Investigations on the Hadamard product of matrices and polynomials

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Doaa Alsaafin

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1. Referee: Prof. Dr. Jürgen Garloff, University of Konstanz.
2. Referee: Prof. Dr. Apoorva Khare, Indian Institute of Science, Bangalore, India.

Abstract

The Hadamard product of two matrices of the same order is obtained by entry-wise multiplication of their coefficients. In a similar way, the Hadamard power of a matrix and a polynomial is formed by real powers of its coefficients. Results for the Hadamard product of some important classes of matrices, e.g., positive definite matrices, conditionally negative definite matrices and matrices with one positive eigenvalue are presented. The results are extended to give sufficient conditions for symmetric matrices to have exactly one positive eigenvalue. A Hurwitz (or stable) polynomial is a real polynomial whose roots are located in the open left half of the complex plane. Results for the Hadamard square root of Hurwitz polynomials of degree five are given. Also, a type of Oppenheim's Inequality for Hurwitz matrices is presented. Finally, interval matrices, i.e., matrices with intervals as entries, are studied, and new results for the interval property of several classes of matrices, e.g., inverse M-matrices, conditionally positive (negative) semidefinite matrices, and infinitely divisible matrices are given.

Dedication

To my parents, Mahmoud and Nada.

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Deutsche Zusammenfassung

Das Hadamard-Produkt von zwei Matrizen gleicher Ordnung erhält man, indem man die beiden Matrizen koeffizientenweise miteinander multipliziert. In ähnlicher Weise erhält man auch die Hadamard-Potenz einer Matrix und eines Polynoms durch Potenzierung der Koeffizienten. In der Dissertation werden Ergebnisse zum Hadamard-Produkt von wichtigen Klassen von Matrizen hergeleitet, und zwar für die positiv definiten Matrizen, die bedingt negativ definiten Matrizen sowie für Matrizen mit einem positiven Eigenwert. Diese Resultate werden erweitert auf hinreichende Bedingungen für symmetrische Matrizen, genau einen positiven Eigenwert zu besitzen.

Es schließt sich eine Untersuchung von Intervallmatrizen an, d. h. von Matrizen, deren Koeffizienten Intervalle sind. Hier werden neue Ergebnisse für die sogenannte Intervalleigenschaft von verschiedenen Klassen von Matrizen nachgewiesen, und zwar für die inversen M-Matrizen, für die bedingt positiv (negativ) definiten Matrizen und für die unendlich teilbaren Matrizen.

Ein Hurwitz-Polynom (auch stabiles Polynom genannt) ist ein Polynom, dessen sämtliche Nullstellen in der offenen linken komplexen Halbebene liegen. Eine hinreichende und notwendige Bedingung dafür, dass die Hadamard-Quadratwurzel eines solchen Polynoms vom Grad fünf selbst wieder ein Hurwitz-Polynom ist, wird hergeleitet. Auch wird die Gültigkeit der Ungleichung von Oppenheim für Hurwitz-Matrizen untersucht.

Contents

1 Introduction

Otherwise than by the usual matrix multiplication, matrices of the same size can be multiplied entry-wise, which is called the *Hadamard product*. Results obtained from both methods are different. The Hadamard product of two matrices is very similar to matrix addition, entries in the same row and column of the given matrices are multiplied together to form a new matrix. The Hadamard product is named after the French mathematician Jacques Hadamard, and is also known as the entry-wise or *Schur product* due to an earlier work of the German mathematician Issai Schur. This product has been used in finite geometry, group theory, number theory, regular graphs, and statistics. For an excellent review of the Hadamard product and its applications, see, e.g., [CLLB12].

1.1 Overview

This thesis contains three main parts. The first looks at the Hadamard powers of some important classes of symmetric matrices. These classes, e.g., symmetric positive definite matrices and symmetric positive matrices with one positive eigenvalue, play an important role in various branches of mathematics and other sciences, see, e.g., [Pen09]. To obtain our main results in this part we use Bernstein functions. A particular instance of the matrices that we study are the conditionally negative semidefinite matrices. It is well-known that the Hadamard inverse of any conditionally negative semidefinite matrix with positive entries is infinitely divisible. We generalize this result to other forms of matrices. Several conditions for symmetric matrices to have exactly one positive eigenvalue are given as well.

The second main part of this thesis looks at Hurwitz polynomials, i.e., real polynomials whose roots are located in the open left half of the complex plane, and their associated Hurwitz matrices. Over the years, various criteria for deciding whether a polynomial is Hurwitz have been obtained in the literature. We give a new formula for the leading principal minors of Hurwitz matrices which leads us to (i) a new criterion for deciding whether a polynomial is Hurwitz, (ii) sufficient and necessary conditions for the Hadamard square root of Hurwitz polynomials of degree five to be Hurwitz, and (iii) a new Oppenheim-type determinantal inequality for Hurwitz matrices associated with Hurwitz polynomials.

Finally, the last main part of this thesis looks at a special type of matrices called the interval matrices, i.e., matrices with intervals as entries. We investigate various properties of interval matrices and prove the interval property of several classes of matrices, e.g., inverse M-matrices, conditionally positive (negative) semidefinite matrices, and infinitely divisible matrices.

1.2 Definitions and Notations

1.2.1 Hadamard Product and Hadamard Power

We use the following standard notation. The set of positive real numbers is denoted by $\mathbb{R}_+$. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n,m}$. A real matrix B is called *nonnegative* (*positive*), if all the entries of B are nonnegative (positive) and this will be denoted by $B \geq (>)0$. Similarly, the strict and non-strict inequalities between vectors are understood entry-wise, e.g., for $\mathbf{x} = [x_i]_{i=1}^n \in \mathbb{R}^n$, $\mathbf{x} > 0$ and $\mathbf{x} \geq 0$ mean $x_i > 0$ and $x_i \geq 0$, respectively, $i = 1, \ldots, n$. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be real $n \times n$ matrices. Their *Hadamard product* (also called *Schur product*) $A \circ B$ is defined as the entry-wise product of A and B, $A \circ B = [a_{ij} b_{ij}]$. The *Hadamard unit* matrix is the matrix E all of whose entries are 1 (the size of E being understood). A matrix is *Hadamard invertible* if all its entries are non-zero, and $A^{\circ -1} = [1/a_{ij}]$ is then called the *Hadamard inverse* of A. If all entries of A are non-negative, then the *r-th Hadamard power* of A is $A^{\circ r} = [a_{ij}^r]$, $r > 0$. We define the *Hadamard exponential* of A by $e^{\circ A} = [e^{a_{ij}}]$ and, if A has all entries positive, the *Hadamard logarithm* of A by $log^{\circ}(A) = [log(a_{ij})]$.

A real $n \times n$ matrix A is said to be *positive semidefinite* if $\mathbf{x}^T A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$; A is *positive definite* if $\mathbf{x}^T A \mathbf{x} > 0$ for every nonzero vector $\mathbf{x} \in \mathbb{R}^n$. Suppose that A is symmetric positive semidefinite and that $A \geq 0$.

We say that A is *infinitely divisible* if the matrix $A^{\circ r}$ is positive semidefinite for every positive r. We consider polynomials with real coefficients, i.e., polynomials of the form

$$p(x) = \sum_{k=0}^{n} a_k x^k, \tag{1.1}$$

where a_k , $k = 0, \ldots, n$, are real numbers. The polynomial p is said to be *Hurwitz* or *stable* if all the roots of p lie in the open left half of the complex plane. Let $p(x), q(x)$ be two polynomials of equal degree n,

$$p(x) = \sum_{k=0}^{n} a_k x^k \ , \quad q(x) = \sum_{k=0}^{n} b_k x^k.$$

Then the *Hadamard product* of the two polynomials is defined by

$$(p \circ q)(x) := \sum_{k=0}^{n} a_k b_k x^k.$$

If p has only real positive coefficients and $t \in \mathbb{R} \setminus \{0\}$, the t-th *Hadamard power* of p is the polynomial $p^{\circ t}(x) = \sum_{k=0}^{n} a_k^t x^k$. By $\boldsymbol{P}_n$, we denote the family of all polynomials of degree n with positive coefficients and by $\boldsymbol{H}_n$ the family of all Hurwitz polynomials in $\boldsymbol{P}_n$.

1.2.2 Conditionally Positive and Conditionally Negative Matrices

Let $\mathbf{e} \in \mathbb{R}^n$ be the vector of all ones and $\mathbf{e}_i \in \mathbb{R}^n$ be the vector which has in its i-th component a one, while all its other components are zero. A real symmetric $n \times n$ matrix A is said to be *conditionally positive (negative) semidefinite* if $\mathbf{x}^T A\mathbf{x} \geq 0$ (≤ 0) for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$ such that $\mathbf{x}^T\mathbf{e} = 0$; A is *conditionally positive (negative) definite* if $\mathbf{x}^T A\mathbf{x} > 0$ (< 0) for all $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \neq 0$ such that $\mathbf{x}^T\mathbf{e} = 0$. If A is a symmetric matrix such that $a_{ij} > 0$ for all i, j and A has exactly one positive eigenvalue, then we say that A is in the class $\mathcal{A}$. The absolute value of vectors and matrices is understood entry-wise.

1.2.3 Matrix Minors

For κ, n we denote by $Q_{\kappa,n}$ the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \ldots, n\}$. Let A be a real $n \times n$ matrix. For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\kappa)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_\kappa) \in Q_{\kappa,n}$, we denote by $A[\alpha|\beta]$ the $\kappa \times \kappa$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_\kappa$. We suppress the brackets when we enumerate the indices explicitly. When $\alpha = \beta$, the *principal submatrix* $A[\alpha \mid \alpha]$ is abbreviated to $A[\alpha]$ and $\det A[\alpha]$ is called a *principal minor*. In the special case where $\alpha = (1, 2, \ldots, \kappa)$, we refer to the principal submatrix $A[\alpha]$ as the *leading principal submatrix* (and to $\det A[\alpha]$ as the *leading principal minor*) *of order* κ.

1.2.4 The Set of P, Z, M, H-Matrices

A real square matrix A is called a *P-matrix* if every principal minor is positive. A real square matrix A is called a *Z-matrix* if all the off-diagonal entries of A are nonpositive. Note that any Z-matrix A can be written as $A = sI - B$, where s is a real number and $B \geq 0$. A Z-matrix A is called an *M-matrix*, if A can be written as $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B)$. Here $\rho(B)$ stands for the spectral radius of the matrix B. A matrix $A \in \mathbb{R}^{n,n}$ is called *inverse nonnegative* if A is nonsingular and $0 \leq A^{-1}$. A Z-matrix is an *M-matrix* if it is inverse nonnegative. If $A = [a_{ij}]$ is a complex square matrix, the *comparison matrix* $\mathcal{M}(A) = [m_{ij}]$ of A is defined by

$$m_{ij} = \begin{cases} |a_{ij}|, \ i = j \\ -|a_{ij}|, \ i \neq j. \end{cases} \tag{1.2}$$

We say that A is an *H-matrix* if $\mathcal{M}(A)$ is an M-matrix. A matrix A is called an *inverse M-matrix* (*H-matrix*) if A is nonsingular and A^{-1} is an M-matrix (H-matrix).

1.2.5 Interval Matrices and Interval Property

We consider (matrix) intervals $[A] := [\underline{A}, \overline{A}]$, such that,

$$[\underline{A}, \overline{A}] = \left\{ A \in \mathbb{R}^{n,n} \mid \underline{A} \leq A \leq \overline{A} \right\},$$

where $\underline{A} \le \overline{A}$ with $(\underline{A})_{ij} = \underline{a}_{ij}$, $(\overline{A})_{ij} = \overline{a}_{ij}$, $i, j = 1, \dots, n$. A *vertex matrix* of $[A]$ is a matrix $A = [a_{ij}]$ with $a_{ij} \in \left\{\underline{a}_{ij}, \overline{a}_{ij}\right\}$; $\underline{A}$ and $\overline{A}$ are called the *corner matrices*. Each matrix interval $[A] = [\underline{A}, \overline{A}]$ can also be represented as an *interval matrix*, i.e., as a matrix with entries taken from the set of the compact nonempty real intervals, i.e.,

$$[A] = [[\underline{a}_{ij}, \overline{a}_{ij}]]_{i,j=1}^{n}.$$

Also, each matrix interval $[A] = [\underline{A}, \overline{A}]$ can be represented as $\{A \in \mathbb{R}^{n,n} \mid \mid A - A_c \mid \le \Delta\}$, where $A_c = \frac{1}{2}(\overline{A} + \underline{A})$ is the *midpoint matrix* and $\Delta = \frac{1}{2}(\overline{A} - \underline{A})$ is the *radius matrix*, in particular, $\underline{A} = A_c - \Delta$ and $\overline{A} = A_c + \Delta$. The matrix interval $[A]$ is called *regular* if A is nonsingular for all $A \in [A]$. If A_c and Δ are symmetric, then we call $[A]$ *symmetric* and define the *symmetric part* of $[A]$ as

$$[A]_{sym} := \left\{B \in [A] : B = B^T\right\}.$$

Let V be a fixed set of vertex matrices. We say that a set S of matrices *has the interval property* (*with respect to* V) if $[A] \subset S$ whenever $V([A]) \subset S$ for any matrix interval $[A]$. Here it is implicitly understood that $S \subset \mathbb{R}^{n,n}$ for an arbitrary, but fixed n. In the sequel we abbreviate 'interval property' by 'IP' when referring to a specified property. We extend properties of real matrices to matrix intervals by saying that a *matrix interval has* a certain *property* if each real matrix contained in it possesses this property.

1.2.6 Linear Complementarity Problem

Let $A \in \mathbb{R}^{n,n}$ and $\mathbf{q} \in \mathbb{R}^n$. The *Linear Complementarity Problem* (LCP)$(\mathbf{q}, A)$ is to find a vector $\mathbf{z} \in \mathbb{R}^n$ such that

$$A\mathbf{z} + \mathbf{q} \ge 0, \;\; \mathbf{z} \ge 0 \;\text{ and }\; \mathbf{z}^T(A\mathbf{z} + \mathbf{q}) = 0.$$

Notice that if $\mathbf{q} \ge 0$, $\mathbf{z} = 0$ is always a solution, called the trivial solution. Thus, given a matrix A, one often wants to know if the only solution when $\mathbf{q} \ge 0$ is the trivial solution, or if there are any others. In this thesis we consider classes of matrices that have been evolved while studying LCP.

2 On the Hadamard Power of Nonnegative Matrices

The d-th *Hadamard power* ($d \in \mathbb{R}_+$) of a matrix is obtained by taking the d-th power of each entry. Entry-wise powers of matrices have been well-studied in the literature, and have recently received renewed attention due to their application in many areas. A direct consequence of Schur's product theorem, see Theorem 2.4 below, is that if A is symmetric positive (semi)definite, then $A^{\circ m}$ is also positive (semi)definite for all positive integers m. It is natural to ask what happens if the integral Hadamard powers are replaced with the fractional Hadamard powers $A^{\circ r}$. There has been much interest in studying the conditions under which these Hadamard power functions preserve properties such as positivity, monotonicity, and convexity for various classes of nonnegative matrices. See for example [FH77], [GKR15], [Hia09]. In this chapter, we study powers of some classes of nonnegative matrices. We obtain our results by using one of the so-called *Bernstein functions*.
The organization of this chapter is as follows. In Section 2.1, completely monotonic functions are studied. In Section 2.2, results for conditionally negative semidefinite matrices and matrices in class $\mathcal{A}$ are given. In Section 2.3, infinitely divisible matrices are considered, examples of such matrices and ways to generate matrices of this class are presented. Finally in Section 2.4, conditions for symmetric matrices to have exactly one positive eigenvalue are given.

2.1 Completely Monotonic Functions

A positive function defined on $\mathbb{R}_+$ such that the sequence of its derivatives alternates signs at every point, is called a *completely monotonic* function. We denote the set of completely monotonic functions by $\mathcal{CM}$. There are interesting topics in probability and statistics where $\mathcal{CM}$ functions play a role; see [Kim74] for one such topic.
The following is a classical definition of $\mathcal{CM}$ functions. See, e.g., in [Ber08].

Definition 2.1. *A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is said to be completely monotonic, denoted by $\mathcal{CM}$, if f has derivatives of all orders and satisfies the inequality*

$$(-1)^n f^{(n)}(x) \geq 0, \; x > 0 \; and \, n = 0, 1, 2, \dots. \tag{2.1}$$

In particular, this implies that each $\mathcal{CM}$ function on $\mathbb{R}_+$ is positive, decreasing, and convex, with concave first derivative.

2.1.1 Bernstein Functions

A differentiable function $f : \mathbb{R}_+ \longrightarrow \mathbb{R} \cup \{0\}$ is called a *Bernstein function*, denoted by $\mathcal{B}$, if f' is completely monotonic. For example, the elementary function $f(x) = x^{\alpha}$, $0 \leq \alpha \leq 1$, is a Bernstein function. The notion of the Bernstein function goes back to the treatment of potential theory in the work of the school of J. Deny. In [Boc55], a Bernstein function is called a completely monotonic function, which was characterized by Bernstein in 1929 as a Laplace transform of positive measure on $\mathbb{R}_+ \cup \{0\}$, see [Ber29]. The literature devoted to this class of functions is impressive since they have remarkable applications in various branches, for instance, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics. A detailed collection of the most important properties as well as lists of Bernstein functions can be found in the survey paper [Ber08] and the monograph [SSV12].
The set $\mathcal{B}$ is a convex cone closed under pointwise convergence. Since a Bernstein function $f(x)$ is nonnegative and increasing, $f(x)$ has a nonnegative limit at zero.

The Bernstein functions on $\mathbb{R}_+$ can be characterized as follows:

Theorem 2.1. *[SSV12, Theorem 3.2] A function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a Bernstein function if and only if it admits the representation*

$$f(x) = \alpha + \beta x + \int_0^{\infty} (1 - e^{-tx})\, \mu(dt), \tag{2.2}$$

where $\alpha, \beta \geq 0$ and μ is a measure on $\mathbb{R}_+$ satisfying $\int_0^{\infty} min\{1, t\}\, \mu(dt) < \infty$.

Example 2.1. *The following are examples of Bernstein functions presented in [Ber08]:*

1. $f(x) = \alpha x + \beta$, *where* $\alpha, \beta \geq 0$.
2. $f(x) = 1 - e^{-xt}$, *where* $t > 0$.
3. $f(x) = log(1 + x)$.
4. $f(x) = x^{\alpha}$, $0 \leq \alpha \leq 1$.
5. $f(x) = (1 + 1/x)^x$, $(1 + x)^{1+1/x}$.

The following theorem gives a relation between the set of the completely monotonic and Bernstein functions.

Theorem 2.2. *[Ber96] Let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \cup \{0\}$. Then $f \in \mathcal{B}$ if and only if $exp(-tf) \in \mathcal{CM}$, for all $t > 0$.*

We end this section by the following composition results, which come from the integral representations of the $\mathcal{CM}$ and $\mathcal{B}$ functions.

Corollary 2.1. *The following composition results hold:*

- $f, g \in \mathcal{B} \Rightarrow f \circ g \in \mathcal{B}$,
- $f \in \mathcal{CM}$, $g \in \mathcal{B} \Rightarrow f \circ g \in \mathcal{CM}$.

2.2 Conditionally Negative Matrices and Matrices with One Positive Eigenvalue

It is well-known that a non-negative conditionally negative semidefinite matrix has exactly one positive eigenvalue, see [BR97, Corollary 4.1.5]. Let $A = [a_{ij}]$ be a real symmetric matrix. When $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \cup \{0\}$ is a Bernstein function, we prove that $[f(a_{ij})]$ has exactly one positive eigenvalue. This result leads us to generalize a remarkable result by [Rea99], and establish more results on conditionally negative semidefinite matrices, infinitely divisible matrices, and matrices with exactly one positive eigenvalue, see [ASG20].

We start this section with the *Perron-Frobenius Theorem* for nonnegative matrices, proved by Oskar Perron and Georg Frobenius, and the *Schur Product Theorem*, proved by Issai Schur. These are two prominent theorems in matrix theory, see, e.g., [HJ13, Theorem 8.4.4, Theorem 7.5.3].

Theorem 2.3. *(Perron-Frobenius Theorem): Suppose $A \in \mathbb{R}^{n,n}$ is an irreducible matrix. Then it has a positive eigenvalue μ such that for any other eigenvalue λ of A, we have $|\lambda| \leq \mu$. Furthermore, there exists a unique eigenvector $\mathbf{v} = [v_i]_{i=1}^n$ of A, which can be taken positive, such that $A\mathbf{v} = \mu\mathbf{v}$ and $\sum_{i=1}^n v_i = 1$.*

Theorem 2.4. *(Schur Product Theorem):*
If A and B are symmetric positive semidefinite matrices of size n, then $A \circ B$ is positive semidefinite, too.

The following are three remarkable existence theorems concerning conditionally positive semidefinite, conditionally negative semidefinite, and infinitely divisible matrices.

Theorem 2.5. *([BR97, Lemma 4.1.4]): If A is a conditionally negative semidefinite matrix, then A has at most one positive eigenvalue.*

Proof. Suppose A has two positive eigenvalues, λ and μ, with $\mathbf{x}$ and $\mathbf{y}$ as the corresponding eigenvectors, respectively, and we assume $\mathbf{x}$ and $\mathbf{y}$ to be orthogonal.
Since $\mathbf{x}^T A\mathbf{x} = \lambda\mathbf{x}^T\mathbf{x} > 0$, then $\mathbf{x}^T\mathbf{e} \neq 0$. Similarly, $\mathbf{y}^T\mathbf{e} \neq 0$, and by a suitable normalization we assume that $\mathbf{x}^T\mathbf{e} = \mathbf{y}^T\mathbf{e}$. Then $(\mathbf{x}-\mathbf{y})^T\mathbf{e} = 0$, but

$$\begin{aligned}(\mathbf{x}-\mathbf{y})^T A(\mathbf{x}-\mathbf{y}) &= \mathbf{x}^T A\mathbf{x} + \mathbf{y}^T A\mathbf{y} \\ &= \lambda\mathbf{x}^T\mathbf{x} + \mu\mathbf{y}^T\mathbf{y} > 0,\end{aligned}$$

and this contradicts the fact that A is conditionally negative semidefinite. Therefore, A has at most one positive eigenvalue, counting multiplicities. □

Theorem 2.6. *([Hor90, p.144]; [Mic86, Corollary 2.1]): The symmetric matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is conditionally positive semidefinite if and only if its Hadamard exponential $e^{\circ tA}$ is positive semidefinite for all $t \geq 0$. Moreover, $e^{\circ tA}$ is positive definite for all $t > 0$ if and only if $2a_{ij} < a_{ii} + a_{jj}$, for all $i \neq j$.*

Remark 2.1. *([Rea99, p. 38]): If $A = [a_{ij}]$ is a symmetric positive semidefinite matrix, then*

$$2a_{ij} < a_{ii} + a_{jj} \text{ for all } i \neq j \;\Leftrightarrow A \text{ has distinct rows.}$$

However, this is not true for conditionally positive semidefinite matrices. For example, consider

$$A = [a_{ij}] = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 5 \end{bmatrix}.$$

Then A is conditionally positive semidefinite and has distinct rows, but $2a_{ij} = a_{ii} + a_{jj}$ when $i = 1$ and $j = 2$.

Necessary and sufficient conditions for a matrix to be conditionally positive semidefinite are also known. One of them says that an $n \times n$ symmetric matrix $B = [b_{ij}]$ is conditionally positive semidefinite if and only if the $(n-1) \times (n-1)$ matrix $D = [d_{ij}]$ with entries

$$d_{ij} = b_{ij} + b_{i+1,j+1} - b_{i,j+1} - b_{i+1,j} \tag{2.3}$$

is positive semidefinite. See [HJ91, pp. 457-458] where these criteria are used to prove the infinite divisibility of some matrices. Several applications of conditionally positive semidefinite matrices may be found in the book [BR97].

Theorem 2.7. *([Rea99, Theorem 2.7]): Let $A \in \mathcal{A}$. Then $A^{\circ -1}$ is positive semidefinite. Moreover, $A^{\circ -1}$ is positive definite if and only if*

$$\frac{a_{ii}}{v_i^2} + \frac{a_{jj}}{v_j^2} < 2\frac{a_{ij}}{v_i v_j}, \quad for\ all\ i \neq j,$$

where $\mathbf{v} = [v_i]_{i=1}^n$ is the Perron eigenvector for A.

Proof. Let the eigenvalues of A be $\lambda_1 \leq \cdots \leq \lambda_{n-1} < r$ with $A\mathbf{u}_i = \lambda_i \mathbf{u}_i$, for $1 \leq i \leq n-1$ and $A\mathbf{v} = r\mathbf{v}$. By writing the matrix A in the form

$$A = r\mathbf{v}\mathbf{v}^T + \lambda_{n-1}\mathbf{u}_{n-1}\mathbf{u}_{n-1}^T + \cdots + \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T,$$

we have

$$\mathbf{x}^T WAW\mathbf{x} = \mathbf{x}^T(r\mathbf{e}\mathbf{e}^T + \lambda_{n-1}(W\mathbf{u}_{n-1})(W\mathbf{u}_{n-1})^T + \cdots + \lambda_1(W\mathbf{u}_1)(W\mathbf{u}_1)^T)\mathbf{x},$$

where $\mathbf{x} \in \mathbb{R}^n$ and $W = \text{diag}(\frac{1}{v_1}, \ldots, \frac{1}{v_n})$. If $\mathbf{x}^T\mathbf{e} = 0$, then

$$\mathbf{x}^T WAW\mathbf{x} \leq 0,$$

i.e. $WAW = B$ is conditionally negative semidefinite, where $B = [b_{ij}] = \frac{a_{ij}}{v_i v_j}$ for all i, j. Next, recall that for $t > 0$,

$$\frac{1}{t} = \int_0^\infty e^{-ts}\, ds.$$

Hence,

$$\mathbf{x}^T \left(\frac{1}{b_{ij}}\right) \mathbf{x} = \int_0^\infty \mathbf{x}^T \left(e^{-b_{ij}s}\right) \mathbf{x}\, ds,$$

and since $[-b_{ij}s]$, for $s > 0$ is conditionally positive semidefinite, by Theorem 2.6 $[e^{-b_{ij}s}]$ is positive semidefinite, so $[1/b_{ij}] = (WAW)^{\circ -1} = W^{-1}A^{\circ -1}W^{-1}$ is positive semidefinite. Therefore $A^{\circ -1}$ is positive semidefinite. Finally, $A^{\circ -1}$ is positive definite if and only if $W^{-1}A^{\circ -1}W^{-1}$ is positive definite if and only if $[e^{-b_{ij}s}]$ is positive definite if and only if $b_{ii} + b_{jj} < 2b_{ij}$, for all $i \neq j$, if and only if

$$\frac{a_{ii}}{v_i^2} + \frac{a_{jj}}{v_j^2} < 2\frac{a_{ij}}{v_i v_j}, \quad \textit{for all } i \neq j.$$

□

Before we move to our results in this section, we need the following well-known lemmata.

Lemma 2.1. *([BR97, Corollary 4.4.5, Theorem 4.4.6]; [Rea99, Corollary 2.8, proof of Theorem 2.7]):*
Let $A = [a_{ij}] \in \mathcal{A}$. Then the following properties hold:

1. *$A^{\circ -1}$ is positive semidefinite. Moreover, it is positive definite if A is invertible.*
2. *If A is conditionally negative definite, then $A^{\circ -1}$ is positive definite if and only if $2a_{ij} > a_{ii} + a_{jj}$, for all $i \neq j$.*
3. *The matrix $A \circ (\mathbf{v}\mathbf{v}^T)^{\circ -1}$ is conditionally negative semidefinite, where $\mathbf{v}$ is the Perron eigenvector of A.*

Lemma 2.2. *([BR97, Lemma 4.3.5]):*
Let A be an $n \times n$ symmetric conditionally negative semidefinite matrix that is not negative semidefinite, and suppose that $\mathbf{x}^T A\mathbf{x} \neq 0$ for every non-zero $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{e} = 0$. Then A is invertible with one positive eigenvalue.

Proof. Suppose $A\mathbf{x} = 0$ for some $\mathbf{x}$. If $\mathbf{x}^T\mathbf{e} = 0$, then $\mathbf{x}^T A\mathbf{x} = 0$ which implies $\mathbf{x} = 0$ by the hypothesis on A.
Without loss of generality, let $\mathbf{x}^T\mathbf{e} > 0$. Since A is not negative semidefinite, there exists $\mathbf{y}$ such that $\mathbf{y}^T A\mathbf{y} > 0$. Again, $\mathbf{y}^T\mathbf{e}$ must be nonzero. Without loss of generality, let $\mathbf{y}^T\mathbf{e} > 0$. Define

$$\alpha = \frac{\mathbf{x}^T\mathbf{e}}{\mathbf{y}^T\mathbf{e}}.$$

Then $(\mathbf{x} - \alpha\mathbf{y})^T\mathbf{e} = 0$ and $(\mathbf{x} - \alpha\mathbf{y})^T A(\mathbf{x} - \alpha\mathbf{y}) = \alpha^2\mathbf{y}^T A\mathbf{y} > 0$, which is a contradiction. Thus $\mathbf{x} = 0$ and hence A must be invertible. By Theorem 2.5, A has at most one positive eigenvalue. Since A is not negative semidefinite then it has at least one positive eigenvalue. Therefore A has one positive eigenvalue, and the proof is complete. □

Lemma 2.3. *(Sylvester's Law of Inertia, e.g., [HJ13, Theorem 4.5.8]):*
Let A and B be Hermitian matrices, then there exists a non-singular matrix S such that $A = SBS^*$ if and only if A and B have the same inertia, i.e., the same number of positive, zero, and negative eigenvalues.

Theorem 2.8. *([ASG20, Theorem 6]): Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a symmetric conditionally negative semidefinite matrix with all entries positive and let $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \cup \{0\}$ be a Bernstein function. Then $[f(a_{ij})] \in \mathcal{A}$. Moreover, it is invertible if A is invertible.*

Proof. Let $f \in \mathcal{B}$, thus by (2.2),

$$f(a_{ij}) = \alpha + \beta a_{ij} + \int_0^\infty (1 - e^{-ta_{ij}}) d\mu(t),$$

where $\alpha, \beta \geq 0$ and $[\alpha + \beta a_{ij}]$ is conditionally negative semidefinite.
Since A is conditionally negative semidefinite, $-tA$ is conditionally positive semidefinite and so by Theorem 2.6, $[e^{\circ -tA}]$ is positive semidefinite for all $t > 0$. Therefore, $[E - e^{\circ -tA}]$ is conditionally negative semidefinite. Whence $[f(a_{ij})]$ is conditionally negative semidefinite, and hence it has one positive eigenvalue.
To prove the sufficient condition on the invertibility of $[f(a_{ij})]$, suppose A is in addition invertible. Then, Lemma 2.1(i), (ii) implies that $2a_{ij} > a_{ii} + a_{jj}$, for all $i \neq j$ and so $-A$ is invertible, conditionally positive semidefinite and has the property $-2a_{ij} < -a_{ii} - a_{jj}$, for all $i \neq j$. By Theorem 2.6, $[e^{\circ -tA}]$ is positive definite and hence $[-e^{\circ -tA}]$ is conditionally negative definite for all $t > 0$. This implies that $[E - e^{\circ -tA}]$ is conditionally negative definite. Therefore, $[f(a_{ij})]$ is conditionally negative definite and since it has a positive eigenvalue, it is not negative semidefinite. By using Lemma 2.2, we conclude that $[f(a_{ij})]$ is invertible. □

Theorem 2.9. *([ASG20, Theorem 7]): Let $A \in \mathcal{A}$ and $0 < \alpha \leq 1$. Then $A^{\circ\alpha}$ belongs also to $\mathcal{A}$, and is invertible if A is invertible.*

Proof. Let $A \in \mathcal{A}$ and let $\mathbf{v} = [v_i]_{i=1}^n$ be the Perron eigenvector of A. By Lemma 2.1(iii), $B := A \circ (\mathbf{v}\mathbf{v}^T)^{\circ -1}$ is conditionally negative semidefinite and hence it has exactly one positive eigenvalue. For $0 < \alpha \leq 1$, the function $f(x) = x^\alpha$ is a Bernstein function. Thus, by Theorem 2.8, $B^{\circ\alpha}$ has exactly one positive eigenvalue and is invertible if B is invertible. Let $W = \operatorname{diag}(v_1^\alpha, \ldots, v_n^\alpha)$, then $A^{\circ\alpha} = WB^{\circ\alpha}W$. By Lemma 2.3, $A^{\circ\alpha}$ has one positive eigenvalue and is invertible if $B^{\circ\alpha}$ is invertible. Finally, by Lemma 2.3, B is invertible if A is invertible.

□

2.3 Infinitely Divisible Matrices

A classical problem in matrix analysis involves the study of functions that act entry-wise on matrices and preserve positivity. In particular, the study of entry-wise power functions $t \longrightarrow t^r$ has been of special interest to various mathematicians; see [BE07], [GKR15],

[Hor69].
For a symmetric matrix A with positive entries, we say that A is *infinitely divisible* if the matrix $A^{\circ r}$ is positive semidefinite for every non-negative r. A lot of work has been done on the real entry-wise powers preserving the positive semidefiniteness of various families of matrices. For more information on this, see [FH77], [Jai17]. For a symmetric positive semidefinite matrix A, it is known that $A^{\circ r}$ is positive semidefinite if r is a positive integer or $r \geq n-2$, see [FH77], [Jai17]. This inequality is sharp [FH77, Theorem 2.2], i.e., for every non integer $r < n-2$, there exists a positive semidefinite matrix A such that $A^{\circ r}$ is not positive semidefinite.
In general, infinitely divisible matrices are not closed under addition and multiplication. For example, let $X = [x_i x_j]$, where $x_1, \ldots, x_n$ are distinct positive real numbers and E be the matrix of order n with each of its entries equals to 1, then X and E are both infinitely divisible, but their sum is not infinitely divisible, see [Jai17, Theorem 1.1]. Also, the Cauchy matrix $C = [c_{ij}] = [\frac{1}{i+j}]$, $1 \leq i, j \leq 3$, is infinitely divisible, see [Bha06], but its square C^2 is not infinitely divisible because $\det(C^2)^{\circ \frac{1}{4}} < 0$. On the other hand, it was shown in [Bap88, p.471, proof of Lemma 6] that if A is in class $\mathcal{A}$, then its Hadamard inverse is infinitely divisible. The converse needs not be true, for example, the Hadamard inverse of the infinitely divisible matrix $\begin{bmatrix} 1 & 3 & 1 \\ 3 & 10 & 4 \\ 1 & 4 & 2.5 \end{bmatrix}$ has two positive eigenvalues.

In this section we will present conditions for a symmetric matrix to be infinitely divisible, see [ASG20]. It is clear that every infinitely divisible matrix is a positive semidefinite matrix, however, the converse needs not be true. for example, the matrix

$$A = \begin{bmatrix} 1.1 & 3 & 1 \\ 3 & 9.1 & 3.9 \\ 1 & 3.9 & 2.9 \end{bmatrix} \tag{2.4}$$

is positive semidefinite (moreover, it is a matrix in which all the minors are positive), while $\det A^{\circ \frac{1}{9}} < 0$, see [AS23].
Note that for a symmetric matrix $A = [a_{ij}]$ with $a_{ij} \geq 0$, the following conditions are equivalent (thanks to the Schur Product Theorem and the obvious continuity argument):

(i) $A^{\circ \frac{1}{m}}$ is positive semidefinite, for each $m \in \mathbb{N}$.

(ii) $A^{\circ r}$ is positive semidefinite, for each $r \in \mathbb{R}_+$.

Some basic examples of infinitely divisible matrices are the nonnegative symmetric positive semidefinite matrices of order 2 and diagonal matrices with nonnegative entries.

2.3.1 Conditions for a Nonnegative Matrix to Be Infinitely Divisible

In this subsection, we give an alternative proof of a result due to Bapat in [Bap88, p.417, proof of Lemma 6], and we provide some results for the conditionally negative semidefinite matrices, as well. Before we do so we present a remarkable theorem by Horn [Hor69].

Theorem 2.10. *([Hor69, Corollary 1.6]): Let A be Hadamard invertible. Then A is infinitely divisible if and only if A is symmetric, positive, and $log^{\circ}(A)$ is conditionally positive semidefinite.*

Theorem 2.11. *([ASG20, Corollary 8]): Let $A \in \mathcal{A}$, then the Hadamard inverse of A is infinitely divisible.*

Proof. Let $A \in \mathcal{A}$, then the proof of the positive semidefiniteness of $A^{\circ -\alpha}$ for all $0 < \alpha \leq 1$ follows directly from Theorem 2.9 and Lemma 2.1(i). The extension to all $\alpha > 1$ is provided by the application of Theorem 2.4. □

Our results in Theorem 2.8 and Theorem 2.11 can be used to obtain the infinite divisibility of some matrices. The following theorem presents three examples.

Theorem 2.12. *[ASG20, Theorem 9] Let A be a symmetric conditionally negative semidefinite matrix with positive entries, then the following matrices are infinitely divisible:*

(i) $E + rA^{\circ -1}$, $r > 0$,

(ii) $(log^{\circ}(E + A))^{\circ -1}$,

(iii) $(A + \alpha E)^{\circ -1} \circ (A + \beta E)$, $0 \leq \alpha < \beta$.

Proof. Assume that $x \in \mathbb{R}_+$. (i): The function $f(x) = \frac{x}{r+x}$, where $r > 0$, is a Bernstein function. Thus, if A is conditionally negative semidefinite with positive entries, Theorem 2.8 and Theorem 2.11 imply (i).
(ii): The proof follows by using the same arguments as in the proof of (i) for the Bernstein function $g(x) = log(1 + x)$.
(iii): By [Ber08, p.9], the following representation holds

$$log\frac{x+\beta}{x+\alpha} = \int_{\alpha}^{\beta} (x+\xi)^{-1} d\xi, \tag{2.5}$$

where $0 \leq \alpha < \beta$ and ξ is a measure on $\mathbb{R}_+$. Now, if a symmetric matrix A is conditionally negative semidefinite with positive entries, then (2.5) implies that the matrix $[log\frac{a_{ij}+\beta}{a_{ij}+\alpha}]$ is positive semidefinite. We complete the proof by using Theorem 2.6. □

2.3.2 Examples

We present now some interesting examples of infinitely divisible matrices, which were discussed in [Bha06].

1. **The Cauchy Matrix**
   Let $0 < p_1 \leq p_2 \leq \cdots \leq p_n$. The matrix C with entries $c_{ij} = \frac{1}{p_i+p_j}$ is called a *Cauchy* matrix. The Hadamard inverse of the Cauchy matrix can be expressed as $C^{\circ -1} = (DE + ED)$, where $D = \text{diag}(p_1, \ldots, p_n)$, and the matrix E is the matrix of ones, i.e., $E := \mathbf{e}^T\mathbf{e}$. Thus, $\mathbf{x}^T C^{\circ -1}\mathbf{x} = 0$, for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{e} = 0$, which

means $C^{\circ -1}$ is both conditionally positive and conditionally negative semidefinite. Therefore, Theorem 2.11 implies that the Cauchy matrix is infinitely divisible.
It was shown in [BJ15a, Theorem 3] that $C^{\circ -r}$ is conditionally negative semidefinite for $0 < r < 1$; and it is conditionally positive semidefinite and nonsingular for $1 < r < 2$.

2. **The Matrix** *min(x,y)*
Let $p_1, \dots, p_n$ be any positive numbers, and let $N = [n_{ij}] \in \mathbb{R}^{n,n}$ such that

$$n_{ij} = min(p_i, p_j).$$

It was shown in [BJ15b, Theorem 2.1] that the matrix $[max(p_i, p_j)]$ is conditionally negative semidefinite. Therefore, it follows from Theorem 2.11 and the observation

$$min(p_i, p_j) = \left(max\left(\frac{1}{p_i}, \frac{1}{p_j} \right) \right)^{-1}$$

that the matrix N is infinitely divisible.

3. **The Pascal Matrix**
The $n \times n$ *Pascal* matrix is the matrix A with entries

$$a_{ij} = \binom{i+j}{i}, \quad i, j = 0, 1, \dots, n-1.$$

The rows of the Pascal triangle occupy the anti-diagonals of A. Thus the 4×4 Pascal matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}.$$

Let L be the lower triangular matrix whose rows are occupied by the rows of the Pascal triangle. Thus for $n = 4$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}.$$

It is well-known that $A = LL^*$, and hence A is positive semidefinite. Using (2.3) and Theorem 2.10 one sees that the $n \times n$ Pascal matrix is infinitely divisible if and only if the $(n-1) \times (n-1)$ matrix D with entries

$$d_{ij} = log \frac{i+j+2}{i+j+1} = log \left(1 + \frac{1}{i+j+1} \right)$$

is positive semidefinite. For $x \geq 0$ we have

$$log(1+x) = \int_1^\infty \frac{tx}{t+x} \, d\mu(t),$$

where μ is the probability measure on $[1, \infty)$ defined as $d\mu(t) = dt/t^2$, see [Bha97, p. 145]. Therefore, we can write d_{ij} as

$$d_{ij} = \int_1^\infty \frac{1}{i + j + 1 + \frac{1}{t}} \, d\mu(t).$$

Thus D is a limit of positive linear combinations of matrices $C(t) = [c_{ij}(t)]$, where

$$c_{ij}(t) = \frac{1}{i + j + 1 + \frac{1}{t}}, \; t \geq 1.$$

Put $p_{ij} = i + \frac{1}{2}\left(1 + \frac{1}{t}\right)$, then

$$c_{ij}(t) = \frac{1}{p_i + p_j}, \; t \geq 1.$$

Thus for each $t \geq 1$, the matrix $C(t)$ is a Cauchy matrix. Hence D is positive semidefinite.

For more examples and results on infinitely divisible matrices, see [Bha06], [BK07], [Hor69].

2.4 Sufficient Conditions for Symmetric Matrices to Have One Positive Eigenvalue

The class of symmetric matrices with exactly one eigenvalue of one sign and the remaining eigenvalues of the other sign presents interesting properties. These matrices arise naturally in many areas such as mathematical programming, matrix theory, numerical analysis, interpolation of scattered data and statistics, see, e.g., [BR97], [Mic86], [Pen09]. In [BR97, Theorem 4.4.6], Bapat and Raghavan gave 18 equivalent conditions for a symmetric matrix with positive entries to have one positive eigenvalue. In this section, other matrices that have one positive eigenvalue are obtained. A consequence of our results will be used to show that if A is a symmetric matrix which has all off-diagonal entries positive, all diagonal entries zero, and has just one positive eigenvalue, then $A^{\circ\alpha}$, $0 < \alpha < 1$, has one positive eigenvalue, too. This result provides a generalization of a theorem given by Reams in [Rea99], see [ASG20].

We will start this section by presenting some sufficient and necessary conditions for a symmetric nonnegative matrix to have exactly one positive eigenvalue.

Theorem 2.13. *([BR97, Theorem 4.4.6]): Let $A = [a_{ij}] \in \mathbb{R}^{n,n}$ be a positive, symmetric matrix. Let $\mathbf{v} := [v_i]_{i=1}^n$ be the Perron eigenvector of A with $\mathbf{v}^T\mathbf{v} = 1$. Define the sets Ω and Φ as follows:*

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^n : \; \mathbf{x}^T A \mathbf{x} > 0 \right\},$$

$$\Phi = \left\{ \mathbf{x} \in \Omega : \ \mathbf{v}^T \mathbf{x} \geq 0 \right\}.$$

Then the following conditions are equivalent:

1. $A \in \mathcal{A}$.
2. *For any* $\mathbf{x}, \mathbf{y} \in \Phi$; $\mathbf{y}^T A \mathbf{x} > 0$.
3. $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \Omega$, $\mathbf{y}^T A \mathbf{x} = 0 \ \Rightarrow \mathbf{x}^T A \mathbf{x} \leq 0$.
4. *A and all its principal minors have exactly one positive eigenvalue.*
5. *For any* $r \times r$ *principal minor* B *of* A, $-1^{r-1}\det B \geq 0$, $r = 1, 2, \ldots, n$.
6. $\left[\frac{a_{ij}}{v_i v_j}\right]$ *is conditionally negative semidefinite.*
7. $[e^{\frac{-ta_{ij}}{v_i v_j}}]$ *is positive semidefinite for any* $t > 0$.
8. $\begin{bmatrix} A & \mathbf{v} \\ \mathbf{v}^T & 0 \end{bmatrix} \in \mathcal{A}$.
9. $[\frac{1}{v_i v_j + \lambda a_{ij}}]$ *is positive semidefinite for any* $\lambda > 0$.

The next theorem obtains a new matrix that is in class $\mathcal{A}$.

Theorem 2.14. *([ASG20, Theorem 10]): Let* $A \in \mathcal{A}$*, then for all* $r > 0$*,* $rA + \mathbf{v}\mathbf{v}^T \in \mathcal{A}$*, where* $\mathbf{v} = [v_i]_{i=1}^n$ *is the Perron eigenvector of A.*

Proof. Let $A = [a_{ij}] \in \mathcal{A}$, then Lemma 2.1(ii) ensures that the matrix $[r\frac{a_{ij}}{v_i v_j} + 1]$ is in class $\mathcal{A}$. We complete the proof in the same way as in the proof of Theorem 2.9. □

The next two lemmata will be used in the proof of Theorem 2.15. Note that, if a symmetric matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is conditionally negative definite, then by $(\mathbf{e}_i - \mathbf{e}_j)^T \mathbf{e} = 0$, we have $0 > (\mathbf{e}_i - \mathbf{e}_j)^T A (\mathbf{e}_i - \mathbf{e}_j) = a_{ii} + a_{jj} - 2a_{ij}$.
It was shown in [BR97, Theorem 4.4.6] and follows from [Pen09, Theorem 3.1] by Lemma 2.3, that a symmetric matrix $A \in \mathbb{R}^{n,n}$ with positive entries belongs to the class $\mathcal{A}$ if and only if, for any $k \times k$ principal submatrix B of A, $(-1)^{k-1}\det B \geq 0$, for all $k = 1, \ldots, n$. The next lemma provides a weaker condition for a symmetric matrix to have exactly one positive eigenvalue by removing the assumption of the positivity of the entries of A.

Lemma 2.4. *([SH10, Lemma 3.8]): Let* $A \in \mathbb{R}^{n,n}$ *be a symmetric matrix and let for each* $k \times k$ *leading principal submatrix* B *of* A*,* $(-1)^{k-1}\det B > 0$*, for* $k = 1, \ldots, n$*. Then A has exactly one positive eigenvalue.*

Proof. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of A. We shall prove the result by induction on n. For $n = 1$, the result trivially holds. Assume the result is true for $n - 1$. That is, if $(-1)^{k-1}\det B > 0$ for any $k \times k$ leading principal submatrix B of A, $k = 1, \ldots, n-1$,

then A has exactly one positive eigenvalue. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$ be the eigenvalues of the $(n-1) \times (n-1)$ leading principal submatrix B of A. By the assumption, μ_{n-1} is the only positive eigenvalue of B.
Now, let $n > 2$. By the interlacing theorem for eigenvalues of Hermitian matrices, we have:
If n is even, then $\det A < 0$ and $\lambda_1\lambda_2\ldots\lambda_{n-2} > 0$, which implies $\lambda_{n-1} < 0$.
If n is odd, then $\det A > 0$ and $\lambda_1\lambda_2\ldots\lambda_{n-2} < 0$, which implies $\lambda_{n-1} < 0$.
Thus, A has one positive eigenvalue. □

The following lemma was proven in [Mic86] for Euclidean distance matrices, where in Remark 3.2 therein it is mentioned that the proof can be extended to arbitrary symmetric conditionally negative semidefinite matrices with nonnegative entries satisfying the inequality condition in Lemma 2.6, cf. [DGM86, pp.163-164].

Lemma 2.5. *([Mic86]):*
If a symmetric matrix $A = [a_{ij}] \in \mathbb{R}^{n,n}$ is conditionally negative semidefinite with nonnegative entries and

$$2a_{ij} > a_{ii} + a_{jj},\ \textit{for all}\ i \neq j,$$

then

$$(-1)^{n-1}\det(A^{\circ\alpha}) > 0,$$

for all $0 < \alpha < 1$.

A *distance* matrix is a square matrix containing the distances, taken pairwise, between the elements of a finite metric space. In general, the distance matrix needs not be conditionally negative semidefinite. For example, the distance matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 2 & 1 & 1 & 1 & 0 \end{bmatrix}$$

is not conditionally negative semidefinite. A famous theorem of Schoenberg in distance geometry says that a symmetric matrix $D = [d_{ij}]$ with zero diagonal entries and positive off-diagonal entries, is conditionally negative semidefinite if and only if there exist distinct vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^s$ for some s such that $\|\mathbf{u}_i - \mathbf{u}_j\|^2 = d_{ij}$ for all i, j, i.e., D is an Euclidean distance matrix, see [BR97, Theorem 4.1.7]. In [Mic86, proof of Theorem 2.3], Micchelli showed that if a conditionally negative semidefinite matrix $A \in \mathbb{R}^{n,n}$ is a distance matrix, then $A^{\circ\alpha}$, $0 < \alpha < 1$, has one positive eigenvalue and is invertible. In the next theorem, we give a weaker condition on A such that $A^{\circ\alpha}$ has one positive eigenvalue.

Theorem 2.15. *([ASG20, Theorem 13]): Let $A \in \mathbb{R}^{n,n}$ be a symmetric conditionally negative semidefinite matrix with nonnegative entries and*

$$2a_{ij} > a_{ii} + a_{jj},\ \textit{for all}\ i \neq j. \tag{2.6}$$

Then $A^{\circ\alpha}, 0 < \alpha < 1$, has one positive eigenvalue and is invertible.

Proof. From the hypothesis, we can conclude that $-A$ is conditionally positive semidefinite and $-2a_{ij} < -a_{ii} - a_{jj}$, for all $i \neq j$. Let B be any leading principal submatrix of A. By Theorem 2.6, $e^{\circ -A}$ is positive definite. Then $e^{\circ -B}$ is also positive definite. By Theorem 2.6 we conclude that B is conditionally negative definite and has the property $2b_{ij} > b_{ii} + b_{jj}$, for all $i \neq j$. Now, Lemma 2.5 and Lemma 2.4 imply that the matrix $A^{\circ\alpha}$, has one positive eigenvalue for all $0 < \alpha < 1$ and is invertible. □

If $A \in \mathbb{R}^{n,n}$ is symmetric with positive off-diagonal entries, zero diagonal entries, and just one positive eigenvalue, then A need not be conditionally negative definite. For example, let $A = \begin{bmatrix} 0 & 1 & 16 \\ 1 & 0 & 4 \\ 16 & 4 & 0 \end{bmatrix}$ and $\mathbf{x} = (1, -2, 1)^T$. Then A has one positive eigenvalue, but $\mathbf{x}^T A \mathbf{x} = 12 > 0$.
In [Rea99, Theorem 2.9], Reams showed that if a symmetric matrix A, with positive off-diagonal entries and all diagonal entries zero, has one positive eigenvalue, then the Hadamard square root of A has also one positive eigenvalue and is invertible. The next theorem gives a generalization of his result which was already proven by Schoenberg in [Sch37] for the special case of distance matrices (of distinct points).

Theorem 2.16. *([ASG20, Theorem 14]): Let $A \in \mathbb{R}^{n,n}$ be symmetric with positive off-diagonal entries, zero diagonal entries, and just one positive eigenvalue. Then $A^{\circ\alpha}$, $0 < \alpha < 1$, has one positive eigenvalue and is invertible.*

Proof. Let $\lambda_1 \leq \cdots \leq \lambda_{n-1} \leq 0 < \lambda_n$ be the eigenvalues of A with $A\mathbf{v} = \lambda_n \mathbf{v}$ and $A\mathbf{u}_j = \lambda_j \mathbf{u}_j$, where $1 \leq j \leq n-1$ and $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$ be the Perron eigenvector of A.
Write

$$A = \lambda_n \mathbf{v}\mathbf{v}^T + \lambda_{n-1}\mathbf{u}_{n-1}\mathbf{u}_{n-1}^T + \cdots + \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T.$$

Then,

$$\mathbf{x}^T WAW\mathbf{x} = \mathbf{x}^T(\lambda_n \mathbf{e}\mathbf{e}^T + \lambda_{n-1}(W\mathbf{u}_{n-1})(W\mathbf{u}_{n-1})^T + \cdots + \lambda_1 (W\mathbf{u}_1)(W\mathbf{u}_1)^T)\mathbf{x},$$

where $W = \mathrm{diag}(\frac{1}{v_1}, \ldots, \frac{1}{v_n})$. So, if $\mathbf{x}^T\mathbf{e} = 0$, then

$$\mathbf{x}^T WAW\mathbf{x} \leq 0,$$

and hence the matrix WAW is conditionally negative semidefinite. By using Theorem 2.15, $(WAW)^{\circ\alpha} = W^{\circ\alpha}A^{\circ\alpha}W^{\circ\alpha}$ has one positive eigenvalue. If $A^{\circ\alpha}$ would not be invertible then $W^{\circ\alpha}A^{\circ\alpha}W^{\circ\alpha}$ would not be invertible by Lemma 2.3 and we obtain by (2.6) a contradiction to the positivity of the off-diagonal entries of A. □

3 Hurwitz Polynomials and Hurwitz Matrices

In many applications, where stability plays a significant role such as systems theory, the problem of deciding whether a given polynomial is Hurwitz or not is very important [Bar83]. This problem was first proposed by the physicist and mathematician James Maxwell in 1868 [Max68], who presented a solution to this problem for polynomials of degree three. At the height of the Industrial Revolution, the mechanism for controlling the speed of every steam engine was plagued by problems of instability and inaccuracy that could apparently not be overcome by either theoretical or practical approaches. In those days, various governors had been newly invented. However, Maxwell's interest in governors was unrelated to their practical utility and instead originated from the desire to address the issue of their stability. Routh in 1887 and Hurwitz in 1895 showed that the stability can be determined directly from the coefficients of the associated characteristic polynomial. The work of Routh and Hurwitz fostered the Routh-Hurwitz Criterion, probably the best known criterion for determining whether or not a polynomial is a Hurwitz polynomial [Lod11]. However, several criteria have been studied to obtain Hurwitz-type polynomials, such as the Routh-Hurwitz criterion, the Liénard conditions, the Hermite-Biehler theorem, the stability test, and the Routh algorithm.
Interesting properties of these polynomials were also found, e.g., total nonnegativity of special matrices associated with this kind of polynomials [Asn70], [Kem82], and the fact that the Hadamard product of Hurwitz polynomials is Hurwitz [GW96].

The organization of this chapter is as follows: In Section 3.1, the Hurwitz matrix and conditions for a real polynomial to be Hurwitz are recalled. In Section 3.2, a new formula for the leading principal minors of Hurwitz matrices is presented. This is used to study the validity of Oppenheim's inequality for the Hurwitz matrix associated with a Hurwitz polynomial. In Section 3.3, conditions under which the Hadamard power of a Hurwitz polynomial is also Hurwitz are given.

3.1 Hurwitz Polynomials

We shall consider only polynomials with positive coefficients, i.e., polynomials of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{3.1}$$

where x is a complex variable, n is a fixed positive integer and a_i , $i = 1, \ldots, n$, are positive numbers.

3.1.1 Hurwitz Matrix Which is Associated with a Real Polynomial

For a polynomial p given by (3.1), the *Hurwitz matrix* $H(p) = [h_{ij}]_{i,j=1}^{n}$ *associated with* p is given by

$$H(p) = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots & 0 & 0 \\ a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots & 0 & 0 \\ 0 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 & 0 \\ 0 & a_{n-0} & a_{n-2} & a_{n-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_2 & a_0 \end{bmatrix}. \quad (3.2)$$

In the Hurwitz matrix the coefficients $a_{n-1}, a_{n-2}, a_{n-3}, \ldots, a_0$ of the polynomial (3.1) are on the main diagonal, and all the elements of the last column are null, except the last element which is a_0. The leading principal minors of the matrix (3.2) are given by the following determinants, called *Hurwitz determinants*,

$$\Delta_1 := a_{n-1},\ \Delta_2 := \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix},\ \Delta_3 := \begin{vmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{vmatrix}, \ldots, \Delta_n := \det(H(p)). \quad (3.3)$$

3.1.2 Criteria for Deciding Whether a Polynomial is Hurwitz

Recall that, a polynomial is said to be *Hurwitz* or *stable* if all its roots lie in the open left half of the complex plane. Many criteria were found for deciding whether a polynomial is Hurwitz, which include the Routh-Hurwitz criterion, the Liénard-Chipart conditions, Routh's algorithm, and the Stability Test in [BK95].

a) Routh-Hurwitz Criterion

Theorem 3.1. *(Routh-Hurwitz Theorem): The polynomial p in* (3.1) *is a Hurwitz polynomial if and only if all leading principal minors $\Delta_1, \Delta_2, \ldots, \Delta_n$ of $H(p)$ are positive.*

b) Liénard-Chipart Criterion

For the polynomial p in (3.1), and the leading principal minors of the matrix (3.2), define the sequences Q_1, Q_2 as

$$Q_1 = (\Delta_1, \Delta_3, \Delta_5, \ldots),$$
$$Q_2 = (\Delta_2, \Delta_4, \Delta_6, \ldots).$$

In [LC14], the French mathematicians Liénard and Chipart established a stability criterion based on the Routh-Hurwitz criterion as follows.

Theorem 3.2. *(Liénard-Chipart Theorem): The polynomial p in* (3.1) *is a Hurwitz polynomial if and only if one of the sequences Q_1, Q_2, has all its members positive.*

Example 3.1. *Consider the polynomial* $p(x) = x^4 + 2x^3 + 9x^2 + x + 4$ *whose coefficients are all positive. Then*

$$\Delta_1 = 2, \quad \Delta_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 9 & 4 \\ 0 & 2 & 1 \end{vmatrix} = 5.$$

Therefore, according to the sequence Q_3*,* p *is a Hurwitz polynomial.*

d) Stability Test in [BK95]

Let $p(x) = \sum_{k=0}^{n} a_k x^k$. We define its *odd* and *even parts* as

$$P_{odd}(x) = a_1 x + a_3 x^3 + \dots,$$

and

$$P_{even}(x) = a_0 + a_2 x^2 + \dots,$$

respectively. In [BK95] the authors have presented methods for testing the Hurwitzness of the polynomial based on the Interlacing Theorem. This test is called the Stability Test, and is described as follows.
Let p be a polynomial with positive coefficients and decomposed into its odd and even parts as

$$p(x) = P_{even}(x) + P_{odd}(x).$$

Now define the polynomial Q of order $n-1$ by:

$$\text{If } n = 2m: \; Q(x) = \left(P_{even}(x) - \frac{a_{2m}}{a_{2m-1}}\, x P_{odd}(x)\right) + P_{odd}(x),$$

$$\text{if } n = 2m+1: \; Q(x) = \left(P_{odd}(x) - \frac{a_{2m+1}}{a_{2m}}\, x P_{even}(x)\right) + P_{even}(x).$$

That is in general, with $\mu = \frac{a_n}{a_{n-1}}$,

$$Q(x) = a_{n-1}x^{n-1} + (a_{n-2} - \mu a_{n-3})\, x^{n-2} + a_{n-3}x^{n-3} + (a_{n-4} - \mu a_{n-5})\, x^{n-4} + \dots.$$

Then the following key result on degree reduction is obtained (for the abbreviations $\boldsymbol{P}_n$ and $\boldsymbol{H}_n$ see Subsection 1.2.1).

Theorem 3.3. *([BK95, p.58]): If* $p \in \boldsymbol{P}_n$*, then* $p \in \boldsymbol{H}_n$ *if and only if* $Q \in \boldsymbol{H}_{n-1}$*.*

Example 3.2. *Let* $p(x) = \sum_{k=0}^{5} a_k x^k \in \boldsymbol{P}_5$*. Then* $p(x) \in \boldsymbol{H}_5$ *if and only if the* $Q(x) = a_4^2 x^4 + (a_4 a_3 - a_5 a_2)\, x^3 + a_4 a_2 x^2 + (a_4 a_1 - a_5 a_0)\, x + a_4 a_0 \in \boldsymbol{H}_4$*.*

Finally, we present a result shows that the stability of a polynomial is not affected by reversing the order of coefficients.

Theorem 3.4. *If a polynomial*

$$p(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1}x + a_n$$

is a Hurwitz polynomial, then so is the polynomial

$$q(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0.$$

3.2 On Hurwitz Determinants and the Validity of Oppenheim's Inequality

In this section, we first present a new formula for the leading principal minors of Hurwitz matrices. In Subsection 3.2.3, this formula used to study the validity of Oppenheim's inequality for the Hurwitz matrix up to order 6 that associated with a Hurwitz polynomial, see [AASG].

3.2.1 A New Formula for Hurwitz Determinants

In Section 3.1.2 we saw that for a polynomial p given by (3.1) to belong to the set $\boldsymbol{H}_n$, it is necessary and sufficient that all the Hurwitz determinants given in (3.3) are positive. We start this subsection by a well-known formula which establishes a useful relation between the roots of a Hurwitz polynomial and the determinant of its associated Hurwitz matrix.
Orlando's Formula ([Gan59, p. 196]): *Let $f(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial with complex coefficients. Then*

$$\triangle_{n-1} = (-1)^{\frac{n(n-1)}{2}} a_n^{n-1} \prod_{1 \le i < k \le n} (r_i + r_k), \tag{3.4}$$

where r_i, $i = 1, \ldots, n$, are the zeros of f.

Theorem 3.5. *([AASG, Theorem 3.1]): Let $A = H(p)$ be the Hurwitz matrix associated with the Hurwitz polynomial p given by (3.1). Then if n is even,* $\det A = \det C_e(A)$*, where the $n/2 \times n/2$ symmetric matrix $C_e(A)$ is defined by*

$$C_e(A) := \begin{bmatrix} \begin{vmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-7} \\ a_n & a_{n-6} \end{vmatrix} & \cdots \\ \begin{vmatrix} a_{n-1} & a_{n-5} \\ a_n & a_{n-4} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-7} \\ a_n & a_{n-6} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-5} \\ a_{n-2} & a_{n-4} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-9} \\ a_n & a_{n-8} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-7} \\ a_{n-2} & a_{n-6} \end{vmatrix} & \cdots \\ \begin{vmatrix} a_{n-1} & a_{n-7} \\ a_n & a_{n-6} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-9} \\ a_n & a_{n-8} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-7} \\ a_{n-2} & a_{n-6} \end{vmatrix} & \begin{vmatrix} a_{n-1} & a_{n-11} \\ a_n & a_{n-10} \end{vmatrix} + \begin{vmatrix} a_{n-3} & a_{n-9} \\ a_{n-2} & a_{n-8} \end{vmatrix} + \begin{vmatrix} a_{n-5} & a_{n-7} \\ a_{n-4} & a_{n-6} \end{vmatrix} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.5}$$

and if n is odd, $\det A = \det C_o(A)$*, where the $(n+1)/2 \times (n+1)/2$ matrix $C_o(A)$ is defined by*

$$C_o(A) := \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ a_n a_{n-3} & a_n a_{n-5} + \begin{vmatrix} a_{n-2} & a_{n-4} \\ a_{n-1} & a_{n-3} \end{vmatrix} & a_n a_{n-7} + \begin{vmatrix} a_{n-2} & a_{n-6} \\ a_{n-1} & a_{n-5} \end{vmatrix} & \cdots \\ a_n a_{n-5} & a_n a_{n-7} + \begin{vmatrix} a_{n-2} & a_{n-6} \\ a_{n-1} & a_{n-5} \end{vmatrix} & a_n a_{n-9} + \begin{vmatrix} a_{n-2} & a_{n-8} \\ a_{n-1} & a_{n-7} \end{vmatrix} + \begin{vmatrix} a_{n-4} & a_{n-6} \\ a_{n-3} & a_{n-5} \end{vmatrix} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{3.6}$$

with the convention that $a_{n-i}=0$*, for all* $i>n$*.*

Proof. Case 1: n is even (for an illustration of $n=8$, see Example 3.3):
First, define the $n\times n$ matrix E_A and the $n/2\times n/2$ matrices $E_A^{(1)}$, $E_A^{(2)}$, all having entries from A, as follows:

$$E_A := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & -a_n & a_{n-1} & \dots & a_3 \\ -a_n & a_{n-1} & -a_{n-2} & a_{n-3} & \dots & a_1 \end{bmatrix},$$

$$E_A^{(1)} := \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & a_1 \\ 0 & a_{n-1} & a_{n-3} & \dots & a_3 \\ 0 & 0 & a_{n-1} & \dots & a_5 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} \end{bmatrix}, \quad E_A^{(2)} := \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ a_1 & 0 & \dots & 0 & 0 \\ a_3 & a_1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-3} & a_{n-5} & \dots & a_1 & 0 \end{bmatrix}.$$

Then $\det E_A = a_{n-1}^{n/2}$ and $E_A A = \left(\begin{array}{c|c} E_A^{(1)} & E_A^{(2)} \\ \hline 0 & C_e(A) \end{array}\right)$.

Thus, $\det(E_A A) = a_{n-1}^{n/2}\det C_e(A)$, and so (3.5) is shown.
Case 2: n is odd:
First, define the $n\times n$ matrix O_A and the $(n-1)/2\times(n-1)/2$, $(n-1)/2\times(n+1)/2$ matrices $O_A^{(1)}$, $O_A^{(2)}$, respectively, as follows:

$$O_A := \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & a_n & -a_{n-1} & \dots & a_3 \\ a_n & -a_{n-1} & a_{n-2} & -a_{n-3} & \dots & a_1 \end{bmatrix},$$

$$O_A^{(1)} := \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \dots & a_0 \\ 0 & a_{n-1} & a_{n-3} & \dots & a_2 \\ 0 & 0 & a_{n-1} & \dots & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} \end{bmatrix}, \quad O_A^{(2)} := \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ a_0 & 0 & \dots & 0 & 0 \\ a_2 & a_0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-3} & a_{n-5} & \dots & a_0 & 0 \end{bmatrix}.$$

Thus, $\det(O_A A) = a_{n-1}^{(n-1)/2}\det C_o(A)$, and so (3.6) is shown. □

Example 3.3. *Let* $n=8$*. We get:*

$$E_A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_8 & a_7 \\ 0 & 0 & 0 & 0 & -a_8 & a_7 & -a_6 & a_5 \\ 0 & 0 & -a_8 & a_7 & -a_6 & a_5 & -a_4 & a_3 \\ -a_8 & a_7 & -a_6 & a_5 & -a_4 & a_3 & -a_2 & a_1 \end{bmatrix},$$

and so,

$$E_A A = \left(\begin{array}{c|c} E_A^{(1)} & E_A^{(2)} \\ \hline 0 & C_e(A) \end{array}\right),$$

where

$$E_A^{(1)} = \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ 0 & a_7 & a_5 & a_3 \\ 0 & 0 & a_7 & a_5 \\ 0 & 0 & 0 & a_7 \end{bmatrix}, \; E_A^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ a_1 & a_2 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 \end{bmatrix},$$

$$C_e(A) = \begin{bmatrix} \begin{vmatrix} a_7 & a_5 \\ a_8 & a_6 \end{vmatrix} & \begin{vmatrix} a_7 & a_3 \\ a_8 & a_4 \end{vmatrix} & \begin{vmatrix} a_7 & a_1 \\ a_8 & a_2 \end{vmatrix} & a_7 a_0 \\ \begin{vmatrix} a_7 & a_3 \\ a_8 & a_4 \end{vmatrix} & \begin{vmatrix} a_7 & a_1 \\ a_8 & a_2 \end{vmatrix} + \begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & a_7 a_0 + \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5 a_0 \\ \begin{vmatrix} a_7 & a_1 \\ a_8 & a_2 \end{vmatrix} & a_7 a_0 + \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5 a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3 a_0 \\ a_7 a_0 & a_5 a_0 & a_3 a_0 & a_1 a_0 \end{bmatrix}.$$

Corollary 3.1. *Let $p \in \boldsymbol{P}_n$ given by* (3.1) *and A be the Hurwitz matrix associated with p . Then*
if $n = 3$,

$$\det A = \begin{vmatrix} a_2 & a_0 \\ a_0 a_3 & a_1 a_0 \end{vmatrix},$$

if $n = 4$,

$$\det A = \begin{vmatrix} \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3 a_0 \\ a_3 a_0 & a_1 a_0 \end{vmatrix},$$

if $n = 5$,

$$\det A = \begin{vmatrix} a_4 & a_2 & a_0 \\ a_5 a_2 & a_5 a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3 a_0 \\ a_5 a_0 & a_3 a_0 & a_1 a_0 \end{vmatrix},$$

if $n = 6$,

$$\det A = \begin{vmatrix} \begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5 a_0 \\ \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5 a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3 a_0 \\ a_5 a_0 & a_3 a_0 & a_1 a_0 \end{vmatrix}.$$

Theorem 3.6. *([AASG, Theorem 3.4]): Let $p \in \boldsymbol{P}_n$ given by* (3.1) *and A be the Hurwitz matrix associated with p. Then the following relations hold for the Hurwitz determinants.*

(i) If n is even, then for $i = 1, \ldots, \frac{n}{2}$,

$$\Delta_{2i} = \det A[1, \ldots, 2i] = \det C_e(A)[1, \ldots, i].$$

(ii) If n is odd, then for $i = 1, \dots, \frac{n+1}{2}$,

$$\Delta_{2i-1} = \det A[1, \dots, 2i-1] = \det C_o(A)[1, \dots, i].$$

Proof. Let $\Delta_1, \Delta_2, \dots, \Delta_n$ be the Hurwitz determinants of A, and n be even. Define the following three index sets

$$\begin{aligned}
\alpha &= \left(\alpha_1, \alpha_2, \dots, \alpha_{\frac{n}{2}}\right) := \left(\frac{n}{2}+1, \frac{n}{2}+2, \dots, n\right), \\
\beta &= \left(\beta_1, \beta_2, \dots, \beta_{\frac{n}{2}}\right) := \left(\frac{n}{2}, \frac{n}{2}-1, \dots, 1\right), \\
\zeta &= (\zeta_1, \zeta_2, \dots, \zeta_n) := (n, n-1, \dots, 1).
\end{aligned}$$

For $i = 1, \dots, \frac{n}{2}$, we have

$$\det\left(E_A\left[(\alpha_1, \alpha_2, \dots, \alpha_i) \cup (\beta_1, \beta_2, \dots, \beta_i) \mid \zeta_1, \zeta_2, \dots, \zeta_{2i}\right]\right) = a_{n-1}^i,$$

and

$$\begin{aligned}
&\det\left(E_A\left[(\alpha_1, \alpha_2, \dots, \alpha_i) \cup (\beta_1, \beta_2, \dots, \beta_i) \mid \zeta_1, \zeta_2, \dots, \zeta_{2i}\right] A[1, \dots, 2i]\right) \\
&= \det\left(\begin{array}{c|c} E_A^{(1)}[\beta_1, \dots, \beta_i] & \star \\ \hline 0 & C_e(A)[1, \dots, i] \end{array}\right) \\
&= \det E_A^{(1)}[\beta_1, \dots, \beta_i] \det C_e(A)[1, \dots, i] = a_{n-1}^i \det C_e(A)[1, \dots, i].
\end{aligned}$$

Thus,

$$\Delta_{2i} = \det A[1, \dots, 2i] = \det C_e(A)[1, \dots, i].$$

The proof of the odd case is similar. □

Example 3.4. *Let $n = 8$. Then the fourth Hurwitz determinant is*

$$\Delta_4 = \det A[1, \dots, 4] = \det \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ a_8 & a_6 & a_4 & a_2 \\ 0 & a_7 & a_5 & a_3 \\ 0 & a_8 & a_6 & a_4 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_8 & a_7 \\ -a_8 & a_7 & -a_6 & a_5 \end{bmatrix} \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ a_8 & a_6 & a_4 & a_2 \\ 0 & a_7 & a_5 & a_3 \\ 0 & a_8 & a_6 & a_4 \end{bmatrix} = \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ 0 & a_7 & a_5 & a_3 \\ 0 & 0 & a_7a_6 - a_8a_5 & a_7a_4 - a_8a_3 \\ 0 & 0 & a_7a_4 - a_8a_3 & a_7a_2 - a_8a_1 + a_5a_4 - a_6a_3 \end{bmatrix}.$$

Thus,

$$\det\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a_8 & a_7 \\ -a_8 & a_7 & -a_6 & a_5 \end{bmatrix} \begin{bmatrix} a_7 & a_5 & a_3 & a_1 \\ a_8 & a_6 & a_4 & a_2 \\ 0 & a_7 & a_5 & a_3 \\ 0 & a_8 & a_6 & a_4 \end{bmatrix}\right) = a_7^2 \det C_e(A)[1, 2]$$

and therefore,

$$\Delta_4 = \det C_e(A)[1,2].$$

By application of Lemma 3.2, we obtain a new necessary and sufficient condition for a polynomial to be Hurwitz.

Theorem 3.7. *([AASG, Theorem 3.6]): Let $p \in \boldsymbol{P}_n$ given by* (3.1)*, with $a_n = 1$ if n is odd* [1]*, and A be the Hurwitz matrix associated with p. Then $p \in \boldsymbol{H}_n$ if and only if $C_e(A) \succ 0$ ($C_o(A) \succ 0$).*

Remark 3.1. *If one checks a symmetric matrix for positive definiteness by the positivity of its leading principal minors, then Theorem 3.7 requires about the same number of minors to be checked as Theorem 3.2. However, the order of minors in Theorem 3.7 is roughly half the order of the respective minors in the Liénard-Chipart Criterion.*

3.2.2 Oppenheim's Inequality

Given two matrices $A = [a_{ij}]$, $B = [b_{ij}]$ that are symmetric positive (semi)definite matrices with the same order, the Hadamard inequality says that

$$\prod_{i=1}^{n} a_{ii} \geq \det A.$$

It is well-known that $A \circ B$ is positive (semi)definite; see Theorem 2.4. Moreover, the celebrated Oppenheim inequality (see [Opp30]) states that

$$\begin{aligned} \det(A \circ B) &\geq \det A \cdot \prod_{i=1}^{n} b_{ii} \\ &\geq \det A \cdot \det B. \end{aligned} \tag{3.7}$$

The last inequality is a consequence of Hadamard's inequality. Over the years, various generalizations for Oppenheim's inequality (3.7) have been obtained in the literature e.g., generalizations of Oppenheim's inequality for positive definite block matrices [Lin14], H-matrices [LL01], and M-matrices [LZ97].

3.2.3 Oppenheim type Determinantal Inequality for Hurwitz Matrices Associated with Hurwitz Polynomials

In this section, we prove that Oppenheim's inequality is valid for the Hurwitz matrices that are associated with Hurwitz polynomials of up to order 5. Then, we give an Oppenheim-type inequality for Hurwitz matrix associated with a Hurwitz polynomial of order 6. Before we start, we present the following useful lemma.

[1] In this case, the matrix $C_o(A)$ in (3.6) is symmetric.

Lemma 3.1. *([Zha11, Theorem 7.7]): Suppose $A, B \succeq 0$ have the same order (> 1). Then*

$$\det(A+B) \geq \det A + \det B$$

with equality if and only if $A+B$ is singular or $A=0$ or $B=0$.

The next theorem presents inequalities of type (3.7) for Hurwitz matrices of order $n \leq 6$. Taking into account that for positive definite matrices, the equality case in inequality (3.7) can occur only in very restricted cases, see, e.g., [ZD09], it is not surprising that the following inequalities are strict.

Theorem 3.8. *([AASG, Theorem 3.8]): Let $f(x) = \sum_{k=0}^{n} a_k x^k$, $g(x) = \sum_{k=0}^{n} b_k x^k$ be Hurwitz polynomials with positive coefficients and let A and B be the Hurwitz matrices associated with f and g, respectively. Then the following statements hold.*

(i) If $n = 3, 4, 5$, then $\det(A \circ B) > \det A \cdot \det B$.

(ii) If $n = 6$, then $\det(A \circ B) + (a_3 a_2 b_5 b_0 + a_5 a_0 b_3 b_2) \det(C_e(A \circ B)[1,3]) > \det A \cdot \det B$.

Proof. In each part, we use the determinant forms in Corollary 3.1.

(i) If $n = 3$, then the result is trivial.
If $n = 4$, then

$$\begin{aligned}
\det(A \circ B) &= \begin{vmatrix} \begin{vmatrix} a_3 b_3 & a_1 b_1 \\ a_4 b_4 & a_2 b_2 \end{vmatrix} & a_3 a_0 b_3 b_0 \\ a_3 a_0 b_3 b_0 & a_1 a_0 b_1 b_0 \end{vmatrix} \\
&= \begin{vmatrix} \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + a_4 a_1 \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + b_4 b_1 \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3 a_0 b_3 b_0 \\ a_3 a_0 b_3 b_0 & a_1 a_0 b_1 b_0 \end{vmatrix}.
\end{aligned}$$

We make use of Theorem 3.7 and Lemma 2.4 to conclude

$$\begin{aligned}
\det(A \circ B) &= \det\left(\begin{pmatrix} \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3 a_0 \\ a_3 a_0 & a_1 a_0 \end{pmatrix} \circ \begin{pmatrix} \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} & b_3 b_0 \\ b_3 b_0 & b_1 b_0 \end{pmatrix} + \operatorname{diag}\left(a_1 a_4 \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + b_1 b_4 \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}, 0 \right) \right) \\
&> \det C_e(A) \cdot \det C_e(B) \text{ (by Lemma 3.1 and (3.7))} \\
&= \det A \cdot \det B.
\end{aligned}$$

Let $n = 5$. Without loss of generality we may assume that $a_5 = b_5 = 1$. Then the matrices A and B are symmetric and

$$\det(A \circ B) = \begin{vmatrix} a_4 b_4 & a_2 b_2 & a_0 b_0 \\ a_2 b_2 & a_0 b_0 + \begin{vmatrix} a_3 b_3 & a_1 b_1 \\ a_4 b_4 & a_2 b_2 \end{vmatrix} & a_3 a_0 b_3 b_0 \\ a_0 b_0 & a_3 a_0 b_3 b_0 & a_1 a_0 b_1 b_0 \end{vmatrix}.$$

Since

$$\begin{aligned} a_0b_0 + \begin{vmatrix} a_3b_3 & a_1b_1 \\ a_4b_4 & a_2b_2 \end{vmatrix} &= \left(a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}\right)\left(b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix}\right) \\ &+ \begin{vmatrix} a_1 & a_0 \\ 1 & a_4 \end{vmatrix}\begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} + \begin{vmatrix} b_1 & b_0 \\ 1 & b_4 \end{vmatrix}\begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}, \end{aligned}$$

we obtain similarly as for $n = 4$

$$\begin{aligned} \det(A \circ B) &\geq \det C_o(A) \cdot \det C_o(B) \\ &= \det A \cdot \det B. \end{aligned}$$

(ii) If $n = 6$, then

$$\begin{aligned} C_e(A \circ B) &= \begin{bmatrix} \begin{vmatrix} a_5 & a_6 \\ a_3 & a_4 \end{vmatrix} & \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 \\ \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & a_3a_0 \\ a_5a_0 & a_3a_0 & a_1a_0 \end{bmatrix} \circ \begin{bmatrix} A\cdot B\begin{vmatrix} b_5 & b_6 \\ b_3 & b_4 \end{vmatrix} & \begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & b_5b_0 \\ \begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & b_5b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix} & b_3b_0 \\ b_5b_0 & b_3b_0 & b_1b_0 \end{bmatrix} \\ &+ \begin{bmatrix} b_6b_3\begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & 0 \\ b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & b_4b_1\left(a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_6a_3\begin{vmatrix} b_5 & b_3 \\ b_6 & b_4 \end{vmatrix} & a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & 0 \\ a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & a_4a_1\left(b_5b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &- \operatorname{diag}(0, c, 0), \text{ where } c := a_3a_2b_5b_0 + a_5a_0b_3b_2. \end{aligned}$$

The Hadamard product on the right-hand side is positive definite because it is the Hadamard product of the two symmetric positive definite matrices $C_e(A)$ and $C_e(B)$. Also, the matrix

$$X := \begin{bmatrix} b_3b_6\begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & 0 \\ b_6b_1\begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & b_4b_1\left(a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} b_6b_3 & b_6b_1 & 0 \\ b_6b_1 & b_4b_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} \begin{vmatrix} a_5 & a_3 \\ a_6 & a_4 \end{vmatrix} & \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & 0 \\ \begin{vmatrix} a_5 & a_1 \\ a_6 & a_2 \end{vmatrix} & a_5a_0 + \begin{vmatrix} a_3 & a_1 \\ a_4 & a_2 \end{vmatrix} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is the Hadamard product of two positive semidefinite matrices because

$$\det \begin{bmatrix} b_6b_3 & b_6b_1 \\ b_6b_1 & b_4b_1 \end{bmatrix} = b_6b_1(b_4b_3 - b_6b_1) > 0,$$

and so it is positive semidefinite with a similar conclusion for the matrix

$$Y := \begin{bmatrix} a_6a_3\begin{vmatrix} b_5 & b_3 \\ b_6 & b_4 \end{vmatrix} & a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & 0 \\ a_6a_1\begin{vmatrix} b_5 & b_1 \\ b_6 & b_2 \end{vmatrix} & a_4a_1\left(b_5b_0 + \begin{vmatrix} b_3 & b_1 \\ b_4 & b_2 \end{vmatrix}\right) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

After rearranging terms, we obtain

$$\det(C_e(A\circ B)+\operatorname{diag}(0,c,0)) \;=\; \det(C_e(A\circ B))+c\det(C_e(A\circ B)[1,3]).$$

On the other side, application of Lemma 3.1 with exclusion of the equality case and (3.7) yields

$$\begin{aligned}\det(C_e(A\circ B)+\operatorname{diag}(0,c,0)) &= \det(A\circ B)+c\,\det C_e(A\circ B)[1,3]\\ &\geq \det(C_e(A)\circ C_e(B)+X+Y)\\ &\geq \det(C_e(A)\circ C_e(B))\\ &\geq \det A\cdot\det B,\end{aligned}$$

from which the statement follows.

□

Example 3.5. *Let $f(x)=(x+1)^6$, then the Hurwitz matrix A associated with the polynomial f is*

$$A=\begin{bmatrix} a_5 & a_3 & a_1 & 0 & 0 & 0\\ a_6 & a_4 & a_2 & a_0 & 0 & 0\\ 0 & a_5 & a_3 & a_1 & 0 & 0\\ 0 & a_6 & a_4 & a_2 & a_0 & 0\\ 0 & 0 & a_5 & a_3 & a_1 & 0\\ 0 & 0 & a_6 & a_4 & a_2 & a_0\end{bmatrix}=\begin{bmatrix} 6 & 20 & 6 & 0 & 0 & 0\\ 1 & 15 & 15 & 1 & 0 & 0\\ 0 & 6 & 20 & 6 & 0 & 0\\ 0 & 1 & 15 & 15 & 1 & 0\\ 0 & 0 & 6 & 20 & 6 & 0\\ 0 & 0 & 1 & 15 & 15 & 1\end{bmatrix},$$

and by Orlando's formula (3.4),

$$\det A=-(-2)^{15}=32768.$$

$$\det A^{\circ 2}=\begin{vmatrix} 36 & 400 & 36 & 0 & 0 & 0\\ 1 & 225 & 225 & 1 & 0 & 0\\ 0 & 36 & 400 & 36 & 0 & 0\\ 0 & 1 & 225 & 225 & 1 & 0\\ 0 & 0 & 36 & 400 & 36 & 0\\ 0 & 0 & 1 & 225 & 225 & 1\end{vmatrix}=19265701888,$$

$$\det(C_e(A^{\circ 2})[1,3])=\det\begin{bmatrix}\begin{vmatrix} a_5^2 & a_6^2\\ a_3^2 & a_4^2\end{vmatrix} & a_5^2a_0^2\\ a_5^2a_0^2 & a_1^2a_0^2\end{bmatrix}=\det\begin{bmatrix}\begin{vmatrix} 36 & 1\\ 400 & 225\end{vmatrix} & 36\\ 36 & 36\end{bmatrix}=\det\begin{bmatrix} 7700 & 36\\ 36 & 36\end{bmatrix}=275904,$$

$$c=2a_3a_2a_5a_0=3600,$$

$$\det A^{\circ 2}+c\,\det(C_e(A^{\circ 2})[1,3])=2.02...\times 10^{10}>\det A^2=1.07...\times 10^9.$$

3.3 Hadamard Product of Polynomials and Its Stability

In this section, we first present conditions under which the Hadamard power of a Hurwitz polynomial is also Hurwitz. Using these, we give a necessary and sufficient condition for the Hadamard square root of a Hurwitz polynomial of degree 5 to be Hurwitz, see [AASG].

3.3.1 Hadamard Powers of Stable Polynomials

Consider the polynomial $f \in \boldsymbol{P}_n$ given as

$$f(x) = \sum_{k=0}^{n} a_k x^k. \tag{3.8}$$

In 1996, Garloff and Wagner proved in [GW96] that $f \in \boldsymbol{H}_n$ implies $f^{\circ t} \in \boldsymbol{H}_n$ for all $t \in \{1, 2, 3, \dots\}$. In general, if $p \in \boldsymbol{H}_n$ then $p^{\circ t}$ need not be a Hurwitz polynomial for every $t \geq 1$. Sufficient conditions for $p^{\circ t}$, $t \in \mathbb{R}$, to be Hurwitz for all $t > t_0$ or $t < t_1$, for some positive t_0 and negative t_1, are given in [BBC17].

For a polynomial f in (3.8), define $\omega_i(t)$ as follows

$$\omega_i(t) = a_{n-i-1}^t a_{n-i}^t - a_{n-i-2}^t a_{n-i+1}^t, \quad i = 1, 2, \dots, n-2. \tag{3.9}$$

Also, let

$$\overline{d} = \max_{1 \leq i \leq n-2} \frac{a_{n-i-2}\; a_{n-i+1}}{a_{n-i-1}\; a_{n-i}}\;, \quad \underline{d} = \min_{1 \leq i \leq n-2} \frac{a_{n-i-2}\; a_{n-i+1}}{a_{n-i-1}\; a_{n-i}}. \tag{3.10}$$

Note that,

- if $\omega_i > 0$ for all i, then $\overline{d} < 1$,
- if $\omega_i < 0$ for all i, then $\underline{d} > 1$.

The following theorem gives sufficient conditions for a Hurwitz polynomial $f^{\circ t}$ of degree $n \geq 3$, to be not Hurwitz for some $t \in \mathbb{R}$.

Theorem 3.9. *([BBC17, Theorem 5]): Let f given in (3.8), $n \geq 3$.*

- *If $\omega_i \geq 0$ for some $i \in \{1, \dots, n-2\}$, then $f^{\circ t} \notin \boldsymbol{H}_n$ for all $t \leq 0$.*
- *If $\omega_i \leq 0$ for some $i \in \{1, \dots, n-2\}$, then $f^{\circ t} \notin \boldsymbol{H}_n$ for all $t \geq 0$.*

Now we present specific cases for the polynomial $f^{\circ t}$ to be Hurwitz.

a) Case $n = 3$

Consider the polynomials of degree $n = 3$. The following theorem gives sufficient conditions for the polynomial $f^{\circ t}$, $t \in \mathbb{R}$, to be Hurwitz.

Theorem 3.10. *([BBC17, Theorem 6]): Let $f(x) = \sum_{k=0}^{3} a_k x^k \in \boldsymbol{H}_3$.*

- *If $\omega_1 > 0$, then $f^{\circ t} \in \boldsymbol{H}_3$ for all $t > 0$.*
- *If $\omega_1 < 0$, then $f^{\circ t} \in \boldsymbol{H}_3$ for all $t < 0$.*

Proof. If $\omega_1 > 0$, then $a_1^t a_2^t - a_0^t a_3^t > 0$ for every $t > 0$. Let $H^{\circ t}(f)$ be the Hurwitz matrix associated to $f^{\circ t}$, and Δ_i, $i = 1, 2, 3$, be the leading principal minors of $H^{\circ t}(f)$. Then by the Routh-Hurwitz criterion we get the stability of $f^{\circ t}$ for $t > 0$, because

$$\Delta_1 = a_2^t > 0\ ,\ \ \Delta_2 = \omega_1(t)\ ,\ \textit{and}\ \ \Delta_3 = a_0^t \omega_1(t).$$

In an analogous manner is the proof of the second statement. □

b) Case $n = 4$

The following two lemmas give a simple characterization of Hurwitz polynomials of degree 4 and a simple consequence for the sign of ω_k, which will be helpful to prove the next theorem.

Lemma 3.2. *([BBC17, Proposition 7]): Let* $f(x) = \sum_{k=0}^{4} a_k x^k \in \boldsymbol{P}_4$. *Then* $f \in \boldsymbol{H}_4$ *if and only if*

$$\frac{a_1 a_4}{a_2 a_3} + \frac{a_0 a_3}{a_1 a_2} < 1. \tag{3.11}$$

Proof. It is easily computed that

$$\Delta_1 = a_3,\ \ \Delta_2 = a_2 a_3 - a_1 a_4,\ \Delta_3 = a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4,\ \textit{and}\ \Delta_4 = a_0 \Delta_3.$$

By the Routh-Hurwitz criterion, $f \in \boldsymbol{H}_4$ implies $\Delta_3 > 0$, i.e.,

$$a_1 a_2 a_3 > a_0 a_3^2 + a_1^2 a_4.$$

Dividing by $a_1 a_2 a_3$, we obtain inequality (3.11). For the reverse implication, we can conclude from (3.11) that

$$\frac{a_1 a_4}{a_2 a_3} < 1.$$

Hence $\Delta_2 > 0$. Moreover, an immediate consequence of (3.11) is $\Delta_3 > 0$, and so $\Delta_4 > 0$. Once again we use the Routh–Hurwitz criterion and get the stability of f. □

Lemma 3.3. *([BBC17, Lemma 1]): Let* $\mu \in (0, 1)$ *and* $f \in \boldsymbol{P}_n$, $n \geq 3$. *Put*

$$t_0 := \frac{log\ \mu}{log\ \overline{d}}\ ,\ \ t_1 := \frac{log\ \mu}{log\ \underline{d}}.$$

Then

- *If* $\omega_i > 0$ *for all* $i = 1, \ldots, n-2$, *then*

$$\mu a_{n-i-1}^t a_{n-i}^t - a_{n-i-2}^t a_{n-i+1}^t > 0\ \ \textit{for all}\ i = 1, \ldots, n-2\ \ \textit{and}\ \ t > t_0 > 0.$$

- *If* $\omega_i < 0$ *for all* $i = 1, \ldots, n-2$, *then*

$$\mu a_{n-i-1}^t a_{n-i}^t - a_{n-i-2}^t a_{n-i+1}^t > 0\ \ \textit{for all}\ i = 1, \ldots, n-2\ \ \textit{and}\ \ t < t_1 < 0.$$

Proof. If ω_i for all i, then $\overline{d} < 1$ and hence for a fixed $t > t_0$ we have $\overline{d}^t < \mu$. Therefore, it is easy to conclude that $\overline{d}a_{n-i-1}a_{n-i} \geq a_{n-i-2}a_{n-i+1}$ for all i. From this we get

$$0 \leq \overline{d}^t a^t_{n-i-1}a^t_{n-i} - a^t_{n-i-2}a^t_{n-i+1} < \mu\, a^t_{n-i-1}a^t_{n-i} - a^t_{n-i-2}a^t_{n-i+1}.$$

In an analogous manner is the proof of the second statement. □

Note that, for any polynomial $f(x) = \sum_{k=0}^{4} a_k x^k$, there are only two $\omega_i^{,}s$ defined by (3.9):

$$\omega_1 = a_2a_3 - a_1a_4\ , \quad \omega_2 = a_1a_2 - a_0a_3$$

and by (3.10):

$$\overline{d} = max\left\{\frac{a_1a_4}{a_2a_3}\ ,\ \frac{a_0a_3}{a_1a_2}\right\}\ , \quad \underline{d} = min\left\{\frac{a_1a_4}{a_2a_3}\ ,\ \frac{a_0a_3}{a_1a_2}\right\}.$$

Theorem 3.11. *([BBC17, Theorem 8]): Let $f(x) = \sum_{k=0}^{4} a_k x^k \in \boldsymbol{P}_n$ and*

$$t_0 := \frac{log\ 0.5}{log\ \overline{d}}, \quad t_1 := \frac{log\ 0.5}{log\ \underline{d}}.$$

Then

- *If $\omega_1,\ \omega_2 > 0$ then $f^{\circ t} \in \boldsymbol{H}_4$ for all $t > t_0 > 0$.*
- *If $\omega_1,\ \omega_2 < 0$ then $f^{\circ t} \in \boldsymbol{H}_4$ for all $t < t_1 < 0$.*

Moreover, the constants t_0 and t_1 are best possible, i.e., for t_0 it means that there exists a polynomial f of degree 4 with positive coefficients and $\omega_1,\ \omega_1 > 0$ such that $f^{\circ t}$ is not stable for every $t \leq t_0$.

Proof. By using Lemma 3.3 for $\mu = \frac{1}{2}$ and $t > t_0$, we have

$$\frac{1}{2}a^t_2a^t_3 - a^t_1a^t_4 > 0\ , \quad \frac{1}{2}a^t_1a^t_2 - a^t_0a^t_3 > 0.$$

From where

$$\frac{a^t_1a^t_4}{a^t_2a^t_3} < \frac{1}{2} \quad and \quad \frac{a^t_0a^t_3}{a^t_1a^t_2} < \frac{1}{2}$$

and therefore

$$\frac{a^t_1a^t_4}{a^t_2a^t_3} + \frac{a^t_0a^t_3}{a^t_1a^t_2} < 1. \tag{3.12}$$

Lemma 3.2 implies that $f^{\circ t} \in \boldsymbol{H}_4$ for $t > t_0$. In an analogous manner is the proof of the second statement. □

The following example shows that the constants t_0 and t_1 cannot be improved.

Example 3.6. *([BBC17, Example 2]): Consider the polynomial*

$$f(x) = 2x^4 + x^3 + 5x^2 + x + 2.$$

Then

$$\omega_1 = 3\ , \quad \omega_2 = 3\ , \quad \overline{d} = 0.4\ , \quad t_0 = \frac{log\ 0.5}{log\ 0.4}.$$

Fix $t \le t_0$. By Lemma 3.2, $f^{\circ t} \in \boldsymbol{H}_4$ if and only if inequality (3.12) *holds. We calculate*

$$\frac{a_1^t a_4^t}{a_2^t a_3^t} + \frac{a_0^t a_3^t}{a_1^t a_2^t} = 2\ (0.4)^t \ge 2\ (0.4)^{t_0} = 1.$$

We see that inequality (3.12) *does not hold and consequently $f^t \notin \boldsymbol{H}_4$. Additionally, by Lemma 3.2, $f \in \boldsymbol{H}_4$.*

c) Case $n = 5$

Theorem 3.12. *([BBCK24, Theorem 2.3]): Let $f(x) = \sum_{k=0}^{5} a_k x^k \in \boldsymbol{H}_5$. Then $f^{\circ t} \in \boldsymbol{H}_5$ for all $t > 1$.*

Let $f(x) = \sum_{k=0}^{n} a_k x^k \in \boldsymbol{P}_n$. In [BG21], the authors give a necessary and sufficient condition for the Hadamard square root of a Hurwitz polynomial f of degree four, $f^{\circ\frac{1}{2}}(x) = \sum_{k=0}^{n} \sqrt{a_k} x^k$, to be Hurwitz. The condition is:

$$(\rho_1 - \rho_2)^2 - 2(\rho_1 + \rho_2) + 1 > 0\ ,$$

where

$$\rho_1 := \frac{a_1 a_4}{a_2 a_3} \quad \textit{and} \quad \rho_2 := \frac{a_0 a_3}{a_2 a_1}.$$

In general, if a polynomial f of degree 5 is Hurwitz, then $f^{\circ\frac{1}{2}}$ does not need to be Hurwitz.

Example 3.7. *([AASG, Example 3.9]): The polynomial*

$$f(x) = 0.1x^5 + 1.5x^4 + x^3 + 3x^2 + x + 1$$

is Hurwitz, but $f^{\circ\frac{1}{2}} \notin \boldsymbol{H}_5$.

We give a necessary and sufficient condition for the Hadamard square root of a Hurwitz polynomial of degree 5 to be Hurwitz.

Theorem 3.13. *([AASG, Theorem 3.10]): Let $f(x) = \sum_{k=0}^{5} a_k x^k \in \boldsymbol{H}_5$, $\omega := \frac{\sqrt{a_4 a_3} - \sqrt{a_5 a_2}}{\sqrt{a_4 a_1} - \sqrt{a_5 a_0}}$. Then $f^{\circ\frac{1}{2}}$ is Hurwitz if and only if*

$$\omega^2 > \frac{a_4}{a_2}\ , \ \textit{and} \quad \sqrt{a_0}\,\omega^2 - \sqrt{a_2}\,\omega + \sqrt{a_4} < 0$$

Proof. By Theorem 3.3, it is enough to show that k is Hurwitz, where

$$k(x) := a_4x^4 + \left(\sqrt{a_4a_3} - \sqrt{a_5a_2}\right)x^3 + \sqrt{a_4a_2}x^2 + \left(\sqrt{a_4a_1} - \sqrt{a_5a_0}\right)x + \sqrt{a_4a_0}.$$

Let K be the Hurwitz matrix associated with the polynomial k. Theorem 3.7 implies that k is Hurwitz if and only if

$$C_e(K) = \sqrt{a_0a_4}\begin{bmatrix} \begin{vmatrix} \sqrt{a_4a_3} - \sqrt{a_5a_2} & \sqrt{a_4a_1} - \sqrt{a_5a_0} \\ a_4 & \sqrt{a_4a_2} \end{vmatrix} & \sqrt{a_0a_4}\left(\sqrt{a_4a_3} - \sqrt{a_5a_2}\right) \\ \sqrt{a_4a_3} - \sqrt{a_5a_2} & \sqrt{a_4a_1} - \sqrt{a_5a_0} \end{bmatrix} \succ 0.$$

The two conditions imply that the two leading principal minors in $C_e(K)$ are positive. □

The following example shows that the result in Theorem 3.12 does not need to hold for polynomials of degree 6 and noninteger t.

Example 3.8. *([BBCK24, Example 3.1]): Consider*

$$f(x) = x^6 + x^5 + 117x^4 + 17.006x^3 + 1772x^2 + 72.051^x + 7200.$$

Then $f \in \boldsymbol{H}_6$ but two of the zeros of the polynomial $f^{\circ 1.05}$ are: 0.00003081 ± 2.68432210. So they have positive real parts. Therefore, $f^{\circ 1.05} \notin \boldsymbol{H}_6$.

4 Related Matrix Intervals

By an interval matrix, we mean a matrix whose elements are interval numbers. Matrix intervals of several classes of matrices have been studied and investigated by some mathematicians. Matrix intervals of P-matrices, positive definite matrices, inverse positive matrices, M-matrices, inverse M-matrices, and stability of interval matrices are considered in [BG84], [Roh87], [Roh94], [RR96], [JS02]. For surveys see [AGT16] and [GASA21]. In this chapter, we study the interval property of several classes of matrices. Many of the classes that are studied in this chapter are related to the linear complementarity problem.

The organization of this chapter depends on the number of vertex matrices that are needed for having the interval property.

4.1 Matrix Properties Which Can Be Inferred from One or Two Vertex Matrices

We start with the class of nonsingular matrices that have a nonnegative inverse. Such matrices are called *inverse nonnegative* matrices.

a) Inverse Nonnegative Matrices

Definition 4.1. *A matrix* $A \in \mathbb{R}^{n,n}$ *is called* inverse nonnegative *if* A *is nonsingular and* $0 \leq A^{-1}$.

IP 4.1. *[Kut71, Corollary 3.5] The inverse nonnegative matrices have the interval property with respect to the set* $V([A]) = \left\{\underline{A}, \overline{A}\right\}$.

IP 4.1 can also be found in [Met72, Bemerkung 1.2 (v) (a), p.15]. It seems that Metelmann found this result independently of Kuttler ([Kut71] appeared in April 1971, Kurt Metelmann submitted his dissertation [Met72] most probably at the end of year 1971 or at the beginning of 1972). In [Roh89, Theorem 4.6] an extension of IP 4.1 to more general sign patterns of the inverse matrix is presented. This interval property involves two vertex matrices of type A_{yz} which will be introduced in Subsection 4.2. For symmetric inverse nonnegative matrices there is also a simple formula for the smallest eigenvalue. The following two theorems give more properties of an inverse nonnegative interval matrix. More about inverse nonnegative matrices can be found in [Hla20].

Theorem 4.1. *([Hla20, Theorem 7,8]): If* $[A]$ *is inverse nonnegative, then*

(1) $\overline{A}^{-1} \leq A^{-1} \leq \underline{A}^{-1}$*, for every* $A \in [A]$.

(2) If both A_c and Δ are symmetric, then $\lambda_{min}([A]_{sym}) = [\lambda_{min}(\underline{A}), \lambda_{min}(\overline{A})]$, where $\lambda_{min}([A]_{sym})$ denotes the minimum eigenvalue of $[A]$.

Notice that for the largest eigenvalue an analogous statement is not valid in general.

Theorem 4.2. *([Hla20, Theorem 10]): Let $[A]$ be inverse nonnegative, then*

$$det[A] = [min(\mathfrak{D}), max(\mathfrak{D})],$$

where $\mathfrak{D} = \left\{\det\underline{A}, \det\overline{A}\right\}$ and $\det[A]$ denotes the range of the determinant function over $[A]$.

An M-matrix is an inverse nonnegative matrix having all its off-diagonal entries nonpositive.

b) Semipositive, Semimonotone, and Copositive Matrices

Definition 4.2. *A matrix $A \in \mathbb{R}^{n,n}$ is called*

(i) semipositive*, if there exists a vector $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq 0$ such that $A\mathbf{x} > 0$;*

(ii) minimally semipositive*, if it is semipositive and no column-deleted submatrix is semipositive;*

and if $m = n$, A is called

(iii) (strictly) semimonotone*, if each nonzero vector $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq 0$ has a component $x_k > 0$ such that $(Ax)_k \geq 0$ (> 0);*

(iv) (strictly) copositive, if $\mathbf{x}^T A\mathbf{x} \geq 0$ (> 0) for all vectors $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \geq 0$ (and $\mathbf{x} \neq 0$).

Notice that the set of strictly copositive (copositive) matrices is a generalization of the set of the positive definite (positive semidefinite) matrices. The classes of matrices in Defintion 4.2 are closely related to the linear complementarity problem. E.g., a matrix $A \in \mathbb{R}^{n,n}$ is (strictly) semimonotone if and only if the (LCP)($\mathbf{q}, A$) has a unique solution for all $\mathbf{q} > 0$ ($\mathbf{q} \geq 0$). Also, if A is a strictly copositive matrix, then for each $\mathbf{q} \in \mathbb{R}^n$ the (LCP)($\mathbf{q}, A$) has a solution. It is easy to see that the (LCP)($\mathbf{q}, A$) need not have a solution if A is only a copositive matrix. For example, if $A = 0$, then A is copositive but the (LCP)($\mathbf{q}, A$) has a solution if and only if $\mathbf{q} > 0$. See [Cot10], [CPS09].
A thorough treatment of properties of the (minimally) semipositive, the strictly semimonotone, and the (strictly) copositive matrices can be found in [JST20, Chapter 3, Section 3.8, and Chapter 6]. Obviously, each of the matrix classes (i), (iii), and (iv) in Definition 4.2 has the interval property, see [Hla19, Propositions 1,3,5].

IP 4.2. *Each of the sets of*

(i) the semipositive matrices,

(ii) the (strictly) semimonotone matrices, and

(iii) the (strictly) copositive matrices

has the interval property with respect to the left corner matrix.

The minimally semipositive matrices possess the interval property with respect to the two corner matrices but a slightly stronger result is valid.

Theorem 4.3. *([CKS18, Corollaries 5.5 and 5.6], [CK21, Theorem 5.1 (b)]): Let* $[A] = [\underline{A}, \overline{A}]$ *be a matrix interval in* $\mathbb{R}^{m,n}$. *Then* $[A]$ *is minimally semipositive if and only if* $\underline{A}$ *is semipositive and* $\overline{A}$ *is minimally semipositive.*

It is known, see, e.g., [JST20, Corollary 3.5.8], that a square matrix A is minimally semipositive if and only if A is inverse nonnegative. Thus, Theorem 4.3 provides a strengthening of IP 4.1. For related results, see [CKS18, Section 5]. Given a (minimally) semipositive matrix B, results on constructing a (minimally) semipositive matrix interval $[A, B]$ can be found in [JST20, Section 3.7.3].

c) Almost P-Matrices

Reversing the sign of the determinant of P-matrices, i.e, of matrices having all their principal minors positive (see Definition 4.5), results in the class of the following matrices which are related to the linear complementarity problem [Mia93].

Definition 4.3. *A matrix* $A \in \mathbb{R}^{n,n}$, $n \geq 2$, *is called an* almost *P-matrix if all its proper principal minors are positive while its determinant is negative.*

In [BP14] Barreras and Peña proved equivalent properties of this class of matrices as follows.

Theorem 4.4. *([BP14, Theorem 4]): Let* $A \in \mathbb{R}^{n,n}$, $n \geq 2$, *be a Z-matrix. Then the following statements are equivalent:*

- *A is an almost P-matrix.*
- $\det A < 0$ *and* $A[1, \ldots, n-1]$ *is a nonsingular M-matrix.*
- $\det A < 0$ *and* $\det A[1, \ldots, k] > 0$ *for all* $k < n$.
- *A is a nonsingular matrix with an odd number of negative eigenvalues, and all eigenvalues of the* $(n-1) \times (n-1)$ *leading principal submatrix have positive real part.*

IP 4.3. *([Fan66, Lemma 3]): The set of almost P-matrices having all their off-diagonal entries nonpositive has the interval property with respect to the two corner matrices.*

Recall that an M-matrix A can be represented as

$$A = tI - B \text{ with } B \geq 0, \tag{4.1}$$

where I is the identity matrix and $t > \rho(B)$ ($\rho(B)$ denotes the spectral radius of B). A characterization of a matrix A appearing in IP 4.3 is that it can be represented as in (4.1) with $\sigma < t < \rho(B)$, where σ denotes the maximum of the spectral radii of all principal submatrices of order $n-1$ of B [Fan66, p. 188]. In [Joh82], the closure of these matrices is considered, i.e., the matrices A having the following properties:

(a) A can be represented as in (4.1),

(b) $\sigma \leq t < \rho(B)$,

(c) the off-diagonal entries of A are nonpositive.

IP 4.4. *([Joh82, Theorem 2.10(i)]): The set of the matrices A with (a)-(c) has the interval property with respect to the two corner matrices.*

d) H-matrices

The set of M-matrices is contained in the set of H-matrices. Also, A is an H-matrix if and only if A is generalized diagonally dominant. Therefore, a given matrix A is an H-matrix if there exists a positive diagonal matrix D such that AD is diagonally dominant. The following theorem gives a link between regularity and the H-matrix property.

Theorem 4.5. *([Neu90, Prop. 4.1.7]): Let A_c be an M-matrix. Then $[A]$ is regular if and only if it is an H-matrix.*

Notice that the assumption cannot be weakened to the assumption that A_c is an H-matrix. For example, the interval matrix

$$\begin{bmatrix} [0,10] & 1 \\ -1 & 10 \end{bmatrix}$$

is regular and its midpoint is an H-matrix. However, $[A]$ itself is not an H-matrix.

Definition 4.4. *Let $[A] = [\underline{a}_{ij}, \overline{a}_{ij}]_{i,j=1}^n$ be a matrix interval in $\mathbb{R}^{n,n}$. Define the vertex matrix $v([A])$ as follows:*

$$v([A])_{ij} := \begin{cases} \min\limits_{a_{ii} \in [a_{ii}]} |a_{ii}| & \text{if } i = j, \\ -\max\limits_{a_{ij} \in [a_{ij}]} |a_{ij}| & \text{if } i \neq j, \end{cases}$$

for all $i, j = 1, \ldots, n$.

IP 4.5. *(E.g., [May17, Theorem 3.3.5(a)]) The set of the H-matrices has the interval property with respect to a vertex matrix at which $v([A])$ is attained.*

Sufficient conditions for a matrix interval to contain only H-matrices can be found in [CS21]. Also, the H-matrix intervals are related to the verified numeric solution of the linear complementarity problem with interval data, see [AS03] and [MXH09] and further references therein.

4.2 Matrix Properties Which Require at Most 2^{n-1} Vertex Matrices

In this section, we study the interval property of several classes of matrices. It was shown that the set of inverse M-matrices has the interval property with respect to at most 2^{n^2-n} of vertex matrices. We reduce this set to $2n^2$ vertices. Also, we prove the interval property for the conditionally positive semidefinite, conditionally negative semidefinite, and infinitely divisible matrices, see [GASA21].

First we recall the definition of the following sets of vertex matrices.

Let $Y_n = \{\mathbf{y} \in \mathbb{R}^n \mid \mid y_i \mid= 1,\ i = 1, \ldots, n\}$ and $T_{\mathbf{y}} = \operatorname{diag}(y_1, y_2, \ldots, y_n)$. We define the set $V_1([A])$ of matrices $A_{\mathbf{yz}} = A_c - T_{\mathbf{y}}\Delta T_{\mathbf{z}}$ for all $\mathbf{y}, \mathbf{z} \in Y_n$. The definition implies that for all $i, j = 1, \ldots, n$,

$$(A_{\mathbf{yz}})_{ij} = (A_c)_{ij} - y_i(\Delta)_{ij}z_j = \begin{cases} \overline{a}_{ij} & \text{if} \quad y_iz_j = -1, \\ \underline{a}_{ij} & \text{if} \quad y_iz_j = 1, \end{cases} \tag{4.2}$$

so that all matrices $A_{\mathbf{yz}}$ are vertex matrices. The cardinality of $V_1([A])$ is at most 2^{2n-1}. In the following subsection, we consider the set $V_2([A])$ which is obtained from $V_1([A])$ if we choose $\mathbf{y} = \mathbf{z}$. The cardinality is then reduced to at most 2^{n-1}.

a) P-matrices:

Definition 4.5. *A matrix is called a $P(P_0)$-*matrix *if all its principal minors are positive (nonnegative).*

Note that, a matrix $A \in \mathbb{R}^{n,n}$ is a P-matrix if and only if $\text{LCP}(\mathbf{q}, A)$ has a unique solution for every $\mathbf{q} \in \mathbb{R}^n$.

IP 4.6. *[BG84, Theorem 1 (i) and Remark (b)]: The set of $P(P_0)$-matrices has the interval property with respect to $V_2([A])$.*

[BG84, Theorem 2] shows that for P-matrices the set $V_2([A])$ cannot be replaced by a nonempty proper subset. For the interval property of matrices with alternating signs of their principal minors see [BG84, Remark (b)].

b) Positive (semi)definiteness

IP 4.7. *([Roh94, Theorem 2]): The set of positive (semi)definite matrices has the interval property with respect to $V_2([A])$.*

In [Kre05, Theorem 2.2] it was shown that in the positive definite case the set $V_2([A])$ cannot be replaced by a nonempty proper subset.

Note that, a sufficient condition for existence and uniqueness of a solution to the LCP problem is that A be symmetric positive-definite.

c) Hurwitz Stability

Definition 4.6. *A matrix is called* Hurwitz stable *if all its eigenvalues have negative real parts.*

It is well-known that the Hurwitz stability of a matrix interval cannot in general be inferred from the Hurwitz stability of all of its vertex matrices, see [GB88, p.395] and [Roh94, p.181]. However, if a matrix A is symmetric then A is Hurwitz stable if and only if $-A$ is positive definite. Using this fact, the following theorem can be shown.

Theorem 4.6. *([Roh94, Theorem 6]): Let $[A]$ be a symmetric matrix interval. Then $[A]$ is Hurwitz stable if and only if each vertex matrix $A_{-\boldsymbol{z},\boldsymbol{z}}$, $\boldsymbol{z} \in Y_n$, is Hurwitz stable.*

In [Kre05, Theorem 2.2] it was shown that a further reduction of the set Y_n is impossible: without checking all 2^{n-1} matrices $A_{-\mathbf{z},\mathbf{z}}$ we cannot guarantee that all $A \in [A]$ are Hurwitz stable. In [YD13] matrices are considered which are connected with mathematical models of ecosystems describing the effects a species may have on itself and its surrounding species. It is demonstrated by some examples that a few vertex matrices of this type may suffice to conclude that the entire matrix interval is Hurwitz stable.

d) Inverse M-Matrices

We consider now the set of the inverse M-matrices, i.e., regular matrices having an M-matrix as inverse matrix. For a comprehensive survey on this class see [JST20, Chapter 5]. This set has the interval property with respect to a set of vertex matrices of cardinality at most 2^{n^2-n}. It has been conjectured in [Hla20, Conjecture 1] that the cardinality of the needed set of vertex matrices can be reduced to $2n^2$. The following interval property reveals that it can be reduced to 2^{n-1} but it leaves the conjecture still open.

IP 4.8. *([GASA21, IP 4.4]): The set of the inverse M-matrices has the interval property with respect to the set V_2 of vertex matrices.*

Proof. We use the following characterization of nonnegative matrices having a special property called the *Minkowski property* [UKW73, Theorem 3], see also [KK78, Chapter 3, Theorem 6]: The inverse matrix of $A \in \mathbb{R}^{n,n}$ has nonnegative diagonal entries and nonpositive off-diagonal entries if and only if for any nonempty proper subset α of $\{1, \ldots, n\}$ of cardinality r and any given positive vector $\mathbf{y} \in \mathbb{R}^{n-r}$, the following system has a positive solution $\mathbf{x} \in \mathbb{R}^r$,

$$(T_{\mathrm{e}^\alpha} A T_{\mathrm{e}^\alpha})\mathbf{w} > 0, \tag{4.3}$$

where the vector $\mathbf{e}^\alpha \in \mathbb{R}^n$ is defined as:

$$(\mathbf{e}^\alpha)_i := 1, \text{ if } i \in \alpha, \textit{ and } (\mathbf{e}^\alpha)_i := -1, \text{ otherwise}, \; i = 1, \ldots, n,$$

where $\mathbf{e}$ is defined in Section 1.2.2, and the components of $\mathbf{w} \in \mathbb{R}^n$ are composed of the r components of $\mathbf{x}$ and the $n-r$ components of $\mathbf{y}$.

Now let $[A]$ be a matrix interval in $\mathbb{R}^{n,n}$ and assume that all its vertex matrices from V_2 are inverse M-matrices. Choose any $A \in [A]$ and consider the system (4.3) with any given positive vector $\mathbf{y} \in \mathbb{R}^{n-r}$. Since A is nonnegative, we obtain for any positive vector $\mathbf{x} \in \mathbb{R}^r$, if $i \in \alpha$:

$$\sum_{j\in\alpha} a_{ij}x_j - \sum_{j\notin\alpha} a_{ij}y_j \geq \sum_{j\in\alpha} \underline{a}_{ij}x_j - \sum_{j\notin\alpha} \overline{a}_{ij}y_j, \tag{4.4}$$

and if $i \notin \alpha$:

$$-\sum_{j\in\alpha} a_{ij}x_j + \sum_{j\notin\alpha} a_{ij}y_j \geq -\sum_{j\in\alpha} \overline{a}_{ij}x_j + \sum_{j\notin\alpha} \underline{a}_{ij}y_j. \tag{4.5}$$

We may conclude that we have found a vertex matrix in V_2 such that the system of the form (4.3) associated with this vertex matrix has a positive solution for the given positive vector $\mathbf{y}$. By (4.4) and (4.5), the system (4.3) has the same positive solution. Therefore, the nonnegative matrix A has the Minkowski property. By [KK78, Chapter 3, Lemma 10], each nonnegative matrix possessing the Minkowski property as well as its inverse matrix are P-matrices. Since A^{-1} is a P-matrix having nonpositive off-diagonal entries, A is an inverse M-matrix. □

For a matrix interval $[A]$ in $\mathbb{R}^{n,n}$, we define the set $V_3([A])$ of its vertex matrices as the set of matrices $A_{\mathbf{zz}}$,

$$A_{\mathbf{zz}} := A_c + T_{\mathbf{z}} \bigtriangleup T_{\mathbf{z}}, \text{ for all } \mathbf{z} \in Y_n;$$

this means that for $\mathbf{y} = -\mathbf{z}$ in the explicit representation (4.2) of the matrices contained in $V_2([A])$, the role of the lower and upper endpoints of the coefficient intervals has to be interchanged.
The next two matrix classes play also a role in mathematical economics.

e) Conditionally Positive and Conditionally Negative Semidefinite Matrices
Recall that, a symmetric matrix $A \in \mathbb{R}^{n,n}$ is said to be conditionally positive (negative) semidefinite if $\mathbf{x}^T A\mathbf{x} \geq 0$ $(\leq 0,)$ for all vectors $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x}^T\mathbf{e} = 0$, see Subsection 1.2.2.

IP 4.9. *([GASA21, IP 4.7]): The set of the conditionally positive (negative) semidefinite matrices has the interval property with respect to the set V_2 (V_3) of vertex matrices.*

Proof. We apply Theorem 2.6. We first prove the statement for conditionally positive semidefinite matrices. Let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval in $\mathbb{R}^{n,n}$ and assume that both corner matrices are symmetric. Furthermore, assume that all vertex matrices in V_2 are conditionally positive semidefinite. Let $A \in [A]$ be symmetric. Then it follows that

$$e^{\circ t\underline{A}} \leq e^{\circ tA} \leq e^{\circ t\overline{A}}, \text{ for all } 0 \leq t.$$

There is a one-to-one correspondence between the matrices in $V_2([A])$ and the matrices in $V_2([e^{\circ t\underline{A}}, e^{\circ t\overline{A}}])$, where the latter vertex matrices are positive semidefinite for all $0 \leq t$. By

IP 4.7, $e^{\circ tA}$ is positive semidefinite for all $0 \leq t$, and we can conclude that A is conditionally positive semidefinite. The statement for conditionally negative semidefinite matrices follows now by using $[-\overline{A}, -\underline{A}]$ instead of $[A]$ and the set V_3 instead of V_2. □

The question whether an analogous interval property is true for the conditionally positive definite matrices is open. Unfortunately, the necessary and sufficient condition for a matrix to be conditionally positive semidefinite used in the proof of IP 4.9 does not carry over to the conditionally positive definite matrices: If a matrix A is conditionally positive definite then, in fact, its Hadamard exponential $e^{\circ A}$ is positive definite, see [Rea99, Corollary 2.6], but the reverse statement is not true. A counterexample is provided by the matrix

$$A := \begin{bmatrix} 6 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 4 & 12 \end{bmatrix},$$

which is not conditionally positive definite while $e^{\circ A}$ is positive definite.

f) Infinitely Divisible Matrices

Recall that, A is infinitely divisible if the matrix $A^{\circ r}$ is symmetric positive semidefinite for every non-negative r.

IP 4.10. *([GASA21, IP 4.8]): The set of positive infinitely divisible matrices has the interval property with respect to the set V_2 of vertex matrices.*

Proof. The proof is similar to the one of IP 4.9. Here we use Theorem 2.10. Let $[A] = [\underline{A}, \overline{A}]$ be a matrix interval in $\mathbb{R}^{n,n}$ and assume that $\underline{A} > 0$ and both corner matrices are symmetric. Furthermore, assume that all vertex matrices in V_2 are infinitely divisible. Let $A \in [A]$ be symmetric. Then it follows that

$$\log^\circ(\underline{A}) \leq \log^\circ(A) \leq \log^\circ(\overline{A}).$$

We conclude by IP 4.9 that $\log^\circ(A)$ is conditionally positive semidefinite, and therefore, A is infinitely divisible. □

Bibliography

[AASG] F. Alsaafin, D. Al-Saafin, and J. Garloff, *A study of the validity of Oppenheim's inequality for Hurwitz matrices associated with hurwitz polynomials*, The Electronic Journal of Linear Algebra, submitted.

[AGT16] M. Adm, J. Garloff, and J. Titi, *A survey of classes of matrices possessing the interval property and related properties*, Reliable Computing **22** (2016), 1–14.

[AS03] G. Alefeld and U. Schäfer, *Iterative methods for linear complementarity problems with interval data*, Computing **70** (2003), no. 3, 235–259.

[AS23] D. Al-Saafin, *On symmetric matrices with one positive eigenvalue and the interval property of some matrix classes*, Mathematics and Computation, Springer Nature Singapore, 2023, pp. 339–344.

[ASG20] D. Al-Saafin and J. Garloff, *Sufficient conditions for symmetric matrices to have exactly one positive eigenvalue*, Special Matrices **8** (2020), no. 1, 98–103.

[Asn70] B. A. Asner, *On the total nonnegativity of the Hurwitz matrix*, SIAM Journal on Applied Mathematics **18** (1970), no. 2, 407–414.

[Bap88] R. Bapat, *Multinomial probabilities, permanents and a conjecture of Karlin and Rinott*, Proceedings of the American Mathematical Society **102** (1988), no. 3, 467–472.

[Bar83] S. Barnett, *Polynomials and Linear Control Systems*, Marcel Dekker, Inc., 1983.

[BBC17] S. Białas and L. Białas-Cież, *Comments on 'on Hadamard powers of polynomials'*, Mathematics of Control, Signals, and Systems **29** (2017), 1–10.

[BBCK24] S. Białas, L. Białas-Cież, and M. Kudra, *On the Hurwitz stability of noninteger Hadamard powers of stable polynomials*, Linear Algebra and its Applications **683** (2024), 111–124.

[BE07] R. Bhatia and L. Elsner, *Positivity preserving Hadamard matrix functions*, Positivity **11** (2007), 583–588.

[Ber29] S. Bernstein, *Sur les fonctions absolument monotones*, Acta Mathematica **52** (1929), no. 1, 1–66.

[Ber96] J. Bertoin, *Lévy Processes*, vol. 121, Cambridge university press Cambridge, 1996.

[Ber08] C. Berg, *Stieltjes-Pick-Bernstein-Schoenberg and their connection to complete monotonicity*, Positive definite functions: From Schoenberg to space-time challenges, Department of Mathematics of University Jaume I of Castellon (2008), 15–45.

[BG84] S. Białas and J. Garloff, *Intervals of P-matrices and related matrices*, Linear Algebra and its Applications **58** (1984), 33–41.

[BG21] S. Białas and M. Góra, *Simple necessary conditions for Hadamard factorizability of Hurwitz polynomials*, The Electronic Journal of Linear Algebra **37** (2021), 671–679.

[Bha97] R. Bhatia, *Matrix Analysis Springer-Verlag New York*, 1997.

[Bha06] R. Bhatia, *Infinitely divisible matrices*, The American Mathematical Monthly **113** (2006), no. 3, 221–235.

[BJ15a] R. Bhatia and T. Jain, *Inertia of the matrix* $[(p_i + p_j)^r]$, Journal of Spectral Theory **5** (2015), no. 1, 71–87.

[BJ15b] ______, *Mean matrices and conditional negativity*, The Electronic Journal of Linear Algebra **29** (2015), 206–222.

[BK95] S. P. Bhattacharyya and L. H. Keel, *Robust control: the parametric approach*, Advances in control education 1994, Elsevier, 1995, pp. 49–52.

[BK07] R. Bhatia and H. Kosaki, *Mean matrices and infinite divisibility*, Linear algebra and its applications **424** (2007), no. 1, 36–54.

[Boc55] S. Bochner, *Harmonic Analysis and the Theory of Probability, University of California Press, Berkeley and Los Angeles*, 1955.

[BP14] Á. Barreras and J. M. Peña, *Characterisations and tests for almost M-matrices*, Monografias Matematicas **39** (2014), 57–63.

[BR97] R. B. Bapat and T. Raghavan, *Nonnegative Matrices and Applications*, vol. 64, Cambridge university press, 1997.

[CK21] P. N. Choudhury and M. R. Kannan, *Interval hulls of N-matrices and almost P-matrices*, Linear Algebra and its Applications **617** (2021), 27–38.

[CKS18] P. Choudhury, R. Kannan, and K. Sivakumar, *New contributions to semipositive and minimally semipositive matrices*, The Electronic Journal of Linear Algebra **34** (2018), 35–53.

[CLLB12] F. J. Caro-Lopera, V. Leiva, and N. Balakrishnan, *Connection between the Hadamard and matrix products with an application to matrix-variate Birnbaum–Saunders distributions*, Journal of Multivariate Analysis **104** (2012), no. 1, 126–139.

[Cot10] R. W. Cottle, *A field guide to the matrix classes found in the literature of the linear complementarity problem*, Journal of Global Optimization **46** (2010), 571–580.

[CPS09] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, SIAM, 2009.

[CS21] M. Chatterjee and K. Sivakumar, *Intervals of H-matrices and inverse M-matrices*, Linear Algebra and its Applications **614** (2021), 24–43.

[DGM86] N. Dyn, T. Goodman, and C. A. Micchelli, *Positive powers of certain conditionally negative definite matrices*, Indagationes Mathematicae (Proceedings) **89** (1986), no. 2, 163–178.

[Fan66] K. Fan, *Some matrix inequalities*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, vol. 29, Springer, 1966, pp. 185–196.

[FH77] C. H. FitzGerald and R. A. Horn, *On fractional Hadamard powers of positive definite matrices*, Journal of Mathematical Analysis and Applications **61** (1977), no. 3, 633–642.

[Gan59] F. Gantmacher, *The Theory of Matrices, vol. 2*, 1959.

[GASA21] J. Garloff, D. Al-Saafin, and M. Adm, *Further matrix classes possessing the interval property*, Reliable Computing **28** (2021), 56–70.

[GB88] J. Garloff and N. K. Bose, *Boundary implications for stability properties,: Ramon E. Moore, editor.* Reliability in Computing, *Academic Press, Boston, San Diego, New York*, 1988, pp. 391–402.

[GKR15] D. Guillot, A. Khare, and B. Rajaratnam, *Complete characterization of Hadamard powers preserving Loewner positivity, monotonicity, and convexity*, Journal of Mathematical Analysis and Applications **425** (2015), no. 1, 489–507.

[GW96] J. Garloff and D. G. Wagner, *Hadamard products of stable polynomials are stable*, Journal of mathematical analysis and applications **202** (1996), no. 3, 797–809.

[Hia09] F. Hiai, *Monotonicity for entrywise functions of matrices*, Linear algebra and its applications **431** (2009), no. 8, 1125–1146.

[HJ91] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge university press, 1991.

[HJ13] R. Horn and C. Johnson, *Matrix Analysis, Second edition*, 2013.

[Hla19] M. Hladík, *Stability of the linear complementarity problem properties under interval uncertainty*, Central European Journal of Operations Research **29** (2019), 875–889.

[Hla20] ———, *An overview of polynomially computable characteristics of special interval matrices*, Beyond Traditional Probabilistic Data Processing Techniques: Interval, Fuzzy etc. Methods and Their Applications (2020), 295–310.

[Hor69] R. A. Horn, *The theory of infinitely divisible matrices and kernels*, Transactions of the American Mathematical Society **136** (1969), 269–286.

[Hor90] ———, *The Hadamard product*, Proceedings of Symposia in Applied Mathematics, vol. 40, 1990, pp. 87–169.

[Jai17] T. Jain, *Hadamard powers of some positive matrices*, Linear Algebra and its Applications **528** (2017), 147–158.

[Joh82] G. A. Johnson, *A generalization of N-matrices*, Linear Algebra and its Applications **48** (1982), 201–217.

[JS02] C. R. Johnson and R. L. Smith, *Intervals of inverse M-matrices*, Reliable Computing **8** (2002), no. 3, 239–243.

[JST20] C. R. Johnson, R. L. Smith, and M. J. Tsatsomeros, *Matrix Positivity*, vol. 221, Cambridge University Press, 2020.

[Kem82] J. H. B. Kemperman, *A Hurwitz matrix is totally positive*, SIAM Journal on Mathematical Analysis **13** (1982), no. 2, 331–341.

[Kim74] C. H. Kimberling, *A probabilistic interpretation of complete monotonicity*, Aequationes mathematicae **10** (1974), 152–164.

[KK78] M. C. Kemp and Y. Kimura, *Introduction to Mathematical Economics*, Springer, New York, Heidelberg, Berlin, 1978.

[Kre05] V. Kreinovich, *Optimal finite characterization of linear problems with inexact data*, Reliable Computing **11** (2005), no. 6, 479–489.

[Kut71] J. R. Kuttler, *A fourth-order finite-difference approximation for the fixed membrane eigenproblem*, Mathematics of Computation **25** (1971), no. 114, 237–256.

[LC14] A. Liénard and A. Chipart, *Sur le signe de la partie réelle des racines d'une équation algébraique*, Journal de Mathématiques Pures et Appliquées **10** (1914), no. 6, 291–346.

[Lin14] M. Lin, *An Oppenheim type inequality for a block Hadamard product*, Linear Algebra and its Applications **452** (2014), 1–6.

[LL01] Y.-t. Li and J.-c. Li, *On the estimations of bounds for determinant of Hadamard product of H-matrices*, Journal of Computational Mathematics (2001), 365–370.

[Lod11] C. Lodero, *Factorización de Hadamard para polinomios Hurwitz*, Mexico: Universidad Autonoma Metropolitana (2011).

[LZ97] J. Liu and L. Zhu, *Some improvement of Oppenheim's inequality for M-matrices*, SIAM Journal on Matrix Analysis and Applications **18** (1997), no. 2, 305–311.

[Max68] J. C. Maxwell, *On Governors*, Proceedings of the Royal Society of London **16** (1868), 270–283.

[May17] G. Mayer, *Interval Analysis: and Automatic Result Verification*, vol. 65, Walter de Gruyter GmbH & Co KG, 2017.

[Met72] K. Metelmann, *Inverspositive Bandmatrizen und totalnichtnegative Green'sche Matrizen*, Ph.D. thesis, University of Cologne, Cologne, Germany, 1972.

[Mia93] J. Miao, *Ky Fan's N-matrices and linear complementarity problems*, Mathematical programming **61** (1993), no. 1-3, 351–356.

[Mic86] C. A. Micchelli, *Interpolation of scattered data: Distance matrices and conditionally positive definite functions*, Constr. Approx **2** (1986), 11–22.

[MXH09] H.-q. Ma, J.-p. Xu, and N.-j. Huang, *An iterative method for a system of linear complementarity problems with perturbations and interval data*, Applied mathematics and computation **215** (2009), no. 1, 175–184.

[Neu90] A. Neumaier, *Interval Methods for Systems of Equations*, no. 37, Cambridge university press, 1990.

[Opp30] A. Oppenheim, *Inequalities connected with definite Hermitian forms*, Journal of the London Mathematical Society **1** (1930), no. 2, 114–119.

[Pen09] J. Pena, *Positive symmetric matrices with exactly one positive eigenvalue*, Linear algebra and its applications **430** (2009), no. 5-6, 1566–1573.

[Rea99] R. Reams, *Hadamard inverses, square roots and products of almost semidefinite matrices*, Linear Algebra and its Applications **288** (1999), 35–43.

[Roh87] J. Rohn, *Inverse-positive interval matrices*, ZAMM Zeitschrift für angewandte Mathematik und Mechanik **67** (1987), no. 5, T492– T493.

[Roh89] J. Rohn, *Systems of linear interval equations*, Linear Algebra and its Applications **126** (1989), 39–78.

[Roh94] ———, *Positive definiteness and stability of interval matrices*, SIAM Journal on Matrix Analysis and Applications **15** (1994), no. 1, 175–184.

[RR96] J. Rohn and G. Rex, *Interval P-matrices*, SIAM Journal on Matrix Analysis and Applications **17** (1996), no. 4, 1020–1024.

[Sch37] I. J. Schoenberg, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Annals of mathematics (1937), 787–793.

[SH10] S.-Q. Shen and T.-Z. Huang, *On symmetric matrices with exactly one positive eigenvalue*, The Electronic Journal of Linear Algebra **20** (2010), 158–167.

[SSV12] R. L. Schilling, R. Song, and Z. Vondracek, *Bernstein Functions*, Bernstein Functions, de Gruyter, 2012.

[UKW73] Y. Uekawa, M. C. Kemp, and L. L. Wegge, *P-and PN-matrices, Minkowski-and Metzler-matrices, and generalizations of the Stolper-Samuelson and Samuelson-Rybczynski theorems*, Journal of International Economics **3** (1973), no. 1, 53–76.

[YD13] R. K. Yedavalli and N. Devarakonda, *Sufficiency of vertex matrix check for robust stability of interval matrices via the concept of qualitative robustness*, *:* Proceedings 2013 European Control Conference*, July 17-19, 2013, Zürich, Switzerland, IEEE*, 2013, pp. 2351–2356.

[ZD09] X.-D. Zhang and C.-X. Ding, *The equality cases for the inequalities of Oppenheim and Schur for positive semi-definite matrices*, Czechoslowak Math. J. **59** (2009), no. 134, 197–206.

[Zha11] F. Zhang, *Matrix Theory: Basic Results and Techniques*, 2nd ed., Springer, New York, NY, 2011.

List of Symbols

$\mathbb{R}^{n,m}$	the set of n-by-m real matrices
$\mathbb{R}^n$	the set of n-real vectors
$\mathbb{R}_+$	the set of positive real numbers
$\mathbf{e}_i$	the vector which has in its *i-th* component a one
$[a_{ij}]$	entries of a matrix
$A \circ B$	the Hadamard product of the matrices A and B
$A^{\circ r}$	the r-th Hadamard power of the matrix A
$f^{\circ r}$	the r-th Hadamard power of the polynomial $f(x)$
$e^{\circ A}$	the Hadamard exponential of the matrix A
$log^{\circ}(A)$	the Hadamard logarithm of the matrix A
$A^{\circ -1}$	the Hadamard inverse of the matrix A
$\mathcal{A}$	the class of matrices that have one positive eigenvalue
$A[\alpha\|\beta]$	submatrix of A
$A[\alpha]$	principal submatrix of A
$\det A$	determinant of A
I	identity matrix
A^T	transpose of a matrix A
E	matrix with all entries are one
$\mathcal{CM}$	the set of completely monotone functions
$\mathcal{B}$	the set of Bernstein functions
$[A]$	matrix interval
$[A]_{sym}$	the set of symmetric matrices in $[A]$
$\boldsymbol{P}_n$	the family of all polynomials of degree n with positive coefficients
$\boldsymbol{H}_n$	the family of all Hurwitz polynomials of degree n with positive coefficients
$H(p)$	the associated Hurwitz matrix with the polynomial $p(x)$