Structure theorems for d-minimal expansions of the real additive ordered group and some consequences

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ABSTRACT

Let $\mathcal{R}$ be an o-minimal expansion of the real additive ordered group. A d-minimal expansion of $\mathcal{R}$ is a structure such that for every definable family of subsets of $\mathbb{R}$ of dimension 0, there is a bound $N \in \mathbb{N}$ such that every element of the family is the union of at most N discrete sets. In other words, there is a uniform bound on the Cantor-Bendixson rank of the elements of the family.

The subject of this thesis is to study d-minimal expansions of $\mathcal{R}$ by a set $P \subseteq \mathbb{R}$ without interior. The main result is:

For every definable set X there is a family $\{X_t : t \in A\}$ definable in $\mathcal{R}$ and a countable set $S \subseteq A$ such that:

$$X = \bigsqcup_{t \in S} X_t.$$

We explore two main consequences of this result. The first is:

Assuming that $\mathcal{R}$ is a reduct of $\overline{\mathbb{R}}$, then a definable $\mathcal{C}^\infty$-function whose domain is open and definable in $\mathcal{R}$ is definable in $\mathcal{R}$.

The second main application concerns the study of the connected components of the sets definable in $\langle \mathcal{R}, P \rangle$. Two questions can be raised:

1. What is the structure generated by $\langle \mathcal{R}, P \rangle$ together with predicates for the connected components of sets definable in $\langle \mathcal{R}, P \rangle$?

2. What is $\mathrm{CC}(\langle \mathcal{R}, P \rangle)$, the smallest structure containing $\langle \mathcal{R}, P \rangle$ such that every connected components of every definable sets is definable?

We answer Question 2 in the case where $\langle \mathcal{R}, P \rangle$ is locally o-minimal:

If $\langle \mathcal{R}, P \rangle$ is locally o-minimal and P has order-type ω, then $\mathrm{CC}(\langle \mathcal{R}, P \rangle)$ is $\langle \mathcal{R}, P, f \rangle$ where $f : P^2 \to P$ is the multiplication on P^2 seen as $\mathbb{N}^2$.

More generally, we show that:

If $\langle \mathcal{R}, P \rangle$ is d-minimal and P has order type ω then $\langle \mathcal{R}, P, f \rangle \subseteq CC(\langle \mathcal{R}, P \rangle)$. Thus $\mathrm{CC}(\langle \mathcal{R}, P \rangle)$ is undecidable.

We also answer Question 1 for different structures and deduce some interesting consequences for the study of definable groups.

DEUTSCHE ZUSAMMENFASSUNG

Lassen Sie $\mathcal{R}$ eine o-minimale Erweiterung der realen additiv geordneten Gruppe sein. Eine d-minimale Expansion von $\mathcal{R}$ ist eine Struktur, so dass fur jede definierbare Familie von Untermengen von $\mathbb{R}$ der Dimension 0 eine gebundene $N \in \mathbb{N}$ vorliegt, so dass jedes Element der Familie die Vereinigung von hochstens N diskreten Mengen ist. Mit anderen Worten, es gibt eine einheitliche Bindung auf dem Cantor-Bendixson-Rang der Elemente der Familie.

Das Thema dieser Arbeit ist die Untersuchung d-minimaler Erweiterungen von $\mathcal{R}$ durch eine Menge $P \subseteq \mathbb{R}$ der Dimension 0. Das Hauptergebnis ist:

Fur jeden definierbaren Satz X gibt es eine Familie $\{X_t : t \in A\}$ definierbar in $\mathcal{R}$ und einen Satz $S \subseteq A$ der Dimension 0, so dass

$$X = \bigsqcup_{t \in S} X_t.$$

Wir untersuchen zwei Hauptfolgen dieses Ergebnisses. Die erste ist:

Angenommen, dass $\mathcal{R}$ eine Reduktion von $\overline{\mathbb{R}}$ ist, dann ist eine definierbare $\mathcal{C}^\infty$-Funktion, deren Domane in $\mathcal{R}$ offen und definierbar ist, in $\mathcal{R}$ definierbar.

Der zweite Hauptantrag betrifft die Untersuchung der verbundenen Komponenten der in $\langle \mathcal{R}, P \rangle$ definierbaren Sets. Es konnen zwei Fragen gestellt werden:

1. Was ist die Struktur, die durch $\langle \mathcal{R}, P \rangle$ zusammen mit Pradikaten fur die verbundenen Komponenten von Mengen erzeugt wird, die in $\langle \mathcal{R}, P \rangle$ definierbar sind?

2. Was ist $\mathrm{CC}(\langle \mathcal{R}, P \rangle)$, die kleinste Struktur, die $\langle \mathcal{R}, P \rangle$ enthalt, so dass jede verbundene Komponente jeder definierbaren Menge definierbar ist?

Wir beantworten Frage 2 fur den Fall, dass $\langle \mathcal{R}, P \rangle$ lokal o-minimal ist:

Wenn $\langle \mathcal{R}, P \rangle$ lokal o-minimal ist und $P \sim \omega$, dann ist $\mathrm{CC}(\langle \mathcal{R}, P \rangle)$ $\langle \mathcal{R}, P, f \rangle$, wobei $f : P^2 \to P$ die Multiplikation von P^2 ist, die als $\mathbb{N}^2$ angesehen wird.

Allgemeiner gesagt, wir zeigen das:

Wenn $\langle \mathcal{R}, P \rangle$ d-minimal ist und $P \sim \omega$ hat, dann $\langle \mathcal{R}, P, f \rangle \subseteq CC(\langle \mathcal{R}, P \rangle)$. Somit ist $\mathrm{CC}(\langle \mathcal{R}, P \rangle)$ unentscheidbar.

Wir beantworten auch die Frage 1 fur verschiedene Strukturen und leiten daraus einige interessante Konsequenzen fur das Studium definierbarer Gruppen ab.

CONTENTS

1. INTRODUCTION

For the last forty years, o-minimality has been a rich and fruitful domain of investigation. Many structures have been shown to be o-minimal and many results in various areas have been proven using o-minimality; to name but two, in real algebraic geometry or in number theory. It was first considered as the natural candidate for the "tame topology" of Grothendieck (the *topologie moderee* of [31]). Tameness is more a metamathematical notion than really a mathematical one, and indeed does not have a proper definition.

Since the mid-eighties, however, this notion of tameness has been applied to many other settings. If we describe the structures' complexity as a bush, with an o-minimal structure at the bottom and the whole real projective hierarchy at the top, there are two ways of exploring the notion of tameness. Let $\mathcal{M}$ be an expansion of a dense linear order with underlying set M.

From the bottom: we consider that $\mathcal{M}$ may be described as tame if the definable sets satisfy some good properties from a model theoretic, a descriptive set theoretic or a geometric/analytic point of view. Most of the time, the good properties are necessary conditions to use some classical technics issued from various fields. For example, if a structure satisfies some kind of cell decomposition, we can derive some technics from o-minimality.

From the top: we define some conditions for $\mathcal{M}$ to be considered not tame (most of the time by showing that it implies that the whole projective hierarchy is then definable). Indeed, from this perspective, Chris Miller and Phillip Hieronymi have recently shown that for an expansion of the real closed field, if the Assouad-dimension and the topological-dimension do not agree on images of closed definable sets under definable continuous maps, then $\mathcal{M}$ defines the whole projective hierarchy (see [32]) (considering the dimension of a set $X \subseteq M^m$ to be defined as the maximal n in such a way that there is some projection onto some n-coordinates that maps X to some set with interior. The Assouad dimension being not only too long, but more importantly outside our present concerns, we won't define it here but refer to the aforementionned paper).

Typical examples of expansions of the real ordered line which are considered as tame include locally o-minimal structures (see [59]), where, by definition, a definable set of topological-dimension 0 is discrete. Other examples include the d-minimal structures (see [46]) where a definable set of dimension 0 is a finite union of discrete sets.

The program of studying the notion of tameness in expansions of the real field has been formalized by Chris Miller (see for example [46]). However, we may consider Lou van den Dries as a precursor in this domain. Among **many** examples, he studied T-convex fields (see [12]), which are locally o-minimal. He also studied $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ where $2^{\mathbb{Z}}$ is a predicate for the integer powers of two (see [10]). It is worthwhile to note that this last structure is the first d-minimal structure that has ever been studied.

All of this being said, our approach is clearly one “from the bottom” and this thesis is a modest contribution toward giving a better understanding of the d-minimal structures of the form $\langle \mathcal{R}, P \rangle$, where $\mathcal{R}$ is an o-minimal expansion of $\langle \mathbb{R}, <, + \rangle$ and $P \subseteq \mathbb{R}$ is a set of dimension 0.

After this introduction, the second chapter is just a quick presentation of some properties of o-minimal structures in general. We there also give a new characterization as well as some properties of semibounded structures.

The first part of the third chapter is a presentation of d-minimality. We expose some tools such as the Cantor-Bendixson rank, useful in the d-minimal setting. We also present some properties of the dimension in this setting. It will have to be remembered that we study only d-minimal expansions of $\langle \mathbb{R}, <, + \rangle$ (but most of the results hold for elementarily equivalent structures) as it is usually done in this field.

In the second part of the third chapter, we further restrict ourselves and study only the d-minimal structures of the form $\langle \mathcal{R}, P \rangle$, where $\mathcal{R}$ is an o-minimal expansion of $\langle \mathbb{R}, <, + \rangle$ and $P \subseteq \mathbb{R}$ is a set of dimension 0. We prove the following theorem (see Theorems 3.3.7 and 3.3.16):

Theorem 1.0.1. *Let $\mathcal{R}$ be an o-minimal expansion of the real ordered additive group and let $P \subseteq \mathbb{R}$ be a set of dimension 0 such that $\langle \mathcal{R}, P \rangle$ is d-minimal. Let X be a set definable in $\langle \mathcal{R}, P \rangle$. Then there is a family of cells $\{X_t : t \in A\}$ definable in $\mathcal{R}$ and a set $S \subseteq A$ of dimension 0 such that $X = \bigcup_{t \in S} X_t$.*

This theorem will prove to be a powerful tool and is somehow similar to the following Theorem of Athipat Thamrongtanyalak:

Fact 1.0.2. *(see [58, Theorem B]) Let $\mathcal{M}$ be a d-minimal expansion of the real field and let $X \subseteq \mathbb{R}^n$ be definable in it. Then there is a finite decomposition of X into definable sets $Y_1, \ldots, Y_m$ such that for every i there is $\pi \in \Pi(d, n)$ (a projection from $\mathbb{R}^n$ to $\mathbb{R}^d$) and for every $x \in \pi(Y_i)$, there is a box $U \ni x$ so that $\pi^{-1}(U) \cap Y_i$ is composed of connected components which are graphs of continuous functions over U.*

Furthermore, the family $\{X_t : t \in S\}$ (of Theorem 1.0.1) is a subfamily of a family definable in an o-minimal structure which is quite useful (see Chapter 5 for some examples of applications), just like having a decomposition into connected sets is, there again, rather useful (see Chapter 6 for some examples of applications).

The fourth chapter of this thesis is a deeper study of some known d-minimal structures of the form $\langle \mathcal{R}, P \rangle$, with $\mathcal{R}$ an o-minimal expansion of $\langle \mathbb{R}, <, + \rangle$ and $P \subseteq \mathbb{R}$ a set of dimension 0. We there mostly focus on three questions:

1. Is $\langle \mathcal{R}, P\rangle$ locally o-minimal?

2. Do we have for every set $X \subseteq \mathbb{R}^k$ of dimension 0 definable in $\langle \mathcal{R}, P\rangle$, a function $f : \mathbb{R}^n \to \mathbb{R}^k$, definable in $\mathcal{R}$ such that X is included in $f(P^k)$?

3. What is the induced structure of $\langle \mathcal{R}, P\rangle$ on P?

In the fifth chapter of this thesis we are interested in studying the $\mathcal{C}^\infty$-functions definable in d-minimal structures of the form $\langle \mathcal{R}, P\rangle$ where $\mathcal{R}$ is an o-minimal expansion of the real additive ordered group and $P \subseteq \mathbb{R}$ has dimension 0. It is not hard to show that, if $\mathcal{R}$ expands $\overline{\mathbb{R}}$, then for every n there are some definable non semialgebraic $\mathcal{C}^n$-functions (see Claim 5.1.6). The main theorem of this chapter will thus be the following:

Theorem 1.0.3. *(see Theorems 5.2.7 and 5.4.4) Let $\mathcal{R}$ be $\langle \mathbb{R}, <, +, \cdot_{\restriction [0,1]^2}, (x \mapsto \lambda x)_{\lambda\in\mathbb{R}}\rangle$ or the real field and let $P \subseteq \mathbb{R}$ such that $\langle \mathcal{R}, P\rangle$ is d-minimal. Let $f : X \to \mathbb{R}$ be a definable smooth function over an open domain definable in $\mathcal{R}$. Then f is definable in $\mathcal{R}$.*

The thesis' sixth chapter contains a discussion and some preliminary results about a problem that has been brought to our attention by Chris Miller: *the one of the connected-components-closure.*

The basic observation that led to this problem is that there are some expansions $\mathcal{M}$ of $\langle \mathbb{R}, <\rangle$ which define some non-connected definably-connected sets (see Fornasiero's [24]). Hence, the connected components of such sets are not definable. Do allow to underline that we have already made a similar observation (before the presentation of the problem by Chris Miller and before hearing about the Fornasiero's paper, but not before the publication of [24]) inasmuch as $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$ defines some sets whose connected components are not definable. This is a slightly different property, because this set is not definably-connected. However, as we show in Section 6.7.2 $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$ defines some non-connected definably connected sets and we discuss further these different properties in that section.

With $\mathcal{M}$ being an expansion of the real ordered additive group that defines some set whose connected components are not definable. The following questions naturally arise :

1. How do the connected components of sets definable in $\mathcal{M}$ look like?

2. What is the smallest structure $\mathcal{M}_1$ which contains $\mathcal{M}$ so that the connected-components of sets definable in $\mathcal{M}$ are definable in $\mathcal{M}_1$?

3. What is the smallest structure $\mathcal{M}_\omega$ that contains $\mathcal{M}$ and so that the connected components of every set definable in $\mathcal{M}_\omega$ is definable in $\mathcal{M}_\omega$?

Regarding the second and the third questions, it is possible to add to $\mathcal{M}$ a predicate for every connected component of every definable set in $\mathcal{M}$ to produce $\mathcal{M}_1$ (this structure answers Question 2). Similarly, we can produce $\mathcal{M}_2$ as $(\mathcal{M}_1)_1$, and for every n, $\mathcal{M}_n = (\mathcal{M}_{n-1})_1$. It is then easy to check that $\bigcup_n \mathcal{M}_n$

does define every connected component of every definable set (this structure answers Question 3). Such a structure is called the Connected-Component-closure of $\mathcal{M}$ (CC($\mathcal{M}$)). Of course, this construction is just a proof of existence and not really a description and the problem of the cc-closure refers to a proper description.

Initially, we hoped to have some control on the complexity of CC($\mathcal{M}$) but the following theorem shows that this is unfortunately not the case:

Theorem 1.0.4. *(see Corollary 6.2.8) If $\mathcal{M}$ defines an infinite discrete set, then* CC($\mathcal{M}$) *is undecidable.*

Moreover, we have a description for locally o-minimal expansions of the real ordered additive group:

Theorem 1.0.5. *(see Theorem 6.3.3) Let $\mathcal{R}$ be an o-minimal expansion of $\langle \mathbb{R}, <, + \rangle$ and let $P \subseteq \mathbb{R}$ be an infinite set of dimension* 0 *with order type $\mathbb{N}$ such that $\langle \mathcal{R}, P \rangle$ is locally o-minimal. Then* $\mathrm{CC}(\langle \mathcal{R}, P \rangle) = \langle \mathcal{R}, P^* \rangle$*, where P^* is the obvious $\langle \mathbb{N}, <, +, \cdot \rangle$-structure on P build via the increasing bijection between P and $\mathbb{N}$.*

It must be added that even if the full connected-component-closure is undecidable in most of the non-o-minimal cases, there are some structures where the connected components are well-behaved. For example, this is the case in expansions of the real additive group by a fast sequence or by an iteration sequence as defined in [28] and [48] (see Section 6.5 and Section 6.6).

The problem for d-minimal expansions of the real field remains open. The solution seems likely to be located at the intersection of computability, model theory and descriptive set theory. The end of Chapter 6 is a long discussion about some conjectures and examples.

1.1 Notations and conventions

We finish this introduction with some notations and basic definitions. Let $\mathcal{R}$ be an expansion of an ordered abelian group and R its underlying set.

For *any* set $X \subseteq R^n$, we define its *dimension* as the maximum k such that some projection of X to k coordinates has non-empty interior (the topological-dimension).

In each section, we fix some structure. By definable we mean definable in this structure. Of course, we specify where a set is definable if some confusion could occur. If $S \subseteq R^n$ is a set, its closure, interior boundary and frontier are denoted by $\overline{S}, \mathrm{int}(S), \mathrm{bd}(S) := \overline{S} \setminus \mathrm{int}(S)$ and $\mathrm{fr}(S) := \overline{S} \setminus S$, respectively. Sole exception to this notation is that of the real field $\overline{\mathbb{R}}$. By an open box $B \subseteq R^n$, we mean a set of the form

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n),$$

for some $a_i < b_i \in R \cup \{\pm\infty\}$. By an open set we always mean a non-empty open set. Let $d : R^2 \to R$ be the usual distance function. For a set $X \subseteq R$ and $x \in R$, we denote by

$$d(x, X) = \min_{y \in X} d(x, y).$$

Let

$$|X| = \sup_{(x,y) \in X^2} (d(x, y)).$$

For $x \in R^n$, $a \in R$, we denote by $\mathcal{B}(x, a)$ the box of radius a and of center x. We denote by $\mathcal{B}(a)$ the box of radius a and of center 0. For $x, y \in R^n$, let

$$[x, y] = \{tx + (1 - t)y \;:\; t \in [0, 1]\}.$$

For $A, B \subseteq R^n$, let

$$[A, B] = \bigcup_{a \in A, b \in B} [a, b].$$

For a set $X \subseteq R$, we denote by $\mathrm{conv}(X)$ the convex hull of X that is

$$\bigcup_{x < y \in X} [x, y].$$

If the underlying set of the structure we are considering is $\mathbb{R}$, $\mathrm{conv}(X)$ is an interval or a point. For $v, a \in R^n$, we denote by av the vector $(a_i v_i)_i$.

Let $\mathcal{M}$ be a structure in the language $\mathcal{L}$ with underlying set M. Let $(A_i)_{i \in I}$ be a sequence of sets in M^n for some n. We denote by $\langle \mathcal{M}, (A_i)_{i \in I} \rangle$ the structure generated by $\mathcal{M}$ and $(A_i)_{i \in I}$. Most of the time, the language in which we generate the structure will be obvious and we will not indicate it. A special case is the one of $\langle v_1, \ldots, v_n \rangle$, the vector space generated by $\{v_1, \ldots, v_n\}$ where $v_i \in M^m$.

We denote by $\mathbb{R}_{gp}$ and $\mathbb{R}_{sb}$ respectively $\langle \mathbb{R}, <, + \rangle$ and $\langle \mathbb{R}, <, +, \cdot_{\restriction [0,1]^2} \rangle$. For $i \leq n$, we denote by $\pi_i : R^n \to R$ the projection on the i-th coordinate. For $n \leq m$, we denote by $\Pi(n, m)$ the set of projections from $\mathbb{R}^m$ to $\mathbb{R}^n$.

For $X \subseteq R^{n+m}$ and A be a property. We say that there is a *uniform decomposition* of the X_x $(x \in R^n)$ into sets which have A if there are finitely many families $X_i \subseteq R^{n+m}$ such that for every $x \in R^n$, $X_x = \bigcup_i X_{i,x}$ and each $X_{i,x}$ has the property A.

Let $\mathcal{M}$ be a first order structure with underlying set M. Let $X \subseteq M$ be a set. We denote by $\mathrm{acl}_{\mathcal{M}}(X)$ the algebraic-closure of X in $\mathcal{M}$; that is, the set of points $y \in M$ that are contained in a finite set definable with parameters in X. We denote $\mathrm{acl}_{\mathcal{M}}(X)$ by $\mathrm{acl}(X)$ if no confusion could occurs.

Let $\mathcal{R}$ be an expansion of the real ordered additive group, we denote by $\mathbb{K}_{\mathcal{R}}$ the field of total linear functions definable in $\mathcal{R}$ (possibly using parameters; for example $\mathbb{K}_{\overline{\mathbb{R}}} = \overline{\mathbb{R}}$). Most of the time, we consider $\mathcal{R}$ such that $\mathbb{K}_{\mathcal{R}} = \langle \mathbb{Q}, +, \cdot \rangle$. Most of the time, the structure over $\mathbb{K}_{\mathcal{R}}$ will be obvious and we omit mentioning it. Ie we only mention the underlying set (for example $\mathbb{K}_{\mathcal{R}} = \mathbb{R}$ or $\mathbb{Q}$).

2. O-MINIMALITY

2.1 Definition

In this section we present the notion of o-minimality. We will not develop too much as the basis can be found in [10].

Definition 2.1.1. An ordered structure $\mathcal{M}$ is *o-minimal* if every definable set $X \subseteq M$ is a finite union of intervals and points.

Definition 2.1.2. We define recursively the notion of a *cell*.

We say that a set $X \subseteq M$ is a cell if it is an interval or a point.

For $X \subseteq M^{n+1}$, we say that it is a cell if:

(1) $\pi(X)$ is a cell, for π the projection on the first n coordinates.

(2) either there are some continuous definable functions $f_1, f_2 : \pi(X) \to M$ such that $X = \{(x, y) : f_1(x) < y < f_2(x)\}$, or there is a continuous definable function $f : \pi(X) \to M$ such that $X = \Gamma_f$.

We fix $\mathcal{M}$ an o-minimal structure.

The following theorem is the very core of what makes o-minimality such an important concept in modern model theory.

Theorem 2.1.3. Uniform cell decomposition theorem in o-minimal structures:

For any $\mathcal{M}$-definable family $\{X_x : x \in A\}$, there are finitely many $\mathcal{M}$-definable families $\{X_{i,x} : x \in A\}$ (for $i \leq m$) such that

- *For every $i \leq m$ and $x \in A$, $X_{i,x}$ is a cell,*
- *for every $x \in a$, $X_x = \bigsqcup_i X_{i,x}$.*

Remark 2.1.4. For every $\mathcal{M}' \equiv \mathcal{M}$, $\mathcal{M}'$ is o-minimal.

The notion of o-minimality is also preserved under taking reducts.

Example 2.1.5. The following structures are o-minimal:

- The real field, the real ordered additive group and the real ordered line are the typical examples of o-minimal structures.
- The real field with exponantiation (see [62]).
- The real field with a predicate for the graph of the restriction to $[0, 1]$ of any analytic function (see [16]).

We finish by just mentioning that in the o-minimal setting, the dimension has all the good properties we could hope for.

Fact 2.1.6. *(see [53, section 1]) Let $X, Y \subseteq M^n$ be some sets, definable in $\mathcal{M}$ and let $f : M^n \to M^m$ be an $\mathcal{M}$-definable function. Then:*

- $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$
- $\dim(\overline{X}) = \dim(X)$
- $\dim(f(X)) \leq \dim(X)$*, and the equality holds if f is injective.*

2.2 Structure theorem for the real field and the real ordered additive group

In this section, we present some consequences of the cell decomposition theorem and some quantifier elimination results for $\overline{\mathbb{R}}$ and $\langle \mathbb{R}, <, + \rangle$. Let $\mathcal{R} \equiv \overline{\mathbb{R}}$ with underlying set R.

Definition 2.2.1. We say that a set $X \subseteq R^n$ is an *algebraic variety* if it is the zero set of a polynomial.

We say that a function $f : X \subseteq R^n \to R^m$ is *algebraic* if its graph is an algebraic variety.

We say that a function $f : R^n \to R$ is *affine* if it is of the form $x \mapsto \sum_i \lambda_i x_i + b$. The restriction of an affine function to any domain is also called affine.

Let $\mathbb{K}$ be a field. We say that a set $X \subseteq R^n$ is an *affine variety (with coefficient in* $\mathbb{K}$*)* if it is the zero set of an affine map $\lambda_i x_i + b$ where $\lambda_i \in \mathbb{K}$.

A set $X \subseteq R^n$ is called *semialgebraic* if it is a finite union of sets of the form

$$\{x \in R^n \;:\; f_i(x) = 0,\, g_j(x) > 0 \text{ for } i \leq m,\, j \leq k\}$$

where $f_i : R^n \to R$ and $g_j : R^n \to R$ are algebraic.

A set is called *semilinear (with coefficient in* $\mathbb{K}$*)* if it is a finite union of sets of the form

$$\{x \in R^n \;:\; f_i(x) = 0,\, g_j(x) > 0 \text{ for } i \leq m \; j \leq k \;\}$$

where $f_i : R^n \to R$ and $g_j : R^n \to R$ are affine with coefficient in $\mathbb{K}$.

Fact 2.2.2. Structure theorem for sets definable in a real closed field, (see [10, Chapter 2]).

Let X be definable in $\mathcal{R} \equiv \overline{\mathbb{R}}$. Then X semialgebraic.

Fact 2.2.3. Structure theorem for ordered abelian groups (see [10, Chapter 1]): *Let $\mathcal{R} \equiv \langle \mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in I \subseteq \mathbb{R}} \rangle$ with underlying set R. Let X be definable in $\mathcal{R}$. Then there is a decomposition of X into finitely many semilinear sets with coefficients in the field generated by I.*

As a consequence, let $\{f_t : X_t \subseteq R^m \to R, :\ t \in C \subseteq R^k\}$ *be an* $\mathcal{R}$*- definable family of functions. Then, there are finitely many* $\mathcal{R}$*-definable sets* $Y_1, \ldots, Y_n$*, linear functions* $A_1, \ldots, A_n : R^n \to R$ *with coefficients in the field generated by* I *and families* $\{b_{(i,t) \in R}\}$ *such that*

$$f_{t \restriction Y_i}(x) = A_i(x) + b_{(i,t)}.$$

2.3 Semibounded structures

In this section we give the definition of a semibounded structure. We give some new characterization and two corollaries.

Definition 2.3.1. We say that an o-minimal structure $\mathcal{R}$ with underlying set R is semibounded if every unary function from R to R is ultimately affine. We say that a set is semibounded if it is definable in $\mathcal{R}$ (there will be no confusion).

Example 2.3.2. An example of a semibounded reduct of $\overline{\mathbb{R}}$ is the pure ordered group $\mathbb{R}_{gp}$, where every definable function is piecewise affine. One can expand $\mathbb{R}_{gp}$ by the restriction of $\cdot$ to a bounded domain (to get $\mathbb{R}_{sb}$), and still remain semibounded. Note that $\langle \mathbb{R}_{sb}, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}} \rangle$ is the unique structure properly between the real ordered vector space and $\overline{\mathbb{R}}$ ([52]).

For the rest of this section, we fix $\mathcal{R}$, an o-minimal expansion of an ordered abelian group with underlying set R.

Definition 2.3.3. We say that a continuous function is a *pole* if it is a bijection between a bounded and an unbounded interval.

We say that a set X is a *cone* if

$$X = B + \sum_{a \in (R^{>0})^l} \sum_i v_i a_i := \{x \in R^n \ :\ \exists b \in B,\, a \in (R^{>0})^l,\ x = b + \sum_i v_i a_i\}$$

where B is a bounded definable set, $v_1, \ldots, v_l \in \mathbb{K}^n_{\mathcal{R}}$ are linearly independent vectors over R and for every $x \in X$, there are unique $b \in B$ and $a \in (R^{>0})^l$ with $x = b + \sum_i v_i a_i$.

Fact 2.3.4. *(see [17, Fact 1.6]) The following are equivalent:*

1. $\mathcal{R}$ *is semibounded.*
2. *Every definable set is already definable in*

$$\langle R, <, +, \{B \subseteq R^n \ :\ B \text{ is definable and bounded}\}, (x \mapsto \lambda x)_{\lambda \in \mathbb{K}_{\mathcal{R}}} \rangle.$$

3. *There is no definable real closed field with domain* R *such that the ordering agrees with* $<$.
4. *There is no definable bijection from a bounded to an unbounded interval.*
5. *Every definable set can be decomposed into finitely many cones.*

As a consequence of Lovey and Peterzil's [43] we get the following fact.

Fact 2.3.5. *(see [17, Fact 1.8])*
If $\mathcal{R}$ is semibounded then it has QE and a universal axiomatization in the language

$$\{0, 1, +, -, <, \{B : B \text{ is definable in } \mathcal{R} \text{ and bounded}\}, (x \mapsto \lambda x)_{\lambda \in \mathbb{K}_{\mathcal{R}}}\}.$$

In the semibounded setting, there is another notion of dimension which characterizes the degree of unboundedness of a definable set.

Definition 2.3.6. Let B and $\{v_1, \ldots, v_l\}$ be as in Fact 2.3.4 and X be the cone $B + \sum_{a \in (R^{>0})^l} v_i a_i$. We define the *long-dimension of* X ($\mathrm{ldim}(X)$) to be l. For X a semibounded set, we define the long dimension of X to be the maximal long dimension of a cone contained in X.

We remark that this notion of dimension is preserved under affine bijections.

We say that an interval is short if it is possible to define a field structure on it.

Fact 2.3.7. *(see [18, Lemma 3.6]) For $\{X_t : t \in A\}$ a semibounded family such that A is bounded, we have*

$$\mathrm{ldim}(\bigcup_{t \in A} X_t) = \max\big(\mathrm{ldim}(X_t),\, t \in A\big).$$

Proposition 2.3.8. *Let $\{X_t \subseteq R^n : t \in A \subseteq R^k\}$ be a semibounded family such that*

$(*)$: there is $\rho \in R$ for every $t \in A$, $x, y \in X_t$,

$$[x, y] \subseteq X_t \Rightarrow d(x, y) < \rho.$$

Then $(**)$: there are $X_1, \ldots, X_m$ ($X_i = \{X_{i,t_i} : t_i \in A_i\}$) some semi bounded families such that for every $t \in A$ there is $(t_1, \ldots, t_m) \in \Pi_i A_i$ with $X_t = \bigcup_i X_{i,t_i}$. For every i, there is a definable family $\{B_t : t \in C \subseteq ¸R^k\}$ and a finite set of vectors $\{u_1, \ldots, u_l\}$ in R^n (possibly empty) such that:

- *there is a bounded definable set B, for every $t \in C$ $B_t \subseteq B$,*
- *C is bounded,*
- *for every $t \in A_i$ there are $t' \in C$ and $(y_1, \ldots, y_l) \in (R^{>0})^l$ with*

$$X_{i,t} = B_{t'} + \sum_j y_j v_j,$$

- *for every $(y_1, \ldots, y_l) \in (R^{>0})^l$ for every $t \in C$ there is $t' \in A_i$ such that*

$$X_{i,t} + \sum_j v_j y_j = X_{i,t'}.$$

Proof. Let
$$X = \bigcup_{t \in A} X_t \times \{t\} \subseteq R^{n+k}.$$
We do a recursion on the long-dimension l of X. For $l = 0$, X is a bounded set and the result follows easily. We assume that the result holds for $l - 1$ and that $\mathrm{ldim}(X) = l$.

By Fact 2.3.4, X is a union of finitely many cones $C_1, \ldots, C_k$. For all $t \in S$,
$$X_t \times \{t\} = R^n \times \{t\} \cap X = \bigcup_i X_t \cap \{t\} \cap C_i.$$
Without loss of generality, we may assume that X is a cone. So,
$$X = B + \sum_{a \in R^l} \sum_i v_i a_i,$$
where B is bounded and $v_1, \ldots, v_l \in R^{n+k}$ are some linearly independent vectors and for every $x \in X$ there are unique $a \in (R^{>0})^l$, $b \in B$ with $x = b + \sum_i v_i a_i$.

Let $\pi_n : R^{n+k} \to R^n$ and $\pi_k : R^{n+k} \to R^k$ be the projections on the first m coordinates and on the last k coordinates. Let $\langle v_1, \ldots, v_l \rangle$ be the R-vector space generated by $v_1, \ldots, v_l$.

Claim 2.3.9. *We have that*
$$\langle v_1, \ldots, v_l \rangle \cap R^n \times \{0\} = \{0\}.$$
Proof. If not then there is a line
$$d \subseteq \langle v_1, \ldots, v_l \rangle \cap R^n \times \{0\}$$
and for every $\alpha \in R^{n+k}$,
$$\alpha + d \subseteq \alpha + \langle v_1, \ldots, v_l \rangle.$$
Without loss of generality, we may assume that $0 \in B$. Let $a \in R^l$ such that
$$0 \neq \sum_i v_i a_i \in d$$
and for $c \in R^{>0}$ let $\alpha_c = \sum_i (c|a_i| + 1) v_i$. We have that for every $\lambda \in (0, c)$,
$$\alpha_c + \lambda \sum_i v_i a_i \in \sum_{b \in (R^{>0})^l} \sum_i v_i b_i,$$
thus
$$0 + \alpha_c + \lambda \sum_i v_i a_i \in X.$$

Moreover, since $d \subseteq R^n \times \{0\}$, and that $\lambda \sum v_i a_i \in d$, there is $t \in R^k$ (which is $\pi_k(\alpha_c)$) such that for every $\lambda \in (0, c)$,

$$\alpha_c + \lambda \sum_i v_i a_i \in R^n \times \{t\}$$

and since it is also in X,

$$\alpha_c + \lambda \sum_i v_i a_i \in X_t \times \{t\}.$$

Let $d_c = \{x \in R^{n+k} \,:\, \exists \lambda \in (0,c),\, x = \lambda \sum_i v_i a_i\}$. We observe that d_c is a piece of a line which lenght is proportional to c. We have that $\alpha_c + d_c \subseteq X_t \times \{t\}$. This is a contradiction with the existence of the bound ρ. □

Therefore there is g, a linear automorphism, such that $g(v_i) = e_{n+i}$ where $(e_1, \ldots, e_{n+k})$ is the canonical basis of R^{n+k} and $g_{\restriction R^n \times \{0\}} = Id$. We note that for $t \in A$,

$$g((X_t \times \{t\}) = g(R^n \times \{t\} \cap X) = g(R^n \times \{t\}) \cap g(X) = R^n \times \{t'\} \cap g(X)$$

for $t' = \pi_k(g((0,t)))$. We have that

$$g(X) = g(B) + \sum_{a \in (R^{>0})^l} \sum_i e_{n+i} a_i.$$

We set $B' = g(B)$ and we note $g(X_t \times \{t\}) = Y_{t'} \times \{t'\}$. We have that $g(X) = \bigcup_{t \in \pi_k(g(X))} Y_t \times \{t\}$ and for every $t \in \pi_k(g(X))$, $R^n \times \{t\} \cap g(X) = Y_t \times \{t\}$.

We define an equivalence relation on $\pi_k(B')$, $\sim$:

$$t \sim t' \text{ iff } \exists a \in R^l,\ R^n \times \{t\} = R^n \times \{t'\} + \sum_i e_{n+i} a_i.$$

For $\bar{t} \in \pi_k(B')/\sim$, we define $\mathcal{B}_{\bar{t}} = \bigcup_{t' \in \bar{t}} Y_{t'} \times \{t'\}$. By definition of $\sim$, for $\bar{t} \neq \overline{t'} \in \pi_k(B')/\sim$, $\pi_k(\mathcal{B}_{\bar{t}}) \cap \pi_k(\mathcal{B}_{\overline{t'}}) = \emptyset$. Observe that we may consider that $\pi_k(B')/\sim$ is a definable set by definable choice in o-minimal structures. Thus, identifying it with its set of representatives, $\pi_k(B')/\sim \subseteq \pi_k(B')$ and is bounded.

Let $\bar{t} \in \pi_k(B)/\sim$. For j, Let π_j be the projection on the j-th coordinates. We note that by definition of g and of $\sim$, we have for $j \geq l+1$ (if $l < k$), $\pi_{n+j}(\mathcal{B}_{\bar{t}})$ is a singleton $\{b_i\}$.

For $n+1 \leq i \leq n+l$, let $\alpha_i = \max(t_i \in \pi_i(\pi_k(\mathcal{B}_{\bar{t}}))) + 1$ and for $i > n+l$, let $\alpha_i = b_i$. Let $\alpha_{\bar{t}} = (\alpha_i)_i$. We note that $\alpha_{\bar{t}}$ exists because $\mathcal{B}_{\bar{t}}$ is bounded.

Claim 2.3.10. *For every $b \in \mathcal{B}_{\bar{t}}$ there is a unique $a \in (R^{>0})^l$ such that $b + \sum_i e_{n+i} a_i \in Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\}$.*

Proof. Let $\pi_n(b) = x$ and $\pi_k(b) = y = (y_1, \ldots, y_k)$. For $n+1 \leq i \leq n+k$, since $y_i < \alpha_i$, $\alpha_i - y_i > 0$. Thus $b + \sum_i (\alpha_i - y_i)e_{n+i} \in Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\}$. The uniqueness comes from the fact that if there were $a, a' \in (R^{>0})^l$ such that

$$b_1 = b + \sum_i e_{n+i}a_i,$$

$$b_2 = b + \sum_i e_{n+i}a'_i \in Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\},$$

$$b_1 - b_2 = \sum_i e_{n+i}(a_i - a'_i) \in R^n \times \{0\}.$$

This is a contradiction. □

Claim 2.3.11. *For every* $t' \in \alpha_{\bar{t}} + \sum_{a \in (R^{>0})^l} \sum_i e_i a_i$, $Y_{t'} = Y_{\alpha_{\bar{t}}}$.

Proof. Let $x \in Y_{t'}$. By definition of a cone and by definition of $\sim$ there is $b \in \mathcal{B}_{\bar{t}}$ such that $(x, t') = b + \sum_i e_{n+i}a_i$. By Claim 2.3.10, there is $y \in Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\}$ such that $y = b + \sum_i e_{n+i}a_i$. Thus

$$x = \pi_n(x, t') = \pi_n(b + \sum_i e_{n+i}a_i) = \pi_n(b) = \pi_n(y - \sum_i e_{n+i}a'_i) = \pi_n(y).$$

Moreover, there is $c \in (R^{>0})^l$ such that $\alpha_{\bar{t}} + \sum e_i c_i = t'$. Since

$$Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\} + \sum_i e_{n+i}c_i \subseteq Y_{t'} \times \{t'\},$$

$$Y_{\alpha_{\bar{t}}} = \pi_n(Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\}) + \pi_n(\sum_i e_{n+i}c_i) = \pi_n(Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\}) \subseteq \pi_n(Y_{t'} \times \{t'\}) = Y_{t'}.$$

□

Claim 2.3.12. *Let* $Y_{\bar{t}} = Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\} + \sum_{a \in (R^{>0})^l} \sum_i e_{n+i}a_i$. *Let* $Y = \bigcup_{\bar{t} \in \pi_k(B')/\sim} Y_{\bar{t}}$. *We have that* $\operatorname{ldim}(X \setminus Y) < l$

Proof. Since $\bigcup_{\bar{t} \in \pi_k(B')/\sim} \mathcal{B}_{\bar{t}}$ is bounded, by Fact 2.3.7, it is sufficient to prove, for a given $\bar{t} \in \pi_k(B')/\sim$ and a given $x \in \mathcal{B}_{\bar{t}}$ that

$$\operatorname{ldim}\left(x + \sum_{a \in (R^{>0})^l} \sum_i e_{n+i}a_i \setminus \left(Y_{\bar{t}}\right)\right) < l.$$

Let $\pi_n(x) = x_1$ and $\pi_k(x) = x_2$. Since $x + \sum_{a \in (R^{>0})^l} \sum_i e_{n+i}a_i \subseteq \{x_1\} \times R^k$ and that, by Claim 2.3.10, $x_1 \in Y_{\alpha_{\bar{t}}}$, we have that

$$\left(x + \sum_{a \in (R^{>0})^l} \sum_i e_{n+i}a_i\right) \cap \left(Y_{\bar{t}}\right)$$

$$= (x_1, \alpha_{\bar{t}}) + \sum_{a\in(R^{>0})^l} \sum_i e_{n+i}a_i.$$

Therefore

$$x + \sum_{a\in(R^{>0})^l} \sum_i e_{n+i}a_i \setminus Y_{\bar{t}}$$

$$= x + \sum_{a\in(R^{>0})^l} \sum_i e_{n+i}a_i \setminus \Big((x, \alpha_{\bar{t}}) + \sum_{a\in(R^{>0})^l} \sum_i e_{n+i}a_i\Big)$$

For $1 \leq i \leq l$, let I_i be the half closed interval $(0, \pi_i(\alpha_{\bar{t}} - x_2)]$ We have that

$$x + \sum_{a\in(R^{>0})^l} \sum_i e_{n+i}a_i \setminus Y_{\bar{t}}$$

$$= x + \bigcup_{1\leq i\leq l} \Big\{ \sum_j e_j y_j \, : \, (y_1, \ldots, y_l) \in (R^{>0})^{i-1} \times I_i \times (R^{>0})^{l-i}\Big\}$$

Since each of the $x + \{\sum_j e_j y_j \, : \, (y_1, \ldots, y_l) \in (R^{>0})^{i-1} \times I_i \times (R^{>0})^{l-i}\}$ are cones of long dimension $l - 1$, we have the result. □

The last thing we need to show is that $g^{-1}(Y)$ together with $\{\pi_n(g^{-1}(Y_{\alpha_{\bar{t}}} \times \alpha_{\bar{t}}) \, : \, \bar{t} \in \pi_k(B')/\sim\}$ and the family $\{\pi_n(v_1), \ldots, \pi_n(v_l)\}$ do satisfy the conditions we are looking for.

For $t \in \pi_k(g^{-1}(Y_{\bar{t}}))$, $X_t = \pi_n(g^{-1}(Y_{t'} \times \{t'\}))$ for some $t' \in \pi_k(Y_{\bar{t}})$. By Claim 2.3.11, for every $t' \in \pi_k(Y_{\bar{t}})$, there is $a \in (R^{>0})^l$ such that

$$Y_{t'} \times \{t'\} = Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\} + \sum_i e_{n+i}a_i.$$

Thus

$$X_t = \pi_n\big(g^{-1}(Y_{t'} \times \{t'\})\big) = \pi_n\big(g^{-1}(Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\} + \sum_i e_{n+i})a_i\big)$$
$$= \pi_n\big(g^{-1}(Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\})\big) + \sum_i \pi_n(v_i)a_i$$

and the first condition is fulfilled.

Moreover, we have that

$$g^{-1}(Y_{\bar{t}}) = g^{-1}(Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\}) + \sum_{a\in(R^{>0})^l} \sum_i v_i a_i.$$

Therefore, for every $a \in (R^{>0})^l$,

$$\pi_n(g^{-1}(Y_{\alpha_{\bar{t}}} \times \{\alpha_{\bar{t}}\})) + \pi_n(\sum_{a\in(R^{>0})^l} v_i a_i) = X_{t'}$$

for some $t' \in A$. Thus the second condition is fulfilled. We can then apply the induction hypothesis to $X \setminus g^{-1}(Y)$ to get the result. □

As a consequence of Proposition 2.3.8 we can state a new characterization of semiboundedness.

Theorem 2.3.13. *For an o-minimal expansion of an ordered group, the condition: "every definable family that satisfies* $(*)$ *of Proposition 2.3.8 satisfies also* $(**)$*" is equivalent to being semibounded.*

Proof. The second direction of the equivalence is Proposition 2.3.8. For the first direction, by contradiction we may assume that $\mathcal{R}$ expand a real closed field. Observing that the family $\{\{(x, x^2)\} : x \in R\}$ does not admit a decomposition of the form of $(**)$ we get a contradiction. □

Corollary 2.3.14. *Let* $\{X_t \subseteq R^n : t \in A \subseteq R^k\}$ *be a semibounded family such that there is a bounded interval* I*, for every* $t \in A$*,* $X_t \subseteq I^n$*. Then there is* $\{Y_t : t \in A' \subseteq I^m\}$ *a semi bounded family such that for every* $t \in A$ *there is* $t' \in A'$ *with* $X_t = Y_{t'}$*.*

Proof. Since for every $t \in A$, $X_t \subseteq I^n$, the condition of Proposition 2.3.8 is fulfilled. Thus, we may assume that there is a semibounded family $\{Y_t : t \in A'\}$ and vectors $v_1, \ldots, v_l \in R^n$ such that A' is bounded, there is $B \subseteq R^n$, a bounded set such that $Y_t \subseteq B$ for every $t \in A'$. Moreover, for every $t \in A$ there are $t' \in A'$ and $a \in R^l$ such that $X_t = \sum_i v_i a_i + Y_{t'}$ and for every $t \in A'$, $a \in (R^{>0})^l$ there is $t' \in A$

$$Y_t + \sum_i v_i a_i = X_t.$$

We show that $\{v_1, \ldots, v_l\} = \emptyset$. Let us assume for contradiction that this set is not empty. Without loss of generality we also assume that the v_i are linearly independent. Thus for $t \in A$ and $x \in X_t$ we have that

$$x + \sum_{\lambda \in (R^{>0})} \sum_i \lambda v_i \subseteq I^n$$

This is a contradiction.

Applying a linear automorphism and a translation, we may assume that $A' \subseteq I^k$ and this gives us the result.

□

Corollary 2.3.15. *If* $\{X_t \subseteq R^n : t \in A\}$ *is a semibounded family satisfying the condition of Proposition 2.3.8, then there is a bound on* $\{|X_t| : t \in A\}$*.*

Proof. By Proposition 2.3.8 we may assume that there are a bounded set $C \subseteq R^n$ and a family $\{Y_t \subseteq R^n : t \in A'\}$ such that A' is bounded, $Y_t \subseteq C$ for every $t \in A'$ and every X_a is a translate of an element of $\{Y_t\}$. The family $\{|Y_t| : t \in A'\}$ is bounded since C is and the result follows from the fact that if for $\alpha \in R^n$, $X_t = Y_{t'} + \alpha$ then $|X_t| = |Y_{t'}|$. □

2.4 Some basic facts from real algebraic geometry

In this section we recall some basic facts from real algebraic geometry. A standard reference is Bochnak-Coste-Roy [2]. This section will be useful only in Chapter 4.

Definition 2.4.1. The *Zariski closure* of a set $V \subseteq R^n$ is the intersection of every algebraic set containing V, denoted by $\overline{V}^{zar}$. Note that $\overline{V}^{zar}$ is algebraic, because $R[X_1, \ldots, X_n]$ is Noetherian.

Let V be an algebraic set. We say that V is *irreducible* if, whenever $V = V_1 \cup V_2$, with each V_i algebraic, we have $V = V_i$, for $i = 1$ or 2.

The *Krull-dimension* of V, denoted by $\dim_K(V)$, is the maximum n such that there is a sequence of irreducible algebraic sets $V_0 \subsetneq \ldots \subsetneq V_n \subseteq V$.

Fact 2.4.2. *Let X be a semialgebraic set. Then*

$$\dim(X) = \dim_K(X) = \dim(\overline{X}^{zar}).$$

Proof. For the first equality: by [2, Proposition 2.8.4], if $A \subseteq R^n$ is an open semialgebraic set, then $\dim_K(A) = n$. By [2, Proposition 2.8.7], the graph of a semialgebraic function has Krull-dimension equal to the Krull-dimension of its domain. By [2, Proposition 2.8.5], the Krull-dimension of a finite union of semialgebraic sets equals the sum of their Krull-dimensions. Now, let X be a semialgebraic set. By cell decomposition, X is a finite union of graphs of semialgebraic functions with open domains, and $\dim X$ equals the maximum of their dimensions. Combining the above three propositions, the first equality follows.

The second equality is by [2, Proposition 2.8.2]. □

Lemma 2.4.3. *Let $U \subseteq V \subseteq R^n$ be two algebraic sets of the same dimension. If V is irreducible, then $U = V$.*

Proof. By Fact 2.4.2, we have $\dim_K(U) = \dim_K(V)$, and hence the result follows from the definition of Krull-dimension. □

Definition 2.4.4. ([2, Definitions 2.9.3, 2.9.9]) A *Nash function* $f : X \subseteq R^n \to R^m$ is a semialgebraic smooth function with open domain (a function that is n-times differentiable for every $n \in \mathbb{N}$). A *Nash-diffeomorphism* $f : X \to Y$ is a Nash function that is a bijection and whose inverse is also Nash.

A semialgebraic set $V \subseteq R^m$ is a *Nash-submanifold of dimension d* if, for every $x \in V$, there is a Nash-diffeomorphism ϕ from an open semialgebraic neighborhood U of the origin in R^m onto an open semialgebraic neighborhood U' of x in R^m, such that $\phi(0) = x$ and $\phi((R^d \times \{0\}) \cap U) = V \cap U'$.

Note that the graph of a Nash function with an open domain is a Nash-submanifold.

We introduce the following notion.

Definition 2.4.5. Let $f : X \subseteq R^n \to R^m$, $g : Y \subseteq R^n \to R^m$ be two semialgebraic maps with open connected domains. We say that f *and* g *are Nash-concatenable*, if the set $A = \operatorname{int}(\overline{X} \cup \overline{Y})$ is connected, and there is a (unique) Nash function $h : A \to R^m$ that extends both f and g. We denote $h = f^\frown g$.

The uniqueness of the above map h follows from its continuity and the fact that $X \cup Y$ is dense in its domain A. Note also that since A is connected, $\Gamma(h)$ is a connected Nash-submanifold.

The key fact from [2] is the following.

Fact 2.4.6. *Let $V \subseteq R^m$ be a connected Nash-submanifold. Then $\overline{V}^{zar}$ is irreducible.*

Proof. See [2, Proposition 8.4.1]. □

Corollary 2.4.7. *Let f and g be two Nash-concatenable functions. Then*

$$\overline{\Gamma(f)}^{zar} = \overline{\Gamma(g)}^{zar}.$$

Proof. Since $\Gamma(f \frown g)$ is a connected Nash-submanifold, by Fact 2.4.6, $\overline{\Gamma(f \frown g)}^{zar}$ is irreducible. Note that $\dim \Gamma(f) = \dim \Gamma(f \frown g) = \dim \Gamma(g)$, and hence, by Lemma 2.4.2,

$$\dim \overline{\Gamma(f)}^{zar} = \dim \overline{\Gamma(f \frown g)}^{zar} = \dim \overline{\Gamma(g)}^{zar}.$$

By Lemma 2.4.3, $\overline{\Gamma(f)}^{zar} = \overline{\Gamma(f \frown g)}^{zar} = \overline{\Gamma(g)}^{zar}$. □

Corollary 2.4.8. *Let $f : X \subseteq R^n \to R$ be a Nash function with connected domain, and let $U \subseteq X$ be open. Then*

$$\overline{\Gamma(f)}^{zar} = \overline{\Gamma(f_{\restriction U})}^{zar}.$$

Proof. By Corollary 2.4.7, for $g = f_{\restriction U}$. □

3. D-MINIMALITY

In this chapter, we study d-minimal structures of the form $\langle \mathcal{R}, P \rangle$ where $\mathcal{R}$ is an o-minimal expansion of the real additive group and P is a subset of $\mathbb{R}$. We show that these expansions have some kind of weak form of model completeness. Namely (Theorems 3.3.16 and 3.3.7): for every definable set X, there is a family $\{X_t : t \in A\}$ which is definable in $\mathcal{R}$ and a set $S \subseteq A$ of dimension 0 such that $X = \bigcup_{t \in S} X_t$.

3.1 Basic tools

3.1.1 Definitions and examples

Let $\mathcal{M}$ be an ordered structure with underlying set M.

Definition 3.1.1. We say that a definable set $X \subseteq M$ is *definably connected* if there is no proper relatively clopen definable subset of X.

We say that $\mathcal{M}$ is *definably complete* (DC) if M is definably connected.

Remark 3.1.2. For $\mathcal{M}$, the property (DC) is equivalent to: Any bounded definable set $Y \subseteq M$ has a supremum.

Proof. See [45, Corollary 1.5]. □

Example 3.1.3. By cell decomposition, any o-minimal structure has (DC). In this case every cell is definably connected.

Any structure with underlying set $\mathbb{R}$ has (DC), since $\mathbb{R}$ is connected and thus definably connected.

Every structure that is elementarily equivalent to a structure whose underlying set is $\mathbb{R}$ has (DC) since being clopen is a first-order property.

The typical example of a structure which does not have (DC) is a T-convex field; that is, a non-standard real closed field with a predicate for a proper convex subring (see [12]).

Definition 3.1.4. We say that $\mathcal{M}$ is *d-minimal* if it is definably complete (DC) and for every $\phi(x, y)$ with for every $y \in \mathbb{R}^k$ $\phi_y \subseteq \mathbb{R}$, there is a natural number N_ϕ which depends only on ϕ, such that for all $y \in \mathbb{R}^k$, either ϕ_y has interior or it is the union of at most N_ϕ discrete sets.

Remark 3.1.5. We could have given a more general definition of d-minimality for any first-order topological structure (that is a topological structure such that a basis for the topology is definable) but we will not use this degree of generality.

Moreover, we only use the notion of d-minimality for expansions of o-minimal structures by a unary set with underlying set $\mathbb{R}$. See [25] or [26], if interested.

Assuming (DC) in the definition of d-minimality is not that usual (for example it is not the case in the papers of Miller (see for example [46], [28]). Since he is only considering expansions of $\mathbb{R}$, it was unecessary). We are taking this definition from Fornasiero (see [24]).

Remark 3.1.6. $\mathcal{M}$ is d-minimal if and only if it has (DC) and, for every definable family of sets of dimension 0, $\{X_x\}$, there is $N \in \mathbb{N}$ such that each X_x is the union of at most N discrete sets.

Example 3.1.7. Let $\mathcal{M}$ be an o-minimal expansion of the real ordered additive group. The following structures are d-minimal:

- $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ (see [13])
- $\langle \mathcal{M}, P \rangle$, where P is an iteration sequence as defined in [48], or a fast sequence as defined in [28].
- $\langle \mathbb{R}, <, +, \sin \rangle$ see [59].

Definition 3.1.8. A structure $\mathcal{M}$ is said to have *definable choice* if for every definable equivalence relation $\sim$ over a set A, there is a definable set $B \subseteq A$ of representatives for $\sim$. That is, for every $a \in A$ there is $b \in B$, $a \sim b$ and for every $b \neq b' \in B$, $b \nsim b'$.

A structure is said to have *the exchange property* if given a, b such that $a \in \mathrm{acl}(b) \setminus \mathrm{acl}(\emptyset)$ then $b \in \mathrm{acl}(a)$.

Fact 3.1.9. *(see [47]) A d-minimal expansion of $\mathbb{R}_{gp}$ has definable choice and the exchange property.*

Remark 3.1.10. The notion of d-minimality is preserved under taking reducts. Thus any reduct of Examples 3.1.7 is d-minimal. In particular, see [8] for a study of $\langle \mathbb{R}, <, +, f : 2^{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto tx \rangle$.

Unless specified, for the rest of this thesis, we fix an o-minimal expansion $\mathcal{R}$ of the real ordered additive group and fix a unary predicate $P \subseteq \mathbb{R}$ of dimension 0 such that $\langle \mathcal{R}, P \rangle$ is d-minimal. We denote by $\widetilde{\mathcal{R}}$ the structure $\langle \mathcal{R}, P \rangle$. We denote by $\mathcal{L}$ the language of $\mathcal{R}$. We say that a set X is $\mathcal{L}$-definable if it is definable in $\mathcal{R}$ and definable if it is definable in $\widetilde{\mathcal{R}}$.

Definition 3.1.11. We denote by $\widetilde{\mathcal{R}}^{\#}$ the structure $\widetilde{\mathcal{R}}$ together with a predicate for any subset of P^k for every $k \in \mathbb{N}$. In general, for a set $P' \subseteq \mathbb{R}$ of dimension 0 we denote by $\langle \mathcal{R}, P' \rangle^{\#}$ the structure $\mathcal{R}$ together with a predicate for any subset of P'^k for any k.

Fact 3.1.12. *(see [29]) The structure $\widetilde{\mathcal{R}}^{\#}$ is d-minimal.*

Definition 3.1.13. For a set $X \subseteq \mathbb{R}$, we denote by $\mathrm{conv}(X)$ the convex hull of X that is $\bigcup_{x<y\in X}[x, y]$. Since we are in $\mathbb{R}$, $\mathrm{conv}(X)$ is an interval or a point.

There are eight cases for the shape of $\mathrm{conv}(P)$ ($a < b \in \mathbb{R}$):

$$\mathbb{R},\ (a,+\infty),\ (-\infty,a), [a,+\infty),\ (-\infty,a],\ (a,b),\ [a,b),\ (a,b],$$

But, in the following, we only use two of these categories: the one where the convex hull of P is $\mathbb{R}$ (for example as in $\langle \mathbb{R},<,+,\mathbb{Z}\rangle$) and the one where it is not (for example as in $\langle \mathcal{R}, P'\rangle$, where P' is an iteration sequence as defined in [48]). In order to define the following function, by replacing P by $-P$ and by doing a translation, we assume in the second case that $0 < P$. We assume also that P is discrete and closed in its convex hull. We will not use it in settings where these conditions are not satisfied.

We define $\lambda : \mathbb{R} \to P \cup \{0\}$, $x \in \mathrm{conv}(P) \mapsto \max(P \cap (-\infty, x])$ (which exists since P is a discrete set and that it is closed in its convex hull) and $x \mapsto 0$ otherwise.

Since P is closed in its convex hull and discrete, its order type is $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$. Furthermore, we assume that P has no maximal element (and thus its order type is $\mathbb{N}$ or $\mathbb{Z}$). Let $A \in \{\mathbb{N}, \mathbb{Z}\}$ be the order type of P and let $\sigma : P \to A$ be the only growing bijection between A and P. We define $s : P \to P$ and $s^{-1} : s(P) \to P$ to be the successor functions on P and its inverse on $s(P)$. That is:

$$s(x) = \sigma(\sigma^{-1}(x) + 1).$$

3.1.2 Cantor-Bendixson-rank

In this section, we define the Cantor-Bendixson-rank of a set. It has first been designed by Cantor for his proof that Borel sets satisfy the Continuum Hypothesis. It has since been a powerful tool in descriptive set theory and it is of particular interest in the d-minimal setting. We will not often use it directly in the rest of this thesis but the next section (Section 3.2) presents the notion of “decomposition into special manifold”, which is, as a tool in the d-minimal setting, a sort of generalization of the Cantor-Bendixson rank.

Definition 3.1.14. Let $X \subseteq \mathbb{R}$.

Let $X^{(1)} = \{x \in X : x \text{ is isolated in } X\}$.

We define recursively $X^{(\alpha+1)} = (X \setminus X^{(\alpha)})^{(1)}$.

If β is a limit ordinal, $X^{(\beta)} = (X \setminus \bigcup_{\alpha<\beta} X^{(\alpha)})^{(1)}$.

We define the *Cantor-Bendixson rank of* X (CB-rank(X)) to be the minimum α such that $X^{(\alpha+1)} = \emptyset$ and ∞ if there is no such α.

Similarly, we say that a point $x \in X$ has CB-rank α over X (CB-rank$(x/X) = \alpha$) if $x \in X^{(\alpha)}$.

Remark 3.1.15. For completeness we gave the definition of the CB-rank for any ordinal α but as we shall see, in the d-minimal setting the CB-rank of definable sets is finite.

For some properties of the CB-rank, see [46]. In particular, that if X has dimension 0 and is definable in a d-minimal structure then it has finite CB-rank. Actually, we can reformulate the definition of d-minimality in terms of CB-rank.

Remark 3.1.16. An expansion of $\langle \mathbb{R}, < \rangle$, $\mathcal{M}$ is d-minimal if and only if for every definable family $\{X_x : x \in M^k\}$ such that for every $x \in \mathbb{R}^k$ $X_x \subseteq \mathbb{R}$ and has dimension 0, there is a uniform finite bound over x on the CB-rank of X_x.

Remark 3.1.17. We note that given a definable set $X \subseteq \mathbb{R}$ of dimension 0 for every $n \in \mathbb{N}$, it follows from the definition of the Cantor-Bendixson rank that $X^{(n)}$ is definable and discrete. Thus, for any definable family of sets of dimension 0, there is a uniform decomposition of each of its elements into discrete sets.

The following lemma will be useful in Chapter 4. It is a direct consequence of the results of Hieronymi in [34].

Lemma 3.1.18. *We assume that $\widetilde{\mathcal{R}}$ expands $\overline{\mathbb{R}}$. Let $\{X_t : t \in S\}$ be a definable family of intervals in $\mathbb{R}$ such that $\dim(S) = 0$ and $\overline{\bigcup_{t\in S} X_t} = [0, \infty)$. Then $\{|X_t| : t \in S\}$ is unbounded.*

Proof. Let Y be the set of left and right endpoints of the X_t's and let $X = \overline{Y}$. Let N be the Cantor-Bendixson rank of X. We define

$$X_1 = X^{(N)}.$$

We observe that X_1 has Cantor-Bendixson rank 1 and since X is closed, X_1 is closed. We define recursively X_n to be

$$X_{n-1} \cup \left(X^{(N-n+1)} \setminus \left(\bigcup_{x \in X_{n-1}} (x-1, x+1) \right) \right).$$

We observe that $CB\text{-rank}(X_n) = 1$ and that X_n is closed. It is direct to show that

$$X \subset \bigcup_{x \in X_{N-1}} [x-1, x+1]$$

and thus, for $y \in [0, \infty)$, if $d(y, X_{N-1}) > 1$ then $y \notin X$. is unbounded.
First, we observe that since $X \subset \bigcup_{x\in X_{N-1}} [x-1, x+1]$, X_{N-1} is unbounded. Moreover, it is closed and without loss of generality, we may assume that it is the range of a strictly increasing unbounded sequence $(x_n)_{n\in\mathbb{N}}$. If Z was bounded by a, we would have that

$$1 = \lim_n \frac{x_n}{x_n} \leq \lim_n \frac{x_{n+1}}{x_n} \leq \lim_n \frac{x_n + a}{x_n} = 1.$$

Since $\widetilde{\mathcal{R}}$ defines $\overline{\mathbb{R}}$, by [34], $\widetilde{\mathcal{R}}$ would define $\mathbb{Z}$ and thus the whole projective hierarchy (see [40]), which is a contradiction with d-minimality.

For $a > 0 \in \mathbb{R}$, let $x \in X_{N-1}$ such that $|x - s(x)| > 2 + a$ and $y \in (x, s(x))$ such that

$$d(y, x) > 1 + a/2 < d(y, s(x)).$$

Therefore $y \notin X$ and since $\bigcup_t X_t \cup X = [0, \infty)$, there is $t \in S$ with $y \in X_t$. We have that $d(y, X) > a/2$ and since the endpoints of X_t are in X, $|X_t| > a$. □

3.1.3 Sparseness

The d-minimal structures of the form $\langle \mathcal{R}, P\rangle$ are part of a larger interesting category of structures.

Definition 3.1.19. We say that $P' \subseteq \mathbb{R}$ is *sparse for* $\mathcal{R}$ (we will just call it sparse if no confusion could occur) if for every $\mathcal{L}$-definable function $f : \mathbb{R}^k \to \mathbb{R}$, $f(P'^k)$ has dimension 0. In such a case we say that $\langle \mathcal{R}, P'\rangle$ is sparse.

The following theorem from Friedman and Miller is a central result for the rest of this thesis.

Fact 3.1.20. *(see [29] last claim of the proof of Theorem A) Let $\langle \mathcal{R}, P'\rangle$ be sparse. Let $A \subseteq \mathbb{R}^{n+1}$ be definable in $\langle \mathcal{R}, P'\rangle^{\#}$ such that for every $x \in \mathbb{R}^n$, A_x has no interior. Then there is an $\mathcal{L}$-definable function $f : \mathbb{R}^{m+n} \to \mathbb{R}$ such that for every $x \in \mathbb{R}^n$,*

$$A_x \subseteq \overline{f(P'^m \times \{x\})}.$$

Remark 3.1.21. It is easy to see that if $P' \subseteq \mathbb{R}$ is countable then P' is sparse. Thus $\widetilde{\mathcal{R}}$ is sparse since a finite union of discrete sets is countable.

3.1.4 Dimension

In this subsection, we just establish an important property of the dimension for the sets definable in $\widetilde{\mathcal{R}}$. Note that this result can be found in Miller's [46, Theorem 3.1] but we take this occasion to develop the proof.

Definition 3.1.22. We say that an ordered structure $\mathcal{M}$ has the *dimension property* (DIM) if for every definable family $\{X_t\}_{t\in S}$, with $\dim S = 0$, we have

$$\dim\big(\overline{\bigcup_{t\in S} X_t}\big) = \max_{t\in S} \dim X_t.$$

Proposition 3.1.23. *The structures $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{R}}^{\#}$ have (DIM).*

Remark 3.1.24. Note that the only things we need for the proof is that these structures are d-minimal and the underlying set of both of these structures is $\mathbb{R}$ (since we use mostly the Baire category theorem and separability). We could probably adapt the proof in a more general setting using [36] but this is none of our concern here. Of course, the property (DIM) is a first order property so any model of $Th(\widetilde{\mathcal{R}})$ has it.

Proof. Let $\{X_t \subseteq \mathbb{R}^m : t \in S\}$ be a definable family of sets with $\dim(S) = 0$. Thus, since S is a finite union of discrete sets and a discrete set is countable, S is countable. Let $X = \overline{\bigcup_{t\in S} X_t}$. We assume that $\dim(X) = n$ and $\max_t(\dim(X_t)) = d$. We do a double induction on $d \leq n$ to prove:

$(*)$: if $d < n$ no definable set of dimension d is dense somewhere.

For any n if $d = 0$ the result follows from d-minimality.

We assume that the result holds for n and $d - 1 < n - 1$ and we prove it for n and d. We assume that there is Y, a definable set of dimension d such that

$\overline{Y}$ has interior. Let $\pi \in \Pi(d,n)$ such that $\pi(X_t)$ has interior. We may assume that π is the projection on the first d coordinates. We first show that

$$\left\{x \in \mathbb{R}^d :\ \pi^{-1}(x) \cap X_t \text{ has dimension bigger than zero}\right\} \text{ is nowhere dense.}$$

If not, by induction hypothesis, there is a box $B \subseteq \mathbb{R}^d$ such that for every $x \in B$, $\pi^{-1}(x) \cap Y$ has dimension bigger than zero. Let μ be the projection on the $d+1$ coordinate. We may assume that

$$\mu(\pi^{-1}(B) \cap Y) \text{ has interior.}$$

For every $a \in \mathbb{Q}$, let

$$Z_a = \{x \in B :\ a \in \text{int}\left(\mu(\pi^{-1}(x) \cap Y)\right)\}.$$

By density of $\mathbb{Q}$ and by assumption on B,

$$\bigcup_{a \in \mathbb{Q}} Z_a = B$$

and by the Baire Category Theorem, there is $a \in \mathbb{Q}$ such that Z_a is dense somewhere. By the induction hypothesis, it must have interior and we may assume that $a \in \text{int}\left(\mu(\pi^{-1}(x) \cap Y)\right)$ for every $x \in B$. For every $x \in B$, there is $b_x \in \mathbb{R}$ such that

$$(a, b_x) \subseteq \mu(\pi^{-1}(x) \cap Y).$$

Let

$$X_n = \{x \in B :\ b_x > a + 1/n\}.$$

By assumption on B, and by the fact that $(a + 1/n)_n \to a$,

$$\bigcup_{n \in \mathbb{N}} X_n = B.$$

Thus, by the Baire Category Theorem, there is $n \in \mathbb{N}$ such that X_n is dense in B and by the induction hypothesis, it must have interior and contains some box B'. Thus $B' \times (a, a + 1/N)$ is a subset of the projection of $\pi^{-1}(B') \cap Y$ on the first $d+1$ coordinates. This is a contradiction with the fact that $\dim(Y) = d$ and we have the result.

We return to the proof of the proposition. By projecting on some n coordinates, we may assume that $n = m$, thus X has interior. By the Baire Category Theorem, some of the X_t's must be dense somewhere and by $(*)$, X_t has interior and $d = n$. □

Definition 3.1.25. A set S is said to be *small* if it is definable and has dimension 0.

Let $X \subseteq Y$ be two definable sets. We say that X *is large in* Y *if* $\dim(Y \backslash X) < \dim(Y)$.

3.2 Decomposition into Special Manifolds

This section is just a presentation of Thamrongtanyalak's [58, Theorem B].

Definition 3.2.1. We say that a set $X \subseteq \mathbb{R}^m$ is a π-*special manifold* (π-SM) for $\pi \in \Pi(n,m)$ if:

1. $\pi(X)$ is open,
2. for every box $U \subseteq \pi(X)$, π maps homeomorphically each connected component of $\pi^{-1}(U) \cap X$ onto U,
3. for every $x \in \pi(X)$, $\pi^{-1}(x) \cap X$ is discrete.

We say that a set X is a *special manifold* (SM) if it is a π-special manifold for some projection π.

Let $\pi \in \Pi(n,m)$. Let $X \subseteq \mathbb{R}^m$ be a set of dimension n. We say that $X_1, \ldots, X_k, X_\omega$ is a π-*maximal decomposition into special manifolds* if

- each X_i is a π-SM for $i \leq k$,
- $\dim(\pi(X_\omega)) < n$.
- $\bigcup_i X_i \cup X_\omega = X$.

We say that a structure $\mathcal{M}$ has *the decomposition into finitely many special manifolds property (DSM)* if for every definable set $X \subseteq \mathbb{R}^{m+n}$, there is N such that for every $a \in \mathbb{R}^m$, there is a decomposition of X_a into at most N SM.

Remark 3.2.2. DSM implies d-minimality since an SM of dimension 0 is a discrete set.

Definition 3.2.3. Let $\mathcal{A}$ be a finite collection of subsets of $\mathbb{R}^n$ and $\mathcal{P}$ be a partition of $\mathbb{R}^n$. We say that $\mathcal{P}$ is *compatible with* $\mathcal{A}$ if every $A \in \mathcal{A}$ is a union of sets in $\mathcal{P}$.

Theorem 3.2.4. *(see [58, Theorem B]) Let $\mathcal{M}$ be a d-minimal expansion of $\overline{\mathbb{R}}$. Let $\mathcal{A}$ be a finite collection of subsets of $\mathbb{R}^n$ definable in $\mathcal{M}$. Then there is π-maximal decomposition of $\mathbb{R}^n$ into SM $X_1, \ldots, X_m$ compatible with $\mathcal{A}$ such that for each i, the frontier of X_i is a finite union of elements of $\mathcal{P}$.*

3.3 Decomposition into $\mathcal{L}$-Embedded Manifolds

In this section, we strengthen Theorem 3.2.4 by showing a decomposition into finitely many $\mathcal{L}$-EM (see Definition 3.3.1 and Theorems 3.3.7 and 3.3.16 below). In the d-minimal setting, that is a mix between what was announced in the introduction of this chapter and the decomposition into finitely many special manifolds of Thamrongthanyalak (Theorem 3.2.4). That is: an $\mathcal{L}$-EM is a special manifold such that the family of its connected components is definable and is even a small subfamily of an $\mathcal{L}$-definable family.

3.3.1 Definition and setting

Definition 3.3.1. Let X be definable and let $\{X_t :\ t \in A\}$ be an $\mathcal{L}$-definable family. For $S \subseteq A$, a small set, we say that $\{X_t :\ t \in S\}$ is an *$\mathcal{L}$-decomposition of* X if $X = \bigcup_{t \in S} X_t$.

A definable set X is called an *$\mathcal{L}$-embedded manifold* ($\mathcal{L}$-EM) if there is $\{X_t :\ t \in S\}$, an $\mathcal{L}$-decomposition of X such that:

1. there is a projection π, for every $t \in S$, X_t is the graph of a continuous function over $\pi(X_t)$ and $\pi(X_t)$ is open and connected.

2. For every $t, t' \in S$, either $\pi(X_t) = \pi(X_{t'})$ or $\pi(X_t) \cap \pi(X'_t) = \emptyset$.

For every $x \in \pi(X)$, $\dim(\pi^{-1}(x) \cap X) = 0$. As a convention, when talking about an $\mathcal{L}$-deomposition of an $\mathcal{L}$-EM, we always mean that this decomposition satisfies 1 and 2.

We say that $\widetilde{\mathcal{R}}$ has *the decomposition into finitely many $\mathcal{L}$-EM property* ($\mathcal{L}$-DEM) if every set definable in $\widetilde{\mathcal{R}}$ admits a decomposition into finitely many $\mathcal{L}$-EM.

The goal of this section is to show $\mathcal{L}$-DEM for $\widetilde{\mathcal{R}}^{\#}$ (Theorems 3.3.7 and 3.3.16).

Let

3.3.2 Expansions of the real field

In this section we prove $\mathcal{L}$-DEM for d-minimal expansions of the real field by a unary set.

We first need to define a new topology, the Hausdorff topology.

Definition 3.3.2. We denote by $\mathcal{K}(\mathbb{R}^n)$ the collection of compact subsets of $\mathbb{R}^n$.

The *Hausdorff distance* on $\mathcal{K}(\mathbb{R}^n)$ is defined by

$$d_H(X, Y) = \sup\{d(x, Y), d(X, y) :\ x \in X, y \in Y\}.$$

It is easy to see that d_H is a metric and it gives us a topology on $\mathcal{K}(\mathbb{R}^n)$, the *Hausdorff topology.* Let (X_n) be a sequence of compact sets which is converging in the Hausdorff topology. We denote by $\lim_H(X_n)$ its limit. For a family of compact sets $\mathcal{C}$, we denote by $\overline{\mathcal{C}}^H$ the closure of $\mathcal{C}$ in the Hausdorff topology.

The following theorem is the technical key to prove $\mathcal{L}$-DEM for d-minimal expansions of the real field by a unary predicate. We thank Erik Walsberg for bringing it to our attention.

Theorem 3.3.3. *(see [15] for an overview of the results on this topic) Let $\mathcal{C}$ be an $\mathcal{L}$-definable family of compact sets contained in $\mathbb{R}^n$. Then*

$$\{Y \in \mathcal{K}(\mathbb{R}^n) :\ Y \in \overline{\mathcal{C}}^H\}$$

is $\mathcal{L}$-definable.

Fact 3.3.4. *(see [42, Theorem 3]) Let Y_j be a sequence in $\mathcal{K}(\mathbb{R}^n)$ and $A \in \mathcal{K}(\mathbb{R}^n)$ such that $\bigcup_j Y_j$ is contained in a compact set. Then (Y_j) converge to A in the Hausdorff topology if and only if the following conditions are satisfied:*

1) Let $y_i \in Y_i$. If (y_i) converge to y then $y \in A$.

2) If $y \in A$ then there is a sequence $(y_i)_i$ with $y_i \in Y_i$ such that $\lim_i y_i = y$.

Fact 3.3.5. *(Dini's Theorem, see for example [54, Theorem 7.13]) Let $\{f_n\}$ be a monotically increasing sequence (that is that for every n, $\mathrm{dom}(f_n) = \mathrm{dom}(f_{n+1})$ and for every $x \in \mathrm{dom}(f_n)$ $f_n(x) < f_{n+1}(x)$) of continuous functions over a compact domain which converges pointwise to a continuous function then the convergence is uniform. The result holds, of course, for a decreasing sequence satisfying the same conditions.*

Corollary 3.3.6. *Let $\{Y_j : j \in \mathbb{N}\}$ be a small subfamily of an $\mathcal{L}$-definable family of graphs of continuous functions $\{f_j : Z \to \mathbb{R} : j \in \mathbb{N}\}$ such that:*

- *Z is an open cell.*
- *For every $i < j$, for every $x \in Z$, $f_i(x) > f_j(x)$.*
- *$\bigcup_j Y_j$ is contained in a compact set.*

Let $A = \{\lim_j(z, f_j(z)) : z \in Z\}$. We assume that A is the graph of a continuous function $f : Z \to \mathbb{R}$. Then

$$\lim_H(\{\overline{Y_j}\}) \cap \pi^{-1}(Z) = A.$$

Proof. First of all, by definition A satisfies the second condition of Fact 3.3.4.

Let $y_j \in Y_j$ such that (y_j) converges to $y \in \pi^{-1}(Z)$. There is a bounded and closed box $B \subseteq Z$ and $N \in \mathbb{N}$ such that for every $j > N$, $\pi(y_j) \in B$. By Fact 3.3.5, the sequence $\{f_j \restriction B : j \in \mathbb{N}\}$ converges uniformly to $f_{\restriction B}$ and thus, $y \in A$. □

Theorem 3.3.7. *Let $X = \{X_x \subseteq \mathbb{R}^m : x \in A \subseteq \mathbb{R}^p\}$ be a family of sets of dimension n definable in $\widetilde{\mathcal{R}}^{\#}$. Then there is a uniform decomposition of the X_x's into finitely many $\mathcal{L}$-embedded manifolds.*

Proof. We are doing a double induction on $n \leq m$. For $m = 0$ or for any m and $n = 0$, the result is obvious. We assume that the result holds for n and $m - 1$. We also assume that the result holds for $n - 1 < m - 1$ and m and prove it for n and m. We deal with the case $n < m - 1$, $n = m - 1$ and $n = m$ separately.

We first prove the following claims:

Claim 3.3.8. *Let Z be a definable set of dimension n and let $\{Z_t : t \in S\}$ be an $\mathcal{L}$-decomposition of Z. Then there is a decomposition of Z into finitely many $\mathcal{L}$-EM.*

Proof. Let π be a projection such that for every $t \in S$, Z_t is the graph of a continuous function over $\pi(Z_t)$ (that exists since the Z_t's are cells). By d-minimality, we may assume that for every $x \in \pi(Z)$, $\pi^{-1}(x) \cap Z$ is discrete. By definable choice, there is $\gamma : Z \to S$, $x \mapsto t$ such that $x \in Z_t$. For $t \in S$ let $Z'_t = \gamma^{-1}(t)$. Note that the Z'_t's are no longer definable in $\mathcal{R}$ but $Z'_t \subseteq Z_t$. By uniform cell decomposition, we may assume that the Z_t's are cells and thus are homeomorphic via π to some open cell in $\mathbb{R}^n$. By induction, there is an uniform decomposition of the Z'_t's into $\mathcal{L}$-EM. For $t \in S$, let Z''_t be the union of the $\mathcal{L}$-EM of dimension n in its decomposition. For $t \in S$ let

$$W_t = \mathrm{bd}(\pi(Z''_t)) \text{ and let } W = \overline{\bigcup_{t \in S} W_t}.$$

Note that, by Proposition 3.1.23, $\dim(W) < n$. By induction there is a decomposition of $\mathbb{R}^n \setminus W$ into $\mathcal{L}$-EM $W_1, \ldots, W_s$ of the form $W_i = \bigcup_{t \in S_i} W_{i,t}$, where there is family $\{W_{i,t} : t \in A_i\}$, definable in $\mathcal{R}$, such that $S_i \subseteq A_i$ and S_i is small.

Let i such that $\dim(W_i) = n$ and let $t \in S_i$. Observe that, by definition of W, $\pi^{-1}(W_{i,t}) \cap Z$ is composed of connected components which are graphs of continuous functions over $W_{i,t}$. Thus, if $W_1, \ldots, W_s$ are the $\mathcal{L}$-EM of dimension n in the decomposition of $\mathbb{R}^n \setminus W$, $\pi^{-1}(W_i) \cap Z$ is an $\mathcal{L}$-EM.

By definition of $W_1, \ldots, W_s$, $\dim(\mathbb{R}^n \setminus \bigcup_{i \leq s} W_i) < n$ and we apply the induction hypothesis to $\pi^{-1}\left(\mathbb{R}^n \setminus \bigcup_{i \leq s} W_i\right) \cap Z$ to get the result. □

Claim 3.3.9. *Let $\{Y_x : x \in A\}$ be a definable family of sets of dimension $s < m$ such that for every $x \in A$ $X_x \subseteq Y_x$. If there is a uniform decomposition of the Y_x's into $\mathcal{L}$-EM then there is such a decomposition for the X_x's.*

Proof. Let $\{Y_{x,t} : t \in S_x\}$ be an $\mathcal{L}$-decomposition of Y_x. Since all the $Y_{x,t}$'s are cells, they are uniformly homeomorphic to some open set via some projection. Thus by applying the induction hypothesis to $Y_{x,t} \cap X_x$ we get some $\mathcal{L}$-decomposition $A_{x,t} = \{A_{x,t,t'} : t' \in S_{x,t}\}$ of $Y_{x,t} \cap X_x$. Thus $\{A_{x,t,t'} : t \in S_x, t' \in S_{x,t}\}$ is an $\mathcal{L}$-decomposition of X_x and we have the result by Claim 3.3.8. □

Case** $n < m - 1$**:

Let π be the projection on the first $m - 1$ coordinates. By the induction hypothesis, there is a decomposition of $\pi(X_x)$ into $\mathcal{L}$-EM. Thus we may assume that $\pi(X_x)$ is an $\mathcal{L}$-EM of dimension n of the form $\bigcup_{t \in S_x} Y_{x,t}$. It is easy to see that $\bigcup_{t \in S_x} \pi^{-1}(Y_{x,t})$ is an $\mathcal{L}$-EM of dimension $n + 1 < m$ and that it contains X_x. We apply Claim 3.3.9 to get the result.

Case** $n = m - 1$**:

Since for every $x \in A$, $y \in \mathbb{R}^n$, $\pi^{-1}(y) \cap X_x$ has dimension 0, By Fact 3.1.20 there is a function $f : \mathbb{R}^{p+n+k} \to \mathbb{R}$, definable in $\mathcal{R}$, such that

$$\pi^{-1}(y) \cap X_x \subseteq \overline{f(x, y, P^k)}.$$

By o-minimality, we may assume that f is continuous (thus we do not have anymore that $\mathrm{dom}(f_t) = \mathbb{R}^n$).

From now, to simplify the notations we do not mention the x in index for $x \in A$. Keep in mind that the following construction can be made uniformly for $x \in A$.

For every $t \in P^k$ let $Y_t = \Gamma(f_t)$ and $Y = \overline{\bigcup_{t \in P^k} Y_t}$. Note that by Proposition 3.1.23, $\dim(Y) = n$ and by Claim 3.3.9, it is sufficient to prove that Y admits a decomposition into $\mathcal{L}$-EM.

Thus, it is sufficient to prove that Y admits an $\mathcal{L}$-decomposition. Let $Z = \bigcup_{t \in S} Y_t$, by Claim 3.3.8, there is a decomposition of Z into finitely many $\mathcal{L}$-EM. Thus, we may assume that Z is an $\mathcal{L}$-EM and let $\{W_t : t \in S'\}$ be a small subfamily of a family definable in $\mathcal{R}$ such that for every $t \in S$ there is $t' \in S'$, $\pi(Y_t) = W_{t'}$. Moreover, we may assume that $Y \cap \pi^{-1}(\bigcup_{t \in S'} W_t)$ has dimension 0.

Claim 3.3.10. *Let $A, B \subseteq Z$ be two graphs of continuous functions $g, h : W_t \to \mathbb{R}$ (for some $t \in S'$). If there is $x \in W_t$ such that $g(x) < h(x)$ then for every $x \in W_t$ $g(x) < h(x)$.*

Proof. We assume that it is not the case. By the Intermediate Value Theorem, there is $x \in Z$ such that $g(x) = h(x)$ and $A \cap B \neq \emptyset$. That contradicts the definition of an $\mathcal{L}$-EM. □

From now, in order to simplify the notations, we fix $W = W_t$ for some $t \in S'$.

Notation 3.3.11. *In the setting of Claim 3.3.10, if there are $A, B \subseteq Z$, two graphs of continuous functions over W such that there is $z \in W$ with $\pi_m(\pi^{-1}(z) \cap A) < \pi_m(\pi^{-1}(z) \cap B)$ then this is true for every $z \in W$ and we denote this fact by $A < B$.*

We prove that there is small subfamily $\{A_t : t \in S'\}$ of a family of cells of dimension n definable in $\mathcal{R}$ such that $X \subseteq \bigcup_{t \in S'} A_t$. Thus applying Claim 3.3.8, we get the result.

from now on, by replacing Y by $\rho(Y)$, where ρ is the homeomorphism

$$\rho : \mathbb{R}^m \to \mathbb{R}^m$$

$$\rho : x \mapsto \big(x_i/(1 + |x_i|)\big)_i,$$

we may assume that Y is bounded. Note that ρ preserves the $\mathcal{L}$-EM and the fact that for every $x \in \pi(Y)$, $\pi^{-1}(x) \cap Y$ has dimension 0.

Claim 3.3.12. *Let $(Y_{t_i})_i$ be a decreasing sequence (of graphs of functions over W) and let A be its pointwise limit (which exists because Y is bounded). Then:*

1. *A is the graph of a continuous function,*
2. *$\overline{\bigcup_i Y_{t_i}} \cap \pi^{-1}(W) \setminus \bigcup_i Y_{t_i} = A$,*

3. $\lim_H(\overline{Y_{t_i}}) = \overline{A}$.

Proof. For the first property, let $x \in W$ and let $a, b \in \overline{A} \cap \pi^{-1}(x)$. We show that $a = b$. Since $(Y_{t_i})_i$ is a decreasing sequence of graphs of continuous functions, by the Intermediate Value Property, $[a, b] \subseteq \overline{\bigcup_i Y_{t_i}} \subseteq Y$. If $a \neq b$, that is a contradiction with the fact that $\dim(\pi^{-1}(x) \cap Y) = 0$. The second property is obvious and the third is just a consequence of Corollary 3.3.6. □

Let

$$H = \overline{\{\overline{Y_t \cap \pi^{-1}(W)} : \ t \in \mathbb{R}^k\}}^H .$$

By Fact 3.3.3, H is a family definable in $\mathcal{R}$, thus it has the form $\{X_a : \ a \in B\}$. Let

$H_P = \{a \in B : \ X_a \cap \pi^{-1}(W) \subseteq Y$ and is the graph of a continuous function over $Z\}$.

Since, by Claim 3.3.12 for $a \in H_P$, $X_a \cap \pi^{-1}(W)$ is uniquely determined by $\pi^{-1}(z) \cap X_a$ for any $z \in W$ and that by assumption, $\pi^{-1}(z)$ has dimension 0, we may assume that H_P has dimension 0. Thus, we have

$$Y \cap \pi^{-1}(W) = \bigcup_{a \in H_P} X_a \cap \pi^{-1}(W)$$

and we have the result for $\pi^{-1}(W) \cap Y$.

It is easy to see that we can do this uniformly $t \in S'$ and $Y \cap \bigcup_{t \in S'} \pi^{-1}(W_t)$ has a decomposition into $\mathcal{L}$-EM.

case $\underline{m = n}$**:**

We assume that $X = \text{int}(X)$. By induction, there is a decomposition of $\text{bd}(X)$ into $\mathcal{L}$-EM. Let $s \in \mathbb{N}$ such that for $i \leq s$, $\bigcup_{t \in S^i} Z_t^i$ is an $\mathcal{L}$-EM of dimension $m - 1$ coming from the family $\{Z_t^i : \ t \in A_i\}$ that is definable in $\mathcal{R}$ and

$$\dim\left(\text{bd}(X) \setminus \bigcup_i Z^i\right) < n - 1.$$

We first need the following claim:

Claim 3.3.13. *Let Z be a definable set of dimension $s < m$. Then there is a decomposition of Z into finitely many $\mathcal{L}$-EM $Z_1, \ldots, Z_a$ such that, for $\{Z_{i,t} : t \in S_i\}$ the $\mathcal{L}$-decomposition of Z_i and π the projection on the first $m-1$-coordinates:*

$(*)$: *for every $i, j, t \in S_i, t' \in S_j$ $\pi(Z_{i,t}) \cap \pi(Z_{j,t'})$ either is empty or is equal.*

Proof. Let $W = \overline{\bigcup_{i,t \in S_i} \text{bd}(\pi(Z_{i,t}))}$. Observe that W has dimension lower than $m-1$. Let $W_1, \ldots, W_l$ be some decomposition into $\mathcal{L}$-EM of $\mathbb{R}^{m-1} \setminus W$. Then for $\{W_{i,t} : \ t \in S_i'\}$ an $\mathcal{L}$-decomposition of W_i, the family $\{\pi^{-1}(W_{i,t} \cap Z_{j,t'}) : i, j, t \in S_i', t' \in S_j\}$ satisfies $(*)$. We deal with $\pi^{-1}(W) \cap Z$ easily by induction. □

Thus, we may assume that the Z_i's satisfies $(*)$ and let $W_1, \dots, W_j$ be an $\mathcal{L}$-decomposition of $\mathbb{R}^{m-1}$ that witnesses it and we may assume that $\bigcup_i W_i = \mathbb{R}^{m-1}$. Moreover, since $\mathrm{bd}(X)$ is closed,

$$\pi^{-1}(\pi(Z_t^i)) \cap \mathrm{bd}(X)$$

is relatively closed in

$$\pi^{-1}(\pi(Z_t^i)).$$

Let B' be the union of the W_i of dimension lower than $m-1$. and let $B = \pi^{-1}(\overline{B'}) \cap \mathrm{bd}(X)$. Note that B has dimension lower than m. We define $\phi : \mathbb{R}^m \setminus \big(\mathrm{bd}(X) \cup B\big) \to \mathbb{R}$ as follow:

$$\phi(y) = \sup \Big\{ \pi_m\big(\{\pi(y)\} \times (-\infty, y_m) \cap \mathrm{bd}(X)\big) \Big\}$$

and $\delta : \mathbb{R}^m \setminus \big(\mathrm{bd}(X) \cup B\big) \to \mathbb{R}$ as follow:

$$y \mapsto s \in S_i \text{ such that } Z_s^i \text{ contains } \phi(y).$$

Note that ϕ is well defined because $\mathrm{bd}(X)$ is relatively closed in $\bigcup_{i \leq s} \pi^{-1}(\pi(Z^i))$.

For $y \in \mathbb{R}^m \setminus \big(\mathrm{bd}(X) \cup B\big)$, we define:

$$\Delta(y) = \delta^{-1}(\delta(y)).$$

We note that $\mathrm{bd}(X) \cup B$ has lower dimension than m and that it contains $\mathrm{bd}(X)$. Therefore, if $y \in X$ then $\Delta(y) \subseteq X$. Moreover, since $\Delta(y)$ only depends on $\delta(y)$, the family

$$\Big\{ \Delta(y) : \ y \in \mathbb{R}^m \setminus \bigcup_i \Big(\mathrm{bd}(X) \cup B \Big) \Big\}$$

is small.

It is also a subfamily of

$$\Big\{ \{y \in \mathbb{R}^m : \ y \in \mathrm{int}\left([Z_{t_1}^i, Z_{t_2}^j]\right)\} : \ t_1 \in A_i, t_2 \in A_j \Big\}$$

which is obviously definable in $\mathcal{R}$.

Since $\bigcup_i \left(\overline{Z^i} \cup B \right)$ has lower dimension than m, we apply the induction hypothesis to get the result. □

3.3.3 Expansions of a semibounded structure

In this section, we prove $\mathcal{L}$-DEM for d-minimal expansions of a semibounded structure by a unary set. We use the notations from the last section, and the proof is a direct continuation of it. First we need a proposition which could have taken place in Section 2.3 but it is so specific that we prefer to put it here.

Proposition 3.3.14. *Let $\{X_t \subseteq \mathbb{R}^n : t \in A \subseteq \mathbb{R}^m\}$ be a semibounded family. Then there are finitely many semibounded families $\{B_{i,t} \subseteq \mathbb{R}^n : t \in C_i\}$, $D_i \subseteq \mathbb{R}^n$ some bounded sets, V_i some vector subspace of $\mathbb{R}^n$ and $\{a_{i,t} : t \in A\}$ such that:*

1. *For every i C_i and D_i are bounded.*
2. *For every $t \in A$, for every i there is $t_i \in C_i$ with $X_t \subseteq \bigcup_i B_{i,t_i} + V_i + a_{i,t}$ and has the same dimension.*
3. *For every $t \in C_i$ $B_{i,t} \subseteq D_i$.*
4. *For every $t \in A$, $x \in B_{i,t} + V_i + a_{i,t}$ there are unique $b \in B_{i,t}$ and $v \in V_i$ such that $x = b + v + a_t$.*

Proof. First, by uniform cell decomposition, we may assume that for every $t \in A$, X_t is a cell of dimension k. Let $X = \bigcup_{t \in A} X_t \times \{t\}$. By Fact 2.3.4 we may assume that X is a cone of the form $B + \sum_{a \in (\mathbb{R}^{>0})^l} \sum_i v_i a_i$ where B is a bounded definable set, the v_i's are independent vectors in $\mathbb{R}^{n+m}$ and $(*)$: for every $x \in X$ there are unique $b \in B$ and $a \in (\mathbb{R}^{>0})^l$ such that $x = b + va$.

Let $V = \langle v_1, \dots, v_l \rangle$ and let

$$Y = B + V.$$

It is clear that $X \subseteq Y$. We first show that Y has $(**)$: for every $x \in Y$, there are unique $b \in B$, $a \in \mathbb{R}^l$ such that $x = b + \sum_i a_i v_i$. Suppose there are $x \in Y$, $b_1, b_2 \in B$, a_1, $a_2 \in \mathbb{R}^l$ such that

$$x = b_1 + va_1 = b_2 + va_2.$$

Let $a = (\max(|(a_1)_i|, |(a_2)_i|) + 1)_i$. We have that

$$b_1 + v(a_1 + a) = b_2 + v(a_2 + a) = x + va \in X$$

and by $(*)$, $b_1 = b_2$ and $a_1 = a_2$.

We show that $\dim(X) = \dim(Y)$. By $(*)$, $\dim(X) = \dim(B) + l$ and by $(**)$, $\dim(Y) = \dim(B) + l$, thus we have the result. For $t \in A$, let $Y_t = \mathbb{R}^n \times \{t\} \cap Y$. Let π_2 be the projection on the last m coordinates and let $E = \pi_2(Y)$. For every $t \in A$, we show that $\dim(X_t) = \dim(Y_t)$. Without loss of generality, we may assume that for every $t, t' \in A$, $\dim(Y_t) = \dim(Y_{t'}) = s$. Since each X_t is a cell of dimension k,

$$\dim(X) = k + \dim(A) = \dim(Y) = s + \dim(E).$$

Moreover, since $X \subseteq Y$, for every $t \in A$,

$$X_t \subseteq Y_t \text{ and } A \subseteq E.$$

Thus

$$\dim(X_t) \leq \dim(Y_t) \text{ and } \dim(A) \leq \dim(E)$$

and we have the equality.

We define an equivalence relation $\sim$ on $\mathbb{R}^m$:

$$x \sim y \text{ if and only if there is } v \in V \text{ such that } \mathbb{R}^n \times \{x\} + v = \mathbb{R}^n \times \{y\}.$$

Let $V_0 = \mathbb{R}^n \times \{0\} \cap V$ and let V_1 be a supplementary of V_0 in V: $V_0 \oplus V_1 = V$. From $(**)$, it is easy to see that for every $t \sim t'$ there is a unique $v \in V_1$ such that

$$\mathbb{R}^n \times \{t\} + v = \mathbb{R}^n \times \{t'\}.$$

Let $\gamma : \pi_2(B) \to \pi_2(B)$ be a choice function for $\sim$ and for $t \in \mathbb{R}^m$ let $B_t = B \cap \mathbb{R}^n \times \{t\}$. Let

$$f : \bigcup_{t \in \pi_2(B)} B_t \to \mathbb{R}^{n+m},$$

$$x \in B_t \mapsto x + v \text{ where } v \in V_1 \text{ is the unique such that } x + v \in B_{\gamma(t)}.$$

Note that f is injective by $(**)$. For $t \in \mathbb{R}^m$, let $B_{\overline{t}} = \bigcup_{t' \sim t} f(B_{t'})$. Note that $B_{\overline{t}} \subseteq \mathbb{R}^n \times \{\gamma(t)\}$. We finish the proof with the following claim:

Claim 3.3.15. *For every $t \in R^m$, $Y_t \times \{t\} = B_{\overline{t}} + V_0 + a_t$ where a_t is the unique element in V_1 such that $\mathbb{R}^n \times \{t\} = \mathbb{R}^n \times \{\gamma(t)\} + a_t$.*

Proof. Let $(x,t) \in Y_t \times \{t\}$. By $(**)$, there are unique $b \in B$ and $v \in V$ such that $(x,t) = b + v$. There are unique $v_0 \in V_0$, $v_1, v_1' \in V_1$ such that $v_1' + f(b) = b$ and $v = v_0 + v_1$. Thus

$$(x,t) = b + v = f(b) + v_0 + v_1 + v_1'.$$

The other inclusion is easy and we have the result. □

It is then easy to check that $\{\pi_1(B_{\overline{t}}) : t \in \pi_2(B)\}$, $\pi_1(V_0)$ and $\{\pi_1(a_t) : t \in A\}$ satisfy the properties $(1) - (5)$. □

Theorem 3.3.16. *Let $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ be d-minimal where $\mathcal{R}$ is semibounded and $P \subseteq \mathbb{R}$ has dimension 0. Let $\{X_x : x \in A\}$ be a family definable in $\widetilde{\mathcal{R}}^{\#}$. Then there is a uniform decomposition of the X_x's into $\mathcal{L}$-EM.*

Proof. The main objections to apply directly the proof of Theorem 3.3.7 in this setting is that we can not assume that Y is bounded and thus, we can not assume anymore that we are working with compact sets.

Following the proof of Theorem 3.3.7, we may reduce the proof to the case $\dim(Y) = m - 1$ and start the proof right after Notation 3.3.11.

By Proposition 3.3.14,we may assume that there are a vector subspace V, and a family $\{B_t : t \in A'\}$ such that:

1. $\{|B_t| : t \in A'\}$ is bounded,
2. for every $t \in A'$, $Y_t + V = B_t + V$,

3. $\dim(Y_t + V) = \dim(Y_t) = n$,
4. For every box $C \subseteq \pi(Y_t)$, $\pi^{-1}(C) \cap Y_t + V \subseteq Y_t$.

Claim 3.3.17. *There is a family $\{B'_t : t \in A'\}$ such that:*

1. *for every $t \in A'$ $\dim(B'_t) = \dim Y_t = n$,*
2. *there is a bound ρ on $\{|B'_t| : t \in A'\}$,*
3. *for every $t, t' \in A'$ if $\pi(Y_t) = \pi(Y_{t'})$ then $\pi(B'_t) = \pi(B'_{t'})$.*
4. *for every $t \in A'$, $Y_t + V$ is the graph of a continuous function over $\pi(Y_t + V)$.*

Proof. Let ρ' be the bound on $\{|B_t| : t \in A'\}$.

By definable choice in o-minimal structures, there is a family $\{x_t : t \in A\}$ such that $x_t \in W_t$. For $t \in A$, let $B''_t = \mathcal{B}(x_t, 2\rho')$. For $t' \in A'$ we denote by $\pi(t')$, $t \in A$ such that $\pi(Y_{t'}) = W_t$. let

$$B'_{t'} = \pi^{-1}(B''_{\pi(t')}) \cap Y_{t'} + V.$$

We show that $\{B'_t : t \in A'\}$ satisfies all the properties we need. The first property is obvious. For the second property, we first observe that since $\dim(Y_t) = \dim(Y_t + V) = n$, $\pi(V) \simeq V$. Let $t \in A'$, let $x, y \in B'_t$. By Proposition 3.3.14, $\pi^{-1}(x_{\pi(t)}) \cap Y_t$ belongs to some translate $B_t + v$ for some $v \in V$. There are also $v_x, v_y \in V$, $b_x, b_y \in B_t + v$ such that $x = b_x + v_x$ and $y = b_y + v_y$. Thus

$$|x - y| \leq |b_x - b_y| + |v_x - v_y|.$$

Since

$$|b_x - b_y| < \rho'$$

and that

$$v_x - v_y \in V \cap \pi(-1)(\mathcal{B}(0, 2\rho'))$$

(which is bounded since $V \simeq \pi(V)$), there is a bound on $\{|B'_t| : t \in A'\}$.

The third property is obvious. For the fourth, since $\pi^{-1}(Y_t) \cap Y_t + V = Y_t$, $\pi(B_t) \simeq B_t$. By proposition 3.3.14 (4), $Y_t + V$ is the disjoint union of translates of B_t by elements of V there are no $x, y \in Y_t + V$ such that $\pi(x) = \pi(y)$. Thus, since $Y_t + V$ is connected (since Y_t and V are) we have the result. □

For every $t \in S$, $\pi^{-1}(W_t) \cap Z$ is composed of connected components which are graphs of continuous functions over W_t. Thus we introduce the notation following Claim 3.3.10. For $Y_t, Y_{t'}$ such that $\pi(Y_t) = \pi(Y_{t'})$. We denote by $Y_t < Y_{t'}$ if there is $z \in \pi(Y_t)$ such that $\pi_m(\pi^{-1}(z) \cap Y_t) < \pi_m(\pi^{-1}(z) \cap Y_{t'})$ (and by Claim 3.3.10 of Theorem 3.3.7 it is true for every $z \in \pi(Y_t)$).

Claim 3.3.18. *Let $(t_n)_n$ be a sequence in S' such that there is $t \in S$ with $\pi(Y_{t_n}) = W_t$ for every n, $(Y_{t_n})_n$ is a decreasing sequence such that there is $a \in \overline{\bigcup_n Y_{t_n}} \setminus \bigcup_n Y_{t_n}$. Then*

1. *$(B'_{t_n})_n$ converges pointwise to A, the graph of a continuous function over $\pi(B'_{t_n})$ (for any n).*

2. *$Y_{t_n} + V \to_n A + V$ pointwise and $A + V$ is the graph of a continuous function.*

Proof. First of all, by translation, we may assume that $\pi(a) \in \pi(B'_{t_n})$. Since there is a uniform bound δ on the B'_{t_n}, it is easy to see that $A \subseteq \mathcal{B}(a, \delta)$. Since $(B'_{t_n})_n$ is a decreasing sequence that is bounded from below, $\bigcup_n B'_{t_n}$ is contained in a bounded set and we may apply Claim 3.3.12 to get the first part.

The second part is coming from the fact that for any $x \in W + \pi(V)$, there is $c \in V$ and an open neighbrohood $x \in C \subseteq W + \pi(V)$ such that for every n:

$$\pi^{-1}(C) \cap Y_{t_n} + V \subseteq c + B_{t_n}.$$

Thus, we have the result by translation. □

As in the case where $\mathcal{R}$ expands the real field, it is sufficient to prove that the family of pointwise limits of $\{Y_t : t \in S'\}$ is definable. Moreover, by Claim 3.3.17.(4),

$$B'_t + V \cap \pi^{-1}(W_{\pi(t))}) = Y_t.$$

Thus it is sufficient to prove that the family of Hausdorff limits of $\{\overline{B'_t} : t \in S'\}$ is definable.

For $t \in A'$, $a \in \mathbb{R}^m$, let $E_{t,a} = B_t \cap \mathcal{B}(a, 3\rho)$. Let

$$\mathcal{W} = \{\overline{E_{t,a} - a} : t \in A', a \in \mathbb{R}^m\}.$$

It is obvious that $\mathcal{W}$ is a semibounded family of compact sets and that all its members are contained in $\mathcal{B}(3\rho)$. Let $H = \{X_t : t \in F\}$ be the set of Hausdorff limits of $\mathcal{W}$ (that is definable in $\mathcal{R}$). Let

$$\mathcal{A} = \{(X_t \cap \pi^{-1}(B''_{t'} - \pi(a)) + a + V \cap \pi^{-1}(W_{t'}) : t' \in A, t \in F, a \in \mathbb{R}^m\}$$

and let

$$\mathcal{A}' = \{(X_t \cap \pi^{-1}(B''_{t'}) - \pi(a)) + a + V \cap \pi^{-1}(W_{t'}) : t \in S', t \in F, a \in \mathbb{R}^m$$

$$\text{such that } X_t + a + V \cap \pi^{-1}(W_{t'}) \subseteq Y\}.$$

It is easy to see that $\mathcal{A}'$ is a small subfamily of $\mathcal{A}$ and that $Y \cap \pi^{-1}(W_t) = \bigcup \mathcal{A}'$ and we have the result. As in the case where $\mathcal{R}$ expands the real field, it is easy to see that we can do this construction uniformly for $t \in S'$ and we have the result. □

3.4 Some consequences and questions

In this section we just take a preliminary look at some consequences of $\mathcal{L}$-DEM.

3.4.1 Generalization of $\mathcal{L}$-DEM to non-d-minimal settings

At an advanced stade of revision, we observed that Theorems 3.3.7 and 3.3.16 could be generalized to structures of the form $\langle \mathcal{R}, P\rangle$ where $\mathcal{R}$ is an o-minimal expansion of $\mathbb{R}_{gp}$, $P \subseteq \mathbb{R}$ is a set of dimension 0 and such that $\langle \mathcal{R}, P\rangle$ has (DIM), is sparse and have definable choice. The proof needs only to replace that for every $x \in \pi(X)$, then $\pi^{-1}(x) \cap X$ has dimension 0 instead of being discrete. Note that the structures that has (DIM) have been named i-minimal by Fornasiero (see [24]) and are just refered to 'structures that satisfy the interior or nowhere dense condition'by Miller or Thamrongthanyalak (see [46] or [58]). A. Fornasiero asked if definable choice was necessary. The answer is no: the only place in the proof where definable choice is being used is at the beginning of the proof of Claim 3.3.8. It can be replaced by the following proof:

Proof. (of Claim 3.3.8) Let π be a projection such that for every $t \in S$, Z_t is the graph of a continuous function over $\pi(Z_t)$. By i-minimality, we may assume that for every $x \in \pi(Z)$, $\dim(\pi^{-1}(x) \cap Z) = 0$.

For $t \in S$ let

$$W_t = \overline{\bigcup_{t' \in S} \mathrm{fr}(\pi(Z_t \cap Z_{t'}))} \text{ and let } W = \overline{\bigcup_{t \in S} W_t}.$$

The rest of the proof is then exactly the same as in Claim 3.3.8: if $W_i = \bigcup_{t \in S_i} W_{i,t}$ is an $\mathcal{L}$-EM of dimension n in the decomposition of $\mathbb{R}^n \setminus W$ then by definition of W, if A is a connected component of $\pi^{-1}(W_i) \cap Z$ then for every $t \in S$, since $A \cap \mathrm{fr}(Z_t) = \emptyset$ $Z_t \cap A \neq \emptyset$ implies $A \subseteq Z_t$ and A is equal to $Z_t \cap \pi^{-1}(W_{i,t'})$ for some $t' \in S_i$.

We conclude with the same argument as in Claim 3.3.8. □

The theorem is then the following:

Theorem 3.4.1. *We assume that $\widetilde{\mathcal{R}}$ is sparse, and has (DIM). Then for every definable set X, there is $\{X_t : t \in S\}$, an $\mathcal{L}$-decomposition of X such that for every t, $t' \in S$, either $\pi(X_t) = \pi(X'_t)$ or $\pi(X_t) \cap \pi(X_{t'}) = \emptyset$.*

We finish this section by giving an example of an i-minimal non-d-minimal structure of the form $\langle \mathcal{R}, P\rangle$.

Example 3.4.2. In [30, Theorem A], H. Friedman, H. Kurdyka, C. Miller and P. Speisseger are building a Cantor set K, such that, for $\mathcal{R}$ an exponentially bounded o-minimal expansion of the real ordered additive group, $\langle \mathcal{R}, P\rangle$ is i-minimal ("satisfies the *interior or nowhere dense condition*" in their paper). Thus, this structure has $\mathcal{L}$-DEM by Theorem 3.4.1.

3.4.2 Near Model Completeness

The property $\mathcal{L}$-DEM is itself close to model completeness (assuming that $\mathcal{R}$ is model complete). We could call it *model completeness up to small sets*. In this subsection, we take a look at the link between model completeness and $\mathcal{L}$-DEM.

Definition 3.4.3. We say that $\widetilde{\mathcal{R}}$ is P-internal if for every definable family of small sets $\{S_x \subseteq \mathbb{R}^n : x \in A \subseteq \mathbb{R}^m\}$, there is an $\mathcal{L}$-definable function $f : \mathbb{R}^{k+m} \to \mathbb{R}^n$ such that $S_x \subseteq f(x, P^k)$.

We denote by P_{ind} the structure induced on P by $\widetilde{\mathcal{R}}$, that is $\langle P, \{X \cap P^k : X \subseteq \mathbb{R}^k,\ \text{definable in } \widetilde{\mathcal{R}}\}\rangle$.

Lemma 3.4.4. *If $\widetilde{\mathcal{R}}$ is P-internal then $\widetilde{\mathcal{R}}^\#$ is too.*

Proof. It is a direct consequence of Fact 3.1.20. □

We can state few easy corollaries:

Corollary 3.4.5. *If $\mathcal{R}$ and P_{ind} are model complete and $\widetilde{\mathcal{R}}$ is P-internal then $\widetilde{\mathcal{R}}$ is model complete. The same holds for 'eliminates quantifiers'. For example, if $\widetilde{\mathcal{R}}$ is P-internal and $\mathcal{R}$ eliminates quantifiers then $\widetilde{\mathcal{R}}^\#$ eliminates quantifiers.*

More can be said even in the non-P-internal setting.

Proposition 3.4.6. *Let $S \subseteq \mathbb{R}^n$ be a small set. Then for any $Y \in \mathcal{P}(S)$, Y is definable in $\widetilde{\mathcal{R}}^\#$.*

Proof. By Fact 3.1.20, there is an $\mathcal{L}$-definable function $f : \mathbb{R}^k \to \mathbb{R}^n$ such that $S \subseteq \overline{f(P^k)}$. Since $Y \in \mathcal{P}(\overline{f(P^k)})$, we may assume that $S = \overline{f(P^k)}$. By d-minimality, there is a decomposition of S into finitely many discrete sets $S_1, \ldots, S_m$. For $x \in S_i$ let B_s be a box such that $B_s \cap S_i = \{s\}$ and let $Z_s = f^{-1}(B_s) \cap P^k$. It is easy to see that $Y = \bigcup_i \bigcup_{t \in Y \cap S_i} \overline{f(Z_s)} \cap S_i$. □

We finish with some interesting property.

Proposition 3.4.7. *Let $\{X_x : x \in \mathbb{R}^n\}$ be a definable family such that for every $x \in \mathbb{R}^n$, $X_x \subseteq P^k$. Then there is a small set of representatives for the equivalence relation on $\mathbb{R}^n$:*

$$x \sim y \text{ if and only if } X_x = X_y.$$

Proof. By d-minimality, we may assume that P is discrete. By definable choice there is a definable set $B \subseteq \mathbb{R}^n$ of representatives for $\sim$. In order to reach a contradiction we assume that $\dim(B) > 0$. Thus, by $\mathcal{L}$-DEM, there is an $\mathcal{L}$-definable set $C \subseteq B$ of dimension one and we may assume that $C = (0, 1)$.

Let $X = \bigcup_{x \in (0,1)} \{x\} \times X_x$. By assumption, $X \subseteq (0, 1) \times P^k$. By $\mathcal{L}$-DEM, we may assume that X is an $\mathcal{L}$-EM. Thus it has the form $\bigcup_{t \in S} X_t$ where the X_t are graphs of continuous functions from (a_t, b_t) to P^k and for every $t, t' \in S$, either $\pi(X_t) = \pi(X_{t'})$ or $\pi(X_t) \cap \pi(X_{t'}) = \emptyset$. Since P^k has dimension 0, for every $t \in S$, the functions which have domain (a_t, b_t) are constant and for every $x, y \in C$, $X_x = X_y$. Since S is small this is a contradiction with the choice of C. □

Remark 3.4.8. It is easy to see that the property of Proposition 3.4.7 holds for any model of $\widetilde{\mathcal{R}}$ since it is a first-order property.

If moreover, we assume that $\widetilde{\mathcal{R}}$ is P-internal, then Proposition 3.4.7 shows that for any definable family $\{X_x \subseteq P^k : x \in \mathbb{R}^n\}$, there is a family $\{Y_x : x \in A\}$ definable in P_{ind} such that for every $x \in \mathbb{R}^n$, there is $y \in A$, $X_x = Y_y$.

3.4.3 NIP

We just say a word on NIP in the P-internal setting. We thank Erik Walsberg for pointing this out.

Definition 3.4.9. We say that a theory T has *IP* (the Independence Property) if there are a structure $\mathcal{M} \models T$, two sequences $(a_I)_{I\in\mathcal{P}(\mathbb{N})}$, $(b_i)_{i\in\mathbb{N}}$ and a formula ϕ such that

$$\mathcal{M} \models \phi(a_I, b_i) \text{ if and only if } i \in I.$$

Moreover, we may assume that $|b_i| = 1$. We say that a theory is *NIP* if it does not have IP. We say that a structure is NIP if its theory is.

Let $\langle \mathcal{A}, B\rangle$ be a structure such that the language of $\mathcal{A}$ is $\mathcal{L}'$ and $B \subseteq A$ where A is the underlying set of $\mathcal{A}$. A formula is said to be B-bounded if it has the form

$$Q_0 x_0 \in B, \dots, Q_n x_n \in B\, \phi(x_0, \dots, x_n)$$

where $Q_i \in \{\exists, \forall\}$ and ϕ is an $\mathcal{L}'$-formula.

We say that the theory of $\langle \mathcal{A}, B\rangle$ is bounded if every formula is B-bounded.

Remark 3.4.10. Any o-minimal theory is NIP (see [55, Example 2.12]).

Proposition 3.4.11. *Let $\widetilde{\mathcal{R}}$ be d-minimal and P-internal. Then $\widetilde{\mathcal{R}}$ is NIP if and only if P_{ind} is.*

Proof. The left-to-right direction is obvious. For the right-to-left direction, it is just an application of Chernikov and Simon's [6, Corollary 2.5] which states that if $Th(\mathcal{R})$ and P_{ind} are NIP and $Th(\langle \mathcal{R}, P\rangle)$ is bounded then $Th(\langle \mathcal{R}, P\rangle)$ is NIP. The boundedness being fulfilled by $\mathcal{L}$-DEM and by P-internality, we have the result. □

3.4.4 Discussion and questions

The structure $\mathcal{R}_2 = \langle \mathbb{R}, <, +, \big(f : 2^{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto tx\big)\rangle$ is model complete (see Delon's [8]). It is also d-minimal since it is a reduct of $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$. It is easy to check that every definable function is almost everywhere locally affine. Of course, it does not have $\mathcal{L}$-DEM because an elementary extension would define some non-standard affine functions (that are not $\mathcal{L}$-definable).

Question 3.4.12. How many intermediate structures lie in between $\langle \mathbb{R}, <, +, 2^{\mathbb{Z}}\rangle$ and $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$? For example is

$$\langle \mathbb{R}, <, +, \big(g : \bigcup_{t\in 2^{\mathbb{Z}}} \{t\} \times [0, t] \to \mathbb{R}, (t, x) \mapsto tx\big)\rangle$$

strictly a reduct of $\mathcal{R}_2$?

We could of course ask the same questions for $\langle \overline{\mathbb{R}}, (2^{\uparrow n})_n\rangle$ and $\langle \mathbb{R}_{exp}, (2^{\uparrow n})_n\rangle$ where $(2^{\uparrow n})_n$ is a tower of exponentiation (which is obviously an iteration sequence for $\mathbb{R}_{exp}$ and thus these structures are d-minimal).

On the other hand, the structure

$$\langle \mathbb{R}, <, +, \big(f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto tx\big)\rangle$$

is not d-minimal. Take:

$$\bigcup_{t\in\mathbb{Z}} f_t^{-1}(\mathbb{Z}) = \mathbb{Q}.$$

Moreover we show that the multiplication is definable on $\mathbb{Q}$. First, the multiplication is definable on $\mathbb{Z}^2$: take $(x, y) \mapsto f_x(y)$. For $(x, y), (z, t) \in \mathbb{Z}^2$ let $(x, y) * (z, t)$ be the multiplication coordinate by coordinate. Let $g : (x, y) \in \mathbb{Z}^2 \mapsto f_y^{-1}(x)$. Let $h : z \in \mathbb{Q} \mapsto (x, y) \in \mathbb{Z}^2$ such that y is the smallest element of $\mathbb{N}$ with $g(x, y) = z$. We define the multiplication on $\mathbb{Q}^2$ by

$$(x, y) \mapsto g(h(z) * h(z')).$$

Therefore the full multiplication on $\mathbb{R}^2$ is definable by taking the closure of the graph of the multiplication on $\mathbb{Q}^2$ and thus

$$\langle \mathbb{R}, <, +, \big(f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto tx\big)\rangle$$

defines the whole projective hierarchy (see [40]).

An other example in the same taste is

$$\langle \overline{\mathbb{R}}, \big(f : 2^{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto x^t\big)\rangle$$

which defines a dense co-dense set by taking

$$\bigcup_{t\in 2^{\mathbb{Z}}} f_t^{-1}(2^{\mathbb{Z}}) = \{2^{n/2^m} : n, m \in \mathbb{Z}\}.$$

Moreover, it is easy to see that this set is ω-orderable (see [33] the definition right after Theorem A in the introduction) and thus

$$\langle \overline{\mathbb{R}}, \big(f : 2^{\mathbb{Z}} \times \mathbb{R} \to \mathbb{R}, (t, x) \mapsto x^t\big)\rangle$$

defines the whole projective hierarchy (see [33] Fact 1.2).

In both cases, there is an o-minimal expansion $\mathcal{R}$ such that f is the restriction of a function definable in it to $S \times \mathbb{R}$ for S a small set. In the first case f is coming from $\overline{\mathbb{R}}$ and in the second case, it is coming from $\langle \overline{\mathbb{R}}, \exp\rangle$. The following question is quite naive and the answer is probably negative, but it could be the starting point of an interesting sequence of questions.

Question 3.4.13. For a d-minimal structure $\mathcal{M}$, is there an o-minimal structure $\mathcal{M}'$ and a set $P \subseteq \mathbb{R}$ of dimension 0 such that $\mathcal{M} \subseteq \langle \mathcal{M}', P\rangle^{\#}$?

We would have liked to have some minimality on $\mathcal{M}'$ (ie that there is no o-minimal structure $\mathcal{M}''$ and set $P' \subseteq \mathbb{R}$ of dimension 0 such that $\mathcal{M}'' \subsetneq \mathcal{M}'$ and $\mathcal{M} \subseteq \langle \mathcal{M}'', P'\rangle^{\#}$). It is not the case as shown by the following example. The notations and definitions are established in Section 4.4.

Example 3.4.14. Let $P \subseteq \mathbb{R}$ be a fast independent sequence for $\overline{\mathbb{R}}$ and let $\widetilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, P \rangle$. We take $(\lambda_i)_i \in \mathbb{R}^{\mathbb{N}}$ such that λ_{i+1} is acl-independent for $\overline{\mathbb{R}}$ over $\{\lambda_j : j < i\} \cup P$. Let $P_i = \lambda_i(P)$. Let $\mathcal{M} = \langle \mathbb{R}, <, +, (P_i)_i \rangle$.

We prove that there is no minimal o-minimal reduct of the real field $\mathcal{R}'$ and a set $P' \subseteq \mathbb{R}$ of dimension 0 such that

$$\langle \mathbb{R}_{gp}, (P_i)_i \rangle \subseteq \langle \mathcal{R}, P' \rangle^{\#} \subseteq \langle \overline{\mathbb{R}}, P \rangle.$$

Let $\mathcal{R}'$ be a reduct of the real field and $P' \subseteq \mathbb{R}$ be a set of dimension 0. Note that since

$$\langle \mathbb{R}, <, +, (P_i)_i \rangle \subseteq \langle \mathbb{R}, <, +, (x \mapsto \lambda_i x)_i, P \rangle$$

and that $\langle \mathbb{R}, <, +, (x \mapsto \lambda_i x)_i \rangle$ is a strict reduct of $\overline{\mathbb{R}}$ and defines the P_i's, we may assume that $\mathcal{R}'$ is semibounded. In a first time, we prove that $\mathbb{K}_{\mathcal{R}'}$, the field of total linear functions definable in $\mathcal{R}'$ has countable index over $\mathbb{Q}$.

By P-internality of $\widetilde{\mathcal{R}}$, we may assume that $P' = g(P^k)$, where g is an $\overline{\mathbb{R}}$-definable function (maybe with finitely many parameters). Let λ_i be acl-independent for $\overline{\mathbb{R}}$ over the parameters needed to define g and P.

By Fact 3.1.20, there is an $\mathcal{R}'$-definable function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$P_i \subseteq \overline{f(P'^n)}.$$

Moreover, by Fact 5.4.7, we may assume that there is a linear function $A : \mathbb{R}^n \to \mathbb{R}$ (with coefficients in $\mathbb{K}_{\mathcal{R}'}$) and a function with bounded image $b : \mathbb{R}^n \to \mathbb{R}$ ($\mathrm{Im}(b) \subseteq \mathcal{B}(a)$) such that $f = A + b$. For $x \in \mathbb{R}$, let

$$X_x = \{y \in \mathbb{R}^{nk} : \ A(g(y)) \in \lambda_i x + \mathcal{B}(a)\}$$

and for $x \in P$, let

$$Y_x = X_x \cap P^{kn}.$$

Note that the Y_x's are definable in $\langle \overline{\mathbb{R}}, P \rangle$. Moreover, by assumption, for every $x \in P$, $Y_x \neq \emptyset$.

Claim 3.4.15. *Let $\{Z_x \subseteq \mathbb{R}^m : x \in \mathbb{R}\}$ be a semialgebraic family such that for every $x \in P$, $Z_x \cap P^m \neq \emptyset$. Then there is a semialgebraic function $h : \mathbb{R} \to \mathbb{R}^m$, definable without parameters such that $h(x) \in X_x$.*

Proof. We do an induction on m. For $m = 0$, the result is trivial. We assume that the result holds for $m - 1$. By unfiorm cell decomposition, we may assume that the Z_x are cells. By induction, there is a semialgebraic function $h_1 : \mathbb{R} \to \mathbb{R}^{m-1}$ definable without parameters, such that $h_1(x) \in \pi(X_x)$ (for π the projection on the first $m-1$ coordinates). By Proposition 4.4.5 and since for every $x \in P$, $Z_x \cap P^m \neq \emptyset$, we may assume that for every $x \in P$, Z_x contains an s-cell. Moreover, by uniform cell-decomposition, we may assume that the Z_x's are cells. Two cases are possible: either Z_x is the graph of a semialgebraic function g_1 over $\pi(Z_x)$ or Z_x has the form

$$\{(y, z) : \, y \in \pi(Z_x), \, z \in (f_1(y), f_2(y))\}$$

for f_1, f_2 two semialgebraic functions.

In the first case, since for $x \in P$, Z_x contains an s-cell and by fastness, we may assume that either $g_1(y_1, y_{m-1}) = y_j$ for some j or g_1 is a constant function in P. The result follows.

In the second case, by partitioning $\mathbb{R}^{m-1}$, we may assume that for $y = (y_1, \dots, y_{m-1}) \in \mathbb{R}^{m-1}$, $y_1 < \dots < y_{m-1}$. By o-minimality it is easy to see that we may reduce the study to four cases:

1. there is $j < m$, for every $y \in \pi(X_x)$:
$$f_1(y) < y_j < f_2(y),$$
2. there is $j < m$, for every $y \in \pi(X_x)$:
$$y_j < f_1(y) < f_2(y) < y_{j+1},$$
3. for every $y \in \pi(X_x)$:
$$y_{m-1} < f_1(y) < f_2(y).$$
4. for every $y \in \pi(X_x)$:
$$f_1(y) < f_2(y) < y_1$$

Case 1*:* the result follows directly by taking
$$h : x \mapsto (h_1(x), \pi_j(h_1(x))).$$

Case 2*:* by fastness, for almost but finitely many $y \in P^{m-1}$,
$$y_j < f_1(y) < f_2(y) < s(y_j).$$
It is a contradiction with $Z_x \cap P^m \neq \emptyset$ for every $x \in P$.

Case 3*:* if $f_2 < +\infty$, then, using the same argument as in Case 2, we get a contradiction. Thus $f_2 = +\infty$ and since $\overline{\mathbb{R}}$ is polynomially bounded, there is $r \in \mathbb{N}$ such that, there is a bounded set A, for $y \in \pi(Z_x) \setminus A$
$$f_1(y) < y_{m-1}^r.$$
The result follows by taking
$$h : x \mapsto (h_1(x), \pi_{m-1}(h_1(x))^r).$$

Case 4*:* similar to Case 3. □

Applying the claim to $\{X_x : x \in \mathbb{R}\}$, we get $h : \mathbb{R} \to \mathbb{R}^m$ such that $h(x) \in X_x$. Thus, $\rho : x \mapsto A(h(x))$ is definable in $\overline{\mathbb{R}}$ and it is easy to see that ρ is equivalent to $x \mapsto \lambda_i x$ in $+\infty$. Thus, since λ_i is independent over the parameters of h, λ_i is dependent over the parameters of A and since A is linear, we have the result: $\mathbb{K}_{\mathcal{R}'}$ has countable index over $\mathbb{Q}$.

Let $\{\mu_i : i \in \mathbb{N}\}$ be a basis of $\mathbb{K}_{\mathcal{R}'}$ over $\mathbb{Q}$. Let λ_i be independent over the parameters of g and let $\mu_{i_1}, \dots, \mu_{i_m}$ such that λ_i is dependent over them. Let $\lambda_{j_1}, \dots, \lambda_{j_s}$ be the elements among the λ_j's that are dependent over $\{\mu_{i_1}, \dots, \mu_{i_m}\}$. Let $\mathcal{M}$ be $\langle \mathbb{R}, <, +\rangle$ if $\mathcal{R}'$ is linear and $\langle \mathbb{R}, <, +, \cdot_{\restriction [0,1]^2}\rangle$ if not. Then we have that

$$\mathcal{M}_1 = \langle \mathcal{M}, \big(x \mapsto \mu_j x\big)_{j \in \mathbb{N} \setminus \{i_1, \dots, i_m\}}\rangle \subsetneq \langle \mathcal{M}, \big(x \mapsto \mu_j x\big)_{j \in \mathbb{N}}\rangle = \mathcal{R}'$$

and

$$\langle \mathcal{M}_1, P' \cup \bigcup_{r \leq s} P_{i_r}\rangle^{\#} \text{ defines every } \lambda_i P.$$

It is a contradiction with the minimality of $\mathcal{R}'$.

We can then give a variation around Question 3.4.13:

Question 3.4.16. For a d-minimal structure $\mathcal{M}$, is there a minimal o-minimal structure $\mathcal{M}'$ and some sequence of small sets $(S_i)_{i \in I}$ such that $\mathcal{M} \subseteq \langle \mathcal{M}', (S_i)_{i \in I}\rangle$ and such that the last structure is d-minimal?

4. EXAMPLES

In this chapter we present some examples of d-minimal structures of the form $\langle \mathcal{R}, P\rangle$ (for $\mathcal{R}$ an o-minimal expansion of the real ordered additive group and $P \subseteq \mathbb{R}$ a set of dimension 0) and, for each structure, we study the following:

- local o-minimality,
- P-internality
- what P_{ind} is.

Definition 4.0.1. We say that an ordered structure $\mathcal{M}$ is *locally o-minimal* if given a definable set $X \subseteq M$, for every $x \in X$, there is an interval I such that $I \cap X$ is a finite union of intervals and points.

Note that, if the underlying set of $\mathcal{M}$ is $\mathbb{R}$, local o-minimality implies d-minimality since a set of dimension 0 needs to be discrete (the assumption that the underlying set is $\mathbb{R}$ is only used to show definable completeness).

As we shall see, P-internality is a quite common property. However there are d-minimal structures of the form $\langle \mathcal{R}, P\rangle$ that are not P-internal as shown by the following example. We would like to thank Philipp Hieronymi for the general idea behind it.

Example 4.0.2. Let $\mathcal{R}$ be an o-minimal expansion of $\overline{\mathbb{R}}$ (in a countable language) and let P be an $\mathcal{R}$-fast sequence (see Definition 4.4.1) and an $\mathcal{L}$-independent sequence (that is x_{n+1} is algebraically independent (for $\mathcal{R}$) over $x_1, \dots, x_n$). The existence of such a sequence is quite easy to show. Indeed, take a fast sequence $(a_n)_n$ in $\mathbb{Q}$ (whom existence is shown in [28]) and a growing sequence $(b_n)_n \in \mathbb{R}^{>0}$ $\mathcal{L}$-independent. Then $(a_n + b_n)_n$ is a fast $\mathcal{L}$-independent sequence.

Now, we build a set P' such that $\langle \mathcal{R}, P'\rangle$ is d-minimal but is not P'-internal. Let

$$f : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto x + 1/y.$$

Let $(P_n)_n$ be a sequence of infinite disjoint subsets of P and let $P_1 = (a_i)_{i\in\mathbb{N}^{>0}}$. Let

$$P' = f\Bigg(\bigcup_{i\geq 2} \{a_i\} \times P_i\Bigg).$$

By Fact 3.1.12, as a reduct of $\langle \mathcal{R}, P\rangle^{\#}$, $\langle \mathcal{R}, P'\rangle$ is d-minimal. Moreover, let $g : \mathbb{R}^k \to \mathbb{R}$ be an $\mathcal{L}$-definable function. Let $x = a_{i_1}, \dots, a_{i_k} \in P_1$ and $y = y_1, \dots, y_k \in \Pi_j P_{i_j}$. Let $z = (x_i + 1/y_i)_i$. It is easy to see that $g(z)$ is $\mathcal{L}$-independent from any element of P_1. Thus $P_1 = \overline{P'} \setminus P'$ is not P'-internal.

We take this occasion to state a question.

Question 4.0.3. If P is discrete and closed in its convex hull, do we have that $\widetilde{\mathcal{R}}$ is P-internal?

4.1 Expansion of a semibounded expansion of the real additive ordered group by the integers

Let $\mathcal{R}$ be a semibounded expansion of $\mathbb{R}_{gp}$ such that $\mathbb{K}_{\mathcal{R}} = \mathbb{Q}$. We first establish local o-minimality for $\langle \mathbb{R}_{gp}, \mathbb{Z}\rangle$ and $\langle \mathcal{R}, \mathbb{Z}\rangle$. In a second step, we establish some QE result for $\langle \mathcal{R}, \mathbb{Z}\rangle$ based on the one for $\langle \mathbb{R}_{gp}, \mathbb{Z}\rangle$ of Miller (see [45]).

Lemma 4.1.1. *The structure $\langle \mathbb{R}_{gp}, \mathbb{Z}\rangle$ is locally o-minimal.*

Proof. It is sufficient to prove that every definable set $X \subseteq \mathbb{R}$ of dimension 0 is discrete.

Since $\mathbb{Z}$ is countable, it is sparse. Thus, by Fact 3.1.20, for every definable set X of dimension 0, there is an $\mathcal{L}$-definable function $f : \mathbb{R}^{k+1} \to \mathbb{R}$ such that $X \subseteq \overline{f(\mathbb{Z}^k)}$. By Fact 2.2.3 we may assume that f is affine and $f : x \mapsto \sum_{i\leq k} a_i(x_i - c_i)$ where $c_i \in \mathbb{R}$ and $a_i \in \mathbb{Q}$. Let $x, x' \in \mathbb{Z}^k$,

$$f(x) - f(x') = \sum_i a_i(x_i - x'_i).$$

Let $a_i = p_i/q_i$ with $p_i, q_i \in \mathbb{Z}$ and let $d = \operatorname{lcm}(q_i)$. We have that

$$d(f(x) - f(x')) = \sum_i da_i(x_i - x'_i) \in \mathbb{Z}.$$

Thus, for every $x \in \mathbb{Z}^k$,

$$\{f(x)\} = X \cap (f(x) - 1/d, f(x) + 1/d)$$

and we have the result. □

Remark 4.1.2. We could have proven Lemma 4.1.1 by observing that it was a reduct of $\langle \mathbb{R}_{gp}, \big(x \mapsto \sin(2\pi x)\big)\rangle$ which is locally o-minimal by [59, Theorem 2.7].

Lemma 4.1.3. *Let $\mathcal{R}$ be a linear o-minimal expansion of $\mathbb{R}_{gp}$. We assume $\langle \mathcal{R}, P\rangle$ to be locally o-minimal. Let $\sim$ be an equivalence relation over $\mathbb{R}^k$ definable in $\mathcal{R}$. Let K be a bounded box. Then*

$$Z = \{A \in \mathbb{R}^k/\sim \colon\ A \cap P^k \neq \emptyset \text{ and } A \cap K \neq \emptyset\}$$

is finite.

Proof. By definable choice in o-minimal structures, there is a definable set $S \subseteq K$ of representatives for $K/\sim$. Moreover, the set

$$S' = \{a \in S : \text{ there is a class } A \in \mathbb{R}^k/\sim \text{ such that } a \in A \text{ and } A \cap P^k \neq \emptyset\}$$

is countable and thus has dimension 0. By local o-minimality of $\langle \mathcal{R}, P\rangle$, since $S' \subseteq S \subseteq K$, S' is finite and thus $Z = \{A \in \mathbb{R}^k/\sim\colon\ \exists a \in S',\, a \in A\}$ is finite too.

□

Lemma 4.1.4. *Let $\mathcal{R}$ be semibounded and let $\mathcal{R}'$ be a semibounded expansion of $\mathcal{R}$ such that $\mathbb{K}_{\mathcal{R}} = \mathbb{K}_{\mathcal{R}'}$ We assume $\langle \mathcal{R}, P\rangle$ to be locally o-minimal. Then $\langle \mathcal{R}', P\rangle$ is locally o-minimal.*

Proof. Let $X \subseteq \mathbb{R}$ be a definable set of dimension 0. We show that X is discrete. Since P is countable, it is sparse and by Fact 3.1.20 there is an $\mathcal{L}$-definable function $f : \mathbb{R}^k \to \mathbb{R}$ such that $X \subseteq f(P^k)$.

Let B be a bounded interval. We show that $B \cap X$ is finite. Let

$$Y = \bigcup_{b \in B} \{b\} \times f^{-1}(b)$$

By Fact 2.3.4, we may assume that Y is a cone of the form $C + \sum_{a \in (\mathbb{R}^{>0})^l} a_i v_i$ with C a bounded set and $v_i \in \mathbb{Q}^k$.

Let us first show that for every $b \in B$,

$$(b, 0) + \sum_{a \in (\mathbb{R}^{>0})^l} \sum_i a_i v_i \subseteq \{b\} \times \mathbb{R}^k.$$

If not, there are $b \neq b' \in B$, $(b, x), (b', x') \in Y$, and $a \in (\mathbb{R}^{>0})^l$ such that

$$(b, x) + \sum_i a_i v_i = (b', x')$$

and δ, the half line starting at (b, x) and passing by (b', x') is included in Y. Let π_1 be the projection on the first coordinate. Since $\pi_1(\delta)$ is unbounded and $\pi_1(Y) = B$ is bounded, we get a contradiction.

Thus, for every $b \in B$, let $C_b = \{b\} \times \mathbb{R}^k \cap C$. we have that

$$\{b\} \times f^{-1}(b) = Y \cap \{b\} \times \mathbb{R}^k = C_b + \sum_{a \in (\mathbb{R}^{>0})^l} \sum_i a_i v_i.$$

Note that V is a vector space. Let $C' = \bigcup_{b \in B} C_b$. Since $\mathrm{ldim}(B) = 0$ and for every $b \in B$ $\mathrm{ldim}(C_b) = 0$, by [18, Lemma 3.6], $\mathrm{ldim}(C') = 0$ and C' is bounded.

Let π_2 be the projection on the last k coordinates. Let

$$V = \pi_2(\sum_{a \in \mathbb{R}^l} \sum_i a_i v_i).$$

We define $\sim$ an equivalence relation on $\mathbb{R}^k$:

$$x \sim x' \text{ if and only if } x \in x' + V.$$

We remark that $\sim$ is definable in $\mathbb{R}_{gp}$ and thus, for every bounded box $K \subseteq \mathbb{R}^k$, the set

$$\{A \in \mathbb{R}^k / \sim\colon\ A \cap P^k \neq \emptyset \text{ and } A \cap K \neq \emptyset\}$$

is finite by Corollary 4.1.3. Moreover, for every bounded set K' since we can find a box K such that $K' \subseteq K$, we have that

$$\{A \in \mathbb{R}^k / \sim\colon\ A \cap P^k \neq \emptyset \text{ and } A \cap K' \neq \emptyset\}$$

is finite too. Thus the set

$$\{A \in \mathbb{R}^k / \sim\colon\ A \cap P^k \neq \emptyset \text{ and } A \cap C' \neq \emptyset\}$$

is finite.

We prove that for every $b \neq b' \in B$, $c \in C_b, c' \in C_{b'}$, $c \nsim c'$. We assume that it is not the case. There is $a \in \mathbb{R}^l$ such that

$$c = c' + \pi_2(av).$$

Let $d = (|a_1| + 1, \dots, |a_l| + 1)$. We have that

$$Y_b \ni c + dv = c' + av + dv \in Y_{b'}.$$

Thus $Y_b \cap Y_{b'} \neq \emptyset$ which is a contradiction.

Therefore, for every class $A \in \mathbb{R}^k / \sim$, there is at most one $b \in B$ such that $f^{-1}(b) \subseteq A$.

Thus the set

$$\{b \in B :\ f^{-1}(b) \cap P^k \neq \emptyset\}$$

is finite and we have the result. □

Lemma 4.1.5. *Let $\mathcal{R}$ be a semibounded expansion of $\mathbb{R}_{gp}$ such that $\mathbb{K}_{\mathcal{R}} = \mathbb{Q}$. Then $\langle \mathcal{R}, \mathbb{Z} \rangle$ is locally o-minimal.*

Proof. The proof follows from Lemma 4.1.4 and Lemma 4.1.1. □

Remark 4.1.6. We could have shown directly that $\langle \mathcal{R}, \mathbb{Z} \rangle$ is locally o-minimal by using the same technics as in [59] and showing that it could be put on the form of simple products of locally o-minimal structures but Lemma 4.1.4 has an interest in itself and will be usefull later.

Note that if we don't assume that $\mathbb{K}_{\mathcal{R}} = \mathbb{Q}$ then the result fails. Take, for example, $\mathcal{R} = \langle \mathbb{R}_{gp}, (x \mapsto \sqrt{2}x) \rangle$ which is not d-minimal (see [35]).

We thank Philipp Hieronymi for pointing out the following theorem.

Theorem 4.1.7. *Let $\mathcal{R}$ be a semibounded expansion of $\mathbb{R}_{gp}$ such that $\mathbb{K}_{\mathcal{R}} = \mathbb{Q}$. The theory of*

$$\mathcal{M} = \langle \mathbb{R}, <, +, \{B \in \mathcal{D}ef(\mathcal{R}) : \ B \text{ is bounded}\}, \mathbb{Z}, (x \mapsto ax)_{a\in\mathbb{Q}}, \lambda\rangle$$

eliminates quantifiers.

Proof. The proof is almost the same as the one of QE for $\langle \mathbb{Q}, <, +, (x \mapsto ax)_{a\in\mathbb{Q}}, \mathbb{Z}\rangle$ given in the appendix of Miller's [45]. We reproduce the begining of it in order to be understandable. Let us recall that $\lambda : \mathbb{R} \to \mathbb{Z}$ is the step-function:

$$\lambda : x \mapsto \sup\{(-\infty, x] \cap \mathbb{Z}\}$$

We prove that this structure is axiomatized by the theory of $\mathcal{M}$ together with axioms expressing that $\mathbb{Z}$ is a subgroup having a least positive element 1, such that every element $x \in R$ is between some $z \in \mathbb{Z}$ and $z + 1$. Let T be the theory of $\mathcal{M}$ and let T' be the theory:

- T,
- for each pair $j, n \in \mathbb{Z}$ with $n > 0$, the axiom $n(\frac{j}{n}x) = jx$,
- $\lambda(\lambda(x) + y) = \lambda(x) + \lambda(y)$,
- $0 \leq x < 1 \Rightarrow \lambda(x) = 0$,
- $\lambda(1) = 1$,
- $\lambda(x) \leq x < \lambda(x) + 1$.

We note that T' is universal since T is (see [17] Theorem 1.8).

As in [45], we use the same QE test that in [13]. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{B}'$ be such that $\mathcal{B}, \mathcal{B}' \models T'$, $\mathcal{B}'$ is $|B|^+$-saturated and $\mathcal{A}$ is a proper substructure of $\mathcal{B}$ embedded in $\mathcal{B}'$. Let A, B, B' be there underlying sets.
Since T' is universal, A is an elementary substructure of B. Thus it is sufficient to find $b \in B \setminus A$ and $b' \in B' \setminus A$ such that the function which sends A to A identically and b to b' can be extended to a partial isomorphism between the structures generated by A and b (in B) and by A and b' (in B').Let $\langle A, b\rangle$ and $\langle A, b'\rangle$ be the semibounded structures genereated by A and b and by A and b'.

Following [45], it is sufficient to find b and b' such that $(*)$:

- $\lambda(\langle A, b\rangle) \subseteq \langle A, b\rangle$
- $\lambda(\langle A, b'\rangle) \subseteq \langle A, b'\rangle$
- If $a \in A^n$ and f is an $\mathcal{L}$-definable function then $\phi(\lambda(f(a, b))) = \lambda(f(a, b))$, where $\phi : \langle A, b\rangle \to \langle A, b'\rangle$ is given by

  $$\phi(g(a, b)) = g(a, b')$$

  for g an $\mathcal{L}$-definable function and $a \in A^n$.

If $\lambda(B) = \lambda(A)$, it is sufficient to take b and b' having the same cut in A.
If $\lambda(B) \neq \lambda(A)$, we first prove the following claim:

Claim 4.1.8. *For $b \gg A$, $\langle A, b\rangle = A + \mathbb{Q}b$.*

Proof. We just need to show that for every semi bounded function $f : X \subseteq B^{n+1} \to B$, $f(a, b) \in A + \mathbb{Q}b$ for $a \in A^n$.

By semiboundedness, for every $a \in \pi_n(X)$, f_a is ultimately affine and we may assume that f_a has the form $x \mapsto g(a) + qx$ where g is a semibounded function and $q \in \mathbb{Q}$ does not depend on a. Therefore the set

$$X_a = \{x \in B : \; f_a(x) = g(a) + qx\}$$

is cofinale in B and by o-minimality, we may assume that X_a is connected.

If $X_a = B$ then $b \in X_a$ and

$$f(a, b) = g(a) + qb$$

and since $\mathcal{A}$ is a substructure, $g(a) \in A$ and

$$f(a, b) \in A + \mathbb{Q}b.$$

We assume that $X_a \neq B$. Let x_a be the infimum of X_a. We have that x_a is algebraic over a and since $\mathcal{A}$ is a substructure, $x_a \in A$. Since $b \gg A$, $b > x_a$ and $b \in X_a$. Thus

$$f(a, b) = g(a) + qb \in A + \mathbb{Q}b.$$

□

Thus, Condition $(*)$ becomes

- $\lambda(A + \mathbb{Q}b) \subseteq A + \mathbb{Q}b$
- $\lambda(A + \mathbb{Q}b') \subseteq A + \mathbb{Q}b'$
- If $a \in A^n$ and f is an affine function then $(\phi(\lambda(f(a, b))) = \lambda(f(a, b))$, where $\phi : \langle A, b\rangle \to \langle A, b'\rangle$ is given by

  $$\phi(g(a, b)) = g(a, b')$$

  for g an affine function and $a \in A^n$.

Which is exactly the conditions of the proof of [45] and the rest of the proof is then exactly the same. □

4.1.1 P-internality

We state a more general result that will be useful in the sequel. Let $\mathcal{R}$ be any o-minimal expansion of the real additive ordered group. As usual, $P \subseteq \mathbb{R}$ is a set of dimension 0 such that $\widetilde{\mathcal{R}}$ is d-minimal.

Proposition 4.1.9. *If $\widetilde{\mathcal{R}}$ is locally o-minimal then $\widetilde{\mathcal{R}}$ is P-internal.*

Proof. For any definable set $X \subseteq \mathbb{R}$ of dimension 0, by Fact 3.1.20 there is an $\mathcal{L}$-definable function $f : \mathbb{R}^k \to \mathbb{R}$ such that $X \subseteq \overline{f(P^k)}$. Since $\widetilde{\mathcal{R}}$ is locally o-minimal, $\overline{f(P^k)} = f(P^k)$ and $X \subseteq f(P^k)$. □

Corollary 4.1.10. *For $\mathcal{R} = \mathbb{R}_{gp}$ or $\mathbb{R}_{sb}$, $\langle \mathcal{R}, \mathbb{Z} \rangle$ is P-internal.*

4.1.2 Induced structure on P

Definition 4.1.11. Let $X \subseteq \mathbb{Z}$. We say that X is a *Presburger-cell* if X either has the form $\{*\}$ or $(a, b) \cap (n\mathbb{Z} + k)$ for $|a - b|$ non-finite and $0 \leq k < n \in \mathbb{N}$. Let $X \subseteq \mathbb{Z}^{k+1}$. We say that X is a Presburger cell if there are $f, g : \mathbb{Z}^k \to \mathbb{Z}$ some affine functions such that there is no uniform bound on $|f(x) - g(x)|$ and $C \subseteq \mathbb{Z}$, a Presburger-cell such that:

- $\pi(X)$ is a Presburger-cell, for π the projection on the first k coordinates.
- X either has the form:
  - $\{(x, f(x)) \in \mathbb{Z}^k : x \in \pi(X)\}$
  - or $\{(x, y) \in \mathbb{Z}^k : x \in \pi(X),\ y \in C \text{ and } f(x) < y < g(x)\}$.

Let $(P_n)_{n \in \mathbb{N}}$ be the predicates $n|x$.

Fact 4.1.12. *(See Cluckers's [5]) Every set definable in $\langle \mathbb{Z}, <, + \rangle$ admits a decomposition into finitely many Presburger cells and $\langle \mathbb{Z}, <, +, (P_n) \rangle$ eliminates quantifiers.*

Proposition 4.1.13. *The structure induced on P by $\widetilde{\mathcal{R}}$ is $\langle \mathbb{Z}, <, + \rangle$.*

Proof. By QE, the result follows easily . □

4.2 Expansion of the real field with a predicate for the integer powers of two

Fact 4.2.1. *(see Tychonievich [60, Chapter 4]) The structure $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ is P-internal and $P_{ind} = \langle 2^{\mathbb{Z}}, <, \cdot \rangle \simeq \langle \mathbb{Z}, <, + \rangle$. Moreover, let*

$$\sigma : \langle \mathbb{Z}, <, + \rangle \to \langle 2^{\mathbb{Z}}, <, \cdot \rangle$$

be an isomorphism, then

$$\langle \overline{R}, \sigma\big(\langle \mathbb{Z}, <, +, (P_n)_n \rangle\big) \rangle$$

eliminates quantifiers (see Van Den Dries's [13]).

4.3 Expansions of an o-minimal structure by an iteration sequence

Let $\mathcal{R}$ be any o-minimal expansion of $\mathbb{R}_{gp}$.

Definition 4.3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be an $\mathcal{L}$-definable bijection. Let f^n be the n-th compositional iterate of f. We say that $\mathcal{R}$ is f-bounded if for every definable function $g : \mathbb{R} \to \mathbb{R}$, there is $N \in \mathbb{N}$ such that $g(x) < f^N(x)$ as $x \to \infty$. Let $c \in \mathbb{R}$ and let f be an $\mathcal{L}$-definable function such that $\mathcal{R}$ is f-bounded. We say that $(f^n(c))_n$ is an *iteration sequence* if it is increasing and unbounded.

For the rest of this section, we fix $\mathcal{R}$ an o-minimal expansion of $\mathbb{R}_{gp}$ and P an iteration sequence.

Definition 4.3.2. We define inductively the notion of an *s-cell* (s here stands for successor).,

- If $X = P^0$.
- For $X \subseteq P^k$, then X is an s-cell if $\pi(X)$ is (for π the projection on the first $k-1$ coordinates) and either X has the form

  $$\{(x, y) : \ x \in \pi(X),\, y = f(x)\}$$

  or the form

  $$\{(x, y) : \ x \in \pi(X),\, f(x) < y < g(x)\}$$

  where $f, g : P^{k-1} \to P$ are functions either of the form $s^N(\pi_i(x))$ (for $N \in \mathbb{Z}$ and π_i the projection onto the i-th coordinate for some i) or constant and such that there is no bound on

  $$\{\#(P \cap (f(x), g(x))) : \ x \in \pi(X)\}.$$

Remark 4.3.3. Given a discrete ordered structure $\langle M, < \rangle$, this structure is o-minimal and the cell decomposition theorem is a decomposition into s-cells.

In the following we set P to be the range of an iteration sequence.

Fact 4.3.4. *The structure $\widetilde{\mathcal{R}}$ is d-minimal (see Miller and Tyne's [48]). Moreover, $\langle \mathcal{R}, P, s, s^{-1}, \lambda \rangle$ has QE up to QE in $\mathcal{R}$ and every definable set $X \subseteq \mathbb{R}^m$ has a decomposition into finitely many sets of the form:*

$$\{x \in \mathbb{R}^m : \ f_i(x) = 0,\ g_i(x) > 0\}$$

where f_i and g_i are compositional iterates of basic functions (that are λ, s, s^{-1} and $\mathcal{L}$-definable functions).

Proposition 4.3.5. *For every definable set X there is a decomposition of X into finitely many sets Y_i of the form $Y_i = \bigcup_{t \in S_i} Y_t$ where $S_i \subseteq P^k$ is an s-cell and there is an $\mathcal{L}$-definable family $\{Y_t : \ t \in A \subseteq \mathbb{R}^k\}$.*

Proof. We begin with a claim.

Claim. *Let h be a composition of basic functions. Then there is an $\mathcal{L}$-definable function f and an s-cell $S \subseteq P'^k$, such that for all $x \in \mathrm{dom}(h)$,*

$$h(x) = z \text{ if and only if there is } y \in S \text{ such that } f(x, y) = z. \qquad (*)$$

Proof of Claim. By induction on the number of iterations of basic functions which compose h.

For $h = \lambda$, let $S = \Gamma_s \subseteq P'^2$ and $f(x, y_1, y_2) = y_1$ if $y_1 \leq x < y_2$, and not defined otherwise. We verify $(*)$. If $\lambda(x) = z$ then $f(x, z, s(z)) = z$. By definition of f, if $f(x, y_1, y_2) = y_1$, then $y_1 \leq x < y_2$, and since $(y_1, y_2) \in \Gamma_s$, we have $y_2 = s(y_1)$ and $\lambda(x) = y_1$. Furthermore, we see that if there is $y \in S$ such that $f(x, y)$ is defined then $f(x, y) = h(x)$. The cases $h = s$, s^{-1} are similar and the case where h is $\mathcal{L}$-definable is straightforward.

Now let $h = h_{n+1}(h_1, \ldots, h_k)$ where h_{n+1} is a basic function and assume that for $1 \leq j \leq k$, there are some $\mathcal{L}$-definable functions $h'_j(x, y)$ and definable $S_j \subseteq P'^{k_j}$ such that for all $x \in \mathrm{dom}(h)$,

$$h_j(x) = z \text{ if and only if there is } y \in S_j \text{ such that } h_j(x) = h'_j(x, y).$$

For $h_{n+1} = \lambda$ (thus $k = 1$), we define f exactly similarly to the last paragraph, namely $f(x, a_1, a_2, y) = a_1$ if $h'_1(x, y)$ is defined and $a_1 \leq h'_1(x, y) < a_2$, and not defined otherwise. We verify $(*)$. If $h(x) = a_1$ then there are $(a_1, a_2) \in \Gamma_s, y_1 \in S_1$ such that $a_1 \leq h'_1(x, y_1) < a_2$ and $h'_1(x, y_1) = h_1(x)$. Thus $f(x, a_1, a_2, y_1) = a_1$. If there is $y \in S_1$ such that $h'_1(x, y)$ is defined then $h'_1(x, y) = h_1(x)$ and if there are $(a_1, a_2) \in \Gamma_s$ such that $f(x, y, a_1, a_2)$ is defined (that is, $a_1 \leq h'_1(x, y) < a_2$) then $h(x) = a_1 = f(x, y, a_1, a_2)$.

Again, the cases $h_{n+1} = s, s^{-1}$ are similar and the case h_{n+1} $\mathcal{L}$-definable is straightforward.

The fact that the small sets produced along this proof are s-cells is straightforward. □

Now let X be a definable set. By Fact 4.3.4, we may assume that there are $f_1, \ldots, f_k$ and $g_1, \ldots, g_{k'}$, which are compositional iterates of basic functions, such that

$$X = \{x \in \mathbb{R}^l : \ \forall i \leq k, j \leq k', \ f_i(x) = 0, \ g_j(x) > 0\}$$

Let f'_i, S_i the maps and sets of dimension 0 given by the claim for $h = f_i$, and g'_j, K_j, for $h = g_j$. Ie for every $i \leq k$, $j \leq k'$, we have that

$$f_i^{-1}(0) = \bigcup_{t \in S_i} f'_i(-, t)^{-1}(0),$$

$$g_j^{-1}(\mathbb{R}^{>0}) = \bigcup_{t \in K_j} g'_j(-, t)^{-1}(\mathbb{R}^{>0}).$$

Note that for every i, j by uniform cell decomposition in o-minimal structures, we may assume that for every $t \in S_i$, $t' \in K_j$ $f_i^{-1}(-, t)(0)$ as well as $g_j^{-1}(-, t')(\mathbb{R}^{>0})$ are cells. Thus X has the form

$$\bigcap_{s \leq m} \bigcup_{t \in S_s} Y_{s,t}$$

where $m = k + k'$, for every $s \leq m$, $t \in S_s$, $Y_{s,t}$ is a cell.

To prove that X has the form $\bigcup_{t \in S} X_t$ where $\{X_t :\ t \in S\}$ is a small subfamily of an $\mathcal{L}$-definable family of cells, by an easy induction it is sufficient to prove that the intersection of two sets of the form $\bigcup_{t \in S'} Y_t$ where there is an $\mathcal{L}$-definable family of cells $\{Y_t :\ t \in A\}$ and $S' \subseteq A$ has dimension 0 is itself a set of this form. Let $X_1 = \bigcup_{t \in S_1} X_{1,t}$ and $X_2 = \bigcup_{t \in S_2} X_{2,t}$ where there are two $\mathcal{L}$-definable families of cells $\{X_{i,t} :\ t \in A_i\}$ and $S_i \subseteq A_i$ being of dimension 0. Then

$$X_1 \cap X_2 = \bigcup_{(t_1, t_2) \in S_1 \times S_2} X_{1,t_1} \cap X_{2,t_2}$$

and the family $\{X_{1,t_1} \cap X_{2,t_2} :\ t_1 \in S_1,\ t_2 \in S_2\}$ is a small subfamily of the $\mathcal{L}$-definable family $Z = \{X_{1,t_1} \cap X_{2,t_2} :\ t_1 \in A_1,\ t_2 \in A_2\}$. Moreover, by uniform cell decomposition in o-minimal structures, we may assume that Z is a family of cells.

It is easy to check that the small sets S, produced all along this proof were s-cells. This proves the result. □

4.3.1 *P-internality*

Proposition 4.3.6. *Let P be an iteration sequence. Then $\widetilde{\mathcal{R}}$ is P-internal.*

Proof. Let S be a small set. By Proposition 4.3.5, we may assume that there is an $\mathcal{L}$-definable function $f : \mathbb{R}^{n+k} \to \mathbb{R}$ and a set $S' \subseteq P^k$ such that

$$\text{either } S = \bigcup_{t \in S'} f_t^{-1}(0)$$

$$\text{or } S = \bigcup_{t \in S'} \{x \in \mathbb{R}^n :\ f_t(x) > 0\}.$$

In the first case, since S is small, for every $t \in S'$, 0_{f_t} is small and thus is finite since f is $\mathcal{L}$-definable. Therefore, by uniform finiteness in o-minimal structures, there are finitely many $\mathcal{L}$-definable functions $g_1, \ldots, g_m : \mathbb{R}^k \to \mathbb{R}$ such that for every $t \in S'$

$$0_{f_t} = \{g_1(t), \ldots, g_m(t)\}$$

and we have the result.

In the second case, by o-minimality, we may assume that f is continuous and that $\mathrm{dom}(f_t)$ is connected. Thus for every $t \in S'$, since

$$\{x \in \mathbb{R}^n :\ f_t(x) > 0\} \subseteq S,$$

$$\{x \in \mathbb{R}^n : f_t(x) > 0\} \text{ is small}$$

and by continuity, f_t is constant and $\mathrm{dom}(f_t)$ is finite. Therefore, $\mathrm{dom}(f_t)$ is a singleton by connectedness. The result follows easily from there. □

4.3.2 Structure induced on P

In this subsection, we prove that $P_{ind} = \langle P, < \rangle$. More precisely, we show that every definable set is a finite union of s-cells.

Theorem 4.3.7. *1. Let $X \subseteq \mathbb{R}^k$ be an $\mathcal{L}$-definable set and $S \subseteq P^k$ being an s-cell. Then there is a finite decomposition of $X \cap S$ into s-cells.*

2. Let $S' \subseteq P^k$ be a definable set. Then there is a finite decomposition of S' into s-cells.

Proof. We first show how 1 easily implies 2. By Proposition 4.3.6 and 1, we may assume that S' has the form $\bigcup_{t \in S''} g(t)$, for some $\mathcal{L}$-definable $g : \mathbb{R}^n \to \mathbb{R}^k$ and some s-cell $S'' \subseteq P^n$. Since the intersection of two s-cells is an s-cell and that the projection of an s-cell is an s-cell, we just have to observe that

$$S' = \pi\big(S'' \times P^k \cap (\Gamma(g) \cap P^{k+n})\big)$$

where π is the projection on the last k coordinates.

We prove 1. Since an s-cell is of the form $P^k \cap Y$ where Y is $\mathcal{L}$-definable, we may assume that $S = P^k$. Moreover, by o-minimality we may assume that X is a cell. We do an induction on k. For $k = 0$, the result is trivial. We assume that the result holds for $k = N$, we prove it for $k = N + 1$. We first prove the folowing claim:

Claim 4.3.8. *Let $f : P^n \to P$ be a definable function such that for every $x \in P^{n-1}$, f_x is unbounded and ultimately growing. Then there are $N_1, \ldots, N_k \in \mathbb{Z}$, for every $x \in P^{n-1}$ there is $i \leq k$ such that $s^{N_i} = f_x$ ultimately.*

Proof. First of all, by definition of an iteration sequence, for every $x \in P^{n-1}$ there is some $N_1 \in \mathbb{Z}$ such that $f_x < s^{N_1}$ ultimately. Note that f_x^{-1} is defined ultimately since f unbounded and growing ultimately, by induction f is equal to s^n ultimately. By observing that there is $N_2 \in \mathbb{Z}$ such that $f^{-1} < s^{N_2}$ ultimately. We assume that $f > Id$ ultimately (the case $f \leq Id$ is similar). We have $f^{-1} < s^{N_2}$ ultimately and thus $f > s^{-N_2}$ ultimately. Therefore

$$s^{-N_2} < f < s^{N_1} \text{ ultimately.}$$

By o-minimality, there is a unique $m \in (-N_2, N_1) \cap \mathbb{Z}$ such that

$$s^m \leq f < s^{m+1} \text{ ultimately.}$$

Since f is from P^k to P,

$$f = s^m \text{ ultimately.}$$

We assume for contradiction that there is no uniform bound m such that for every $x \in P^n$, $f_x < s^m$ ultimately. We define $g : P^n \to \mathbb{R}$, $x \mapsto y$ where y is a choice among

$$X_x = \{y' \in P : \text{ for every } x' <_{lex} x \; s(f_{x'})(z) < f_x(z)\}$$

and $g(x) = 0$ if there is no such y.

We prove that
$$Z = \{x \in P^{n-1} : \; X_x \neq \emptyset\}$$
is non-empty and unbounded in P^n. Since for $x \in P^n$,

$$\{x' \in P^n : \; x' <_{lex} x\} = \bigcup_{1 \leq i \leq n-1} \bigcup_{y < x_i} \{(x_1, \ldots, x_{i-1}, y)\} \times P^{n-i},$$

by considering
$$f_{i,y} = f \restriction \{(x_1, \ldots, x_{i-1}, y)\} \times P^{n-i+1}$$
for i and $y \in P^{<x_i}$ (there are only finitely many such y) and by applying the induction hypothesis, we get that the $f_{i,y}$'s are ultimately equal to $s^m \circ \pi_j$ (for some $m \in \mathbb{Z}$ and $j \leq n-1$). Therefore, there is a bound m_x such that for every $x' <_{lex} x$, $f_{x'} < s^{m_x}$ ultimately. Thus, if Z is bounded, we take a the maximal element of Z (for the lexicographical order). By definition of the X_x's, for every $x \in P^n$, $f_x < s^{m_a}$ ultimately and this is a contradiction.

We define
$$h : x \in P^{n-1} \mapsto f_x(g(x)).$$
Since $\Gamma_h \subseteq P^n$, by induction we may assume that $h = s^a \circ \pi_j$ (for some $a \in \mathbb{Z}$, $j \leq n-1$). This is a contradiction since by construction, h does not have such a bound. Therefore, there is an upper bound $m_1 \in \mathbb{Z}$ such that for every $x \in P^{n-1}$, $f_x < s^{m_1}$ ultimately.

Without loss of generality, we may assume that for every $x \in P^{n-1}$ f_x is ultimately injective. Therefore, f_x is ultimately equal to s^a (for some $a \in \mathbb{Z}$). Thus f_x is ultimately bijective and f_x^{-1} is defined ultimately. Using exactly the same argument as in the first paragraph, we have that there are finitely many $m_i \in \mathbb{Z}$ such that for every $x \in P^{n-1}$

$$\bigvee_i f_x = s^{m_i}(x) \text{ ultimately}$$

and we have the result. □

We first assume that $X = \{(x, y) \in \mathbb{R}^k : \; x \in B, y = f(x)\}$ where $f : \mathbb{R}^{k-1} \to \mathbb{R}$ is $\mathcal{L}$-definable and $B \subseteq \mathbb{R}^{k-1}$ is a cell. By induction, we may assume that $B \cap P^{k-1}$ is an s-cell. There are three cases for the behaviour of f_x (for $x \in \mathbb{R}^{k-2}$) that we need to look at:

- <u>case 1</u>: $\mathrm{dom}(f_x)$ is of the form $(a, +\infty)$ (for some $a \in \mathbb{R}$), f_x is ultimately increasing and $\mathrm{Im}(f_x)$ is unbounded,

- <u>case 2</u>: $\mathrm{dom}(f_x)$ is of the form $(a, +\infty)$ (for $a \in \mathbb{R}$), f_x and $\mathrm{Im}(f_x)$ is bounded
- <u>case 3</u>: $\mathrm{dom}(f_x)$ is bounded or is of the form $(a, +\infty)$ (for $a \in \mathbb{R}$) but is decreasing ultimately and $\mathrm{Im}(f_x)$ is unbounded.

<u>*case 1:*</u> By Claim 4.3.8 applied to $f_{\restriction P^n}$, we may assume that there is $n \in \mathbb{Z}$ such that ultimately

$$s^n \leq f_x < s^{n+1}.$$

Thus by o-minality, there are two cases: either ultimately $f_x = s^n$ or it is not the case and by shrinking $\mathrm{dom}(f_x)$, we may assume that one or the other holds for every $x \in \mathbb{R}^{k-1}$, for every $y \in \mathrm{dom}(f_x)$. In the first case, it is easy to see that S is an s-cell and in the second case, since for $y \in P$,

$$s^n(y) < f_x(y) < s^{n+1}(y),$$

$$f_x(y) \notin P$$

and thus

$$X \cap P^k = \emptyset.$$

<u>case 2:</u> By o-minimality, for every $x \in \mathbb{R}^{k-1}$ there are three possible cases for the ultimate behaviour of f_x and by shrinking the domain of f_x, we may assume that for every x, f_x is increasing, decreasing or constant. In the first and the second cases, f_x is injective and by considering f_x^{-1} and by permuting the last two coordinates, we will handle this in *case* 3. In the third case, let $c_x = f_x(y)$ for any $y \in \mathrm{dom}(f_x)$. Let

$$g : \mathbb{R}^{k-2} \to \mathbb{R},\ x \mapsto c_x.$$

By Induction we may assume that either for every $x \in \mathbb{R}^{k-2}$, $g_x = s^n(\pi_i(x))$ (for some $n \in \mathbb{Z}$ and some i) or $g_x(P) \cap P = \emptyset$. In the first case, it is easy to see that S is an s-cell and in the second case, $S = \emptyset$.

<u>case 3:</u> We first observe that if $\mathrm{dom}(f_x)$ is cofinal but f_x is ultimately decreasing and $\mathrm{Im}(f_x)$ is unbounded then there is $b \in \mathbb{R}$ such that $f_x(y) < 0$ and thus $f_x(y) \notin P$ for $y \in (b, +\infty)$. Therefore we may assume that $\mathrm{dom}(f_x)$ is bounded. Let

$$g : P^{k-2} \to P,\ x \mapsto \lambda(|\mathrm{dom}(f_x)|).$$

By Induction, we may assume that Γ_g is an s-cell and $g = s^n(\pi_i(x))$ for some $n \in \mathbb{Z}$ and some $i \leq k-2$. Thus

$$\text{if } y \in \mathrm{dom}(f_x) \text{ then } y \leq s^n(\pi_i(x))$$

By a simple induction, we can do this decomposition for every coordinate (we just did it for the last one) and thus, for $x \in S'$ we have $k-1$ inequalities of the form:

$$I_i : x_i \leq s_{n_i}(\pi_{j_i}(x))$$

for some $n_i \in \mathbb{Z}$, $i \neq j_i \leq k-1$. By combining these inequalities, we get:

$$x_1 \leq s^{n_1}(\pi_{i_1}(x)) \leq s^{n_{i_1}}(\pi_{i_{i_1}}(x)) \leq \dots$$

which is an expression with k terms. Observe that from the form of the I_i, this inequality does not contain a subsequence of the form $s^N(x_i) \leq s^{N'}(x_i)$. Thus by a simple application of the pigeonhole principle, it contains a subinequality of the form

$$s^N(x_i) \leq s^{N''}(x_j) \leq s^{N'}(x_i).$$

For $d \in \mathbb{N}$, let

$$S'_d = \{x \in S' : \ \pi_i(x) = s^d(\pi_j(x))\}$$

and

$$P^{k-1}_d = \{x \in P^{k-1} : \ \pi_i(x) = s^d(\pi_j(x))\}.$$

We have that

$$S' = \bigcup_{N-N'' \leq d \leq N'-N''} S'_d.$$

Since each of the P^{k-1}_d is in bijection with P^{k-2} (bijection that is definable in $\langle P, <\rangle$) and that $S'_d \subseteq P^{k-1}_d$, by induction $\Gamma\big(f_{\restriction S'_d}\big)$ has a decomposition into finitely many s-cells and we have the result.

The case where $X = \{(x,y) \in \mathbb{R}^k : \ x \in \pi(X),\ f(x) < y < g(x)\}$ follows easily from the last case by replacing f and g by $s(\lambda(f))$ and $\lambda(g)$. □

4.3.3 Local o-minimality for expansions of semibounded structures

If $\mathcal{R}$ is semibounded, then since any unary function is ultimately affine, $\widetilde{\mathcal{R}}$ is interdefinable with $\langle \mathcal{R}, (a^n x + \sum_{i<n} a^i b)_{n\in\mathbb{N}}\rangle$ for some $a \in \mathbb{Q}^{>1}$, $b, x \in \mathbb{R}$. Moreover,

$$a^n x + \sum_{i<n} a^i b = a^n x + \frac{a^n - 1}{a-1} b = \frac{a^n(ax - x + b)}{a-1} - \frac{b}{a-1}.$$

Thus $\widetilde{\mathcal{R}}$ is interdefinable with $\langle \mathcal{R}, (ca^n)_n\rangle$ for $c = (ax - x + b)$.

Lemma 4.3.9. *Let P be an iteration sequence for $\mathcal{R}$. Then $\widetilde{\mathcal{R}}$ is locally o-minimal.*

Proof. As we saw before, we may assume that $P = (a^n)$ for some $a \in \mathbb{Q}^{>1}$. Thus, P is also an iteration sequence for $\mathbb{R}_{gp}$ and by Lemma 4.1.4, we may assume that $\mathcal{R} = \mathbb{R}_{gp}$.

Let $\Lambda : \mathbb{R}^k \to \mathbb{R}$ be a linear function. By P-internality, it is sufficient to prove that $\lambda(P^k)$ is closed and discrete. We do a recursion on k. For $k = 0$, the result is trivial. We assume that the result holds for $k < N$ and prove it for N.

We first observe that for any permutation of coordinates σ from $\mathbb{R}^N$ to $\mathbb{R}^N$ and for $x \in P$,

$$f\big(\sigma(\{x\} \times P^{N-1})\big)$$

is closed and discrete by induction.

For $m \in \mathbb{N}$, we set $x_m \in P^N$ the smallest element in P^N for the lexicographical order such that $|f(x)| < 1/m$. As we saw in the last paragraph, there is $b \in \mathbb{N}$ such that for every $i \leq N$, $s > b$, $\pi_i(x_s) > 1$. Let $x_s = (a^{n_i})_i$ for $s > b$. We have that

$$\Lambda(x_s) = \sum_i \Lambda_i a^{n_i} = a\big(\sum_i \Lambda_i (a^{n_i - 1})\big)$$

and since $a > 1$,

$$|\Lambda(x_s)| > \big|\sum_i \Lambda_i (a^{n_i-1})\big|$$

which is a contradiction with the choice of x_s. □

Remark 4.3.10. We could have adapted the proof of [38, Proposition 25]. The Claim A is exactly what we are proving. Anyway, they did not provide a proper proof.

4.4 Expansions of an o-minimal structures by a fast sequence

Definition 4.4.1. We say that a sequence $(x_n)_n$ is $\mathcal{R}$-fast if for every $\mathcal{L}$-definable function $f : \mathbb{R} \to \mathbb{R}$, there is $N \in \mathbb{N}$ such that for every $k > N$, $f(x_k) < x_{k+1}$.

Actually a QE result similar to the one for expansions by iteration sequences holds for expansions by fast sequences. Unfortunately it is an unpublished result of Trent Ohl (a student of Chris Miller). Even if it is unlikely to be published any time soon, we still have to respect the author and we can not provide a proof. Thus we will use it as a black box. Anyway, the only properties we need in chapter 4 and 5 are d-minimality and local o-minimality for expansions of semibounded structures which are not coming from this result.

Black Box 4.4.2. *The structure $\langle \mathcal{R}, P, s, s^{-1}, \lambda \rangle$ eliminates quantifiers up to QE in $\mathcal{R}$.*

Thus every set $X \subseteq \mathbb{R}^m$ has a decomposition into finitely many sets of the form:

$$\{x \in \mathbb{R}^m :\ f_i(x) = 0,\ g_i(x) > 0\}$$

where f_i and g_i are compositional iterates of basic functions (that are λ, s, s^{-1} and $\mathcal{L}$-definable functions).

Fact 4.4.3. *The structure $\widetilde{\mathcal{R}}$ is d-minimal and if $\mathcal{R} = \mathbb{R}_{gp}$ or $\mathbb{R}_{sb}$ then $\widetilde{\mathcal{R}}$ is locally o-minimal (see Friedman and Miller's [28]).*

Proposition 4.4.4. *Let P be an $\mathcal{R}$-fast sequence. Then $\widetilde{\mathcal{R}}$ is P-internal.*

Proof. It is exactly what Friedman and Miller are showing in [28, 2.2 and 3.6]. □

Proposition 4.4.5. *The induced structure on P by $\widetilde{\mathcal{R}}$ is $\langle P, < \rangle$.*

Proof. It is almost the same proof than for expansions of o-minimal structures by an iteration sequence. □

5. NO NEW SMOOTH FUNCTIONS

In this chapter, we prove that under some assumptions (which cover d-minimality) an expansion of $\mathbb{R}_{gp}$ does not define any non-$\mathcal{L}$-definable smooth function over an $\mathcal{L}$-definable domain.

5.1 Assumptions

Let $\mathcal{R}$ be an o-minimal expansion of $\mathbb{R}_{gp}$ and let $P \subseteq \mathbb{R}$ be a set of dimension 0.

Definition 5.1.1. Let $Y \subseteq X \subseteq \mathbb{R}^n$ be two sets. We say that Y is an *$\mathcal{L}$-chunk of* X if it is an $\mathcal{L}$-definable cell, $\dim Y = \dim X$, and for every $y \in Y$, there is an open box $B \subseteq \mathbb{R}^n$ containing y such that $B \cap X \subseteq Y$. In other words, Y is relatively open in X.

Definition 5.1.2. We say that $\widetilde{\mathcal{R}}$ has the *decomposition property* (DP) if for every definable set $X \subseteq \mathbb{R}^n$,

DP(I) there is an $\mathcal{L}$-decomposition of X,

DP(II) X contains an $\mathcal{L}$-chunk.

Definition 5.1.3. We say that $\widetilde{\mathcal{R}}$ has the *dimension property* (DIM) if for every definable family $\{X_t\}_{t\in S}$, with $\dim S = 0$, we have

$$\dim \overline{\bigcup_{t\in S} X_t} = \max_{t\in S} \dim X_t.$$

If $\widetilde{\mathcal{R}}$ is d-minimal then it satisfies (DP) and (DIM). The property (DIM) is just Proposition 3.1.23. The property DP(I) is coming from Theorem 3.3.7 for expansions of $\overline{\mathbb{R}}$ and of Theorem 3.3.16 for expansions of $\mathbb{R}_{gp}$ and $\mathbb{R}_{sb}$. We just have to show DP(II).

Proposition 5.1.4. *If $\langle \mathcal{R}, P\rangle$ is d-minimal then it has DP(II).*

Proof. Let $X \subseteq \mathbb{R}^m$ be a definable set of dimension n. We are doing a recursion on the number k of $\mathcal{L}$-EM of dimension n in the decomposition of X. For $k = 1$, $X = \bigcup_{t\in S} X_t \cup LD$ where $\{X_t :\ t \in S\}$ is a small family of cells such that for every $t' \in S$, $X_{t'}$ is an $\mathcal{L}$-chunk of $\bigcup_{t\in S} X_t$ and LD has lower dimension than n. Let $t \in S$ and B a box such that $B \cap \bigcup_{t'\in S} X_{t'} \subseteq X_t$. By (DIM), LD is nowhere dense in $B \cap X_t$ and thus, we may assume that $B \cap LD = \emptyset$. It gives us the result.

We assume that the result holds for $X = \bigcup_{i<k} Y_i \cup LD$ where the Y_i's are $\mathcal{L}$-EM of dimension m and LD has lower dimension. Let $X = \bigcup_{i\leq k} Y_i \cup LD$ where the Y_i's are $\mathcal{L}$-EM of dimension m and LD has lower dimension. Let $Y_k = \bigcup_{t\in S} X_t$ where every X_t is an $\mathcal{L}$-chunk of Y_k and B be a box such that $B \cap Y_k \subseteq Y_t$ for some $t \in S$. Using the same argument as in the previous paragraph, we may assume that $B \cap LD = \emptyset$.

There are two cases: either $B \cap \bigcup_{i<k} Y_i = \emptyset$ and we have the result, or it is not the case. Since $B \cap Y_i$ is an $\mathcal{L}$-EM of dimension n for every i, $Y = \bigcup_{i<k} Y_i \cap B \setminus X_t$ admits a decomposition into less than k $\mathcal{L}$-EM. We apply the induction hypothesis to get a box B' such that $B' \cap Y$ is an $\mathcal{L}$-chunk of Y of dimension n. Moreover, by taking B' small enough, we may assume that $B' \cap X_t = \emptyset$ and we have the result. □

Remark 5.1.5. Note that if $\langle \mathcal{R}, P\rangle$ has (DIM) then P is sparse.

Proof. Let $X = f(P^k)$. By (DIM),

$$\dim(X) = \dim\big(\bigcup_{t\in P^k} \{f(t)\}\big) = \max_{t\in P^k}\big(\dim(\{f(f)\})\big) = 0.$$

□

We finish this section by showing that in most d-minimal settings, it is possible to define new $\mathcal{C}^n$ functions for every $n \in \mathbb{N}$.

Claim 5.1.6. *We assume that $\mathcal{R}$ expands $\mathbb{R}_{sb}$ and $\widetilde{\mathcal{R}}$ is d-minimal non o-minimal. Then $\widetilde{\mathcal{R}}$ defines new $\mathcal{C}^n$-functions for every $n \in \mathbb{N}$*

Proof. Let $S = (t_n)_{n\in\mathbb{N}}$ be definable in $\widetilde{\mathcal{R}}$. By d-minimality, we may assume that S is the range of a strictly growing sequence. We only study two cases, but any other situation boils down to these cases.

Case 1: $(t_{n+1} - t_n)_n$ is bounded
Let $n \in \mathbb{N}$

$$f_n : (t_n, t_{n+1}) \to \mathbb{R},\ x \mapsto (x - t_n)^m (x - t_{n+1})^m.$$

Since $(t_{n+1} - t_n)_{n\in\mathbb{N}}$ is bounded, the f_n's are uniformly $\mathcal{L}$-definable and we can define

$$h : \mathrm{conv}(S) \to \mathbb{R},\ \Gamma_h = \overline{\bigcup_{n\in\mathbb{N}} \Gamma(f_n)}$$

Case 2: $(t_{n+1} - t_n)_n$ is strictly growing and unbounded
First, we may assume that for all $t_n \in S$, $(t_{n+1} - t_n) > 1$. For $n \in \mathbb{N}$ let

$$f_n : (t_n, t_n + 1)) \to \mathbb{R},\ x \mapsto (x - t_n)^m (x - t_n - 1)^m$$

and

$$g_n : (t_n + 1, t_{n+1}) \to \mathbb{R},\ x \mapsto 0.$$

Let

$$h : \mathrm{conv}(S) \to \mathbb{R},\ \Gamma_h = \overline{\bigcup_{n\in\mathbb{N}} \Gamma(f_n^\frown g_n)}$$

In both cases, $h \in \mathcal{C}^{m-1}$. □

Remark 5.1.7. The above examples are similar to the examples of Miller and Thamrongthanyalak's [50].

The goal of this chapter is to prove the following Theorem:

Theorem 5.1.8. *Let $\mathcal{R} = \mathbb{R}_{gp}$, $\mathbb{R}_{sb}$ or $\overline{\mathbb{R}}$ and let $P \subseteq \mathbb{R}$ be a set of dimension 0 such that $\widetilde{\mathcal{R}}$ satisfies (DP) and (DIM). Let f be a smooth definable function with an open semialgebraic domain. Then f is definable in $\mathcal{R}$.*

5.2 Proof of Theorem 5.1.8 for expansions of the real field

Before proving Theorem 5.1.8 in the case $\mathcal{R} = \overline{\mathbb{R}}$, we need some preliminaries.

For the rest of this section we fix $P \subseteq \mathbb{R}$ a set of dimension 0 such that $\langle \overline{\mathbb{R}}, P \rangle$ satisfies (DP) and (DIM).

If $\mathcal{C}$ is a collection of subsets in $\mathbb{R}^{n+1}$, by $\pi(\mathcal{C})$ we denote the collection of their projections on $\mathbb{R}^n$.

Lemma 5.2.1. *Let $X \subseteq \mathbb{R}^n$ be an open semialgebraic set and let $Y \subseteq \overline{X}$ be a definable set, such that $\dim(X \setminus Y) < n$. Then $\overline{Y} = \overline{X}$.*

Proof. Let $x \in \overline{X}$. Let $B \subseteq \mathbb{R}^n$ be any open box containing x. Since X is open, there is an open box $B' \subseteq B \cap X$. Since $\dim(X \setminus Y) < n$, $B' \cap Y \neq \emptyset$, and hence $B \cap Y \neq \emptyset$. Thus $x \in \overline{Y}$, as needed. □

Lemma 5.2.2. *Let $Y \subseteq \mathbb{R}^n$ be an open set. Then $\dim \operatorname{bd}(Y) < n$.*

Proof. Let us assume for contradiction that $\dim \operatorname{bd}(Y) = n$. By definition of the dimension, $\operatorname{bd}(Y)$ contains an open box B. Since $B \subseteq \operatorname{bd}(Y)$, we have $B \cap Y = \emptyset$. But then $X \cap \overline{Y} = \emptyset$, a contradiction. □

Lemma 5.2.3. *Let A_1, $A_2 \subseteq \mathbb{R}^n$ be two open, connected and disjoint semialgebraic sets. Suppose there is an open box $B \subseteq \operatorname{int}(\overline{A_1} \cup \overline{A_2})$, such that $B \cap A_i \neq \emptyset$, for $i = 1, 2$. Then $\operatorname{int}(\overline{A_1} \cup \overline{A_2})$ is connected.*

Proof. Denote $X = \operatorname{int}(\overline{A_1} \cup \overline{A_2})$, and take any $x, y \in X$. We need to find a semialgebraic path in X connecting x and y. Take an open box $B_1 \subseteq X$ containing x. Clearly, B_1 must intersect A_i, for some $i = 1, 2$, and since B_1 is connected, we may assume that $x \in A_i$. Similarly, we may assume $y \in A_i$, for some $i = 1, 2$. By assumption, there is an open box $B \subseteq X$ with each $B \cap A_i \neq \emptyset$. Since all of B, A_1 and A_2 are connected and contained in X, the result easily follows. □

Proposition 5.2.4. *Assume* (DP)(II). *Let $\{W_t : t \in S\}$ be a definable family of open sets in $\mathbb{R}^n$, such that:*

(i) *for every $t \in S$, $W_t = \operatorname{int}(\overline{W_t})$,*

(ii) *there is an open cell $X \subseteq \mathbb{R}^n$, such that $\overline{X} = \overline{\bigcup_{t \in S} W_t}$,*

(iii) *for every $t, t' \in S$, $W_t \neq W_{t'} \Rightarrow W_t \cap W_{t'} = \emptyset$,*

(iv) there are $t, t' \in S$, $W_t \neq W_{t'}$.

Then there are an open box $B \subseteq X$ and $W_{t_1} \neq W_{t_2}$, such that

1. *$B \cap W_{t_i}$ is semialgebraic, open and connected, $i = 1, 2$,*
2. *$\operatorname{int}(\overline{B \cap W_{t_1}} \cup \overline{B \cap W_{t_2}})$ is connected.*

Proof. By (iii) and since each W_t is open, we have $\bigcup_{t\in S} \operatorname{bd}(W_t) \subseteq \operatorname{bd}(\bigcup_{t\in S} W_t)$. By Lemma 5.2.2, $\dim \bigcup_{t\in S} \operatorname{bd}(W_t) \leq n-1$.

Claim. $\dim \bigcup_{t\in S} \operatorname{bd}(W_t) \cap X = n-1$.

Proof of Claim. It suffices to find $t \in S$, such that $\dim \operatorname{bd}(W_t) \cap X = n-1$. By induction on n. For $n = 1$, X is an open interval and the statement follows easily from (iv) and (DC). Let $n > 0$. We split into two cases.

Case I: For all $t \in S$, $\pi^{-1}(\pi(W_t)) \cap X \subseteq W_t$. It is then not hard to see that for all $t \in S$ and $x \in \operatorname{bd}(\pi(W_t))$ (which is not empty by (DC)),

$$\pi^{-1}(x) \cap X \subseteq \operatorname{bd}(W_t) \cap X,$$

and the family $\{\pi(W_t) : t \in S\}$ satisfies Conditions (i)-(iv). By inductive hypothesis, there is $t \in S$, such that $\operatorname{bd}(\pi(W_t)) \cap \pi(X)$ has dimension $n-2$. Thus, since X is an open cell, the set

$$\pi^{-1}(\operatorname{bd}(\pi(W_t)) \cap \pi(X)) \cap X \subseteq \operatorname{bd}(W_t) \cap X$$

has dimension $n-1$.

Case II: Otherwise, there are $t_1, t_2 \in S$ with $W_{t_1} \neq W_{t_2}$ and $x_i \in W_{t_i}$, such that $\pi(x_1) = \pi(x_2)$. Since each W_{t_i} is open, the set $A \subseteq \pi(W_{t_1}) \cap \pi(W_{t_2}) \subseteq \mathbb{R}^{n-1}$ is open. It suffices to show that for all $x \in A$, there is $y \in \operatorname{bd}(W_{t_1}) \cap X$, such that $\pi(y) = x$. Since X is an open cell, by (DC) $\pi^{-1}(x) \cap X$ is connected. The sets W_{t_i} being disjoint by (iii), and each $\pi^{-1}(x) \cap W_{t_i} \neq \emptyset$ open, we obtain the result. □

By the claim and (DP)(II), there is a semialgebraic set $Y \subseteq \bigcup_{t\in S} \operatorname{bd}(W_t) \cap X$ of dimension $n-1$, and an open box $B \subseteq \mathbb{R}^n$, such that

$$\emptyset \neq B \cap \bigcup_{t\in S} \operatorname{bd}(W_t) \cap X \subseteq Y.$$

Since X is open, we may assume $B \subseteq X$. Moreover, since Y is semialgebraic and has dimension $n-1$, by taking B small enough, we may assume that $B \setminus Y$ has two open semialgebraic connected components, B_1 and B_2. By (ii), Lemma 5.2.2, and since each $B_i \subseteq X$, we can find $t_1, t_2 \in S$, such that $B_i \cap W_{t_i} \neq \emptyset$, $i = 1, 2$.

We next show that $W_{t_1} \neq W_{t_2}$. First observe that since $B \cap \bigcup_{t\in S} \operatorname{bd}(W_t) \subseteq Y$ and $B_i \cap Y = \emptyset$, we have $B_i \cap \bigcup_{t\in S} \operatorname{bd}(W_t) = \emptyset$. In particular, $B_i \cap \operatorname{bd}(W_{t_i}) = \emptyset$.

Therefore, $B_i \subseteq W_{t_i}$, $i = 1, 2$. Since B is an open box and $\dim(B \setminus (B_1 \cup B_2)) < n$, we have

$$(*) \quad B \subseteq \text{int}(\overline{B_1} \cup \overline{B_2}) \subseteq \text{int}(\overline{W_{t_1}} \cup \overline{W_{t_2}}).$$

If $W_{t_1} = W_{t_2}$, then by (i), we obtain

$$B \subseteq \text{int}(\overline{B_1} \cup \overline{B_2}) = \text{int}(\overline{W_{t_1}} \cup \overline{W_{t_2}}) = \text{int}(\overline{W_{t_1}}) = W_{t_1},$$

contradicting $B \cap \bigcup_{t \in S} \text{bd}(W_t) \neq \emptyset$, (iii) &(iv).

Now we are ready to show (1). It suffices to prove that $B \cap W_{t_i} = B_i$, for $i = 1, 2$. As shown earlier, we have $B_i \subseteq W_{t_i} \cap B$. On the other hand, observe that

$$B \cap W_{t_1} = (B_1 \cap W_{t_1}) \cup (B_2 \cap W_{t_1}) \cup (B \cap Y \cap W_{t_1}).$$

Since $B_2 \subseteq W_{t_2}$ and $W_{t_1} \cap W_{t_2} = \emptyset$, the second part of the above union is empty. Moreover, since $Y \subseteq \bigcup_{t \in S} \text{bd}(W_t)$, we have $Y \cap W_{t_1} = \emptyset$. Therefore, the third part of the above union is also empty. Hence $B \cap W_{t_1} = B_1$. Similarly, one shows $B \cap W_{t_2} = B_2$, as needed.

Finally, (2) follows from (1), $(*)$ and Lemma 5.2.3, for $A_i = B \cap W_{t_i}$. $\square$

Lemma 5.2.5. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuous map, such that X is a cell of dimension k and $\Gamma(f) \subseteq Y$, with Y a semialgebraic set of dimension k. Then f is semialgebraic.*

Proof. We may clearly assume that $\pi(Y) = X$. Let $\mathcal{D}$ be a cell decomposition of $\mathbb{R}^{n+1}$ partitioning Y, and $\mathcal{C} \subseteq \mathcal{D}$ the collection of those cells that intersect Y. So $\pi(\mathcal{C})$ is a finite partition of X. Since f is continuous, it suffices to show that for every $S \in \pi(\mathcal{C})$ of dimension k, $f_{\restriction S}$ is semialgebraic. Fix such S. By definition of cell decomposition, we have

$$Y \cap (S \times \mathbb{R}) = Y_1 \cup \cdots \cup Y_m,$$

for some disjoint cells $Y_i \subseteq \mathbb{R}^{n+1}$, with $\pi(Y_i) = S$. Of course, since $\Gamma(f) \subseteq Y$, we have $\Gamma(f_{\restriction S}) \subseteq Y \cap (S \times \mathbb{R})$. If there is i such that $\Gamma(f_{\restriction S}) \cap Y_i \neq \emptyset$ and is not equal to it, then, by (DC) there is $x \in S$ such that for every box $x \in B \subseteq S$, there is $j \neq i$ with $\Gamma(f_{\restriction B}) \cap Y_k \neq \emptyset$, for $k = i, j$. Since the Y_i's are disjoint and $f_{\restriction S}$ is continuous, this is a contradiction and $\Gamma(f_{\restriction S})$ must equal one of the Y_i's. Hence it is semialgebraic. $\square$

Lemma 5.2.6. *Let $X \subseteq \mathbb{R}^{m+l}$ be a semialgebraic set and A its projection onto the last l coordinates. Then there are finitely many polynomials $P_i \in \mathbb{R}[X, T]$ and semialgebraic sets A_i, $i = 1, \ldots, s$, such that $A = \bigcup_i A_i$, and for every i and $t \in A_i$,*

$$(*) \quad X_t \subseteq P_i(-, t)^{-1}(0) \quad \textit{and} \quad \dim X_t = \dim P_i(-, t)^{-1}(0).$$

Proof. By partitioning A into finitely many sets, we may assume that there is n with each $\dim X_t = n$. We perform induction on $k = \dim A$. For $k = 0$, the result follows from Fact 2.4.2, applied to each X_t. Let $k > 0$. By Fact

2.4.2 again, there is $P_0 \in \mathbb{R}[X,T]$ such that $X \subseteq P_0^{-1}(0)$ and both sets have dimension $n+k$. Let

$$A_0 = \{t \in A : \dim P_0(-,t)^{-1}(0) > n\}.$$

Then clearly, for every $t \in A \setminus A_0$, $(*)$ holds, with $i = 0$. Moreover, by o-minimality, $\dim A_0 < k$. Hence, by inductive hypothesis, the statement of the lemma also holds for $X \cap (\mathbb{R}^m \times A_0)$, and hence it holds for X. $\square$

We are now ready to prove our main theorem.

Theorem 5.2.7. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a definable smooth function, where X is an open semialgebraic set. Then f is semialgebraic.*

Proof. By cell decomposition, there are $C_1, \dots, C_k$ some cells such that $X = \bigcup_i C_i$. Let us assume that there is j such that $\dim(C_j) < n$. Since X is open and connected, for every $x \in C_j$, there is i, with $\dim(C_i) = n$ and $x \in \overline{C_i}$. Since f is continuous and defined at x, $f(x) = \lim_{y \to x} f_{\restriction C_i}(y)$. Therefore, if f is semialgebraic on each cell of dimension n, then it is semialgebraic. Thus, we may assume that X is an open cell.

By (DP)(I), there is a semialgebraic family $\{Y_t\}_{t \in \mathbb{R}^m}$ of sets $Y_t \subseteq \mathbb{R}^{n+1}$, and a definable $S \subseteq \mathbb{R}^m$ with $\dim S = 0$, such that $\Gamma(f) = \bigcup_{t \in S} Y_t$. By cell decomposition again, we may assume that each Y_t is a cell. For every $t \in S$, write $Z_t = \pi(Y_t)$ and $f_t = f_{\restriction Z_t}$. So $Y_t = \Gamma(f_t)$. Observe that $\dim Z_t = n$ if and only if Z_t is an open cell. Let

$$A = \{t \in S : \dim Z_t = n\}.$$

Since $X = \bigcup_{t \in S} Z_t$, and for every $t \in S \setminus A$, we have $\dim Z_t < n$, by (DIM), we obtain

$$\dim \left(X \setminus \bigcup_{t \in A} Z_t \right) < n. \tag{5.1}$$

Since X is an open cell, by Lemma 5.2.1, we have $X \subseteq \overline{\bigcup_{t \in A} Z_t}$. By continuity of f,

$$\Gamma(f) \subseteq \overline{\bigcup_{t \in A} \Gamma(f_t)}. \tag{5.2}$$

By Lemma 5.2.6, there are finitely many polynomials $P_i \in \mathbb{R}[X,T]$ and semialgebraic sets C_i, $i = 1, \dots, s$, such that $\mathbb{R}^n = \bigcup_i C_i$, and for every i and $t \in C_i$,

$$Y_t \subseteq P_i(-,t)^{-1}(0) \text{ and } \dim P_i(-,t)^{-1}(0) = \dim Y_t.$$

In particular, for every i and $t \in C_i \cap A$,

$$\Gamma(f_t) = Y_t \subseteq P_i(-,t)^{-1}(0) \text{ and } \dim P_i(-,t)^{-1}(0) \leq n.$$

For every i, denote $A_i = C_i \cap A$. So $A = \bigcup_i A_i$. For every $t \in A$, define

$$B_t = \bigcup_i \{t' \in A_i : \Gamma(f_t) \subseteq P_i(-,t')^{-1}(0)\} \subseteq A.$$

Observe that for every $t \in A$, we have $t \in B_t$, and hence $\bigcup_{t\in A} B_t = A$.

Our goal is to show that for every $t, t' \in A$, $B_t = B_{t'}$. This will imply the conclusion of the theorem. Indeed, in this case, we obtain that for every $t \in A$, $B_t = A$. Now, fix $a \in A$, say $a \in A_i$. Take any $t \in A$. So $a \in B_t$. That is, $\Gamma(f_t) \subseteq P_i(-,a)^{-1}(0)$. Together with (5.2), this implies that

$$\Gamma(f) \subseteq \overline{\bigcup_i P_i(-,a)^{-1}(0)},$$

and since the set on the right is semialgebraic and has dimension n, the conclusion of the theorem follows from Lemma 5.2.5.

We achieve our goal in Claim 3 below. First, we need two preliminary claims.

Claim 1. *Let $t, t' \in A$. Then*

$$Z_t \cap Z_{t'} \neq \emptyset \;\Rightarrow\; \overline{\Gamma(f_t)}^{zar} = \overline{\Gamma(f_{t'})}^{zar} \;\Rightarrow\; B_t = B_{t'}.$$

Proof of Claim 1. For the first implication, let $U = Z_t \cap Z_{t'}$. Since U is open, by Corollary 2.4.8, $\overline{\Gamma(f_t)}^{zar} = \overline{\Gamma(f_{\restriction U})}^{zar} = \overline{\Gamma(f_{t'})}^{zar}$.

For the second implication, let $x \in B_t$. Hence $\Gamma(f_t) \subseteq P_i(-,x)^{-1}(0)$. By assumption $\Gamma(f_{t'}) \subseteq P_i(-,x)^{-1}(0)$. Hence $x \in B_{t'}$. □

Let us define the equivalence relation $\sim$ on A, by $t \sim t'$ if and only if $B_t = B_{t'}$. For $t \in A$, let

$$W_t = \operatorname{int}\left(\overline{\bigcup_{t\sim t'\in A} Z_{t'}}\right).$$

Clearly, $\{W_t : t \in A\}$ is a definable family of open sets in $\mathbb{R}^n$.

Claim 2. *We have:*

(i) for every $t \in A$, $W_t = \operatorname{int}(\overline{W_t})$,

(ii) $\overline{X} = \overline{\bigcup_{t\in T_i} W_t}$,

(iii) for every $t, t' \in A$,

$$W_t \neq W_{t'} \;\Rightarrow\; t \not\sim t' \in A \;\Rightarrow\; W_t \cap W_{t'} = \emptyset.$$

Proof of Claim 2. (i) Clear, by definition of W_t.

(ii) For every $t \in A$, since X contains $\bigcup_{t\sim t'\in A} Z_{t'}$, we have that $W_t \subseteq \overline{X}$. Moreover, since Z_t is open, by definition of W_t we have $Z_t \subseteq W_t$. Hence, by (5.1),

$$\dim\left(X \setminus \bigcup_{t\in A} W_t\right) < n,$$

and by Lemma 5.2.1, $\overline{X} = \overline{\bigcup_{t\in A} W_t}$.

(iii) The first implication is clear. So we need to prove that if $t \not\sim t' \in A$, then $W_t \cap W_{t'} = \emptyset$. Assume otherwise. Note that $W_t \cap W_{t'}$ is open. Hence the intersection

$$\overline{\bigcup_{t\sim s\in A} Z_s} \cap \overline{\bigcup_{t'\sim s'\in A} Z_{s'}}$$

contains an open set. But then, by (DIM), there must be $s \not\sim s'$ with $Z_s \cap Z_{s'} \neq \emptyset$, contradicting Claim 1. □

In particular, the family $\{W_t : t \in A\}$ satisfies the properties of Lemma 5.2.4(i)-(iii).

Claim 3. *For every $t, t' \in A$, $B_t = B_{t'}$.*

Proof of Claim 3. Assume not. By Claim 2(iii), the family $\{W_t : t \in A\}$ satisfies all properties of Lemma 5.2.4. By that lemma, there are an open box $B \subseteq \mathbb{R}^n$ and $t_1, t_2 \in A$ with $W_{t_1} \neq W_{t_2}$, such that each $B \cap W_{t_j}$ is semialgebraic, open and connected, and $\mathrm{int}(\overline{B \cap W_{t_1}} \cup \overline{B \cap W_{t_2}})$ is connected. That is, $f_{\restriction B\cap W_{t_1}}$ and $f_{\restriction B\cap W_{t_2}}$ are Nash-concatenable. By Corollary 2.4.7, $\overline{\Gamma(f_{\restriction B\cap W_{t_1}})}^{zar} = \overline{\Gamma(f_{\restriction B\cap W_{t_2}})}^{zar}$. Since each $B \cap W_{t_j}$ is open, by (DIM), there are $a_1, a_2 \in A$ with $a_j \sim t_j$, such that $B \cap Z_{a_j}$ is open, $j = 1, 2$. Since $B \cap W_{t_j}$ is connected by (DC), open and semialgebraic, by Lemma 2.4.8, we obtain $\overline{\Gamma(f_{\restriction B\cap W_{t_j}})}^{zar} = \overline{\Gamma(f_{\restriction B\cap Z_{a_j}})}^{zar}$, $j = 1, 2$, and hence

$$\overline{\Gamma(f_{\restriction B\cap Z_{a_1}})}^{zar} = \overline{\Gamma(f_{\restriction B\cap Z_{a_2}})}^{zar}.$$

By Claim 1, $t_1 \sim a_1 \sim a_2 \sim t_2$, contradicting Claim 2(iii). □

This completes the proof of the theorem. □

5.3 d-minimal expansions of the real additive group by a discrete set does not define non-affine $\mathcal{C}^1$ functions

In this section we assume $\widetilde{\mathcal{R}}$ to be $\langle \mathbb{R}, <, +, P\rangle$ which satisfies (DP) and (DIM).

Note that if $\widetilde{\mathcal{R}}$ is d-minimal, then this result is a consequence of Hieronymi and Walsberg's [33, Corollary 4.9] which state that if $\widetilde{\mathcal{R}}$ is not of field type and does not interpret $\langle \mathbb{N}, \mathcal{P}(N), \in, +, \cdot\rangle$ then every $\mathcal{C}^2$ definable function is linear.

It is easy to see that $\widetilde{\mathcal{R}}$ is not of field type by DP(I) and that it is type (A) by DP(II) (or by d-minimality).

We do not know if (DP) and (DIM) implies that $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, , +, \cdot\rangle$ is not interpretable in $\widetilde{\mathcal{R}}$. In this section, we establish that $\widetilde{\mathcal{R}}$ does not define non affine $\mathcal{C}^1$-functions over $\mathbb{R}$.

Fact 5.3.1. *Let $f : X_1 \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : X_2 \subseteq \mathbb{R}^n \to \mathbb{R}$ be two affine functions which are $\mathcal{C}^1$-concatenable. Then Γ_f and Γ_g are both contained in a hyperplane of $\mathbb{R}^{n+1}$.*

Theorem 5.3.2. *Let $\mathcal{R} = \langle \mathbb{R}, +, < \rangle$ and $P \subseteq \mathbb{R}$ such that $\langle \mathcal{R}, P \rangle$ satisfies (DP) and (DIM). Then there are no non-affine definable $\mathcal{C}^1$ functions over an open semialgebraic domain.*

Proof. We may assume that X is an open cell.

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ such that X is $\mathcal{L}$-definable and $f \in \mathcal{C}^1$. By DP(I), there are $\{X_t\}$ an $\mathcal{L}$-definable family of cells of dimension n and S of dimension 0 such that

$$\Gamma_f = \bigcup_{t \in S} X_t \cup Y$$

where $\dim(Y) < n$.

Let $\sim$ be the equivalence relation on S defined by

$$t \sim t' \text{ iff } X_t \text{ and } X_{t'} \text{ are contained in the same hyperplane}$$

Let π be the projection on the first n-th coordinates and let $Y_t = \pi(X_t)$. For $t \in S$, let

$$W_t = \operatorname{int}(\overline{\bigcup_{t' \sim t} Y_{t'}})$$

By Fact 2.2.3, there are finitely many hyperplanes $H_1, \dots, H_k$ such that for every $t \in S$, there is $a \in \mathbb{R}^{n+1}$ with

$$\bigvee_i X_t \subseteq H_i + a.$$

Thus

$$t \sim t' \text{ if and only if there is } a \in \mathbb{R}^{n+1} \text{ and } i,\ X_t,\ X_{t'} \subseteq H_i + a.$$

and $\sim$ is definable. Therefore $\{W_t : \ t \in S\}$ is definable too and the rest of the proof is then exactly as in Theorem 5.2.7 using Fact 5.3.1 instead of Fact 2.4.6. □

5.4 Proof of Theorem 5.1.8 for expansions of the real ordered vector space with restricted multiplication

Let $\mathcal{K}$ be a subfield of $\overline{\mathbb{R}}$. For the rest of this section, we fix $\mathcal{R} = \langle \mathbb{R}_{sb}, (x \mapsto \lambda x)_{\lambda \in \mathcal{K}} \rangle$. As usual, we fix $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ where $P \subseteq \mathbb{R}$ is a set of dimension 0 and $\widetilde{\mathcal{R}}$ satisfies (DP) and (DIM).

The goal of this section is to prove an analogue of Theorem 5.1.8 for $\widetilde{\mathcal{R}}$.

5.4.1 No non semialgebraic functions

In this subsection, we show that a smooth function over an open semibounded domain is semi-algebraic. The proof splits into two steps, the first being when X is bounded. This case is handled by reduction to the semialgebraic case. We first need some preparation.

For the rest of this section, we fix the interval $I = (-1,1) \subseteq \mathbb{R}$ and the order-preserving semialgebraic diffeomorphism $\tau : \mathbb{R} \to I : x \mapsto \frac{x}{\sqrt{x^2+1}}$. We let $\mathcal{I} = \langle I, <, \oplus, \odot \rangle$ be the field structure induced on I from $\mathcal{R}$ via τ. Namely, for every $x, y \in I$,

$$x \oplus y = \tau(\tau^{-1}(x) + \tau^{-1}(y))$$

and

$$x \odot y = \tau(\tau^{-1}(x) \cdot \tau^{-1}(y)).$$

Clearly, $\mathcal{I}$ is a real closed field. We write $(\frac{x}{y})_{\mathcal{I}}$ for the division in $\mathcal{I}$. Since the order-topology on I coincides with the subspace topology from R, the dimension of a subset of I^n with respect to either structure is the same. Moreover, if $f : X \subseteq I^n \to I$ is any function, then continuity of f is invariant with respect to the two structures. We next prove that smoothness of f is also invariant. Let us write $f \in \mathcal{C}^\infty(\mathcal{R})$ if f is smooth in the sense of $\mathcal{R}$, and $f \in \mathcal{C}^\infty(\mathcal{I})$ if it is smooth in the sense of $\mathcal{I}$. For $n = 1$, we denote

$$\left(\frac{df}{dx}\right)_{\mathcal{I}} = \lim_{h \to 0} \left(\frac{f(x \oplus t) \ominus f(x)}{t}\right)_{\mathcal{I}}.$$

Lemma 5.4.1. *Let $f : X \subseteq I^n \to I$ be any function with open domain. Then $f \in \mathcal{C}^\infty(\mathcal{R})$ if and only if $f \in \mathcal{C}^\infty(\mathcal{I})$.*

Proof. We only prove the left-to-right direction, since the other direction is similar. Working inductively on the m-th partial derivatives of f with respect to $\mathcal{I}$, it is enough to show that each partial derivative of f is in $\mathcal{C}^\infty(\mathcal{R})$. For this, it is enough to show that if $n = 1$ and $f \in \mathcal{C}^\infty(\mathcal{R})$, then there is an $\mathcal{L}$-definable function $g \in \mathcal{C}^\infty(\mathcal{R})$ such that $g = \left(\frac{df}{dx}\right)_{\mathcal{I}}$. We have:

$$\begin{aligned}
\left(\frac{df}{dx}\right)_{\mathcal{I}} &= \lim_{t \to 0} \left(\frac{f(x \oplus t) \ominus f(x)}{t}\right)_{\mathcal{I}} \\
&= \lim_{t \to 0} \left(\frac{\tau(\tau^{-1} f \tau(\tau^{-1}(x) + \tau^{-1}(t)) - \tau^{-1} f(x))}{t}\right)_{\mathcal{I}} \\
&= \lim_{t \to 0} \tau \left(\frac{\tau^{-1}\tau(\tau^{-1} f \tau(\tau^{-1}(x) + \tau^{-1}(t)) - \tau^{-1} f(x))}{\tau^{-1}(t)}\right) \\
&= \lim_{t \to 0} \tau \left(\frac{\tau^{-1} f \tau(\tau^{-1}(x) + \tau^{-1}(t)) - \tau^{-1} f \tau(\tau^{-1} x)}{\tau^{-1}(t)}\right)
\end{aligned}$$

Letting $F = \tau^{-1} f \tau \in \mathcal{C}^\infty(\mathcal{R})$, $X = \tau^{-1}(x)$ and $T = \tau^{-1}(t)$, the above equals

$$\begin{aligned}
&= \lim_{T \to 0} \tau \left(\frac{F(X+T) - F(X)}{T}\right) \\
&= \tau \frac{dF}{dX} \in \mathcal{C}^\infty(\mathcal{R}),
\end{aligned}$$

as needed. $\square$

For the rest of this section, we also fix $\widetilde{\mathcal{I}}$ to be the structure on I induced from $\widetilde{\mathcal{R}}$. Namely,

$$\widetilde{\mathcal{I}} = \langle I, \{X\}_{X\subseteq I^n \text{ definable}}\rangle.$$

Clearly, $\widetilde{\mathcal{I}}$ expands $\mathcal{I}$. For a set $X \subseteq I^n$, we say that X is *$\widetilde{\mathcal{I}}$-definable* if it is definable in $\widetilde{\mathcal{I}}$ and *$\mathcal{I}$-definable* if it is definable in $\mathcal{I}$. We recall the following fact.

Fact 5.4.2 ([44, Corollary 3.6])**.** *If $X \subseteq I^n$ is semialgebraic, then X is $\mathcal{I}$-definable.*

Lemma 5.4.3. *Suppose $\widetilde{\mathcal{R}}$ has* (DP) *and* (DIM). *Then so does $\widetilde{\mathcal{I}}$.*

Proof. Let $X \subseteq \mathbb{R}^n$ be an $\widetilde{\mathcal{I}}$-definable set.

(DP)(I): Observe that X is also definable in $\widetilde{\mathcal{R}}$. By (DPI) for $\widetilde{\mathcal{R}}$ there is an $\mathcal{L}$-definable family $\{Y_t\}_{t\in\mathbb{R}^m}$ of subsets of $\mathbb{R}^n$, and a definable set $S \subseteq \mathbb{R}^m$ with $\dim S = 0$, such that $X = \bigcup_{t\in S} Y_t$. By Corollary 2.3.14, we may assume that $S \subseteq I^m$. By Fact 5.4.2, the $\{Y_t\}_{t\in S}$ family is $\mathcal{I}$-definable, as needed.

(DP)(II): Let X be an $\widetilde{\mathcal{I}}$-definable set and let Y be an $\mathcal{R}$-chunk of X. Since the topology on $\widetilde{\mathcal{I}}$ and on $\widetilde{\mathcal{R}}$ are the same and, by Fact 5.4.2, Y is $\mathcal{I}$-definable, Y is also an $\mathcal{I}$-chunk of X.

(DIM): Straightforward. □

We are ready to prove that there is no non semialgebraic smooth function over a semialgebraic domain definable in $\widetilde{\mathcal{R}}$.

Theorem 5.4.4. *Assume that $\mathcal{R} = \mathbb{R}_{sb}$, $\widetilde{\mathcal{R}}$ satisfies* (DP) *and* (DIM). *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a definable smooth function, where X is an open semialgebraic set. Then f is semialgebraic.*

Proof. We proceed in two steps:

Step I. Γ_f is bounded. We handle this case by reduction to the semialgebraic case, Theorem 5.2.7. First, after multiplying all coordinates of Γ_f by a small enough $r \in R$, we obtain the graph of a function contained in I^{n+1}. It is clearly enough to prove the theorem for that function instead. We may thus assume that $\Gamma_f \subseteq I^{n+1}$. In particular, Γ_f is $\widetilde{\mathcal{I}}$-definable. Also, since $f \in \mathcal{C}^\infty(\mathcal{R})$, by Lemma 5.4.1 we obtain $f \in \mathcal{C}^\infty(\mathcal{I})$. Now, by Lemma 5.4.3, $\widetilde{\mathcal{I}}$ has (DP) and (DIM), and hence, by Theorem 5.2.7, f is $\mathcal{I}$-definable. In particular, it is semialgebraic, as needed.

Step II. General case. By [19], every open semialgebraic set is a finite union of open cells. Hence we may assume X is an open cell. Let B be a bounded open box contained in X, and denote $g = f_{\restriction B}$. By Step I, g is semialgebraic, and hence Nash. Therefore Γ_g is a connected Nash-submanifold. By Fact 2.4.6, the set $Y = \overline{\Gamma_g}^{zar}$ is irreducible, and by Fact 2.4.2 it has dimension n.

Claim. $\Gamma_f \subseteq Y$.

Proof of Claim. Let
$$Z = \{x \in X : (x, f(x)) \in Y\}.$$
It is enough to show $X \subseteq Z$. Note that $B \subseteq Z$, and hence $\dim Z = n$. Assume towards a contradiction that $X \not\subseteq Z$. Since X is connected and open, it is easy to find an open bounded box $D \subseteq X$ with

1. $\overline{D} \subseteq X$
2. $\dim(D \cap Z) = n$, and
3. $D \setminus Z \neq \emptyset$.

By (1), $\Gamma_{f_{\restriction D}}$ is bounded and by Step I, we have that $f_{\restriction D}$ is semialgebraic. Since $\dim D = n$, the Zariski closure $Y' = \overline{\Gamma_{f_{\restriction D}}}^{zar}$ has dimension n (Fact 2.4.2). Moreover, the intersection $Y \cap Y'$ contains $\Gamma_{f_{\restriction D \cap Z}}$ and hence also has dimension n. By Lemma 2.4.3, $Y = Y'$. It follows that for every $d \in D$,
$$(d, f(d)) \in \Gamma_{f_{\restriction D}} \subseteq Y' = Y.$$
This implies $D \subseteq Z$, which contradicts (2). □

Since X is an n-cell, $\Gamma_f \subseteq Y$ and $\dim Y = k$, by Lemma 5.2.5, we obtain that f is semialgebraic. □

5.4.2 No non-semibounded functions

In this section, we prove that under our assumptions, $\widetilde{\mathcal{R}}$ does not define non semibounded smooth functions. Equivalently, the multiplication is not definable (see [44]).

In private communication, Philipp Hieronymi asked the following question:

Question 5.4.5. Does $\langle \mathcal{R}, \mathbb{Z}\rangle$ define new $\mathcal{C}^\infty$ functions?

In this case, the answer is quite simple, since a $\mathcal{C}^\infty$ function f definable in this structure is semialgebraic (either linear or non linear), and that if f is not semibounded then $\langle \widetilde{\mathcal{R}}, f\rangle = \overline{\mathbb{R}}$ (see [44]), we have that
$$\langle \widetilde{\mathcal{R}}, \mathbb{Z}\rangle = \langle \widetilde{\mathcal{R}}, f, \mathbb{Z}\rangle = \langle \overline{\mathbb{R}}, \mathbb{Z}\rangle = PH.$$
Which is a contradiction with (DIM) (or with (DP)). If f is linear, $f \notin \{(x \mapsto \lambda x) : \lambda \in \mathbb{Q}\}$ and since $\langle \mathcal{R}, \mathbb{Z}, \lambda\mathbb{Z}\rangle$ defines a dense-co-dense set, contradicting (DP2).

Anyway, this question is answered by Theorem 5.4.10.

Lemma 5.4.6. *Let* $P' \subseteq \mathbb{R}$ *be a set of dimension* 0*. If* $\langle \mathcal{R}, P'\rangle$ *has (DIM) then* P' *is sparse.*

Proof. By (DIM), for $f : \mathbb{R}^k \to \mathbb{R}$ a function definable in $\mathcal{R}$,
$$\dim(f(P'^k)) = \dim\big(\bigcup_{t \in P'^k} f(f)\big) = \max_{t \in P'^k} \dim(f(t)) = 0.$$
□

We recall that $\mathbb{K}_{\widetilde{\mathcal{R}}}$ denotes the field of total linear functions definable in $\widetilde{\mathcal{R}}$. Note that we do not know if $\mathbb{K}_{\widetilde{\mathcal{R}}} = \mathbb{K}_{\mathcal{R}}$ (see Section 5.4.3 for a discussion about this question).

Proposition 5.4.7. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{L}$-definable function. Then there is an interval $B \subseteq \mathbb{R}$ and an affine function $\lambda : \mathbb{R}^n \to \mathbb{R}$ with coefficient in $\mathbb{K}_{\widetilde{\mathcal{R}}}$ such that for every $x \in X$, $f(x) \in \lambda(x) + B$.*

Proof. By Fact 2.3.4, we may assume that Γ_f is a cone of the form $A + \sum_{a\in(\mathbb{R}^{>0})^l} \sum a_i v_i$ where A is a bounded set and the $v_i \in \mathbb{R}^{n+1}$ are linearly independent. By translation, we may also assume that $0 \in A$. We note that since Γ_f has dimension n, $\dim(\sum_{a\in\mathbb{R}^l} a_i v_i) \leq n$. Let $H \subseteq \mathbb{R}^{n+1}$ be an hyperplane containing $\sum_{a\in\mathbb{R}^l} a_i v_i$ which is not perpendicular to $\mathbb{R}^n \times \{0\}$. Let $\lambda : \mathbb{R}^n \to \mathbb{R}^{n+1}$ be the projection on H.

We first observe that since A is bounded and that H is not perpendicular to $\mathbb{R}^n \times \{0\}$, there is a uniform bound on $|(A + H)_x|$ for $x \in \mathbb{R}^n$. Since $\Gamma_f \subseteq A+H$, there is also a uniform bound on $d\big((x, f(x)), (x, \lambda(x))\big)$ and we have the result. □

Proposition 5.4.8. *Let $P' \subseteq \mathbb{R}$ be a set of dimension 0. If $\langle \mathcal{R}, P' \rangle$ has (DIM), then the multiplication is not definable in $\langle \mathcal{R}, P' \rangle^{\#}$.*

Proof. Let us assume for contradiction that the multiplication is definable. Thus, by the existence of a pole, there is S, a definable unbounded 0-dimensional set. We consider the family $\{xS :\ x \in \mathbb{R}\}$. By Fact 3.1.20 and Lemma 5.4.6, there is function definable in $\mathcal{R}$ $f : \mathbb{R}^{k+1} \to \mathbb{R}$ such that $xS \subseteq \overline{f_x(P'^k)}$. By Proposition 5.4.7, there is a bounded set B and a linear function $\lambda : \mathbb{R}^k \to \mathbb{R}$ such that

$$f_x(P'^k) \subseteq B + \lambda(\{x\} \times P'^k).$$

Since λ is linear, we have that $\lambda(x, y) = \lambda'(y) + b(x)$ where λ' and b are linear.

For every $t \in S$,

$$x \in \Big\{y :\ \exists a \in B,\, z \in \lambda'(P'^k),\ y = \frac{a + b(x) + z}{t}\Big\}.$$

Since B is bounded and $b(x)$ is a constant, for every $\epsilon > 0$, there is $t \in S$ such that $\frac{|B+b(x)|}{t} < \epsilon$. It gives us that for every $x \in \mathbb{R}$

$$x \in Z = \overline{\{y/t\ \ y \in \lambda'(P'^k),\, t \in S\}}.$$

Therefore Z has interior. Moreover, by (DIM)

$$\dim(Z) = \dim\big(\{y/t :\ y \in \lambda'(P'^k), t \in S\}\big) = \max_{(y,t)\in P'^k\times S} \dim\big(\{\lambda'(y)/t\}\big) = 0.$$

This is a contradiction. □

Remark 5.4.9. The above proposition holds for $\mathcal{R}$ being any semibounded expansion of $\mathbb{R}_{gp}$, $P \subseteq \mathbb{R}$ of dimension 0 and $\widetilde{\mathcal{R}}$ having (DIM).

Theorem 5.4.10. *Let f be a $\mathcal{C}^\infty$-function over an open semialgebraic domain definable in $\langle \mathcal{R}, P\rangle$. Then f is definable in $\langle \mathcal{R}, (x \mapsto \lambda x)_{\lambda \in \mathbb{K}_{\widetilde{\mathcal{R}}}}\rangle$.*

Proof. By Theorem 5.4.4, we have that f is semialgebraic. By contradiction, we may assume that f is not definable in $\langle \mathcal{R}, (x \mapsto \lambda x)_{\lambda \in \mathbb{K}_{\widetilde{\mathcal{R}}}}\rangle$.

First we assume that f is a non-semibounded semialgebraic function. By [44], it implies that $\widetilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, P\rangle$, this is a contradiction with Proposition 5.4.8.

Now, we assume that f is definable in a semibounded structure. Then, it is easy to see that we may define a linear function that does not belong to $\mathbb{K}_{\widetilde{\mathcal{R}}}$, which is absurde since $\langle \mathcal{R}, (x \mapsto \lambda x)_{\lambda \in \mathbb{K}_{\widetilde{\mathcal{R}}}}\rangle$ defines every linear functions. □

Remark 5.4.11. Note that even if Proposition 5.4.8 is true for expansions of any semibounded structure by a 0 dimensional unary set which satisfies (DIM), the conclusion of Theorem 5.4.10 fails. Take for example $\langle \mathbb{R}_{gp}, \sin_{\restriction[0,2\pi]}, 2\pi\mathbb{Z}\rangle$ which defines sin.

5.4.3 Defining new linear functions

Let $\mathcal{R}$ be semibounded and $P \subseteq \mathbb{R}$ a set of dimension 0 such that $\widetilde{\mathcal{R}}$ has (DP) and (DIM).

This subsection is a sum of preliminary results and questions related to the question:

Question 5.4.12. *Does $\widetilde{\mathcal{R}}$ defines a total linear function that is not definable in $\mathcal{R}$?*

Let us start with some obvious proposition.

Proposition 5.4.13. *If $\mathcal{R}$ is of the form $\langle \mathbb{R}_{gp}, \left(x \mapsto \lambda x\right)_{\lambda \in I}\rangle$ for some $I \subseteq \mathbb{R}$ and $\widetilde{\mathcal{R}}$ as usual, then a $\mathcal{C}^1$ definable function with $\mathcal{R}$-definable domain is definable in $\mathcal{R}$.*

Proof. By DP(I), f is locally linear and definable in $\mathcal{R}$. By Theorem 5.4.4, f is semialgebraic and since f is $\mathcal{C}^1$, it needs to be linear. Let $Y \subseteq \mathbb{R}^{n+1}$ be a set of dimension n, definable in $\mathcal{R}$ such that

$$\Gamma_f \cap \pi^{-1}(\pi(Y)) = Y$$

(for π the projection on the first n-coordinates). Since f is linear,

$$\Gamma_f = \overline{Y}^{zar} \cap \pi^{-1}(X)$$

and since $\overline{Y}^{zar}$ is definable in $\mathcal{R}$, we have the result. □

From now on, $\mathcal{R}$ expands $\mathbb{R}_{sb}$. We also fix a smooth definable function $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ such that X is $\mathcal{R}$-definable.

Let us start with an easy corollary from Theorem 5.4.4.

Corollary 5.4.14. *If $\mathcal{R} = \langle \mathbb{R}_{sb}, \left(x \mapsto \lambda x\right)_{\lambda \in \mathbb{R}}\rangle$ then f is $\mathcal{R}$-definable.*

For the rest of this section, we assume that there is $\lambda \notin \mathbb{K}_{\mathcal{R}}$.

Let us start a discussion about some restrictions that $\widetilde{\mathcal{R}}$ should satisfy in order to define $x \mapsto \lambda x$. First, a general lemma.

Lemma 5.4.15. *Let $\{g_t : t \in A\}$ be a family of functions definable in $\mathcal{R}$ such that for every $t \in A$,*

$$g_t = (x \mapsto \lambda x) \restriction (0, x_t).$$

Then there is a uniform bound over the x_t's.

Proof. If there was not, $x \mapsto \lambda x$ would be definable in $\mathcal{R}$. □

Proposition 5.4.16. *We assume that $f : x \mapsto \lambda x$ is definable in $\widetilde{\mathcal{R}}$. Let $\{f_t : (a_t, b_t) \to \mathbb{R} : t \in S\} \subseteq \{f_t : t \in A\}$ be the families of functions given by DP(I) applied to f (the second one being definable in $\mathcal{R}$). Then there is a uniform bound over $|\mathrm{dom}(f_t)|$ for $t \in S$.*

Proof. First of all, we may assume that for every $t \in A$, f_t is linear and that f_t is the restriction of $x \mapsto \lambda x$. We define $g_t : (0, b_t - a_t)$ by

$$g_t : x \mapsto f_t(x + a_t) - f_t(a_t).$$

We have the result by applying Lemma 5.4.15. □

The following proposition shows that more can be said in the d-minimal setting.

Proposition 5.4.17. *Let $P' \subseteq \mathbb{R}$ be a set of dimension 0 such that $\langle \overline{\mathbb{R}}, P' \rangle$ is d-minimal and P is definable in $\langle \overline{\mathbb{R}}, P' \rangle$. Then $x \mapsto \lambda x$ is not definable in $\widetilde{\mathcal{R}}$.*

Proof. The result follows easily from Lemma 3.1.18 and Proposition 5.4.16. □

We can then give some necessary conditions for $\widetilde{\mathcal{R}}^{\#}$ to define new linear functions.

Proposition 5.4.18. *Let $P \subseteq \mathbb{R}$ be an unbounded set of dimension 0 so that $\langle \mathcal{R}, (x \mapsto \lambda x), P \rangle$ has (DP) and (DIM) and such that $\{d(x, P \setminus \{x\}) : x \in P\}$ is bounded. Then $x \mapsto \lambda x$ is definable in $\langle \mathcal{R}, P \cup \lambda P \rangle^{\#}$.*

Proof. First of all, we may assume that P is closed. Let $\{(a_t, b_t)\}$ be the (bounded) family of intervals of $(0, \infty) \setminus P$. We first observe that the function $g : x \mapsto \lambda x \restriction P$ is definable in $\langle \mathcal{R}, P \cup \lambda P \rangle^{\#}$. We define $x \mapsto \lambda x$ on $(0, \infty)$ by

$$x \in (a_t, b_t) \mapsto g(a_t) + \lambda(x - a_t).$$

□

Thus, we can rephrase Question 5.4.12 into:

Question 5.4.19. *Is there a set $P \subseteq \mathbb{R}$ such that P is closed unbounded, of dimension 0, $\langle \mathcal{R}, (x \mapsto \lambda x), P \rangle$ has (DP) and (DIM) and $\{d(x, P \setminus \{x\}) : x \in P\}$ is bounded?*

Here are some examples of structures that does not define new total linear functions:

- $\langle \mathcal{R}, \mathbb{Z} \rangle$, since for $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ $\langle \mathcal{R}, \mathbb{Z}, \lambda Z \rangle$ defines a dense-codense set.
- $\langle \mathcal{R}, P \rangle$ where P is an iteration, a fast sequence for $\overline{\mathbb{R}}$ or $\alpha^{\mathbb{Z}}$ for $\alpha \in \mathbb{R}$, by applying Proposition 5.4.18

5.4.4 Alternative proofs in the d-minimal seting

This section has only a technical interest and provides alternative proofs for Theorem 5.4.4 and for Theorem 5.4.8 in the d-minimal setting. Thus we assume that $\langle \mathcal{R}, P \rangle$ is d-minimal.

Theorem 5.4.20. *Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a smooth function definable in $\widetilde{\mathcal{R}}^{\#}$ over an $\mathcal{L}$-definable open domain. Then f is definable in a semibounded structure.*

Proof. Following the proof of Theorem 5.2.7, we build the families $\{Z_t : t \in S\}$, $\{B_t : t \in A\}$ and $\{W_t : t \in A\}$. The only difference is that these families are definable in $\langle \overline{\mathbb{R}}, P \rangle$ but not necessarily in $\widetilde{\mathcal{R}}$ (except for $\{Z_t\}$). By Proposition 3.4.11 we have that any subset of A^n is definable in $\widetilde{\mathcal{R}}^{\#}$.

We show that the family $\{B_t\}$ and therefore $\{W_t\}$ are definable in $\widetilde{\mathcal{R}}^{\#}$. Let $C = \bigcup_{t \in A} \{t\} \times B_t \subseteq A^2$. We have that C is definable in $\widetilde{\mathcal{R}}^{\#}$ and for any $t \in A$, $B_t = C_t$.

Then we apply the rest of the proof of Theorem 5.2.7 to get the result. □

Proof. (of Theorem 5.4.8) Let us assume for contradiction that $\widetilde{\mathcal{R}}^{\#}$ defines the multiplication. Thus by $\mathcal{L}$-DEM, we have that the graph of $f : x \mapsto x^2$ has the form $\bigcup_{t \in S} X_t$ where S is a small set and $\{X_t : t \in A\}$ is a semibounded family. Since all the X_t's of dimension 1 are part of the graph of a nowhere affine function, we may assume that the family $\{X_t : t \in A\}$ satisfies the condition of Corollary 2.3.15. Thus, applying it, we have that $\{|X_t| : t \in A\}$ is bounded. Moreover, By Lemma 3.1.18 this family is unbounded and we get a contradiction. □

5.5 Generalization and questions

We finish this chapter by asking some questions and finding some limits to the possible generalizations.

The following example, shows that if we remove the assumption that $\mathcal{R}$ is the real field from Theorem 5.2.7, then the theorem fails. Note that the structure $\langle \mathbb{R}_{an}, e^{2\pi\mathbb{Z}} \rangle$ is d-minimal. It has been brought to us by P. Hieronymi but is essentially due to Miller and Speissegger (see the first corollary after [46, Theorem 3.4.2]).

Example 5.5.1. The structure $\widetilde{\mathcal{R}} = \langle \overline{\mathbb{R}}_{an}, e^{2\pi\mathbb{Z}} \rangle$ defines new smooth functions with domain $\mathbb{R}$.

Proof. Let $f : \mathbb{R}^{>0} \to \mathbb{R}$ be the map $f(x) = \sin\log x$. Clearly, f is not definable in the o-minimal $\mathbb{R}_{an}$, since its zero set is an infinite discrete set. We show that f is definable in $\widetilde{\mathcal{R}}$. For every $x \in \mathbb{R}$, we have:

$$\begin{aligned} f(x) &= \sin\log\left(\lambda(x)\frac{x}{\lambda(x)}\right) \\ &= \sin\log\lambda(x) + \sin\log\left(\frac{x}{\lambda(x)}\right) \\ &= \sin\log\left(\frac{x}{\lambda(x)}\right). \end{aligned}$$

But $\frac{x}{\lambda(x)} \in [1, e^{2\pi}]$ and $\log([1, e^{2\pi}]) = [0, 2\pi]$. Therefore, f is definable in $\langle \mathbb{R}_{an}, e^{2\pi\mathbb{Z}} \rangle$. □

Here is an other example, where the function is not analytic.

Example 5.5.2. Let $\mathcal{R}$ be an expansion of $\langle \overline{\mathbb{R}}, \exp \rangle$. Let P be an iteration sequence for $\mathcal{R}$. Let s be the successor function on P. For $t \in P$, let $f_t : (t, s(t)) \to \mathbb{R}$, $x \mapsto e^{\frac{1}{(x-t)(x-s(t))}}$. $(\mathcal{R}, P)$ defines

$$\bigcup_{t\in P} \Gamma_{f_t} \cup \bigcup_{t \in P} (t, 0)$$

which is the graph of a $\mathcal{C}^\infty$ function not definable in $\mathcal{R}$.

The referee of [21] asked the following question:

Question 5.5.3. Let $\mathcal{M} \subseteq \mathbb{R}_{an}$ be a proper expansion of $\overline{\mathbb{R}}$. Does $\langle \mathcal{M}, 2^{\mathbb{Z}} \rangle$ define new $\mathcal{C}^\infty$-functions?

We conclude with a list of open questions that arise naturally from this work. To introduce the first question, we prove the following corollary:

Corollary 5.5.4. *Let $\widetilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, \mathbb{R}_{alg}, 2^{\mathbb{Z}} \rangle$, where $\mathbb{R}_{alg}$ is the field of real algebraic numbers. Then every definable smooth function with open semialgebraic domain is semialgebraic.*

Proof. Let $Y = \overline{\Gamma(f)}$. It is easy to see that $Y \cap (X \times \mathbb{R}) = \Gamma(f)$. By [39, Theorem 4.41 (2)], every open definable set in $\widetilde{\mathcal{R}}$ is definable in $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$. We can thus apply Theorem 5.2.7 to f. □

Question 5.5.5. **Necessary conditions.** Clearly, (DP) and (DIM) are not necessary conditions for Theorem 5.2.7. For example, in the dense pair $\langle \overline{\mathbb{R}}, \mathbb{R}_{alg} \rangle$, or in the example of Corollary 5.5.4, there are again no new smooth definable functions, but (DP)(II) fails. What could be other sufficient conditions for the conclusion of Theorem 5.2.7 to be true?

We give a necessary condition for this phenomenon to happen.

Definition 5.5.6. We say that a structure $\mathcal{M}$ has *d-minimal open core* if there is a d-minimal structure $\mathcal{M}'$ such that every open definable set in $\mathcal{M}$ is definable in $\mathcal{M}'$.

Corollary 5.5.7. *Let $\mathcal{M}$ be a d-minimal open core expansion of $\mathbb{R}_{gp}$ such that every open definable set is definable in a d-minimal structure of the form $\langle \mathcal{R}, P\rangle$ where $\mathcal{R}$ is either $\mathbb{R}_{sb}$ or $\overline{\mathbb{R}}$. Then a smooth definable function over an open domain definable in $\mathcal{R}$ is definable in $\mathcal{R}$.*

Question 5.5.8. **Smooth functions with arbitrary domains.** Here we relax the assumption on the domain of f in Theorem 5.2.7.

Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a definable smooth function, with open connected domain X. Is it true that f is the restriction to X of a semialgebraic function?

The additional connectedness assumption is necessary. Indeed, let $\widetilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$, and $f : \bigcup_{t\in 2^{\mathbb{Z}}}(t, 2t) \to \mathbb{R}$ be given via: $f(x) = 0$, if $x \in (t, 2t)$, for some $t \in 2^{2\mathbb{Z}}$, and $f(x) = 1$ if $x \in (t, 2t)$ for some $t \in 2^{2\mathbb{Z}+1}$. Then f is definable in $\widetilde{\mathcal{R}}$, with open domain, and it is smooth but not semialgebraic.

One of the issue is that we do not have Lemma 5.2.5 anymore.

Question 5.5.9. Is there an expansion of the real closed field by a Cantor set that satisfy $(DP1)$ and which defines a new $\mathcal{C}^\infty$ function?

Question 5.5.10. Do we have a dichotomy between expansions of $\mathcal{R} = \mathbb{R}_{gp}, \mathbb{R}_{sb}$ or $\overline{\mathbb{R}}$ by $P \subseteq \mathbb{R}$, a set of dimension 0 that defines no new smooth functions over a semialgebraic domain and the whole projective hierarchy?

We would like to finish with some general discussion about Proposition 5.4.8 and some applications. In [33], Hieronymi and Walsberg are doing a dichotomy between the *field type expansions of the real oredered groups* (the one which defines a real closed field) and the others. How fruitful would it be to do also a dichotomy between the structures which define the full multiplication and the ones which define only the multiplication restricted to some bounded set? As we saw in Theorem 5.4.8, if a structure of the form $\langle \mathcal{R}, P\rangle$ (with $\mathcal{R} = \mathbb{R}_{gp}$ or $\mathbb{R}_{sb}$) has (DIM) then it does not define the full multiplication. On the other hand, some o-minimal open-core expansions of $\mathcal{R}$ by a dense subset of $\mathbb{R}$ do not define the full multiplication; for example $\langle \mathbb{R}_{gp}, \mathbb{Q}\rangle$, see [14]. See [20] for some other examples. Of course, if a structure defines an ultimately growing $\mathcal{C}^2$ function over $\mathbb{R}$, it would define a field structure on $\mathbb{R}$ (using the technics of [44]).

Question 5.5.11. Is there a sparse o-minimal open-core expansion of $\mathbb{R}_{sb}$ by a dense-codense subset of $\mathbb{R}$ that defines the full multiplication?

We finish with some short discussion about the difference between DP(II) and d-minimality. The first question is:

Question 5.5.12. Is there an expansion of $\mathbb{R}_{gp}$ satisfying DP(II) but is not d-minimal?

Is there such one of the form $\langle \mathcal{R}, P\rangle$ with $\mathcal{R}$ an o-minimal expansion of $\mathbb{R}_{gp}$ and $P \subseteq \mathbb{R}$ of dimension 0?

We give some preliminary observation to explore this question. We assume that $\langle \mathcal{R}, P\rangle$ has DP(II) but is not d-minimal. To explore this difference, the first idea that comes naturally in mind is to define the notion of local d-minimality:

Definition 5.5.13. We say that an expansion $\mathcal{M}$ of a dense linear order without endpoints $\langle M, <\rangle$ is *locally d-minimal* if for every definable set $X \subseteq M$, and $x \in M$ there is an interval $I \ni x$ such that $\langle M, <, I \cap X\rangle$ is d-minimal.

We assume that $\widetilde{\mathcal{R}}$ is locally d-minimal. First, we observe that by local d-minimality, P is countable. Since $\langle \mathcal{R}, P\rangle$ is not d-minimal, either there is a definable set $X \subseteq \mathbb{R}$ of dimension 0 but with non finite CB-rank, or there is a definable family $\{X_x :\ x \in A\}$ of sets of dimension 0 each of finite CB-rank but with no uniform bound on the CB-rank.

In the first case, it is easy to see that in $X - X$, 0 has CB-rank at least ω. Thus $\langle \mathcal{R}, I \cap X - X\rangle$ is not d-minimal for every interval $I \ni 0$ and thus, $\widetilde{\mathcal{R}}$ is not locally d-minimal.

In the second case, if $\dim(A) = 0$, then $\bigcup_{x\in A} X_x$ has dimension 0, infinite CB-rank and we are back to case one.

Thus we can formulate the following question:

Question 5.5.14. Is there a locally d-minimal non d-minimal expansion of $\mathbb{R}_{gp}$ of the form $\langle \mathcal{R}, P\rangle$ where $\mathcal{R}$ is o-minimal and $P \subseteq \mathbb{R}$ is of dimension 0? In other words, does $\langle \mathcal{R}, P\rangle$ define a family $\{X_x :\ x \in A\}$ such that each X_x has finite CB-rank but there is no uniform bound on this CB-rank and $\dim(A) > 0$?

A positive answer to Question 5.5.14 would, for sure answer Question 5.5.12. Anyway, it does not seem to be a necessary condition.

6. THE PROBLEM OF THE CONNECTED-COMPONENT-CLOSURE

The problem of the connected-component-closure (cc-closure for short. $\mathbf{Q}_1$ below) is the problem of knowing what is the smallest structure that contains a given structure and such that the connected components of any definable set is definable. This chapter is more of a discussion about $\mathbf{Q}_1$ than a sum of results. We first define the problem, then we present a solution for locally o-minimal structures of the form $\langle \mathcal{R}, P\rangle$ ($\mathcal{R}$ an o-minimal expansion of $\langle \mathbb{R}, <\rangle$ and $P \subset \mathbb{R}$) and the undecidability of the theory of the cc-closure of d-minimal structures in general. Then we adress $\mathbf{Q}_1$ for general d-minimal expansions of an o-minimal structure by a unary predicate, we discuss the difficulties and propose some tools. We finish with some discussion about related questions.

For the records, $\mathbf{Q}_1$ has been brought to us by Chris Miller in 2017 while visiting Konstanz. We already observed that $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$ was not cc-closed and we could also mention that the first person to have described a non-cc-closed structure is Antongiulio Fornasiero in [27]. The definitions of the cc-closure as well as the problem in its present form are due to Chris Miller. We would like to thank him for bringing this problem to us. We would like to thank Serge Randriambololona and Francoise Point for useful discussions.

6.1 The problems

The first observation that led to formulate these questions ($\mathbf{Q}_1$-$\mathbf{Q}_3$ below) is that there are some expansions of the real line where it is possible to define some set that is non-connected but definably connected (see [27]). In such sets, the connected components are of course not definable and we already made the observation that in $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$ such phenomenon could happend (as shown in the following example).

Example 6.1.1. Let

$$X = \bigcup_{t\in 2^{\mathbb{N}}} [(0,t)(t,0)] \cup [(t,0),(0,-t)] \cup [(0,-t),(-t,0)] \cup [(-t,0),(0,t^2)] \subseteq \mathbb{R}^2.$$

It is easy to see that, for Y the connected-component of $(0,1)$,

$$Y \cap \{0\} \times 2^{\mathbb{N}} = \{(0, 2^{2^n}) : \ n \in \mathbb{N}\}.$$

By Fact 4.2.1, this set is not definable in $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}\rangle$. Note also that X is not definably connected (see Subsection 6.7.2 for a discussion about this difference).

Definition 6.1.2. We say that an expansion of the real line $\mathcal{M}$ is *connected-component-closed* (cc-closed) if for every definable set X, the connected components of X are definable.

We denote by $\mathrm{CC}(\mathcal{M})$ the smallest cc-closed structure that contains $\mathcal{M}$.

Let $\mathcal{M}_0 = \mathcal{M}$, we define recursively $\mathcal{M}_n$ to be the structure generated by $\mathcal{M}_{n-1}$ together with a predicate for every connected component of every set definable in $\mathcal{M}_{n-1}$. It is easy to see that $\mathrm{CC}(\mathcal{M}) = \bigcup_{n\in\mathbb{N}} \mathcal{M}_n$.

Let X be a set definable in $\mathcal{M}$ and let $x \in X$, we denote by $\mathrm{cc}_X(x)$ the connected component of x in X.

The main questions that arise from there can now be formulated. As usual, we fix $\mathcal{R}$ an o-minimal expansion of $\mathbb{R}_{gp}$ and a unary set $P \subseteq \mathbb{R}$ of dimension 0 such that $\widetilde{\mathcal{R}}$ is d-minimal.

The questions:

1. What is the cc-closure of $\widetilde{\mathcal{R}}$?
2. What can be said about the connected components of sets definable in $\widetilde{\mathcal{R}}$?
3. What is $\widetilde{\mathcal{R}}_1$?

In the following, we refer to these questions as $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$.

6.1.1 An upper bound

We can give, as a preliminary, an upper bound for $\mathrm{CC}(\widetilde{\mathcal{R}})$.

Proposition 6.1.3. *The structure $\widetilde{\mathcal{R}}^{\#}$ is cc-closed.*

Proof. Let X be definable in $\widetilde{\mathcal{R}}^{\#}$. By Theorems 3.3.7 and 3.3.16, there is an $\mathcal{L}$-definable family of cells $\{X_t : t \in A\}$ and $S \subseteq A$ a small set such that $X = \bigcup_{t\in S} X_t$. Thus, since a cell is connected, the connected components of X have the form $\bigcup_{t\in S'} X_t$ for some $S' \subseteq S$. By Proposition 3.4.6, S' is definable in $\widetilde{\mathcal{R}}^{\#}$ and we have the result. □

6.1.2 Motivations

In a first place, the reason that led us to get interested in $\mathbf{Q}_2$ and $\mathbf{Q}_3$ was to study definable groups in $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ and in general d-minimal expansions of the real field. The fact to be able to assume that a definable group is connected was meaningful in a more general problem. Note that for this purpose, only $\mathbf{Q}_2$ and $\mathbf{Q}_3$ are relevant (see Subsection 6.6 for some preliminary results on this topic).

After the talk of Chris Miller, it appeared that other motivations led them (him, Athipat Tamrongthanyalak and, later, Alf Dolich) to this problem. Namely: some reachability problems and some control theory problems. (see [1], [56], [57])

Anyway, this topic seems to present connexions with many different fields of mathematics. For example with: model theory, descriptive set theory, some higher recursion theory (Turing machine with infinite time (see for example [3])). We will develop a little bit these connections in the rest of this chapter.

6.2 The theory of the cc-closure of a d-minimal non-o-minimal structure is undecidable

In the following, we prove that for any expansion $\mathcal{M}$ of $\mathbb{R}_{gp}$ such that its CC-closure defines a closed discrete set then its cc-closure interprets $(\mathbb{Z}, +, \cdot)$.

Let P be an infinite closed discrete set definable in the CC-closure of $\mathcal{M}$.

Definition 6.2.1. Let $G \subseteq P^{2k}$ be a graph. We always assume that G is reflexive, symetric and that for every $x \in P^k$, $(x, x) \in G$. From now, when defining a graph with A a non symetric and reflexive relation we will just note $G = \{(x, y) : A(x, y)\}$ instead of $\{(x, y), (y, x) : A(x, y)\} \cup \{(x, x)\}$. For x a vertice of G, we note $\mathrm{cc}_G(x)$ the set

$$\{y \in P^k : \ y \text{ is in the transitive closure of } x\}$$

ie there are $y_1, \ldots, y_n$ such that $(x, y_1) \in G$, $(y_i, y_{i+1}) \in G$ and $(y_n, y) \in G$.

Definition 6.2.2. Let $G \subseteq P^{2k}$ be a graph definable in $\mathrm{CC}(\mathcal{M})$. We define X_G to be

$$\{0\} \times P^k \times \mathbb{R}^k \cup \{1\} \times \mathbb{R}^k \times P^k \cup$$
$$\bigcup_{x \in P^k} [(0, x, x), (1, x, x)] \cup \bigcup_{(x,y) \in G} [(0, x, y), (1, x, y)].$$

Observe that this graph is definable in $(\mathbb{R}, <, P)^{\#}$. Observe also that this definition is redundant since $(x, x) \in G$ but it was just to emphasize this fact.

From now on, when mentioning the connected components of an element we refer to both the connected component of an element in a set and to the transitive closure of an element in a graph. It will be clear from the context which one of these notions is used.

Lemma 6.2.3. *Let $G \subseteq P^{2k}$ be a graph definable in P_{ind}. For every $x \in P^k$, the connected component of x (in G) is exactly the projection on the coordinates between 1 and $k + 1$ of $\mathrm{cc}_{X_G}((0, x, x)) \cap \{0\} \times P^k \times \mathbb{R}^k$.*

Thus, the connected components of G are definable in $\mathrm{CC}(\mathcal{M})$.

Using the last Lemma, the only thing left to prove is that we can interpret $(\mathbb{Z}, <, +, \cdot)$ on the structure on P induced by the cc-closure of $\mathcal{M}$.

Lemma 6.2.4. *Let $G \subseteq P^{2k}$ be a definable graph (in* $\mathrm{CC}(\mathcal{M})$*). There is a* $\mathrm{CC}(\mathcal{M})$*-formula ϕ such that for every $x \in P^k$, $\phi(x) = \mathrm{cc}_G(x)$.*

Proof. Let $0 \neq 1 \in P$ and let

$$G' = \{((0, x, x), (1, x, x)) : \ x \in P^k\} \cup \{((0, x, x), (0, y, y)) : \ x, y \in P^k\}$$
$$\cup \{((1, x, y), (1, x, z)) : \ x \in P^k, (y, z) \in G\} \subseteq P^{4k+2}$$

The connected component of any $(0, x, x)$ is

$$C = \bigcup_{y \in P^k} (0, y, y) \cup \bigcup_{y \in P^k} \{(1, y)\} \times \mathrm{cc}_G(y).$$

It is then not hard to see that $\pi(C \cap \{(1,x)\} \times P^k) = \mathrm{cc}_G(x)$ (for π the projection on the last k coordinates). □

Proposition 6.2.5. *Let $P \subseteq \mathbb{R}$ be a definable set with $\langle P, <\rangle \simeq \langle \mathbb{N}, <\rangle$. Then $P_{Pres} = \langle P, <, \oplus\rangle \simeq \langle \mathbb{N}, <, +\rangle$ is definable in $\mathcal{M}_1$.*

Proof. Let $G_+ \subseteq P^4$ be the graph defined by

$$G_+ = \{((x,y),(s(x), s^{-1}(y)) : \ y > 0\}.$$

We define $x + y$ to be the (unique) $a \in P$ such that $(a, 0) \in \mathrm{cc}_{G_+}((x,y))$. It is not hard to see that $(P, <, \oplus) \simeq (\mathbb{N}, <, +)$. □

Theorem 6.2.6. *Let $P \subseteq \mathbb{R}$ be a definable set with $\langle P, <\rangle \simeq \langle \mathbb{N}, <\rangle$. Then $P^* = \langle P, <, \oplus, \odot\rangle \simeq \langle \mathbb{N}, <, +, \cdot\rangle$ is definable in $\mathcal{M}_2$. Moreover, if P_{Pres} is definable in $\mathcal{M}$ then P^* is definable in $\mathcal{M}_1$.*

Proof. By Proposition 6.2.5, we may assume that $+$ is definable on P and we use the Presburger notation from now. Ie the successor function will be denoted by $x \mapsto x + 1$ and $x \mapsto x + \ldots + x$ (i times) will be denoted by $x \mapsto i \cdot x$. 0 will denote the least element of P.

By [61], it is sufficient to define V_i (the i-valuation) and V_j for $i \wedge j = 1$. We show that we can define V_i for every $i \in \mathbb{N}$. Let $G \subseteq P^4$ be the graph defined by $((x,y),(x+1, i \cdot y))$. For $x \in P$, we have that $\mathrm{cc}_G((0,x) = \{(n, i^n \cdot x)\}$.

For $x \in P$, there is a unique $z \in P \setminus iP$ such that there is $t \in P$, $(t, x) \in \mathrm{cc}_G((0,z))$. We define $V_i(x)$ to be this t. □

Remark 6.2.7. We could have proven that $P^* \subseteq \langle \mathrm{CC}(\mathcal{R}, P\rangle)$ directly by taking

$$G_{\cdot} = \{((x,y,z),(x-1, y+z, z)) : \ x \in \mathbb{N}^{>0}, y, z \in \mathbb{N}\}.$$

We define

$$x \cdot y \text{ is the unique point in } \pi_2(\mathrm{cc}((x,0,y)) \cap \{0\} \times \mathbb{N}^2) \subseteq N^6$$

where π_2 is the projection on the second coordinates.

Corollary 6.2.8. *Let $\mathcal{M}$ be any expansion of the real line. If $\mathrm{CC}(\mathcal{M})$ defines an infinite discrete set then it is undecidable.*

6.3 The locally o-minimal case

In this section we prove that if $\widetilde{\mathcal{R}}$ is locally o-minimal and non-o-minimal and if the order type of P is $\mathbb{N}$ then $\mathrm{CC}(\widetilde{\mathcal{R}}) = \langle \mathcal{R}, P^*\rangle$ where $P^* = \langle P, <, \oplus, \odot\rangle \simeq \langle \mathbb{N}, <, +, \cdot\rangle$. We denote by $\widetilde{\mathcal{R}}^*$ the structure $\langle \mathcal{R}, P^*\rangle$. We recall that if $\widetilde{\mathcal{R}}$ is locally o-minimal then it is P-internal (Proposition 4.1.9).

Definition 6.3.1. Let X be a definable set and let $\{X_t : t \in S\}$ be an $\mathcal{L}$-decomposition of X. We define the cc_0 of $x \in X_{t_0}$ in X for $\{X_t : t \in S\}$ to be

$$\mathrm{cc}_{0,X,\{X_t\}}(x) = \{y \in X : \exists n \in \mathbb{N} \text{ and } t_1, \ldots, t_n \text{ such that}$$
$$\bigwedge_{0 \le i \le n} \overline{X_{t_i}} \cap \overline{X_{t_{i+1}}} \cap X \neq \emptyset \text{ and } y \in X_{t_n}\}.$$

As we shall see, in the locally o-minimal case, the mention of the family $\{X_t\}$ is unecessary. Note also that presently, the mention of X is not necessary since $X = \bigcup_{t \in S} X_t$.

Lemma 6.3.2. *If $\widetilde{\mathcal{R}}$ is locally o-minimal then for every definable set $X = \bigcup_{t \in S} X_t$ (for $\{X_t : t \in S\}$ some $\mathcal{L}$-decompositions of X) and every $x \in X$,*

$$\mathrm{cc}_{0,X,\{X_t\}}(x) = \mathrm{cc}_X(x).$$

Proof. By local o-minimality, we have that for every $S' \subseteq S$

$$\overline{\bigcup_{t \in S'} X_t} = \bigcup_{t \in S'} \overline{X_t}.$$

The result follows easily. □

Thus, $\mathrm{cc}_{0X,\{X_t\}}$ does not depend on $\{X_t : t \in S\}$ and in the following we denote by $\mathrm{cc}_X(x)$ the set $cc_{0X,\{X_t :\, t \in S\}}(x)$ for any $\mathcal{L}$-decomposition $\{X_t : t \in S\}$ of X.

Theorem 6.3.3. *If $\widetilde{\mathcal{R}}$ is locally o-minimal then* $\mathrm{CC}(\widetilde{\mathcal{R}}) = \widetilde{\mathcal{R}}^*$.

Proof. By Proposition 6.2.6, $\widetilde{\mathcal{R}}^* \subseteq \mathrm{CC}(\widetilde{\mathcal{R}})$. Moreover, since $\widetilde{\mathcal{R}}$ is locally o-minimal $\widetilde{\mathcal{R}}^\#$ is locally o-minimal and since $\mathrm{CC}(\widetilde{\mathcal{R}}) \subseteq \widetilde{\mathcal{R}}^\#$, $\mathrm{CC}(\widetilde{\mathcal{R}})$ is locally o-minimal too.

Let X be a definable set and let $\{X_t : t \in S\}$ be an $\mathcal{L}$-decomposition of X. For $t \in S$ let

$$Y_t = \{s \in S : X_t \cap \overline{X_s} \text{ or } \overline{X_t} \cap X_s \neq \emptyset\}.$$

Applying Lemma 6.3.2, we get:

$$\mathrm{cc}_X(x) = \bigcup_{t \in S'} X_t$$

where

$$S' = \{t \in S : \exists t_1, \ldots, t_n,\ x \in X_{t_1},\ y \in X_{t_n},\ \bigwedge_{0 \le i \le n} t_{i+1} \in Y_{t_i}\}.$$

We finish by showing that this set is definable in $\widetilde{\mathcal{R}}^*$. We do an induction on the level $(\widetilde{\mathcal{R}})_n$ in which we take a definable set. Let X be definable in $\widetilde{\mathcal{R}}$. Let us show that the connected components of X are definable in $\widetilde{\mathcal{R}}^*$. The family $\{Y_t\}$

is definable in $\widetilde{\mathcal{R}}$ and by classical recursion theory methods, S' is definable in $\widetilde{\mathcal{R}}^*$ and thus, the connected components of X are definable in $\widetilde{\mathcal{R}}^*$. We assume that the result holds for X definable in $(\widetilde{\mathcal{R}})_n$ and let X be definable in $(\widetilde{\mathcal{R}})_{n+1}$. The same argument applies here too and we have the result. □

Remark 6.3.4. If, moreover, we add the condition $P_{ind} \subseteq P^*$ then the structure induced on P by $\mathrm{CC}(\widetilde{\mathcal{R}})$ is P^*.

This condition is of course, not trivial: take for example: $\mathcal{R} = \mathbb{R}_{gp}$ and P a fast sequence. Let $A \subseteq P$ be a set that is not definable in P^*. Let

$$P' = P \cup \{x + 1 : \ x \in A\}.$$

As a subset of $P \cup P + 1$, P' is definable in $\widetilde{\mathcal{R}}^{\#}$ and $\langle \mathcal{R}, P' \rangle$ is d-minimal. Thus, A is definable in P'_{ind} but A is not definable in P'^*.

6.4 An approach of the cc-closure's problem for d-minimal non-locally o-minimal expansions

In this section, we are giving some tools and first observations to study the general problem of the cc-closure ($\mathbf{Q}_1$).

Question 6.4.1. Let $P \subseteq \mathbb{R}$ be a set of dimension 0 such that $\overline{P} \neq P$. What is the cc-closure of $\langle \mathcal{R}, P \rangle$?

Of course, the condition on P is not necessary if $\mathcal{R}$ expands the real field. Anyway, it is an open question and the following section is just a list of tools and examples, that we hope to be useful for the future resolution of the problem.

For the rest of this section: $P \subseteq \mathbb{R}$ is a set of dimension 0, $\mathcal{R}$ is an o-minimal expansion of $\mathbb{R}_{gp}$ and $\langle \mathcal{R}, P \rangle$ is d-minimal and non-locally o-minimal. Moreover, we assume that $\langle \mathcal{R}, P \rangle$ is P-internal.

As we saw, $\mathrm{CC}(\langle \mathcal{R}, P \rangle) \subseteq \langle \mathcal{R}, P \rangle^{\#}$. Thus, we can at least deduce some properties of the cc-closure.

Lemma 6.4.2. *The cc-closure of $\langle \mathcal{R}, P \rangle$ is d-minimal and if $\mathcal{R}$ is semibounded then the multiplication is not definable in it.*

Proof. By Fact 3.1.12, $\langle \mathcal{R}, P \rangle^{\#}$ is d-minimal. By Lemma 6.1.3

$$\mathrm{CC}(\langle \mathcal{R}, P \rangle) \subseteq \langle \mathcal{R}, P \rangle^{\#}$$

and since d-minimality is stable by taking a reduct, $\mathrm{CC}(\langle \mathcal{R}, P \rangle)$ is d-minimal. By Proposition 5.4.8, we have the second part. □

6.4.1 The cc-degree

What made things easy in the locally o-minimal case was that we just had to "follow the connections"from cell to cell. We generalize this idea:

Definition 6.4.3. For $X = \bigcup_{t\in P^k} X_t$. Let $y_0 \in X_{t_0}$ and let

$$\mathrm{cc}_{0,X,\{X_t\}}(y_0) = \{y \in X :\ \exists t_1, \ldots, t_{n-1},\, y \in X_{t_n} \text{ and for every } i$$

$$\overline{X_{t_i}} \cap X_{t_{i+1}} \text{ or } X_{t_i} \cap \overline{X_{t_{i+1}}} \neq \emptyset\}$$

(This is exactly the same definition as in Definition 6.3.1).

For $\beta \in \mathcal{O}rd$:

$$\mathrm{cc}_{\beta+1,X,\{X_t\}}(y_0) = \big\{y_n \in X :\ \exists y_1, \ldots, y_{n-1} \text{ such that}$$

$$y_n \in X_{t_n} \wedge \bigwedge_{1\le i\le n} \big(\overline{\mathrm{cc}_{\beta,X,\{X_t\}}(y_{i-1})} \cap \overline{\mathrm{cc}_{\beta,X,\{X_t\}}(y_i)} \cap X \neq \emptyset\big)\big\}.$$

If α is a limit ordinal:

$$\mathrm{cc}_{\alpha,X,\{X_t\}}(y_0) = \bigcup_{\beta<\alpha} \mathrm{cc}_{\beta,X,\{X_t\}}(y_0).$$

Let X be a definable set. We say that X has cc-degree $\alpha \in \mathcal{O}rd$ (cc-deg$(X) = \alpha$) if α is the least ordinal so that there is an $\mathcal{L}$-decomposition $\{X_t :\ t \in S\}$ of X, for every $x \in X$ $\mathrm{cc}_{\alpha,X,\{X_t\}}(x) = \mathrm{cc}_{\alpha+1,X,\{X_t\}}(x)$. For $\{X_t :\ t \in S\}$ an $\mathcal{L}$-decomposition of X and $x \in X$, we denote by $\text{cc-deg}_{X,\{X_t\}}(x) = \alpha$ if α is the least ordinal so that $\mathrm{cc}_{\alpha,X,\{X_t\}}(x) = \mathrm{cc}^*_X(x)$.
Let X be a definable set, $\{Y_t :\ t \in S'\}$ an $\mathcal{L}$-decomposition of X such that for every $x \in X$ $\mathrm{cc}_{\text{cc-deg}(X),X,\{Y_t\}}(x) = \mathrm{cc}_{\text{cc-deg}(X)+1,X,\{Y_t\}}(x)$. Let $x \in X$, we say that x has cc-degree $\alpha \in \mathcal{O}rd$ in X ($\text{cc-deg}_X(x) = \alpha$) if

$$\alpha = \inf\{\beta :\ \text{there is an } \mathcal{L}\text{-decomposition } \{X_t :\ t \in S\} \text{ of } X$$

$$\mathrm{cc}_{\beta,X,\{X_t:\, t\in S\}}(x) = \mathrm{cc}_{\text{cc-deg}(X),X,\{Y_t\}}(x)\}.$$

Note that for any $\mathcal{L}$-decomposition of X $\{X_t :\ t \in S\}$, since the X_t's are connected and that S is countable $\text{cc-deg}_X(x) < \omega_1$.
For $x \in X$ and some $\mathcal{L}$-decomposition of X $\{X_t :\ t \in S\}$, we denote by $\mathrm{cc}^*_X(x)$ the set $\mathrm{cc}_{\omega_1,X,\{X_t\}}$. Note that this set does not depend on the decomposition $\{X_t :\ t \in S\}$. Note also that since the cc-degree of a set according to some $\mathcal{L}$-decomposition is an ordinal, for every $x \in X$ there is a family $\{X_t :\ t \in S\}$ such that $\text{cc-deg}_X(x) = \text{cc-deg}_{X,\{X_t:\, t\in S\}}(x)$.
We say that X has *reached* cc-*degree* α if there is $x \in X$ with $\text{cc-deg}_X(x) = \alpha$.

Remark 6.4.4. This notion does not directly help to define the connected components of definable sets. There is some set X such that there is $x \in X$ with $\mathrm{cc}^*_X(x) \neq \mathrm{cc}_X(x)$ (see Example 6.4.5). However, it is easy to see that for every definable set X and $x \in X$ $\mathrm{cc}^*_X(x) \subseteq \mathrm{cc}_X(x)$.

Example 6.4.5. Let $(t_n)_{n\in\mathbb{N}} = S$ be a growing sequence with limit 1. We set

$$X \subseteq \mathbb{R}^3 = \bigcup_{t,t'\in S} [(0,t,t'),(t,t,t')] \cup \bigcup_{t\in S} [(0,t,t),(1,t,t)]$$

$$\cup \bigcup_{t \in S} [(1,0,t),(1,1,t)] \cup \bigcup_{n \in \mathbb{N}, t \in S} [(0,t,t_n),(0,t,t_{n+1})].$$

for $t \in S$, it is easy to see that

$$\mathrm{cc}^*_X((0,t,t)) = \bigcup_{n \in \mathbb{N}} [(0,t,t_n),(0,t,t_{n+1})] \cup \bigcup_{t' \in S} [(0,t,t'),(t,t,t')]$$

$$\cup [(0,t,t),(1,t,t)] \cup [(1,0,t),(1,1,t)] \neq X = \mathrm{cc}_X((0,t,t))$$

Definition 6.4.6. We say that a definable set X is cc*-*good* if for every $x \in X$, $\mathrm{cc}^*_X(x) = \mathrm{cc}_X(x)$.

We can now formulate a new question ($\mathbf{Q}_4$ below):

Question 6.4.7. What is the smallest structure $\mathcal{M}$ that expands $\widetilde{\mathcal{R}}$ and such that for every set definable in $\widetilde{\mathcal{R}}$, its cc* are definable in $\mathcal{M}$.

Proposition 6.4.8. *There is $\delta < \omega_1$ such that for any definable set X, any $\mathcal{L}$-decomposition of X $\{X_t : t \in S\}$ and any $x \in X$:*

$$cc_{\delta,X,\{X_t\}}(x) = cc^*_X(x).$$

Proof. There are only countably many definable sets and as we saw, $cc^*_X(x)$ is a countable union of some of the X_t's. Thus, since ω_1 is regular, it is suffcient to observe that by definition of the cc_β, if there is $\alpha \in \mathcal{O}rd$ for every $x \in X$:

$$\mathrm{cc}_{\alpha,X,\{X_t\}}(x) = \mathrm{cc}_{\alpha+1,X,\{X_t\}}(x)$$

then

$$\mathrm{cc}^*_X(x) = \mathrm{cc}_{\alpha,X,\{X_t\}}(x).$$

□

Question 6.4.9. Is the bound of the previous proposition the Church-Kleene ordinal ω_1^{CK}?

In order to avoid mentionning the decomposition into cells in the cc-degree, we would have liked to answer positively to the following question:

Question 6.4.10. Is there an $\mathcal{L}$-decomposition $\{X_t\}$ of X such that for every $x \in X$:

$$\text{cc-deg}_{X,\{X_t\}}(x) = \text{cc-deg}_X(x)?$$

The following proposition shows that we do not really need to answer this question.

Proposition 6.4.11. *Let $X \subseteq \mathbb{R}^n$ be a definable set. Let $\{X_t : t \in S\}$, $\{X'_t : t \in S'\}$ be some $\mathcal{L}$-decomposition of X. Then there is $N \in \mathbb{N}$ such that for every $x \in X$,*

$$\text{cc-deg}_{X,\{X_t\}}(x) \leq \text{cc-deg}_{X,\{X'_t\}}(x) + N.$$

Proof. Let $A = \{X_{t,t'} = X_t \cap X'_{t'} : t \in S, t' \in S'\}$. It is easy to see that for every $x \in X$ $\text{cc-deg}_{X,\{X_t\}}(x) \leq \text{cc-deg}_{X,A}(x)$. The only thing to prove is that there is $N \in \mathbb{N}$, for every $t \in S$, $t' \in S'$ if

$$X_t \cap X'_{t'} \neq \emptyset$$

then

$$X_t \subseteq \text{cc}_{N,X,A}(X_t \cap X'_{t'}).$$

For every $t \in S$, let

$$K_t = \{t' \in S' : X'_{t'} \cap X_t \neq \emptyset \text{ and has dimension lower than } X_t\}.$$

Let

$$Z_t = \overline{\bigcup_{t' \in K_t} X'_{t'} \cap X_t}.$$

By Theorem 3.3.16, there is a uniform decomposition of the Z_t's into finitely many $\mathcal{L}$ -EM of dimension lower than X_t $Y_i = \bigcup_{t \in S_i} Y_{i,t}$. Moreover there are some $\mathcal{L}$-EM $\bigcup_{t \in S'_i} A_{t,i}$ such that for every $t \in S_i$ there is j, $t' \in S'_j$ such that

$$\pi(Y_{i,t}) = A_{j,t'}.$$

Thus, for every j, $t' \in S'_j$ $\pi^{-1}(A_{j,t'}) \cap Z_t$ is composed of connected components which are graphs of continuous functions over $A_{j,t'}$.

By induction, there is a uniform bound on the cc-degree of ´$\pi(X_t \cap X'_{t'})$ for the decomposition $\{A_{i,t'} : i, t' \in S'_i\}$ and thus it is the same uniform bound for X_t for the decomposition $\{\pi^{-1}(A_{i,t'}) \cap X_t : i, t' \in S'_i\}$. Thus it is sufficient to prove that there is a uniform bound on $\pi^{-1}(A_{i,t'}) \cap X_t$ for the decomposition

$$\{\pi^{-1}(A_{i,t'}) \cap Y_{j,t''} \cap X_t : i, j, t' \in S_i, t'' \in S_j\}$$

for every $t \in S$, i, $t' \in S'_i$.

We finish by proving the following claim:

Claim 6.4.12. *For $z \in Z_{i,t'}$, let*

$$A = \{B : B \text{ is a maximal interval in } \pi^{-1}(z) \setminus Z_t \text{ or is a point in } Z_t \cap \pi^{-1}(z)\}.$$

Then

$$\text{cc-deg}(\pi^{-1}(z), A)$$

is bounded by N, the Cantor-Bendixson-rank of $\pi^{-1}(z) \cap Y$.

Proof. We do an induction on N. For $N = 0$, $Y \cap \pi^{-1}(z)$ is discrete and the result follows easily. We assume that the result holds for N and prove it for $N+1$. Let $Z = \pi^{-1}(z) \cap Z_t$. We have that $Z^{(N+1)}$ is discrete and that the CB-rank of $Z_t \cap \{z\} \times (a,b)$ where a, b are two consecutive points of $Z^{(N+1)}$, have CB-rank lower or equal to N. We apply the induction hypothesis to every subsets of $\pi^{-1}(z)$ of the form $\{z\} \times (a,b)$ for the decomposition A (restricted to $\{z\} \times (a,b)$) where a, b are two consecutive points of $Z^{(N+1)}$, that brings us back to the discrete case and gives us the result. ∎

By d-minimality, there is a uniform bound on the CB-rank of

$$\{\pi^{-1}(z) \cap Z_t : z \in A_{i,t'}, i, t' \in S_i\}.$$

Thus, for every $z \in A_{i,t'}$, there is a uniform bound N on the cc-degree of $X \cap \pi^{-1}(z)$ for the decomposition $\{X_t \cap X_{t'} \cap \pi^{-1}(z) : t \in St' \in S'\}$. Moreover, since for every $Y_{j,t''}$, $\pi(Y_{j,t''}) = A_{i,t'}$,

$$\mathrm{cc}_{N,\{Y_{j,t''}: t'' \in S'_j\}, X_t \cap \pi^{-1}(A_{i,t'})}(Y_{j,t''}) = X_t \cap \pi^{-1}(A_{i,t'}),$$

and we have the result. □

Corollary 6.4.13. *Let X be a definable set. Let $X = \bigcup_{t \in S} X_t = \bigcup_{t \in S'} X'_t$ such that $\{X_t : t \in S\}$ and $\{X'_t : t \in S'\}$ are $\mathcal{L}$-decomposition of X. If* cc-deg$(X, \{X_t\}) = \alpha \geq \omega$ *then*

$$\text{cc-deg}(X, \{X'_t : t \in S'\}) = \alpha.$$

Proof. The proof of Lemma 6.4.11 shows that for any $x \in X$,

$$\mathrm{cc}_{\omega,X,\{X_t\}}(x) = \mathrm{cc}_{\omega,X,\{X'_t\}}(x).$$

This gives us the result. □

Remark 6.4.14. The notion of cc-degree of a set according to some decomposition makes only sens in the d-minimal setting. Take for example a Cantor set A and the decomposition of $\mathbb{R}$ into A and the family of maximal intervals in $\mathbb{R} \setminus A$.

We could have hoped that by d-minimality, the cc-degree of a definable set X according to any decomposition of X would be finite (as it is the case for X an open cell). It is unfortunately not the case in general and in the next following subsections, we exhibit some examples of various infinite cc-degree.

6.4.2 An example of cc-degree ω

Let us begin with an example that shows that there should be no finite bound on the cc-degree (in order to lightened the notations we do not mention anymore the family for which we calculate the cc-degree. Anyway, in the following examples, the $\mathcal{L}$-decomposition is quite obvious).

Remark 6.4.15. Given a set $X = \bigcup_{t \in S} X_t$ (an $\mathcal{L}$-decomposition of X), assuming that the cc-degree of X is equal to $\alpha \in \mathcal{O}rd$, we have that the $(\mathrm{cc}_\beta)_{\beta < \alpha}$ is a sequence of equivalence relations satisfying for every $\beta < \alpha$:

(1)$\mathrm{cc}_\beta(x) \subseteq \mathrm{cc}_{\beta+1}(x)$,

(2): $|X/\mathrm{cc}_\beta| = \infty$,

(3): There are infinitely many classes of $\mathrm{cc}_{\beta+1}$ which contains infinitely many classes of cc_β.

Question 6.4.16. Let $(R_n)_{n\in\omega}$ be a sequence of equivalence relations on $\mathbb{N}$. At which conditions can we produce a good set $X = \bigcup_{t\in S} X_t$ such that there is a bijection $\sigma : \mathbb{N} \to S$ and such that for every $t \in S$

$$cc_{n,X,\{X'_t\}}(X_t) = \bigcup_{a\in\sigma^{-1}(R_n(t))} X_a \text{ ?}$$

The next example has been inspired by the following sequence of equivalence relations:
Let R_n be the equivalence relation on $\omega^{<\omega}$ defined by

$$xR_ny \text{ iff } |x| = |y| = m > n,\ (x_n, \dots, x_m) = (y_n, \dots, y_m).$$

It is easy to show that (1)-(3) of Remark 6.4.15 are satisfied.

We assume for now that $P = \mathbb{N}$. Let $\phi : \mathbb{N}^2 \to \mathbb{N} \setminus \{0, 1\}$ be a bijection definable in $\langle \mathbb{Z}, +, \cdot\rangle$. In $\mathbb{R}^2$ we define

$$X = \bigcup_{x,y\in\mathbb{N},z\in\omega^{<\omega}} (1/\phi(x, y), z) \cup \bigcup_{z\in\omega^{<\omega}} (0, z).$$

Note that for any non ultimately repetitive sequence (x_n, y_n), $1/\phi(x_n, y_n) \to 0$.

Of course This set is totally disconnected and we need to create some "links". That will be $(*)$:

1. some paths between two points of two copies of X, X_1 and X_2
2. passing by no other point of X_1 and X_2,
3. such that two such paths does not cross,
4. if we take some sequence (A_n) of such paths if A is a path such that $\overline{\bigcup_n A_n} \cap A \neq \emptyset$ then the extremities (in X_1 and X_2) of A belongs to the closure of the extremities of $\{A_n\}$.

(note that the second and third properties are a special case of the fourth).

We set

$$Z \subseteq X^2 = \big\{\big((1/\phi(x, y), 0^\frown z), (1/\psi(x + 1, y), 0^\frown z)\big) :\ x, y \in \mathbb{N}, z \in \omega^{<\omega}\}$$

$$\cup\{\big((1/\phi(0, y), n^\frown z), (1/\phi(y, z_m), (n + 1)^\frown z_1^\frown \dots^\frown z_{m-1})\big) :$$

$$n, y \in \mathbb{N}, z = z_1 \dots z_m \in \omega^{<\omega},\ m \geq 1\big\}.$$

Using the same definition than in Definition 6.2.2, we set $Y = X_Z$ (assuming of course that Z is reflexive and symetric on X^2). It is easy to see that the set of paths in Y between points of $X_1 = \{0\} \times \{(x, x) :\ x \in X\}$ and $X_2 = \{1\} \times \{(x, x) :\ x \in X\}$ satisfies $(*)$: (1)-(4).

We analyse the traces of the connected components of Y on X_1 and in order to avoid the notations to be heavy, we denote (for the next few lines) by $\mathrm{cc}_{n,Y}(x)$

the set $\pi(\mathrm{cc}_{n,Y}(x) \cap X_1)$, where π is the projection on the coordinates between 2 and $k+1$. Let $a = (1/\phi(x,y), 0^z) \in X$ (for $x, y \in \mathbb{N}$ and $z = z_1 \dots z_m \in \omega^{<\omega} \setminus \{\emptyset\}$). We have:

$$\mathrm{cc}_{0,Y}(a) = \big\{(1/\phi(x',y), 0^\frown z) : \ x' \in \mathbb{N}\big\} \cup \big\{(1/\phi(y, z_m), 1^\frown z_1 \frown \dots z_{m-1})\big\}$$

$$\mathrm{cc}_{1,Y}(a) = \big\{(1/n, 0^\frown z) : \ n \in \mathbb{N}\big\} \cup \big\{(0, 0^\frown z)\big\} \cup \big\{(1/\phi(y, z_m), 1^\frown z_1 \dots z_{m-1}) : \ y \in \mathbb{N}\big\},$$

If $|z| > 1$

$$\begin{aligned}\mathrm{cc}_{2,Y}(a) = \big\{(1/n, 0^\frown z_1 \dots z_{m-1}^\frown z') : \ n, z' \in \mathbb{N}\big\} \cup \big\{(0, 0^\frown z_1 \dots z_{m-1}^\frown z') : \ z' \in \mathbb{N}\big\} \\ \cup \big\{(1/n, 1^\frown z_1 \dots z_{m-1}) : \ n \in \mathbb{N}\big\} \cup \big\{(0, 1^\frown z_1 \dots z_{m-1})\big\} \cup \\ \big\{(1/\phi(y, z_{m-1}), 2^\frown z_1 \dots z_{m-2}) : \ y \in \mathbb{N}\big\}.\end{aligned}$$

By an easy induction we get that, for $|z| \geq k$:

$$\begin{aligned}\mathrm{cc}_{k,Y}(a) = \big\{(1/n, 0^\frown z_1 \dots z_{m-k+1}^\frown z') : \ n \in \mathbb{N}, z' \in \mathbb{N}^{k-1}\big\} \cup \\ \big\{(0, 0^\frown z_1 \dots z_{m-k+1}^\frown z') : \ z' \in \mathbb{N}^{k-1}\big\} \\ \cup \dots \\ \cup \big\{(1/n, (k-1)^\frown z_1 \dots z_{m-k+1} z') : \ n, z' \in \mathbb{N}\big\} \cup \big\{(0, (k-1)^\frown z_1 \dots z_{m-k+1}^\frown z') : \ z' \in \mathbb{N}\big\} \\ \cup \big\{(1/\phi(y, z_{m-k+1}), k^\frown z_1 \dots z_{m-k}) : \ y \in \mathbb{N}\big\}.\end{aligned}$$

It has of course cc-degree the length of z and thus there is no finite bound on the cc-degree over $x \in X$ and cc-deg$(X) = \omega$.

Remark 6.4.17. The set build in this section is cc*-good.

Moreover, we can build from the previous example an example of reached cc-degree ω. Let $Y' = X_{Z'}$ where

$$Z' = Z \cup \bigcup_{n \in \mathbb{N}} \big((0, n^\frown \emptyset), (0, (n+1)^\frown \emptyset)\big).$$

It is easy to see that cc-deg$(0, \emptyset) = \omega$.

6.4.3 An example of cc-degree ω^2

Following the idea of the previous section, we can build examples of various cc-degrees.

Let $k \in \mathbb{N}$, for $i \leq k$, let $C(i,k) =$

$$\begin{aligned}\Big\{\big((k, 0, (\Pi_{j \leq k-i} 0^\frown z_j)^\frown n^\frown \emptyset), (k, 1/\phi(n, (z_{k-i})_m), (\Pi_{j \leq k-i-1} 0^\frown z_j)^\frown 0^\frown (z_{k-i})_1 \dots (z_{k-i})_{m-1})\big) : \\ z_{k-i} = z'_1 \dots z'_m \in \omega^{<\omega} \setminus \{\emptyset\}, \ z = \Pi_{i \leq k-i+1} 0^\frown z_i, \ z_i \in \omega^{<\omega} \setminus \{\emptyset\}\Big\}\end{aligned}$$

$$\cup\big\{\big((k,1/\phi(0,y),z^\frown n^\frown z'),(k,1/\phi(y,z'_m),z^\frown(n+1)^\frown z'_1\dots z'_{m-1})\big):$$
$$y,\in\mathbb{N},\ z'=z'_1\dots z'_m\in\omega^{<\omega}\setminus\{\emptyset\},\ z=\Pi_{i\le k-i+1}0^\frown z_i,\ z_i\in\omega^{<\omega}\setminus\{\emptyset\}\big\}$$

For $i\le k$, let

$$D(i,k)=\big\{\big((k,0,(\Pi_{j\le k-i}0^\frown z_j)^\frown 0^\frown(1^s)^\frown(\Pi_{j\le i-1})0^\frown 1),$$
$$(k,0,(\Pi_{j\le k-i}0^\frown z_j)^\frown 0^\frown(1^{s+1})^\frown(\Pi_{j\le i-1}0^\frown 1))\big):\ z_j\in\omega^{<\omega}\setminus\{\emptyset\},\ s\in\mathbb{N}\big\}$$

Let

$$K_k=\bigcup_{i\le k}(C(k,i)\cup D(k,i))\cup\big\{\big((k,1/\phi(x,y),z),(k,1/\phi(x+1,y),z)\big):$$
$$y,\in\mathbb{N},\ z=\Pi_{i\le k}0^\frown z_i,\ z_i\in\omega^{<\omega}\setminus\{\emptyset\}\big\}.$$

It is easy to see that for every $k\in\mathbb{N}$, K_k has cc-degree $k\omega$ and thus

$$K_\omega=\bigcup_k K_k$$

has cc-degree ω^2. Moreover, it is easy to see that this set is cc*-good.

We could, of course, generalize this example to build some examples of cc-degree ω^n for every n, $\bigcup_n\omega^n$, $\bigcup_n\left(\bigcup_m\omega^m\right)^n$ and so on.

6.4.4 A problem in $\mathbb{Z}$ related to the problem of the cc-closure

In this subsection, we exhibit a correspondance between the problem of defining the cc*'s of sets definable in $\widetilde{\mathcal{R}}$ and some problem expressible in $\langle\mathbb{Z},<,+,\cdot,P_{ind}\rangle$. We assume that $\widetilde{\mathcal{R}}$ is P-internal, that $\mathcal{R}$ expands $\overline{\mathbb{R}}$ and that $P\simeq\mathbb{N}$.

Let X be a definable set and let $\{X_t:\ t\in S\subseteq P^k\}$ be an $\mathcal{L}$-decomposition of X. Let $P'\subseteq\mathbb{R}$ be a small set such that

$$0\notin P',\ \overline{P'}=P'\cup\{0\}.$$

The existence of such P' is easy to show by d-minimality (and of course because $\widetilde{\mathcal{R}}$ is not o-minimal). By P-internality, there is an $\mathcal{L}$-definable function $f:\mathbb{R}^n\to\mathbb{R}$ such that $P'\subseteq f(P^n)$. Thus, we may see P' as a set definable in P_{ind}. For $t\in S,x\in X_t,t'\in P'$ let

$$Y_{x,t'}=\{a\in S:\ X_a\cap\mathcal{B}(x,t')\neq\emptyset\}.$$

Let $\sim$ be the equivalence relation on X defined by:

$x\sim y$ if and only if $x,y\in X_t$ for some $t\in S$ and for every $t'\in P$, $Y_{x,t'}=Y_{y,t'}$.

By Proposition 3.4.7, there is a small number of classes $A_X = \{A_{t,t'} : t \in S, t' \in K_t\}$. For every $t \in S, t' \in K_t$, we set $W_{t,t'} = \{Y_{x,a} : a \in P'\}$ for some , $x \in A_{t,t'}$. Let $W_X = \{W_{t,t'} : t \in S, t' \in K_t\}$. Finally, for $t \in S$ let

$$Z_t = \{t' \in S : \overline{X_t} \cap \overline{X_{t'}} \cap X \neq \emptyset\}.$$

Note that the families $K = \{K_t : t \in S\}$ as well as $Z = \{Z_t : t \in S\}$ and W_X are definable in P_{ind}. Moreover,

$$Z_t = \bigcup_{t' \in K_t} \bigcap_{a \in P'} W_{t,t',a}.$$

Definition 6.4.18. For $x_0 \in S$, Let

$$Z_{0,W_X}(x_0) = Z_{x_0},$$

$$\mathrm{cc}_{0,W_X}(x_0) = \{x_n \in S : \exists x_1, \dots, x_{n-1} \in S \bigwedge_{i \leq n-1} x_{i+1} \in Z_{x_i}\}.$$

For $\beta \in \mathcal{O}rd$:

$$Z_{\beta+1,W_X}(x_0) = \{x_1 \in S : \text{ there is } i \neq j \in \{0,1\},\ z \in \mathrm{cc}_{\beta,W_X}(x_i) \text{ such that}$$

$$\text{there is } t' \in K_z, \text{ for all } a \in P',\ \mathrm{cc}_{\beta,W_X}(x_j) \cap W_{t,t',a} \neq \emptyset\}$$

$$\mathrm{cc}_{\beta+1,W_X}(x_o) = \{x_n \in S : \exists x_1, \dots, x_{n-1} \in S \text{ for every } i\ x_i \in Z_{\beta+1,W_X}(x_{i+1})\}.$$

For α a limit ordinal, we set

$$Z_{\alpha,W_X}(x) = \mathrm{cc}_{\alpha,W_X}(x) = \bigcup_{\beta<\alpha} \mathrm{cc}_{\beta,W_X}(x).$$

We define the cc-degree exactly in the same way than previously.

Lemma 6.4.19. *Let X be a definable set and $\{X_t : t \in S\}$ be an $\mathcal{L}$-decomposition of X. Then for every $\alpha \in \mathcal{O}rd$ for every $t \in S$ $x \in X_t$,*

$$\mathrm{cc}_{\alpha,X,\{X_t\}}(x) = \bigcup_{t' \in \mathrm{cc}_{\alpha,W_X}(t)} X_{t'}.$$

Proof. The proof is straigthforward. □

Thus, $\mathbf{Q}_4$ can be related to:

Question 6.4.20. Let $X = \bigcup_{t \in S} X_t$ be a set definable in $\widetilde{\mathcal{R}}^*$ and W_X, Z_X, A_X be defined as previously. Is $\mathrm{cc}_{W_X}(t)$ definable in $\langle \mathbb{N}, <, +, \cdot, P_{ind} \rangle$?

As it was the case in the locally-o-minimal setting, we would like to have a correspondance between the sets definable in $\widetilde{\mathcal{R}}$ and the graph definable in P^*_{ind}. We set $P' \subseteq \mathbb{R}$, a bounded discrete set with accumulation $\{0\}$.

Proposition 6.4.21. *We assume $\widetilde{\mathcal{R}}$ to expand the real field. For a triplet (definable in the induced structure on P by $\widetilde{\mathcal{R}}^*$) $K = \{t, t' : t \in S, t' \in K_t\}$, $Z = \{Z_t \subseteq S : t \in S\}$, $W = \{W_{t,t'} = \{W_{t,t',a} : a \in P'\} : t \in S, t' \in K_t\}$ of the form $(**)$:*

1. *for every $t, t' \in S$ if $t \in Z_{t'}$ then $t' \in Z_t$.*
2. *For every $t \in S$, $t' \in K_t$, $a \in P'$, $W_{t,t',a} \subseteq S$,*
3. *for every $t \in S$, $t' \in K_t$, $a' < a \in P'$, $W_{t,t',a'} \subseteq W_{t,t',a}$.*
4. *for every $t \in S$ $Z_t = \bigcup_{t' \in K_t} \bigcap_{a \in P'} W_{t,t',a}$.*

Then there is set X definable in $\widetilde{\mathcal{R}}^*$, an $\mathcal{L}$-decomposition $X = \bigcup_{t \in K} X_t$ and a definable embedding $\sigma : S \to X$ such that for every $\alpha \in \mathcal{O}rd$, for every $t \in S$

$$\mathrm{cc}_{\alpha,X,\{X_t\}}(\sigma(t)) \cap \sigma(S) = \mathrm{cc}_{\alpha,W}(t).$$

Proof. Without loss of generality, by applying an homeomorphism definable in $\overline{\mathbb{R}}$, we may assume $S \subseteq \mathbb{R}^k$ to be discrete, bounded and closed in its convex hull. Thus $P' \simeq -\mathbb{N}$ and we use the Presburger notation for the rest of the proof. We first define

$$X = \{(0, t, t', t'', a) : t \in S,\ t' \in K_t,\ a \in P',\ t'' \in W_{t,t',a}\} \cup$$
$$\{(0, t, t', \mathbb{R}^k, 0) : t \in S, t' \in K_t\} \cup \{(1, 0, 0, 0, t) : t \in S\}.$$

Let

$$G \subseteq X^2 = \big\{\big((1, 0, 0, 0, t), (1, 0, 0, 0, t')\big) : t' \in Z_t\big\} \cup$$
$$\big\{\big((0, t, t', t'', a), (0, t, t', t'', a-1)\big) : t \in S, t' \in K_t, t'' \in W_{t,t',a-1}\big\}$$
$$\big\{\big((0, t, t', t'', a), (1, 0, 0, 0, t'') : t \in S, t' \in K_t,\ t'' \in W_{t,t',a}\big)\big\}$$
$$\cup \big\{\big((0, t, t', 0, 0), (1, 0, 0, 0, t)\big) : t \in S, t' \in K_t\big\}.$$

As usual, we need to create some links between the points of X. We take

$$X' = X \setminus \{(0, t, t', \mathbb{R}^k \setminus \{0\}, 0) : t \in S, t' \in K_t\},$$

we consider G to be a graph on X' and we take X_G, the set build as in Definition 6.2.2. It is easy to see that the set of links between the two copies of X' satisfies (1)-(4) of $(*)$ in Section 6.4.2. In order to stay understandable, we do not work in X_G but in X and consider the "links"as virtual. Ie We consider the connected components of X_G but we denote by $\mathrm{cc}_X(x)$ the set $\mathrm{cc}_{X_G}(\rho(x)) \cap X$ where $\rho : X \to X_G$ is an embedding that sends X onto one of its copies in X_G.

Let $X_1 = \{(1, 0, 0, 0, t) : t \in S\}$. We finish with the following claim:

Claim 6.4.22. *For every $t \in S$, every $\beta \in \mathcal{O}rd$,*

$$Z_{\beta,W}(t) = X_1 \cap Z_{\beta,X}((1, 0, 0, 0, t)).$$

Thus, $\mathrm{cc}_{\beta,W}(t) = X_1 \cap \mathrm{cc}_{\beta,X}((1, 0, 0, 0, t))$.

Proof. The second part follows directly from the first part.

For the first part, we do an induction on $\beta \in \mathcal{O}rd$. For $\beta = 0$, $Z_{0,W}(t) = Z_t$ and

$$Z_{0,X}((1,0,0,0,t)) = \{(1,0,0,0,t') : t' \in Z_t\} \cup$$
$$\{(0,t',t'',t''',a) : t' \in S, t'' \in K_{t'}, t'' \in Z_t \cap W_{t,t',a}\}.$$

The result follows directly.

We assume that the result holds for $\beta \in \mathcal{O}rd$. We show it for $\beta+1$. We first show that

$$Z_{\beta,W}(t) \subseteq X_1 \cap Z_{\beta,X}((1,0,0,0,t)).$$

By definition,

$$Z_{\beta+1,W}(t) = \{t' : \exists t'' \in K_{t'},\ \text{for every } a \in P',\ W_{t',t'',a} \cap \mathrm{cc}_{\beta,W}(t) \neq \emptyset\}$$

Let $t' \in Z_{\beta+1,W}(t)$ and for $n \in \mathbb{N}$ let $t_n \in \mathrm{cc}_{\beta,W}(t)$ such that there is $t'' \in K_{t'}$, for every $a \in P$, there is N_a with

$$t_{N_a} \in W_{t',t'',a}.$$

Since S is bounded, the sequence

$$\big((0,t',t'',t_{N_a},a)\big)_{a \in P'}$$

has an accumulation point in $(0,t,t',\mathbb{R}^k,0)$ and

$$(0,t',t'',0,0) \in \mathrm{cc}_{\beta+1,X}(1,0,0,0,t).$$

Thus:

$$(1,0,0,0,t') \in \mathrm{cc}_{\beta+1,X}(1,0,0,0,t)$$

and we have the result.

Finaly, we prove

$$X_1 \cap Z_{\beta+1,X}((1,0,0,0,t)) \subseteq Z_{\beta+1,W}(t).$$

Let $(1,0,0,0,t') \in Z_{\beta+1,X}((1,0,0,0,t))$ By construction, it is easy to see that there is $t'' \in K_{t'}$ and $x \in R^k$ such that $(0,t',t'',x,0)$ is an accumulation point of

$$\mathrm{cc}_{\beta,X}((1,0,0,0,t)) \cap \{(0,t',t'',t''',a) : t''' \in S, a \in P'\}.$$

Thus for every $a \in P'$ there is t''' such that $(0,t',t'',t''',a) \in \mathrm{cc}_{\beta,X}(1,0,0,0,t)$. By definition, $t''' \in W_{t',t'',a}$ and

$$t' \in Z_{\beta+1,W}(t).$$

The case where β is a limit ordinal follows directly and we have the result. □

□

6.5 An answer to Question 2 and 3 for expansions of a semibounded structure by an iteration sequence or a fast sequence

In this section we take a look at some questions about the connected components of expansions of a semibounded structure by a fast or an iteration sequence. As usual, let $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P\rangle$ where $\mathcal{R}$ is a semibounded expansion of $\mathbb{R}_{gp}$ and P a fast or an iteration sequence.

6.5.1 $\widetilde{\mathcal{R}}_1 = \langle \mathcal{R}, P_{Pres}\rangle$

In this subsection, we show that $\widetilde{\mathcal{R}}_1 = \langle \mathcal{R}, P_{Pres}\rangle$ where P_{Pres} is the Presburger-arithmetic put on P seen as $\mathbb{N}$.

We first need some preliminary lemmas. For the rest of this section, we use the Presburger notations to refer to operations on P. In this subsection, we only look at sets in P^k, thus when using the notation $\mathcal{B}(x,a)$ (for $x \in P^k$, $a \in \mathbb{N}$), we mean the set $\{y \in P^k : \exists -a \leq m_i \leq a,\ \bigwedge_{i\leq k} y_i = s^{m_i}(x_i)\}$.

Definition 6.5.1. Let $S \subseteq P^k$, we say that S has s-dimension k and we denote by $\dim_s(S) = k$ the fact that for every $n \in \mathbb{N}$ there is $x \in S$

$$\mathcal{B}(x,n) \subseteq S.$$

We say that an s-cell S has s-dimension m and we denote by $\dim_s(S) = m < k$ the fact that S is in bijection (via some projection on some coordinates) with some s-cell of dimension m in P^m. We say that a definable set $X \subseteq P^k$ has s-dimension n if it contains an s-cell of dimension n and no cell of dimension $n+1$.

Remark 6.5.2. Note that if $X \subseteq P^k$ is a definable set, for any $n \in \mathbb{N}$:

$$\dim\big(\{x \in X : \mathcal{B}(x,n) \nsubseteq X\}\big) < k.$$

Lemma 6.5.3. *Let $S \subseteq P^k$ be an n s-cell. Then S is convex: for every $b \in P^k$ if there is $x \in S$ and $m \in \mathbb{N}$ such that $x + mb \in S$ then for every $m > n$ $x + nb \in S$.*

Proof. By construction of an s-cell. □

Lemma 6.5.4. *Let $S_1, \dots, S_n \subseteq P^k$ be some definable sets. Then*

$$\dim_s(\bigcup_i S_i) = \max(\dim_s(S_i)).$$

Moreover, for a definable family $\{X_t \subseteq P^k : t \in P\}$ of disjoint sets of dimension n, $\dim_s(\bigcup_{t\in P} X_t) > n$.

Proof. The first part is by convexity of an s-cell. The second part is by a simple analysis on the structure of an s-cell. □

Definition 6.5.5. Let $G \subseteq S^2$ be a graph. Let $A \subseteq S$, we denote by G_A the set

$$\{(x, y) \in A^2 : (x, y) \in G\}.$$

Theorem 6.5.6. *The structure $\widetilde{\mathcal{R}}_1$ is equal to $\langle \mathcal{R}, P_{Pres} \rangle$. Moreover, let $X = \bigcup_{t \in S} X_t$ be definable in $\widetilde{\mathcal{R}}$ and $\{X_t : t \in S\}$ an $\mathcal{L}$-decomposition of X. Then the connected components of X are a finite union of sets of the form $\bigcup_{t \in S'} X_t$ where S' is the intersection of an s-cell with $x + \langle (b_i)_{i \leq n} \rangle$ (the translate by $x \in P^k$ of the $\mathbb{Z}$-module generated by $(b_i)_{i \leq n}$ for some $b_i \in P^k$).*

Proof. We first prove that $\widetilde{\mathcal{R}}_1 \subseteq \langle \mathcal{R}, P_{Pres} \rangle$.

Let X be a set definable in $\widetilde{\mathcal{R}}$. By Theorem 3.3.16, there is an $\mathcal{L}$-decomposition of X $\{X_t : t \in S\}$. Moreover, by Theorem 4.3.6, we may assume that $S \subseteq P^k$. By Proposition 6.3.2, for every $x \in X$, $\mathrm{cc}_X(x) = \mathrm{cc}_{0,X}(x)$ and as usual, we may see X as a graph $X' \subseteq S^2$. Ie for any $x \in X$ (and $t \in S$ such that $x \in X_t$) the connected component of x in X is equal to

$$\bigcup_{t \in \mathrm{cc}_{X'}(t)} X_t.$$

We do an induction on k. For $k = 0$, the result is trivial. We assume that the result holds for $i < k$.

For $t \in S$ let

$$Z_t = \{s \in S : (t, s) \in X'\} = \{s \in S : \overline{X_s} \cap X_t \text{ or } X_s \cap \overline{X_t} \neq \emptyset\}.$$

By Theorem 3.4.7, there is a decomposition of $\bigcup_{t \in S} Z_t$ into finitely many s-cells and we may assume that Z is an s-cell. By a simple analysis we see that if $\{\#(Z_t) : t \in S\}$ is bounded, then Z is the graph of a function $f : S' \to S$ over $S' \subseteq S$ and for every $t \in S'$, $\#(Z_t) = 1$. Moreover, by definition of an s-cell, it is easy to see that there is $b \in \mathbb{N}^d$ such that for every $t \in S'$

$$f(t) = t + b.$$

If there is no such bound, we replace S by S^2 and X' by an other graph with bounded branching but before defining it, we first need the following claim:

Claim 6.5.7. *Let $\{Z_t : t \in S\}$ be a Presburger-definable family of sets of the form $S \cap x + \langle (b_i)_{i \leq n} \rangle$ for $S \subseteq P^k$ an s-cell and $b_i \in P^k$. Then there is a definable family of graphs $\{Z_t^* \subseteq P^{2k} : t \in S\}$ definable in $\langle P, < \rangle$ such that:*

1. *$\{Z_t^* : t \in S\}$ is definable in $\langle P, < \rangle$,*
2. *for every $t \in S$ for every $y \in Z_t$*

$$\mathrm{cc}_{Z_t^*}(y) \cap S = Z_t,$$

3. *there is a uniform bound on the branching of Z_t^*.*

Proof. We set

$$Z_t^* = \{(x, y) \in P^{2k} :\ \exists i, x = y + b_i \text{ or } x = y - b_i\}.$$

Properties (1)-(3) are easy to check. □

By s-cell decomposition, we may assume that the Z_t's are s-cells and we replace X' by

$$X'' = \{((x, y), (x, t)) :\ (y, t) \in Z_x^*\} \cup \{((x, y), (y, x)) :\ x \in Z_y,\ y \in S\}.$$

It is easy to see that, by Claim 6.5.7 X'' has bounded branching and that for every $x \in S$

$$\mathrm{cc}_{X'}(x) = \pi_1(\mathrm{cc}_{X''}((x, x))),$$

where π_1 is the projection on the first d coordinates.

We finish by proving that the connected components of a graph definable in $\widetilde{\mathcal{R}}$ with bounded branching is Presburger-definable.

Let $b_1, \dots, b_n \in \mathbb{N}^k$ such that the branches of X'' either have the form $(x, x + b_i)$ or $(x, x - b_i)$ for some i (there are finitely many because X'' has bounded branching). For $x \in S$, let

$$E_x = \{i :\ (x, x - b_i) \in X''\},$$

and let $F_x = \{i :\ (x, x + b_i) \in X''\}$. We can decompose S into finitely many sets A_j such that

$$(*) : \forall x, y \in A_j :\ E_x = E_y,\ F_x = F_y.$$

Moreover, by uniform s-cell-decomposition, we may assume that A_j is an s-cell. In the following whenever an s-cell A satisfies $(*)$ we denote by E_A and F_A the sets E_x and F_x for any $x \in A$. We first prove the following claim:

Claim 6.5.8. *Let A be an s-cell of s-dimension k that satisfies $(*)$. Then $E_A = F_A$.*

Proof. Since A has s-dimension k then there is $x \in A$ such that

$$\mathcal{B}(x, \max_i(|b_i| + 1) \subseteq A.$$

Thus,

$$\forall i \in E_A,\ x + b_i \in A$$

and $i \in F_{x+b_i}$. Since

$$F_{x+b_i} = F_x = F_A,$$

$$i \in F_x$$

and we have the resut. □

Note that for every $x \in A$

$$\mathrm{cc}_{X''_A}(x) = x + \langle (b_i)_{i \in E_A} \rangle \cap A.$$

From Claim 6.5.8 we can deduce that

$$A' = \{x \in A : \exists i, \, x + b_i \notin A\}$$

has lower dimension than k. We recall that

$$\{\mathrm{cc}_{X''_A}(x) \cap A' : \, x \in A'\} \text{ is definable in Presburger.}$$

Moreover, $\mathrm{cc}_{X''_A}(x) = x + \langle b_i : i \in E_A \rangle \cap A$. Let

$$A'' = \{(x, y) \in A'^2 : \, (x, y) \in \big(\mathrm{cc}_{X''_A}(x)\big)^* \cap A'\}.$$

Claim 6.5.9. *Let*

$$G = X''_{(S \setminus A) \cup A'} \cup A''.$$

Then for $x \in (S \setminus A) \cup A'$

$$\mathrm{cc}_{X'}(x) \cap \big((S \setminus A) \cup A'\big) = \mathrm{cc}_G(x).$$

Moreover, for $x \in A$*, if there is* $z \in \mathrm{cc}_{X''_A}$*:*

$$cc_{X''}(x) \cap A = \bigcup_{y \in \mathrm{cc}_G(z)} y + \langle b_i : \, i \in E_A \rangle \cap A.$$

Proof. By definition of all the terms involved. □

Thus, we may replace X'' by G and assume that S is a finite union of s-cells $A'_1, \dots, A'_a$ of s-dimension lower than k.

We do an induction on a. For $a = 1$, we just apply the induction hypothesis (the one on k). Let $A_1, \dots, A_a$ be an s-cell-decomposition of A. For $c, d < a$, let

$$K = \{(x, y) : \, \exists c, d \leq a, \, \exists z \in A_a, \, x \in A_c, \, y \in A_d, b \in E_z, b' \in F_z, \, x{+}b = z, \, z{+}b' = y\}.$$

By induction, the connected components of X''_{A_a} are a finite union of sets of the form $\langle b'i \rangle \cap Q$ where Q is an s-cell. Thus, we may assume that they have this form. Let

$$W = \{(x, y) : \, \exists i, j \leq ax, y \notin A_a, \, x + b_i, y - b_j \in A_a, \, b_i \in F_{x+b_i}, b_j \in E_{x-b_j},$$
$$(x + b_i, y - b_j) \in (\mathrm{cc}_{A_a}(x + b_i))^* \cap Q\}.$$

It is easy to see that $G' = W \cup K \cup X''_{\bigcup_{i<a} A_i}$ has bounded branching. By definition of W and K, for every $x \in \bigcup_{i<a} A_i$:

$$\mathrm{cc}_{G'}(x) = \mathrm{cc}_{X''}(x) \cap \bigcup_{i<a} A_i.$$

Moreover, for $z \in \mathrm{cc}_{X''}(x) \setminus A_a$ (if there is such z):

$\mathrm{cc}_{X''}(x) \cap A_a = \{y \in A_a :\ \exists z' \in \mathrm{cc}_{G'}(z),\ z + b_i \text{ or } z - b_i = y \text{ and } i \in E_y \text{ or } F_y\}.$

Thus, we apply the induction hypothesis (the one on a) to G' to get the result.

Since by Proposition 6.2.5, P_{Pres} is definable in $\widetilde{\mathcal{R}}_1$, we have the result. □

Remark 6.5.10. Note that, for example the graph of $+$ does not have the shape of a connected component of a set definable in $\widetilde{\mathcal{R}}$.

6.6 Some preliminary results for the study of definable groups

In this subsection, we take a quick look at some preliminary result of applications to the study of definable groups in $\widetilde{\mathcal{R}}$. For the rest of this subsection, we fix $G = (G, \oplus)$, a group definable in $\widetilde{\mathcal{R}}$ (we denote by G both the group and its domain).

Fact 6.6.1. *(see Fornasiero's [23]) There is a definable topology τ on G that makes G a topological group. Moreover, there are some definable sets $G_1, \ldots, G_m \subseteq G$ such that:*

1. *$\tau_{\restriction G_i}$ is equal to the ambiant topology*
2. *G_{i+1} is relatively open (for τ) and large in $G \setminus \bigcup_{j \leq i} G_j$.*

Note that each G_i admits an $\mathcal{L}$-decomposition and thus there is an $\mathcal{L}$-decomposition of G $G = \bigcup_{t \in S} X_t$ such that for every $t \in S$, there is i, $X_t \subseteq G_i$. We denote by $\mathrm{cc}_\tau(x)$ the τ-connected component of x in G. For $x \in G$, $\alpha \in \mathcal{O}rd$ we define $\mathrm{cc}_{\tau,\alpha}(x)$ exactly in the same way that in Definition 6.4.3. Note that since (G, τ) is a topological group, the τ-connected component of 1_G is a subgroup.

Proposition 6.6.2. *If $\widetilde{\mathcal{R}}$ expands the real field, then the connected components of G for τ are uniformly definable in $\widetilde{\mathcal{R}}_1$.*

Proof. Let $X_0 \subseteq G_1$ be a cell of dimension $\dim(G)$ and let $x \in X_0$. We have that around x, the τ-topology and the ambient topology coincide.

As in Section 6.4.4, we build the families A, Z, W. Let $P' \subseteq (0,1)$ be a discrete bounded set such that $\overline{P'} = P' \cup \{0\}$. We may assume that for every $t \in P'$,

$$\mathcal{B}(x, t') \cap G \subseteq X_0.$$

By P-internality, there is an $\mathcal{L}$-definable function $f : \mathbb{R}^n \to \mathbb{R}$ such that $P' \subseteq f(P^n)$. Thus, we may see P' as a set definable in P_{ind}. For $t \in S, y \in X_t, t' \in P'$ let

$$Y_{y,t'} = \{a \in S :\ X_a \ominus x \oplus y \cap \mathcal{B}(t') \neq \emptyset\}.$$

Let $\sim$ be the equivalence relation on G defined by

$z \sim y$ if and only if $z, y \in X_t$ for some $t \in S$ and for every $t' \in P$, $Y_{z,t'} = Y_{y,t'}$.

By Proposition 3.4.7, there is a small number of classes $A = \{A_{t,t'} : t \in S, t' \in K_t\}$. For every $t \in S, t' \in K_t$, we set $W_{t,t'} = \{Y_{z,a} : a \in P', z \in A_{t,t'}\}$ and $W = \{W_{t,t'} : t \in S, t' \in K_t\}$. As in Lemma 6.4.19, it is easy to see that for every $\alpha \in \mathcal{O}rd$ for every $t \in S$ $zx \in X_t$,

$$\mathrm{cc}_{\tau,\alpha,G,\{X_t\}}(z) = \bigcup_{t' \in \mathrm{cc}_{\alpha,A,Z,W}(t)} X_{t'}.$$

It is easy to see that Z, A, W satisfy the conditions of Proposition 6.4.21 and let X_G be given by it. Since the connected components of X_G are definable in $\widetilde{\mathcal{R}}_1$, $\{\mathrm{cc}_{Z,A,W}(t) : t \in S\}$ is definable in $\mathbb{R}_{sb}$ and we have the result. □

We have the same result in the locally o-minimal setting.

Proposition 6.6.3. *We assume $\widetilde{\mathcal{R}}$ to be locally o-minimal. Then the τ-connected components of G are uniformly definable in $\widetilde{\mathcal{R}}_1$.*

Proof. We just have to show that for $x \in G$, $\mathrm{cc}_{\tau,0}(x) = \mathrm{cc}_\tau(x)$. That is equivalent to:

$$\text{if } x \in \overline{\bigcup_{t \in S'} X_t}^{\tau} \text{ then } x \in \overline{X_t}^{\tau} \text{ for some } t \in S'.$$

Let $B \ni y$ be a box such that the τ-topology and the ambient topology agree on $B \cap G$. Let

$$f : B \to S,\ z \mapsto t \text{ such that } z \oplus x \ominus y \in X_t.$$

By local o-minimality, there are $B_1, \dots, B_n$ such that $\bigcup_i B_i = B$ and $f_{\restriction B_i}$ is $\mathcal{L}$-definable and continuous. Thus, the image of f is finite and we have the result. □

Question 6.6.4. Do we have the uniform definability of the connected components of G in $\widetilde{\mathcal{R}}_1$ for $\widetilde{\mathcal{R}}$ a d-minimal expansions of $\mathbb{R}_{gp}$ in general?

Question 6.6.5. Let X be a definable set and let τ be a definable topology on X. Do we have the uniform definability of the τ-connected components of X in $\widetilde{\mathcal{R}}_1$?

Proposition 6.6.6. *If $\widetilde{\mathcal{R}}$ is locally o-minimal. The τ-connected component of 1_G has finite index.*

Proof. We assume that there is an infinite set $S' \subseteq S$ such that for every $t \neq t' \in S'$ X_t and $X_{t'}$ are not in the same τ-connected component. Let $\{x_t \in X_t : t \in S\}$ be a definable family (that is possible by definable choice) and let

$$f : (t,t') \in S^2 \mapsto t'' \text{such that } x_t \oplus x_{t'} \in X_{t''}.$$

f is definable in $\langle P, < \rangle$ and thus, by decomposition into s-cells, we may assume that the graph of f is an s-cell and that $f(x,y) = s^n(\pi_i(x))$ (for some $n \in \mathbb{N}$ and

i). Moreover, we may assume that there is x, y, $y' \in S'$ such that (x,y), $(x,y') \in \mathrm{dom}(f)$. Thus
$$f(x,y) = f(x,y'),$$
$x_x \oplus x_y$ and $x_x \oplus x_{y'}$ are in the same τ-cc and thus x_y and $x_{y'}$ too which is a contradiction with the choice of S'. □

As we saw in Section 5.5.14, it is not because $\widetilde{\mathcal{R}}_1$ is complex that the connected components of sets definable in $\widetilde{\mathcal{R}}$ are as complex. For example, it is easy to see that the connected components of sets definable in $\langle \mathbb{R}_{gp}, \mathbb{Z}\rangle$ are of the form

$$\big\{y \in P^k : \exists i_1, \dots, i_n \in \{1, \dots, m\}, \bigwedge_j \big(\Pi_{l \leq j} f_{i_j}\big)(x) \in \mathrm{dom}(f_{i_{j+1}}) \text{ and } (\Pi_{j, \leq n} f_{i_j})(x) = y\big\}$$

for some affine functions f_i. Moreover, it is easy to see that the graph of the multiplication on $\mathbb{Z}$ is definable in $\widetilde{\mathcal{R}}_1$ but does not have this form. Thus studying the properties of the τ-connected component of 1_G could be simpler.

6.7 Questions and discussions about possible generalizations

6.7.1 General d-minimal setting

In this subsection, we just ask some questions in line of $\mathbf{Q}_1$ for d-minimal expansions of $\mathbb{R}_{gp}$ (that are not of the form $\langle \mathcal{R}, P\rangle$ for $P \subseteq \mathbb{R}$). The main objection is that we do not know d-minimal expansions of $\mathbb{R}_{gp}$ that are not reducts of structures of the form $\langle \mathcal{R}, P\rangle^{\#}$. Note also that we do not have anymore notions like P-internality, P^* or $\mathcal{L}$-decomposition. About the last one, we can ask the following questions:

Question 6.7.1. Let $\mathcal{M}$ be a d-minimal expansion of $\mathbb{R}_{gp}$. Is

$$\langle \mathcal{M}, \big\{\{X_t : t \in S\} : \text{there is an } \mathcal{M}\text{-definable set } X,\ X = \bigcup_{t\in S} \text{ and the}$$

$$X_t\text{'s are the connected components of a special manifold in the decomposition of } X\big\}\rangle$$

d-minimal ?

Question 6.7.2. Is the expansion of $\mathcal{M}$ by one connected component of a special manifold in the decomposition of a definable set d-minimal ?

6.7.2 Some questions about the topology generated by the definably-connected components of some definable set

In the example of a non-cc-closed structure of Fornasiero ([27], the structure being roughly expansion of $\mathbb{R}_{an}$ by a an iteration sequence), the double spirale is non-connected but definably-connected. Let $\mathcal{M}$ be an expansion of $\mathbb{R}_{gp}$.

Definition 6.7.3. Let X be a set definable in $\mathcal{M}$. Let τ_X be the topology generated by the definable clopen subsets of X. Let τ_X/cc be the quotient-topology of τ_X by the equivalence relation:

$$x \sim y \text{ if and only if } \mathrm{cc}_X(x) = \mathrm{cc}_X(y).$$

Note that this topology is on $X/\sim$. We say that X is cc-*Hausdorff* if X is not definably connected but for every $x, y \in X$ such that $\mathrm{cc}_X(x) \neq \mathrm{cc}_X(y)$ there are two definable sets $Z \ni x$ and $Y \ni y$ such that

$$Z \cap Y = \emptyset \text{ and } Z, Y \text{ are clopen in X.}$$

In other words, if τ_X/cc is Hausdorff.

We could inspect the different properties of τ_X/cc such as T_2. In this subsection we prove that any non cc-closed locally o-minimal structure defines some non-connected definably-connected sets. In the following, we say that such a set is *ncdc*.

We assume that $\mathcal{M} \subseteq \widetilde{\mathcal{R}}^{\#}$ where $\widetilde{\mathcal{R}}$ is the expansion of an o-minimal structure and $P \subseteq \mathbb{R}$ is a set of dimension 0 with order-type $\mathbb{N}$.

Proposition 6.7.4. *If $\mathcal{M}$ defines no ncdc set then it defines P_{Pres}.*

Proof. As usual we use the Presburger notations. Let

$$G_+ = \big\{\big((x,y,z),(x+1,y+1,z)\big) :\ y,z,\in\mathbb{N},\ x \geq y \in \mathbb{N}\big\} \cup$$

$$\big\{\big((x,0,x),(y,0,y)\big) :\ x,y \in \mathbb{N}\big\} \cup \big\{\big((x,0,y),(x',0,y')\big)\ x \neq y,\ x' \neq y'\big\} \subseteq \mathbb{N}^6.$$

We assume that G_+ is a graph over $Z = \{(x,y,z) :\ x \geq y, z \in \mathbb{N}\}$. It is easy to see that G_+ has two connected components:

$$X_1 = \big\{(x+y,y,z) :\ x \neq z \in \mathbb{N},\ y \in \mathbb{N}\big\}$$

and

$$X_2 = \big\{(x+y,y,x) :\ x,y \in \mathbb{N}\big\}.$$

Since $\mathcal{M}$ defines no ncpc set, X_2 is definable in P_{ind} and we define $x+y$ to be the unique $z \in \mathbb{N}$ such that $(z,y,x) \in X_2$. □

Proposition 6.7.5. *If $\mathcal{M}$ defines no ncdc set then it defines P^*.*

Proof. Let

$$G_\cdot = \big\{\big((x,y,z),(x+y,y,z+1)\big) :\ x \in \mathbb{Z},\ y,z \in \mathbb{N}\big\} \cup \big\{\big((0,y,0),(0,x,0)\big) :\ x,y \in \mathbb{N}\big\}$$

$$\cup \big\{\big((x,y,0),(z,t,0)\big) : x,y \neq 0, x,y \in \mathbb{Z},\ z,t \in \mathbb{N}\big\} \subseteq \big(\mathbb{Z} \times \mathbb{N}^2\big)^2$$

It is easy to see that $G_\cdot$ has two connected components

$$X_1 = \{(xy,y,x) :\ x,y \in \mathbb{N}\}$$

and

$$X_2 = \{(x + zy, y, z) : x \in \mathbb{Z}, y, z \in \mathbb{N}\}.$$

Since $\mathcal{M}$ defines no ncdc set, X_1 is definable and we define the multiplication by taking $x \cdot y = z$ where $z \in \mathbb{N}$ is the unique element in $\mathbb{N}$ such that $(z, x, y) \in X_1$. □

Theorem 6.7.6. *If $\widetilde{\mathcal{R}}$ is locally o-minimal then either $\mathcal{M}$ is* cc*-closed or it defines some ncdc set.*

Proof. By Proposition 6.7.5 and Theorem 6.3.3. □

Question 6.7.7. If $\mathcal{M}$ is not locally o-minimal and non-cc-closed does it define some ncdc set?

If X is ncdc then $\tau_X/\sim$ is just $\{\emptyset, X/\sim\}$. On the other hand, if $\mathcal{M}$ is cc-closed, then for every definable set X, $\tau_X/\sim$ is the discrete topology. We could ask questions about the range of behaviour of $\tau_X/\sim$ for X definable. We finish with an example of question of this kind.

Definition 6.7.8. We say that a definable set X is *purely non-definably-connected* (pndc) if it is not definably connected and for every definable set $\emptyset \neq Y \subseteq X$ such that Y is clopen in X, Y is not definably-connected.

Example 6.7.9. Example 6.1.1 is pndc.

Question 6.7.10. Is there a structure that defines a pndc set X such that for every definable clopen subset of X, $\emptyset \neq Y \subseteq X$, $\tau_Y/\sim$ is not Hausdorff?

Remark 6.7.11. Example 6.1.1 is not an example for Question 6.7.10.

Of course, we could take the product of a pndc set and a ncdc set to get an example for Question 6.7.10. Thus we could ask for an example that is not definably homeomorphic to a product of such sets.

6.7.3 Expansions by a dense-codense set

We finish this thesis by saying few words about the cc-closure of expansions of $\mathcal{R}$ (an o-minimal expansion of $\mathbb{R}_{gp}$) by $P \subseteq \mathbb{R}$, a dense-codense set. Some of these structures are considered as tame. For example dense pairs of o-minimal structures (see [14]), expansions of o-minimal structures by an independent set (see [9]) or expansions of o-minimal structures by a multiplicative group with the Mann Property (see [11]). See [20] for a structure theorem that holds for the three of them.

However, their cc-closure could not behave as well. For example:

Lemma 6.7.12. *Let $\mathcal{M} = \langle \mathcal{R}, \mathcal{R}' \rangle$ be a dense pair of o-minimal expansions of $\mathbb{R}_{gp}$. Then, the cc-closure of $\mathcal{M}$ defines the whole projective hierarchy.*

Proof. Let

$$K = \mathbb{R}^2 \setminus \big[[(-1,0),(0,1)],[(0,-1),(1,0)]\big],$$

that is the plane without a diamond in the center. We first define $\mathbb{Z}$ by taking

$$X = \bigcup_{t \in R'^{>0}} [(0,t)(t,0)] \cup [(t,0),(0,-t)] \cup [(0,-t),(-t,0)] \cup [(-t,0),(0,t{+}1)] \cap K \subseteq \mathbb{R}^2.$$

We show that

$$\mathrm{cc}_X((0,1)) \cap \mathbb{R} \times \{0\} = \mathbb{Z} \setminus \{0\}.$$

We just have to show that

$$\mathrm{cc}_X((0,1)) = Y := \bigcup_{x \in \mathbb{N}^{>0}} [(0,x),(x,0)] \cup [(x,0),(0,-x)] \cup [(0,-x),(-x,0)] \cup [(-x,0),(0,x{+}1)].$$

First of all, since Y is path connected, it is contained in $\mathrm{cc}((0,1))$.

Since $\mathcal{R}'$ is a substructure, for every $x \notin \mathcal{R}'$, for every $n \in \mathbb{N}$, $x+n \notin R'$ and thus

$$\bigcup_{n \in \mathbb{N}} [(0,x+n),(x+n,0)] \cup [(x+n,0),(0,-x-n)] \cup [(0,-x-n),(-x-n,0)] \cup$$

$$[(-x-n,0),(0,x+n+1)] \subseteq \mathbb{R}^2 \setminus X.$$

Thus, for $x \in \mathbb{R} \setminus R'$ close to 0:

$$\bigcup_{n \in \mathbb{N}} \big[[(0,1-x+n),(1-x+n,0)],[(0,1+x+n),(1+x+n,0)]\big] \cup$$

$$\big[[(1-x+n,0),(0,x-1-n)],[(1+x+n,0),(0,-1-x-n)]\big] \cup$$

$$\big[[(0,x-1-n),(x-1-n,0)][(0,-x-1-n),(-x-1-n,0)]\big] \cup$$

$$\big[[(x-1-n,0),(0,1-x+1+n)][(-1-x-n,0),(0,1+x+1+n)]\big] \cap X$$

is clopen in X. Thus $Y = \mathrm{cc}_X((0,1))$.

In the case where $\mathcal{R}$ expands $\overline{\mathbb{R}}$, the proof finishes. If $\mathcal{R}$ is semibounded, we define

$$X = \bigcup_{x \in R'^{>0}, y,z \in \mathbb{N}} [(0,x,y,z),(x,0,y,z)] \cup [(x,0,y,z),(0,-x,y,z)] \cup [(0,-x,y,z),(-x,0,y,z)] \cup$$

$$[(-x,0,y,z),(0,x/2,y,z+1)] \cup \bigcup_{x \in \mathbb{N}} [(0,x,x,0),(0,x+1,x+1,0)].$$

Using the same arguments as in the first paragraph, it is easy to see that

$$\mathrm{cc}_X((0,0,0,0)) \cap \{0\} \times \mathbb{R}^3 = \{(0,x/2^z,x,z) : x \in \mathbb{N}, z \in \mathbb{N}\}.$$

Thus we define $f : \mathbb{N}^2 \to \mathbb{Q}$,

$$(x,z) \mapsto x/2^z.$$

Let $g : \mathrm{Im}(f) \to \mathbb{N}^2$,

$$y \mapsto \inf_{<_{lex}} (f^{-1}(y)).$$

Let $* : \mathbb{N}^4 \to \mathbb{N}^2$,

$$\big((x,y),(z,t)\big) \mapsto (xz, y+t).$$

We define the multiplication on $\mathrm{Im}(f)^2$ by taking:

$$(x,y) \mapsto f(g(x) * g(y)).$$

Since $\mathrm{Im}(f)$ is dense in $\mathbb{R}$, by taking the closure of the graph of the multiplication on $\mathrm{Im}(f)^2$, we get the full multiplication on $\mathbb{R}^2$ and since we have defined $\mathbb{Z}$, we have the result. □

Remark 6.7.13. The argument of the first paragraph of the proof of Lemma 6.7.12 is based on the fact that R' is dcl-closed and that $\mathbb{R} \setminus R'$ is closed by adding one and by dividing by two. This is why we would believe that structures of the form $\langle \mathcal{R}, P \rangle$ where $P \subseteq \mathbb{R}$ is a dense-codense independent set is cc-closed.

Of course, by replacing the notion of CC-closure by path-CC-closure (the smallest structure such that every definable set has its path-connected components definable) we would have that the path-CC-closure of $\langle \mathcal{R}, P \rangle$ is PH by replacing R' by $\mathbb{R} \setminus P$ in the proof of Lemma 6.7.12.

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