University of Konstanz
Department of Economics

Portfolio Choice for HARA Investors: When Does $1/\gamma$ (not) Work?

Günter Franke and Ferdinand Graf

Working Paper Series
2010-11

Portfolio Choice for HARA Investors: When Does $1/\gamma$ (not) Work? [1]

Günter Franke[2] and Ferdinand Graf[3]
University of Konstanz

November 4, 2010

In the continuous time-Merton-model the instantaneous stock proportions are inversely proportional to the investor´s local relative risk aversion γ. This paper analyses the conditions under which a HARA-investor can use this $1/\gamma$-rule to approximate her optimal portfolio in a finite time setting without material effects on the certainty equivalent of the portfolio payoff. The approximation is of high quality if approximate arbitrage opportunities do not exist and if the investor´s relative risk aversion is higher than that used for deriving the approximation portfolio. Otherwise, the approximation quality may be bad.

Keywords:	HARA-utility, portfolio choice, certainty equivalent, approximated choice
JEL-classification:	G10, G11, D81

---

[1] We are grateful to Jeanette Roth, Julia Hein, Harris Schlesinger, Richard Stapleton and Thomas Weber for very helpful insights and for valuable comments to Alexandra Hansis and other participants of the German Finance Association meeting (2009).

[2] Department of Economics, Box D-147, 78457 Konstanz, Germany, Phone: +49 7531 88 2545, E-Mail: guenter.franke@uni-konstanz.de

[3] Department of Economics, Box D-147, 78457 Konstanz, Germany, Phone: +49 7531 88 3620, E-Mail: ferdinand.graf@uni-konstanz.de

1 Introduction

Merton (1971) showed that in a continuous-time model with i.i.d. asset returns the optimal instantaneous stock proportions of an investor are proportional to $1/\gamma$ with γ being the local relative risk aversion of the agent. The $1/\gamma$-rule can also serve as a rule of thumb for portfolio choice in a finite period setting. This paper asks how good this rule of thumb is for a HARA-investor who invests money for a given finite period and does not adjust the portfolio within the period. More precisely, we derive the optimal portfolio for some HARA-investor with exponent ϕ and for another HARA-investor with exponent γ and check how well the γ-optimal risky portfolio is approximated by the ϕ-optimal risky portfolio multiplied by ϕ/γ.

We measure the approximation quality by the approximation loss. This is defined as the relative increase in initial endowment required for the approximation portfolio to generate the same certainty equivalent as the optimal portfolio. If, for example, an initial endowment of 100 \$ is invested in the optimal portfolio and the approximation loss is 5%, then the investor needs to invest 105 \$ in the approximation portfolio to equalize the certainty equivalents of both portfolios.

The main findings of the paper can be summarized as follows. For a given investment opportunity set the approximation loss depends not only on γ and ϕ, but also on the elasticity of the pricing kernel, θ. To illustrate, assume a stock market such that the elasticity of the pricing kernel with respect to the market return is a constant θ. Investors buy stocks and borrow/lend at the risk-free rate. According to the $1/\gamma$-rule, the approximation stock portfolio of the γ-investor equals the optimal stock portfolio of the ϕ-investor multiplied by ϕ/γ. If $\gamma < \phi$, then the γ-investor invests more in stocks than the ϕ-investor. Hence the γ-investor may have to borrow at the risk-free rate. But then he may end up with negative terminal wealth which is infeasible. Hence, we require $\gamma \geq \phi$. We find a very small approximation loss for $\gamma \geq \phi \geq \theta$. Then the $1/\gamma$ -rule works very well.

But we find possibly high approximation losses for $\theta \geq \gamma \geq \phi$ and for $\gamma \geq \theta \geq \phi$. Whenever the elasticity of the pricing kernel, θ, is much higher than the relative risk aversion ϕ used for the approximation, the ϕ-investor will be very aggressive in her risk taking. This is a situation which provides approximate arbitrage opportunities as defined by Bernardo and Ledoit (2000). A high pricing kernel elasticity implies a high equity premium. An investor with low relative risk aversion would then buy a very risky approximation portfolio. This implies a high approximation loss for an investor whose relative risk aversion γ is clearly higher than ϕ. The reason is that the γ-investor would be more conservative in her risk taking than the $1/\gamma$-rule suggests. Therefore this rule does a poor job. If, however, γ is very large, then the γ-investor takes a very small risk anyway so that the approximation loss is rather small. Hence, the $1/\gamma$-rule works quite well when the equity premium is rather small, but it may be seriously misleading in case of a high equity premium.

Fortunately the problem of a high equity premium can be resolved by replacing the market return by a transformed market return with a low pricing kernel elasticity. Also, if the pricing kernel elasticity of the market return is not constant, a transformed market return with low constant pricing kernel elasticity can easily be derived. This transformed market return is the payoff of a special exchange traded fund (ETF). Then the approximation portfolio would invest in this ETF and the risk-free asset. The approximation loss would be quite small then for a wide range of HARA-investors so that the $1/\gamma$-rule works quite well. This approximation can be viewed as a generalization of the two-fund separation result of Cass and Stiglitz (1970). Their result only holds for HARA-functions with a given exponent. Our approximation generalizes it to a wide range of exponents.

The practical relevance of our findings is easily illustrated. A portfolio manager has many different customers investing in different risky funds and the risk-free asset. Their preferences may be characterized by increasing, constant or declining relative risk aversion (RRA) and can be approximated by a HARA-function. The portfolio manager proceeds as follows. First, she derives the optimal portfolio for some low constant RRA ϕ. Second, she allocates the customer's initial endowment to the same portfolio and the risk-free asset, using the $1/\gamma$-rule for the risky investment and putting the rest in the risk-free asset. Hence, the allocations for different customers only differ by the amount invested in that risky portfolio and the amount invested risk-free.

Our analysis refers to static portfolio choice. We do not address dynamic portfolio strategies which may try to exploit predictability of asset returns. As a caveat, our results should not be applied to risk management which focuses on tail risks. Our results are based on the certainty equivalent of portfolio payoffs covering the full distribution of payoffs.

The rest of the paper is organized as follows. Section 2 gives a literature review. Section 3 and 4 describe the general approximation approach and the measurement of the approximation quality. Section 5 analyzes the approximation quality in a perfect market with a continuous state space and long investment horizons. In section 6, we consider a market with very few states. Section 7 concludes.

2 Literature Review

Over the last decades a sophisticated theory of decision making under uncertainty, based on the expected utility paradigm, has been developed. Following the seminal papers by Arrow (1974), Pratt (1964), Rothschild and Stiglitz (1970), Diamond and Stiglitz (1974), many papers investigated optimal decisions and showed how they depend on the utility function. Portfolio choice is perhaps the most important application.

There is an extensive literature about portfolio choice. Hakansson (1970) derives the optimal portfolio for a HARA-investor in a complete market. Regarding dynamic strategies, Merton (1971) was one of the first to look into these strategies in a continuous time model. Later on, Karatzas et. al. (1986) provide a rigorous mathematical treatment of these strategies. They pay attention, in particular, to non-negativity constraints for consumption. Viceira (2001) discusses dynamic strategies in the presence of uncertain labor income. He uses an approximation approach to derive a simplified strategy which, however, deviates very little in terms of the certainty equivalent from the optimal strategy. Other papers, for example, Balduzzi and Lynch (1999), Brandt et. al. (2005), look for optimal strategies in the case of asset return predictability, Chacko and Viceira (2005) analyze the impact of stochastic volatility in incomplete markets. Brandt et. al. (2009) derive optimal portfolios using stock characteristics like the firm's capitalization and book-to-market ratio.

Black and Littermann (1992) show that the optimal portfolio for a (μ, σ)-investor reacts very sensitively to changes in asset return parameters. Yet, the Sharpe-ratio may vary only little. Then an intensive discussion on shrinkage-models started. Recently, DeMiguel, Garlappi and Uppal (2009) compare several portfolio strategies to the simple $1/n$ strategy that gives equal weight to all risky investments. These strategies are compared out-of-sample in the presence of uncertainties about asset return parameters. Using the certainty equivalent return for an investor with a quadratic utility function, the Sharpe-ratio and the turnover volume of each strategy, they find that no strategy consistently outperforms the $1/n$ strategy. In a related paper, DeMiguel, Garlappi, Nogales and Uppal (2009) solve for minimum-variance-portfolios under additional constraints. They find that a partial minimum-variance portfolio calibrated by optimizing the portfolio return in the previous period performs best out-of-sample. Jacobs, Müller and Weber (2009) compare various asset allocation strategies including stocks, bonds and commodities and find that a broad class of asset allocation strategies with fixed weights for the asset classes performs out-of-sample equally well in terms of the Sharpe-ratio as long as strong diversification is maintained. Hodder, Jackwerth and Kolokolova (2009) find that portfolios based on second order stochastic dominance perform best out-of-sample. Our approximation results will be shown to hold also under parameter uncertainty.

3 The Approximation Approach

To explain our approximation approach, first derive the optimal portfolio of a HARA-investor. We consider a market with n risky assets and one risk-free asset. The gross return of asset i is denoted R_i for $i \in \{1, \dots, n\}$. We denote the vector $(R_1, \dots, R_n)'$ by $\mathbf{R}$. The gross risk-free rate is R_f. An investor with initial endowment W_0 maximizes her expected utility of payoff V, given by

$$V := V(\alpha, W_0) = (W_0 - \alpha'\mathbf{1})R_f + \alpha\mathbf{R} = W_0 R_f + \alpha\mathbf{r},$$

where α_i denotes the dollar-amount invested in asset i, $\alpha = (\alpha_1, \dots, \alpha_n)$, and $\mathbf{1}$ is the n-dimensional vector consisting only of ones. $r_i = R_i - R_f$ denotes the random excess return of asset i and $\mathbf{r} = (r_1, \dots, r_n)'$. The investor has a utility function with hyperbolic absolute risk aversion

$$u(V) = \frac{\gamma}{1-\gamma}\left(\frac{\eta + V}{\gamma}\right)^{1-\gamma}, \tag{1}$$

where the parameters η and γ assure that u is increasing and concave in V. Moreover, $0 < \gamma < \infty$ indicates decreasing absolute risk aversion. For $\gamma = 1$, we obtain log-utility. The well-known first order condition for this optimization problem is

$$\mathbb{E}\left[r_i\left(\frac{\eta + W_0 R_f + \alpha^+\mathbf{r}}{\gamma}\right)^{-\gamma}\right] = 0,\ \forall i \in \{1, \dots, n\}. \tag{2}$$

The optimal solution is denoted α^+.

Our approximation approach consists of the following three steps. First, we transform the decision problem to an equivalent problem under constant RRA. Define $\tilde{W}_0 = \frac{\eta}{R_f} + W_0$ as the enlarged initial endowment. Then after substituting $\tilde{W}_0$ in (2), this condition remains the same, but the investor is constant relative risk averse. Second, we restrict the enlarged initial endowment to the artificial initial endowment γ/R_f. This leaves the structure of the optimal portfolio unchanged. Without loss of generality, we multiply the first order condition (2) by $(\tilde{W}_0 R_f/\gamma)^\gamma$. This gives

$$\mathbb{E}\left[r_i\left(1+\frac{\hat{\alpha}^+\mathbf{r}}{\gamma}\right)^{-\gamma}\right] = 0,\ \forall i \in \{1,\dots,n\}. \tag{3}$$

The terminal wealth implied by (3) is

$$\hat{V}^+ = V\left(\hat{\alpha}^+, \frac{\gamma}{R_f}\right) = \gamma + \hat{\alpha}^+\mathbf{r} > 0. \tag{4}$$

Positivity follows from $u'(\hat{V}^+) \to \infty$ for $\hat{V}^+ \to 0$.

The solution of the optimization problem for an investor with enlarged initial endowment $\tilde{W}_0$ is proportional to that with artificial endowment $\frac{\gamma}{R_f}$: $V^+ = \hat{V}^+\tilde{W}_0 R_f/\gamma$ with $\alpha^+ = \hat{\alpha}^+\frac{\tilde{W}_0 R_f}{\gamma} = \hat{\alpha}^+\frac{\eta + W_0 R_f}{\gamma}$.

Third, we define some low level of constant relative aversion ϕ to approximate the optimal portfolio. We approximate the solution of equation (3), $\hat{\alpha}^+$, by $\hat{\alpha}^-$, the solution of

$$\mathbb{E}\left[r_i\left(1+\frac{\hat{\alpha}^-\mathbf{r}}{\phi}\right)^{-\phi}\right] = 0,\ \forall i \in \{1,\dots,n\}. \tag{5}$$

The terminal wealth implied by (5) is $\phi + \hat{\alpha}^-\mathbf{r} > 0$, given the artificial endowment ϕ/R_f. To make up for the difference in artificial initial endowment in our approximation, the difference, $\gamma/R_f - \phi/R_f$, is simply invested in the risk-free asset adding $\gamma - \phi$ to the terminal wealth $\phi + \hat{\alpha}^-\mathbf{r}$,

$$\hat{V}^- = \phi + \hat{\alpha}^-\mathbf{r} + (\gamma - \phi) = \gamma + \hat{\alpha}^-\mathbf{r}. \tag{6}$$

Since $\phi + \hat{\alpha}^- r > 0$, $\hat{V}^-$ is also positive for $\gamma \geq \phi$.

Comparing $\hat{\alpha}^+$ and $\hat{\alpha}^-$ reveals two effects, a structure effect and a volume effect. The structure of α is defined by $\alpha_1 : \alpha_2 : \alpha_3 : \ldots : \alpha_n$. This structure changes with the level of RRA used for optimization. This structure change is denoted the structure effect. The volume is defined as the amount of money invested in all risky assets together. Hence the volume equals $\sum_{i=1}^n \alpha_i$. This volume also changes when RRA ϕ replaces RRA γ. The volume change is denoted the volume effect.

The $1/\gamma$-rule suggests that the stock proportions are inversely proportional to the investor's local relative risk aversion.

$$\frac{\hat{\alpha}^+}{\frac{\gamma}{R_f}} \sim \frac{1}{\gamma} \qquad \text{or} \qquad \hat{\alpha}^+ \sim \frac{1}{R_f}.$$

Similarly,

$$\frac{\hat{\alpha}^-}{\frac{\phi}{R_f}} \sim \frac{1}{\phi} \qquad \text{or} \qquad \hat{\alpha}^- \sim \frac{1}{R_f}.$$

Hence if the $1/\gamma$-rule is absolutely correct, $\hat{\alpha}^+ = \hat{\alpha}^-$. As a consequence, the volume and the structure effect would disappear. Doubling γ doubles the artificial initial endowment and the relative risk aversion so that the $1/\gamma$-rule implies unchanged risky investments. Therefore, one benchmark for evaluating the quality of our approximation approach is a zero volume effect and a zero structure effect.

The approximation (6) assures $\hat{V}^- > 0$ for $\gamma \geq \phi$. For $\gamma < \phi$, the investor would borrow $(\phi - \gamma)/R_f$ at the risk-free rate. Then, $\hat{V}^-$ might turn negative since $\phi + \hat{\alpha}^- \mathbf{r}$ can be very close to zero. $\hat{V}^-$ would be infeasible and is ruled out if $\gamma \geq \phi$. Therefore our approximation requires $\gamma \geq \phi$. This will be assumed in the following.

4 The Approximation Quality

4.1 The General Argument

First, we present some arguments which support our conjecture of a strong approximation quality. Comparing (4) and (6) gives the difference between the optimal and the approximation portfolio payoff, $\hat{V}^+ - \hat{V}^- = (\hat{\alpha}^+ - \hat{\alpha}^-)\,\mathbf{r}$. Hence we expect a good approximation if the vectors $\hat{\alpha}^+$ and $\hat{\alpha}^-$ are similar. Essential for this is that both utility functions display similar patterns of absolute risk aversion in the range of relevant terminal wealth. The utility functions

$$\frac{\gamma}{1-\gamma}\left(\frac{\gamma + \alpha\mathbf{r}}{\gamma}\right)^{1-\gamma} \text{ and } \frac{\phi}{1-\phi}\left(\frac{\phi + \alpha\mathbf{r}}{\phi}\right)^{1-\phi}$$

give absolute risk aversion functions

$$\frac{1}{1+\alpha\mathbf{r}/\gamma} \text{ and } \frac{1}{1+\alpha\mathbf{r}/\phi}.$$

Hence, if the portfolio excess return αr is zero, both utility functions display absolute risk aversion of 1. As long as the portfolio excess return does not differ much from 0, absolute risk aversion is similar for both functions implying similar portfolio choice. Figure 1 illustrates the absolute risk aversion functions for different levels of γ. The smaller is γ, the steeper the curve is. For exponential utility, the curve is horizontal at a level of 1. The similarity of the absolute risk aversion patterns suggests small volume and structure effects.

[Figure 1 about here.]

The first order conditions (3) and (5) allow us to derive more precisely market settings of high approximation quality. Let $u^i(\cdot)$ denote the i-th derivative of the utility function. Then a Taylor series for the first derivative of the utility function around an excess return of zero yields

$$u'(\hat{\alpha}^+\mathbf{r}) = \sum_{i=0}^{\infty} \frac{u^{(i+1)}(0)}{i!}(\hat{\alpha}^+\mathbf{r})^i \tag{7}$$

so that

$$\left(1 + \frac{\hat{\alpha}^+\mathbf{r}}{\gamma}\right)^{-\gamma} = 1 + \sum_{i=1}^{\infty}(-1)^i \frac{(\hat{\alpha}^+\mathbf{r})^i}{i!} \prod_{j=0}^{i-1}\left(\frac{j}{\gamma}+1\right). \tag{8}$$

Hence, the first order condition (3), multiplied by $\hat{\alpha}_i^+$ and summed over i, yields

$$
\begin{aligned}
\mathbb{E}\left[\hat{\alpha}^+\mathbf{r}\left(1+\sum_{i=1}^{\infty}(-1)^i\frac{(\hat{\alpha}^+\mathbf{r})^i}{i!}\prod_{j=0}^{i-1}\left(\frac{j}{\gamma}+1\right)\right)\right] &= 0 \\
\Leftrightarrow \quad \mathbb{E}[\hat{\alpha}^+\mathbf{r}]+\sum_{i=1}^{\infty}(-1)^i\frac{\mathbb{E}[(\hat{\alpha}^+\mathbf{r})^{i+1}]}{i!}\prod_{j=0}^{i-1}\left(\frac{j}{\gamma}+1\right) &= 0.
\end{aligned}
$$

Denoting the i-th non-centered moment of the optimal portfolio excess return by m_i and rearranging the last equation, the previous equation can be rewritten as

$$\frac{m_1}{m_2}+\frac{1}{2}\frac{m_3}{m_2}\left(\frac{1}{\gamma}+1\right)-\frac{1}{6}\frac{m_4}{m_2}\left(\frac{2}{\gamma}+1\right)\left(\frac{1}{\gamma}+1\right)+\ldots=1. \tag{9}$$

From the first order condition (5) we have

$$\frac{n_1}{n_2}+\frac{1}{2}\frac{n_3}{n_2}\left(\frac{1}{\phi}+1\right)-\frac{1}{6}\frac{n_4}{n_2}\left(\frac{2}{\phi}+1\right)\left(\frac{1}{\phi}+1\right)+\ldots=1, \tag{10}$$

where n_i is the i-th non-centered moment of $\hat{\alpha}^-\mathbf{r}$. Absolute portfolio excess returns below 1 imply $|\alpha\mathbf{r}|^{i+1} < |\alpha\mathbf{r}|^i$. Then it follows for the non-centered moments: $|m_{i+2}| << |m_i|,\ i \geq 2$. Also, $|m_3| << m_2$. Therefore, we may neglect the terms $m_i,\ i \geq 5$, in the Taylor series and focus on the first four moments. The same is true for n_i.

Whenever the excess return distributions of the optimal and the approximation portfolio have non-centered third and fourth moments close to zero, both first order conditions are very similar implying a very good approximation quality[1]. Otherwise, equations (9) and (10) indicate that the approximated return distribution derived from (10) attaches too much weight to the skewness and the kurtosis relative to (9) for $\gamma > \phi$. Hence, we expect the approximated return distribution to have fatter tails, but less skewness than the optimal return distribution. This follows because a HARA-investor with declining absolute risk aversion likes positive skewness, but dislikes kurtosis.

We summarize our findings in the following lemma:

Lemma 1 *Let $\gamma \geq \phi$. The approximation is of high quality even for large differences between ϕ and γ if the non-centered moments of the optimal and of the approximation portfolio excess return decline fast, i.e. if $m_{i+2} << m_i$, $i \geq 2$, $m_3 << m_2$ and $n_{i+2} << n_i$, $i \geq 2$, $n_3 << n_2$.*

4.2 The Approximation Loss

We measure the economic impact of the approximation by the approximation loss. Compare the certainty equivalent of the optimal portfolio α^+ and that of the approximation portfolio α^-. In both cases, the certainty equivalent is based on the investor's HARA-function (1). For that utility function, given a portfolio α, the certainty equivalent, CE, is defined by

$$
\begin{aligned}
\left(\frac{\eta+CE}{\gamma}\right)^{1-\gamma} &= \mathbb{E}\left[\frac{(\eta/R_f+W_0)R_f+\alpha\mathbf{r}}{\gamma}\right]^{1-\gamma} \\
&= \left(\tilde{W}_0\frac{R_f}{\gamma}\right)^{1-\gamma}\mathbb{E}\left[1+\frac{\hat{\alpha}\mathbf{r}}{\gamma}\right]^{1-\gamma} = \left(\frac{ce}{\gamma}\right)^{1-\gamma}.
\end{aligned} \tag{11}
$$

Expected utility is the same for an investor with utility function (1) and endowment W_0 and an investor with constant relative risk aversion and enlarged initial endowment $\tilde{W}_0 = \eta/R_f + W_0$. Therefore, we consider the enlarged certainty equivalent $ce = \eta + CE$. Define ε as the ratio of the enlarged certainty equivalent, ce^+, of the optimal portfolio $\alpha^+ = \hat{\alpha}^+ \tilde{W}_0 R_f/\gamma$, and the enlarged certainty equivalent, ce^-, of the approximated optimal portfolio $\alpha^- = \hat{\alpha}^- \tilde{W}_0 R_f/\gamma$. Then

$$\varepsilon = \frac{ce^+}{ce^-} = \left(\frac{\mathbb{E}\left[\left(\frac{(\eta/R_f + W_0)R_f + \alpha^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]}{\mathbb{E}\left[\left(\frac{(\eta/R_f + W_0)R_f + \alpha^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]} \right)^{1/(1-\gamma)} = \left(\frac{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]}{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]} \right)^{1/(1-\gamma)}. \tag{12}$$

Hence, ε is the same for the enlarged initial endowment $\eta/R_f + W_0$ and the artificial initial endowment γ/R_f. This is stated in:

Lemma 2 *For a given market setting, the certainty equivalent ratio ε depends on the exponent γ, but not on the initial endowment nor on the parameter η.*

The lower boundary of ε is one, since the optimal portfolio $\hat{\alpha}^+$ yields the highest possible certainty equivalent. For a HARA-investor there exists a second interpretation of ε. $k = (\varepsilon - 1) \geq 0$ is the relative increase in the enlarged initial endowment $\tilde{W}_0$, that is required for the approximation portfolio to generate the same expected utility as the optimal portfolio generates with initial endowment $\tilde{W}_0$. To see that $\varepsilon = 1 + k$, note

$$\left(\frac{\tilde{W}_0 R_f}{\gamma} \right)^{1-\gamma} \mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right] = \left(\frac{(1+k)\tilde{W}_0 R_f}{\gamma} \right)^{1-\gamma} \mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right].$$

Rearranging yields

$$1 + k = \left(\frac{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^+ \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]}{\mathbb{E}\left[\left(1 + \frac{\hat{\alpha}^- \mathbf{r}}{\gamma} \right)^{1-\gamma} \right]} \right)^{1/(1-\gamma)} = \frac{ce^+}{ce^-} = \varepsilon.$$

We call k the approximation loss. If $k = 0.02$, for example, then the investor needs to invest additionally 2% of his enlarged initial endowment in the approximation portfolio to achieve the same expected utility as her optimal portfolio does.

For $\gamma = \phi$, the approximation loss is 0, by definition. If we increase γ, the approximation loss will be positive. But it does not increase monotonically. Instead, for $\gamma \to \infty$, $k \to 0$ again. For $\gamma \to \infty$, the investor's utility is exponential and the artificial initial endowment tends to infinity. The exponential utility investor buys a risky portfolio independently of her initial endowment. Given an infinite artificial initial endowment, this risky portfolio turns out to be irrelevant for the optimal payoff $\hat{V}^+$. The same is true for the approximated payoff $\hat{V}^-$. Hence, both certainty equivalents converge for $\gamma \to \infty$ so that $k \to 0$.

In the following, we illustrate the approximation loss k by looking, first, at a complete market with a continuous state space and different distributions of the market return. Thereafter, we consider a discrete state space with few states only.

5 Approximation in a Continuous State Space

5.1 Demand Functions for State-Contingent Claims

5.1.1 Characterization of Demand Functions

We start from a perfect market with a continuous state space. First, assume a complete market. Then state-contingent claims for all possible states $s \in \mathcal{S}$ exist. Hakansson (1970) was the first to investigate investment and consumption strategies of HARA investors in a complete market. Consider an investor with constant relative risk aversion γ and artificial initial endowment γ/R_f. The investor's demand for state-contingent claims, $\hat{\alpha} = (\hat{\alpha}_s)_{s\in\mathcal{S}}$, is optimized

$$\max_{\hat{\alpha}} \mathbb{E}\left[\frac{\gamma}{1-\gamma}\left(\frac{\gamma+\hat{\alpha}}{\gamma}\right)^{1-\gamma}\right] \; s.t. \; \mathbb{E}[\pi V] = \gamma/R_f, \tag{13}$$

where $\pi = (\pi_s)_{s\in\mathcal{S}}$ denotes the pricing kernel and $\hat{\alpha}_s$ is the demand for claims with payoff one in state s and zero otherwise. Differentiating the corresponding Lagrangian with respect to α_s gives the well-known optimality condition for each state

$$\left(\frac{\gamma+\hat{\alpha}_s}{\gamma}\right)^{-\gamma} = \lambda\pi_s, \; s \in \mathcal{S}. \tag{14}$$

First, we assume that the pricing kernel is a power function of the market portfolio return

$$\pi_s = \frac{1}{R_f}\frac{R_{M,s}^{-\theta}}{\mathbb{E}[R_M^{-\theta}]}, \tag{15}$$

where $R_{M,s}$ denotes the gross market return in state s and θ is the constant relative risk aversion of the market, i.e. the constant elasticity of the pricing kernel. Hence, we assume a pricing kernel as implied by the Black-Scholes setting.

Replacing π_s by (15) and solving (14) for $\hat{V}_s^+ = \gamma + \alpha_s$ yields for finite γ

$$V_s^+ = R_{M,s}^{\theta/\gamma}\exp\{a(\gamma)\}. \tag{16}$$

$a(\gamma)$ depends on the investor's relative risk aversion and is determined by the budget constraint: $\mathbb{E}[R_M^{\theta/\gamma}\exp\{a(\gamma)\}\pi] = \frac{\gamma}{R_f}$. We have

$$\exp\{a(\gamma)\} = \gamma\frac{\mathbb{E}[R_M^{-\theta}]}{\mathbb{E}\left[R_M^{-\theta+\theta/\gamma}\right]} = \frac{\gamma}{\mathbb{E}^Q\left[R_M^{\theta/\gamma}\right]}, \tag{17}$$

with $\mathbb{E}^Q[\cdot]$ being the expectation operator under the risk neutral probability measure using the pricing kernel $\pi(R_M)$.

The optimal terminal wealth, $\hat{V}^+$, is approximated by $\hat{V}^-$. For $\gamma \geq \phi$, $\hat{V}^-$ is the optimal terminal wealth of an investor with CRRA ϕ and artificial endowment ϕ/R_f, supplemented by the risk-free payoff $(\gamma - \phi)$,

$$\hat{V}_s^- = R_{M,s}^{\theta/\phi}\exp\{a(\phi)\} + (\gamma - \phi). \tag{18}$$

How does $(\hat{V}_s^+ - \hat{V}_s^-)$ depend on $(\gamma - \phi)$ for $\gamma > \phi$? The functions $\hat{V}^+(R_M)$ and $\hat{V}^-(R_M)$ intersect twice, given a sufficiently large domain of R_M. Both functions have to intersect at least once to rule out arbitrage opportunities. For $R_M \to 0$, $\hat{V}^- \to \gamma - \phi > \hat{V}^+ \to 0$. Also $\hat{V}^- > \hat{V}^+$ for $R_M \to \infty$ (this follows from $\theta/\gamma < \theta/\phi$). Since $\hat{V}^+(R_M)$ is more concave than $\hat{V}^-(R_M)$, both functions intersect twice. The demand for state contingent claims is overestimated by the approximation in the bad states and in the good states and underestimated in between, as Figure 2 illustrates. This range-dependent over-/underestimation of the optimal demand characterizes the structure effect.

Consider the special case $\phi = \theta$. This implies that $\hat{V}^-$ is linear in R_M and, hence, $\exp\{a(\theta)\} = \theta/\mathbb{E}^Q[R_M] = \theta/R_f$. Then (18) yields

$$\hat{V}_s^- = \frac{\theta}{R_f} R_{M,s} + \gamma - \theta, \ \gamma \geq \theta. \tag{19}$$

The approximation portfolio policy is very simple. The investor invests θ/R_f in the market portfolio and $(\gamma - \theta)/R_f$ in the risk-free asset.

[Figure 2 about here.]

5.1.2 Approximation Quality and Shape of the Probability Distribution

Next, we illustrate the effect of the shape of the market return distribution on the approximation quality. A change in the probability distribution of R_M implies an adjustment in the intersection point(s) of $\hat{V}^+(R_M)$ and $\hat{V}^-(R_M)$. This adjustment tends to stabilize the approximation quality. To characterize the adjustment, we state the following Lemma:

Lemma 3 *Assume $\gamma > \phi$. Let p be a changing parameter of the market return distribution and $R_M^j = R_M^j(p), j \in \{l, u\}$, denote the lower respectively upper market return where $\hat{V}^+(R_M|p)$ and $\hat{V}^-(R_M|p)$ intersect. Then, holding $\mathbb{E}^Q[R_M] = R_f$ and θ constant, $\frac{\partial \ln R_M^j}{\partial p}$ is given by*

$$\frac{\partial \ln R_M^j}{\partial p} \frac{\theta}{\phi} \frac{1}{\gamma} \left[\hat{V}^+(R_M^j) - \gamma \right] = \left[\frac{\partial a(\gamma)}{\partial p} - \frac{\partial a(\phi)}{\partial p} \right] \frac{\hat{V}^+(R_M^j)}{(\gamma - \phi)} + \frac{\partial a(\phi)}{\partial p}, \tag{20}$$

with

$$\gamma \frac{\partial a(\gamma)}{\partial p} = - \int_0^\infty \hat{V}^+(R_M) \frac{\partial F^Q(R_M)}{\partial p} \tag{21}$$

and

$$\phi \frac{\partial a(\phi)}{\partial p} = - \int_0^\infty \hat{V}^-(R_M) \frac{\partial F^Q(R_M)}{\partial p}. \tag{22}$$

$F^Q(R_M)$ is the cumulative probability distribution of R_M under the risk-neutral measure.

The proof of this lemma is given in the appendix. The lemma relates changes in the risk-neutral probability distribution of the market return to changes in the intersection points of $\hat{V}^+(R_M)$ and $\hat{V}^-(R_M)$. For simplicity, assume $\theta = \phi$. Then $\exp\{a(\phi)\} = \phi/R_f$ so that $\partial a(\phi)/\partial p = 0$. Then, by (20), since $\hat{V}^+ > 0$ and $\hat{V}^+(R_M^l) - \gamma < 0$, $\hat{V}^+(R_M^u) - \gamma > 0$, a marginal change in the underlying probability distribution function of R_M (1) either lowers R_M^l and raises R_M^u or (2) raises R_M^l and lowers R_M^u, or (3) leaves both unchanged.

For illustration, consider a mean preserving spread in the market return, such that $\mathbb{E}^Q[R_M] = R_f$ stays the same. Lemma 1 suggests that the approximation loss increases. However, Lemma 3 implies that an increase in the volatility lowers R^l and increases R^u. To see this, subtract $\phi \frac{\partial a(\phi)}{\partial p} = 0$ from equation (21),

$$\gamma \frac{\partial a(\gamma)}{\partial p} = \int_0^\infty \left[\hat{V}^-(R_M) - \hat{V}^+(R_M)\right] \frac{\partial F^Q(R_M)}{\partial p}.$$

Increasing the volatility reallocates probability mass from the center to the tails, so that the integral is positive. Then, by (20), $\frac{\ln R_M^l}{\partial p} < 0$ and $\frac{\ln R_M^u}{\partial p} > 0$. This reduces the claim difference $(\hat{V}^+ - \hat{V}^-)$ in the tails and raises it in the center so that the approximation quality is stabilized.

Alternatively, consider a reduction in the skewness of the market return distribution. It is not clear whether the intersection points are spreading. Relocating probability mass from the right to the left tail of the market return distribution would lower $\mathbb{E}^Q[R_M]$ and, therefore, is infeasible. In order to keep $\mathbb{E}^Q[R_M] = R_f$ unchanged, the probability mass needs to go up in some range of R_M with $R_M > R_f$. Therefore, $a(\gamma)$ can change in either direction, stabilizing the approximation quality.

5.2 Simulation Results for $\gamma \geq \phi = \theta$

5.2.1 Normal Distribution

Now we illustrate the approximation loss numerically for various probability distributions of R_M and various time horizons. The investor buys state-contingent claims due at the time horizon. He does not readjust the portfolio over time. First assume that $\ln R_M$ is normally distributed with mean μ and variance σ^2. Then $\ln \mathbb{E}[R_M] = \mu + \frac{\sigma^2}{2}$ so that the annual Sharpe-ratio is

$$\frac{\mathbb{E}[R_M] - R_f}{\sigma(R_M)} = \left[1 - \exp\left\{r_f - \left(\mu + \frac{\sigma^2}{2}\right)\right\}\right] \left(\exp\{\sigma^2\} - 1\right)^{-1/2}.$$

The elasticity of the pricing kernel is $\theta = \frac{\mu + \sigma^2/2 - r_f}{\sigma^2}$, the certainty equivalent of $\hat{V}^+$ has a closed form representation

$$ce(\hat{V}^+) = \gamma \exp\left\{\frac{1}{2}\frac{\sigma^2\theta^2}{\gamma}\right\}.$$

For $\gamma \geq \phi$, the approximation portfolio is given by (19). To compute its certainty equivalent, we have to rely on numerical integration techniques.

Consider the case $\phi = \theta$. To calibrate our analysis to observable market returns, we first use an annual expected logarithmic market return $\mu = 6\%$, an annual market volatility $\sigma = 25\%$ and an instantaneous risk-free rate $r_f = 3\%$. This implies a pricing kernel elasticity of $\theta = 0.98$, an annual equity premium of 6.51% and an annual Sharpe-ratio of 23.4%. We consider investors with constant relative risk aversion in the range $[0.98; 8]$, an investment horizon between three month and 5 years and assume an i.i.d. market return. Hence, the expected logarithmic market return for t years is $\mu_t = t\mu$ and the standard deviation of the t-year logarithmic market return is $\sigma_t = \sqrt{t}\sigma$.

Figure 3 shows the approximation loss. For $\gamma = 0.98$, the approximation portfolio equals the optimal portfolio so that there is no approximation loss. For $\gamma > \phi = \theta$, the approximation loss increases with a longer investment horizon because the market return distribution becomes wider implying higher risk. Yet, the approximation quality still remains very good. The highest approximation loss in Figure 3, left, is about 0.3% for an investor with γ about 3 and an investment horizon of 5 years, or, about 0.06% per year. In other words, the investor would need to raise her initial endowment by 0.3% to make up for the approximation loss.

[Figure 3 about here.]

The impact of γ and the investment horizon can also be seen in Figure 3, right, which depicts isoquants of the approximation loss, i.e. combinations of γ and investment horizon yielding the same loss. For an investment horizon of 2.5 years, for example, the loss always remains below 0.08%. For all horizons the loss has a maximum at some γ between 2 and 4 and then monotonically declines to zero with increasing γ.

To illustrate the relation between the approximation loss and the chosen parameters, let $\mu = 0.075$ and $\sigma = 0.15$, retaining $r_f = 3\%$. Then the Sharpe-ratio is 36.3% and the elasticity of the pricing kernel is 2.5. Let $\gamma \in (2.5; 15)$. In this scenario the highest approximation loss is 0.1% for an investment horizon of 5 years and γ about 8.

Next, consider a somewhat extreme case with $\mu = 0.08$, $\sigma = 0.10$ and $r_f = 3\%$. This yields a Sharpe-ratio of 53.4% and a pricing kernel elasticity of 5.5. Let $\gamma \in (5.5; 20)$. Then the highest approximation loss is 0.04% for an investment horizon of 5 years and γ about 17.

The results indicate that the highest approximation loss is inversely related to the pricing kernel elasticity respectively the Sharpe-ratio, provided $\gamma \geq \phi = \theta$. This is not surprising because a higher pricing kernel elasticity has only small effects on the shape of the $\hat{V}^+(R_M)$- and $\hat{V}^-(R_M)$-curves, but is associated with a strong decline in $\sigma(R_M)$ so that the monents m_j and n_j, $j > 1$, of the portfolio excess return decline (Lemma 1). Thus, the $1/\gamma$-rule works quite well.

5.2.2 Symmetric, Fat-tailed Distributions

Next, we analyze fat-tailed distributions. Consider a t-distribution to account for excess kurtosis (fat tails) in logarithmic market returns. The density for a t-year investment period is given by

$$f(\ln R_{M,t}|\mu_t, \sigma_{\nu,t}, \nu_t) = \frac{\Gamma\left(\frac{\nu_t+1}{2}\right)}{\sigma_{\nu,t}\sqrt{\nu_t \pi}\,\Gamma\left(\frac{\nu_t}{2}\right)} \left(1 + \frac{\left(\frac{\ln R_{M,t}-\mu_t}{\sigma_{\nu,t}}\right)^2}{\nu_t}\right)^{-(\nu_t+1)/2}, \qquad (23)$$

where $\sigma_{\nu,t} = \sigma_t(\nu_t/(\nu_t - 2))^{-1/2}$. The mean of the distribution is $\mu_t = t\mu$, the standard deviation is $\sigma_t = \sqrt{t}\sigma$ and the excess kurtosis is $\frac{6}{\nu_t-4}$ for $\nu_t > 4$. Empirical studies, for example Corrado and Su (1997), report a kurtosis of about 12 for the monthly logarithmic returns of the S&P 500 between 1986 and 1995. Assuming i.i.d. returns, this translates into an annual kurtosis $\kappa_1 = 3.75$. Independent increments imply $\kappa_t = 3 + \frac{(\kappa_1 - 3)}{t}$ for t-years. For robustness, we stress the calculation of the approximation loss with an annual kurtosis of 4.5. This gives the simple rule for ν_t: $\nu_t = 4t + 4$. Using the initial parameter values, $\mu = 0.06$ and $\sigma = 0.25$, we derive the Sharpe-ratio and the approximation loss for t-distributed logarithmic market returns. The Sharpe-ratio is 23%. The approximation loss is shown in Figure 4, left. We assume $\phi = \theta = \frac{\mu - r + \frac{1}{2}\sigma^2}{\sigma^2} = 0.98$. The fat tails raise the approximation loss, as predicted by Lemma 1. However, the approximation loss is still remarkably low, even for an investment horizon of five years. For $\gamma = 3$ and a five year horizon, the highest approximation loss is about 0.35%, i.e. about 0.07% per year.

[Figure 4 about here.]

5.2.3 Left-skewed, Fat-tailed Distributions

As a final example of a complete market we consider a distribution with fat tails and negative skewness. Since 1987 stock returns up to one year are mostly skewed to the left. This is also true for stock index returns. For the simulation we use the skewed, fat tailed normal distribution to model the logarithmic market return. The density function is given by

$$f(\ln R_{M,t}|\lambda_t, \omega_t, \xi_t) = \left(\frac{2}{\sigma_t}\right) n\left(\frac{\ln R_{M,t} - \lambda_t}{\omega_t}\right) \mathcal{N}\left(\xi_t\left(\frac{\ln R_{M,t} - \lambda_t}{\omega_t}\right)\right), \tag{24}$$

where $n(\cdot)$ is the density of the standard normal density function and $\mathcal{N}(\cdot)$ is the standard normal distribution function. The mean is given by $\mu_t = \lambda_t + \omega_t \delta_t \sqrt{2/\pi}$, the standard deviation is $\sigma_t = \omega_t \sqrt{1 - 2\delta_t^2/\pi}$, where $\delta_t = \xi_t / \sqrt{1 + \xi^2}$ [2]. Corrado and Su (1997) find that the monthly logarithmic stock returns of the S&P 500 are skewed to the left at -1.67. Assuming i.i.d. returns, this translates to an annual skewness of about -0.5[3]. We stress this number and use an annual skewness of -0.6 together with an annual excess kurtosis of 0.4426. For each investment horizon, we choose the parameters λ_t, ω_t and ξ_t such that $\mu_t = 0.06t, \sigma_t = 0.25\sqrt{t}$, $sk_t = -0.6/\sqrt{t}$ and the excess kurtosis over t years is $(3.4426 - 3)/t$. This yields a somewhat higher annual Sharpe-ratio of 24.6%. The approximation loss is shown in Figure 4, right, for $\gamma \geq \phi = \theta$, and $t \in [0.3; 5]$. The highest loss is about 0.36% for $\gamma = 3$ and 5 years. Figure 4, left, and Figure 4, right, indicate very similar loss levels. Skewness and excess kurtosis do not affect the approximation loss substantially. This is driven also by the adjustment of the intersection points of the optimal and the approximate demand functions to the shape of the probability distribution. Hence the $1/\gamma$-rule works well also for skewed, leptokurtic distributions for $\gamma \geq \phi = \theta$.

5.3 Simulation Results for $\gamma \geq \phi \neq \theta$

So far the simulations were based on $\gamma \geq \phi = \theta$. If the pricing kernel elasticity is lowered, then the smaller equity premium induces investors to take less risk. This is true for the optimal and for the approximation portfolio. Hence the approximation loss should decline with a smaller θ. Since the last section shows that the approximation loss is rather insensitive to skewness and leptokurtosis, we use again the lognormal distribution with

$\sigma = 0.25$ and $\mu + \frac{\sigma^2}{2} - r_f = \theta\sigma^2$ to simulate the impact of θ on the approximation loss. Figure 5 shows the approximation loss for an investment horizon of one year and different combinations of γ and θ. In Figure 5, left, $\phi = 1$, in Figure 5, right, $\phi = 2$. Both figures clearly show that the approximation loss monotonically grows with θ, holding γ and ϕ constant. Hence, if $\theta < \phi$, the approximation loss is smaller than that for $\theta = \phi$. Therefore, the $1/\gamma$-rule works even better for $\theta < \phi$, as expected.

Conversely, the approximation loss strongly increases with the pricing kernel elasticity, given $\theta \gg \phi$. For $\phi = 1$, the highest approximation loss in Figure 5, left, is about 1.1% for the highest $\theta = 11$ and $\gamma \approx 5$. For $\phi = 2$ (Figure 5, right), it is about 0.16% for $\theta = 11$ and $\gamma \approx 6.5$. These findings nicely illustrate the approximate arbitrage-story of Bernado and Ledoit (2000). They show that a market with a very high pricing kernel elasticity offers approximate arbitrage opportunities. An investor with low relative risk aversion would then take very much risk through a strongly convex demand function for state-contingent claims, see Figure 2, right. Claims in states of high market returns are very cheap, they almost offer a free lunch to investors with low relative risk aversion. Therefore these investors buy a large number of these claims. Investors with $\gamma \geq \theta$ benefit much less from this effect because they buy a linear or a concave demand function. This implies a high approximation loss as long as γ is not very high.

Clearly, the approximation loss is smaller for a higher ϕ. This is due to the fact that an investor with higher ϕ would take less risk and, thus, benefit less from the approximate arbitrage opportunity. The effect on the approximation loss can be seen by comparing Figures 5, left, and 5, right.

Summarizing, the $1/\gamma$-rule does a very good job when the approximation is based on a linear or concave demand function ($\phi \geq \theta$). But it does a poor job when the approximation is based (1) on a strongly convex demand function ($\theta \gg \phi$), (2) the investor's optimal demand function is much less convex ($\gamma \gg \phi$) and (3) γ is not so high that risk taking becomes negligible.

[Figure 5 about here.]

5.4 Non-Constant Elasticity of the Pricing Kernel

So far we assumed constant elasticity of the pricing kernel for the market return. Ait-Sahalia and Lo (2000), Jackwerth (2000), Rosenberg and Engle (2002), Bliss and Panigirtzoglou (2004), Barone-Adesi, Engle and Mancini (2008) estimate the elasticity of the pricing kernel using prices of options on the S&P 500 and the FTSE 100. They conclude that the pricing kernel elasticity is declining, perhaps with a local maximum in between.

If the pricing kernel of the market portfolio does not have constant elasticity, we derive a transformed market portfolio such that its pricing kernel has low constant elasticity. Then, instead of the market portfolio, we use this transformed market portfolio for the approximation. Assume that the elasticity $\nu(R_M) = -\partial \ln \pi(R_M)/\partial \ln R_M$ is positive and non-constant. Let $R_{TM} := g(R_M) := \exp\left\{\frac{1}{\tilde{\theta}}\int_{\varepsilon}^{R_M} \nu(R_M^0) d\ln R_M^0\right\}$, where $\tilde{\theta}$ is a small, positive constant. ε is a positive lower bound of R_M. g is invertible and yields a pricing kernel $\tilde{\pi}$ of constant elasticity $\tilde{\theta}$ with respect to R_{TM}. This follows since

$$
\begin{aligned}
-\ln \pi(R_M) + \ln \pi(\varepsilon) &= \int_{\varepsilon}^{R_M} \nu(R_M^0) d \ln R_M^0 \\
&= \tilde{\theta} \ln g(R_M) \\
&= \tilde{\theta} \ln R_{TM} \\
&= -\ln \tilde{\pi}(R_{TM}) + \ln \pi(\varepsilon). \qquad (25)
\end{aligned}
$$

The per unit-probability price for a claim contingent on R_{TM}, $\tilde{\pi}(R_{TM})$, equals $\pi(R_M)$, the price for a claim contingent on $R_M = g^{-1}(R_{TM})$. By definition, R_{TM} is a one-to-one transformation of the market return so that constant elasticity $\tilde{\theta}$ of the new pricing kernel $\tilde{\pi}(R_{TM})$ is assured. This is true regardless of the sign of $\nu'(R_M)$. Moreover, the level of the constant pricing kernel elasticity of the transformed market return can be chosen freely. Therefore, we can always create an exchange traded fund (ETF) on the market return with return R_{TM} and low constant elasticity $\tilde{\theta}$ of its pricing kernel. This ETF is a suitable candidate for the approximation portfolio so that

$$
\hat{V}^{-}(R_{TM}) = \frac{\tilde{\theta}}{R_f} R_{TM} + (\gamma - \tilde{\theta}); \qquad \gamma \geq \tilde{\theta}.
$$

This approximation assures a low approximation loss.

5.5 Incomplete Markets

So far we considered complete markets. In an incomplete market, the pricing kernel is no longer unique. Suppose, first, that a pricing kernel on the market with low constant elasticity is feasible. For this case the preceding analysis has shown that buying the market portfolio and the risk-free asset provides a very good approximation to the optimal portfolio for a large variety of settings. Actually, in an incomplete market the approximation quality is even better. This follows because incompleteness does not affect the availability of the market portfolio and, hence, the approximation portfolio, but the optimal portfolio in a complete market may not be available. This reduces the the approximation loss.

Second, suppose that that a pricing kernel with low constant elasticity is not feasible. Then we can use an ETF. If both, the ETF and the portfolio which would be optimal in a complete market, cannot be replicated exactly in an incomplete market, then the approximation loss might go up or down. But, given a large number of available risky assets, the incompleteness effect should be small.

5.6 Extension to Parameter Uncertainty

So far all parameters are assumed to be known precisely. The discussion on parameter uncertainty focusses on the probability with which one portfolio is preferable to another one in the presence of parameter uncertainty. As discussed before, several papers conclude that, given a set of well-diversified portfolio strategies, no strategy significantly outperforms the other strategies. To address this issue, consider the following setting. At date 0 the investor buys a portfolio of state-contingent claims based on the a priori probability distribution of the market return. The date-0-pricing kernel is consistent with the a priori distribution. The investor liquidates her portfolio at the investment horizon. Then she also knows the a posteriori probability distribution of the market return.

In the spirit of the papers on parameter uncertainty, we derive the a priori probability that ex post, i.e. given the a posteriori distribution, the ex ante optimal portfolio is still preferred to the approximation portfolio. Let $\mathcal{I}$ denote the parameter vector of the a posteriori distribution of the market return. Hence, we check

$$\mathcal{P} = \text{Prob}\left[\, \mathcal{I} \;\middle|\; ce(\hat{V}^{+}|\mathcal{I}) \geq ce(\hat{V}^{-}|\mathcal{I}) \,\right], \tag{26}$$

where $ce(\hat{V}|\mathcal{I})$ is the certainty equivalent of portfolio payoff $\hat{V}$, given the a posteriori distribution $\mathcal{I}$.

To illustrate, assume that each a posteriori distribution of the market return is log normal, $n(\ln R_M|\mathcal{I})$. Then the a priori probability density of R_M is given by $\int n(\ln R_M|\mathcal{I})dF(\mathcal{I})$, with $F(\mathcal{I})$ being the cumulative probability distribution of $\mathcal{I}$. We use a symmetric truncated normal distribution for $\mathcal{I} = [\mathbb{E}[R_M], \sigma(R_M)]$ with bounds $[0.8955; 1.2955]$ for $\mathbb{E}[R_M|\mathcal{I}]$ and $[0.1782; 0.3782]$ for $\sigma(R_M|\mathcal{I})$. We assume that $\mathbb{E}[R_M|\mathcal{I}]$ and $\sigma(R_M|\mathcal{I})$ are uncorrelated and that the standard deviation of both parameters is 0.1. This yields an a priori probability distribution of R_M with simulated expected market return of 1.0974 and standard deviation of 0.2942. This distribution is not log normal. Given the a priori distribution, we derive $\exp\{a(\gamma)\}$ by simulation and obtain the optimal demand $\hat{V}^{+}(R_M)$. The linear demand function $\hat{V}^{-}(R_M)$ is based on $\phi = \theta = 1$ and is independent of the distribution. Figure 6 plots the certainty equivalent difference, $ce(\hat{V}^{+}|\mathcal{I}) - ce(\hat{V}^{-}|\mathcal{I})$, for a time horizon of 1 year and for $\gamma = 3$, $\gamma = 8$ and $\gamma = 50$. Also, the plot illustrates the $\mathcal{I}$-range for which this difference is positive, i.e. it is above the black zero hyper-plane. The a priori probability of this range is $\mathcal{P} \approx 0.75$ for all γ-values. Hence, the optimal portfolio does not outperform the approximation portfolio at any conventional significance level. Therefore, the regret probability $(1 - \mathcal{P})$ of not having chosen the approximation portfolio is substantial.

This finding is not surprising because the approximation portfolio payoff is linear in the market return while the optimal payoff is concave. The certainty equivalent of the linear payoff is less sensitive to parameter variations than that of the concave payoff. Since, a priori, the certainty equivalent of the optimal portfolio exceeds that of the approximation portfolio only by a small percentage, we can only expect a high probability $\mathcal{P}$ if the a posteriori certainty equivalent of the optimal portfolio is as stable as that of the approximation portfolio with respect to parameter variations. But as illustrated in Figure 6, this is not true. Therefore the investor faces a rather high regret probability $(1-\mathcal{P})$. This may be viewed as another argument for choosing the simple approximation portfolio instead of the complicated optimal portfolio.

[Figure 6 about here.]

6 Approximation in a Discrete State Space

In the following, we analyze the approximation quality in a discrete state space with few states. In a continuous state space the probability mass of the optimal and the approximation portfolio payoff may be concentrated around the zero excess payoff inducing a strong approximation quality. This quality might be weaker for portfolio returns with more probability mass in the tails. Lemma 1, however, suggests that the approximation loss is similar in a continuous and a discrete state space whenever the non-central moments are.

As an example, consider a bank which only invests in loans. The loan market is arbitrage free. Loans either are fully paid or go into default paying a non-random recovery amount. If the bank can invest in many different loans, it can achieve strong portfolio diversification. Then the loan portfolio payoff can be approximated quite well by a continuous unimodal probability distribution. This suggests again a high approximation quality in the absence of approximate arbitrage opportunities. Critical might be cases in which the number of loans is small. In the following, we present examples with one and two loans.

6.1 One Risky Asset

In the case of only one risky asset, there is no structure effect. Yet, the volume effect $(\hat{\alpha}^+ - \hat{\alpha}^-)$ remains and determines the approximation loss. A volume effect would not exist if the $1/\gamma$-rule would work perfectly. $\hat{\alpha}^+ = \hat{\alpha}(\gamma)$ and $\hat{\alpha}^- = \hat{\alpha}(\phi)$ denote the optimal and the approximate amount invested in the single risky asset, derived from the first order conditions (3) respectively (5).

We analyze a negatively and a positively skewed binomial distribution. Both distributions have the same expected return of 10.5% and standard deviation of 30% so that the Sharpe-ratio is the same. The risk-free rate is 3%. Let u (d) be the gross return in the up-state (down-state). p is the up-state probability for the distribution R skewed to the right and also the down-state probability for the distribution L skewed to the left. For example, let $p = 0.25$, $u_R = 1.42$ and $d_R = 1$, $u_L = 1.21$ and $d_L = 0.79$. Hence the distribution R has a skewness of 0.191, while distribution L has a skewness of -0.165. The approximated investment in the risky asset is the optimal investment using $\phi = 1$. The optimal investment, the volume effect and the approximation loss are shown in Table 1 for an investor with constant relative risk aversion $\gamma = 2$, $\gamma = 3$ and $\gamma = 10$.

[Table 1 about here.]

The volume effect is negative (positive) for the positively (negatively) skewed return distribution. The volume effect relative to the optimal investment in the risky asset is rather large (small) for the positively (negatively) skewed distribution. Yet, the approximation loss is rather small in all cases. It is larger for the positively skewed distribution, and declines with increasing γ for high values of γ.

The intuition for the sign of the volume effect can be derived from Figure 1. The optimal volume depends on the absolute risk aversion levels in the down-state and the up-state. Given $\gamma > \phi$, the absolute risk aversion is higher [smaller] for the utility function with parameter ϕ than for that with parameter γ in the down-state [up-state].

For the positively skewed distribution the absolute difference in risk aversion is much higher in the up-state than in the down-state. Therefore we expect a negative volume effect, $\hat{\alpha}^+ < \hat{\alpha}^-$. For a negatively skewed distribution we expect a positive volume effect.

This is confirmed in Table 1. For a symmetric distribution, the absolute difference in risk aversion is about the same in the down- and in the up-state. Therefore $\hat{\alpha}^{+}(\gamma) \approx \hat{\alpha}^{-}(\phi)$ implying a very small volume effect.

There is an easy way to understand the strong volume effect for the positively skewed distribution. For a binomial distribution, the first order condition yields

$$p_u r_u \left(1 + \frac{\hat{\alpha}^{+} r_u}{\gamma}\right)^{-\gamma} = (1 - p_u)|r_d| \left(1 + \frac{\hat{\alpha}^{+} r_d}{\gamma}\right)^{-\gamma}$$

$$\Leftrightarrow \frac{p_u r_u}{(1 - p_u)|r_d|} = \left(\frac{\gamma + \hat{\alpha}^{+} r_u}{\gamma + \hat{\alpha}^{+} r_d}\right)^{\gamma}. \tag{27}$$

The left hand side of (27) denotes the gain/loss- ratio of Bernado and Ledoit (2000). The higher it is, the closer is an approximate arbitrage opportunity. For the positively (negatively) skewed distribution the gain/loss- ratio is 4.33 (2.25). Hence, the positively skewed distribution is much closer to approximate arbitrage. An investor with low relative risk aversion benefits more from approximate arbitrage than an investor with higher risk aversion by choosing a more aggressive portfolio. This explains for $\phi = 1$ the high value of $\hat{\alpha}^{-} = 6.41$, the strong negative volume effect and the relatively high approximation loss.

As argued by Bernado and Ledoit, a high elasticity of the pricing kernel also indicates an approximate arbitrage opportunity[4]. The pricing kernel elasticity is 4.18 (1.90) for the positively (negatively) skewed distribution. Hence, the higher elasticity for the positively skewed distribution also motivates a higher approximation loss.

6.2 Two Risky Assets with Dependent Returns

Next, consider two risky loans with correlated binomial returns. In this case there exist only 4 states of nature. If there are two loans with different expected returns and perfectly negatively correlated returns, then there exists an arbitrage opportunity. If the returns are strongly negatively correlated, then there exists an approximate arbitrage opportunity. Hence, investors with low relative risk aversion will take very large positions in the risky assets which should raise the approximation loss.

For illustration, let the marginal distribution of each risky asset have a binomial distribution with equal probability for both outcomes, the up-state and the down-state. The gross return of asset 1 is $R_1 = (1.2; 0.925)$ and of asset 2 is $R_2 = (1.3; 0.85)$, respectively. The risk-free rate is 3%. This implies an expected excess return of 3.25% for asset 1 and 4.5% for asset 2. The standard deviation is 13.75% for the first asset and 22.5% for the second asset. Holding the marginal distributions for both asset returns constant, we change the return correlation by the following procedure. Let $P_{s,t} := \text{Prob}(R_1 = s, R_2 = t)$ denote the probability that asset 1 is in the s-state and asset 2 is in the t-state, $s, t \in \{\text{up, down}\}$. Then, the joint probability is

$$[P_{s,t}]_{s,t \in \{\text{up, down}\}} = \begin{pmatrix} 0.5 - x & x \\ x & 0.5 - x \end{pmatrix},$$

with $x \in [0; 0.5]$. Reducing $P_{\text{up, up}}$ and $P_{\text{down, down}}$ by x and adding x to $P_{\text{down, up}}$ and $P_{\text{up, down}}$, decreases the correlation without affecting the marginal distributions.

The approximation portfolio is based on $\phi = 0.98$. For relative risk aversion $\gamma \in [0.98; 8]$ and for a return correlation between -0.8 and 0.8, Figure 7, left, shows the approximation loss. It is very low for correlations above -0.5, but increases strongly for lower correlations. Given negatively correlated assets, the investor can buy a hedged portfolio with long positions in both assets and earn a high portfolio return with little downside potential. Consider the case with correlation -0.6 and $\gamma = 2.5$. The optimal portfolio invests about 3.57$ of the initial endowment in asset 1 and about 2.06$ in asset 2. This gives an expected excess return of the optimal portfolio of 8.61% and a standard deviation of 17.64%. The approximation portfolio invests 3.02$ in asset 1 and 1.70$ in asset 2 implying an approximation loss of about 0.15%. The volume effect is $(3.57+2.06)-(3.02+1.70) = 0.91$$, it is quite strong. The structure effect $\frac{3.57}{2.06} - \frac{3.02}{1.70} = -0.04$ is, however, very weak. For higher correlations, the approximation quality is excellent.

[Figure 7 about here.]

Figure 8 shows the volume and the structure effect. The volume effect is quite strong for strongly negative correlation, while the structure effect is always quite modest. This indicates that the approximation quality is impaired primarily by the volume effect.

[Figure 8 about here.]

The example shows that the approximation loss is substantial whenever the asset correlation supports an approximate arbitrage. Then the investor with low RRA ϕ takes large positions in both risky assets and borrows a lot. If we exclude short selling, then approximate arbitrage opportunities cannot be used extensively so that the approximation loss is much smaller. This is illustrated in Figure 7, right. Compared to Figure 7, left, the restriction lowers the approximation loss strongly in the area $[-0.8, -0.4] \times [1.75, 4]$, where the first dimension is the asset correlation and the second the relative risk aversion γ.

7 Conclusion

HARA-utility functions cover a wide spectrum of utility functions with declining, constant and increasing relative risk aversion. We constrain our analysis to declining absolute risk aversion and ask whether the parameters of the HARA-utility function really matter for optimal investment decisions. The paper shows that the optimal portfolio can be approximated by the simple $1/\gamma$-rule if the exponent of the investor's utility function is higher than that used for the approximation and if approximate arbitrage opportunities do not exist. If these opportunities exist and the approximation portfolio takes high risk while the optimal portfolio does not, then the $1/\gamma$-rule leads to high approximation losses. Whenever the pricing kernel of the market return displays low constant elasticity, an investor with higher relative risk aversion may simply buy the market portfolio and the risk-free asset without noticeable harm. Otherwise, the investor may buy a transformed market portfolio with low constant elasticity of the pricing kernel, again without noticeable harm.

Critical for a good approximation quality is that the investor's relative risk aversion is higher than that used for the approximation and that approximate arbitrage opportunities are ruled out.

If there exists uncertainty about the parameters of the asset returns, then our examples demonstrate that the approximation portfolio turns out to be better than the optimal portfolio with a substantial probability. This also supports the use of a simple approximation portfolio.

Further research might analyse the approximation quality in market settings in which investors use dynamic trading strategies to benefit from stock return predictability. Also the set of utility functions should be widened beyond HARA.

A Footnotes

[1]For small portfolio risk, $m_i \to 0$ for $i > 2$. Then the optimal portfolio satisfies $m_1/m_2 \to 1$ rendering γ irrelevant. This is the case in a continuous time model with i.i.d. returns. Then the volume and the structure effect disappear.

[2]The skewness is $sk_t = \frac{4-\pi}{2}\frac{\left(\delta_t\sqrt{2/\pi}\right)^3}{(1-2\delta_t^2/\pi)^{3/2}}$ and the excess kurtosis is $2(\pi-3)\frac{\left(\delta_t\sqrt{2/\pi}\right)^4}{(1-2\delta_t^2/\pi)^2}$.

[3]Independent increments imply $sk_t = sk_1/\sqrt{t}$, where sk_1 denotes the skewness for one year and sk_t is the skewness for t-years.

[4]Consider the Arrow-Debreu prices in this complete market setting. For a binomial return there always exists a pricing kernel with constant elasticity. The two Arrow-Debreu prices are

$$\pi_u = \frac{1}{R_f}\frac{p_u R_u^{-\theta}}{\mathbb{E}[R^{-\theta}]} \text{ and } \pi_d = \frac{1}{R_f}\frac{(1-p_u)R_d^{-\theta}}{\mathbb{E}[R^{-\theta}]}.$$

The ratio $\frac{\pi_u}{\pi_d}$ can be used to solve for the pricing kernel elasticity θ,

$$\theta = \frac{\ln\left(\text{gain-loss-ratio}\right)}{\ln\left(\frac{R_u}{R_d}\right)}.$$

B Proof of Lemma 3

At an intersection point, $\hat{V}^+(R_M^j) = \hat{V}^-(R_M^j)$. Then, we have at an intersection with $R = R_M^j$,

$$\frac{\partial \hat{V}^+(R)}{\partial p} = \frac{\partial \hat{V}^-(R)}{\partial p}$$

$$\Leftrightarrow \frac{\theta}{\gamma}\exp\{a(\gamma)\}R^{(\theta/\gamma)-1}\frac{\partial R}{\partial p}+\frac{\partial a(\gamma)}{\partial p}\hat{V}^+(R) = \frac{\theta}{\phi}\exp\{a(\phi)\}R^{(\theta/\phi-1)}\frac{\partial R}{\partial p}+\frac{\partial a(\phi)}{\partial p}\left[\hat{V}^-(R)-(\gamma-\phi)\right]$$

Since $\hat{V}^+(R) = \hat{V}^-(R)$, we have

$$\frac{\theta}{\gamma}\hat{V}^+(R)\frac{\partial \ln R}{\partial p} + \frac{\partial a(\gamma)}{\partial p}\hat{V}^+(R) = \left[\frac{\theta}{\phi}\frac{\partial \ln R}{\partial p} + \frac{\partial a(\phi)}{\partial p}\right][\hat{V}^+(R) - (\gamma-\phi)]$$

$$\Leftrightarrow \frac{\partial \ln R}{\partial p}\frac{\theta}{\phi}\frac{1}{\gamma}(\gamma-\phi)[\hat{V}^+(R)-\gamma] = \left[\frac{\partial a(\gamma)}{\partial p} - \frac{\partial a(\phi)}{\partial p}\right]\hat{V}^+(R) + \frac{\partial a(\phi)}{\partial p}(\gamma-\phi).$$

Dividing by $(\gamma-\phi)$ yields equation (20). Equations (21) and (22) follow from differentiating the budget constraint with respect to p.

References

[1] Ait-Sahalia, Y. and A. W. Lo, 2000, "Nonparametric risk management and implied risk aversion", *Journal of Econometrics* 94, 9-51.

[2] Arrow, K. J., 1974, "Essays in the Theory of Risk-Bearing", *Markham Publishing Co.*

[3] Balduzzi, P. and A. Lynch, 1999, "Transaction Costs and Predictability: Some Utility Cost Calculations", *Journal of Financial Economics* 52, 47-78.

[4] Barone-Adesi G., R. F. Engle and L. Mancini, 2008, "A GARCH Option Pricing Model with Filtered Historical Simulation", *Review of Financial Studies* 21, 1223-1258.

[5] Bernado, A. and O. Ledoit, 2000, "Gain, Loss and Asset Pricing", *Journal of Political Economy* 108, 144-172.

[6] Black, F., R. Littermann, 1992, "Global Portfolio Optimization", *Financial Analysts Journal*, Sept/Oct. 28-43.

[7] Bliss, R. R. and N. Panigirtzoglou, 2004, "Option-Implied Risk Aversion Estimates", *Journal of Finance* 59, 407-446.

[8] Brandt, M., A. Goyal, P. Santa-Clara and J. Stroud, 2005, "A Simulation Approach to Dynamic Portfolio Choice with an Application to Learning About Return Predictability", *Review of Financial Studies* 18, 831-873.

[9] Brandt, M., P. Santa-Clara and R. Valkanov, 2009, "Parametric Portfolio Policies: Exploiting Characteristics in the Cross Section of Equity Returns", *Review of Financial Studies* 22, 3411-3447.

[10] Cass, D., and J. E. Stiglitz, 1970, "The structure of investor preferences and asset returns, and separability in portfolio allocation: A contribution to the pure theory of mutual funds", *Journal of Economic Theory* 2, 122-160.

[11] Chacko, G. and L. Viceira, 2005, "Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets", *Review of Financial Studies*, 18, 1369-1402.

[12] Corrado, C. J. and T. Su, 1997 "Implied Volatility Skews and Stock Index Skewness and Kurtosis in S&P 500 Index Option Prices.", *Journal of Derivatives*, 4, 8-19.

[13] DeMiguel, V., L. Garlappi and R. Uppal, 2009, "Optimal versus Naive Diversification: How Inefficient is the $1/N$ Portfolio Strategy?", *Review of Financial Studies* 22, 1915-1953.

[14] DeMiguel, V., L. Garlappi, J. Nogales and R. Uppal, 2009, "A Generalized Approach to Portfolio Optimization: Improving Performance By Constraining Portfolio Norms", *Management Science* 55, 798-812.

[15] Diamond, P. A. and J. E. Stiglitz, 1974, "Increase in Risk and Risk Aversion", *Journal of Economic Theory* 66, 522-535.

[16] Hakansson, N. H. 1970, "Optimal Investment and Consumption Strategies Under Risk for a Class of Utility Functions", *Econometrica*, 38, 587-607.

[17] Hodder, J., J. Jackwerth and O. Kokolova, 2009, "Improved Portfolio Choice Using Second Order Stochastic Dominance", Working paper, University Konstanz.

[18] Jackwerth, J., 2000, "Recovering Risk Aversion from Option Prices and Realized Returns", *Review of Financial Studies* 13, 433-451.

[19] Jacobs, H., S. Mueller and M. Weber, 2009, "How Should Private Investors Diversify? An Empirical Evaluation of Alternative Asset Allocation Policies to Construct a 'World Market Portfolio", Working paper, University of Mannheim.

[20] Karatzas, I., J. Lehoczky, S. Sethi, S. Shreve, 1986, "Explicit Solution of a General Consumption/Investment Problem", *Mathematics of Operations Research* 11, 261-294.

[21] Merton, R. C. 1971, "Optimal Consumption and Investment Rules in a Continuous Time Model", *Journal of Economic Theory* 3, 373-413.

[22] Pratt, J. W., 1964, "Risk Aversion in the Small and in the Large", *Econometrica* 32, 122-136.

[23] Rosenberg, J. V. and R. F. Engle, 2002, "Option Hedging Using Empirical Pricing Kernels", *Journal of Financial Economics* 64, 341-372.

[24] Rothschild, M. and J. E. Stiglitz, 1970, "Increasing Risk I: A Definition", *Journal of Economic Theory* 2, 225-243.

[25] Viceira, L. (2001), "Optimal Portfolio Choice for Lang-Horizon Investors with Non-tradable Labor Income", *Journal of Finance* 61,433-470.

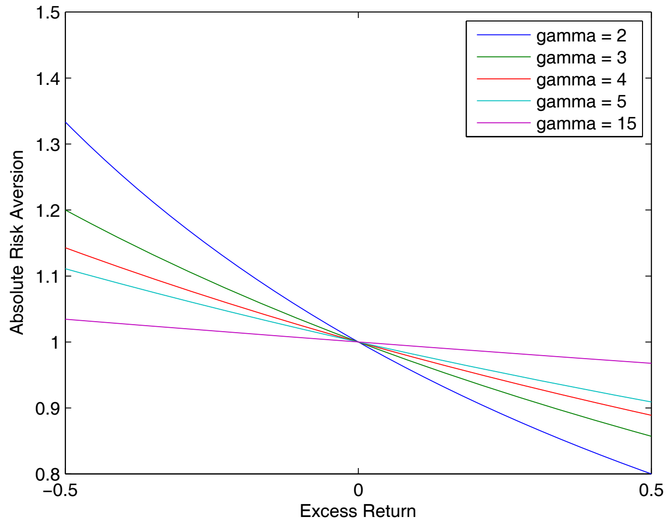

Figure 1: The absolute risk aversion of the HARA-function with endowment γ/R_f declines in the portfolio excess return. For increasing γ the difference between the absolute risk aversion of the HARA-function and that of the exponential utility function, being 1 everywhere, decreases.

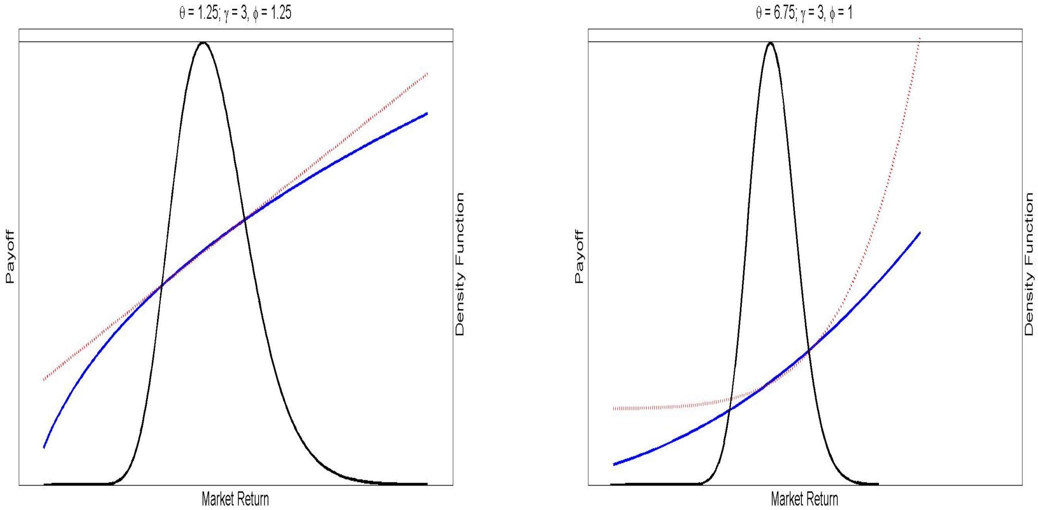

Figure 2: **Left:** The figure shows the optimal demand for state contingent claims (blue solid curve) and the approximation demand (red dotted line) for $\gamma = 3 \geq \phi = \theta = 1.25$. In addition, on a different scale the graph shows the probability density of the market return. **Right:** $\gamma = 3$, $\theta = 6.75$ and $\phi = 1$. This implies a strongly convex approximation demand function while the optimal demand function is only moderately convex.

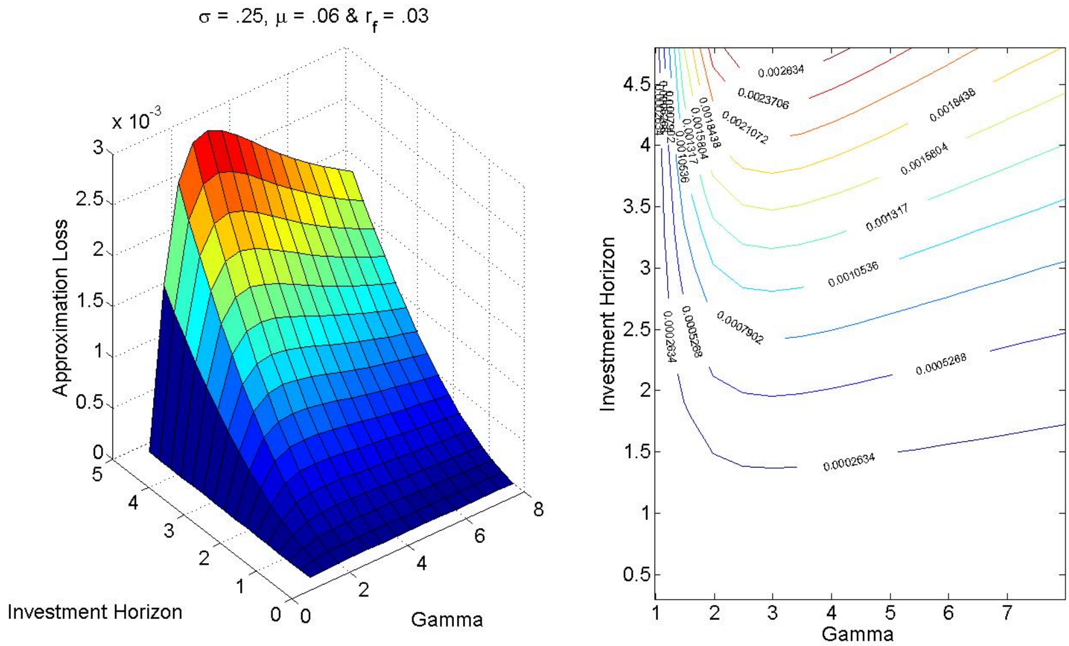

Figure 3: **Left:** The surface shows the approximation loss for $\gamma \in [0.98; 8]$, $\phi = \theta = 0.98$ and an investment horizon between 3 months and 5 years. For this setting, the highest loss in certainty equivalent is obtained for γ between 3 and 4 and an investment of five years. The investor would have lost about 0.3% of the optimal certainty equivalent or 0.06% per year. **Right:** Each isoquant shows the combination of γ and investment horizon with the same approximation loss k depicted in the curve.

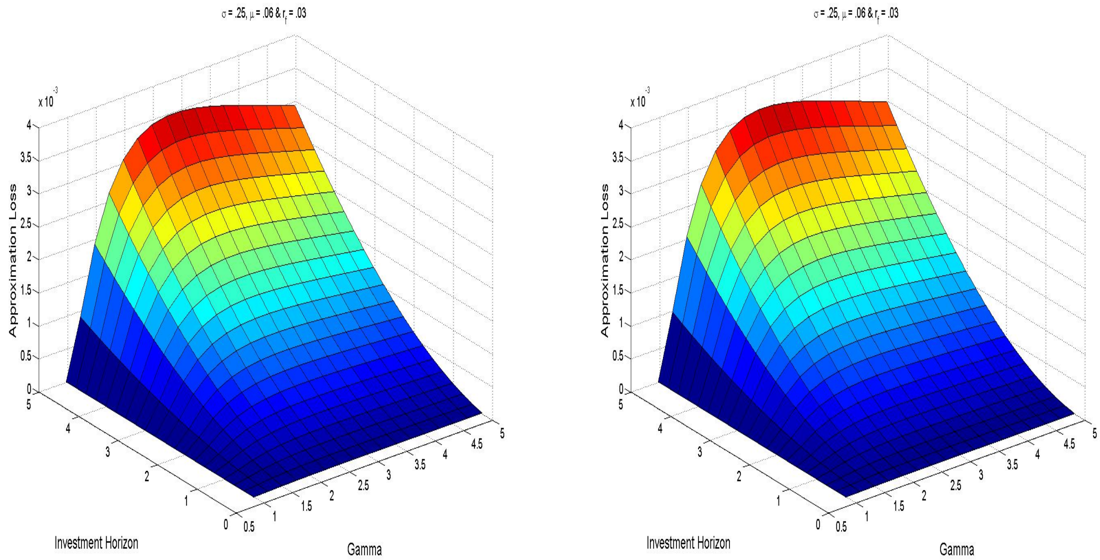

Figure 4: The surface shows the approximation loss for $\gamma \in [0.98; 8]$, $\phi = \theta = 0.98$ and an investment horizon between 3 months and 5 years. **Left:** The logarithmic market return is t-distributed. We assume independent and identically distributed increments, hence, $\mu_t = 0.06t, \sigma_t = 0.25\sqrt{t}$ and $\nu_t = 4t{+}4$. For $\gamma \approx 3$ and an investment horizon of five years, the highest approximation loss is about 0.4%. **Right:** The logarithmic market return is left-skewed, fat tailed distributed with independent and identically distributed increments.

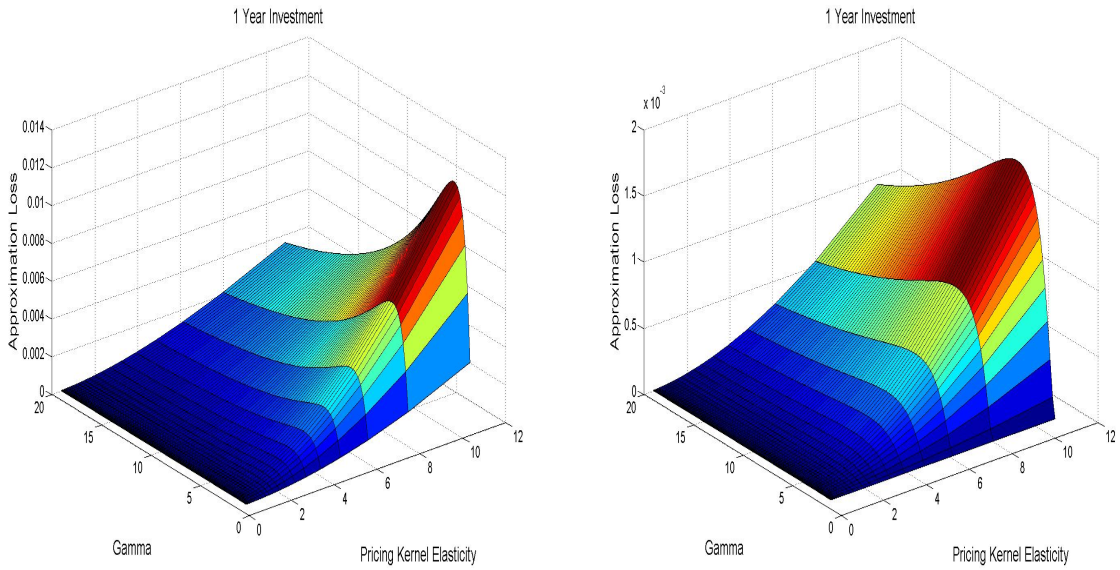

Figure 5: **Left:** The approximation loss for an investment horizon of one year as a function of θ and γ, assuming a lognormal market return with $\sigma = 0.25$. $r_f = 0.03$ and $\phi = 1$. $\gamma \in [\phi; 20]$, $\theta \in [0.44; 11]$. **Right:** The approximation loss for the same setting as in Figure 5, left, but with $\phi = 2$.

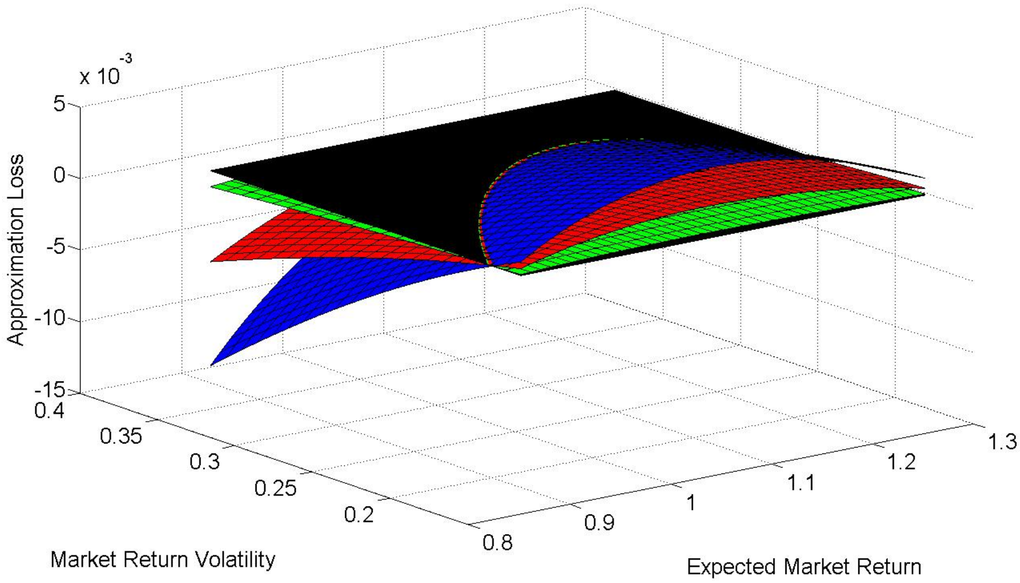

Figure 6: The plot shows the a posteriori-approximation loss assuming parameter uncertainty. The expected market return and the market volatility are a posteriori-realisations of both variables. The blue (red) [green] surface shows the loss assuming $\gamma = 3$ ($\gamma = 8$) [$\gamma = 50$], the black hyper-plane marks zero everywhere.

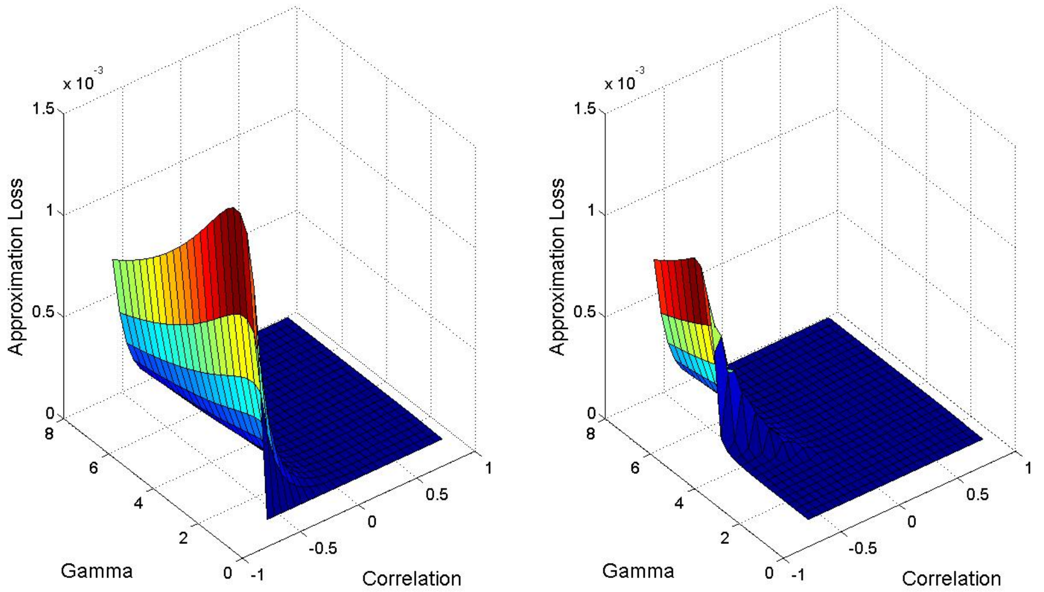

Figure 7: $\gamma > \phi = 0.98$, correlation of binomial returns between -0.8 and 0.8. **Left:** The figures show the approximation loss in a market with two binomial assets for different return correlations and γs. The expected excess return for asset 1 is 3.25% and 4.5% for asset 2. The volatility is 13.75% and 22.5%, respectively. **Right:** The figure shows the approximation loss for the same market setting with borrowing being prohibited.

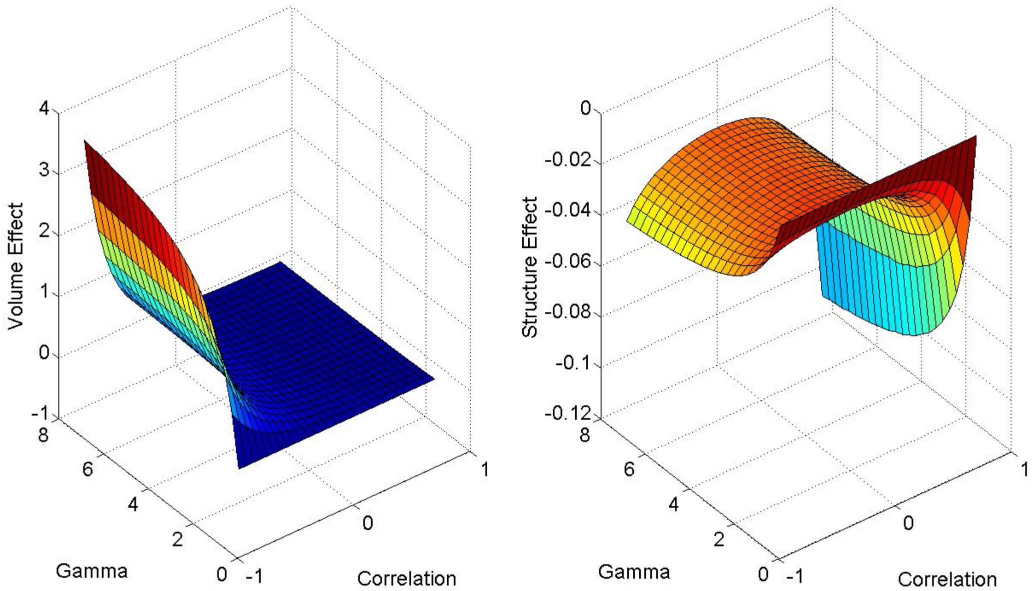

Figure 8: **Left:** The volume effect for a market with two binomial assets as in Figure 7. Only for strongly negative asset correlation there is a substantial volume effect. **Right:** The structure effect is remarkably small.

Distribution	$\gamma = 2$ R	L	$\gamma = 3$ R	L	$\gamma = 10$ R	L
$\hat{\alpha}^+$	4.7813	1.8519	4.3083	1.8830	3.7183	1.9187
$(\hat{\alpha}^+ - \hat{\alpha}^-)$	-1.6290	0.1158	-2.1020	0.1469	-2.6920	0.1826
k	0.0038	0.0002	0.0048	0.0002	0.0028	0.0001

Table 1: It shows the optimal investment in the risky asset for $\gamma = 2, 3$ and 10 and the volume effect $(\hat{\alpha}^+ - \hat{\alpha}^-)$. The approximated investment based on $\phi = 1$ is $\hat{\alpha}^- = 6.4103$ for R and $\hat{\alpha}^- = 1.9677$ for L. k is the approximation loss. R (L) denotes the probability distribution skewed to the right (left)