Zuerst ersch. in : Mathematical Methods in the Applied Sciences ; 34 (2011), 5. - pp. 520-531
DOI : 10.1002/mma.1377

Viscous quantum hydrodynamics and parameter-elliptic systems

Li Chen[a] and Michael Dreher[b*†]

Communicated by S. Jiang

The viscous quantum hydrodynamic model derived for semiconductor simulation is studied in this paper. The principal part of the vQHD system constitutes a parameter-elliptic operator provided that boundary conditions satisfying the Shapiro–Lopatinskii criterion are specified. We classify admissible boundary conditions and show that this principal part generates an analytic semigroup, from which we then obtain the local in time well-posedness. Furthermore, the exponential stability of zero current and large current steady states is proved, without any kind of subsonic condition. The decay rate is given explicitly.

Keywords: semiconductor model; boundary conditions; analytic semigroup; exponential stability; decay rate

1. Introduction

The fast developments of computer sciences and telecommunications require the size of semiconductor devices to reach the nanometer dimension. Then the quantum mechanical effects can no longer be neglected in designing mathematical models. In quantum physics, the motion of an electron ensemble is described by a many-body Schrödinger system, which is hard to investigate both analytically and numerically as the number of particles goes to infinity. One reasonable attempt to deal with this problem is to find macroscopic fluid dynamic models for the motion of the electrons. By using the density matrix for rewriting the Schrödinger system into the Heisenberg equation and applying the Wigner transform, a so-called quantum Boltzmann equation can be formally derived. Then by further introducing various collision operators, several macroscopic models can be obtained, such as the quantum drift-diffusion model by the entropy minimizing method or the quantum hydrodynamic model by the momentum method. For a thorough presentation of derivations of semiconductor models, we refer the reader to [1, 2]. In this paper, we will study a so-called viscous quantum hydrodynamic model derived from the quantum Boltzmann equation with a Fokker–Planck collision operator describing the interaction of the electrons with crystal phonons. More precisely, the system is

$$\begin{cases} \partial_t n - \operatorname{div} J = \nu_1 \triangle n, \\ \partial_t J - \operatorname{div}\left(\dfrac{J\otimes J}{n}\right) - \nabla p(n) + n\nabla V + \dfrac{\varepsilon^2}{2} n \nabla\left(\dfrac{\triangle\sqrt{n}}{\sqrt{n}}\right) = \nu_2 \triangle J - \dfrac{J}{\tau}, \\ \lambda_D^2 \triangle V = n - C(x), \end{cases} \tag{1}$$

for $(t,x)\in(0,T_0)\times\Omega$, where the spatial domain Ω is an open subset of $\mathbb{R}^d$, $d=1,2,3$, with smooth boundary. We prescribe initial values

$$n(0,x)=n_0(x), \quad J(0,x)=J_0(x), \quad x\in\Omega, \tag{2}$$

and certain boundary conditions on the unknown functions (n,J,V). One of the key results of this paper will be a list of admissible boundary conditions that lead to a well-posed initial-boundary value problem. Two other key results will be the analyticity of the semigroup and the exponential stability of a steady state with explicit description of the decay rate.

[a]Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China
[b]Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany
*Correspondence to: Michael Dreher, Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany.
†E-mail: michael.dreher@uni-konstanz.de

The unknown functions are the electron density $n=n(t,x):[0,T_0)\times\Omega\to\mathbb{R}_+$, the density of electrical currents $J=J(t,x):[0,T_0)\times\Omega\to\mathbb{R}^d$, and the electric potential $V=V(t,x):[0,T_0)\times\Omega\to\mathbb{R}$. The smooth function $p=p(n)$ denotes the pressure; typical examples are $p(n)=Tn^{\gamma}$ with $\gamma\geqslant 1$. The scaled physical constants are the electron temperature T (in case $\gamma=1$), the Planck constant ε, the Debye length λ_D, and constants ν_1, ν_2, τ describing the interaction of the electrons with crystal phonons. The parameter τ is also called momentum relaxation time. The known function $C=C(x)$ is the so-called doping profile that describes the density of positively charged background ions.

The macroscopic quantum models have been derived only recently, and only a small number of analytical results are to be found in the literature. Among them, the quantum drift diffusion model, which is basically a nonlinear fourth-order parabolic equation, was studied extensively by a series of works in one-space dimension, on existence of weak solution, long time behavior and semiclassical limits; we only list here [3–7]. The quantum drift diffusion model only contains the conservation of particle density, whereas the quantum hydrodynamic model could provide more information for the particles in the semiconductor simulation, having one more equation for the current density. Concerning the quantum hydrodynamic model without viscous terms, the existence of smooth solutions and their long-time asymptotic behavior for small initial data were investigated in [8, 9]. The viscous quantum hydrodynamic model has been used for numerical simulation on typical quantum semiconductor devices, the resonant tunnel diode (RTD), see [10, 11]. From the numerical point of view, this model seems to be relatively easier to handle than the inviscid model. The viscous model in the higher-dimensional case for the first time was studied by the authors in [12, 13], and results on the time evolution with the Cauchy data or insulating boundary conditions, on the local existence of smooth solutions and exponential decay to the thermal equilibrium state by the entropy dissipation method, and on the inviscid limit have been obtained. We also mention [14], where the exponential decay toward a constant steady state in a one-dimensional setting with a certain boundary condition was proved, and [15, 16] for the global existence of weak solutions to variants of the viscous model.

One of the main obstacles for an analytic study of the quantum hydrodynamic models is the Bohm potential term

$$B(n)=\frac{\Delta\sqrt{n}}{\sqrt{n}},$$

which introduces a third-order perturbation to the Euler Poisson system, making maximum principles and related tools inaccessible. In this paper, we will show how to treat this term as a part of a mixed-order elliptic system in the sense of Douglis and coworkers [17, 18] in a natural way. For greater flexibility in the mathematical analysis, we allow for two different viscosity constants in the differential equations for n and J, and we point out that most of our results will remain valid for $\nu_1=0$, which is a case that seems physically quite intuitive.

The principal part of the $(1+d)\times(1+d)$ matrix differential operator from (1) is

$$A(x,D_x)=\begin{pmatrix}\nu_1\Delta & \mathrm{div}\\ -\frac{\varepsilon^2}{4}\nabla\Delta & \nu_2\Delta I_d\end{pmatrix}, \tag{3}$$

where the viscosity entries lie on the diagonal. Under certain generic conditions on the parameters, this principal part will turn out to be parameter-elliptic of mixed order. Then a discussion on the Shapiro–Lopatinskii condition will give the possible candidates for the boundary conditions of the whole system. We will pursue this matter in Section 3, where the theory of parameter-elliptic mixed-order matrix operators and their associated *a priori* estimates will be first recalled.

In Section 4 we then show that this principal part even generates an analytic semigroup, from which we then directly obtain the local in time well-posedness of (1).

In the last part, Section 5, we will give two stability results in one-space dimension. The first one is on the stability of the zero current steady state with suitable boundary conditions, and we will prove that the solution of the linearized system decays exponentially fast. The second result is on the exponential stability of a large current steady state, where we do not need any subsonic condition. The decay rate turns out to be directly related to the momentum relaxation time.

2. Main results

The linear principal part of the steady-state problem of quantum hydrodynamics is

$$A\begin{pmatrix}n(x)\\ J(x)\end{pmatrix}=\begin{pmatrix}f_0(x)\\ f'(x)\end{pmatrix},\quad x\in\Omega,$$

where A is given in (3). We assume that the parameters satisfy

$$\begin{cases}\nu_1,\nu_2\geqslant 0, & \nu_1+\nu_2>0, \quad :d=1,\\ \nu_1\geqslant 0, & \nu_2>0, \quad :d\geqslant 2.\end{cases} \tag{4}$$

Theorem 2.1
Suppose (4) and $\varepsilon\neq 0$. Then for $d\geqslant 1$, each of the following boundary conditions on n and J satisfies the Shapiro–Lopatinskii criterion:

$$n_{|\partial\Omega}=n_\Gamma,\quad J_{|\partial\Omega}=J_\Gamma,$$
$$\partial_\nu n_{|\partial\Omega}=n_\Gamma,\quad J_{|\partial\Omega}=J_\Gamma,$$
$$n_{|\partial\Omega}=n_\Gamma,\quad (J_\parallel)_{|\partial\Omega}=J_{\parallel,\Gamma},\quad (\partial_\nu J_\perp)_{|\partial\Omega}=J_{\perp,\Gamma}\quad (\nu_1>0),$$

where $J_\parallel$ and $J_\perp$ are the components of J tangential and perpendicular to $\partial\Omega$.
Additionally, for $d=1$, the following collection of boundary conditions is admissible:

$$(n,n_x)_{|\partial\Omega}=(n_{\Gamma,0},n_{\Gamma,1}).$$

There is a sector Σ_ϑ with

$$\Sigma_\vartheta=\{z\in\mathbb{C}\setminus\{0\}:|\arg z|<\vartheta\},\quad \frac{\pi}{2}<\vartheta<\pi, \tag{5}$$

and for each $p\in(1,\infty)$ there is a positive $\lambda_0(p)$ such that: for zero boundary values and $\lambda\in\Sigma_\vartheta$ with $|\lambda|\geqslant\lambda_0(p)$, the solution (n,J) to the problem

$$A\begin{pmatrix}n\\J\end{pmatrix}-\lambda\begin{pmatrix}n\\J\end{pmatrix}=\begin{pmatrix}f_0\\f'\end{pmatrix}\in W_p^1(\Omega)\times(L^p(\Omega))^d,$$

with the above homogeneous boundary conditions, exists in $W_p^3(\Omega)\times(W_p^2(\Omega))^d$ and fulfills the *a priori* estimate

$$\|n\|_{W_p^3(\Omega)}+|\lambda|^{3/2}\|n\|_{L^p(\Omega)}+\|J\|_{W_p^2(\Omega)}+|\lambda|\|J\|_{L^p(\Omega)}\leqslant C(\|f_0\|_{W_p^1(\Omega)}+|\lambda|^{1/2}\|f_0\|_{L^p(\Omega)}+\|f'\|_{L^p(\Omega)}). \tag{6}$$

The proof is given in Section 3.

Remark 2.1
We note that in the one-dimensional case with $\nu_1>0$, boundary conditions on the values $(n,J_x)_{|\partial\Omega}$ give us boundary values of n_{xx}, by the equation for n. This observation seems helpful for difference methods in numerical simulations, since then an additional boundary condition becomes available. Note that taking traces of time derivatives at the boundary is possible in case of analytic semigroups.

This *a priori* estimate then will enable us to verify the operator A as the generator of a semigroup. Our approach is as follows:
Let B_n and B_J be boundary condition operators, with either $B_n=1$ or $B_n=\partial_\nu$, and either $B_J=I_d$, $B_J=I_d\partial_\nu$, or $B_J=(P_\parallel,\partial_\nu P_\perp)$, with $P_\parallel$ and $P_\perp$ being the projectors onto the tangential and normal parts of a vector field.
For $1<p<\infty$, we consider the operator A from (3) with domain

$$D(A)=\{(n,J)\in W_p^3(\Omega)\times(W_p^2(\Omega))^d:\ (B_n n)_{|\partial\Omega}=0,\ (B_J J)_{|\partial\Omega}=0\}.$$

Theorem 2.2
Assume (4) and $\varepsilon\neq 0$. Then the operator A generates an analytic semigroup on the space

$$X=\{(f_0,f')\in W_p^1(\Omega)\times(L^p(\Omega))^d:\ (B_n f_0)_{|\partial\Omega}=0\}.$$

This brings us in a position to study local well-posedness of the system (1) with initial conditions (2) and boundary conditions

$$\begin{cases}(B_n n)(t,x)=n_\Gamma(x),\\ (B_J J)(t,x)=J_\Gamma(x)\qquad (t,x)\in(0,T_0)\times\partial\Omega,\\ V(t,x)=V_\Gamma(x),\end{cases} \tag{7}$$

where B_n and B_J are the above-mentioned boundary operators.

Theorem 2.3
We suppose that the initial data possess the regularity $n_0\in W_p^3(\Omega)$, $J_0\in W_p^2(\Omega)$; and for the boundary data we assume $n_\Gamma\in W_p^{3-\mathrm{ord}B_n-1/p}(\partial\Omega)$, $J_\Gamma\in W_p^{2-\mathrm{ord}B_J-1/p}(\partial\Omega)$ as well as $V_\Gamma\in W_p^{2-1/p}(\partial\Omega)$, where $p>d$. The doping profile C is assumed to be an $L^p(\Omega)$ function. Moreover, suppose

$$\inf_{x\in\Omega} n_0(x)>0$$

and the compatibility conditions

$$(B_n n_0)(x)=n_\Gamma(x),\quad (B_J J_0)(x)=J_\Gamma(x)\quad x\in\partial\Omega.$$

Then the problem (1), (2), (7) has a unique time-local classical solution (n,J,V) with

$$\begin{aligned} n&\in C([0,T_0],W_p^3(\Omega)), & \partial_t n&\in C([0,T_0],W_p^1(\Omega)),\\ J&\in C([0,T_0],W_p^2(\Omega)), & \partial_t J&\in C([0,T_0],L^p(\Omega)),\\ V&\in C([0,T_0],W_p^2(\Omega)), & \partial_t V&\in C([0,T_0],W_p^3(\Omega)),\end{aligned}$$

for some positive T_0.

Remark 2.2
Of course, the existence of solutions to (at least a linearized version of the) system (1) could also be shown by traditional methods like a Galerkin scheme. However, in this way one only obtains *a priori* estimates of the solution in L^2-based Sobolev spaces, and handling the nonlinearities then still requires a big effort, in particular in the higher-dimensional case. The key advantage of the semigroup approach is, of course, that these difficulties disappear after choosing a large p.

Our final two main results are about exponential stability.
First, we consider the viscous model in the interval $\Omega=(0,L)$,

$$\begin{cases} \partial_t n = \nu_0 n_{xx} + J_x,\\ \partial_t J = -\dfrac{\varepsilon^2}{4} n_{xxx} + \nu_0 J_{xx} + \left(\dfrac{1}{n}\left(J^2+\dfrac{\varepsilon^2}{4}n_x^2\right)\right)_x + T n_x - n V_x - \dfrac{1}{\tau}J,\\ \lambda_D^2 V_{xx} = n - C_0, \end{cases} \tag{8}$$

together with insulating boundary conditions

$$\begin{cases} n_x(t,x) = 0,\\ J(t,x) = 0 \qquad (t,x)\in\mathbb{R}_+\times\partial\Omega\\ V_x(t,x) = 0, \end{cases} \tag{9}$$

and the initial data $(n_0,J_0)\in H^3(\Omega)\times H^2(\Omega)$. The doping profile C_0 is a positive constant, and we suppose

$$\inf_{x\in\Omega} n_0(x)>0, \qquad \int_\Omega (n_0(x)-C_0)\,dx=0.$$

To linearize around the steady state $(n,J,V)=(C_0,0,0)$, we set $n(t,x)=C_0+m(t,x)$ and obtain the linearized version

$$\begin{cases} \partial_t m = \nu_0 m_{xx} + J_x,\\ \partial_t J = -\dfrac{\varepsilon^2}{4} m_{xxx} + \nu_0 J_{xx} + T m_x - C_0 V_x - \dfrac{1}{\tau}J,\\ \lambda_D^2 V_{xx} = m. \end{cases} \tag{10}$$

We put $u=\binom{m}{J}$ and write this system as $\partial_t u = Au$. By decomposing into eigenfunctions of the Dirichlet–Laplacian and the Neumann–Laplacian, we observe that A generates a C_0 semigroup (even an analytic one) on the space

$$X=\{(m,J)\in \dot H^1(\Omega)\times L^2(\Omega)\}, \quad \dot H^1(\Omega):=\left\{w\in H^1(\Omega)\colon \int_\Omega w\,dx=0\right\}.$$

The domain of A is

$$D(A)=\{(m,J)\in (H^3(\Omega)\cap \dot H^1(\Omega))\times H^2(\Omega)\colon (m_x,J)_{|\partial\Omega}=0\}.$$

Theorem 2.4
Suppose $u_0=\binom{m_0}{J_0}\in D(A^2)$, $\varepsilon>0$, $\nu_0>0$ and

$$\frac{C_0}{\lambda_D^2}-\frac{1}{4\tau^2}>0. \tag{11}$$

Then the solution (m,J) to (10) decays exponentially to the zero state,

$$\|m(t,\cdot)\|_{H^2(\Omega)}+\|J(t,\cdot)\|_{H^1(\Omega)}\leqslant C\exp\left(-\frac{t}{2\tau}\right)(\|n_0\|_{H^2(\Omega)}+\|J_0\|_{H^1(\Omega)}), \tag{12}$$

and the steady state of the nonlinear problem (8) is asymptotically stable.

Remark 2.3
We note that the condition (11) seems reasonable in practical applications: if we follow the scaling of [10], we get $C_0=1$, $\tau\approx 1$ and $\lambda_D^2\approx 10^{-4}$, and (11) is satisfied with a wide margin.

The next result is about linearized exponential stability in the case of arbitrarily large currents in the bulk material. The system under consideration is again (8), but now for $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, with standard initial data (n_0,J_0) for $t=0$.

Since only derivatives of the electric potential V appear, but not V itself, we introduce the electric field $E(t,x)=V_x(t,x)$ and consider (n,J,E) as our new set of unknown functions. Then a constant stationary state is given by

$$(n,J,E)=(C_0,J_0,E_0),$$

provided that $C_0>0$ is constant and that Ohm's law is satisfied:

$$C_0E_0+\frac{1}{\tau}J_0=0. \tag{13}$$

To linearize around the steady state, we put

$$n(t,x)=C_0+m(t,x),$$
$$J(t,x)=J_0+K(t,x),$$
$$E(t,x)=E_0+D(t,x).$$

Then the linearized version of (8) reads

$$\begin{cases} \partial_t m = \nu_0 m_{xx}+K_x, \\ \partial_t K = -\dfrac{\varepsilon^2}{4}m_{xxx}+\nu_0K_{xx}+\dfrac{2J_0}{C_0}K_x-\dfrac{J_0^2}{C_0^2}m_x+Tm_x-C_0D-mE_0-\dfrac{1}{\tau}K, \\ \lambda_D^2D_x = m. \end{cases} \tag{14}$$

Theorem 2.5
Suppose $\varepsilon\geqslant 0$, $\nu_0\geqslant 0$, (11) and (13). Let (m,K,D) be a solution to (14), with the regularity

$$m\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^4(\mathbb{R})),\quad m_t\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^3(\mathbb{R})),$$
$$K\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^3(\mathbb{R})),\quad K_t\in L^2_{\mathrm{loc}}(\mathbb{R}_+,H^2(\mathbb{R})).$$

Then the following decay estimate holds in the case when $\varepsilon>0$:

$$\|m(t,\cdot)\|_{H^2(\mathbb{R})}+\|K_x(t,\cdot)\|_{L^2(\mathbb{R})}\leqslant C\exp\left(-\frac{t}{2\tau}\right)(\|m(0,\cdot)\|_{H^2(\mathbb{R})}+\|K_x(0,\cdot)\|_{L^2(\mathbb{R})}),$$

and in the case when $\varepsilon=0$, we have

$$\|m(t,\cdot)\|_{H^1(\mathbb{R})}+\|K_x(t,\cdot)\|_{L^2(\mathbb{R})}\leqslant C\exp\left(-\frac{t}{2\tau}\right)(\|m(0,\cdot)\|_{H^1(\mathbb{R})}+\|K_x(0,\cdot)\|_{L^2(\mathbb{R})}).$$

Moreover, the stationary state (C_0,J_0,E_0) to the nonlinear problem (8) is asymptotically stable.

We emphasize that we do not need any kind of subsonic condition: the current density J_0 can be arbitrarily large. The reason of this phenomenon will become clear during the proof, see Remark 5.1.

3. Parameter-elliptic mixed-order systems

We consider an $N\times N$ matrix differential operator $A(x,D_x)$ consisting of entries $a_{jk}(x,D_x)$. Here we have put $D_{x_j}=-\mathrm{i}\partial/\partial x_j$ and $D_x=-\mathrm{i}\nabla_x$. We suppose that there are integers $s_1,\dots,s_N$ and $m_1,\dots,m_N$ such that $s_j+m_j=:m\in\mathbb{N}_+$ is independent of j, with the property that the order of a_{jk} is no more than s_j+m_k for all $j,k=1,\dots,N$. We do not lose generality if we suppose that $m_1\geqslant m_2\geqslant\cdots\geqslant m_N=0$. Additionally, we assume that $a_{jk}\equiv 0$ in case of $s_j+m_k<0$.

We wish to solve the system of partial differential equations

$$(A(x,D_x)-\lambda I_N)u(x)=f(x),\quad x\in\Omega, \tag{15}$$

for all λ in a certain sector $\mathscr{L}$ of the complex plane. This interior problem is complemented with boundary conditions

$$B_j(x,D_x)u(x)=g_j(x),\quad x\in\partial\Omega,\ j=1,\dots,mN/2\in\mathbb{N}, \tag{16}$$

where B_j is a $1\times N$ matrix differential operator with entries b_{jk}, $k=1,\dots,N$, whose order does not exceed r_j+m_k. Here we assume that such numbers $r_1,\dots,r_{mN/2}\in\mathbb{Z}$ exist with $r_j<m$ and that $b_{jk}\equiv 0$ in case of $r_j+m_k<0$.

Through this section, we assume that the coefficients of A and B_j belong to $C^\infty(\overline{\Omega})$ and

$$\Omega\subset\mathbb{R}^n,\quad \partial\Omega\in C^{\max m_j+m-1,1}.$$

Definition 3.1

Let $\mathscr{L}$ be a closed sector in the complex plane with vertex at the origin. Write $a^0_{jk}(x,D_x)$, $b^0_{jk}(x,D_x)$ for the principal parts of a_{jk} and b_{jk}, with $\operatorname{ord} a^0_{jk}=s_j+m_k$ and $\operatorname{ord} b^0_{jk}=r_j+m_k$. Let A^0 and B^0_j be the $N\times N$ and $1\times N$ matrices with entries a^0_{jk} and b^0_{jk}.

The boundary value problem (15), (16) is called *elliptic with parameter in the sector* $\mathscr{L}$ if the following conditions hold:

Interior ellipticity condition: $\det(A^0(x,\xi)-\lambda I_N)\neq 0$ for all $(x,\xi,\lambda)\in\overline{\Omega}\times\mathbb{R}^n\times\mathscr{L}$ with $|\xi|+|\lambda|>0$.

Shapiro–Lopatinskii condition: Let $x^0\in\partial\Omega$ and system (15), (16) be rewritten in local coordinates near x^0 (using a translation and a rotation), in such a way that the boundary at x^0 corresponds to $x_n=0$, and the interior normal vector corresponds to the half-axis with $x_n>0$. Then the boundary value problem on the half-line

$$\begin{cases} A^0(0,\xi',D_{x_n})v(t)-\lambda v(t)=0, & 0<t=x_n<\infty,\\ B^0_j(0,\xi',D_{x_n})v(t)=0, & t=0,\ j=1,\dots,mN/2,\\ \lim\limits_{t\to+\infty} v(t)=0 \end{cases}\tag{17}$$

has only the trivial solution $v\equiv 0$, for all $(x^0,\xi',\lambda)\in\partial\Omega\times\mathbb{R}^{n-1}\times\mathscr{L}$ with $|\xi'|+|\lambda|>0$.

In [19], it has been shown that the interior ellipticity condition implies $mN\in 2\mathbb{N}$.

For a vector-valued function u on Ω of regularity $W^s_p(\Omega)$, where $1<p<\infty$ and $s\in\mathbb{N}_0$, we define a parameter-dependent norm,

$$\|u\|_{s,p,\Omega,\lambda}=\|u\|_{W^s_p(\Omega)}+|\lambda|^{s/m}\|u\|_{L^p(\Omega)},\quad \lambda\in\mathbb{C}\setminus\{0\}.$$

Similarly, for a function v living on the boundary $\partial\Omega$ with regularity $W^{s-1/p}_p(\partial\Omega)$, where $1<p<\infty$ and $s\in\{1,2,\dots,m\}$, we define a norm

$$\|v\|_{s-1/p,p,\partial\Omega,\lambda}=\|v\|_{W^{s-1/p}_p(\partial\Omega)}+|\lambda|^{s-1/p/m}\|v\|_{L^p(\partial\Omega)},\quad \lambda\in\mathbb{C}\setminus\{0\}.$$

We quote a well-posedness result from [20], see also [21, Theorem 6.4.1].

Theorem 3.1

Suppose that the boundary value problem (15), (16) is elliptic with parameter in the sector $\mathscr{L}$. Then there exists a $\lambda_0=\lambda_0(p)$ such that for $\lambda\in\mathscr{L}$ with $|\lambda|\geqslant\lambda_0$, the boundary value problem has a unique solution $(u_1,\dots,u_N)\in\prod_{j=1}^N W_p^{m_j+m}(\Omega)$ for any right-hand side $f=(f_1,\dots,f_N)\in\prod_{j=1}^N W_p^{m_j}(\Omega)$ and all boundary values $g=(g_1,\dots,g_{mN/2})\in\prod_{j=1}^{mN/2} W_p^{m-r_j-1/p}(\partial\Omega)$, and the *a priori* estimate

$$\sum_{j=1}^N \|u_j\|_{m_j+m,p,\Omega,\lambda}\leqslant C\left(\sum_{j=1}^N \|f_j\|_{m_j,p,\Omega,\lambda}+\sum_{j=1}^{mN/2}\|g_j\|_{m-r_j-1/p,p,\partial\Omega,\lambda}\right)$$

holds, where the constant C does not depend upon f,g and λ.

The stationary vQHD system can be written as

$$\begin{cases} \nu_1\Delta n+\operatorname{div}J=0,\\ -\dfrac{\varepsilon^2}{4}\nabla\Delta n+\nu_2\Delta J=-\operatorname{div}\left(\dfrac{J\otimes J+\frac{\varepsilon^2}{4}(\nabla n)\otimes(\nabla n)}{n}\right)-\nabla p(n)+n\nabla V+\dfrac{1}{\tau}J,\\ \Delta V=\dfrac{1}{\lambda_D^2}(n-C). \end{cases}$$

We put $u=\binom{n}{J}$ and define a $(1+d)\times(1+d)$ matrix differential operator A as in (3) with pseudodifferential symbol

$$A(x,\xi)=\begin{pmatrix} -\nu_1|\xi|^2 & i\xi_1 & i\xi_2 & \dots & i\xi_d\\ i\frac{\varepsilon^2}{4}\xi_1|\xi|^2 & -\nu_2|\xi|^2 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ i\frac{\varepsilon^2}{4}\xi_d|\xi|^2 & 0 & 0 & \dots & -\nu_2|\xi|^2 \end{pmatrix}.\tag{18}$$

Now we are in a position to prove Theorem 2.1.

Proof
The operator A has families of orders $(s_1,s_2,\dots,s_{d+1})=(1,2,\dots,2)$ and $(m_1,m_2,\dots,m_{d+1})=(1,0,\dots,0)$, and we have $N=d+1$, $m=2$ as well as $A=A^0$. Then the eigenvalues of A^0 are the solutions λ to

$$(\nu_1|\xi|^2+\lambda)(\nu_2|\xi|^2+\lambda)^d-(\nu_2|\xi|^2+\lambda)^{d-1}\frac{\varepsilon^2}{4}|\xi|^4=0,$$

hence

$$\lambda_{1,\dots,d-1}=-\nu_2|\xi|^2,$$
$$\lambda_{d,d+1}=-\tfrac{1}{2}(\nu_1+\nu_2)|\xi|^2\pm\tfrac{1}{2}\sqrt{(\nu_1-\nu_2)^2-\varepsilon^2}|\xi|^2.$$

Recalling that the parameters satisfy (4), we then can find an angle ϑ (even if $\nu_1=0$) with $\pi/2<\vartheta<\pi$ in such a way that the closure of the sector Σ_ϑ as in (5) contains none of the values $\lambda_1,\dots,\lambda_{d+1}$, provided that $|\xi|>0$. This will be a first condition on the choice of the sector $\mathscr{L}$ as in Definition 3.1.

In order to discuss the Shapiro–Lopatinskii condition, we pick a point $x^0\in\partial\Omega$, and then we rotate and shift the coordinates in such a way that the interior normal direction at x^0 is given by $(0,\dots,0,1)\in\mathbb{R}^d$. We consider the boundary value problem on the half-line

$$\begin{cases}(A^0(\xi',D_{x_d})-\lambda I_{d+1})v(x_d)=0, & 0<x_d<\infty,\\ B_j^0(\xi',D_{x_d})v(x_d)=0, & x_d=0,\ j=1,\dots,d+1,\\ \lim\limits_{x_d\to+\infty}v(x_d)=0, & \end{cases}\tag{19}$$

where $\xi'\in\mathbb{R}^{d-1}$ and $A^0=A^0(\xi',D_{x_d})$ is given as follows:

$$A^0=\begin{pmatrix}-\nu_1(|\xi'|^2+D^2_{x_d}) & i\xi_1 & \dots & iD_{x_d}\\ i\frac{\varepsilon^2}{4}\xi_1(|\xi'|^2+D^2_{x_d}) & -\nu_2(|\xi'|^2+D^2_{x_d}) & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ i\frac{\varepsilon^2}{4}D_{x_d}(|\xi'|^2+D^2_{x_d}) & 0 & \dots & -\nu_2(|\xi'|^2+D^2_{x_d})\end{pmatrix}.$$

Our intention is to show that the choice of boundary condition operators $B_j^0(\xi',D_{x_d})$ listed in Theorem 2.1 implies $v(x_d)\equiv0$ for all $\xi'=(\xi_1,\dots,\xi_{d-1})$ and all $\lambda\in\mathscr{L}$, but $|\xi'|+|\lambda|>0$.

The first line of (19) is a system of ODEs with constant coefficients and of mixed order, and it is clear that any decaying solution $v=v(x_d)$ of this system must decay exponentially for $x_d\to\infty$, as well as all derivatives of v. A thorough description of the structure of the solutions to mixed-order ODE systems can be found in [18]. We have

$$-\nu_1(|\xi'|^2+D^2_{x_d})n+i\xi_1J_1+\dots+i\xi_{d-1}J_{d-1}+iD_{x_d}J_d=\lambda n,$$
$$i\frac{\varepsilon^2}{4}\xi_k(|\xi'|^2+D^2_{x_d})n-\nu_2(|\xi'|^2+D^2_{x_d})J_k=\lambda J_k,\quad k\leqslant d-1,$$
$$i\frac{\varepsilon^2}{4}D_{x_d}(|\xi'|^2+D^2_{x_d})n-\nu_2(|\xi'|^2+D^2_{x_d})J_d=\lambda J_d.$$

Write $\langle\cdot,\cdot\rangle$ for the scalar product on $L^2(\mathbb{R}_+;\mathbb{C})$: $\langle u,v\rangle:=\int_0^\infty u\bar{v}\,dx_d$ and $\|u\|^2:=\langle u,u\rangle$. We take this scalar product of the equations for J_k with J_k and perform appropriate integrations by parts (which produce no boundary terms due to the choice of B_n and B_J):

$$-\frac{\varepsilon^2}{4}\langle(|\xi'|^2+D^2_{x_d})n,i\xi_kJ_k\rangle-\nu_2|\xi'|^2\|J_k\|^2-\nu_2\|D_{x_d}J_k\|^2=\lambda\|J_k\|^2,$$
$$-\frac{\varepsilon^2}{4}\langle(|\xi'|^2+D^2_{x_d})n,iD_{x_d}J_d\rangle-\nu_2|\xi'|^2\|J_d\|^2-\nu_2\|D_{x_d}J_d\|^2=\lambda\|J_d\|^2.$$

Summing up and plugging in the equation for n then give

$$-\frac{\varepsilon^2}{4}\nu_1\|(|\xi'|^2+D^2_{x_d})n\|^2-\nu_2\sum_{k=1}^d(|\xi'|^2\|J_k\|^2+\|D_{x_d}J_k\|^2)=\lambda\sum_{k=1}^d\|J_k\|^2+\bar{\lambda}\frac{\varepsilon^2}{4}(|\xi'|^2\|n\|^2+\|D_{x_d}n\|^2).$$

The left-hand side is a non-positive real number, which enforces $n\equiv0$ and $J\equiv0$ in the case when $\Re\lambda\geqslant0$. There is even a closed sector $\mathscr{L}$, strictly larger than the right complex half-plane, such that $\lambda\in\mathscr{L}$ implies $n\equiv0$ and $J\equiv0$. This can be seen as follows. By scaling arguments, we can assume $|\xi'|=1$, or $\xi'=0$ and $|\lambda|=1$. Keep ξ' fixed. The Shapiro–Lopatinskii criterion is violated exactly for those λ, for which the Lopatinskii determinant vanishes. These values of λ form a discrete set in $\mathbb{C}$, which continuously depends on $\xi' \in S^{d-1}$, the unit sphere. But S^{d-1} is compact, which ensures the existence of a sector $\mathscr{L}$ with the desired properties.

This completes the proof of Theorem 2.1. □

4. Semigroup properties

The purpose of this section is to prove Theorems 2.2 and 2.3.

Proof

First we derive a resolvent estimate for A, improving the *a priori* estimates of (6). Let $\lambda \in \Sigma_\vartheta$ with $|\lambda| \geqslant \lambda_0(p)$ as in Theorem 2.1, and consider the problem

$$(A-\lambda I_{d+1})\begin{pmatrix} n \\ J \end{pmatrix} = \begin{pmatrix} f_0 \\ f' \end{pmatrix} \in X. \tag{20}$$

We define an operator $P=\Delta$ with domain $D(P)=\{v \in W_p^2(\Omega): (B_n v)_{|\partial\Omega}=0\}$, and set $n^*=(P-\lambda)^{-1}f_0$. For this function we have, by classical results,

$$|\lambda|\|n^*\|_{L^p(\Omega)} + \|n^*\|_{W_p^2(\Omega)} \leqslant C\|f_0\|_{L^p(\Omega)}, \quad f_0 \in L^p(\Omega),$$

$$|\lambda|\|n^*\|_{W_p^2(\Omega)} + \|n^*\|_{W_p^4(\Omega)} \leqslant C\|f_0\|_{D(P)}, \quad f_0 \in D(P).$$

Interpolating between these estimates, we then find

$$|\lambda|\|n^*\|_{W_p^1(\Omega)} + |\lambda|^{1/2}\|n^*\|_{W_p^2(\Omega)} + \|n^*\|_{W_p^3(\Omega)} \leqslant C\|f_0\|_{W_p^1(\Omega)}, \quad f_0 \in D(P).$$

By density, this estimate holds for all $f_0 \in W_p^1(\Omega)$ with $(B_n f_0)_{|\partial\Omega}=0$.

Now we put $n=n^*+m$ and apply the inequality (6) to the problem

$$A\begin{pmatrix} m \\ J \end{pmatrix} - \lambda\begin{pmatrix} m \\ J \end{pmatrix} = \begin{pmatrix} f_0 - \nu_1 \Delta n^* + \lambda n^* \\ f' + \frac{\varepsilon^2}{4}\nabla \Delta n^* \end{pmatrix},$$

and deduce that

$$\begin{aligned}\|m\|_{W_p^3(\Omega)} + |\lambda|^{3/2}\|m\|_{L^p(\Omega)} + \|J\|_{W_p^2(\Omega)} + |\lambda|\|J\|_{L^p(\Omega)} &\leqslant C(\|\Delta n^*\|_{W_p^1(\Omega)} + |\lambda|^{1/2}\|\Delta n^*\|_{L^p(\Omega)} + \|f'\|_{L^p(\Omega)} + \|n^*\|_{W_p^3(\Omega)}) \\ &\leqslant C(\|f_0\|_{W_p^1(\Omega)} + \|f'\|_{L^p(\Omega)}).\end{aligned}$$

Summing up and using $|\lambda|\|m\|_{W_p^1(\Omega)} \leqslant C(\|m\|_{W_p^3(\Omega)} + |\lambda|^{3/2}\|m\|_{L^p(\Omega)})$, we then find

$$\|n\|_{W_p^3(\Omega)} + |\lambda|\|n\|_{W_p^1(\Omega)} + \|J\|_{W_p^2(\Omega)} + |\lambda|\|J\|_{L^p(\Omega)} \leqslant C(\|f_0\|_{W_p^1(\Omega)} + \|f'\|_{L^p(\Omega)}), \tag{21}$$

which can be expressed, for $|\lambda| \geqslant \max(1, \lambda_0(p))$, as

$$|\lambda| \cdot \|(A-\lambda I_{d+1})^{-1}\|_{\mathscr{L}(X,X)} + \|(A-\lambda I_{d+1})^{-1}\|_{\mathscr{L}(X,D(A))} \leqslant C.$$

Put $\lambda_1 = \lambda_0(p)+1$. Then we have

$$\sup_{\lambda \in \Sigma_\vartheta} \|\lambda(A-(\lambda+\lambda_1)I_{d+1})^{-1}\|_{\mathscr{L}(X,X)} < \infty.$$

Since $D(A)$ is dense in X, the operator $A-\lambda_1 I_{d+1}$ then is a sectorial operator with spectral angle greater than $\pi/2$. Consequently, the operator $A-\lambda_1 I_{d+1}$ (and then also A) generates an analytic semigroup on X. This completes the proof of Theorem 2.2. □

Remark 4.1

It is clear that this technique allows to prove semigroup properties also for a certain subclass of all parameter-elliptic boundary value problems.

Now we demonstrate Theorem 2.3.

Proof
We write the system as $\partial_t u = Au + F(u)$, $u(0) = u_0$ with $u = \binom{n}{J}$, $u_0 = \binom{n_0}{J_0}$, A as in (3) and

$$F(u) = \begin{pmatrix} 0 \\ \operatorname{div}\left(\frac{1}{n}\left(J \otimes J + \frac{\varepsilon^2}{4}(\nabla n) \otimes (\nabla n)\right)\right) + \nabla p(n) - n\nabla V - \frac{1}{\tau} J \end{pmatrix}.$$

With $u = u^* + u_0$, we then wish to solve

$$u^*(t) = \int_{s=0}^{t} \exp(A(t-s))(Au_0 + F(u_0 + u^*)(s))\,ds, \tag{22}$$

by means of the iteration scheme

$$u_0^*(t) = 0,$$
$$u_{k+1}^*(t) = \int_{s=0}^{t} \exp(A(t-s))(Au_0 + F(u_0 + u_k^*)(s))\,ds.$$

The analytic semigroup $(\exp(At))_{t \geqslant 0}$ on the space X enjoys the estimate

$$\|\exp(At)v\|_{D(A)} \leqslant \frac{C(T_0)}{t}\|v\|_X, \quad 0 < t \leqslant T_0, \tag{23}$$

for all $v \in X$. Define by complex interpolation

$$Y = [D(A), X]_{1/2} = \{(n, J) \in W_p^2(\Omega) \times (W_p^1(\Omega))^d : (B_n n)_{|\partial\Omega} = (B_J J)_{|\partial\Omega} = 0\}.$$

Then the representation formula of u_{k+1}^* gives us, since $W_p^1(\Omega) \subset L^\infty(\Omega)$ because of $p > d$,

$$\|u_{k+1}^*(t)\|_Y \leqslant Ct^{1/2} \sup_{[0,t]} \|Au_0 + F(u_0 + u_k^*)(s)\|_X \leqslant Ct^{1/2}\left(1 + \sup_{[0,t]} \|u_k^*(s)\|_Y^3\right),$$

and the convergence $u_{k+1}^* \to u^*$ in the space $C([0, T_0], Y)$ can be shown by the contraction mapping principle for small T_0. This function $u^* \in C([0, T_0], Y)$ then is a mild solution to the problem

$$\begin{cases} \partial_t u^* = Au^* + Au_0 + F(u_0 + u^*) =: Au^* + f(t), \\ u^*(0) = 0. \end{cases}$$

Now we bring the standard regularity theory into play: since A is the infinitesimal generator of an analytic semigroup on X, and since $f \in L^q((0, T_0), X)$ for any $q \in (1, \infty)$ (we have even $f \in C([0, T_0], X)$), it follows that

$$u^* \in C^\theta([0, T_0], X), \quad \theta = (q-1)/q.$$

From (23) and the representation of u^* we also get $u^* \in L^\infty((0, T_0), [D(A), X]_\gamma)$ for any $\gamma \in (0, 1)$, and interpolating once more we then find

$$u^* \in C^{1/3}([0, T_0], Y).$$

This implies $f \in C^{1/3}([0, T_0], X)$. Then, by standard theory, u^* is a classical solution with regularity

$$Au^*, \partial_t u^* \in C^{1/3}([\gamma, T_0], X) \quad \forall \gamma > 0,$$
$$Au^*, \partial_t u^* \in C([0, T_0], X).$$

The proof of Theorem 2.3 is complete. □

5. Stability

We show Theorem 2.4.

Proof
From $u_0 \in D(A^2)$ we get $u(t) \in D(A^2)$ for all $t > 0$, and in particular

$$m \in L^2_{\text{loc}}(\mathbb{R}_+, H^4(\Omega)), \qquad m_t \in L^2_{\text{loc}}(\mathbb{R}_+, H^3(\Omega)),$$
$$J \in L^2_{\text{loc}}(\mathbb{R}_+, H^3(\Omega)), \qquad J_t \in L^2_{\text{loc}}(\mathbb{R}_+, H^2(\Omega)),$$
$$(m_x, m_{xxx}, J, J_{xx})_{|\partial\Omega} = 0.$$

With $\triangle$ being the Neumann–Laplacian, we define an operator P by the spectral theorem,

$$P=\sqrt{\frac{\varepsilon^2}{4}\triangle^2-T\triangle+\frac{C_0}{\lambda_D^2}-\frac{1}{4\tau^2}},$$

$$D(P)=D(\triangle)=\{w\in H^2(\Omega):w_{x|\partial\Omega}=0\}.$$

By our assumption (11), we have $P=P^*>0$ on $L^2(\Omega)$. Define a function $v=v(t,x)$ by

$$v(t,x)=\left(\frac{1}{2\tau}+iP\right)m(t,x)+J_x(t,x).$$

Then $v\in L^2_{loc}(\mathbb{R}_+,H^2(\Omega))$ with vanishing Neumann boundary values. By calculation, we check that

$$\left(\partial_t-\nu_0\triangle+\frac{1}{2\tau}-iP\right)v=0.$$

Then, we conclude that

$$\partial_t\|v(t,\cdot)\|^2_{L^2(\Omega)}=2\Re\langle\partial_t v,v\rangle_{L^2(\Omega)}=-2\nu_0\|\nabla v(t,\cdot)\|^2_{L^2(\Omega)}-\frac{1}{\tau}\|v(t,\cdot)\|^2_{L^2(\Omega)},$$

and therefore

$$\|v(t,\cdot)\|_{L^2(\Omega)}\leqslant e^{-t/2\tau}\|v(0,\cdot)\|_{L^2(\Omega)}.$$

Since m, Pm and J_x are real-valued, we have $\|Pm\|_{L^2(\Omega)}\leqslant\|v\|_{L^2(\Omega)}$. By the spectral theorem, we also have $\|m(t,\cdot)\|_{H^2(\Omega)}\leqslant C\|Pm(t,\cdot)\|_{L^2(\Omega)}$, which completes the proof of (12).

Going back to the nonlinear system (8), we note that the nonlinear terms can be bounded as follows:

$$\left\|\left(\frac{1}{n}\left(J^2+\frac{\varepsilon^2}{4}n_x^2\right)\right)_x\right\|_{L^2(\Omega)}+\|(n-C_0)V_x\|_{L^2(\Omega)}\leqslant C(\|J\|^2_{H^1(\Omega)}+\|J\|^3_{H^1(\Omega)}+\|m\|^2_{H^2(\Omega)}+\|m\|^3_{H^2(\Omega)}).$$

Then the exponential decay of the linearized problem leads to the asymptotic stability of the steady state $(C_0,0,0)$ to (8). □

And finally, we show Theorem 2.5.

Proof

Taking derivatives of the first two equations of (14) and exploiting (13) gives

$$m_{tt}=2\nu_0 m_{txx}-\left(\nu_0^2+\frac{\varepsilon^2}{4}\right)m_{xxxx}-2\tau E_0 m_{tx}-\frac{1}{\tau}m_t+2\nu_0\tau E_0 m_{xxx}+\left(T-\tau^2E_0^2+\frac{\nu_0}{\tau}\right)m_{xx}-E_0 m_x-\frac{C_0}{\lambda_D^2}m. \tag{24}$$

Next, we put

$$B(\partial_x)=-\nu_0\partial_x^2+\tau E_0\partial_x+\frac{1}{2\tau},$$

$$A(\partial_x)=(B(\partial_x))^2+\frac{\varepsilon^2}{4}\partial_x^4-T\partial_x^2+\left(\frac{C_0}{\lambda_D^2}-\frac{1}{4\tau^2}\right),$$

and rewrite (24) as

$$m_{tt}+2B(\partial_x)m_t+A(\partial_x)m=0. \tag{25}$$

By the assumption (11), we can define a pseudodifferential operator P,

$$P(\partial_x)=\sqrt{\frac{\varepsilon^2}{4}\partial_x^4-T\partial_x^2+\left(\frac{C_0}{\lambda_D^2}-\frac{1}{4\tau^2}\right)},$$

that is the pseudodifferential symbol of P is

$$p=p(\xi)=\sqrt{\frac{\varepsilon^2}{4}\xi^4+T\xi^2+\left(\frac{C_0}{\lambda_D^2}-\frac{1}{4\tau^2}\right)},\quad \xi\in\mathbb{R}.$$

Then P is defined as

$$(Pu)(x)=\mathscr{F}^{-1}_{\xi\to x}\left(p(\xi)(\mathscr{F}_{x\to\xi}u)(\xi)\right)=\int_{\mathbb{R}_\xi}\left(\int_{\mathbb{R}_y}e^{i(x-y)\xi}p(\xi)u(y)\,dy\right)\frac{d\xi}{2\pi},$$

for $u\in\mathscr{S}'$, the Schwartz space of tempered distributions, and $\mathscr{F}$ being the Fourier transform. The operator P is a norm isomorphism between $D(P)$ and $L^2(\mathbb{R})$, where the domain is

$$D(P)=\begin{cases}H^2(\mathbb{R}) & :\varepsilon>0,\\ H^1(\mathbb{R}) & :\varepsilon=0.\end{cases}$$

Moreover, $P=P^*$ and $P>0$ on the Hilbert space $L^2(\mathbb{R})$. Then we can write $A=B^2+P^2$. Fix

$$B_1(\partial_x)=B(\partial_x)+iP(\partial_x),\quad B_2(\partial_x)=B(\partial_x)-iP(\partial_x),$$

and set $m_j(t,x)=(\partial_t+B_{3-j}(\partial_x))m(t,x)$ for $j=1,2$. Then we have

$$(\partial_t+B_j(\partial_x))m_j(t,x)=0,\quad j=1,2\ \ (t,x)\in\mathbb{R}_+\times\mathbb{R},$$

from which we deduce that

$$\begin{aligned}\partial_t\|m_j(t,\cdot)\|^2_{L^2(\mathbb{R})}&=2\Re\langle\partial_t m_j,m_j\rangle_{L^2(\mathbb{R})}\\&=2\Re\langle -B_jm_j,m_j\rangle_{L^2(\mathbb{R})}\\&=-\langle(B+B^*)m_j,m_j\rangle_{L^2(\mathbb{R})}\\&=-2\nu_0\|\partial_x m_j(t,\cdot)\|^2_{L^2(\mathbb{R})}-\frac{1}{\tau}\|m_j(t,\cdot)\|^2_{L^2(\mathbb{R})},\end{aligned}$$

which implies the decay estimate $\|m_j(t,\cdot)\|_{L^2(\mathbb{R})}\leqslant\exp(-t/(2\tau))\|m_j(0,\cdot)\|_{L^2}$. Now we have $m_j=(\partial_t+B)m\pm iPm$, and Pm is real-valued. Therefore, it follows that

$$\begin{aligned}\|m(t,\cdot)\|_{D(P)}&\leqslant C\|Pm(t,\cdot)\|_{L^2(\mathbb{R})}\leqslant Ce^{-t/(2\tau)})\|m_1(0,\cdot)\|_{L^2}\\&\leqslant Ce^{-\frac{t}{2\tau}}(\|K_x(0,\cdot)\|_{L^2(\mathbb{R})}+\|m(0,\cdot)\|_{D(P)}).\end{aligned}$$

From $K_x=(\partial_t+B)m-(\tau E_0\partial_x+1/2\tau)m$, we then also find

$$\|K_x(t,\cdot)\|_{L^2(\mathbb{R})}\leqslant Ce^{-t/(2\tau)}(\|K_x(0,\cdot)\|_{L^2(\mathbb{R})}+\|m(0,\cdot)\|_{D(P)}).$$

These are the desired estimates. □

Remark 5.1
The reason for this stability (independent of the size of J_0) is neither the viscosity nor the quantum effects, since $\nu_0=\varepsilon=0$ are allowed. And also $\tau=\infty$ would still lead to a stable linearized system (although not asymptotically stable). However, the situation changes completely if we flip the sign in the Poisson equation: replacing λ_D^2 by $-\lambda_D^2$ implies that (11) can never hold, and indeed, a spectral analysis yields exponential instability of the linearized problem for small wave numbers $|\xi|$. The physical meaning would be to push away the electrons from their normal positions, which clearly makes that state instable.

Acknowledgements

The first author has been partially supported by NSFC 10571101, and the second author has been supported by DFG (446 CHV 113/170/0-2) and by a grant from the Ministry of Science, Research and the Arts of Baden-Württemberg (Az:21-655.042-5-2/1). Part of the work has been carried out during a stay of the second author at the Tsinghua University, Beijing, and we express our gratitude for the nice hospitality. We also thank Robert Denk for helpful remarks.

References

1. Degond P, Méhats F, Ringhofer C. Quantum energy-transport and drift-diffusion models. *Journal of Statistical Physics* 2005; **118**:625–667.
2. Jüngel A. Quasi-hydrodynamic semiconductor equations. *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser: Basel, 2001.
3. Jüngel A, Pinnau R. A positivity preserving numerical scheme for a nonlinear fourth-order parabolic system. *SIAM Journal on Numerical Analysis* 2001; **39**(2):385–406.
4. Jüngel A, Pinnau R. Convergent semidiscretization of a nonlinear fourth order parabolic system. *Mathematical Modelling and Numerical Analysis* 2003; **37**(2):277–289.

5. Chen L, Ju Q. Existence of weak solution and semiclassical limit for quantum drift-diffusion model. *Zeitschrift für Angewandte Mathematik und Physik* 2007; **58**(1):1–15.
6. Chen X, Chen L, Jian H. The existence and long-time behavior of weak solution to bipolar quantum drift-diffusion model. *Chinese Annals of Mathematics Series B* 2007; **28**(6):651–664.
7. Chen X, Chen L, Jian H. The Dirichlet problem of the quantum drift-diffusion model. *Nonlinear Analysis* 2008; **69**:3084–3092.
8. Jüngel A, Li H-L. Quantum Euler–Poisson systems: global existence and exponential decay. *Quarterly of Applied Mathematics* 2004; **62**:569–600.
9. Zhang B, Jerome J. On a steady-state quantum hydrodynamic model for semiconductors. *Nonlinear Analysis* 1996; **26**:845–856.
10. Jüngel A, Milisić J. Physical and numerical viscosity for quantum hydrodynamics. *Communications in Mathematical Physics* 2007; **5**(2):447–471.
11. Jüngel A, Tang S. Numerical approximation of the viscous quantum hydrodynamic model for semiconductors. *Applied Numerical Mathematics* 2006; **56**(7):899–915.
12. Chen L, Dreher M. The viscous model of quantum hydrodynamics in several dimensions. *Mathematical Models and Methods in Applied Sciences* 2007; **17**(7):1065–1093.
13. Dreher M. The transient equations of viscous quantum hydrodynamics. *Mathematical Models and Methods in Applied Sciences* 2008; **31**:391–414.
14. Gualdani M, Jüngel A, Toscani G. Exponential decay in time of solutions of the viscous quantum hydrodynamic equations. *Applied Mathematics Letters* 2003; **16**(8):1273–1278.
15. Gamba IM, Jüngel A, Vasseur A. Global existence of solutions to one-dimensional viscous quantum hydrodynamic equations. *Journal of Differential Equations* 2009; **247**(11):3117–3135.
16. Jüngel A. Global weak solutions to compressible Navier–Stokes equations for quantum fluids. *Preprint*, 2009.
17. Douglis A, Nirenberg L. Interior estimates for elliptic systems of partial differential equations. *Communications on Pure and Applied Mathematics* 1955; **8**:503–538.
18. Agmon S, Douglis A, Nirenberg L. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Communications on Pure and Applied Mathematics* 1964; **17**:35–92.
19. Agranovich MS, Vishik MI. Elliptic problems with a parameter and parabolic systems of general form. *Russian Mathematical Surveys* 1964; **19**:53–157.
20. Faierman M. Eigenvalue asymptotics for a boundary problem involving an elliptic system. *Mathematische Nachrichten* 2006; **279**(11):1159–1184.
21. Agranovich MS. Elliptic boundary problems. *Partial Differential Equations IX*. Encyclopaedia of Mathematical Sciences, vol. 79. Springer: Berlin, 1997; 1–145.