

Long-time asymptotics for a coupled thermoelastic plate–membrane system

Bienvenido Barraza Martínez¹  | Robert Denk²  | Jonathan González Ospino¹  | Jairo Hernández Monzón¹  | Sophia Rau²

¹Departamento de Matemáticas y Estadística, Universidad del Norte, Barranquilla, Colombia

²Fachbereich Mathematik und Statistik, Universität Konstanz, Konstanz, Germany

Correspondence

Robert Denk, Fachbereich Mathematik und Statistik, Universität Konstanz, Germany.

Email: robert.denk@uni-konstanz.de

Communicated by: J. Muñoz Rivera

Funding information

Minciencias (formerly Colciencias), Grant/Award Number: 121571250194

In this paper, we consider a transmission problem for a system of a thermoelastic plate with (or without) rotational inertia term coupled with a membrane with different variants of damping for the plate and/or the membrane. We prove well-posedness of the problem and higher regularity of the solution and study the asymptotic behavior of the solution, depending on the damping and on the presence of the rotational term.

KEYWORDS

exponential stability, polynomial stability, thermoelastic plate-membrane system, transmission problem

MSC CLASSIFICATION

35M33; 35B35; 35B40; 74K15; 74K20

1 | INTRODUCTION

Transmission problems appear frequently in different fields of physics and engineering, for example, in solid mechanics of composed materials, in processes of electromagnetism on ferromagnetic materials with dielectric constants, in the vibration of membranes, and in coupled plates (see Borsuk¹). Structures compound by a finite number of interconnected flexible elements, such as beams, plates, shells, membranes, and combinations of them were studied in different settings in Ammari and Nicaise,² Balmès and Germès,³ Denk and Kammerlander,⁴ Kleinmann and Martin,⁵ Lagnese and Leugering⁶, Leissa,⁷ Muñoz Rivera and Naso,⁸ and Roy et al.⁹

The classical linear model of thermoelastic plates due to Kirchhoff is given by

$$\begin{aligned}\rho_1 u_{tt} - \gamma \Delta u_{tt} + \beta_1 \Delta^2 u + \mu \Delta \theta &= 0 \quad \text{in } \Omega_1 \times (0, \infty), \\ \rho_0 \theta_t - \beta_0 \Delta \theta - \mu \Delta u_t &= 0 \quad \text{in } \Omega_1 \times (0, \infty),\end{aligned}$$

where ρ_0 , ρ_1 , β_0 , β_1 , and μ are positive constants and $\gamma \geq 0$. Here, $\Omega_1 \subset \mathbb{R}^2$ is the region occupied by the middle surface of the plate, u represents the transverse displacements of the points on the middle surface of the plate, and θ is the difference of temperature on the plate with respect to a reference temperature. For the physical model and the deduction of these equations, see Lagnese and Lions.¹⁰

Konstanzer Online-Publikations-System (KOPS)

URL: <http://nbn-resolving.de/urn:nbn:de:bsz:352-2-10lxm5fywvph6>

This is an open access article under the terms of the Creative Commons Attribution-NonCommercial License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited and is not used for commercial purposes.

© 2021 The Authors. Mathematical Methods in the Applied Sciences published John Wiley & Sons, Ltd.

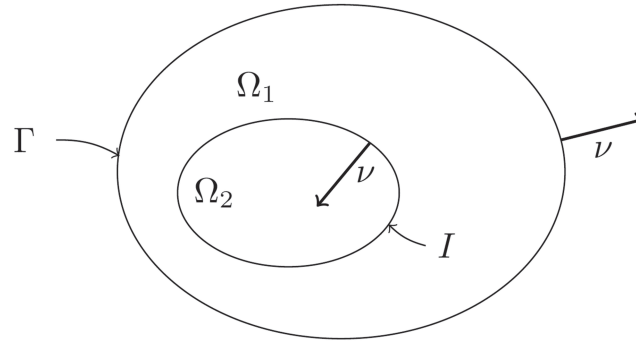


FIGURE 1 Plate-membrane system

In the present paper, we study the situation where the thermoelastic plate in Ω_1 is coupled with an elastic membrane in the region Ω_2 . To fix the geometric situation of the problem, we consider two non-empty, open, connected, and bounded subsets Ω and Ω_2 of \mathbb{R}^2 , with boundary of class C^4 such that $\overline{\Omega_2} \subset \Omega$. We denote $\Omega_1 := \Omega \setminus \overline{\Omega_2}$, $\Gamma := \partial\Omega$ and $I := \partial\Omega_2$. Note that $\partial\Omega_1 = \Gamma \cup I$. The plate-membrane system of interest is composed of a thermoelastic plate in Ω_1 and a membrane, occupying in equilibrium the region Ω_2 , as shown in Figure 1.

Denoting by $u(x, t)$, $v(x, t)$, the vertical displacements of the points on the middle surface of the plate and on the membrane with coordinates x at time t , respectively, and by $\theta(x, t)$, the temperature difference on the plate, the mathematical model for the structure is given by the equations

$$\rho_1 u_{tt} - \gamma \Delta u_{tt} + \beta_1 \Delta^2 u - \rho \Delta u_t + \mu \Delta \theta = 0 \quad \text{in } \Omega_1 \times (0, \infty), \quad (1)$$

$$\rho_0 \theta_t - \beta_0 \Delta \theta - \mu \Delta u_t = 0 \quad \text{in } \Omega_1 \times (0, \infty), \quad (2)$$

$$\rho_2 v_{tt} - \beta_2 \Delta v + m v_t = 0 \quad \text{in } \Omega_2 \times (0, \infty), \quad (3)$$

where ρ_i , β_i ($i = 0, 1, 2$) and μ are positive constants depending on the properties of the materials and γ , ρ , and m are non-negative constants. The coefficient $\gamma > 0$ represents the rotational inertia of the filaments of the plate and is proportional to the square of the plate thickness. Consequently, it is usual to consider this thickness very small (the case $\gamma = 0$ corresponds to a thin plate). The coefficient $m \geq 0$ describes the damping (or the absence of damping) for the wave equation (3), whereas ρ in (1) describes a structural damping on the plate. We will also include the situation when thermal effects for the plate are not taken into account by setting $\mu = 0$ in (1) and omitting (2).

We will assume that the plate is clamped at the exterior boundary Γ , namely,

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (4)$$

where ν represents the outward pointing unit normal vector to the boundary of Ω_1 and, consequently, $-\nu$ is the corresponding outward unit normal vector to the boundary of Ω_2 .

Due to the lack of thermal effects in the membrane, we will assume that the difference of temperature in the interface is zero. We will also assume that the plate satisfies Newton's cooling law. This leads to the boundary conditions

$$\theta = 0 \quad \text{on } I \times (0, \infty) \quad \text{and} \quad \frac{\partial \theta}{\partial \nu} + \kappa \theta = 0 \quad \text{on } \Gamma \times (0, \infty), \quad (5)$$

for some constant $\kappa > 0$.

In addition, we consider the following transmission conditions on the interface:

$$u = v, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } I \times (0, \infty), \quad (6)$$

$$\beta_1 \frac{\partial \Delta u}{\partial \nu} + \beta_2 \frac{\partial v}{\partial \nu} + \mu \frac{\partial \theta}{\partial \nu} = 0 \quad \text{on } I \times (0, \infty), \quad (7)$$

and the initial conditions

$$\begin{aligned} u(\cdot, 0) &= u_0, \quad v(\cdot, 0) = v_0, \quad \theta(\cdot, 0) = \theta_0, \\ u_t(\cdot, 0) &= u_1, \quad v_t(\cdot, 0) = v_1. \end{aligned} \quad (8)$$

In (6), the condition $u = v$ on I is necessary for the continuity of the solution, and the condition $\partial u / \partial \nu = 0$ on I has the following interpretation: the transversal force caused by the tension and the one originated by the shear stress between the plate and the membrane cancel each other, which forces the horizontal displacements on the interface to be zero (compare with Hassine¹¹). The aim of the present paper is to study well-posedness, regularity, and asymptotic behavior of the solution of (1)–(8), in dependence on the parameters m, ρ, γ , and μ .

For the case $\gamma = 0$, we refer to Ammari and Nicaise,² Hassine,¹¹ Barraza Martínez et al.,¹² Hernández Monzón,¹³ and Liu and Su¹⁴ where structures formed by a plate and a membrane were studied. In Hernández Monzón,¹³ the author models a system composed by an elastic thin plate coupled with an elastic membrane and shows existence and uniqueness of weak solutions. In Liu and Su,¹⁴ the authors study undamped plate–membrane systems, where the plate and the membrane are two layers occupying the same region in the plane. In Ammari and Nicaise,² a coupled system of a wave equation and a plate equation with damping on the boundary without thermal effects is studied. Hassine¹¹ studies a transmission problem with the configuration presented in Figure 1, but with the plate being surrounded by the membrane. In Barraza Martínez et al.,¹² the plate is isothermal. Most of these references study some kind of stability for the solution, but only a few of them deal with regularity.

Regarding the rotational inertia term ($\gamma > 0$), we mention Avalos and Lasiecka,¹⁵ Chueshov and Lasiecka,¹⁶ Dell’Oro et al.,¹⁷ Fernández Sare et al.,¹⁸ Lasiecka et al.,¹⁹ Lasiecka and Triggiani,²⁰ Muñoz Rivera and Portillo Oquendo,²¹ and Muñoz Rivera and Vega²². From these, only in Muñoz Rivera and Portillo Oquendo,²¹ a transmission problem is analyzed, and it is of the thermoelastic plate–plate type. There seem to be few results for the structure (1)–(8), even for the case $\gamma = 0$.

The principal results of this work state the existence and uniqueness of the solution of problem (1)–(8) and its continuous dependence on the data (i.e., the well-posedness). It is also proved that the solution for the case $\gamma \geq 0$ (i.e., with or without rotational inertia term) and $m \geq 0$ (i.e., with or without damping over the membrane) has higher regularity. In particular, the boundary and transmission conditions hold in the strong sense of traces if the initial values are smooth enough. Furthermore, we study the asymptotic behavior of the solution in terms of the stability of the associated semigroup in different situations. For a damped membrane ($m > 0$), we show that exponential stability holds if $\rho > 0$ (Theorem 4.1) or if $\rho = \gamma = 0$ (Theorem 4.4). For the undamped membrane ($m = 0$), we have no exponential stability (Theorem 5.1) but polynomial stability (Theorem 5.2) under some geometric condition.

The paper is organized as follows: In Section 2, we define the basic spaces and operators and prove the generation of a C_0 -semigroup of contractions, which implies the well-posedness of problem (1)–(8). In Section 3, we show some spectral properties of the operator defined by the weak formulation of the transmission problem, and we also show the regularity of the solution using the theory of parameter-elliptic boundary value problems and some ideas from Barraza Martínez et al.,¹² where the authors study a plate–membrane transmission problem with $\gamma = 0$ and without thermal effect over the plate. In Section 4, we prove exponential stability for the damped membrane, whereas in Section 5, we study the undamped membrane.

In the following, the letter C stands for a generic constant which may vary in each time of appearance. We will also use the notation χ_A for the characteristic function of the set A , that is, $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ else. If X and Y are Banach spaces, we write $X \subset Y$ for the continuous embedding, that is, if X is a subset of Y and if $\text{id} : X \rightarrow Y$ is continuous. The space of all bounded linear operators in \mathcal{H} will be denoted by $\mathcal{L}(\mathcal{H})$.

2 | WELL-POSEDNESS

Following the standard approach, we will formulate (1)–(8) as an abstract Cauchy problem and study the associated semigroup. We define $w = (w_j)_{j=1, \dots, 5} := (u, u_t, \theta, v, v_t)^\top$ and write (1)–(3), with the initial conditions (8), formally as a first order system

$$M(D)\partial_t w(t) - A(D)w(t) = 0 \quad (t > 0), \quad w(0) = w_0 \quad (9)$$

with the diagonal matrix

$$M(D) := \begin{pmatrix} 1 & & & & \\ & \rho_1 - \gamma \Delta & & & \\ & & \rho_0 & & \\ & & & 1 & \\ & & & & \rho_2 \end{pmatrix}$$

and with

$$A(D) := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -\beta_1 \Delta^2 & \rho \Delta & -\mu \Delta & 0 & 0 \\ 0 & \mu \Delta & \beta_0 \Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \beta_2 \Delta & -m \end{pmatrix}$$

and $w_0 = (u_0, u_1, \theta_0, v_0, v_1)^\top$.

We start with the definition of the related operators in a weak Hilbert space setting. Here, we have to distinguish the case $\gamma > 0$ (presence of rotational inertia term) from the case $\gamma = 0$. For $\gamma > 0$, the Hilbert space \mathcal{H}_γ is defined as the space of all complex-valued functions

$$w = (w_1, \dots, w_5)^\top \in H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$$

satisfying the boundary and transmission conditions

$$\begin{aligned} w_1 &= \partial_\nu w_1 = w_2 = 0 \text{ on } \Gamma, \\ w_1 &= w_4, \partial_\nu w_1 = 0 \text{ on } I. \end{aligned}$$

For $\gamma = 0$, we modify this definition by replacing the condition $w_2 \in H^1(\Omega_1)$ by $w_2 \in L^2(\Omega_1)$ and omitting the boundary condition $w_2 = 0$ on Γ . We endow the space \mathcal{H}_γ for all $\gamma \geq 0$ with a scalar product, which is adapted to the transmission problem. For $w, \phi \in \mathcal{H}_\gamma$, we set

$$\begin{aligned} \langle w, \phi \rangle_{\mathcal{H}_\gamma} &:= \beta_1 \langle \Delta w_1, \Delta \phi_1 \rangle_{L^2(\Omega_1)} + \rho_1 \langle w_2, \phi_2 \rangle_{L^2(\Omega_1)} + \gamma \langle \nabla w_2, \nabla \phi_2 \rangle_{L^2(\Omega_1)} \\ &+ \rho_0 \langle w_3, \phi_3 \rangle_{L^2(\Omega_1)} + \beta_2 \langle \nabla w_4, \nabla \phi_4 \rangle_{L^2(\Omega_2)} + \rho_2 \langle w_5, \phi_5 \rangle_{L^2(\Omega_2)}. \end{aligned}$$

Lemma 2.1. For $\gamma > 0$, the norm in \mathcal{H}_γ is equivalent to the standard norm in $H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$. For $\gamma = 0$, the norm in \mathcal{H}_0 is equivalent to the standard norm in $H^2(\Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$.

Proof. Let $\gamma > 0$. Obviously, all terms in the norm $\|\cdot\|_{\mathcal{H}_\gamma}$ can be estimated by the standard norm in $\mathcal{H} := H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$, so we only have to show $\|\cdot\|_{\mathcal{H}_\gamma} \geq C \|\cdot\|_{\mathcal{H}}$.

Let $w \in \mathcal{H}_\gamma$. Then w_1 is a solution of the boundary value problem

$$\begin{aligned} \Delta u &= f \text{ in } \Omega_1, \\ \partial_\nu u &= 0 \text{ on } \partial\Omega_1 \end{aligned}$$

with $f := \Delta w_1 \in L^2(\Omega_1)$. By elliptic regularity (see Agranovich,²³ Theorem 7.1.3) we get

$$\|w_1\|_{H^2(\Omega_1)} \leq C (\|\Delta w_1\|_{L^2(\Omega_1)} + \|w_1\|_{L^2(\Omega_1)}). \quad (10)$$

Due to the boundary and transmission conditions $w_1 = 0$ on Γ and $w_1 = w_4$ on I , the function $\chi_{\Omega_1} w_1 + \chi_{\Omega_2} w_4$ belongs to $H_0^1(\Omega)$. With Poincaré's inequality, we obtain

$$\|w_1\|_{L^2(\Omega_1)}^2 + \|w_4\|_{L^2(\Omega_2)}^2 \leq C \left(\|\nabla w_1\|_{L^2(\Omega_1)}^2 + \|\nabla w_4\|_{L^2(\Omega_2)}^2 \right). \quad (11)$$

As $w_1 = 0$ on Γ and $\partial_\nu w_1 = 0$ on $\partial\Omega_1$, we can apply Poincaré's inequality, integration by parts, and Young's inequality to see that for every $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$\begin{aligned} \|w_1\|_{L^2(\Omega_1)}^2 &\leq C \|\nabla w_1\|_{L^2(\Omega_1)}^2 = -C \langle \Delta w_1, w_1 \rangle_{L^2(\Omega_1)} \\ &\leq \varepsilon C \|w_1\|_{L^2(\Omega_1)}^2 + C C_\varepsilon \|\Delta w_1\|_{L^2(\Omega_1)}^2. \end{aligned}$$

Choosing ε with $\varepsilon C \leq 1/2$, we can estimate $\|w_1\|_{L^2(\Omega_1)} \leq C\|\Delta w_1\|_{L^2(\Omega_1)}$. Combining this with (11) and (10), we obtain

$$\|w_1\|_{H^2(\Omega_1)}^2 + \|w_4\|_{H^1(\Omega_2)}^2 \leq C \left(\|\Delta w_1\|_{L^2(\Omega_1)}^2 + \|\nabla w_4\|_{L^2(\Omega_2)}^2 \right).$$

By definition of the norms in \mathcal{H}_γ and the standard norm in \mathcal{H} , this yields $\|w\|_{\mathcal{H}} \leq C\|w\|_{\mathcal{H}_\gamma}$.

The same arguments show that also for $\gamma = 0$, the norm in the space \mathcal{H}_0 is equivalent to the standard norm in $H^2(\Omega_1) \times L^2(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$. \square

To formulate the transmission problem (9) in a weak setting, we formally apply the operator $\beta_1 \Delta^2$ to the first component and $-\beta_2 \Delta$ to the fourth component. We obtain

$$\tilde{M}(D)\partial_t w(t) - \tilde{A}(D)w(t) = 0 \quad (t > 0), \quad w(0) = w_0$$

with

$$\tilde{M}(D) := \begin{pmatrix} \beta_1 \Delta^2 & & & & \\ & \rho_1 - \gamma \Delta & & & \\ & & \rho_0 & & \\ & & & -\beta_2 \Delta & \\ & & & & \rho_2 \end{pmatrix}$$

and

$$\tilde{A}(D) := \begin{pmatrix} 0 & \beta_1 \Delta^2 & 0 & 0 & 0 \\ -\beta_1 \Delta^2 & \rho \Delta & -\mu \Delta & 0 & 0 \\ 0 & \mu \Delta & \beta_0 \Delta & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_2 \Delta \\ 0 & 0 & 0 & \beta_2 \Delta & -m \end{pmatrix}.$$

In this way, the weak formulation is adapted to the definition of the Hilbert space \mathcal{H}_γ and to the boundary and transmission conditions. Let \mathcal{H}'_γ denote the antidual space of \mathcal{H}_γ , that is, the space of all continuous conjugate linear functionals on \mathcal{H}_γ . We define the operator $\mathbb{M} : \mathcal{H}_\gamma \rightarrow \mathcal{H}'_\gamma$ by

$$\langle \mathbb{M}w, \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} := \langle w, \phi \rangle_{\mathcal{H}_\gamma} \quad (w, \phi \in \mathcal{H}_\gamma).$$

To define the operator related to $\tilde{A}(D)$ in a weak setting, we introduce the space

$$H^{2,1} := \{(u, v)^\top \in H^2(\Omega_1) \times H^1(\Omega_2) : u = \partial_\nu u = 0 \text{ on } \Gamma, u = v, \partial_\nu u = 0 \text{ on } I\}$$

with inner product

$$\langle (u, v), (u', v') \rangle_{H^{2,1}} := \beta_1 \langle \Delta u, \Delta u' \rangle_{L^2(\Omega_1)} + \beta_2 \langle \nabla v, \nabla v' \rangle_{L^2(\Omega_2)}.$$

We have seen in the proof of Lemma 2.1 that this norm is equivalent to the standard norm in $H^2(\Omega_1) \times H^1(\Omega_2)$. Note that in the definition of \mathcal{H}_γ , we have $(w_1, w_4)^\top \in H^{2,1}$. We define the subspace $\mathcal{V} \subset \mathcal{H}_\gamma$ by

$$\mathcal{V} := \{w \in \mathcal{H}_\gamma : (w_2, w_5)^\top \in H^{2,1}, w_3 \in H^1(\Omega_1), w_3 = 0 \text{ on } I\}$$

with inner product

$$\begin{aligned} \langle w, \phi \rangle_{\mathcal{V}} &:= \langle (w_1, w_4), (\phi_1, \phi_4) \rangle_{H^{2,1}} + \langle (w_2, w_5), (\phi_2, \phi_5) \rangle_{H^{2,1}} \\ &\quad + \beta_0 \langle \nabla w_3, \nabla \phi_3 \rangle_{L^2(\Omega_1)} + \beta_0 \kappa \langle w_3, \phi_3 \rangle_{L^2(\Gamma)}. \end{aligned}$$

Now we can define $\mathbb{A} : \mathcal{V} \rightarrow \mathcal{V}'$ by

$$\begin{aligned} \langle \mathbb{A}w, \phi \rangle_{\mathcal{V}' \times \mathcal{V}} &:= - \langle (w_1, w_4), (\phi_2, \phi_5) \rangle_{H^{2,1}} + \langle (w_2, w_5), (\phi_1, \phi_4) \rangle_{H^{2,1}} \\ &\quad - \mu \langle w_3, \Delta \phi_2 \rangle_{L^2(\Omega_1)} + \mu \langle \Delta w_2, \phi_3 \rangle_{L^2(\Omega_1)} - \rho \langle \nabla w_2, \nabla \phi_2 \rangle_{L^2(\Omega_1)} \\ &\quad - \beta_0 \langle \nabla w_3, \nabla \phi_3 \rangle_{L^2(\Omega_1)} - \beta_0 \kappa \langle w_3, \phi_3 \rangle_{L^2(\Gamma)} - m \langle w_5, \phi_5 \rangle_{L^2(\Omega_2)} \end{aligned} \tag{12}$$

for $w, \phi \in \mathcal{V}$.

Remark 2.2. (a) The norm in \mathcal{V} is equivalent to the standard norm in $H^2(\Omega_1) \times H^2(\Omega_1) \times H^1(\Omega_1) \times H^1(\Omega_2) \times H^1(\Omega_2)$. In fact, we have already seen that the norm in $H^{2,1}$ is equivalent to the norm in $H^2(\Omega_1) \times H^1(\Omega_2)$, and for the component w_3 , we have $\|w_3\|_{H^1(\Omega_1)} \leq C\|\nabla w_3\|_{L^2(\Omega_1)}$ by Poincaré’s inequality and $\|w_3\|_{L^2(\Gamma)} \leq \|w_3\|_{H^{1/2}(\Gamma)} \leq C\|w_3\|_{H^1(\Omega_1)}$ by trace results.

(b) From the definition, we immediately see that $\mathbb{M} \in \mathcal{L}(\mathcal{H}_\gamma, \mathcal{H}'_\gamma)$ and $\mathbb{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$. Moreover, \mathbb{M} is defined as the scalar product in the Hilbert space \mathcal{H}_γ and therefore is an isometric isomorphism from \mathcal{H}_γ to \mathcal{H}'_γ .

Based on Remark 2.2(b), we can define the \mathcal{H}_γ -realization of the transmission problem as the operator $\mathcal{A} : \mathcal{H}_\gamma \supset D(\mathcal{A}) \rightarrow \mathcal{H}_\gamma$ by

$$D(\mathcal{A}) := \{w \in \mathcal{V} : \mathbb{A}w \in \mathcal{H}'_\gamma\}, \quad \mathcal{A}w := \mathbb{M}^{-1}\mathbb{A}w.$$

We consider the abstract Cauchy problem

$$\begin{aligned} \partial_t w(t) - \mathcal{A}w(t) &= 0 \quad (t > 0), \\ w(0) &= w_0 \end{aligned} \tag{13}$$

with $w_0 := (u_0, u_1, \theta_0, v_0, v_1)^\top$. The following remark shows that this Cauchy problem is in fact the weak formulation of the transmission problem (1)–(8).

Remark 2.3. (a) We have $\mathbb{A}w = \tilde{A}(D)w$ for all $w \in D(\mathcal{A})$ and $\mathbb{M}w = \tilde{M}(D)w$ for all $w \in \mathcal{H}_\gamma$ in the sense of distributions. This follows immediately from the definitions of the operators and integration by parts, when we choose $\phi \in \mathcal{D}(\Omega_1)^3 \times \mathcal{D}(\Omega_2)^2$, where $\mathcal{D}(\Omega_1)$ stands for the infinitely smooth functions with compact support in Ω_1 . Consequently, a function $w \in C^1([0, \infty), D(\mathcal{A}))$ is a classical solution of (13) if and only if w satisfies (1)–(3) in the distributional sense.

(b) Let $w \in D(\mathcal{A})$ be of the higher regularity $w \in H^4(\Omega_1) \times H^2(\Omega_1)^2 \times H^2(\Omega_2) \times H^1(\Omega_2)$. Then w satisfies the boundary and transmission conditions (4)–(7) in the strong sense, that is, as equality of the traces of the functions on Γ and I , respectively.

To see this, we only have to show that the second equality in (5) and equality (7) holds, as the other conditions are already included in the definition of \mathcal{V} . Setting $\phi = (0, 0, \phi_3, 0, 0)^\top$, we obtain by (a)

$$\langle \mathbb{A}w, \phi \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle \mu \Delta w_2 + \beta_0 \Delta w_3, \phi_3 \rangle_{L^2(\Omega_1)}.$$

Comparing this with the definition of \mathbb{A} , we obtain, using integration by parts, that

$$\int_\Gamma (\kappa w_3 + \partial_\nu w_3) \overline{\phi_3} dS = 0$$

holds for all $\phi_3 \in H^1(\Omega_1)$ with $\phi_3 = 0$ on I . Therefore, $\kappa w_3 + \partial_\nu w_3 = 0$ holds on Γ in the strong sense, that is, as equality in the trace space $H^{1/2}(\Gamma)$. In the same way, one can prove that (7) holds in the strong sense.

To show well-posedness, we will also need the following result.

Lemma 2.4. *The space \mathcal{V} is dense in \mathcal{H}_γ , and therefore, we have the dense embeddings*

$$\mathcal{V} \subset \mathcal{H}_\gamma \subset (L^2(\Omega_1))^3 \times (L^2(\Omega_2))^2 \subset \mathcal{H}'_\gamma \subset \mathcal{V}'.$$

Proof. (i) In a first step, we show that

$$V(\Omega_1) := \{\phi \in H^2(\Omega_1) : \phi = \partial_\nu \phi = 0 \text{ on } \Gamma, \partial_\nu \phi = 0 \text{ on } I\}$$

is dense in $H^1_\Gamma(\Omega_1) := \{u \in H^1(\Omega_1) : u = 0 \text{ on } \Gamma\}$. For this, let $u \in H^1_\Gamma(\Omega_1)$. We choose a function $\tilde{\phi} \in C^\infty(\Omega_1)$ with $\tilde{\phi} = 1$ near I , $\tilde{\phi} = 0$ near Γ , and $0 \leq \tilde{\phi} \leq 1$ in Ω_1 . We set $\phi := \tilde{\phi}^2$. Note that $(1 - \phi)u \in H^1_0(\Omega_1)$ and $\tilde{\phi}u \in H^1(\Omega_1)$. As the test functions are dense in $H^1_0(\Omega_1)$, there exists a sequence $(\phi_n^{(1)})_{n \in \mathbb{N}} \subset \mathcal{D}(\Omega_1)$ such that $\phi_n^{(1)} \rightarrow (1 - \phi)u$ in $H^1(\Omega_1)$ for $n \rightarrow \infty$. Moreover, as the domain of the Neumann Laplacian

$$D(\Delta_N) := \{u \in H^2(\Omega_1) : \partial_\nu u = 0 \text{ on } \partial\Omega_1\}$$

is dense in $H^1(\Omega_1)$ (see Ouhabaz,²⁴ Lemma 1.25) there exists a sequence $(\tilde{\phi}_n^{(2)})_{n \in \mathbb{N}} \subset D(\Delta_N)$ with $\tilde{\phi}_n^{(2)} \rightarrow \tilde{\phi}u$ in $H^1(\Omega_1)$. Now, setting $\phi_n^{(2)} := \tilde{\phi}\tilde{\phi}_n^{(2)} \in V(\Omega_1)$ for $n \in \mathbb{N}$, we get $\phi_n := \phi_n^{(1)} + \phi_n^{(2)} \rightarrow (1 - \phi)u + \tilde{\phi}^2u = u$ in $H^1(\Omega_1)$.

(ii) Now we show that \mathcal{V} is dense in \mathcal{H}_γ . Comparing the definitions of \mathcal{V} and \mathcal{H}_γ and noting that test functions are dense in L^2 spaces, we only have to consider the case $\gamma > 0$ and to show that the embedding

$$H^{2,1} \subset H^1_\Gamma(\Omega_1) \times L^2(\Omega_2)$$

is dense. Therefore, we fix $u \in H^1_\Gamma(\Omega_1)$, $v \in L^2(\Omega_2)$, and $\varepsilon > 0$. Using step (i), we find a function $\phi_1 \in V(\Omega_1)$ with $\|u - \phi_1\|_{H^1(\Omega_1)} < \varepsilon/2$. Now, let $\tilde{\phi}_1 \in H^1(\Omega_2)$ such that $\phi_1 = \tilde{\phi}_1$ on I and choose $\phi_2, \phi_3 \in \mathcal{D}(\Omega_2)$ with $\|\tilde{\phi}_1 - \phi_2\|_{L^2(\Omega_2)} < \varepsilon/4$ and $\|v - \phi_3\|_{L^2(\Omega_2)} < \varepsilon/4$. Then we obtain $(\phi_1, \tilde{\phi}_1 - \phi_2 + \phi_3) \in H^{2,1}$ and

$$\|(u, v) - (\phi_1, \tilde{\phi}_1 - \phi_2 + \phi_3)\|_{H^1(\Omega_1) \times L^2(\Omega_2)} < \varepsilon.$$

Note that the embedding $\mathcal{V} \subset \mathcal{H}_\gamma$ is dense and injective, and the same holds for the embedding of \mathcal{H}_γ into $(L^2(\Omega_1))^3 \times (L^2(\Omega_2))^2$. Therefore, all embeddings stated in the lemma are dense. \square

Theorem 2.5. For all $\gamma, \rho, m \geq 0$, the operator \mathcal{A} generates a C_0 -semigroup $(S(t))_{t \geq 0}$ of contractions on \mathcal{H}_γ . Therefore, for any $w_0 \in D(\mathcal{A})$, there exists a unique classical solution $w \in C^1([0, \infty), \mathcal{H}_\gamma) \cap C([0, \infty), D(\mathcal{A}))$ of (13).

Proof. Following a standard approach, we show that \mathcal{A} is dissipative and $1 - \mathcal{A}$ is surjective and apply the theorem of Lumer–Phillips.

Let $w \in D(\mathcal{A})$. As $\mathbb{M} : \mathcal{H}_\gamma \rightarrow \mathcal{H}'_\gamma$ is defined by $\langle \mathbb{M}w, \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = \langle w, \phi \rangle_{\mathcal{H}_\gamma}$, we get

$$\langle \mathcal{A}w, w \rangle_{\mathcal{H}_\gamma} = \langle \mathbb{M}^{-1}\mathbb{A}w, w \rangle_{\mathcal{H}_\gamma} = \langle \mathbb{A}w, w \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma}. \tag{14}$$

By the definition of \mathbb{A} in (12), we immediately obtain

$$\text{Re}\langle \mathbb{A}w, w \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = -\rho \|\nabla w_2\|_{L^2(\Omega_1)}^2 - \beta_0 \|\nabla w_3\|_{L^2(\Omega_1)}^2 - \beta_0 \kappa \|w_3\|_{L^2(\Gamma)}^2 - m \|w_5\|_{L^2(\Omega_2)}^2 \leq 0, \tag{15}$$

which shows that \mathcal{A} is dissipative.

To show that $1 - \mathcal{A}$ is surjective, it suffices by Remark 2.2 (b) to show that $\mathbb{M} - \mathbb{A} : D(\mathcal{A}) \rightarrow \mathcal{H}'_\gamma$ is surjective. Let $f \in \mathcal{H}'_\gamma$. We have to find $w \in D(\mathcal{A})$ such that

$$(\mathbb{M} - \mathbb{A})w = \left(\tilde{M}(D) - \tilde{A}(D) \right) w = f \tag{16}$$

holds in \mathcal{H}'_γ (cf. Remark 2.3). From (16), we obtain

$$\begin{aligned} \beta_1 \Delta^2 w_1 &= \beta_1 \Delta^2 w_2 + f_1, \\ \beta_2 \Delta w_4 &= \beta_2 \Delta w_5 - f_4 \end{aligned} \tag{17}$$

as equality in $(H^{2,1})'$. Replacing this into (16), we get

$$\begin{pmatrix} \rho_1 + \beta_1 \Delta^2 - (\gamma + \rho) \Delta & \mu \Delta & 0 \\ -\mu \Delta & \rho_0 - \beta_0 \Delta & 0 \\ 0 & 0 & \rho_2 + m - \beta_2 \Delta \end{pmatrix} \begin{pmatrix} w_2 \\ w_3 \\ w_5 \end{pmatrix} = \begin{pmatrix} f_2 - f_1 \\ f_3 \\ f_5 - f_4 \end{pmatrix} =: \tilde{f}. \tag{18}$$

We will solve this weakly with respect to the dual pairing $\mathcal{V}'_0 \times \mathcal{V}_0$, where \mathcal{V}_0 is the projection of \mathcal{V} to the components (w_2, w_3, w_5) , that is,

$$\mathcal{V}_0 := \{(w_2, w_3, w_5)^\top : (0, w_2, w_3, 0, w_5)^\top \in \mathcal{V}\}.$$

So we define the sesquilinear form $b : \mathcal{V}_0 \times \mathcal{V}_0 \rightarrow \mathbb{C}$ by

$$\begin{aligned} b((w_2, w_3, w_5), (\phi_2, \phi_3, \phi_5)) &:= \rho_1 \langle w_2, \phi_2 \rangle_{L^2(\Omega_1)} + \beta_1 \langle \Delta w_2, \Delta \phi_2 \rangle_{L^2(\Omega_1)} \\ &\quad + (\gamma + \rho) \langle \nabla w_2, \nabla \phi_2 \rangle_{L^2(\Omega_1)} - \mu \langle \nabla w_2, \nabla \phi_3 \rangle_{L^2(\Omega_1)} \\ &\quad + \mu \langle \nabla w_3, \nabla \phi_2 \rangle_{L^2(\Omega_1)} + \rho_0 \langle w_3, \phi_3 \rangle_{L^2(\Omega_1)} + \beta_0 \langle \nabla w_3, \nabla \phi_3 \rangle_{L^2(\Omega_1)} \\ &\quad + (\rho_2 + m) \langle w_5, \phi_5 \rangle_{L^2(\Omega_2)} + \beta_2 \langle \nabla w_5, \nabla \phi_5 \rangle_{L^2(\Omega_2)}. \end{aligned}$$

Obviously, b is continuous, and a computation of $b((w_2, w_3, w_5), (w_2, w_3, w_5))$ shows that

$$\operatorname{Re} b((w_2, w_3, w_5), (w_2, w_3, w_5)) \geq C \|(w_2, w_3, w_5)\|_{\mathcal{V}_0}^2.$$

As the right-hand side of (18) belongs to \mathcal{V}'_0 , we may apply the theorem of Lax–Milgram to obtain a unique solution $(w_2, w_3, w_5) \in \mathcal{V}_0$ of

$$b((w_2, w_3, w_5), (\phi_2, \phi_3, \phi_5)) = \tilde{f}((\phi_2, \phi_3, \phi_5)).$$

By definition of \mathcal{V}_0 , we have $(w_2, w_5) \in H^{2,1}$. Because

$$\begin{pmatrix} \beta_1 \Delta^2 & 0 \\ 0 & -\beta_2 \Delta \end{pmatrix} : H^{2,1} \rightarrow (H^{2,1})' \quad (19)$$

is an isomorphism due to Remark 2.2 (b), the right-hand side of (17) belongs to $(H^{2,1})'$. By the same reason, there exists a unique $(w_1, w_4) \in H^{2,1}$ such that (17) holds in $(H^{2,1})'$.

Altogether, we have found $w \in \mathcal{V}$ such that (16) holds in \mathcal{V}' , that is,

$$((\mathbb{M} - \mathbb{A})w)(\phi) = f(\phi) \quad (\phi \in \mathcal{V}).$$

As the right-hand side belongs to \mathcal{H}'_γ and \mathcal{V} is dense in \mathcal{H}_γ by Lemma 2.4, also the left-hand side belongs to \mathcal{H}'_γ , and (16) holds in \mathcal{H}'_γ . In particular, $\mathbb{A}w = \mathbb{M}w - f \in \mathcal{H}'_\gamma$, which shows that $w \in D(\mathcal{A})$. Therefore, $1 - \mathcal{A}$ is surjective, and an application of the theorem of Lumer–Phillips finishes the proof. \square

3 | SPECTRAL PROPERTIES AND REGULARITY OF THE SOLUTION

In this section, we study properties of the spectrum of the operator \mathcal{A} defined above and show that functions in its domain have higher regularity. We denote by $\sigma(\mathcal{A})$ and $\rho(\mathcal{A})$ the spectrum and the resolvent set of \mathcal{A} , respectively. Note that due to Theorem 2.5, the operator \mathcal{A} is closed and densely defined.

Proposition 3.1. *For all $\gamma, m, \rho \geq 0$, we have $0 \in \rho(\mathcal{A})$.*

Proof. We show that $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}_\gamma$ is bijective. Let $f \in \mathcal{H}_\gamma$. Then $\mathcal{A}w = f$ is equivalent to

$$\langle \mathbb{A}w, \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = \langle \mathbb{M}f, \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} \quad (\phi \in \mathcal{H}_\gamma). \quad (20)$$

Choosing $\phi = (\phi_1, 0, 0, 0, 0)^\top$ and $\phi = (0, 0, 0, \phi_4, 0)^\top$, we obtain $\beta_1 \Delta^2 w_2 = \beta_1 \Delta^2 f_1$ and $-\beta_2 \Delta w_5 = -\beta_2 \Delta f_4$, respectively, which has the unique solution $w_2 := f_1$ and $w_5 := f_4$ (see (19)). Now choosing $\phi = (0, 0, \phi_3, 0, 0)$, we obtain

$$\beta_0 \langle \nabla w_3, \nabla \phi_3 \rangle_{L^2(\Omega_1)} + \beta_0 \kappa \langle w_3, \phi_3 \rangle_{L^2(\Gamma)} = \langle \mu \Delta f_1 - \rho_0 f_3, \phi_3 \rangle_{L^2(\Omega_1)} \quad (21)$$

for all $\phi_3 \in H^1(\Omega_1)$ with $\phi_3 = 0$ on I . As $\mu \Delta f_1 - \rho_0 f_3 \in L^2(\Omega_1)$, the right-hand side is a continuous conjugate linear functional of ϕ_3 . Let us denote the left-hand side of (21) by $b(w_3, \phi_3)$. Then b is a continuous sesquilinear form in the Hilbert space $\{w_3 \in H^1(\Omega_1) : w_3 = 0 \text{ on } I\}$. From Remark 2.2 (a), we know that the left-hand side is equivalent to the $H^1(\Omega_1)$ -norm, which shows that $b(\cdot, \cdot)$ is coercive. Now an application of the theorem of Lax–Milgram yields the existence of a unique solution w_3 of (21).

For the remaining components w_1 and w_4 , we choose $\phi = (0, \phi_2, 0, 0, \phi_5)^\top$ in (20) and obtain

$$\begin{aligned}
 -\langle (w_1, w_4), (\phi_2, \phi_5) \rangle_{H^{2,1}} &= \mu \langle w_3, \Delta \phi_2 \rangle_{L^2(\Omega_1)} \\
 &+ \rho \langle \nabla w_2, \nabla \phi_2 \rangle_{L^2(\Omega_1)} + m \langle w_5, \phi_5 \rangle_{L^2(\Omega_2)} \\
 &+ \rho_1 \langle f_2, \phi_2 \rangle_{L^2(\Omega_1)} + \gamma \langle \nabla f_2, \nabla \phi_2 \rangle_{L^2(\Omega_1)} + \rho_2 \langle f_5, \phi_5 \rangle_{L^2(\Omega_2)} \\
 &=: R(\phi_2, \phi_5).
 \end{aligned} \tag{22}$$

Because of $(\phi_2, \phi_5) \in H^{2,1}$, the conjugate linear functional $R : H^{2,1} \rightarrow \mathbb{C}$ is well-defined and continuous. By the theorem of Riesz, there exists a unique solution $(w_1, w_4) \in H^{2,1}$ of (22). Setting $w := (w_1, \dots, w_5)^\top$, we obtain $w \in \mathcal{V}$ (note here that $(w_2, w_5)^\top = (f_1, f_4)^\top \in H^{2,1}$), and w is a solution of (20). In particular, $\mathbb{A}w \in \mathcal{H}'_\gamma$ by construction, so we have $w \in D(\mathcal{A})$, and \mathcal{A} is surjective. As the solution w constructed above is unique, we also obtain the injectivity of \mathcal{A} . As $\mathcal{A} : D(\mathcal{A}) \rightarrow \mathcal{H}_\gamma$ is bijective and closed, we get $0 \in \rho(\mathcal{A})$. \square

For the proof of higher regularity of the solution w , we need a priori estimates from the theory of parameter-elliptic boundary value problems as developed, for example, in Agranovich and Vishik.²⁵ We recall the main definitions and results (see Agranovich,²³ Section 7.1). Let $(A(D), B_1(D), \dots, B_m(D))$ be a boundary value problem in some domain $\Omega \subset \mathbb{R}^n$ with $A(D) = \sum_{|\alpha| \leq 2m} a_\alpha \partial^\alpha$ and $B_j(D) = \sum_{|\beta| \leq m_j} b_{j\beta} \partial^\beta$, where $a_\alpha, b_{j\beta} \in \mathbb{C}$ and $m_j < 2m$. Then the principal symbols of A and B_j are defined by $A(i\xi) := \sum_{|\alpha|=2m} a_\alpha (i\xi)^\alpha$ and $B_j(i\xi) := \sum_{|\beta|=m_j} b_{j\beta} (i\xi)^\beta$, respectively. The operator $A(D)$ is called parameter-elliptic if its principal symbol satisfies

$$\lambda - A(i\xi) \neq 0 \quad (\text{Re } \lambda \geq 0, \xi \in \mathbb{R}^n, (\lambda, \xi) \neq 0).$$

The boundary value problem (A, B_1, \dots, B_m) is called parameter-elliptic if $A(D)$ is parameter-elliptic and if the following Shapiro–Lopatinskii condition holds:

Let $x_0 \in \partial\Omega$, and rewrite the boundary value problem in the coordinate system associated with x_0 , which is obtained from the original one by a rotation after which the positive x_n -axis has the direction of the interior normal vector to $\partial\Omega$ at x_0 . Then the trivial solution $w = 0$ is the only stable solution of the ordinary differential equation on the half-line

$$\begin{aligned}
 (\lambda - A(i\xi', \partial_n)) w(x_n) &= 0 \quad (x_n \in (0, \infty)), \\
 B_j(i\xi', \partial_n) w(0) &= 0 \quad (j = 1, \dots, m)
 \end{aligned}$$

for all $\xi' \in \mathbb{R}^{n-1}$ and $\text{Re } \lambda \geq 0$ with $(\xi', \lambda) \neq 0$.

In Agranovich and Vishik,²⁵ Theorem 5.1, the following result was shown:

Theorem 3.2. *Let (A, B_1, \dots, B_m) be parameter-elliptic in Ω . Then for sufficiently large $\lambda_0 > 0$, the boundary value problem*

$$\begin{aligned}
 (\lambda_0 - A(D))u &= f \quad \text{in } \Omega, \\
 B_j(D)u &= g_j \quad \text{on } \partial\Omega, \quad j = 1, \dots, m,
 \end{aligned}$$

has a unique solution $u \in H^{2m}(\Omega)$, and the a priori estimate

$$\|u\|_{H^{2m}(\Omega)} \leq C \left(\|f\|_{L^2(\Omega)} + \sum_{j=1}^m \|g_j\|_{H^{2m-m_j-1/2}(\partial\Omega)} \right)$$

holds with a constant $C > 0$ which depends on λ_0 but not on u or on the data.

Remark 3.3. (a) We will apply this also in the case $\Omega = \Omega_1$, where $\partial\Omega_1 = I \cup \Gamma$. It was shown in Barraza Martínez et al,¹² Remark 4.4, that we may also consider different boundary operators (even with different orders) in I and Γ , respectively. One obtains unique solvability and the above a priori estimate, where now the boundary norm for g_j is given as the sum $\|g_j\|_{H^{2m-m'_j-1/2}(I)} + \|g_j\|_{H^{2m-m''_j-1/2}(\Gamma)}$ with m'_j and m''_j being the order of B_j on I and Γ , respectively.

(b) It is well-known (see, e.g. Agranovich,²³ Subsection 7.1) that the Laplace operator is parameter-elliptic with Dirichlet boundary condition and with Neumann boundary condition. As only the principal part is involved in the

definition of parameter ellipticity, also Δ with mixed boundary condition $\partial_\nu u + \kappa u = 0$ is parameter-elliptic. The same holds for $-\Delta^2$ with boundary conditions $u = \partial_\nu u = 0$ (Agranovich,²³ Remark 7.1.2).

Lemma 3.4. *The operator $A(D) := -\Delta^2$ in Ω_1 , supplemented with the boundary operators $B_1(D)u := \partial_\nu u$ and $B_2(D)u := \partial_\nu \Delta u$, is parameter-elliptic.*

Proof. Let $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}^2$ with $\text{Re} \lambda \geq 0$ and $(\lambda, \xi) \neq 0$. Because of $\lambda - A(i\xi) = \lambda + |\xi|^4 \neq 0$, the operator $-\Delta^2$ is parameter-elliptic. For the Shapiro–Lopatinskii condition, we have to solve the ordinary differential equation

$$(\lambda + (\partial_2^2 - \xi_1^2)^2)w(x_2) = 0 \quad (x_2 > 0), \tag{23}$$

$$\partial_2 w(0) = 0, \tag{24}$$

$$\partial_2^3 w(0) - \xi_1^2 \partial_2 w(0) = 0. \tag{25}$$

Note that by (24), we can replace (25) by $\partial_2^3 w(0) = 0$. Let $\tau_{1,2} = -\sqrt{\xi_1^2 \pm \sqrt{-\lambda}}$ be the two roots of the polynomial $\lambda - A(i\xi_1, \cdot)$ with negative real part. For $\lambda \neq 0$, we have $\tau_1 \neq \tau_2$, and therefore, every stable solution of (23) has the form $w(x_2) = c_1 e^{\tau_1 x_2} + c_2 e^{\tau_2 x_2}$. Inserting this into the initial conditions, we obtain

$$\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_1^3 & \tau_2^3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

As the determinant of this matrix equals $\tau_1 \tau_2 (\tau_2^2 - \tau_1^2) \neq 0$, we get $c_1 = c_2 = 0$ and therefore $w = 0$.

If $\lambda = 0$, we have $\tau_1 = \tau_2 = -|\xi_1|$, and $w(x_2) = (c_1 + c_2 x_2) e^{\tau_1 x_2}$. Now the initial conditions yield

$$\begin{pmatrix} \tau_1 & 1 \\ \tau_1^3 & 3\tau_1^2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0,$$

which implies $w = 0$ again. □

In the following, we will show that $D(\mathcal{A})$ is embedded into a tuple of Sobolev spaces of higher regularity. For the continuity of the embedding, we use the following observation.

Lemma 3.5. *Let $A : H \supset D(A) \rightarrow H$ be a closed operator in the Hilbert space H and let V be a Hilbert space. If $D(A)$ is a subset of V , then we have the continuous embedding $D(A) \subset V$.*

Proof. As A is closed, $D(A)$ with the graph norm is a Hilbert space. We show that $\text{id} : D(A) \rightarrow V \cap H$ is a closed operator. For this, let $(x_n)_{n \in \mathbb{N}} \subset D(A)$ be a sequence with $x_n \rightarrow x$ in $D(A)$ and $x_n \rightarrow y$ in $V \cap H$. Then we obtain $x_n \rightarrow x$ in H by the definition of the graph norm and also $x_n \rightarrow y$ in H by the definition of the norm in $V \cap H$. This yields $x = y$, and $\text{id} : D(A) \rightarrow V \cap H$ is closed and, by the closed graph theorem, continuous. As the embedding $V \cap H \rightarrow V$ is continuous by the definition of the norms, we obtain the continuity of $\text{id} : D(A) \rightarrow V$. □

The elliptic regularity results above are the key for the strong solvability of the transmission problem, that is, for higher regularity of the weak solution.

Theorem 3.6. *Let $\gamma, \rho, m \geq 0$. Then the following embeddings are continuous.*

- (i) $D(\mathcal{A}) \subset H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2)$,
- (ii) $D(\mathcal{A}^2) \subset H^4(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2)$,
- (iii) $D(\mathcal{A}) \subset H^4(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2)$ for $\gamma = 0$.

In consequence, if $w_0 \in D(\mathcal{A}^2)$, then $w(t) := S(t)w_0$ ($t \geq 0$) is the unique solution of problem (1)–(8) and satisfies the boundary and transmission conditions in the strong sense of traces. In the case $\gamma = 0$, we get the same result even for $w_0 \in D(\mathcal{A})$.

Proof. (i) Let $w \in D(\mathcal{A})$ and $f := \mathcal{A}w$. First, we show $w_3 \in H^2(\Omega_1)$. As in (20), we get

$$\langle \mathbb{A}w, \phi \rangle_{H'_\gamma \times \mathcal{H}_\gamma} = \langle \mathbb{M}f, \phi \rangle_{H'_\gamma \times \mathcal{H}_\gamma} \quad (\phi \in \mathcal{H}_\gamma). \tag{26}$$

As we already have seen in (21), for $\phi_3 \in H^1(\Omega_1)$ with $\phi_3 = 0$ on I , we have

$$\langle g, \phi_3 \rangle_{L^2(\Omega_1)} = \langle \nabla w_3, \nabla \phi_3 \rangle_{L^2(\Omega_1)} + \langle \kappa w_3, \phi_3 \rangle_{L^2(\Gamma)},$$

where $g := 1/\beta_0 (\mu \Delta w_2 - \rho_0 f_3)$. By Theorem 3.2 and Remark 3.3, there exists some $\lambda_0 > 0$ such that the problem

$$\begin{aligned} (\lambda_0 - \Delta) \widetilde{w}_3 &= \lambda_0 w_3 + g \quad \text{in } \Omega_1, \\ \partial_\nu \widetilde{w}_3 &= -\kappa w_3 \quad \text{on } \Gamma, \\ \widetilde{w}_3 &= 0 \quad \text{on } I \end{aligned}$$

has a unique solution $\widetilde{w}_3 \in H^2(\Omega_1)$. Integration by parts shows that $z := \widetilde{w}_3 - w_3$ satisfies

$$0 = \langle (\lambda_0 - \Delta) \widetilde{w}_3 - \lambda_0 w_3 - g, \phi_3 \rangle_{L^2(\Omega_1)} = \lambda_0 \langle z, \phi_3 \rangle_{L^2(\Omega_1)} + \langle \nabla z, \nabla \phi_3 \rangle_{L^2(\Omega_1)}$$

for all $\phi_3 \in H^1(\Omega_1)$ with $\phi_3 = 0$ on I . Choosing $\phi_3 = z$, we get $w_3 = \widetilde{w}_3 \in H^2(\Omega_1)$.

Now, we prove $w_4 \in H^2(\Omega_2)$. We choose $\phi = (0, 0, 0, 0, \phi_5)$ with $\phi_5 \in H_0^1(\Omega_2)$ in (26). As in (22), we obtain

$$\langle \nabla w_4, \nabla \phi_5 \rangle_{L^2(\Omega_2)} = \langle \tilde{g}, \phi_5 \rangle_{L^2(\Omega_2)},$$

where $\tilde{g} := -1/\beta_2 (m w_5 + \rho_2 f_5)$. By Theorem 3.2 and Remark 3.3 (b), there exists a unique $\widetilde{w}_4 \in H^2(\Omega_2)$ such that

$$\begin{aligned} -\Delta \widetilde{w}_4 &= \tilde{g} \quad \text{in } \Omega_2 \\ \widetilde{w}_4 &= w_1 \quad \text{on } I. \end{aligned}$$

Therefore, $z := \widetilde{w}_4 - w_4 \in H_0^1(\Omega_2)$ fulfills

$$0 = \langle -\Delta \widetilde{w}_4 - \tilde{g}, \phi_5 \rangle_{L^2(\Omega_2)} = \langle \nabla z, \nabla \phi_5 \rangle_{L^2(\Omega_2)}$$

for all $\phi_5 \in H_0^1(\Omega_2)$. By choosing $\phi_5 = z$, we obtain $w_4 = \widetilde{w}_4 \in H^2(\Omega_2)$.

(ii) Now, let $w \in D(\mathcal{A}^2)$. We show $w_1 \in H^4(\Omega_1)$. In (26), we can choose $\phi = (0, \phi_2, 0, 0, 0)$ for all $\phi_2 \in H^2(\Omega_1)$ with $\phi_2 = \partial_\nu \phi_2 = 0$ on Γ and $\partial_\nu \phi_2 = 0$ on I . Integration by parts yields to

$$\begin{aligned} \langle \Delta w_1, \Delta \phi_2 \rangle_{L^2(\Omega_1)} &= \frac{1}{\beta_1} (-\mu \langle w_3, \Delta \phi_2 \rangle_{L^2(\Omega_1)} - \langle \nabla(\rho w_2 + \gamma f_2), \nabla \phi_2 \rangle_{L^2(\Omega_1)} - \rho_1 \langle f_2, \phi_2 \rangle_{L^2(\Omega_1)}) \\ &= \langle g^*, \phi_2 \rangle_{L^2(\Omega_1)} - \langle h, \phi_2 \rangle_{L^2(I)}, \end{aligned}$$

where $g^* := 1/\beta_1 (\Delta(-\mu w_3 + \rho w_2 + \gamma f_2) - \rho_1 f_2)$ and $h := 1/\beta_1 \partial_\nu (-\mu w_3 + \rho w_2 + \gamma f_2)$. By Theorem 3.2 and Lemma 3.4, there is a $\lambda_0 > 0$ such that there exists a unique solution $\widetilde{w}_1 \in H^4(\Omega_1)$ of the boundary value problem

$$\begin{aligned} (\lambda_0 + \Delta^2) \widetilde{w}_1 &= \lambda_0 w_1 + g^* \quad \text{in } \Omega_1, \\ \widetilde{w}_1 &= \partial_\nu \widetilde{w}_1 = 0 \quad \text{on } \Gamma, \\ \partial_\nu \widetilde{w}_1 &= 0 \quad \text{on } I, \\ \partial_\nu (\Delta \widetilde{w}_1) &= h \quad \text{on } I. \end{aligned}$$

Note that $g^* \in L^2(\Omega_1)$ and $h \in H^1(\Omega_1)$ since $w \in D(\mathcal{A}^2)$. Therefore, all boundary conditions hold in the trace sense. Using integration by parts, $z := \widetilde{w}_1 - w_1$ fulfills

$$0 = \langle (\lambda_0 + \Delta^2) \widetilde{w}_1 - \lambda_0 w_1 - g^*, \phi_2 \rangle_{L^2(\Omega_1)} = \lambda_0 \langle z, \phi_2 \rangle_{L^2(\Omega_1)} + \langle \Delta z, \Delta \phi_2 \rangle_{L^2(\Omega_1)}$$

for all $\phi_2 \in H^2(\Omega_1)$ with $\phi_2 = \partial_\nu \phi_2 = 0$ on Γ and $\partial_\nu \phi_2 = 0$ on I . By choosing $\phi_2 = z$, we obtain $w_1 = \widetilde{w}_1 \in H^4(\Omega_1)$.

(iii) Let $\gamma = 0$ and $w \in D(\mathcal{A})$. Following the proof of (ii), we get $w_1 \in H^4(\Omega_1)$.

Due to Lemma 3.5, all embeddings are continuous. □

Remark 3.7. By the last proof, we see that the corresponding assertions of Theorem 3.6 hold true if the plate is isothermal.

Corollary 3.8. *For all $\gamma, \rho, m \geq 0$, we have the continuous embedding*

$$D(\mathcal{A}) \subset H^3(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2).$$

Proof. From Theorem 2.5, we know that $\mathcal{A} : \mathcal{H}_\gamma \supset D(\mathcal{A}) \rightarrow \mathcal{H}_\gamma$ is the generator of a C_0 -semigroup of contractions on \mathcal{H}_γ . So, $-\mathcal{A}$ is an m -accretive operator (see Section 4.3 from Lunardi²⁶). By Corollary 4.30 and Corollary 4.37 from Lunardi,²⁶ we obtain $D(\mathcal{A}) = (\mathcal{H}_\gamma, D(\mathcal{A}^2))_{\frac{1}{2}, 2}$. Due to Theorem 3.6, it holds

$$\begin{aligned} D(\mathcal{A}) &\subset H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2), \\ D(\mathcal{A}^2) &\subset H^4(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2). \end{aligned}$$

By Proposition 5.12 from Barraza Martínez et al,²⁷ we have

$$D(\mathcal{A}) \subset (H^2(\Omega_1), H^4(\Omega_1))_{\frac{1}{2}, 2} \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2).$$

By Theorem 1 of Section 4.3.1. from Triebel,²⁸ we get $(H^2(\Omega_1), H^4(\Omega_1))_{\frac{1}{2}, 2} = H^3(\Omega_1)$. □

The following result allows us to affirm that the spectrum $\sigma(\mathcal{A})$ of \mathcal{A} coincides with its point spectrum $\sigma_p(\mathcal{A})$.

Proposition 3.9. *The operator $\mathcal{A}^{-1} : \mathcal{H}_\gamma \rightarrow \mathcal{H}_\gamma$ is compact.*

Proof. By Corollary 3.8 and the Rellich–Kondrachov theorem, we have

$$\begin{aligned} D(\mathcal{A}) &\subset H^3(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_1) \times H^2(\Omega_2) \times H^1(\Omega_2) \\ &\stackrel{c}{\subset} H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2). \end{aligned}$$

As \mathcal{H}_γ is a closed subspace of $H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2)$, we get $D(\mathcal{A}) \stackrel{c}{\subset} \mathcal{H}_\gamma$. Therefore, the identity operator $\text{id} : D(\mathcal{A}) \rightarrow \mathcal{H}_\gamma$ is compact. Proposition 3.1 implies the continuity of the operator $\mathcal{A}^{-1} : \mathcal{H}_\gamma \rightarrow D(\mathcal{A})$. In consequence, $\mathcal{A}^{-1} = \text{id} \circ \mathcal{A}^{-1} : \mathcal{H}_\gamma \rightarrow \mathcal{H}_\gamma$ is a compact operator. □

Proposition 3.10. *If $\gamma, \rho \geq 0$ and $m > 0$, then $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. Let us suppose $\gamma \geq 0$ and $m > 0$. Since \mathcal{A}^{-1} is compact, the spectrum of \mathcal{A} consists of eigenvalues only. Thus, we have to establish that there are no purely imaginary eigenvalues. Let $0 \neq \lambda \in \mathbb{R}$ and $w \in D(\mathcal{A})$ with $\mathcal{A}w = i\lambda w$. By (14), we have

$$\langle \mathbb{A}w, \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = i\lambda \langle w, \phi \rangle_{\mathcal{H}_\gamma} \quad (\phi \in \mathcal{H}_\gamma). \tag{27}$$

Using Remark 2.3, we see that $i\lambda w_1 = w_2$ and $i\lambda w_4 = w_5$. Choosing $\phi = w$ in (27), we obtain

$$\lambda \text{Im} \langle \mathbb{A}w, w \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = \lambda^2 \|w\|_{\mathcal{H}_\gamma}^2, \tag{28}$$

and

$$\begin{aligned} 0 &= \text{Re} \langle \mathbb{A}w, w \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} \\ &= -\rho \|\nabla w_2\|_{L^2(\Omega_1)}^2 - \beta_0 \|\nabla w_3\|_{L^2(\Omega_1)}^2 - \beta_0 \kappa \|w_3\|_{L^2(\Gamma)}^2 - m \|w_5\|_{L^2(\Omega_2)}^2 \end{aligned} \tag{29}$$

as in the proof of Theorem 2.5. For $\rho > 0$, we get $w_2 = w_3 = w_5 = 0$ due to (29) and Poincaré’s inequality. We conclude $w = 0$.

If $\rho = 0$, we have $w_3 = w_5 = 0$, and therefore, $w_4 = 0$, that is, $w = (w_1, w_2, 0, 0, 0)^\top$. Equality (28) leads to

$$\begin{aligned} 2\beta_1 \|\Delta w_2\|_{L^2(\Omega_1)}^2 &= \lambda \operatorname{Im} \langle \mathbb{A}w, w \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} \\ &= \beta_1 \|\Delta w_2\|_{L^2(\Omega_1)}^2 + \lambda^2 \left(\rho_1 \|w_2\|_{L^2(\Omega_1)}^2 + \gamma \|\nabla w_2\|_{L^2(\Omega_1)}^2 \right); \end{aligned}$$

therefore, we have

$$\beta_1 \|\Delta w_2\|_{L^2(\Omega_1)}^2 = \lambda^2 \left(\rho_1 \|w_2\|_{L^2(\Omega_1)}^2 + \gamma \|\nabla w_2\|_{L^2(\Omega_1)}^2 \right). \quad (30)$$

Now, we choose $\phi = (0, 0, \phi_3, 0, 0) \in \mathcal{H}_\gamma$ in (27) and obtain $0 = \mu \langle \Delta w_2, \phi_3 \rangle_{L^2(\Omega_1)}$ for all $\phi_3 \in H^1(\Omega_1)$ with $\phi_3 = 0$ on I . In consequence, we get $\Delta w_2 = 0$. From (30), it follows that $w_2 = 0$. Finally, for any case of ρ , we have shown that $w = 0$. \square

Remark 3.8. (a) Proposition 3.10 also holds true if the plate is isothermal and $\rho > 0$. In the case $\rho = 0$, the above proof does not work in its present form in the isothermal case.

(b) We will see in the proof of Theorem 5.2 that $i\mathbb{R} \subset \rho(\mathcal{A})$ holds also for $m = 0$.

4 | EXPONENTIAL STABILITY IN THE CASE OF DAMPED MEMBRANE

In this section, we will prove the exponential stability of the solution of system (1)–(8) when the membrane is damped and when $\rho > 0$ or $\rho = \gamma = 0$.

Theorem 4.1. *If $m > 0$ and $\rho > 0$, then for all $\gamma \geq 0$, the semigroup $(S(t))_{t \geq 0}$ generated by \mathcal{A} is exponentially stable, that is, there exist constants $C \geq 1$ and $\delta > 0$ such that $\|S(t)\|_{\mathcal{L}(\mathcal{H}_\gamma)} \leq Ce^{-\delta t}$ for all $t \geq 0$.*

Proof. Let $\gamma \geq 0$, $m, \rho > 0$, and $\lambda \in \mathbb{R}$. For the proof, we use the characterization of exponential stability by Gearhart and Prüss (see Prüss²⁹) which tells us that the semigroup is exponentially stable if $i\mathbb{R} \subset \rho(\mathcal{A})$, and there is a constant $C > 0$, which does not depend on $\lambda \in \mathbb{R}$, such that

$$\|(i\lambda - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H}_\gamma)} \leq C. \quad (31)$$

As $i\mathbb{R} \subset \rho(\mathcal{A})$ by Proposition 3.10, we have to show (31). To see this, let $w \in D(\mathcal{A})$, $\lambda \in \mathbb{R}$, and

$$(i\lambda - \mathcal{A})w =: f. \quad (32)$$

To prove (31), it is sufficient to establish that there is a constant $C > 0$ such that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ with

$$\|w\|_{\mathcal{H}_\gamma}^2 \leq C_\varepsilon \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \varepsilon C \|w\|_{\mathcal{H}_\gamma}^2. \quad (33)$$

Multiplying the resolvent Equation (32) by w , we obtain

$$i\lambda \|w\|_{\mathcal{H}_\gamma}^2 - \langle \mathcal{A}w, w \rangle_{\mathcal{H}_\gamma} = \langle f, w \rangle_{\mathcal{H}_\gamma}.$$

In consequence,

$$-\operatorname{Re} \langle \mathcal{A}w, w \rangle_{\mathcal{H}_\gamma} = \operatorname{Re} \langle f, w \rangle_{\mathcal{H}_\gamma} \leq \left| \langle f, w \rangle_{\mathcal{H}_\gamma} \right|.$$

As we have seen in the proof of Theorem 2.5, this means

$$\rho \|\nabla w_2\|_{L^2(\Omega_1)}^2 + \beta_0 \|\nabla w_3\|_{L^2(\Omega_1)}^2 + \beta_0 \kappa \|w_3\|_{L^2(\Gamma)}^2 + m \|w_5\|_{L^2(\Omega_2)}^2 \leq \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma}. \quad (34)$$

From Remark 2.2 and (32), it follows that

$$\begin{aligned} (i\lambda w_1 - w_2, i\lambda w_4 - w_5) &= (f_1, f_4), \\ \langle \mathbb{M}(i\lambda w - f), \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} &= \langle \mathbb{A}w, \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma}. \end{aligned} \quad (35)$$

Choosing $\phi = (0, w_1, 0, 0, w_4)$, we get

$$\begin{aligned} & \rho_1 \langle i\lambda w_2 - f_2, w_1 \rangle_{L^2(\Omega_1)} + \gamma \langle \nabla(i\lambda w_2 - f_2), \nabla w_1 \rangle_{L^2(\Omega_1)} + \rho_2 \langle i\lambda w_5 - f_5, w_4 \rangle_{L^2(\Omega_2)} \\ &= - \|(w_1, w_4)\|_{H^{2,1}}^2 - \mu \langle w_3, \Delta w_1 \rangle_{L^2(\Omega_1)} - \rho \langle \nabla w_2, \nabla w_1 \rangle_{L^2(\Omega_1)} - m \langle w_5, w_4 \rangle_{L^2(\Omega_2)}. \end{aligned}$$

Taking (35) into account, we get

$$\begin{aligned} & \rho_1 \langle i\lambda w_2, w_1 \rangle_{L^2(\Omega_1)} + \gamma \langle i\lambda \nabla w_2, \nabla w_1 \rangle_{L^2(\Omega_1)} + \rho_2 \langle i\lambda w_5, w_4 \rangle_{L^2(\Omega_2)} \\ &= - \rho_1 \langle w_2, w_2 + f_1 \rangle_{L^2(\Omega_1)} - \gamma \langle \nabla w_2, \nabla(w_2 + f_1) \rangle_{L^2(\Omega_1)} - \rho_2 \langle w_5, w_5 + f_4 \rangle_{L^2(\Omega_2)} \\ &= - \rho_1 \|w_2\|_{L^2(\Omega_1)}^2 - \gamma \|\nabla w_2\|_{L^2(\Omega_1)}^2 - \rho_2 \|w_5\|_{L^2(\Omega_2)}^2 - \rho_1 \langle w_2, f_1 \rangle_{L^2(\Omega_1)} - \gamma \langle \nabla w_2, \nabla f_1 \rangle_{L^2(\Omega_1)} - \rho_2 \langle w_5, f_4 \rangle_{L^2(\Omega_2)}. \end{aligned}$$

Using the last equality, Poincaré's inequality, and inequality (34), we obtain

$$\begin{aligned} \|(w_1, w_4)\|_{H^{2,1}}^2 &= \rho_1 \|w_2\|_{L^2(\Omega_1)}^2 + \gamma \|\nabla w_2\|_{L^2(\Omega_1)}^2 + \rho_2 \|w_5\|_{L^2(\Omega_2)}^2 \\ &\quad + \rho_1 \langle w_2, f_1 \rangle_{L^2(\Omega_1)} + \gamma \langle \nabla w_2, \nabla f_1 \rangle_{L^2(\Omega_1)} + \rho_2 \langle w_5, f_4 \rangle_{L^2(\Omega_2)} \\ &\quad + \rho_1 \langle f_2, w_1 \rangle_{L^2(\Omega_1)} + \gamma \langle \nabla f_2, \nabla w_1 \rangle_{L^2(\Omega_1)} + \rho_2 \langle f_5, w_4 \rangle_{L^2(\Omega_2)} \\ &\quad - \mu \langle w_3, \Delta w_1 \rangle_{L^2(\Omega_1)} - \rho \langle \nabla w_2, \nabla w_1 \rangle_{L^2(\Omega_1)} - m \langle w_5, w_4 \rangle_{L^2(\Omega_2)} \\ &\leq C \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \mu \left| \langle w_3, \Delta w_1 \rangle_{L^2(\Omega_1)} \right| + \rho \left| \langle \nabla w_2, \nabla w_1 \rangle_{L^2(\Omega_1)} \right| + m \left| \langle w_5, w_4 \rangle_{L^2(\Omega_2)} \right|. \end{aligned}$$

Now, we estimate the remaining terms on the right-hand side. Due to Young's and Poincaré's inequality, we have

$$\begin{aligned} \left| \langle w_3, \Delta w_1 \rangle_{L^2(\Omega_1)} \right| &\leq C_\varepsilon \|w_3\|_{L^2(\Omega_1)}^2 + \varepsilon \|\Delta w_1\|_{L^2(\Omega_1)}^2 \\ &\leq C_\varepsilon \|\nabla w_3\|_{L^2(\Omega_1)}^2 + \varepsilon C \|w\|_{\mathcal{H}_\gamma}^2 \\ &\leq C_\varepsilon \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \varepsilon C \|w\|_{\mathcal{H}_\gamma}^2 \end{aligned}$$

for all $\varepsilon > 0$. In the last step, we used again inequality (34). Similarly, we get

$$\left| \langle \nabla w_2, \nabla w_1 \rangle_{L^2(\Omega_1)} \right| \leq C_\varepsilon \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \varepsilon C \|w\|_{\mathcal{H}_\gamma}^2 \quad \text{and} \quad \left| \langle w_5, w_4 \rangle_{L^2(\Omega_2)} \right| \leq C_\varepsilon \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \varepsilon C \|w\|_{\mathcal{H}_\gamma}^2$$

for all $\varepsilon > 0$. Altogether, we have shown inequality (33). Therefore, the semigroup is exponentially stable. \square

Remark 4.2. Our plate–membrane system also has exponential stability when the plate is isothermal.

We will now show that the thermoelastic plate–membrane system without rotational inertia has exponential stability if the membrane is damped ($m > 0$) even without structural damping ($\rho = 0$). Under this situation, the thermal effect on the plate is enough for exponential decay. For the proof of this result, the following lemma (Theorem 1.4.4 in Liu and Zheng³⁰) will be useful.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^1 -boundary. Then, for any function $u \in H^1(\Omega)$, the following estimate holds:*

$$\|u\|_{L^2(\partial\Omega)} \leq C \|u\|_{H^1(\Omega)}^{1/2} \|u\|_{L^2(\Omega)}^{1/2}.$$

Below, we will also apply this lemma to $\partial_\nu u$, Δu , and $\partial_\nu \Delta u$ with u being sufficiently smooth.

Theorem 4.4. *If $\gamma = 0$, $\rho = 0$, and $m > 0$, then the semigroup $(S(t))_{t \geq 0}$ generated by the operator \mathcal{A} is exponentially stable.*

Proof. Again we use the Gearhart–Prüss criterion, so we study the resolvent in \mathcal{H}_0 (note $\gamma = 0$) on the imaginary axis. From Proposition 3.10, we have $i\mathbb{R} \subset \rho(\mathcal{A})$. Let us suppose (31) is not true. Then, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}$

and a sequence $(w^n)_{n \in \mathbb{N}} \subset D(\mathcal{A})$ with $\|w^n\|_{\mathcal{H}_0} = 1$ such that

$$\|(i\lambda_n - \mathcal{A})w^n\|_{\mathcal{H}_0} \rightarrow 0 \quad (n \rightarrow \infty). \quad (36)$$

As the resolvent is holomorphic and therefore bounded on compact subsets of the imaginary axis, we see that the sequence $(\lambda_n)_{n \in \mathbb{N}}$ is unbounded, so we may assume $\lim_{n \rightarrow \infty} |\lambda_n| = \infty$. For $f^n := (i\lambda_n - \mathcal{A})w^n$, we obtain

$$i\lambda_n w_1^n - w_2^n = f_1^n, \quad (37)$$

$$i\lambda_n \rho_1 w_2^n + \beta_1 \Delta^2 w_1^n + \mu \Delta w_3^n = \rho_1 f_2^n, \quad (38)$$

$$i\lambda_n \rho_0 w_3^n - \mu \Delta w_2^n - \beta_0 \Delta w_3^n = \rho_0 f_3^n, \quad (39)$$

$$i\lambda_n w_4^n - w_5^n = f_4^n, \quad (40)$$

$$i\lambda_n \rho_2 w_5^n - \beta_2 \Delta w_4^n + m w_5^n = \rho_2 f_5^n, \quad (41)$$

and since

$$\begin{aligned} \|(i\lambda_n - \mathcal{A})w^n\|_{\mathcal{H}_0}^2 &= \|f^n\|_{\mathcal{H}_0}^2 = \beta_1 \|\Delta f_1^n\|_{L^2(\Omega_1)}^2 + \beta_2 \|\nabla f_4^n\|_{L^2(\Omega_2)}^2 \\ &\quad + \rho_1 \|f_2^n\|_{L^2(\Omega_1)}^2 + \rho_2 \|f_5^n\|_{L^2(\Omega_2)}^2 + \rho_0 \|f_3^n\|_{L^2(\Omega_1)}^2, \end{aligned} \quad (42)$$

we obtain from (36)–(42)

$$i\lambda_n \Delta w_1^n - \Delta w_2^n \rightarrow 0 \quad \text{in } L^2(\Omega_1), \quad (43)$$

$$i\lambda_n \rho_1 w_2^n + \beta_1 \Delta^2 w_1^n + \mu \Delta w_3^n \rightarrow 0 \quad \text{in } L^2(\Omega_1), \quad (44)$$

$$i\lambda_n \rho_0 w_3^n - \mu \Delta w_2^n - \beta_0 \Delta w_3^n \rightarrow 0 \quad \text{in } L^2(\Omega_1), \quad (45)$$

$$i\lambda_n \nabla w_4^n - \nabla w_5^n \rightarrow 0 \quad \text{in } L^2(\Omega_2), \quad (46)$$

$$i\lambda_n \rho_2 w_5^n - \beta_2 \Delta w_4^n + m w_5^n \rightarrow 0 \quad \text{in } L^2(\Omega_2) \quad (47)$$

for $n \rightarrow \infty$. From (15), it follows that

$$\begin{aligned} \operatorname{Re} \langle (i\lambda_n - \mathcal{A})w^n, w^n \rangle_{\mathcal{H}_0} &= \operatorname{Re} \left[i\lambda_n \|w^n\|_{\mathcal{H}_0}^2 - \langle \mathcal{A}w^n, w^n \rangle_{\mathcal{H}_0} \right] \\ &= -\operatorname{Re} \langle \mathcal{A}w^n, w^n \rangle_{\mathcal{H}_0} \\ &= m \|w_5^n\|_{L^2(\Omega_2)}^2 + \beta_0 \|\nabla w_3^n\|_{L^2(\Omega_1)}^2 + \beta_0 \kappa \|w_3^n\|_{L^2(\Gamma)}^2. \end{aligned}$$

As $(i\lambda_n - \mathcal{A})w^n$ converges to zero in \mathcal{H}_0 and $(w^n)_{n \in \mathbb{N}}$ is a bounded sequence in \mathcal{H}_0 , the right-hand side of the last equality tends to zero. Therefore,

$$w_5^n \rightarrow 0 \quad \text{in } L^2(\Omega_2) \quad \text{and} \quad w_3^n \rightarrow 0 \quad \text{in } H^1(\Omega_1). \quad (48)$$

Note also that the sequences $(w_1^n)_{n \in \mathbb{N}} \subset H^2(\Omega_1)$, $(w_2^n)_{n \in \mathbb{N}} \subset L^2(\Omega_1)$ and $(w_4^n)_{n \in \mathbb{N}} \subset H^1(\Omega_2)$ are bounded because of $\|w^n\|_{\mathcal{H}_0} = 1$. As $f^n \rightarrow 0$ in \mathcal{H}_0 , we obtain $f_1^n \rightarrow 0$ in $H^2(\Omega_1)$.

The convergences (47), (45), and (48) imply

$$\frac{1}{|\lambda_n|} \Delta w_4^n \rightarrow 0 \quad \text{in } L^2(\Omega_2) \quad (49)$$

and

$$\frac{\mu}{|\lambda_n|} \Delta w_2^n + \frac{\beta_0}{|\lambda_n|} \Delta w_3^n \rightarrow 0 \quad \text{in } L^2(\Omega_1). \quad (50)$$

As $(\nabla w_4^n)_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(\Omega_2)$, we get from (46) that

$$\langle i\lambda_n \nabla w_4^n - \nabla w_5^n, \nabla w_4^n \rangle_{L^2(\Omega_2)} \rightarrow 0.$$

Using integration by parts, we have

$$\langle i\lambda_n \nabla w_4^n - \nabla w_5^n, \nabla w_4^n \rangle_{L^2(\Omega_2)} = i\lambda_n \|\nabla w_4^n\|_{L^2(\Omega_2)}^2 + \langle w_5^n, \Delta w_4^n \rangle_{L^2(\Omega_2)} + \langle w_5^n, \partial_\nu w_4^n \rangle_{L^2(I)}. \quad (51)$$

With the interpolation inequality, we obtain

$$\begin{aligned} \|w_2^n\|_{H^1(\Omega_1)} &\leq C \|w_2^n\|_{H^2(\Omega_1)}^{1/2} \|w_2^n\|_{L^2(\Omega_1)}^{1/2} \\ &= C \|i\lambda_n w_1^n - f_1^n\|_{H^2(\Omega_1)}^{1/2} \|w_2^n\|_{L^2(\Omega_1)}^{1/2} \\ &\leq C \left(|\lambda_n|^{1/2} \|w_1^n\|_{H^2(\Omega_1)}^{1/2} + \|f_1^n\|_{H^2(\Omega_1)}^{1/2} \right) \|w_2^n\|_{L^2(\Omega_1)}^{1/2} \end{aligned}$$

and thus

$$\frac{\|w_2^n\|_{H^1(\Omega_1)}}{|\lambda_n|^{1/2}} \leq C \left(\|w_1^n\|_{H^2(\Omega_1)}^{1/2} + \frac{\|f_1^n\|_{H^2(\Omega_1)}^{1/2}}{|\lambda_n|^{1/2}} \right) \|w_2^n\|_{L^2(\Omega_1)}^{1/2} \leq C. \quad (52)$$

By trace theorem, Lemma 4.3 applied to $\partial_\nu w_4^n$ and $w_5^n = w_2^n$ on I , we get

$$\begin{aligned} \left| \langle w_5^n, \partial_\nu w_4^n \rangle_{L^2(I)} \right| &\leq \|w_5^n\|_{L^2(I)} \|\partial_\nu w_4^n\|_{L^2(I)} \\ &\leq C \|w_2^n\|_{H^1(\Omega_1)} \|w_4^n\|_{H^2(\Omega_2)}^{1/2} \|w_4^n\|_{H^1(\Omega_2)}^{1/2} \end{aligned}$$

and therefore

$$\begin{aligned} \left| \frac{1}{\lambda_n} \langle w_5^n, \partial_\nu w_4^n \rangle_{L^2(I)} \right| &\leq C \frac{\|w_2^n\|_{H^1(\Omega_1)}}{|\lambda_n|^{1/2}} \frac{\|w_4^n\|_{H^2(\Omega_2)}^{1/2}}{|\lambda_n|^{1/2}} \|w_4^n\|_{H^1(\Omega_2)}^{1/2} \\ &\leq C \frac{\|w_4^n\|_{H^2(\Omega_2)}^{1/2}}{|\lambda_n|^{1/2}}. \end{aligned} \quad (53)$$

By (41), we have

$$\Delta w_4^n = \frac{1}{\beta_2} (i\lambda_n \rho_2 w_5^n + m w_5^n - \rho_2 f_5^n)$$

and as $w_4^n = w_1^n$ on I , elliptic regularity for the Dirichlet Laplacian (Theorem 3.2) implies

$$\begin{aligned} \|w_4^n\|_{H^2(\Omega_2)} &\leq C \left(\|i\lambda_n \rho_2 w_5^n + m w_5^n - \rho_2 f_5^n\|_{L^2(\Omega_2)} + \|w_1^n\|_{H^{3/2}(I)} \right) \\ &\leq C \left(|\lambda_n| \|w_5^n\|_{L^2(\Omega_2)} + \|w_5^n\|_{L^2(\Omega_2)} + \|f_5^n\|_{L^2(\Omega_2)} + \|w_1^n\|_{H^2(\Omega_1)} \right). \end{aligned}$$

Thus,

$$\frac{\|w_4^n\|_{H^2(\Omega_2)}}{|\lambda_n|} \leq C \left(\|w_5^n\|_{L^2(\Omega_2)} + \frac{\|w_5^n\|_{L^2(\Omega_2)}}{|\lambda_n|} + \frac{\|f_5^n\|_{L^2(\Omega_2)}}{|\lambda_n|} + \frac{\|w_1^n\|_{H^2(\Omega_1)}}{|\lambda_n|} \right).$$

Because of $\lim_{n \rightarrow \infty} \|w_5^n\|_{L^2(\Omega_2)} = 0$, $\lim_{n \rightarrow \infty} \|f_5^n\|_{L^2(\Omega_2)} = 0$ and $\|w_1^n\|_{H^2(\Omega_1)} \leq C$, we get

$$\frac{\|w_4^n\|_{H^2(\Omega_2)}}{|\lambda_n|} \rightarrow 0 \quad (n \rightarrow \infty).$$

From (49), (51), (53), and (54), we obtain

$$\nabla w_4^n \rightarrow 0 \quad \text{in } L^2(\Omega_2).$$

Dividing (43) by $|\lambda_n|$, we have

$$\pm i\Delta w_1^n - \frac{1}{|\lambda_n|}\Delta w_2^n \rightarrow 0 \text{ in } L^2(\Omega_1)$$

and given that $\|\Delta w_1^n\|_{L^2(\Omega_1)} \leq C$, then $(1/|\lambda_n|\Delta w_2^n)_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(\Omega_1)$. Consequently, the limit (50) implies that $(1/|\lambda_n|\Delta w_3^n)_{n \in \mathbb{N}}$ is a bounded sequence in $L^2(\Omega_1)$. From (44), it follows that

$$\pm i\rho_1 w_2^n + \frac{\beta_1}{|\lambda_n|}\Delta^2 w_1^n + \frac{\mu}{|\lambda_n|}\Delta w_3^n \rightarrow 0 \text{ in } L^2(\Omega_1).$$

Hence,

$$\frac{1}{|\lambda_n|}\|\Delta^2 w_1^n\|_{L^2(\Omega_1)} \leq C.$$

Due to (38) and $w^n \in D(\mathcal{A})$ (see Theorem 3.6), w_1^n satisfies the problem

$$\begin{cases} (\eta_0 + \Delta^2)w_1^n = \eta_0 w_1^n + \beta_1^{-1}z^n & \text{in } \Omega_1, \\ w_1^n = 0, \partial_\nu w_1^n = 0 & \text{on } \Gamma, \\ \partial_\nu w_1^n = 0, \partial_\nu(\Delta w_1^n) = \beta_1^{-1}(-\beta_2 \partial_\nu w_4^n - \mu \partial_\nu w_3^n) & \text{on } I, \end{cases}$$

with

$$z^n := \lambda_n^2 \rho_1 w_1^n - \mu \Delta w_3^n + i\lambda_n \rho_1 f_1^n + \rho_1 f_2^n.$$

Then, Theorem 3.2 implies

$$\begin{aligned} \|w_1^n\|_{H^4(\Omega_1)} &\leq C \left(\|\eta_0 w_1^n + \Delta^2 w_1^n\|_{L^2(\Omega_1)} + \|\beta_1^{-1}(-\beta_2 \partial_\nu w_4^n - \mu \partial_\nu w_3^n)\|_{H^{1/2}(I)} \right) \\ &\leq C \left(\|w_1^n\|_{L^2(\Omega_1)} + \|\Delta^2 w_1^n\|_{L^2(\Omega_1)} + \|\partial_\nu w_4^n\|_{H^{1/2}(I)} + \|\partial_\nu w_3^n\|_{H^{1/2}(I)} \right). \end{aligned} \tag{57}$$

By the trace theorem and (54), we have

$$\frac{\|\partial_\nu w_4^n\|_{H^{1/2}(I)}}{|\lambda_n|} \leq C \frac{1}{|\lambda_n|} \|w_4^n\|_{H^2(\Omega_2)} \rightarrow 0. \tag{58}$$

Note that w_3^n is a solution to the following problem:

$$\begin{cases} \Delta w_3^n = \frac{\rho_0}{\beta_0}(i\lambda_n w_3^n - \rho_0^{-1}\mu \Delta w_2^n - f_3^n) =: h_n^* \in L^2(\Omega_1), \\ \partial_\nu w_3^n + \kappa w_3^n = 0 & \text{on } \Gamma, \\ w_3^n = 0 & \text{on } I. \end{cases}$$

In consequence,

$$\begin{aligned} \|w_3^n\|_{H^2(\Omega_1)} &\leq C \|\eta_0 w_3^n - h_n^*\|_{L^2(\Omega_1)} \leq C \left(\|w_3^n\|_{L^2(\Omega_1)} + |\lambda_n| \|w_3^n\|_{L^2(\Omega_1)} \right. \\ &\quad \left. + |\lambda_n| \|\Delta w_1^n\|_{L^2(\Omega_1)} + \|\Delta f_1^n\|_{L^2(\Omega_1)} + \|f_3^n\|_{L^2(\Omega_1)} \right); \end{aligned}$$

here, we have used equality (37). Thus,

$$\frac{1}{|\lambda_n|} \|w_3^n\|_{H^2(\Omega_1)} \leq C. \tag{59}$$

Consequently,

$$\frac{\|\partial_\nu w_3^n\|_{H^{1/2}(I)}}{|\lambda_n|} \leq C \frac{1}{|\lambda_n|} \|w_3^n\|_{H^2(\Omega_1)} \leq C. \tag{60}$$

The estimates (56), (57), (58), and (60) imply $1/|\lambda_n| \|w_1^n\|_{H^4(\Omega_1)} \leq C$. By interpolation inequality,

$$\frac{1}{|\lambda_n|^{1/2}} \|w_1^n\|_{H^3(\Omega_1)} \leq C \frac{\|w_1^n\|_{H^4(\Omega_1)}^{1/2}}{|\lambda_n|^{1/2}} \|w_1^n\|_{H^2(\Omega_1)}^{1/2}.$$

Therefore, $(1/|\lambda_n|^{1/2}w_1^n)_{n \in \mathbb{N}}$ is a bounded sequence in $H^3(\Omega_1)$. From (43) and (50), it follows that

$$\pm i\mu\Delta w_1^n + \frac{\beta_0}{|\lambda_n|}\Delta w_3^n \rightarrow 0 \text{ in } L^2(\Omega_1),$$

and then

$$\left\langle \pm i\mu\Delta w_1^n + \frac{\beta_0}{|\lambda_n|}\Delta w_3^n, \Delta w_1^n \right\rangle_{L^2(\Omega_1)} \rightarrow 0.$$

So,

$$\pm i\mu \|\Delta w_1^n\|_{L^2(\Omega_1)}^2 + \frac{\beta_0}{|\lambda_n|} (\Delta w_3^n, \Delta w_1^n)_{L^2(\Omega_1)} \rightarrow 0. \tag{61}$$

Using integration by parts,

$$\langle \Delta w_3^n, \Delta w_1^n \rangle_{L^2(\Omega_1)} = \langle w_3^n, \Delta^2 w_1^n \rangle_{L^2(\Omega_1)} - \langle w_3^n, \partial_\nu \Delta w_1^n \rangle_{L^2(\Gamma)} + \langle \partial_\nu w_3^n, \Delta w_1^n \rangle_{L^2(\partial\Omega_1)}. \tag{62}$$

By the trace theorem and Lemma 4.3 applied to $\partial_\nu \Delta w_1^n$, we have

$$\begin{aligned} \left| \langle w_3^n, \partial_\nu \Delta w_1^n \rangle_{L^2(\Gamma)} \right| &\leq \|w_3^n\|_{L^2(\Gamma)} \|\partial_\nu \Delta w_1^n\|_{L^2(\Gamma)} \\ &\leq C \|w_3^n\|_{H^1(\Omega_1)} \|w_1^n\|_{H^4(\Omega_1)}^{1/2} \|w_1^n\|_{H^3(\Omega_1)}^{1/2}. \end{aligned}$$

Then,

$$\begin{aligned} \left| \frac{1}{\lambda_n} \langle w_3^n, \partial_\nu \Delta w_1^n \rangle_{L^2(\Gamma)} \right| &\leq C \frac{\|w_3^n\|_{H^1(\Omega_1)}}{|\lambda_n|^{1/4}} \frac{\|w_1^n\|_{H^4(\Omega_1)}^{1/2}}{|\lambda_n|^{1/2}} \frac{\|w_1^n\|_{H^3(\Omega_1)}^{1/2}}{|\lambda_n|^{1/4}} \\ &\leq C \frac{\|w_3^n\|_{H^1(\Omega_1)}}{|\lambda_n|^{1/4}}. \end{aligned}$$

Therefore,

$$\frac{1}{|\lambda_n|} \langle w_3^n, \partial_\nu \Delta w_1^n \rangle_{L^2(\Gamma)} \rightarrow 0. \tag{63}$$

Lemma 4.3 implies

$$\begin{aligned} \left| \langle \partial_\nu w_3^n, \Delta w_1^n \rangle_{L^2(\partial\Omega_1)} \right| &\leq \|\partial_\nu w_3^n\|_{L^2(\partial\Omega_1)} \|\Delta w_1^n\|_{L^2(\partial\Omega_1)} \\ &\leq C \|w_3^n\|_{H^2(\Omega_1)}^{1/2} \|w_3^n\|_{H^1(\Omega_1)}^{1/2} \|w_1^n\|_{H^3(\Omega_1)}^{1/2} \|w_1^n\|_{H^2(\Omega_1)}^{1/2}. \end{aligned}$$

By (59), we have

$$\begin{aligned} \left| \frac{1}{\lambda_n} \langle \partial_\nu w_3^n, \Delta w_1^n \rangle_{L^2(\partial\Omega_1)} \right| &\leq C \frac{\|w_3^n\|_{H^2(\Omega_1)}^{1/2}}{|\lambda_n|^{1/2}} \|w_3^n\|_{H^1(\Omega_1)}^{1/2} \frac{\|w_1^n\|_{H^3(\Omega_1)}^{1/2}}{|\lambda_n|^{1/4}} \frac{\|w_1^n\|_{H^2(\Omega_1)}^{1/2}}{|\lambda_n|^{1/4}} \\ &\leq C \|w_3^n\|_{H^1(\Omega_1)}^{1/2}. \end{aligned}$$

Hence,

$$\frac{1}{|\lambda_n|} \langle \partial_\nu w_3^n, \Delta w_1^n \rangle_{L^2(\partial\Omega_1)} \rightarrow 0. \tag{64}$$

From (48), (56), and (61)–(64), it follows that

$$\Delta w_1^n \rightarrow 0 \text{ in } L^2(\Omega_1). \tag{65}$$

The limit (44) implies $\langle i\lambda_n \rho_1 w_2^n + \beta_1 \Delta^2 w_1^n + \mu \Delta w_3^n, w_2^n \rangle_{L^2(\Omega_1)} \rightarrow 0$. Using integration by parts, we have

$$\begin{aligned} \langle i\lambda_n \rho_1 w_2^n + \beta_1 \Delta^2 w_1^n + \mu \Delta w_3^n, w_2^n \rangle_{L^2(\Omega_1)} &= i\lambda_n \rho_1 \|w_2^n\|_{L^2(\Omega_1)}^2 \\ &+ \beta_1 \langle \Delta w_1^n, \Delta w_2^n \rangle_{L^2(\Omega_1)} + \beta_1 \langle \partial_\nu \Delta w_1^n, w_2^n \rangle_{L^2(\Gamma)} + \mu \langle \Delta w_3^n, w_2^n \rangle_{L^2(\Omega_1)}. \end{aligned} \tag{66}$$

From (43) and (65), we get

$$\frac{1}{|\lambda_n|} \Delta w_2^n \rightarrow 0 \text{ in } L^2(\Omega_1). \tag{67}$$

By Theorem 3.6, the trace theorem, and Lemma 4.3, we get

$$\begin{aligned} \left| \langle \beta_1 \partial_\nu \Delta w_1^n, w_2^n \rangle_{L^2(I)} \right| &= \left| \langle -\beta_2 \partial_\nu w_4^n - \mu \partial_\nu w_3^n, w_2^n \rangle_{L^2(I)} \right| \\ &\leq C \left(\|\partial_\nu w_4^n\|_{L^2(I)} + \|\partial_\nu w_3^n\|_{L^2(I)} \right) \|w_2^n\|_{L^2(I)} \\ &\leq C \left(\|w_4^n\|_{H^2(\Omega_2)}^{1/2} \|w_4^n\|_{H^1(\Omega_2)}^{1/2} + \|w_3^n\|_{H^2(\Omega_1)}^{1/2} \|w_3^n\|_{H^1(\Omega_1)}^{1/2} \right) \|w_2^n\|_{H^1(\Omega_1)}. \end{aligned}$$

Then,

$$\left| \frac{1}{\lambda_n} \langle \beta_1 \partial_\nu \Delta w_1^n, w_2^n \rangle_{L^2(I)} \right| \leq C \left(\frac{\|w_4^n\|_{H^2(\Omega_2)}^{1/2}}{|\lambda_n|^{1/2}} \|w_4^n\|_{H^1(\Omega_2)}^{1/2} + \frac{\|w_3^n\|_{H^2(\Omega_1)}^{1/2}}{|\lambda_n|^{1/2}} \|w_3^n\|_{H^1(\Omega_1)}^{1/2} \right) \frac{\|w_2^n\|_{H^1(\Omega_1)}}{|\lambda_n|^{1/2}}.$$

From (52), (54), and (59), it follows that

$$\frac{1}{|\lambda_n|} \langle \partial_\nu \Delta w_1^n, w_2^n \rangle_{L^2(I)} \rightarrow 0. \tag{68}$$

The limits (50) and (67) imply

$$\frac{1}{|\lambda_n|} \Delta w_3^n \rightarrow 0 \text{ in } L^2(\Omega_1). \tag{69}$$

From (66)–(69), we obtain

$$w_2^n \rightarrow 0 \text{ in } L^2(\Omega_1). \tag{70}$$

Finally, the limits (48), (55), (65), and (70) allow us to write $\|w^n\|_{\mathcal{H}_0} \rightarrow 0$, which is a contradiction to $\|w^n\|_{\mathcal{H}_0} = 1$ for all $n \in \mathbb{N}$. \square

5 | THE CASE OF UNDAMPED MEMBRANE

We now consider the situation when the membrane is undamped, that is, $m = 0$. First, we will prove that the solution of our system is not exponentially stable. However, under certain geometric assumptions, we will prove that in this case, the system is polynomially stable. The proof of the next theorem follows the ideas from Theorem 3.5 in Muñoz Rivera and Racke.³¹

Theorem 5.1. *For $\rho \geq 0$, $\gamma > 0$ and $m = 0$, the system (1)–(8) is not exponentially stable.*

Proof. We set $\tilde{\mathcal{H}} := \{0\} \times \{0\} \times \{0\} \times H_0^1(\Omega_2) \times L^2(\Omega_2)$. Note that $\tilde{\mathcal{H}}$ is a Hilbert subspace of \mathcal{H}_γ . We define the operator $\tilde{\mathcal{A}}$ given by $D(\tilde{\mathcal{A}}) = \{0\} \times \{0\} \times \{0\} \times (H^2(\Omega_2) \cap H_0^1(\Omega_2)) \times H_0^1(\Omega_2) \subset \tilde{\mathcal{H}}$ and

$$\tilde{\mathcal{A}}\tilde{w} = (0, 0, 0, \tilde{w}_5, \beta_2/\rho_2 \Delta \tilde{w}_4)^\top.$$

With respect to the fourth and fifth component, $\tilde{\mathcal{A}}$ is the first-order system related to the nondamped wave equation for $\tilde{v} := \tilde{w}_4$

$$\begin{cases} \rho_2 \tilde{v}_{tt} - \beta_2 \Delta \tilde{v} = 0 & \text{in } \Omega_2 \times (0, \infty), \\ \tilde{v} = 0 & \text{on } I \times (0, \infty), \\ \tilde{v}(\cdot, 0) = \tilde{v}^0, \tilde{v}_t(\cdot, 0) = \tilde{v}^1 & \text{in } \Omega_2 \end{cases} \tag{71}$$

with appropriate initial values \tilde{v}^0, \tilde{v}^1 . Let $(\tilde{S}(t))_{t \geq 0}$ be the C_0 -semigroup generated by $\tilde{\mathcal{A}}$ on $\tilde{\mathcal{H}}$. As (71) contains no damping term, this is a unitary semigroup. Thus, the essential spectral radius $r_{\text{ess}}(\tilde{S}(t))$ is equal to 1.

We will show that $S(t) - \tilde{S}(t) : \tilde{\mathcal{H}} \rightarrow \mathcal{H}_\gamma$ is compact, where $(S(t))_{t \geq 0}$ stands for the C_0 -semigroup generated by \mathcal{A} (Theorem 2.5). It is enough to prove that $S(t) - \tilde{S}(t) : \mathcal{W} \rightarrow \mathcal{H}_\gamma$ is compact for some dense subspace \mathcal{W} of $\tilde{\mathcal{H}}$. We define

$$\mathcal{W} := \{0\} \times \{0\} \times \{0\} \times \mathcal{D}(\Omega_2) \times \mathcal{D}(\Omega_2).$$

Then \mathcal{W} is dense in \tilde{H} , and obviously, $\mathcal{W} \subset D(\mathcal{A}) \cap D(\tilde{\mathcal{A}})$. For $w_0 \in \mathcal{W}$, we consider

$$E(t) := \frac{1}{2} \|S(t)w_0 - \tilde{S}(t)w_0\|_{\mathcal{H}_\gamma}^2 \quad (t \geq 0).$$

Let $w(t) := S(t)w_0$ and $\tilde{w}(t) := \tilde{S}(t)w_0$. Then

$$\begin{aligned} E'(t) &= \frac{1}{2} \frac{d}{dt} \|w(t) - \tilde{w}(t)\|_{\mathcal{H}_\gamma}^2 = \operatorname{Re} \langle w'(t) - \tilde{w}'(t), w(t) - \tilde{w}(t) \rangle_{\mathcal{H}_\gamma} \\ &= \operatorname{Re} \langle \mathcal{A}w(t) - \tilde{\mathcal{A}}\tilde{w}(t), w(t) - \tilde{w}(t) \rangle_{\mathcal{H}_\gamma} \\ &= \operatorname{Re} \langle \mathcal{A}w(t), w(t) \rangle_{\mathcal{H}_\gamma} + \operatorname{Re} \langle \tilde{\mathcal{A}}\tilde{w}(t), \tilde{w}(t) \rangle_{\mathcal{H}_\gamma} - \operatorname{Re} \langle \mathcal{A}w(t), \tilde{w}(t) \rangle_{\mathcal{H}_\gamma} - \operatorname{Re} \langle \tilde{\mathcal{A}}\tilde{w}(t), w(t) \rangle_{\mathcal{H}_\gamma}. \end{aligned} \quad (72)$$

From (15), we know $\operatorname{Re} \langle \mathcal{A}w(t), w(t) \rangle_{\mathcal{H}_\gamma} \leq 0$, and for the undamped wave equation, we obtain $\operatorname{Re} \langle \tilde{\mathcal{A}}\tilde{w}(t), \tilde{w}(t) \rangle_{\mathcal{H}_\gamma} = 0$. Moreover, by the definition of \mathcal{A} and (14), we see that

$$\begin{aligned} \langle \mathcal{A}w(t), \tilde{w}(t) \rangle_{\mathcal{H}_\gamma} &= \langle \mathbb{A}w(t), \tilde{w}(t) \rangle_{\mathcal{H}_\gamma \times \mathcal{H}_\gamma} \\ &= -\beta_2 \langle \nabla w_4(t), \nabla \tilde{w}_5(t) \rangle_{L^2(\Omega_2)} + \beta_2 \langle \nabla w_5(t), \nabla \tilde{w}_4(t) \rangle_{L^2(\Omega_2)}. \end{aligned}$$

With integration by parts, we obtain

$$\begin{aligned} \langle \tilde{\mathcal{A}}\tilde{w}(t), w(t) \rangle_{\mathcal{H}_\gamma} &= \beta_2 \langle \nabla \tilde{w}_5(t), \nabla w_4(t) \rangle_{L^2(\Omega_2)} + \rho_2 \langle \frac{\beta_2}{\rho_2} \Delta \tilde{w}_4(t), w_5(t) \rangle_{L^2(\Omega_2)} \\ &= \beta_2 \langle \nabla \tilde{w}_5(t), \nabla w_4(t) \rangle_{L^2(\Omega_2)} - \beta_2 \langle \nabla \tilde{w}_4(t), \nabla w_5(t) \rangle_{L^2(\Omega_2)} - \beta_2 \langle \partial_\nu \tilde{w}_4(t), w_5(t) \rangle_{L^2(I)}. \end{aligned}$$

Taking the real part in the last two equalities and inserting this into (72), we see that

$$E'(t) \leq \beta_2 \operatorname{Re} \langle \partial_\nu \tilde{w}_4(t), w_5(t) \rangle_{L^2(I)} = \beta_2 \operatorname{Re} \langle \partial_\nu \tilde{w}_4(t), w_2(t) \rangle_{L^2(I)},$$

where we used in the last equality that $w_5(t) = w_2(t)$ on I because $w(t) \in D(\mathcal{A})$. Therefore, noting $E(0) = 0$, we have

$$E(t) \leq \beta_2 \operatorname{Re} \int_0^t \langle \partial_\nu \tilde{w}_4(s), w_2(s) \rangle_{L^2(I)} ds. \quad (73)$$

Let $(w_0^k)_{k \in \mathbb{N}} \subset \mathcal{W}$ be a bounded sequence in \tilde{H} , and let $w^k(t) := S(t)w_0^k$ and $\tilde{w}^k(t) := \tilde{S}(t)w_0^k$.

We estimate the terms appearing on the right-hand side of (73) separately:

(i) We first show that the sequence $(\partial_\nu \tilde{w}_4^k)_{k \in \mathbb{N}} \subset L^2((0, t), L^2(I))$ is uniformly bounded. For this, we first note that $\tilde{v}^k := \tilde{w}_4^k$ is a solution of (71). We fix a vector field $\sigma \in C^1(\overline{\Omega_2})^2$ satisfying $\sigma = -\nu$ on I . Straight-forward calculations using integration by parts (note that $\tilde{v}^k(s) = 0$ on I) yield for $s \in (0, t)$

$$\int_{\Omega_2} \tilde{v}_{tt}^k(s) \left(\sigma \cdot \overline{\nabla \tilde{v}^k(s)} \right) dx = \frac{1}{2} \int_{\Omega_2} |\tilde{v}_t^k(s)|^2 \operatorname{div}(\sigma) dx + \frac{d}{ds} \int_{\Omega_2} \tilde{v}_t^k(s) \left(\sigma \cdot \overline{\nabla \tilde{v}^k(s)} \right) dx \quad (74)$$

and

$$\int_{\Omega_2} \Delta \tilde{v}^k(s) \left(\sigma \cdot \overline{\nabla \tilde{v}^k(s)} \right) dx = \frac{1}{2} \|\partial_\nu v(s)\|_{L^2(I)}^2 - \int_{\Omega_2} \nabla \tilde{v}^k(s) \cdot \left(D\sigma \overline{\nabla \tilde{v}^k(s)} \right) dx + \frac{1}{2} \int_{\Omega_2} |\nabla \tilde{v}^k(s)|^2 \operatorname{div}(\sigma) dx. \quad (75)$$

Multiplying the differential equation $\rho_2 \widetilde{v}_t^k - \beta_2 \Delta \widetilde{v}^k = 0$ with $\sigma \cdot \overline{\nabla \widetilde{v}^k}$ and integrating over Ω_2 , we obtain from (74) and (75)

$$\begin{aligned} \beta_2 \|\partial_v \widetilde{v}^k(s)\|_{L^2(I)}^2 &= \int_{\Omega_2} (\rho_2 |\widetilde{v}_t^k(s)|^2 - \beta_2 |\nabla \widetilde{v}^k(s)|^2) \operatorname{div}(\sigma) dx \\ &\quad + 2\beta_2 \int_{\Omega_2} \nabla \widetilde{v}^k(s) \cdot (D\sigma \overline{\nabla \widetilde{v}^k(s)}) dx + 2\rho_2 \frac{d}{ds} \int_{\Omega_2} \widetilde{v}_t^k(s) (\sigma \cdot \overline{\nabla \widetilde{v}^k(s)}) dx. \end{aligned}$$

Now we integrate the last equality with respect to $s \in (0, t)$ and obtain

$$\begin{aligned} \|\partial_v \widetilde{v}^k\|_{L^2((0,t),L^2(I))}^2 &\leq C \sup_{s \in [0,t]} \left(\|\nabla \widetilde{v}^k(s)\|_{L^2(\Omega_2)}^2 + \|\widetilde{v}_t^k(s)\|_{L^2(\Omega_2)}^2 \right) \\ &\leq C \sup_{s \in [0,t]} \|\widetilde{w}^k(s)\|_{\widetilde{\mathcal{H}}}^2 \leq C \|w_0^k\|_{\widetilde{\mathcal{H}}}^2 \leq C, \end{aligned}$$

which finishes the proof of (i).

(ii) In the next step, we show that there exists a subsequence of $(w_2^k|_I)_{k \in \mathbb{N}}$ which converges in $L^2((0, t), L^2(I))$. For this, we apply the lemma of Aubin–Lions and the theory of extrapolation–interpolation scales as developed in Amann,³² Chapter V. To define this scale, one sets $X_0 := \mathcal{H}_\gamma$ and defines X_{-1} as the completion of X_0 with respect to the norm $\|f\|_{-1} := \|\mathcal{A}^{-1} f\|_{X_0}$ (see³², Theorem V.1.3.2). It is known that X_{-1} is the dual space of $D(\mathcal{A}')$, where $\mathcal{A}' : \mathcal{H}'_\gamma \supset D(\mathcal{A}') \rightarrow \mathcal{H}'_\gamma$ denotes the adjoint operator to \mathcal{A} (see³², Corollary V.1.4.7). We define $X_{-\alpha} := [X_0, X_{-1}]_\alpha$ for $\alpha \in (0, 1)$, where $[\cdot, \cdot]$ stands for the complex interpolation functor. Then the general theory of extrapolation–interpolation scales tells us that we have the dense and compact embeddings

$$\mathcal{H}_\gamma = X_0 \subset X_{-\alpha} \subset X_{-1}$$

for all $\alpha \in (0, 1)$; see Amann.³², Theorem V.1.5.1 The compactness of both embeddings follows from the fact that $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H}_\gamma)$ is compact due to Proposition 3.9.

To describe X_{-1} , let

$$\phi \in Y_0 := H_0^3(\Omega_1) \times H_0^2(\Omega_1) \times H_0^2(\Omega_1) \times H_0^2(\Omega_2) \times H_0^1(\Omega_2),$$

and let $w \in D(\mathcal{A})$. By definition of \mathbb{M} , we have

$$\langle \mathcal{A}w, \phi \rangle_{\mathcal{H}_\gamma} = \langle \mathbb{M}^{-1} \mathbb{A}w, \phi \rangle_{\mathcal{H}_\gamma} = \langle \mathbb{A}w, \phi \rangle_{\mathcal{H}'_\gamma \times \mathcal{H}_\gamma} = \langle \mathbb{A}w, \phi \rangle_{\mathcal{V}' \times \mathcal{V}},$$

where the last equality follows from the fact that $\phi \in \mathcal{V}$. By the definition of \mathbb{A} (equality (12)) and integration by parts, we obtain

$$\left| \langle \mathcal{A}w, \phi \rangle_{\mathcal{H}_\gamma} \right| \leq C_\phi \|w\|_{\mathcal{H}_\gamma},$$

where the constant C_ϕ can be estimated by a constant times the norm of ϕ in the space Y_0 . Therefore, the map $(D(\mathcal{A}), \|\cdot\|_{\mathcal{H}_\gamma}) \rightarrow \mathbb{C}, w \mapsto \langle \mathcal{A}w, \phi \rangle_{\mathcal{H}_\gamma}$ is a continuous linear functional, and its continuous extension to \mathcal{H}_γ is an element of \mathcal{H}'_γ . This shows that $\phi \in D(\mathcal{A}')$, and therefore, $Y_0 \subset D(\mathcal{A}')$. As X_{-1} is the dual space of $D(\mathcal{A}')$, this yields

$$X_{-1} \subset Y'_0 = H^{-3}(\Omega_1) \times H^{-2}(\Omega_1) \times H^{-2}(\Omega_1) \times H^{-2}(\Omega_2) \times H^{-1}(\Omega_2). \tag{76}$$

On the other hand, by definition of \mathcal{H}_γ , we have

$$\mathcal{H}_\gamma = X_0 \subset H^2(\Omega_1) \times H^1(\Omega_1) \times L^2(\Omega_1) \times H^1(\Omega_2) \times L^2(\Omega_2). \tag{77}$$

With complex interpolation, we obtain from (76) and (77)

$$X_{-\alpha} \subset H^{2-5\alpha}(\Omega_1) \times H^{1-3\alpha}(\Omega_1) \times H^{-2\alpha}(\Omega_1) \times H^{1-3\alpha}(\Omega_2) \times H^{-\alpha}(\Omega_2). \tag{78}$$

Now we apply the Aubin–Lions lemma to the compact embedding $\mathcal{H}_\gamma = X_0 \subset X_{-\alpha} \subset X_{-1}$. For this, we first note that $(w^k)_{k \in \mathbb{N}}$ is bounded in $C([0, t], \mathcal{H}_\gamma)$ and therefore in $L^2((0, t), \mathcal{H}_\gamma)$ because of $\|S(s)\|_{\mathcal{L}(\mathcal{H}_\gamma)} \leq 1$ for all $s \in [0, t]$. For the time derivative, we use $\partial_t w^k = \mathcal{A}w^k$ and obtain

$$\sup_{s \in [0, t]} \|\partial_s w^k(s)\|_{X_{-1}} = \sup_{s \in [0, t]} \|\mathcal{A}w^k(s)\|_{X_{-1}} = \sup_{s \in [0, t]} \|w^k(s)\|_{\mathcal{H}_\gamma} \leq C.$$

So $(\partial_t w^k)_{k \in \mathbb{N}}$ is bounded in $L^2((0, t), X_{-1})$, and the application of the Aubin–Lions lemma yields that there exists a subsequence $(w^{k_j})_{j \in \mathbb{N}}$ which is convergent in $L^2((0, t), X_{-\alpha})$. From the embedding (78), we see that the second component $(w_2^{k_j})_{j \in \mathbb{N}}$ converges in $L^2((0, t), H^{1-3\alpha}(\Omega_1))$. Now we choose $\alpha < 1/6$ and take the trace on I and obtain convergence in $L^2((0, t), H^{\frac{1}{2}-3\alpha}(I))$ and therefore in $L^2((0, t), L^2(I))$ for $(w_2^{k_j})_{j \in \mathbb{N}}$, which finishes the proof of (ii).

We know from (i) and (ii) that, after passing to a subsequence, we have that $(\partial_v \tilde{w}_4^k)_{k \in \mathbb{N}}$ is bounded in $L^2((0, t), L^2(I))$ and $(w_2^k|_I)_{k \in \mathbb{N}}$ converges strongly in $L^2((0, t), L^2(I))$. For $k, \ell \in \mathbb{N}$ we now denote by

$$E^{k\ell}(t) := \frac{1}{2} \|S(t)(w_0^k - w_0^\ell) - \tilde{S}(t)(w_0^k - w_0^\ell)\|_{\mathcal{H}_\gamma}^2 \quad (t \geq 0).$$

By (73), we have that

$$E^{k\ell}(t) \leq \beta_2 \left| \langle \partial_v \tilde{w}_4^{k\ell}, w_2^{k\ell} \rangle_{L^2((0, t), L^2(I))} \right| \rightarrow 0 \quad (k, \ell \rightarrow \infty),$$

where $w^{k\ell}(t) := S(t)(w_0^k - w_0^\ell)$ and $\tilde{w}^{k\ell}(t) := \tilde{S}(t)(w_0^k - w_0^\ell)$ for $k, \ell \in \mathbb{N}$. Therefore, $((S(t) - \tilde{S}(t))w_0^k)_{k \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{H}_γ and thus convergent. This shows the compactness of $S(t) - \tilde{S}(t) : \mathcal{W} \rightarrow \mathcal{H}_\gamma$. Therefore, $S(t) - \tilde{S}(t) : \tilde{\mathcal{H}} \rightarrow \mathcal{H}_\gamma$ is compact. As $r_{\text{ess}}(\tilde{S}(t)) = 1$, Theorem 3.3 in Muñoz Rivera and Racke³¹ implies that $r_{\text{ess}}(S(t)) = 1$, and thus, $(S(t))_{t \geq 0}$ is not exponentially stable. \square

Although the last result tells us that there is no exponential stability in the case of an undamped membrane, we will now show that the system decays polynomially under the following geometric condition: there exists some $x_0 \in \mathbb{R}^2$ such that

$$q(x) \cdot v(x) \leq 0 \quad (x \in I), \quad \text{where } q(x) := x - x_0 \quad (x \in \overline{\Omega_2}). \tag{79}$$

Theorem 5.2. *Let $m = 0, \rho > 0, \gamma \geq 0$ and assume that the geometrical condition (79) is satisfied. Then, the semigroup $(S(t))_{t \geq 0}$ generated by \mathcal{A} decays polynomially; that is, there exist constants $\alpha, C > 0$ such that*

$$\|S(t)w_0\|_{\mathcal{H}_\gamma} \leq Ct^{-\alpha} \|w_0\|_{D(\mathcal{A})}$$

for all $t > 0$ and $w_0 \in D(\mathcal{A})$.

Proof. By Lemma 5.2 in Muñoz Rivera and Racke,³¹ the semigroup is polynomially stable if $i\mathbb{R} \subset \rho(\mathcal{A})$ and if there exist $C > 0, \lambda_0 > 0$, and $\beta > 0, \beta' \geq 0$ with

$$\|(i\lambda - \mathcal{A})^{-1}f\|_{\mathcal{H}_\gamma} \leq C|\lambda|^\beta \|\mathcal{A}^{\beta'}f\|_{\mathcal{H}_\gamma} \quad (f \in D(\mathcal{A}^{\beta'}), \lambda \in \mathbb{R}, |\lambda| > \lambda_0). \tag{80}$$

First, let $\gamma > 0$. We will show (80) with $\beta' = 1$. Let $\lambda_0 > 0$ and $\lambda \in \mathbb{R}$ with $|\lambda| > \lambda_0$. Let $w \in D(\mathcal{A}^2)$ and $f := (i\lambda - \mathcal{A})w$. Then $f \in D(\mathcal{A})$, and

$$i\lambda w_1 - w_2 = f_1, \tag{81}$$

$$i\lambda \rho_1 w_2 - i\lambda \gamma \Delta w_2 + \beta_1 \Delta^2 w_1 + \mu \Delta w_3 - \rho \Delta w_2 = \rho_1 f_2 - \gamma \Delta f_2, \tag{82}$$

$$i\lambda \rho_0 w_3 - \mu \Delta w_2 - \beta_0 \Delta w_3 = \rho_0 f_3, \tag{83}$$

$$i\lambda w_4 - w_5 = f_4, \tag{84}$$

$$i\lambda \rho_2 w_5 - \beta_2 \Delta w_4 = \rho_2 f_5. \tag{85}$$

Replacing (81) into (82) and (83), we have

$$-\lambda^2 \rho_1 w_1 + \lambda^2 \gamma \Delta w_1 + \beta_1 \Delta^2 w_1 + \mu \Delta w_3 - i\lambda \rho \Delta w_1 = i\lambda \rho_1 f_1 - i\lambda \gamma \Delta f_1 - \rho \Delta f_1 + \rho_1 f_2 - \gamma \Delta f_2, \tag{86}$$

and

$$i\lambda\rho_0w_3 - i\lambda\mu\Delta w_1 - \beta_0\Delta w_3 = -\mu\Delta f_1 + \rho_0f_3, \quad (87)$$

respectively. Replacing (84) into (85), we get

$$-\lambda^2\rho_2w_4 - \beta_2\Delta w_4 = i\lambda\rho_2f_4 + \rho_2f_5. \quad (88)$$

Multiplying (86) by $-\overline{w_1}$, (87) by $i/\lambda\overline{w_3}$, and (88) by $-\overline{w_4}$, integrating and adding the resulting equalities, we obtain

$$\begin{aligned} & \lambda^2 \left(\rho_1 \|w_1\|_{L^2(\Omega_1)}^2 + \rho_2 \|w_4\|_{L^2(\Omega_2)}^2 \right) - \rho_0 \|w_3\|_{L^2(\Omega_1)}^2 - \beta_1 \langle \Delta^2 w_1, w_1 \rangle_{L^2(\Omega_1)} \\ & + i\lambda\rho \langle \Delta w_1, w_1 \rangle_{L^2(\Omega_1)} - \lambda^2\gamma \langle \Delta w_1, w_1 \rangle_{L^2(\Omega_1)} - \mu \langle \Delta w_3, w_1 \rangle_{L^2(\Omega_1)} \\ & + \beta_2 \langle \Delta w_4, w_4 \rangle_{L^2(\Omega_2)} + \mu \langle \Delta w_1, w_3 \rangle_{L^2(\Omega_1)} - i\beta_0\lambda^{-1} \langle \Delta w_3, w_3 \rangle_{L^2(\Omega_1)} \\ = & -i\lambda\rho_1 \langle f_1, w_1 \rangle_{L^2(\Omega_1)} + i\lambda\gamma \langle \Delta f_1, w_1 \rangle_{L^2(\Omega_1)} + \rho \langle \Delta f_1, w_1 \rangle_{L^2(\Omega_1)} - \rho_1 \langle f_2, w_1 \rangle_{L^2(\Omega_1)} \\ & + \gamma \langle \Delta f_2, w_1 \rangle_{L^2(\Omega_1)} - \langle i\lambda\rho_2f_4 + \rho_2f_5, w_4 \rangle_{L^2(\Omega_2)} - \langle i\mu\lambda^{-1}\Delta f_1 - i\rho_0\lambda^{-1}f_3, w_3 \rangle_{L^2(\Omega_1)}. \end{aligned}$$

Integrating by parts and using the transmission conditions, we obtain

$$\begin{aligned} & \lambda^2 \left(\rho_1 \|w_1\|_{L^2(\Omega_1)}^2 + \rho_2 \|w_4\|_{L^2(\Omega_2)}^2 + \gamma \|\nabla w_1\|_{L^2(\Omega_1)}^2 \right) - \beta_1 \|\Delta w_1\|_{L^2(\Omega_1)}^2 \\ & - \rho_0 \|w_3\|_{L^2(\Omega_1)}^2 - i\lambda\rho \|\nabla w_1\|_{L^2(\Omega_1)}^2 + i2\mu\text{Im} \langle \Delta w_1, w_3 \rangle_{L^2(\Omega_1)} - \beta_2 \|\nabla w_4\|_{L^2(\Omega_2)}^2 \\ & + i\frac{\beta_0}{\lambda} \|\nabla w_3\|_{L^2(\Omega_1)}^2 + i\frac{\kappa\beta_0}{\lambda} \|w_3\|_{L^2(\Gamma)}^2 \\ = & -(\rho + i\lambda\gamma) \langle \nabla f_1, \nabla w_1 \rangle_{L^2(\Omega_1)} - \gamma \langle \nabla f_2, \nabla w_1 \rangle_{L^2(\Omega_1)} - \langle i\lambda\rho_1f_1 + \rho_1f_2, w_1 \rangle_{L^2(\Omega_1)} - \langle \rho_2f_5 + i\lambda\rho_2f_4, w_4 \rangle_{L^2(\Omega_2)} \\ & + i\frac{\mu}{\lambda} \langle \nabla f_1, \nabla w_3 \rangle_{L^2(\Omega_1)} + i\frac{\rho_0}{\lambda} \langle f_3, w_3 \rangle_{L^2(\Omega_1)}. \end{aligned}$$

Taking real part in the previous equation, we get

$$\begin{aligned} & \beta_1 \|\Delta w_1\|_{L^2(\Omega_1)}^2 + \beta_2 \|\nabla w_4\|_{L^2(\Omega_2)}^2 + \rho_0 \|w_3\|_{L^2(\Omega_1)}^2 \\ & \leq \lambda^2 \left(\rho_1 \|w_1\|_{L^2(\Omega_1)}^2 + \rho_2 \|w_4\|_{L^2(\Omega_2)}^2 + \gamma \|\nabla w_1\|_{L^2(\Omega_1)}^2 \right) \\ & + (\rho \|\nabla f_1\|_{L^2(\Omega_1)} + |\lambda|\gamma \|\nabla f_1\|_{L^2(\Omega_1)} + \gamma \|\nabla f_2\|_{L^2(\Omega_1)}) \|\nabla w_1\|_{L^2(\Omega_1)} \\ & + \rho_1 (|\lambda| \|f_1\|_{L^2(\Omega_1)} + \|f_2\|_{L^2(\Omega_1)}) \|w_1\|_{L^2(\Omega_1)} + \frac{\mu}{|\lambda|} \|\nabla f_1\|_{L^2(\Omega_1)} \|\nabla w_3\|_{L^2(\Omega_1)} \\ & + \rho_2 (\|f_5\|_{L^2(\Omega_2)} + |\lambda| \|f_4\|_{L^2(\Omega_2)}) \|w_4\|_{L^2(\Omega_2)} + \frac{\rho_0}{|\lambda|} \|f_3\|_{L^2(\Omega_1)} \|w_3\|_{L^2(\Omega_1)}. \end{aligned}$$

By Lemma 2.1, (34), and $|\lambda| \geq \lambda_0$, we obtain

$$\begin{aligned} & \beta_1 \|\Delta w_1\|_{L^2(\Omega_1)}^2 + \beta_2 \|\nabla w_4\|_{L^2(\Omega_2)}^2 + \rho_0 \|w_3\|_{L^2(\Omega_1)}^2 \leq C \left(\lambda^2 \|w_1\|_{H^1(\Omega_1)}^2 \right. \\ & \left. + \lambda^2 \|w_4\|_{L^2(\Omega_2)}^2 + |\lambda| \|\omega\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + |\lambda|^{-1} \|\omega\|_{\mathcal{H}_\gamma}^{1/2} \|f\|_{\mathcal{H}_\gamma}^{3/2} \right). \end{aligned} \quad (89)$$

Since

$$\begin{aligned} & \|w_2\|_{L^2(\Omega_1)}^2 \leq 4(\lambda^2 \|w_1\|_{L^2(\Omega_1)}^2 + \|f_1\|_{L^2(\Omega_1)}^2) \leq C(\lambda^2 \|w_1\|_{H^1(\Omega_1)}^2 + \|f\|_{\mathcal{H}_\gamma}^2), \\ & \|\nabla w_2\|_{L^2(\Omega_1)}^2 \leq \frac{1}{\rho} \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma}, \\ & \|w_5\|_{L^2(\Omega_2)}^2 \leq 4(\lambda^2 \|w_4\|_{L^2(\Omega_2)}^2 + \|f_4\|_{L^2(\Omega_2)}^2) \leq C(\lambda^2 \|w_4\|_{L^2(\Omega_2)}^2 + \|f\|_{\mathcal{H}_\gamma}^2), \end{aligned}$$

we get

$$\rho_1 \|w_2\|_{L^2(\Omega_1)}^2 + \gamma \|\nabla w_2\|_{L^2(\Omega_1)}^2 + \rho_2 \|w_5\|_{L^2(\Omega_2)}^2 \leq C \left(\lambda^2 \|w_1\|_{H^1(\Omega_1)}^2 + \lambda^2 \|w_4\|_{L^2(\Omega_2)}^2 + \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma}^2 \right). \quad (90)$$

Poincaré's inequality implies

$$\begin{aligned} \lambda^2 \|w_1\|_{H^1(\Omega_1)}^2 &= \|w_2 + f_1\|_{H^1(\Omega_1)}^2 \leq C \left(\|\nabla w_2\|_{L^2(\Omega_1)}^2 + \|f_1\|_{H^1(\Omega_1)}^2 \right) \\ &\leq C \left(\|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma}^2 \right). \end{aligned} \quad (91)$$

From (89)–(91), it follows that

$$\|w\|_{\mathcal{H}_\gamma}^2 \leq C \left(\lambda^2 \|w_4\|_{L^2(\Omega_2)}^2 + |\lambda| \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + |\lambda|^{-1} \|w\|_{\mathcal{H}_\gamma}^{1/2} \|f\|_{\mathcal{H}_\gamma}^{3/2} + \|f\|_{\mathcal{H}_\gamma}^2 \right). \quad (92)$$

Now we will prove that

$$\lambda^2 \|w_4\|_{L^2(\Omega_2)}^2 \leq C \left(|\lambda| \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma}^2 + \beta_2 \int_I |\partial_\nu w_4(q\nabla\bar{w}_4)| dS \right). \quad (93)$$

In fact, using Rellich's identity (see eq. 2.5 from Mitidieri³³), we have the following equality:

$$\operatorname{Re} \int_{\Omega_2} \Delta w_4(q\nabla\bar{w}_4) dx = -\operatorname{Re} \int_I \left[\partial_\nu w_4(q\nabla\bar{w}_4) - \frac{1}{2}(q \cdot \nu) |\nabla w_4|^2 \right] dS. \quad (94)$$

Multiplying (88) by $q\nabla\bar{w}_4$ and integrating, we get

$$-\lambda^2 \rho_2 \int_{\Omega_2} w_4(q\nabla\bar{w}_4) dx - \beta_2 \int_{\Omega_2} \Delta w_4(q\nabla\bar{w}_4) dx = \int_{\Omega_2} (i\lambda\rho_2 f_4 + \rho_2 f_5)(q\nabla\bar{w}_4) dx.$$

Taking real part and using (94), we see that

$$-\lambda^2 \rho_2 \operatorname{Re} \int_{\Omega_2} w_4(q\nabla\bar{w}_4) dx + \beta_2 \operatorname{Re} \int_I \left[\partial_\nu w_4(q\nabla\bar{w}_4) - \frac{1}{2}(q \cdot \nu) |\nabla w_4|^2 \right] dS = \operatorname{Re} \int_{\Omega_2} (i\lambda\rho_2 f_4 + \rho_2 f_5)(q\nabla\bar{w}_4) dx.$$

Using integration by parts and the identity $q\nabla w_4 = \operatorname{div}(qw_4) - 2w_4$, it holds

$$\begin{aligned} \int_{\Omega_2} w_4(q\nabla\bar{w}_4) dx &= \int_{\Omega_2} w_4 (\operatorname{div}(q\bar{w}_4) - 2\bar{w}_4) dx \\ &= \int_{\Omega_2} w_4 \operatorname{div}(q\bar{w}_4) dx - 2 \int_{\Omega_2} w_4 \bar{w}_4 dx \\ &= - \int_{\Omega_2} \nabla w_4 \cdot q\bar{w}_4 dx - \int_I w_4 q\bar{w}_4 \cdot \nu dS - 2 \|w_4\|_{L^2(\Omega_2)}^2 \end{aligned}$$

and therefore

$$\int_{\Omega_2} w_4(q\nabla\bar{w}_4) dx + \overline{\int_{\Omega_2} w_4(q\nabla\bar{w}_4) dx} = - \int_I (q\nu) |w_4|^2 dS - 2 \|w_4\|_{L^2(\Omega_2)}^2.$$

Thus,

$$\operatorname{Re} \int_{\Omega_2} w_4(q\nabla\bar{w}_4) dx = - \|w_4\|_{L^2(\Omega_2)}^2 - \frac{1}{2} \int_I (q\nu) |w_4|^2 dS.$$

In consequence,

$$\begin{aligned} \lambda^2 \rho_2 \|w_4\|_{L^2(\Omega_2)}^2 &= -\lambda^2 \rho_2 \operatorname{Re} \int_{\Omega_2} w_4(q\nabla\bar{w}_4) dx - \frac{1}{2} \lambda^2 \rho_2 \int_I (q\nu) |w_4|^2 dS \\ &= \operatorname{Re} \int_{\Omega_2} (i\lambda\rho_2 f_4 + \rho_2 f_5)(q\nabla\bar{w}_4) dx - \beta_2 \operatorname{Re} \int_I \partial_\nu w_4(q\nabla\bar{w}_4) dS \\ &\quad + \frac{1}{2} \beta_2 \int_I (q\nu) |\nabla w_4|^2 dS - \frac{1}{2} \lambda^2 \rho_2 \int_I (q\nu) |w_4|^2 dS. \end{aligned}$$

Due to $q \cdot \nu \leq 0$ on I , $w_1 = w_4$ on I , trace theorem, and (91), we obtain that

$$\begin{aligned} \lambda^2 \rho_2 \|w_4\|_{L^2(\Omega_2)}^2 &\leq C \left(|\lambda| \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \beta_2 \left| \int_I \partial_\nu w_4 (q \nabla \overline{w_4}) dS \right| + \lambda^2 \|w_1\|_{L^2(I)}^2 \right) \\ &\leq \left(|\lambda| \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \beta_2 \int_I |\partial_\nu w_4 (q \nabla \overline{w_4})| dS + \lambda^2 \|w_1\|_{H^1(\Omega_1)}^2 \right) \\ &\leq C \left(|\lambda| \|w\|_{\mathcal{H}_\gamma} \|f\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma}^2 + \beta_2 \int_I |\partial_\nu w_4 (q \nabla \overline{w_4})| dS \right). \end{aligned}$$

Next, we will show that for any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

$$\beta_2 \int_I |\partial_\nu w_4 (q \nabla \overline{w_4})| dS \leq \varepsilon \|w\|_{\mathcal{H}_\gamma}^2 + C_\varepsilon |\lambda|^{48} \|Af\|_{\mathcal{H}_\gamma}^2. \quad (95)$$

Indeed, the transmission conditions imply

$$\begin{aligned} \beta_2 \int_I |\partial_\nu w_4 (q \nabla \overline{w_4})| dS &\leq \|\beta_1 \partial_\nu (\Delta w_1) + \mu \partial_\nu w_3\|_{L^2(I)} \|q \nabla \overline{w_4}\|_{L^2(I)} \\ &\leq C (\|\partial_\nu (\Delta w_1)\|_{L^2(I)} + \|\partial_\nu w_3\|_{L^2(I)}) \|\nabla w_4\|_{L^2(I)}. \end{aligned} \quad (96)$$

From (85), it holds that $\Delta w_4 = 1/\beta_2(i\lambda\rho_2 w_5 - \rho_2 f_5)$. Because of $w_1 = w_4$ on I , elliptic regularity for the Dirichlet–Laplace operator (Theorem 3.2) yields

$$\begin{aligned} \|w_4\|_{H^2(\Omega_2)} &\leq C (\|i\lambda\rho_2 w_5 - \rho_2 f_5\|_{L^2(\Omega_2)} + \|w_1\|_{H^{3/2}(I)}) \\ &\leq C (|\lambda| \|w_5\|_{L^2(\Omega_2)} + \|f_5\|_{L^2(\Omega_2)} + \|w_1\|_{H^2(\Omega_1)}) \\ &\leq C (|\lambda| \|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma}). \end{aligned} \quad (97)$$

Applying Lemma 4.3 to ∇w_4 , we see that

$$\begin{aligned} \|\nabla w_4\|_{L^2(I)} &\leq C \|w_4\|_{H^2(\Omega_2)}^{1/2} \|w_4\|_{H^1(\Omega_2)}^{1/2} \\ &\leq C (|\lambda| \|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma})^{1/2} \|w\|_{\mathcal{H}_\gamma}^{1/2} \\ &\leq C (|\lambda|^{1/2} \|w\|_{\mathcal{H}_\gamma} + \|w\|_{\mathcal{H}_\gamma}^{1/2} \|f\|_{\mathcal{H}_\gamma}^{1/2}). \end{aligned} \quad (98)$$

Note that w_3 belongs to $H^2(\Omega_1)$ and is a solution of the problem

$$\begin{cases} \Delta w_3 = \frac{\rho_0}{\beta_0} (i\lambda w_3 - \rho_0^{-1} \mu \Delta w_2 - f_3) =: h^* \in L^2(\Omega_1), \\ \partial_\nu w_3 + \kappa w_3 = 0 \text{ on } \Gamma, \\ w_3 = 0 \text{ on } I. \end{cases}$$

By Remark 3.3 and Theorem 3.2, there exist $\eta_0 > 0$ such that

$$\begin{aligned} \|w_3\|_{H^2(\Omega_1)} &\leq C \|\eta_0 w_3 - h^*\|_{L^2(\Omega_1)} \\ &\leq C (\|w_3\|_{L^2(\Omega_1)} + |\lambda| \|w_3\|_{L^2(\Omega_1)} + \|\Delta w_2\|_{L^2(\Omega_1)} + \|f_3\|_{L^2(\Omega_1)}). \end{aligned}$$

From (35) and the last inequality, it follows that

$$\|w_3\|_{H^2(\Omega_1)} \leq C (\|w_3\|_{L^2(\Omega_1)} + |\lambda| \|w_3\|_{L^2(\Omega_1)} + |\lambda| \|\Delta w_1\|_{L^2(\Omega_1)} + \|\Delta f_1\|_{L^2(\Omega_1)} + \|f_3\|_{L^2(\Omega_1)}).$$

This and (34) imply

$$\|w_3\|_{H^2(\Omega_1)} \leq C \left[|\lambda| \|w\|_{\mathcal{H}_\gamma}^{1/2} \|f\|_{\mathcal{H}_\gamma}^{1/2} + |\lambda| \|w\|_{\mathcal{H}_\gamma} + \|f\|_{\mathcal{H}_\gamma} \right]. \quad (99)$$

Due to (86), w_1 satisfies the equation

$$(\eta_0 + \Delta^2)w_1 = \eta_0 w_1 + \beta_1^{-1} z$$

with $z := (\lambda^2 \rho_1 - \lambda^2 \gamma \Delta + i \lambda \rho \Delta)w_1 - \mu \Delta w_3 + (i \lambda \rho_1 - i \lambda \gamma \Delta - \rho \Delta)f_1 + (\rho_1 - \gamma \Delta)f_2$ and $\eta_0 > 0$. Note that, if $g := \mathcal{A}f$, then $g_1 = f_2$ and therefore

$$\|\Delta f_2\|_{L^2(\Omega_1)} = \|\Delta g_1\|_{L^2(\Omega_1)} \leq C \|g\|_{\mathcal{H}_\gamma} = C \|\mathcal{A}f\|_{\mathcal{H}_\gamma}. \quad (100)$$

From Theorem 3.2, the transmission conditions, inequalities (99), (97), (100), and $0 \in \rho(\mathcal{A})$, we obtain

$$\begin{aligned} \|w_1\|_{H^4(\Omega_1)} &\leq C \left(\|\eta_0 w_1 + \beta_1^{-1} z\|_{L^2(\Omega_1)} + \|\beta_1^{-1}(-\beta_2 \partial_\nu w_4 - \mu \partial_\nu w_3)\|_{H^{1/2}(I)} \right) \\ &\leq C \left(\lambda^2 \|w\|_{\mathcal{H}_\gamma} + |\lambda| \|w\|_{\mathcal{H}_\gamma}^{1/2} \|f\|_{\mathcal{H}_\gamma}^{1/2} + |\lambda| \|f\|_{\mathcal{H}_\gamma} + |\lambda| \|\mathcal{A}f\|_{\mathcal{H}_\gamma} \right) \\ &\leq C |\lambda| \left(|\lambda| \|w\|_{\mathcal{H}_\gamma} + \|w\|_{\mathcal{H}_\gamma}^{1/2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/2} + \|\mathcal{A}f\|_{\mathcal{H}_\gamma} \right). \end{aligned}$$

We have from (91) that

$$\|w_1\|_{H^1(\Omega_1)} \leq C |\lambda|^{-1} \left(\|w\|_{\mathcal{H}_\gamma}^{1/2} \|f\|_{\mathcal{H}_\gamma}^{1/2} + \|f\|_{\mathcal{H}_\gamma} \right).$$

Applying Lemma 4.3 to $\partial_\nu(\Delta w_1)$, using interpolation inequality and $0 \in \rho(\mathcal{A})$, we obtain that

$$\begin{aligned} \|\partial_\nu(\Delta w_1)\|_{L^2(I)} &\leq C \|w_1\|_{H^4(\Omega_1)}^{5/6} \|w_1\|_{H^1(\Omega_1)}^{1/6} \\ &\leq C |\lambda|^{5/6} \left(|\lambda| \|w\|_{\mathcal{H}_\gamma} + \|w\|_{\mathcal{H}_\gamma}^{1/2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/2} + \|\mathcal{A}f\|_{\mathcal{H}_\gamma} \right)^{5/6} |\lambda|^{-1/6} \left(\|w\|_{\mathcal{H}_\gamma}^{1/2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/2} + \|\mathcal{A}f\|_{\mathcal{H}_\gamma} \right)^{1/6} \\ &\leq C |\lambda|^{2/3} \left(|\lambda|^{5/6} \|w\|_{\mathcal{H}_\gamma}^{5/6} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/6} + |\lambda|^{5/6} \|w\|_{\mathcal{H}_\gamma}^{11/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/12} + \|\mathcal{A}f\|_{\mathcal{H}_\gamma} \right. \\ &\quad \left. + \|w\|_{\mathcal{H}_\gamma}^{5/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{7/12} + \|w\|_{\mathcal{H}_\gamma}^{1/2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/2} + \|w\|_{\mathcal{H}_\gamma}^{1/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{11/12} \right). \end{aligned} \quad (101)$$

From (98) and (101), it follows that

$$\begin{aligned} \|\partial_\nu(\Delta w_1)\|_{L^2(I)} \|\nabla w_4\|_{L^2(I)} &\leq C \left(|\lambda|^2 \|w\|_{\mathcal{H}_\gamma}^{11/6} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/6} + |\lambda|^{3/2} \|w\|_{\mathcal{H}_\gamma}^{4/3} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{2/3} \right. \\ &\quad \left. + |\lambda|^2 \|w\|_{\mathcal{H}_\gamma}^{23/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/12} + |\lambda|^{3/2} \|w\|_{\mathcal{H}_\gamma}^{17/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{7/12} \right. \\ &\quad \left. + |\lambda|^{2/3} \|w\|_{\mathcal{H}_\gamma}^{11/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{13/12} + |\lambda|^{7/6} \|w\|_{\mathcal{H}_\gamma}^{3/2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/2} \right. \\ &\quad \left. + |\lambda|^{7/6} \|w\|_{\mathcal{H}_\gamma} \|\mathcal{A}f\|_{\mathcal{H}_\gamma} + |\lambda|^{2/3} \|w\|_{\mathcal{H}_\gamma}^{1/2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{3/2} \right. \\ &\quad \left. + |\lambda|^{7/6} \|w\|_{\mathcal{H}_\gamma}^{13/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{11/12} + |\lambda|^{2/3} \|w\|_{\mathcal{H}_\gamma}^{7/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{17/12} \right). \end{aligned} \quad (102)$$

Applying Lemma 4.3 to $\partial_\nu w_3$ and using (34) and (99), we get

$$\begin{aligned} \|\partial_\nu w_3\|_{L^2(I)} &\leq C \|w_3\|_{H^2(\Omega_1)}^{1/2} \|w_3\|_{H^1(\Omega_1)}^{1/2} \\ &\leq C \left(|\lambda|^{1/2} \|w\|_{\mathcal{H}_\gamma}^{1/2} \|f\|_{\mathcal{H}_\gamma}^{1/2} + |\lambda|^{1/2} \|w\|_{\mathcal{H}_\gamma}^{3/4} \|f\|_{\mathcal{H}_\gamma}^{1/4} + \|w\|_{\mathcal{H}_\gamma}^{1/4} \|f\|_{\mathcal{H}_\gamma}^{3/4} \right). \end{aligned} \quad (103)$$

Then, from (98) and (103), we obtain

$$\begin{aligned} \|\partial_\nu w_3\|_{L^2(I)} \|\nabla w_4\|_{L^2(I)} &\leq C \left(|\lambda| \|w\|_{\mathcal{H}_\gamma}^{3/2} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/2} + |\lambda|^{1/2} \|w\|_{\mathcal{H}_\gamma} \|\mathcal{A}f\|_{\mathcal{H}_\gamma} + |\lambda| \|w\|_{\mathcal{H}_\gamma}^{7/4} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/4} \right. \\ &\quad \left. + |\lambda|^{1/2} \|w\|_{\mathcal{H}_\gamma}^{5/4} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{3/4} + \|w\|_{\mathcal{H}_\gamma}^{3/4} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{5/4} \right). \end{aligned} \quad (104)$$

Considering (102), (104), and Young's inequality, we observe that the worst term in the estimate of the right side of (96) is

$$|\lambda|^2 \|w\|_{\mathcal{H}_\gamma}^{23/12} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^{1/12} = \|w\|_{\mathcal{H}_\gamma}^{23/12} (|\lambda|^{24} \|\mathcal{A}f\|_{\mathcal{H}_\gamma})^{1/12} \leq \varepsilon \|w\|_{\mathcal{H}_\gamma}^2 + C_\varepsilon |\lambda|^{48} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^2.$$

This implies that (95) holds. Now, by (92), (93), (95), and Young's inequality, we obtain

$$\|w\|_{\mathcal{H}_\gamma}^2 \leq \varepsilon \|w\|_{\mathcal{H}_\gamma}^2 + C_\varepsilon |\lambda|^{48} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}^2 \quad \text{for any } \varepsilon > 0.$$

Hence,

$$\|w\|_{\mathcal{H}_\gamma} \leq C |\lambda|^{24} \|\mathcal{A}f\|_{\mathcal{H}_\gamma}, \quad (105)$$

which shows (80) with $\beta' = 1$ and $\beta = 24$ for the case $\gamma > 0$. Note that (105) implies that $i\lambda - \mathcal{A}$ is injective. In fact, let $w \in D(\mathcal{A})$ with $f := (i\lambda - \mathcal{A})w = 0$. Then $\mathcal{A}w = i\lambda w \in D(\mathcal{A})$, which shows $w \in D(\mathcal{A}^2)$, and we can apply (105) to see $w = 0$. Therefore, $i\mathbb{R} \cap \sigma(\mathcal{A}) = \emptyset$. Now, the assertion follows from Muñoz Rivera and Racke,³¹ Lemma 5.2.

For the case $\gamma = 0$, we can argue analogously and obtain (80) with $\beta' = 0$ and $\beta = 24$. \square

Remark 4.3. (a) Note that the proof gives no information on the optimal decay rate.

(b) It was shown in Barraza Martínez et al,¹² Theorem 3.2 and Theorem 5.2, that also in the isothermal situation, we have polynomial but no exponential stability if the membrane is undamped.

ACKNOWLEDGEMENT

Financial support through Minciencias (formerly Colciencias) via Project 121571250194 is gratefully acknowledged.

Open access funding enabled and organized by Projekt DEAL.

ORCID

Bienvenido Barraza Martínez  <https://orcid.org/0000-0002-5284-0031>

Robert Denk  <https://orcid.org/0000-0002-5406-4486>

Jonathan González Ospino  <https://orcid.org/0000-0002-2748-7715>

Jairo Hernández Monzón  <https://orcid.org/0000-0002-7841-134X>

REFERENCES

1. Borsuk M. Transmission Problems for Elliptic Second-Order Equations in Non-Smooth Domains. *Frontiers in Mathematics*. Basel: Springer Science & Business Media; 2010.
2. Ammari K, Nicaise S. Stabilization of a transmission wave/plate equation. *J Differ Equ*. 2010;249(3):707-727.
3. Balmès E, Germès S. Tools for viscoelastic damping treatment design. Application to an Automotive Floor Panel. *International Seminar on Modal Analysis*. Belgium: Leuven; 2002.
4. Denk R, Kammerlander F. Exponential stability for a coupled system of damped-undamped plate equations. *IMA J Appl Math*. 2018;83(2):302-322.
5. Kleinman RE, Martin PA. On single equations for the transmission problem of acoustics. *SIAM J Appl Math*. 1988;48(2):307-325.
6. Lagnese J, Leugering G. *Modelling, Analysis and Control of Dynamic Elastic Multi-Link Structures*. Birkhäuser Basel: Systems & Control: Foundations & Applications; 1994.
7. Leissa AW. *Vibration of Plates*. NASA SP. Scientific and Technical Information Division, National Aeronautics and Space Administration; 1969.
8. Muñoz Rivera JE, Naso MG. About asymptotic behavior for a transmission problem in hyperbolic thermoelasticity. *Acta Applicandae Math*. 2007;99(1):1-27.
9. Roy N, Germès S, Lefebvre B, Balmès E. Damping allocation in automotive structures using reduced models. In: *Proceedings of ISMA*; 2006:3915-3924.
10. Lagnese JE, Lions JL. *Modelling analysis and control of thin plates*. Masson: Recherches en mathématiques appliquées; 1988.
11. Hassine F. Asymptotic behavior of the transmission Euler-Bernoulli plate and wave equation with a localized Kelvin-Voigt damping. *Discrete Contin Dyn Syst - B*. 2016;21:1757-1774.
12. Barraza Martínez B, Denk R, Hernández Monzón J, Kammerlander F, Nendel M. Regularity and asymptotic behavior for a damped plate-membrane transmission problem. *J Math Anal Appl*. 2019;474(2):1082-1103.

13. Hernández Monzón J. A system of semilinear evolution equations with homogeneous boundary conditions for thin plates coupled with membranes. In Proceedings of the 2003 Colloquium on Differential Equations and Applications, volume 13 of Electron. J. Differ. Equ. Conf., pages 35–47. Southwest Texas State Univ., San Marcos, TX; 2005.
14. Liu H, Su N. Existence and uniform decay of solutions for a class of generalized plate-membrane-like systems. *Int J Math Math Sci*. 2006;2006:1-24.
15. Avalos G, Lasiecka I. Exponential stability of a thermoelastic system with free Boundary Conditions without Mechanical Dissipation. *SIAM J Math Anal*. 1998;29(1):155-182.
16. Chueshov I, Lasiecka I. Attractors and long time behavior of von Karman Thermoelastic Plates. *Appl Math Optim*. 2008;58(2):195-241.
17. Dell'Oro F, Muñoz Rivera JE, Pata V. Stability properties of an abstract system with applications to linear thermoelastic plates. *J Evol Equ*. 2013;13(4):777-794.
18. Fernández Sare HD, Liu Z, Racke R. Stability of abstract thermoelastic systems with inertial terms. *J Differ Equ*. 2019;267(12):7085-7134.
19. Lasiecka I, Pokojovy M, Wan X. Global existence and exponential stability for a non-linear thermoelastic Kirchhoff–Love plate. *Nonlinear Anal Real World Appl*. 2017;38:184-221.
20. Lasiecka I, Triggiani R. Sharp trace estimates of solutions to Kirchhoff and Euler-Bernoulli equations. *Appl Math Optim*. 1993;28(3):277-306.
21. Muñoz Rivera JE, Portillo Oquendo H. A transmission problem for thermoelastic plates. *Q Appl Math*. 2004;62(2):273-293.
22. Muñoz Rivera JE, Vega JC. Large time behaviour for non-simple thermoelasticity with second sound. *Electron J Differ Equ*. 2017;2017(259):1-7.
23. Agranovich MS. *Sobolev Spaces, their Generalizations and Elliptic Problems in Smooth and Lipschitz Domains*, Springer Monographs in Mathematics. Cham: Springer; 2015. Revised translation of the 2013 Russian original.
24. Ouhabaz EM. *Analysis of heat equations on domains*, volume 31 of London Mathematical Society Monographs Series. Princeton, NJ: Princeton University Press; 2005.
25. Agranovich MS, Vishik MI. Elliptic problems with a parameter and parabolic problems of general type. *Russ Math Surv*. 1964;19(3):53-157.
26. Lunardi A. *Interpolation Theory*, volume 16 of Lecture Notes (Scuola Normale Superiore). Edizioni della Normale, Third edition; 2018.
27. Barraza Martínez B, González Ospino J, Hernández Monzón J. *Vector-valued function and distribution spaces on the torus*. Barranquilla: Editorial Universidad del Norte; 2019.
28. Triebel H. *Interpolation Theory, Function Spaces, Differential Operators*, volume 18 of North-Holland Mathematical Library. Berlin: North-Holland Publishing Company; 1978.
29. Prüss J. On the spectrum of C_0 -semigroups. *Trans Amer Math Soc*. 1984;284(2):847-857.
30. Liu Z, Zheng S. *Semigroups associated with dissipative systems*: Chapman and Hall/CRC; 1999.
31. Muñoz Rivera JE, Racke R. Transmission problems in (thermo)viscoelasticity with Kelvin–Voigt damping Nonexponential, strong, and polynomial stability. *SIAM J Math Anal*. 2017;49(5):3741-3765.
32. Amann H. *Linear and quasilinear parabolic problems*, volume 89 of Monographs in Mathematics, vol. I. Boston, MA: Birkhäuser Boston, Inc.; 1995. Abstract linear theory.
33. Mitidieri E. A Rellich type identity and applications. *Commun Partial Differ Equ*. 1993;18(1-2):125-151.

How to cite this article: Barraza Martínez B, Denk R, González Ospino J, Hernández Monzón J, Rau S. Long-time asymptotics for a coupled thermoelastic plate–membrane system. *Math Meth Appl Sci*. 2021;44: 12881-12908. <https://doi.org/10.1002/mma.7592>