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Jan Sprenger

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Universität Konstanz
Fach D 188, 78457 Konstanz, Germany
E-Mail: preprints@informatik.uni-konstanz.de
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ON A CONTACT PROBLEM IN THERMOELASTICITY WITH SECOND SOUND

JAN SPRENGER

ABSTRACT. We investigate the existence and stability of a thermoelastic contact problem with second sound. Previous results established the existence and stability of a solution of the corresponding classical system in the case of radial symmetry. However, recent works have shown that sometimes stability can be lost when the classical Fourier heat conduction is substituted by Cattaneo's Law. We show that also in this case this substitution does indeed lead to a loss in regularity that proves to be a major problem prohibiting the transfer of the existence proof for the classical problem to the problem with second sound, leaving the existence of a solution an open question. We then prove that, if a viscoelastic term is added to the equations providing additional regularity, existence and exponential stability - the second, as can be expected, only in the case of radial symmetry - follow.

1. INTRODUCTION

We consider a thermoelastic system that can come into contact with a rigid foundation. In particular, consider the equations of thermoelasticity with second sound on a bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega = \Gamma_C \cup \Gamma_N \cup \Gamma_D$. On Γ_D, the body is held fix, while on Γ_N tractions are zero. On Γ_C, the body is free, albeit its extension is limited by a rigid foundation. The temperature is held fixed at the entire boundary. If $u = u(t, x)$, $\theta = \theta(t, x)$ and $q = q(t, x)$ describe the displacement, relative temperature and heat flow respectively, our equations take the form

$$\partial_t^2 u_i - (C_{ijkl}u_{k,l})_{,j} - \mu\partial_t u_{i,jj} + m_{ij}\theta_{,j} = 0 \quad (1)$$

$$\partial_t\theta + \operatorname{div} q + m_{ij}\partial_t u_{i,j} = 0 \quad (2)$$

$$\tau_0\partial_t q_i + q_i + K_{ij}\theta_{,j} = 0 \quad (3)$$

On $[0, T] \times \Omega$, with initial values

$$u(0, \cdot) = u_0; \quad u_t(0, \cdot) = u_1; \quad \theta(0, \cdot) = \theta_0; \quad q(0, \cdot) = q_0 \quad (4)$$

satisfying

$$u_0 \in (H^1_{\Gamma_D}(\Omega))^n; \ u_1 \in (L^2(\Omega))^n; \ \theta_0 \in L^2(\Omega); \ q_0 \in (L^2(\Omega))^n \quad (5)$$

and boundary conditions

$$\theta|_{\partial\Omega} = 0 \quad (6)$$

$$u|_{\Gamma_D} = 0; \qquad \sigma_T|_{\Gamma_N} = 0;$$
$$\sigma_\nu \le 0; \quad u_\nu \le 0; \quad \sigma_\nu(u_\nu - g) = 0; \qquad \sigma_T = 0 \text{ on } \Gamma_C \quad (7)$$

Where

$$\sigma_{ij} = C_{ijkl}u_{k,l} + \mu u_{i,j} - m_{ij}\theta$$

is the stress tensor and (with ν being the exterior normal vector)

$$\sigma_\nu = \sigma_{ij}\nu_i\nu_j; \quad \sigma_T = \sigma\nu - \sigma_\nu\nu$$

The author would like to thank Prof. Dr. Racke for the opportunity to work on this interesting topic and helpful suggestions.

its normal and tangential components. We assume the elasticity module $C = (C_{ijkl})_{i,j,k,l}$, the thermal expansion tensor m and the heat conduction tensor K to satisfy

$$C_{ijkl} \in L^\infty(\Omega); \quad \exists d_C > 0 \forall \eta \in \mathbb{R}^{n\times n} : \eta_{ij} C_{ijkl} \eta_{kl} \geq d_C |\eta|^2; \quad C_{ijkl} = C_{jikl} = C_{klij}$$

$$k_{ij} \in L^\infty(\Omega); \quad \exists d_k > 0 \forall \xi \in \mathbb{R}^n : \xi_i k_{ij} \xi_j \geq d_k |\xi|^2; \quad k_{ij} = k_{ji}$$

$$m_{ij} \in L^\infty(\Omega); \quad m_{ij} \geq 0; \quad m_{ij} = m_{ji}$$

where $k = K^{-1}$ in the sense of matrix inversion and $\mu \geq 0$ is (for now) an arbitrary constant.

A few remarks on notation: We denote $\partial_j u = u_{,j}$, $||\cdot|| := ||\cdot||_{(L^2)^m}$, where m is either 1, n or n^2, which will be clear from the context. In addition, $L^\infty(H^1) := L^\infty([0,T], H^1(\Omega))$ and likewise. $H^1_{\Gamma_C}(\Omega)$ denotes the space of weakly differentiable functions satisfying $u|_{\Gamma_C} = 0$ in a weak sense. The technical problem in the handling of these equations lies in the boundary conditions for u on Γ_C, which do not allow the well-known semi-group theoretic approach. Problems of this form arise naturally in the manufacturing of casts and pistons, cf. [8].

On the classical problem, i.e $\tau_0 = \mu = 0$, there are a number of papers available. In particular, Munõz-Rivera and Racke [6] studied the corresponding classical problem and derived existence and stability under the condition of radial symmetry. In the case of one space dimension, there are several results: Elliot and Tang [2] gave an existence result for more general boundary conditions; Muñoz Rivera and Jiang [5] gave an existence and stability result for a contact problem of two rods, and Gao and Muñoz Rivera [3] gave an existence and stability result for the semilinear case. Dropping the $\partial_t^2 u$ term in the first equation, one arrives at the quasi-statical case, where Shi and Shillor [8] proved the existence of a solution and Ames and Payne [1] gave a uniqueness result. Munõz-Rivera and Racke [6] also prove the existence of a unique solution to the corresponding classical quasi-static problem and its exponential stability. One would - and, in fact, has for quite some time - expect these results to carry over to the fully hyperbolic problem, especially as the critical equation for the displacement u where the difficult boundary conditions arise remains unchanged. However, in a recent work, Racke and Fernández Sare [7] showed that for a damped Timoshenko system, exponential stability is lost when substituting the Fourier Law of heat conduction by Cattaneo's. In this light, the investigation of the behaviour of this particular system under a transition from classical to hyperbolic heat conduction poses an interesting question. We shall indeed see that this transition leads to a loss in regularity that is not easily compensated, thus requiring the additional viscoelastic term ($\mu > 0$).

This paper is organized as follows: In Section 2, we will give a proof for the existence of a weak solution. We will start following the approach of Muñoz Rivera and Racke [6] and then show why it can not be extended to this problem. To this end we will approximize the difficult boundary conditions on Γ_C and obtain a penalized problem. We will then show that this penalized problem has a solution and give a sufficient condition for the convergence of this solution to a solution of our original problem - this is where the loss of regularity from the changed heat equation leaves its mark, as the conditions derived by Muñoz Rivera and Racke will no longer be sufficient. Finally, in section 3, we will prove a stability result in the radially symmetrical case, that is, the solutions to our problem decay to 0 exponentially. We will use a Lyapunov functional, similar to [6], although some changes are required to compensate for the different heat equation.

2. Existence

We will prove the existence of a solution in the following sense:

Definition 2.1. *(u, θ, q) is a solution to (1)-(7) iff*

$$u \in W^{1,\infty}((L^2)^n) \cap L^\infty((H^1_{\Gamma_D})^n), \quad \theta \in L^\infty(L^2), \quad q \in L^\infty((L^2)^n) \tag{8}$$

$$\partial_t u(T,\cdot),\ q(T,\cdot) \in (L^2(\Omega))^n; \quad \theta(T,\cdot) \in L^2(\Omega); \quad \nabla u(T,\cdot) \in L^2(\Omega) \tag{9}$$

$$\begin{aligned}
&\forall w \in W^{1,\infty}((L^2)^n) \cap L^\infty((H^1_{\Gamma_D})^n), w_\nu \le g \text{ on } \Gamma_C :\\
&-\int_0^T \langle \partial_t u, \partial_t w\rangle\, dt + \langle u(T,\cdot), \partial_t w(T,\cdot) - \partial_t u(T,\cdot)\rangle - \langle u_0, \partial_t w(0,\cdot) - u_1\rangle\\
&+\mu \int_0^T \langle \partial_t u_{i,j}, w_{i,j}\rangle\, dt + \int_0^T \langle C_{ijkl} u_{k,l}, w_{i,j}\rangle\, dt - \int_0^T \langle m_{ij}\theta, w_{i,j}\rangle\, dt\\
&+\int_0^T \langle \partial_t u, \partial_t u\rangle\, dt - \int_0^T \langle C_{ijkl} u_{k,l}, u_{i,j}\rangle\, dt - \frac{\mu}{2}(||\nabla u(T,\cdot)||^2 - ||\nabla u_0||^2)\\
&+\int_0^T \langle m_{ij}\theta, u_{ij}\rangle\, dt \ge 0
\end{aligned} \tag{10}$$

$$\begin{aligned}
&\forall z \in W^{1,\infty}(H^1_0):\\
&-\int_0^T \langle \theta, z\rangle\, dt + \langle \theta(T,\cdot), z(T,\cdot)\rangle - \langle \theta_0, z(0,\cdot)\rangle - \int_0^T \langle q_i, z_{,i}\rangle\, dt\\
&-\int_0^T \langle m_{ij} u_{i,j}, \partial_t z\rangle\, dt + \langle m_{ij} u_{i,j}(T,\cdot), z(T,\cdot)\rangle - \langle m_{ij} u_{0i,j}, z(0,\cdot)\rangle = 0
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\forall y \in W^{1,\infty}((H^1)^n):\\
&-\tau_0 \int_0^T \langle k_{ij} q_i, \partial_t y_j\rangle\, dt + \tau_0 \langle k_{ij} q_i(T,\cdot), y_j(T,\cdot)\rangle - \tau_0 \langle k_{ij} q_{0i}, y_j(0,\cdot)\rangle\\
&+\int_0^T \langle k_{ij} q_i, y_j\rangle\, dt - \int_0^T \langle \theta, y_{i,i}\rangle\, dt = 0
\end{aligned} \tag{12}$$

$$u|_{\Gamma_C} \le g \ a.e. \tag{13}$$

We remark that all boundary conditions are represented in a weak sense in the above definiton. Also, we need the unusual condition (9) for condition (10) to make sense. This will be seen from the context in section 2.
To better handle the difficult boundary conditions in u, we consider the following penalized problem:

$$\partial_t^2 u_i^\epsilon - (C_{ijkl} u^\epsilon_{k,l})_{,j} - \mu u^\epsilon_{i,jj} + m_{ij}\theta^\epsilon_{,j} = 0 \tag{14}$$

$$\partial_t \theta^\epsilon + \text{div } q^\epsilon + m_{ij}\partial_t u^\epsilon_{i,j} = 0 \tag{15}$$

$$\tau_0 \partial_t k_{ij} q^\epsilon_j + k_{ij} q^\epsilon_j + \theta^\epsilon_{,i} = 0 \tag{16}$$

with initial conditions

$$u(0,\cdot) = u_0; \quad u_t(0,\cdot) = u_1; \quad \theta(0,\cdot) = \theta_0; \quad q(0,\cdot) = q_0 \tag{17}$$

and boundary conditions

$$\begin{aligned} \theta|_{\partial\Omega} &= 0 \\ u|_{\Gamma_D} = 0; &\qquad \sigma_T|_{\Gamma_N} = 0 \\ \sigma_\nu^\epsilon = -\frac{1}{\epsilon}(u_\nu^\epsilon - g)^+ - \epsilon\partial_t u_\nu &\qquad \sigma_T = 0 \text{ on } \Gamma_C \end{aligned} \tag{18}$$

Note that only the boundary conditions on Γ_C have been changed, everything else is identical to the original problem. We will see that σ_ν^ϵ is bounded and therefore by (18) $(u_\nu^\epsilon - g)^+ \to 0$ as $\epsilon \to 0$, satisfying (13). Next, we give a definition of a solution to the penalized problem. Let $w^p{}_p, y^p{}_p \subset H^1(\Omega)$ be bases of $(L^2(\Omega))^n$ and $z^p{}_p \subset H_0^1(\Omega)$ be a basis of $L^2(\Omega)$.

Definition 2.2. *Let*

$$\begin{aligned} u_0^\varepsilon, u_1^\varepsilon &\in (H^{2,2}(\Omega) \cap H_0^1(\Omega))^n \\ q_0^\varepsilon &\in (H^1(\Omega))^n \\ \theta_0^\epsilon &\in H_0^1(\Omega) \end{aligned}$$

Then $(u^\epsilon, \theta^\epsilon, q^\epsilon)$ is a solution to (14)-(18) iff

$$\begin{aligned} &u^\varepsilon \in W^{2,\infty}((L^2)^n) \cap W^{1,\infty}((H^1_{\Gamma_D})^n); \quad \theta^\varepsilon \in W^{1,\infty}(L^2) \cap L^\infty(H_0^1); \\ &q^\varepsilon \in W^{1,\infty}((L^2)^n) \cap L^\infty(D^1) \end{aligned} \tag{19}$$

$$u^\varepsilon(0,\cdot) = u_0^\varepsilon; \quad \partial_t u^\varepsilon(0,\cdot) = u_1^\varepsilon; \quad \theta^\varepsilon(0,\cdot) = \theta_0^\varepsilon; \quad q^\varepsilon(0,\cdot) = q_0^\varepsilon \tag{20}$$

and for allmost all $t \in [0,T]$

$$\begin{aligned} \forall p \in \mathbb{N} : &\langle \partial_t^2 u^\varepsilon(t,\cdot), w^p \rangle + \mu \langle \partial_t u_{i,j}^\varepsilon(t,\cdot), w_{i,j}^p \rangle + \langle C_{ijkl} u_{k,l}^\varepsilon(t,\cdot), w_{i,j}^p \rangle \\ &- \langle m_{ij}\theta^\varepsilon(t,\cdot), w_{i,j}^p \rangle = -\frac{1}{\varepsilon}\int_{\Gamma_C} (u_\nu^\varepsilon(t,\cdot) - g)^+ w^p \mathrm{d}\Gamma - \varepsilon \int_{\Gamma_C} \partial_t u_\nu^\varepsilon(t,\cdot) w^p \mathrm{d}\Gamma \end{aligned} \tag{21}$$

$$\forall p \in \mathbb{N} : \ \langle \partial_t \theta^\varepsilon(t,\cdot), z^p \rangle + \langle \text{div } q^\varepsilon(t,\cdot), z^p \rangle + \langle m_{ij}\partial_t u_{i,j}^\varepsilon(t,\cdot), z^p \rangle = 0 \tag{22}$$

$$\forall p \in \mathbb{N} : \ \tau_0 \langle k_{ij}\partial_t q_i^\varepsilon(t,\cdot), y_j^p \rangle + \langle k_{ij} q_i^\varepsilon(t,\cdot), y_j^p \rangle + \langle \nabla\theta^\varepsilon(t,\cdot), y_i^p \rangle = 0 \tag{23}$$

To construct a solution to the penalized problem, we will use a Faedo-Galerkin-method. Note that, if (v, ψ, h) satisfy

$$v(0,\cdot) = \partial_t v(0,\cdot) = \psi(0,\cdot) = h(0,\cdot) = 0$$

$$\begin{aligned} &\langle \partial_t^2 v, w^p \rangle + \langle (C_{ijkl} v_{k,l}), w_{i,j}^p \rangle + \mu \langle \partial_t v_{i,j}, w_{i,j}^p \rangle - \langle m_{ij}\psi, w_{i,j}^p \rangle \\ &= \langle f, w^p \rangle - \frac{1}{\varepsilon}\int_{\Gamma_C} (v_\nu - g)^+ w_\nu^p \mathrm{d}\Gamma - \varepsilon \int_{\Gamma_C} (\partial_t v_\nu) w_\nu^p \mathrm{d}\Gamma \end{aligned} \tag{24}$$

$$\langle \partial_t \psi, z^p \rangle + \langle \text{div } h, z^p \rangle + \langle m_{ij}\partial_t v_{i,j}, z^p \rangle = \langle b, z^p \rangle \tag{25}$$

$$\langle \tau_0 k_{ij} \partial_t h_j, y_i \rangle + \langle k_{ij} h_j, y_i \rangle + \langle \nabla\psi, y \rangle = \langle e, y^p \rangle \tag{26}$$

with

$$\begin{aligned} f_i &:= C_{ijkl}(u_{0k,l}^\varepsilon - t u_{1k,l}^\varepsilon))_{,j} + \mu u_{1i,jj}^\varepsilon - (m_{ij}\theta_0^\varepsilon)_{,j}; \quad (i = 1, \ldots, n) \\ b &:= -q_{0i,i}^\varepsilon + m_{ij} u_{1i,j}^\varepsilon \\ e &:= -k_{ij} q_{0j}^\varepsilon - \theta_{0,j}^\varepsilon \end{aligned}$$

then $u := v + u_0 + tu_1$, $\theta := \psi + \theta_0$ and $q := h + q_0$ are a solution to the penalized problem. To find such (v, ψ, h), consider the following set of equations on $[0, T]$

$$\begin{aligned}&\langle \partial_t^2 v^m, w^p \rangle_n + \langle C_{ijkl} v^m_{k,l}, w^p_{i,j} \rangle + \mu \langle \partial_t v^m_{i,j}, w^p_{i,j} \rangle - \langle m_{ij} \psi, w^p_{i,j} \rangle \\ =&\langle f, w^p \rangle_n - \frac{1}{\varepsilon} \int\limits_{\Gamma_C} (v^m_\nu - g)^+ w^p_\nu \mathrm{d}\Gamma - \varepsilon \int\limits_{\Gamma_C} \partial_t v^m_\nu w^p_\nu \mathrm{d}\Gamma \quad (p = 1, ..., m)\end{aligned} \tag{27}$$

$$\langle \partial_t \psi^m, z^p \rangle + \langle \mathrm{div}\ h^m, z^p \rangle + \langle m_{ij} \partial_t v^m_{i,j}, z^p \rangle = \langle b, z^p \rangle \ (p = 1, ..., m) \tag{28}$$

$$\tau_0 \langle k_{ij} \partial_t h^m_i, y^p_j \rangle + \langle k_{ij} h^m_i, y^p_j \rangle + \langle \nabla \psi^m, y^p \rangle_n = \langle e, y^p \rangle_n \ (p = 1, ..., m) \tag{29}$$

$$v(0, \cdot) = \partial_t v(0, \cdot) = \psi(0, \cdot) = h(0, \cdot) = 0 \tag{30}$$

where $v^m(t, x) = a^m_p(t) w^p(x)$, $\psi^m(t, x) = b^m_p(t) z^p(x)$ and $h^m(t, x) = c^m_p(t) y^p(x)$ with unknown coefficents (a^m_p, b^m_p, c^m_p). Then (27)-(30) is a set of ordinary differential equations for (a^m_p, b^m_p, c^m_p), thus possessing a solution with the regularity

$$v^m \in W^{3,\infty}((H^1_{\Gamma_D})^n),\ \psi^m \in W^{2,\infty}(H^1_0),\ h^m \in W^{2,\infty}((H^1)^n)$$

Note that the initial conditions are arbitrarily smooth and f, g, e are polynomial in t, allowing for a solution with the required smoothness.

Proposition 2.1. *There exist (v, ψ, h) such that*

$$\begin{aligned}(v^m)_m &\overset{*}{\rightharpoonup} v \ in\ W^{2,\infty}((L^2)^n) \cap W^{1,\infty}((H^1_{\Gamma_C})^n) \\ (\psi^m)_m &\overset{*}{\rightharpoonup} \psi \ in\ W^{1,\infty}(L^2) \\ (h^m)_m &\overset{*}{\rightharpoonup} h \ in\ W^{1,\infty}((L^2)^n)\end{aligned}$$

Proof: Multiplying (27) by $\frac{\mathrm{d}}{\mathrm{d}t} a^m_p$, (28) by b^m_p and (29) by c^m_p respectively, we obtain after summarizing from 1 to m:

$$\begin{aligned}&\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left(||\partial_t v^m||^2 + \langle C_{ijkl} v^m_{k,l}, v^m_{i,j} \rangle + \frac{1}{\varepsilon} \int\limits_{\Gamma_C} |(u^m_\nu - g)^+|^2 \mathrm{d}\Gamma + ||\psi^m||^2 + \tau_0 ||h^m||^2 \right) \\ &+ \mu ||\partial_t \nabla v^m||^2 + \varepsilon \int\limits_{\Gamma_C} |\partial_t v^m_\nu|^2 \mathrm{d}\Gamma + \langle k_{ij} h^m_i, h^m_j \rangle + \langle \mathrm{div}\ h^m, \psi^m \rangle + \langle \nabla \psi^m, h^m \rangle \\ =& \langle f, \partial_t v^m \rangle + \langle b, \psi^m \rangle + \langle e, h^m \rangle\end{aligned} \tag{31}$$

where we used that

$$\int\limits_{\Gamma_C} (v^m_\nu - g)^+ \partial_t v^m \mathrm{d}\Gamma = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma_C} |(v^m_\nu - g)^+|^2 \mathrm{d}\Gamma$$

As one easily checks by partial integration,

$$\langle \mathrm{div}\ h^m, \psi^m \rangle + \langle \nabla \psi^m, h^m \rangle = 0$$

and therefore, after integrating (31) on $(0,t)$, we obtain by Gronwall's inequality

$$\begin{aligned}
||\partial_t v^m(t,\cdot)||_n &\leq C \\
\langle C_{ijkl} v^m_{k,l}(t,\cdot), v^m_{i,j}(t,\cdot)\rangle &\leq C \\
\frac{1}{\varepsilon}\int\limits_{\Gamma_C} |(v^m_\nu(t,\cdot)-g)^+|^2 \mathrm{d}\Gamma &\leq C \\
||h^m(t,\cdot)||_n &\leq C \\
\varepsilon \int\limits_0^t \int\limits_{\Gamma_C} |\partial_t v^m_\nu(t,\cdot)|^2 \mathrm{d}\Gamma \mathrm{d}t &\leq C \qquad (32)\\
\int\limits_0^t \langle k_{ij} h^m_i(t,\cdot), h^m_j(t,\cdot)\rangle \mathrm{d}t &\leq C \qquad (33)
\end{aligned}$$

Using the smoothness of the functions (v^m, ψ^m, h^m), we see that they satisfy the time-derivated system

$$\begin{aligned}
&\langle \partial_t^3 v^m, w^p\rangle_n + \langle C_{ijkl}\partial_t v^m_{k,l}, w^p_{i,j}\rangle + \mu\langle \partial_t^2 v^m_{i,j}, w^p_{i,j}\rangle - \langle m_{ij}\partial_t\psi, w^p_{i,j}\rangle \\
= &-\frac{1}{\varepsilon}\int\limits_{\Gamma_C} \partial_t(v^m_\nu - g)^+ w^p_\nu \mathrm{d}\Gamma - \varepsilon\int\limits_{\Gamma_C} \partial_t^2 v^m_\nu w^p_\nu \mathrm{d}\Gamma + \langle \partial_t f, w^p\rangle_n
\end{aligned} \qquad (34)$$

$$\langle \partial_t^2 \psi^m, z^p\rangle + \langle \partial_t \mathrm{div}\ h^m, z^p\rangle + \langle m_{ij}\partial_t^2 v^m_{i,j}, z^p\rangle = 0 \qquad (35)$$

$$\tau_0\langle k_{ij}\partial_t^2 h^m_i, y^p_j\rangle + \langle k_{ij}\partial_t h^m_i, y^p_j\rangle + \langle \partial_t \nabla\psi^m, y^p\rangle_n = 0 \qquad (36)$$

Multiplying (34) by $\frac{\mathrm{d}^2}{\mathrm{d}t^2}a^m_p$, (35) by $\frac{\mathrm{d}}{\mathrm{d}t}b^m_p$ and (36) by $\frac{\mathrm{d}}{\mathrm{d}t}c^m_p$ respectively, we obtain similar to (31)

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[||\partial_t^2 v^m||_n^2 + \langle C_{ijkl}\partial_t v^m_{k,l}, \partial_t v^m_{i,j}\rangle + ||\partial_t\psi^m||^2 + \tau_0||\partial_t h^m||_n^2\right] \\
&+ \mu||\partial_t^2 \nabla v^m||^2_{n\times n} + \langle k_{ij} h^m_i, h^m_j\rangle \\
= &\ \langle \partial_t f, \partial_t^2 v^m\rangle - \frac{1}{\varepsilon}\int\limits_{\Gamma_C} \partial_t(v^m_\nu - g)^+ \partial_t^2 v^m \mathrm{d}\Gamma - \varepsilon\int\limits_{\Gamma_C} \partial_t^2 v^m_\nu \partial_t^2 v^m_\nu \mathrm{d}\Gamma
\end{aligned}$$

Observe that in general it is not

$$\partial_t\ |\partial_t(v^m_\nu - g)^+|^2 = 2\partial_t(v^m_\nu - g)^+\partial_t^2 v^m \text{ a.e.}$$

since the distributional second derivative of $(v^m_\nu - g)^+$ need not be regular. However, using (32),

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[||\partial_t^2 v^m||_n^2 + \langle C_{ijkl}\partial_t v^m_{k,l}, \partial_t v^m_{i,j}\rangle + ||\partial_t\psi^m||^2 + \tau_0||\partial_t h^m||_n^2\right] \\
&+ \mu||\partial_t^2 \nabla v^m||^2_{n\times n} + \langle k_{ij} h^m_i, h^m_j\rangle \\
&\leq \langle \partial_t f, \partial_t^2 v^m\rangle_n + \frac{1}{2\varepsilon^3}\int\limits_{\Gamma_C} |\partial_t v^m_\nu|^2 \mathrm{d}\Gamma - \frac{\varepsilon}{2}\int\limits_{\Gamma_C} |\partial_t^2 v^m_\nu|^2 \mathrm{d}\Gamma \\
&\leq \langle \partial_t f, \partial_t^2 v^m\rangle_n + C_\varepsilon
\end{aligned} \qquad (37)$$

where $C^\epsilon \to \infty$ as $\epsilon \to 0$. For constant ϵ we conclude, using Gronwall's inequality again, that

$$\begin{array}{lll} (v^m)_m & \text{is bounded in} & W^{2,\infty}((L^2)^n) \cap W^{1,\infty}((H^1_{\Gamma_D})^n) \\ (\psi^m)_m & \text{is bounded in} & W^{1,\infty}(L^2) \\ (h^m)_m & \text{is bounded in} & W^{1,\infty}((L^2)^n) \end{array}$$

from which the claimed convergence follows. □

We can now show

Theorem 2.1. *There is a solution to the penalized problem.*

Proof: Take (v, ψ, h) as in Proposition 2.1. Define

$$u := v + u_0 + tu_1$$
$$\theta := \psi + \theta_0$$
$$q := h + q_0$$

Then it is clear that (u, θ, q) have the desired regularity (19) and fulfill the initial conditions (20). Using Lemma 1.4 from [4], we obtain the convergence

$$u^\varepsilon \to u \text{ in } C^1([0, T], (L^2(\Gamma_C))^n)$$

It then follows from the convergence proved in Theorem 2.1 that (u, θ, q) satisfy (21)-(23). □

Now we will prove the convergence of solutions to the penalized problem. As we can see in the proof of Proposition (2.1), we can not use the second energy level to gain estimates on the convergence of $(u^\epsilon, \theta^\epsilon, q^\epsilon)$, as ϵ is now no longer constant. Therefore, we lose one level of regularity in time. This loss is grave, since we will no longer have convergence of some terms in the equations, i.e. it is generally unknown if the limits (u, ψ, q) are solutions to the original problem. However, if $\mu > 0$, the viscoelastic term will provide us with the missing regularity and an existence proof is possible. This will be shown in detail in the proof of Theorem 2.2.

Proposition 2.2. *There exist* (u, θ, q) *such that*

$$\begin{array}{lll} u^\varepsilon \overset{*}{\rightharpoonup} u & in & W^{1,\infty}((L^2)^n) \cap L^\infty(H^1_{\Gamma_C}) \\ \theta^\varepsilon \overset{*}{\rightharpoonup} \theta & in & L^\infty(L^2) \\ q^\varepsilon \overset{*}{\rightharpoonup} q & in & L^\infty((L^2)^() \end{array}$$

If $\mu > 0$*, then*

$$u^\varepsilon \rightharpoonup u \text{ in } W^{1,2}((H^1_{\Gamma_C}))$$

Proof: By the regularity of $(u^\varepsilon, \theta^\epsilon, q^\epsilon)$, we can subsitute them for (w^p, z^p, y^p) in (21), (22) and (23) respectively and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(||\partial_t u^\varepsilon||^2 + \langle C_{ijkl} u^\varepsilon_{k,l}, u^\varepsilon_{i,j}\rangle + ||\theta^\varepsilon||^2 + \langle k_{ij} q^\varepsilon_i, q^\varepsilon_j\rangle + \frac{1}{\varepsilon}\int_{\Gamma_C} |(u^\varepsilon_\nu - g)^+|^2 \mathrm{d}\Gamma \right)$$
$$+ \mu||\partial_t \nabla u^\varepsilon||^2 + \langle k_{ij} q^\varepsilon_i, q^\varepsilon_j\rangle + \varepsilon \int_{\Gamma_C} |\partial_t u^\varepsilon_\nu|^2 \mathrm{d}\Gamma = 0 \tag{38}$$

where we again used that

$$< \text{div } q, \theta > + < \nabla\theta, q >= 0$$

Integrating from 0 to t and using Gronwall's inequality, we conclude the existence of a constant $C = C(||u_0^\varepsilon||, ||\theta_0^\varepsilon||, ||q_0^\varepsilon||)$ such that for all $t > 0$

$$
\begin{aligned}
||\partial_t u^\varepsilon(t,\cdot)||_n &\leq C \\
\langle C_{ijkl} u^\varepsilon_{k,l}(t,\cdot), u^\varepsilon_{i,j}(t,\cdot)\rangle &\leq C \\
||\theta^\varepsilon(t,\cdot)|| &\leq C \\
\langle k_{ij} q_i^\varepsilon(t,\cdot), q_j^\varepsilon(t,\cdot)\rangle &\leq C \\
\frac{1}{\varepsilon}\int_{\Gamma_C} |(u_\nu^\varepsilon(t,\cdot) - g(\cdot))^+|^2 \mathrm{d}\Gamma &\leq C \\
\mu \int_0^t ||\partial_t \nabla u^\varepsilon(s,\cdot)||_n^2 \mathrm{d}s &\leq C \\
\int_0^t \langle k_{ij} q_i^\varepsilon(s,\cdot), q_j^\varepsilon(s,\cdot)\rangle \mathrm{d}s &\leq C \\
\varepsilon \int_0^t \int_{\Gamma_C} |\partial_t u_\nu^\varepsilon(s,\cdot)|^2 \mathrm{d}\Gamma \mathrm{d}s &\leq C
\end{aligned}
$$

This implies the desired convergence. □

Proposition 2.3. *Let (u, θ, q) be the functions from Proposition 2.2. Then*

$$
\begin{aligned}
u^\varepsilon(T,\cdot) &\overset{*}{\rightharpoonup} u(T,\cdot) \quad in \quad L^\infty(H^1_{\Gamma_C}) \\
u_t^\varepsilon(T,\cdot) &\overset{*}{\rightharpoonup} u_t(T,\cdot) \quad in \quad L^\infty(L^2) \\
\theta^\varepsilon(T,\cdot) &\overset{*}{\rightharpoonup} \theta(T,\cdot) \quad in \quad L^\infty(L^2) \\
q^\varepsilon(T,\cdot) &\overset{*}{\rightharpoonup} q(T,\cdot) \quad in \quad L^\infty(L^2)
\end{aligned}
$$

Proof: Note that due to the regularity of the solutions to the penalized problem, $u^\epsilon, u_t^\epsilon, \theta^\epsilon$ and q^ϵ are continuous in time by Sobolev's Imbedding Theorem. Therefore, the asserted convergence holds by the estimates gained in the proof for Theorem 2.2. □

Theorem 2.2. *Let $\mu > 0$. Let $(u_0, u_1, \theta_0, q_0) \in (H^1_{\Gamma_D}(\Omega))^n \times (L^2(\Omega))^{2n+1}$. Then there exists a solution to (1)-(7).*

Proof: Let

$$
\begin{aligned}
(u_0^\varepsilon)_\varepsilon,\ (u_1^\varepsilon)_\varepsilon &\subset (H_0^1(\Omega) \cap H^{2,2}(\Omega))^n \\
(q_0^\varepsilon)_\varepsilon &\subset (H^1(\Omega))^n \\
(\theta_0^\varepsilon)_\varepsilon &\subset H^1(\Omega)
\end{aligned}
$$

with

$$u_0^\varepsilon \longrightarrow u_0 \quad \text{in} \quad (H^1_{\Gamma_D})^n \tag{39}$$

$$u_1^\varepsilon \longrightarrow u_1 \quad \text{in} \quad (L^2(\Omega))^n \tag{40}$$

$$\theta_0^\varepsilon \longrightarrow \theta_0 \quad \text{in} \quad L^2(\Omega) \tag{41}$$

$$q_0^\varepsilon \longrightarrow q_0 \quad \text{in} \quad (L^2(\Omega))^n \tag{42}$$

Let $(u^\epsilon, \theta^\epsilon, q^\epsilon)$ be the solutions to the penalized problem for each $\epsilon > 0$ and (u, θ, q) be the limits from Proposition 2.2. Then (u, θ, q) will satisfy (8).

We can substitute z^p in (22) for any $z \in W^{1,\infty}(H^1_0)$ and obtain

$$\langle \partial_t \theta^\varepsilon, z\rangle + \langle \text{div } q^\varepsilon, z\rangle + \langle m_{ij}\partial_t u^\varepsilon_{i,j}, z\rangle = 0$$

Integrating from 0 to T we arrive at

$$\begin{aligned} &\langle \theta(T,\cdot)^\varepsilon, z(T,\cdot)\rangle - \langle \theta^\varepsilon_0, z(0,\cdot\rangle - \int_0^T \langle \theta^\varepsilon, \partial_t z\rangle \mathrm{d}t - \int_0^T \langle q^\varepsilon_i, z_{,i}\rangle \mathrm{d}t \\ &+ \langle m_{ij} u^\varepsilon_{i,j}(T,\cdot), z(T,\cdot)\rangle - \langle m_{ij} u^\varepsilon_{0i,j}, z(0,\cdot)\rangle - \int_0^T \langle m_{ij} u^\varepsilon_{i,j}, z\rangle \mathrm{d}t = 0 \end{aligned} \tag{43}$$

Using Propositions 2.2 and 2.3, we conclude by taking the limit $\epsilon \to 0$ that (u, θ, q) fulfill (11).
Similarly, substituting y^p in (23) for any $y \in W^{1,\infty}(H^1)$ and integrating yields

$$\begin{aligned} &\langle k_{ij} q^\varepsilon_i(T,\cdot), y(T,\cdot)\rangle - \langle k_{ij} q_0 i^\varepsilon, y(0,\cdot)\rangle - \int_0^T \langle k_{ij} q^\varepsilon_i(t,\cdot), \partial_t y_j(t,\cdot)\rangle \mathrm{d}t \\ &+ \int_0^T \langle k_{ij} q^\varepsilon_i(t,\cdot), y_j(t,\cdot)\rangle + \int_0^T \langle \theta^\varepsilon(t,\cdot), y_{i,i}(t,\cdot)\rangle \mathrm{d}t = 0 \end{aligned} \tag{44}$$

Again, taking the limit $\epsilon \to 0$ and using Propositions 2.2 and 2.3 we conclude that (u, θ, q) fulfill (12).
From Proposition 2.3, it is immediately clear that (u, θ, q) satisfy (9). Using Lemma 1.4 from [4] again, it follows from Proposition 2.2 that

$$u^\varepsilon \longrightarrow u \;\text{ in } C^0([0,T], (L^2(\Gamma_C))^n)$$

therefore, since

$$\frac{1}{\varepsilon}\int_{\Gamma_C} |(u^\varepsilon_\nu(t,\cdot) - g(\cdot))^+|^2 \mathrm{d}\Gamma \;\le C,$$

we conclude that

$$\int_{\Gamma_C} |(u_\nu(t,\cdot) - g(\cdot))^+|^2 \mathrm{d}\Gamma = 0$$

and therefore (13) is satisfied.
Note that we did not use $\mu > 0$ yet, therefore everything we proved so far will also hold if $\mu = 0$. The critical part is in fact the convergence of quadratic terms that appear in (10), as we will see in the following calculations.
For any $w \in L^\infty(H^1_{\Gamma_D}) \cap W^{1,\infty}(L^2)$ we substitute w_p in (23) by $w - u$ and obtain

$$\begin{aligned} &\langle \partial^2_t u^\varepsilon, w - u^\varepsilon\rangle + \langle C_{ijkl} u^\varepsilon_{k,l}, w_{i,j} - u^\varepsilon_{i,j}\rangle \\ &+ \mu\langle \partial_t u^\varepsilon_{i,j}, w_{i,j} - u^\varepsilon_{i,j}\rangle + \langle m_{ij}\theta^\varepsilon, w_{i,j} - u^\varepsilon_{i,j}\rangle \\ = \; & -\frac{1}{\varepsilon}\int_{\Gamma_C} (u^\varepsilon_\nu - g)^+(w_\nu - u^\varepsilon_\nu)\mathrm{d}\Gamma - \varepsilon\int_{\Gamma_C} \partial_t u^\varepsilon_\nu(w_\nu - u^\varepsilon_\nu)\mathrm{d}\Gamma \end{aligned}$$

Integrating from 0 to T we arrive at

$$
\begin{aligned}
&\langle\partial_t u^\varepsilon(T,\cdot), w(T,\cdot)\rangle - \langle\partial_t u^\varepsilon(T,\cdot), u^\varepsilon(T,\cdot)\rangle - \langle u_1^\varepsilon, w(0,\cdot) - u_0^\varepsilon\rangle \\
&- \int_0^T \langle\partial_t u^\varepsilon(t,\cdot), \partial_t w(t,\cdot)\rangle \mathrm{d}t + \int_0^T \langle C_{ijkl} u_{k,l}^\varepsilon(t,\cdot), w_{i,j}(t,\cdot)\rangle \mathrm{d}t \\
&+ \int_0^T \langle\partial_t u^\varepsilon(t,\cdot), \partial_t u^\varepsilon(t,\cdot)\rangle \mathrm{d}t - \int_0^T \langle C_{ijkl} u_{k,l}^\varepsilon(t,\cdot), u_{i,j}^\varepsilon(t,\cdot)\rangle \mathrm{d}t \\
&+ \mu \int_0^T \langle\partial_t u_{i,j}^\varepsilon(t,\cdot), w_{i,j}(t,\cdot)\rangle \mathrm{d}t - \frac{\mu}{2}\left(||\nabla u^\varepsilon(T,\cdot)||^2 - ||\nabla u_0^\varepsilon||^2\right) \\
&+ \int_0^T \langle m_{ij}\theta^\varepsilon(t,\cdot), w_{i,j}(t,\cdot)\rangle \mathrm{d}t - \int_0^T \langle m_{ij}\theta^\varepsilon(t,\cdot), u_{i,j}^\varepsilon(t,\cdot)\rangle \mathrm{d}t \\
&= \frac{1}{\varepsilon} \int_0^T \int_{\Gamma_C} (u_\nu^\varepsilon(t,\cdot) - g)^+ (u_\nu^\varepsilon(t,\cdot) - g) - (u_\nu^\varepsilon(t,\cdot) - g)^+ (w_\nu(t,\cdot) - g) \mathrm{d}\Gamma \mathrm{d}t \\
&- \varepsilon \int_0^T \int_{\Gamma_C} \partial_t u_\nu^\varepsilon(t,\cdot) w_\nu(t,\cdot) \mathrm{d}\Gamma \mathrm{d}t + \frac{\varepsilon}{2} \int_{\Gamma_C} |u_\nu^\varepsilon(T,\cdot)|^2 - |u_{0\nu}^\varepsilon|^2 \mathrm{d}\Gamma \\
&\geq - \varepsilon \left(\int_0^T \int_{\Gamma_C} \partial_t u_\nu^\varepsilon(t,\cdot) w_\nu(t,\cdot) \mathrm{d}\Gamma \mathrm{d}t + \frac{1}{2} \int_{\Gamma_C} |u_\nu^\varepsilon(T,\cdot)|^2 - |u_{0\nu}^\varepsilon|^2 \mathrm{d}\Gamma \right)
\end{aligned}
\tag{45}
$$

Using Propositions 2.2 and 2.3, we see that the right hand side of (45) will converge to 0 as $\epsilon \to 0$, since weak-* convergent series are bounded in norm. For the left hand side we can again conlude the convergence of all terms that are linear in $(u^\epsilon, \theta^\epsilon, q^\epsilon)$. However, the convergence of the quadratic terms, namely the $L^2(0,T)$-Norms of $||u_t^\epsilon||, \langle C_{ijkl} u_{i,j}, u_{k,l}\rangle$ and $\langle m_{ij}\theta, u_{i,j}\rangle$ remains an issue. While we know the terms will be bounded, we can not conclude their convergence to the respective terms for u, as weak-* convergence does not imply norm convergence.
Note that it is not possible to circumvent this problem by simply taking estimates for the second order energy and giving a strong solution, since the second order energy is not (trivially) bounded in ϵ. We remark that Munõz Rivera and Racke [6] encountered a similar problem, which could be circumvented by reducing the problem to the radially symmetrical case and using an estimate obtained via compensated compactness. However, it is not possible to utilize this for our problem, since we do not have a bound on $\nabla\theta$, which is a necessary component of the proof in [6].
Therefore, we shall use $\mu > 0$, which will yield

$$
u^\varepsilon \rightharpoonup u \text{ in } W^{1,2}((H^1_{\Gamma_C}))
$$

by Proposition 2.2. From this we can conclude the uniform convergence of u_t^ϵ as well as ∇u. It is then possible to take the limit $\epsilon \to 0$ in (45) and conclude that (u, θ, q) will satisfy (10). $\square$

3. Stability

In general, one can not expect the exponential stability of a thermoelastic problem that is not radially symmetric. Therefore, we shall restrict our problem to the radially symmetric, isotropic and homogenous case, i.e. we assume that the following conditions hold:
The domain Ω is radially symmetric, in this case annular:
$\Omega = B(0,1)\backslash B(0,r_0)$, $1 > r_0 > 0$; $\quad \Gamma_D = \partial B(0,r_0)$; $\quad \Gamma_C = \partial B(0,1)$; $\quad \Gamma_N = \emptyset$
The coefficients satisfy the following symmetry conditions:

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \nu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$$

$$m_{ij} = \overline{m}\delta_{ij}, \quad K_{ij} = \kappa\delta_{ij}, \quad g(x) = \overline{g} \geq 0 \text{ f.a. } x \in \Gamma_C$$

Additionally, we shall assume that the solution to the problem as derived in the previous section is unique in this case, which implies that with radially symmetric initial data and the above assumptions on the coefficients, the solution itself will be radially symmetric. We shall first investigate the stability of the penalized problem, which will transfer to the original problem by a simple continuity argument.
With our assumptions, the equations take the form

$$\partial_t^2 u^\varepsilon - \mu\partial_t\triangle u^\varepsilon - \lambda_1\triangle u^\varepsilon - (\lambda_1+\lambda_2)\nabla\text{div } u^\varepsilon + \overline{m}\nabla\theta^\varepsilon = 0 \tag{46}$$

$$\partial_t\theta^\varepsilon + \text{div } q^\varepsilon + \overline{m}\text{div } \partial_t u^\varepsilon = 0 \tag{47}$$

$$\tau_0\partial_t q^\varepsilon + q^\varepsilon + \kappa\theta^\varepsilon = 0 \tag{48}$$

with Lame-Moduli λ_1, λ_2 satisfying $2\lambda_1 + n\lambda_2 > 0$ and constants $\kappa > 0$ and $m \neq 0$.
The boundary conditions to the penalized problem then read

$$\begin{gathered} \theta^\varepsilon|_{\partial\Omega} = 0, \; u^\varepsilon|_{\Gamma_D} = 0 \\ \mu\partial_t\frac{\partial u^\varepsilon}{\partial\nu}\cdot\nu + \lambda_1\frac{\partial u^\varepsilon}{\partial\nu}\cdot\nu + (\lambda_1+\lambda_2)\text{div } u^\varepsilon = -\frac{1}{\varepsilon}(u_\nu^\varepsilon - \overline{g})^+ - \varepsilon\partial_t u_\nu^\varepsilon \quad \text{on } \Gamma_C \end{gathered} \tag{49}$$

As mentioned above, solutions to this problem will also be radially symmetric, so we can write

$$u^\varepsilon(t,x) = xw(t,|x|), \; \theta^\varepsilon(t,x) = \psi(t,|x|), \; q^\varepsilon(t,x) = xh(t,|x|)$$

Writing $r := |x|$, (w,ψ,h) will then satisfy the equations

$$\partial_t^2 w - \mu\partial_t w_r r - \mu\partial_t\frac{1}{r}w_r - \nu_1 w_{rr} - \frac{\nu_2}{r}w_r + \frac{\overline{m}}{r}\theta_r = 0 \tag{50}$$

$$\partial_t\psi + nh + rh_r + \overline{m}n\partial_t w + \overline{m}r\partial_t w_r = 0 \tag{51}$$

$$\tau_0\partial_t h + h + \frac{\kappa}{r}\psi_r = 0 \tag{52}$$

We will now show that the energy of the penalized problem, defined by

$$E^\varepsilon(t) := ||\partial_t u^\varepsilon||^2 + \lambda_1||\nabla u^\varepsilon||^2 + \kappa||\theta^\varepsilon||^2 + ||q^\varepsilon||^2 + \frac{1}{\varepsilon}\int\limits_{\Gamma_C}|(u_\nu^\varepsilon - \overline{g})^+|^2 \mathrm{d}\Gamma$$

decays exponentially as time goes to infinity, i.e.

$$E^\varepsilon(t) \leq \alpha E_0^\varepsilon e^{-\beta t}$$

We will use the technique of a Lyapunov functional, constructing the negative terms of the energy and combining the respective functionals in a final estimate. First, one easily sees by multiplying (46) with $\partial_t u$, (47) with $\kappa\theta$ and (48) with q and integrating over Ω, that the energy satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E^\varepsilon(t) \leq -C_2\ (\mu||\partial_t u^\varepsilon||_n^2 + ||q^\varepsilon||_n^2) \tag{53}$$

Proposition 3.1. *Let*

$$F_1(t) := \langle \partial_t u^\varepsilon, u^\varepsilon\rangle_n + \varepsilon \int\limits_{\Gamma_C} |u_\nu^\varepsilon|^2\, d\Gamma - \mu||\nabla u^\varepsilon||_{n\times n}^2$$

then for any $\delta_1 > 0$

$$\frac{d}{dt}F_1(t) \leq -(C_3-\delta_1)||\nabla u^\varepsilon||_{n\times n}^2 - \frac{1}{\varepsilon}\int\limits_{\Gamma_C} |(u_\nu^\varepsilon - \overline{g})^+|^2\, d\Gamma + ||\partial_t u^\varepsilon||_n^2 + \frac{C_4}{\delta_1}||\theta^\varepsilon||^2 \tag{54}$$

Proof:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle \partial_t u^\varepsilon, u^\varepsilon\rangle \\ &= ||\partial_t u^\varepsilon||^2 + \langle \partial_t^2 u^\varepsilon, u^\varepsilon\rangle \\ &= ||\partial_t u^\varepsilon||^2 - \lambda_1||\nabla u^\varepsilon||^2 - \mu\langle \partial_t \nabla u^\varepsilon, \nabla u^\varepsilon\rangle - \langle \overline{m}\theta^\varepsilon, \mathrm{div}\ u^\varepsilon\rangle \\ &\quad - (\lambda_1+\lambda_2)||\mathrm{div}\ u^\varepsilon||^2 - \frac{1}{\varepsilon}\int\limits_{\Gamma_C}(u_\nu^\varepsilon - \overline{g})^+ u_\nu \mathrm{d}\Gamma - \varepsilon\int\limits_{\Gamma_C}\partial_t u_\nu^\varepsilon u_\nu^\varepsilon \mathrm{d}\Gamma \end{aligned} \tag{55}$$

Estimating

$$|\langle \overline{m}\theta^\varepsilon, \mathrm{div}\ u^\varepsilon\rangle| \leq C_4\left(\delta_1||\nabla u^\varepsilon||^2 + \frac{1}{\delta_1}||\theta^\varepsilon||^2\right)$$

for any $\delta_1 > 0$ and

$$\begin{aligned} &-\frac{1}{\varepsilon}\int\limits_{\Gamma_C}(u_\nu^\varepsilon - \overline{g})^+ u_\nu^\varepsilon \mathrm{d}\Gamma \\ \leq\ &-\frac{1}{\varepsilon}\int\limits_{\Gamma_C}|(u_\nu^\varepsilon - \overline{g})^+|^2 \mathrm{d}\Gamma \end{aligned}$$

we obtain the desired result. □

Proposition 3.2. *Let*

$$\Psi(t,r) := \int\limits_{r_0}^{r}\psi(t,s)\,ds$$

and

$$F_2(t) := -\tau_0\int\limits_{r_0}^{1}\Psi(t,r)h(t,r)\,dr$$

Then for any $\delta_2, \delta_3 > 0$

$$\frac{d}{dt}F_2(t) \leq \frac{C}{\delta_2+\delta_3}\int\limits_{r_0}^{1}|h(t,r)|^2\,dr - \frac{\kappa-\delta_2}{r_0}\int\limits_{r_0}^{1}|\psi(t,r)|^2\,dr + \delta_3\int\limits_{r_0}^{1}|\partial_t w(t,r)|^2\,dr \tag{56}$$

Proof: By (51), Ψ satisfies

$$\partial_t \Psi + \int_{r_0}^{r} nh \mathrm{d}s + \int_{r_0}^{r} sh \mathrm{d}s + \int_{r_0}^{r} \overline{m} n \partial_t w \mathrm{d}s + \int_{r_0}^{r} \overline{m} s \partial_t w_s \mathrm{d}s = 0 \tag{57}$$

Multiplying (57) with hr and integrating, we obtain for any $\delta_3 > 0$

$$\begin{aligned} -\int_{r_0}^{1} (\partial_t \Psi) hr \mathrm{d}r &= n \int_{r_0}^{1} hr \int_{r_0}^{r} h \mathrm{d}s \mathrm{d}r + \int_{r_0}^{1} hr \int_{r_0}^{r} sh_s \mathrm{d}s \mathrm{d}r \\ &\quad + \overline{m} n \int_{r_0}^{1} hr \int_{r_0}^{r} \partial_t w(t,s) \mathrm{d}s \mathrm{d}r + \overline{m} \int_{r_0}^{1} h \int_{r_0}^{r} s \partial_t w_s \mathrm{d}s \mathrm{d}r \\ &\leq \frac{C}{\delta_3} \int_{r_0}^{1} |h(t,r)|^2 \mathrm{d}r + \delta_3 \int_{r_0}^{1} |w(t,r)|^2 \mathrm{d}r \end{aligned} \tag{58}$$

Multiplying (52) by Ψr and integrating, we obtain

$$\tau_0 \int_{r_0}^{1} \Psi (\partial_t h) r \mathrm{d}r + \int_{r_0}^{1} \Psi h r \mathrm{d}r + \int_{r_0}^{1} \kappa \Psi_{rr} \Psi \mathrm{d}r = 0$$

We have, by definition of Ψ,

$$\int_{r_0}^{1} \kappa \Psi_{rr}(t,r) \Psi(t,r) \mathrm{d}r = -\kappa \int_{r_0}^{1} |\Psi_r(t,r)|^2 \mathrm{d}r = -\kappa \int_{r_0}^{1} |\psi(t,r)|^2 \mathrm{d}r$$

Using Poincaré's Inequalitiy for Ψ, this implies for any $\delta_2 > 0$

$$-\int_{r_0}^{1} \Psi (\partial_t h) r \mathrm{d}r \leq \frac{C}{\delta_2} \int_{r_0}^{1} |h|^2 \mathrm{d}r - (\kappa - \delta_2) \int_{r_0}^{1} |\psi|^2 \mathrm{d}r \tag{59}$$

Combining (58) and (59), we obtain the desired result. □
Defining

$$L(t) := N E^{\varepsilon}(t) + F_1(t) + \delta_4 F_2(t)$$

where δ_4 will be chosen later, we easily see that for large enough N there exist C_1, $C_2 > 0$ such that

$$C_1 E(t) \leq L(t) \leq C_2 E(t) \tag{60}$$

Now, we can prove the essential theorem of this section.

Theorem 3.1. *Let $\mu > 0$. Then the system is exponentially stable, i.e. there is a $\beta > 0$ such that*

$$E^{\varepsilon}(t) \leq \alpha E_0^{\varepsilon} e^{-\beta t}$$

Proof: Using (53), (54) and (56), we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}L(t) \leq & -NC_2(\mu||\partial_t u^\varepsilon||_n^2 + ||q^\varepsilon||_n^2) - (C_3 - \delta_1)||\nabla u^\varepsilon||_{n\times n}^2 \\ & - \frac{1}{\varepsilon}\int\limits_{\Gamma_C} |(u_\nu^\varepsilon - \overline{g})^+|^2 \mathrm{d}\Gamma + ||\partial_t u^\varepsilon||_n^2 + \frac{C_4}{\delta_1}||\theta^\varepsilon||^2 \\ & + \delta_4\left(\frac{C_5}{\delta_2+\delta_3}\int\limits_{r_0}^{1}|h(t,r)|^2\mathrm{d}r - \frac{\kappa-\delta_2}{r_0}\int\limits_{r_0}^{1}|\psi|^2\mathrm{d}r + \delta_3\int\limits_{r_0}^{1}|\partial_t w|^2\mathrm{d}r\right) \\ \leq & \,(1 + C_6\delta_3\delta_4 - NC_2\mu)||\partial_t u^\varepsilon||_n^2 + (\frac{\delta_4 C_7}{\delta_2+\delta_3} - NC_2)||q^\varepsilon||_n^2 + (\delta_1 - C_3)||\nabla u^\varepsilon||_{n\times n}^2 \\ & - \frac{1}{\varepsilon}\int\limits_{\Gamma_C} |(u_\nu^\varepsilon - \overline{g})^+|^2 \mathrm{d}\Gamma + (\frac{C_4}{\delta_1} - \delta_4\frac{\kappa-\delta_2}{r_0})||\theta^\varepsilon||^2\end{aligned}$$

Choosing $\delta_1 < C_3$ and $\delta_2 < \kappa$, then $\delta_4 > \frac{r_0 C_4}{\delta_1(\kappa-\delta_2)}$ and (arbitrarily) $\delta_3 = 1$ we conclude that, for sufficiently large N, there is a $C > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}L(t) \leq -CE^\varepsilon(t)$$

Using (60), this proves our theorem. □

Note that δ_3 is not really needed for the construction of the Lyapunov functional and could have been left as 1. However, we want to point out that the positive u_t term arising from F_2 is not a problem; the problem requiring $\mu > 0$ is the positive u_t term arising from F_1, which can not be made arbitrarily small without losing the negative terms for the derivatives of u. For the classical problem, Munõz-Rivera and Racke [6] showed that this term can be handled by adding additional functions to the Lyapunov functional; however, this gives rise to a positive $\nabla\theta$ term. While $\nabla\theta$ is given as a negative term from the energy itself in the classical case, this does not hold for $\tau_0 > 0$; in fact we do not know anything about derivatives of θ. It is therefore necessary to gain the negative u_t term by other means, one of them being the viscoelastic term. If we define the energy of the original problem as

$$E(t) := ||\partial_t u||^2 + ||\nabla u||^2 + ||\theta||^2 + ||q||^2$$

we see by the lower semicontinuity of the norms of weak*-convergent series, using Proposition 2.2, that

$$\liminf_{\varepsilon\to 0} E^\varepsilon(t) \geq E(t)$$

Using the strong convergence of initial data, we obtain

$$E(t) \leq E^\varepsilon(t) \leq \alpha E^\varepsilon(0)\exp(-\beta t) \to \alpha E(0)\exp(-\beta t)$$

This proves our final theorem:

Theorem 3.2. *Let $\mu > 0$. Then there are α, $\beta > 0$ such that*

$$E(t) \;\leq\; \alpha E(0)\exp(-\beta t)$$

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Jan Sprenger, Institute for Analysis and Scientific Computing, Vienna University of Technology, Wiedner Hauptstraße 8-10, 1040 Wien, Austria
E-Mail: jsprenger@asc.tuwien.ac.at