant for an extension to infinite structures, or to a relativistic version employing *branching space-times*.^(2,16,17)

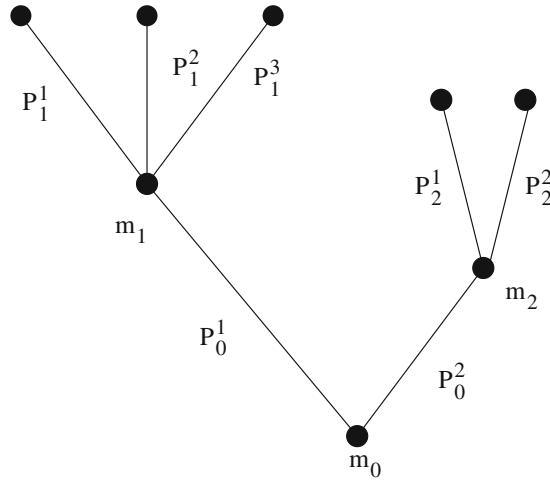


Fig. 2. An example branching family of histories. See text for details.

Lemma 1. (*Branching versus product families*)

Every product family of histories is a branching family of histories, but not conversely.

Proof. In order to see that product families are branching families, let a product family corresponding to n decompositions of the identity $\{P_1^i, \dots, P_{n_i}^i\}$ at times t_1, \dots, t_n be given. An equivalent branching family is constructed in $n + 1$ steps as follows: We start with a single node, $M_0 = \{m_0\}$ (stage 0). Then at each stage i ($1 \leq i \leq n$), we add new nodes and enlarge the structure. We assign the time t_i to all of the maximal elements of M_{i-1} , and we add the decomposition $\{P_1^i, \dots, P_{n_i}^i\}$ above all these maxima by introducing n_i new elements above *each* maximum, thus arriving at the new set of nodes M_i . When M_n has been constructed, we finally assign some time $t^* > t_n$ to all the maximal elements in M_n . This construction yields a symmetrically growing tree that in the end (at stage n) corresponds to the original product family of histories.

For a branching family that is not a product family, let \mathcal{H} have dimension 2, and let $\{|\phi_1\rangle, |\phi_2\rangle\}$ and $\{|\psi_1\rangle, |\psi_2\rangle\}$ be two different orthonormal bases of \mathcal{H} . The family

$$h_1 = |\phi_1\rangle\langle\phi_1| \odot |\phi_1\rangle\langle\phi_1|, \tag{22}$$

$$h_2 = |\phi_1\rangle\langle\phi_1| \odot |\phi_2\rangle\langle\phi_2|, \tag{23}$$

$$h_3 = |\phi_2\rangle\langle\phi_2| \odot |\psi_1\rangle\langle\psi_1|, \quad (24)$$

$$h_4 = |\phi_2\rangle\langle\phi_2| \odot |\psi_2\rangle\langle\psi_2| \quad (25)$$

describes the system at two times t_1 and t_2 , yielding a branching family, but not a product family. \square

Branching families of histories $\mathfrak{F} = \langle M, \preceq, \tau, P \rangle$ thus yield a natural generalization of product histories while retaining a strong link between the formalism and the intended temporal interpretation of the histories. Furthermore, the inductive definition of the structure makes it easy to prove two key properties of branching families of histories. First, the construction immediately shows that \mathfrak{F} is exclusive and exhaustive: The one-element family has that property, and it is retained by adding inductively further (exclusive and exhaustive) decompositions of the identity at a maximal node.¹⁰ Secondly, one can show that the weights of histories in a branching family \mathfrak{F} always add up to one. Thus, the weights immediately induce probabilities on the power set Boolean algebra of \mathfrak{F} (cf. Sec. 2.2).

Lemma 2. (*Weights in branching families*)

In a branching family of histories \mathfrak{F} , the weights sum to one:

$$\sum_{h \in \mathfrak{F}} W(h) = 1. \quad (26)$$

Proof. Assume that an initial state of the system is given by a density matrix ρ_0 .¹¹ For the trivial family consisting of only one history $\{m_0\}$, the sum of weights reduces to

$$W(\{m_0\}) = Tr[\rho_0] = 1. \quad (27)$$

Now assume that the property in question holds for the family \mathfrak{F} corresponding to $\langle M, \preceq, \tau, P \rangle$, let m be a maximal node, m^- its (unique) direct predecessor, and let

$$P_i^m P_j^m = \delta_{ij} P_i^m, \quad \sum_{i=1}^n P_i^m = I \quad (28)$$

¹⁰ “Exhaustive” does not mean “maximally detailed”. The latter notion is quite dubious anyway, as for every finite description of a quantum system’s dynamics one can give a more detailed one.

¹¹ In the case of complete ignorance, $\rho_0 = I/\dim(\mathcal{H})$.

be the decomposition of the identity that is to be employed at the n new maximal elements m_1^*, \dots, m_n^* that are to be added after m . Suppose that \mathfrak{F} had N elements h_1, \dots, h_N , whose weights add to one. In order to facilitate book-keeping, suppose further that m is the final node of h_N . To the new quantum branching structure $\langle M', \preceq', \tau', P' \rangle$ there corresponds a family \mathfrak{F}' of $N + n - 1$ histories, where for $i = 1, \dots, N - 1$, $h'_i = h_i$, whereas h_N is replaced by the n new histories h'_N, \dots, h'_{N+n-1} ending in the new elements m_1^*, \dots, m_n^* . In order to show that in \mathfrak{F}' , the weights still add to one, we only need to show that

$$W(h_N) = \sum_{i=1}^n W(h'_{N+i-1}), \quad (29)$$

i.e., the histories replacing old h_N must together have the same weight as h_N . Now in terms of the chain operator $K(h_N)$ for h_N , the chain operators for the new histories are

$$K(h'_{N+i-1}) = P_i^m \cdot T(\tau(m), \tau(m^-)) \cdot K(h_N). \quad (30)$$

The initial density matrix ρ_0 , evolved along h_N , becomes

$$\rho_m = K(h_N) \rho_0 K^\dagger(h_N) \quad (31)$$

and the weight of h_N can be expressed as

$$W(h_N) = \text{Tr}[K(h_N) \rho_0 K^\dagger(h_N)] = \text{Tr}[\rho_m]. \quad (32)$$

The weights for the new histories can then be written as

$$\begin{aligned} W(h'_{N+i-1}) &= \text{Tr}[P_i^m T(\tau(m), \tau(m^-)) \rho_m T(\tau(m^-), \tau(m)) (P_i^m)^\dagger] \\ &= \text{Tr}[T(\tau(m^-), \tau(m)) P_i^m T(\tau(m), \tau(m^-)) \rho_m], \end{aligned} \quad (33)$$

where we used the cyclic property of the trace and $P^\dagger = P^2 = P$. Now as T is unitary, the n operators

$$\tilde{P}_i^m = T(\tau(m^-), \tau(m)) P_i^m T(\tau(m), \tau(m^-)) \quad (34)$$

are again projectors forming a decomposition of the identity, so that by the linearity of the trace,

$$\sum_{i=1}^n W(h'_{N+i-1}) = \sum_{i=1}^n \text{Tr}[\tilde{P}_i^m \rho_m] \quad (35)$$

$$= \text{Tr} \left[\sum_{i=1}^n \tilde{P}_i^m \rho_m \right] \quad (36)$$

$$= \text{Tr}[I \cdot \rho_m] = W(h_N), \quad (37)$$

which was to be proved. \square

4. COARSE GRAINING, PROBABILITIES AND THE CONSISTENCY CONDITION

The general idea of coarse graining is that it should be possible to move from a more to a less detailed description of a given system in a coherent way. If probabilities are attached to a fine-grained description, then an obvious requirement is that the probability of a coarse-grained alternative should be the sum of the probabilities of the corresponding fine-grained alternatives. Considerations of coarse graining are important for quantum histories because of the interplay between weights of histories and probability measures in a family of histories. Our discussion will show that one needs to distinguish two notions of coarse graining.

One notion of coarse graining comes for free in any probability space: Due to the additivity of the measure, if b^* is the disjoint union of b_1, \dots, b_n in the event algebra, then $\mu(b^*) = \sum_{i=1}^n \mu(b_i)$.¹² By Lemma 2, for any branching family (and thus, by Lemma 1, for any product family) of quantum histories, the weights $W(h)$ of the histories $h \in \mathfrak{F}$ induce a probability measure on \mathfrak{F} via $\mu(\{h\}) = W(h)$, which is extended to the power set Boolean algebra of \mathfrak{F} via (7):

$$\mu(\{h_1, \dots, h_n\}) = \sum_{i=1}^n \mu(\{h_i\}) = \sum_{i=1}^n W(h_i). \quad (38)$$

If h_1, \dots, h_n are fine-grained descriptions of a system's dynamics, Eq. (38) shows that the coarse-grained description $\{h_1, \dots, h_n\}$ automatically is assigned the correct probability. Thus, branching families of histories unconditionally and naturally support this notion of coarse graining.

¹² In the infinite case, that equation holds for countable unions.

If all branching families support probabilities and coarse graining, then what is behind the consistency condition? In the literature it is often suggested that the possibility of defining probabilities or the possibility of coarse graining for a family of histories is conditional upon the so-called *consistency condition*,

$$\langle K(Y^\alpha), K(Y^\beta) \rangle_\rho = 0 \quad \text{if } \alpha \neq \beta, \quad (39)$$

which demands that the chain operators of different histories must be orthogonal. Condition (39) is also called “medium decoherence”.⁽⁷⁾

The above considerations show that for branching families, both probabilities and one notion of coarse graining are unproblematic, independent of any condition like (39). However, that condition does play an important role with respect to a second, different notion of coarse graining.

That second notion of coarse graining is based on the idea of constructing from the histories $h \in \mathfrak{F}$ not histories *sets of histories*, as in Eq. (38), but *new histories*, by something like addition. Whether such additive combination of histories is possible at all, generally depends on what histories are mathematically. We will see below that additive combination is not always possible for histories in a branching family. Accordingly, in order not to suggest that addition of histories is always unproblematic, we will use the formal notation “ $\text{sum}(h_1, h_2)$ ” when we wish to leave open the question whether that sum is in fact defined.

If we consider the additive combination of two histories, h_1 and h_2 ,

$$h = \text{sum}(h_1, h_2) \quad (40)$$

the idea behind coarse graining suggests that for h , which is a less detailed description than the two fine-grained histories, the probabilities should just add:

$$\mu(h) = \mu(\text{sum}(h_1, h_2)) = \mu(h_1) + \mu(h_2). \quad (41)$$

Even apart from the question of whether $\text{sum}(h_1, h_2)$ can be defined, Eq. (41) is problematic as it stands: No family of histories can contain both two histories h_1 and h_2 and their sum h , as that would violate the requirement of exclusiveness. Thus, μ in (41) cannot be a probability measure in a single family of histories. At this point the idea of weights $W(h)$ as probabilities enters. Assuming that $h = \text{sum}(h_1, h_2)$ is indeed a history,

W is defined for all three of h_1 , h_2 , and h , and the main idea of (41) can be reformulated as

$$W(h) = W(\text{sum}(h_1, h_2)) = W(h_1) + W(h_2). \quad (42)$$

The validity of (42) is indeed linked to the consistency condition (39). However, depending on which type of family of histories one considers, there are some subtle issues, as the following sections point out.

4.1. Coarse Graining in Product Families

If \mathfrak{F} is a product family of histories, the sketched idea of coarse graining makes immediate sense, as the sum of any two histories in a given product family can be defined. To consider the basic case, let two histories $h_1, h_2 \in \mathfrak{F}$ be given,

$$h_\alpha = P_\alpha^1 \odot \cdots \odot P_\alpha^n, \quad \alpha = 1, 2 \quad (43)$$

such that they coincide everywhere except for the j th position: $P_1^i = P_2^i$ for $i \neq j$, $P_1^j \neq P_2^j$. In this case, one can define their sum

$$h = \text{sum}(h_1, h_2) := P_1^1 \odot \cdots \odot P_1^{j-1} \odot (P_1^j + P_2^j) \odot P_1^{j+1} \odot \cdots \odot P_1^n, \quad (44)$$

i.e., at each time t_i for which the histories h_1 and h_2 are defined, their sum, h , specifies either the same projector as each of h_1 and h_2 , or gives a less detailed description in terms of the projector $P_1^j + P_2^j$. The assumption of a product family is crucial in this definition, as it guarantees that the P_1^i and P_2^i are both defined at the same times, and that P_1^j and P_2^j commute—for a branch-dependent family, $P_1^j + P_2^j$, even if defined, would not normally be a projector.

For a history like h in (44), the weight function W is naturally defined even though $h \notin \mathfrak{F}$, and it appears natural to demand that

$$W(h) = W(\text{sum}(h_1, h_2)) = W(h_1) + W(h_2). \quad (45)$$

This equation does not hold in general in product families of histories. A family of histories \mathfrak{F} must satisfy the above-mentioned condition of *consistency* if it is to satisfy (45) for all $h_1, h_2 \in \mathfrak{F}$, and it was along these lines

that Griffiths⁽⁸⁾ originally motivated the consistency condition for product families of histories.¹³ However, as the next section shows, the symmetric nature of product families hides an important asymmetry in adding histories.

4.2. Coarse Graining in Branching Families

We have already seen that in product families \mathfrak{F} , the formal addition $\text{sum}(h_1, h_2)$ can be defined for any $h_1, h_2 \in \mathfrak{F}$. For branching families, this is not always possible. In fact, we will see that with respect to formula (45) one should distinguish two types of coarse graining, which we call *intra-branch* and *trans-branch* coarse graining. *Intra-branch* coarse graining means that maximal nodes from an otherwise shared branch are added, whereas *trans-branch* coarse graining means adding “across branches”. The notion of intra-branch coarse graining and the respective summation of histories for the basic case can be defined as follows:

Definition 3. (*Intra-branch coarse graining*)

In a branching family \mathfrak{F} , the formal summation $h = \text{sum}(h_1, h_2)$ of two histories $h_1, h_2 \in \mathfrak{F}$,

$$h_\alpha = P_\alpha^1 \odot \cdots \odot P_\alpha^{n_\alpha}, \quad \alpha = 1, 2 \tag{46}$$

is called *intra-branch coarse graining* iff $n_1 = n_2$, the histories are defined at the same times, and for $1 \leq i < n_1$, $P_1^i = P_2^i$. In that case, the sum is defined to be

$$h = \text{sum}(h_1, h_2) := P_1^1 \odot \cdots \odot P_1^{n_1-1} \odot (P_1^{n_1} + P_2^{n_1}). \tag{47}$$

Intra-branch coarse graining is both well-defined and probabilistically unproblematic for all branching families:

Lemma 3. For intra-branch coarse graining $h = \text{sum}(h_1, h_2)$ as in (47) in a branching family of histories, the weights add according to (45), i.e.,

$$W(h) = W(\text{sum}(h_1, h_2)) = W(h_1) + W(h_2). \tag{48}$$

¹³ An extended discussion of questions of uniqueness conditions for probability assignments in product families is given in Ref. 18.—The notion of consistency in Ref. 8, which corresponds to (45), is weaker than the consistency condition (39) formulated above: For (45) to hold, it is sufficient that the *real part* of $\langle K(Y^\alpha), K(Y^\beta) \rangle_p$ vanish for $\alpha \neq \beta$. The latter condition is known as *weak consistency*. In what follows, we will not differentiate between medium and weak consistency.

Proof. We can follow the lines of the inductive proof of Lemma 2: Let two histories h_1 and h_2 fulfilling Definition 3 be given, and assume that m is the immediate predecessor of maximal nodes m_1^* and m_2^* of histories h_1 and h_2 , respectively. Let m^- be the unique direct predecessor of m . Let $K(h_m)$ be the chain operator for the path from the root node to m , and set $T = T(\tau(m), \tau(m^-))$. Then the chain operators for the histories h_α are:

$$K(h_\alpha) = P(m_\alpha^*) \cdot T \cdot K(h_m). \quad (49)$$

The sum of the two final projectors, $P(m_1^*) + P(m_2^*)$, is again a projector in virtue of (18). Accordingly, the weight of the coarse-grained history $h = h_1 + h_2$ is

$$\begin{aligned} W(h) &= \langle K(h), K(h) \rangle_\rho = \langle K(h_1 + h_2), K(h_1 + h_2) \rangle_\rho \\ &= Tr[(P(m_1^*) + P(m_2^*)) T K(h_m) \rho \\ &\quad \times K^\dagger(h_m) T^\dagger (P(m_1^*) + P(m_2^*))^\dagger] \\ &= Tr[T^\dagger P(m_1^*) T K(h_m) \rho K^\dagger(h_m)] \\ &\quad + Tr[T^\dagger P(m_2^*) T K(h_m) \rho K^\dagger(h_m)] \\ &= W(h_1) + W(h_2), \end{aligned} \quad (50)$$

where we employed $P^\dagger(m_\alpha^*) = P(m_\alpha^*) = (P(m_\alpha^*))^2$, the fact that $P(m_1^*) \cdot P(m_2^*) = 0$ (18), and linearity and the cyclic property of the trace. \square

So all is well probabilistically if histories are formed as sums of histories that differ only at the last node, i.e., via intra-branch coarse graining. Note that this result carries over to product families of histories: for intra-branch coarse graining in branching families and in product families, Eq. (45) holds automatically, without having to presuppose a consistency condition like (39).

What about trans-branch coarse graining? For a product family, Eq. (44) shows how to build histories from other histories quite generally, and we have mentioned the fact that the validity of Eq. (45) for trans-branch coarse graining in a product family generally depends on a consistency condition like (39).^(8,10) For the more general case of branching families, so far the summation of histories for trans-branch coarse graining has not been defined. It is possible to define that type of summation in the extended framework of Isham's HPO formalism, but this amounts to discarding the intuitive idea that histories are temporal sequences of one-time descriptions (cf. the next subsection).

The problem of defining trans-branch coarse graining while holding on to the intuitive interpretation of a history may be illustrated by considering a two-dimensional Hilbert space and the two histories h_1 and h_3 from Eq. (22) and Eq. (24), respectively, taken to be defined at the two times t_1 and t_2 . How should one define $\text{sum}(h_1, h_3)$? Surely one can coarse-grain by considering the *set* of histories $\{h_1, h_3\}$, and the sum of h_1 and h_3 in Isham’s formalism amounts to this exactly. But no temporal interpretation in terms of a single history is forthcoming, as the projectors involved at t_2 do not commute.

Thus, for trans-branch coarse graining, the formal summation $\text{sum}(h_1, h_2)$ generally must remain undefined. With respect to the coarse-graining criterion (45), this means that in all cases in which it generally makes sense to ask whether it is satisfied, it is satisfied automatically: Eq. (45) is generally defined only for intra-branch coarse graining, and for that case, Lemma 3 has shown the equation to hold unconditionally.

This result does of course not mean that all branching families of histories are consistent. The test of Eq. (39) can still be applied for any branching family, and many branching families will be classified as inconsistent. However, in these cases, the link with Eq. (45), which holds for product families, can no longer be made in our branching histories framework. This points to a somewhat different interpretation of the consistency condition: In a branching family, that condition should not be read as a precondition for the assignment of probabilities via weights (which is unproblematic in view of Lemma 2), but rather as the condition that the different descriptions of the system’s dynamics given by the histories in the family amount to wholly separate, interference-free alternatives. This is just what it means for the chain operators to be orthogonal, i.e., to satisfy Eq. (39). We hold it to be an advantage of the branching family formalism proposed here that by leaving trans-branch coarse graining undefined, it forces us to rethink the interpretation of the consistency condition. The formalism of product histories is deceptively smooth in treating all times in the same way, thus blurring the distinction between intra-branch and trans-branch coarse graining.

4.3. Coarse Graining in Isham’s HPO

The comparison between branching families and Isham’s HPO scheme is illuminating in another respect. HPO is more general and more abstract than branching families. However, that abstractness comes at a price: we

will argue that much of the intuitiveness of branching families is lost by moving to the HPO scheme.

In Isham's HPO scheme, histories are themselves represented as projectors on the (large) history Hilbert space, not as chain operators on the (much smaller) system Hilbert space. As an HPO-family \mathfrak{F} must correspond to a decomposition of the history identity operator, sums of histories in \mathfrak{F} will again correspond to history projectors (even though not from the given family); the formal addition $h = \text{sum}(h_1, h_2)$ has a direct interpretation as the literal addition of history projectors. Thus one can form a Boolean algebra with elements

$$Y = \sum_{Y^\alpha \in \mathfrak{F}} \pi_\alpha Y^\alpha, \quad \pi_\alpha \in \{0, 1\} \quad (51)$$

that is isomorphic to the power set algebra of \mathfrak{F} (the π_α playing the role of characteristic functions)—just like in the first type of coarse graining considered at the beginning of this section. From one point of view, addition here always forms like objects from like objects: sums of history projectors are again history projectors. From another point of view, however, addition remains problematic: Even if the Y^α are homogeneous histories, i.e., have an intuitive interpretation as temporally ordered sequences of projectors, that will generally not be so for their sums. Thus in general, the elements (51) of the Boolean algebra are inhomogeneous histories without an intuitive interpretation.

At the level of abstraction of Eq. (51), one need not distinguish between different kinds of coarse graining. Accordingly, it seems more natural to demand additivity of weights,

$$W \left(\sum_{Y^\alpha \in \mathfrak{F}} \pi_\alpha Y^\alpha \right) = \sum_{Y^\alpha \in \mathfrak{F}} \pi_\alpha W(Y^\alpha), \quad (52)$$

which corresponds to satisfaction of the consistency condition (39). However, at the same level of abstraction, one can note that W is a quadratic function, whereas (52) demands linearity—not a natural demand at all. The motivation for additivity was, after all, given in terms of the time-ordered sequences of projectors that formed the basis of the history framework, not in terms of some abstract algebra. Furthermore, even when HPO is restricted to families of homogeneous histories, a probability interpretation of the weights is not forthcoming generally; families of HPO histories can violate a number of seemingly straightforward assumptions.⁽¹²⁾ As an example, consider the following family of histories: Let \mathcal{H} have

dimension 2, and let $\{|\phi\rangle, |\psi\rangle\}$ be an orthonormal basis, so that $\langle\phi|\psi\rangle = 0$. Then, define

$$|\chi\rangle = \frac{1}{\sqrt{2}}(|\phi\rangle + |\psi\rangle), \quad |\chi'\rangle = \frac{1}{\sqrt{2}}(|\phi\rangle - |\psi\rangle). \quad (53)$$

Note that $\langle\chi|\chi'\rangle = 0$. We now construct a family of two-time histories; the corresponding history Hilbert space $\tilde{\mathcal{H}}$ has dimension 4:

$$h_1 = |\chi\rangle\langle\chi| \odot |\psi\rangle\langle\psi|, \quad (54)$$

$$h_2 = |\chi'\rangle\langle\chi'| \odot |\psi\rangle\langle\psi|, \quad (55)$$

$$h_3 = |\phi\rangle\langle\phi| \odot |\phi\rangle\langle\phi|, \quad (56)$$

$$h_4 = |\psi\rangle\langle\psi| \odot |\phi\rangle\langle\phi|. \quad (57)$$

These four histories are pairwise orthogonal, and their sum is the identity operator in $\tilde{\mathcal{H}}$. Thus, $\{h_1, \dots, h_4\}$ is a homogeneous family of histories. Now, taking the initial density matrix ρ to be the pure state $\rho = |\phi\rangle\langle\phi|$, one can compute the following:

$$K(h_1) \rho K^\dagger(h_1) = \frac{1}{4} |\psi\rangle\langle\psi|, \quad W(h_1) = \frac{1}{4}, \quad (58)$$

$$K(h_2) \rho K^\dagger(h_2) = \frac{1}{4} |\psi\rangle\langle\psi|, \quad W(h_2) = \frac{1}{4}, \quad (59)$$

$$K(h_3) \rho K^\dagger(h_3) = |\phi\rangle\langle\phi|, \quad W(h_3) = 1, \quad (60)$$

$$K(h_4) \rho K^\dagger(h_4) = 0, \quad W(h_4) = 0. \quad (61)$$

Thus, the sum of weights in this HPO family of histories is $3/2$, barring any straightforward probability interpretation. This family is *not* a branching family of histories in the sense of Sec. 3, proving that branching families of histories are a subclass even of homogeneous HPO families.¹⁴

4.4. Discussion

The consistent history approach to quantum mechanics offers a view of quantum mechanics that honours many classical intuitions while remaining, of course, faithful to the empirical predictions of orthodox

¹⁴Note that the temporally reversed family *is* a branching family, and the weights do sum to unity (as only the mirror image of h_3 , which is h_3 itself, contributes a non-zero weight).

quantum mechanics. While the initial motivation of the approach in terms of product families makes good pedagogical sense, it is too narrow for applications. Furthermore, as we have shown in Sec. 4.2, product families may be misleading because they fail to distinguish between two importantly different notions of coarse-graining.

A number of applications demand branching families of histories. However, so far there has not been available a formally rigorous definition of that class of families. Isham's HPO formalism, while offering the necessary generality, is in danger of losing touch with the intuitive motivation of the history approach. To be sure, this does not amount to any fundamental criticism, but it gives additional support for providing a less general definition that stays closely tied to the intuitive motivation of branch-dependent histories.

Through our definition of branching families of histories we have here provided the sought-for formal framework. Branching families are more general than product families, and they are general enough for applications while retaining a natural interpretation.

Figure 3 gives a graphical overview of the various kinds of families of histories that we considered in this paper. Branching families always admit an interpretation of weights in terms of probabilities, as the weights of the histories in such a family add to one. Consistent branching families in addition are free from interference effects. The formal framework presented here is neutral with respect to the question of whether families of

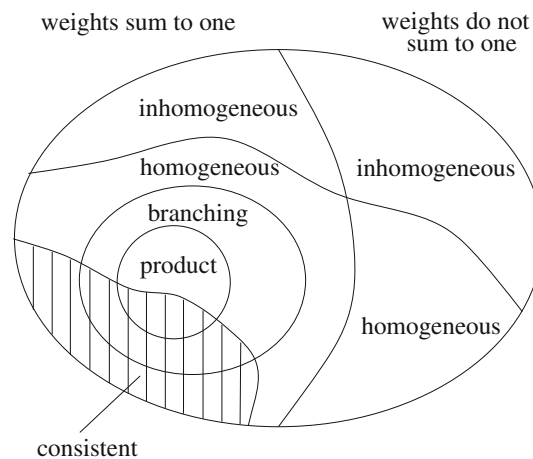


Fig. 3. The relation of the various notions of families of histories considered in this paper.

histories that are not consistent in this sense can be put to good physical use or not.

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