Elliptic-Parabolic Systems with Applications to Lithium-Ion Battery Models

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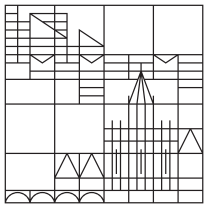

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1. Referent: Prof. Dr. Robert Denk
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For Dagmar
and
my parents
Brigitte & Alfred

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Introduction

The aim of the present thesis is to develop several useful tools for the treatment of elliptic and parabolic partial differential equations which arise in the context of certain lithium-ion battery models, for instance. Typically, such model equations exhibit the structure of a coupled elliptic-parabolic system in a domain $\Omega \subset \mathbb{R}^n$:

$$\left\{ \begin{array}{rcll} -\nabla \cdot (\kappa \nabla \phi + \lambda \nabla c) & = & F & \text{in } (0,T) \times \Omega, \\ \frac{\partial c}{\partial t} - \nabla \cdot (\alpha \nabla \phi + \beta \nabla c) & = & G & \text{in } (0,T) \times \Omega, \end{array} \right. \tag{1}$$

with different kinds of – possibly nonlinear – boundary conditions. In the system (1) the coefficients as well as the functions on the right-hand side may depend on the unknown functions ϕ and c which will lead to a nonlinear coupling of the equations.

Regarding the geometry of the domain Ω, our results cover two application-oriented situations: First, we will consider the case when Ω is a bounded $C^{1,1}$-domain which is divided by M disjoint, closed $C^{1,1}$-surfaces Σ_i, which do not intersect with the boundary, into $M+1$ subdomains $\Omega_0, \ldots, \Omega_M$ with $\partial\Omega_i = \Sigma_i$ for $i = 1, \ldots, M$. In this situation, general boundary conditions on the exterior boundary $\partial\Omega$ can be imposed. Secondly, we will consider a paraxial rectangular domain which is divided by parallel surfaces Σ_i into several subdomains. In this case, the boundary $\partial\Omega$ is the union of $2n$ smooth parts, and on every smooth part, Dirichlet or Neumann boundary conditions are imposed. Whereas the required results on linear equations for the $C^{1,1}$-domain will be obtained from considering the model problems in the half space $\mathbb{R}^n_+$, our starting point for the case of a rectangular domain will be a model problem in a k-corner of the form $\mathbb{K}^n_k := \mathbb{R}^{n-k} \times (0,\infty)^k$ with $1 \leq k \leq n$.

We will focus on the following version of (1) with one parabolic and two elliptic equations:

$$\left\{ \begin{array}{rcll} -\nabla \cdot (\kappa_1(u_3) \nabla u_1) & = & F(u_2 - u_1, u_3) & \text{in } (0,T) \times \Omega, \\ -\nabla \cdot (\kappa_2(u_3) \nabla u_2) & = & -F(u_2 - u_1, u_3) & \text{in } (0,T) \times \Omega, \\ \frac{\partial u_3}{\partial t} - \nabla \cdot (\kappa_3 \nabla u_3) & = & G(u_1, u_2, u_3) & \text{in } (0,T) \times \Omega, \end{array} \right. \tag{2}$$

with linear boundary conditions. Here, the nonlinear coupling is essentially conveyed by functions $F : \mathbb{R} \times (0,\infty) \times \Omega \to \mathbb{R}$ and $G : \mathbb{R}^2 \times (0,\infty) \times \Omega \to \mathbb{R}$ where we assume that the following properties are satisfied:

(i) F and G are sufficiently smooth in each subdomain Ω_i, but may have jumps at the interfaces Σ_i. In addition, F is strictly monotonically increasing with respect to the first variable.

(ii) For large p and $0 < s < \frac{1}{p}$, the nonlinear operator

$$T_F : W^2_p(\Omega) \times W^{2-3/p}_{p,+}(\Omega) \to W^s_p(\Omega), \quad (u,v) \mapsto F(u(\cdot), v(\cdot), \cdot)$$

is well-defined and bounded.

(iii) For $T \in (0,\infty)$, $J := (0,T)$, and $K > 0$ let $\mathbb{X} := C(\overline{J}; W^2_p(\Omega)) \times C(\overline{J}; W^2_p(\Omega)) \times W^{1,2}_{p,K}(J \times \Omega)$. Then the nonlinear operator

$$T_G : \mathbb{X} \to L_p(J; L_p(\Omega)), \quad (u_1, u_2, u_3) \mapsto G(u_1(\cdot), u_2(\cdot), u_3(\cdot), \cdot)$$

is well-defined, continuous, and bounded.

We refer to Assumption 6.2 for the more precise formulation. The examples

$$F(u_1, u_2) := u_2^\alpha \cdot \sinh(u_1) \quad \text{and} \quad G(u_1, u_2, u_3) := u_3^\alpha \cdot \sinh(u_2 - u_1)$$

for some $\alpha \in (-1, 1)$ as well as our applications in Chapter 7 demonstrate that the mapping properties (ii) and (iii) do not necessarily imply polynomial growth conditions on F and G as they are often imposed in the theory of semi- and quasilinear elliptic equations (cf. Chapter 8 in [GT77]).

The main mathematical challenges in the course of our considerations can be summarized as follows:

- We deal with a coupled elliptic-parabolic system where the coupling is effected by nonlinear functions on the right-hand side of the equations as well as by the coefficients, making the coupling quasilinear.
- The nonlinear functions do not satisfy mild growth conditions with respect to the unknown functions in general. On the contrary, they even may involve exponential growth or singular behaviour (e.g., for $u_3 \to 0$) with respect to different variables. As mentioned above, this renders a direct application of standard results on semi- and quasilinear elliptic equations to the present situation impossible.
- The coefficients κ_i occurring in (2) can possibly have discontinuities. To be precise, we shall only assume $C^{1,\tau}$-regularity of κ_1 and κ_2 with respect to c as well as of κ_3 with respect to x in each subdomain Ω_i, but not on the whole domain. A standard assumption in the L_p-theory for elliptic and parameter-elliptic equations is the continuity of the highest-order coefficients (cf. [AV63] or [ADF97]) and therefore these results will not be applicable on the whole of Ω directly.
- A further difficulty due to the lack of smoothness of the boundary occurs when Ω possesses the above-mentioned rectangular geometry and the boundary conditions are of different type, different order, and partially inhomogeneous on the smooth parts of $\partial\Omega$. In view of the L_p-theory for parabolic equations with periodic, homogeneous boundary conditions, such problems have been considered in the article [DN13] as well as the dissertation [Nau12] using a Fourier-series approach and reflection techniques. We will adapt the latter idea to the situation of inhomogeneous boundary conditions in the Chapters 2 and 5.

Nevertheless, our main results state the local in time existence of a strong solution to the system (2) in the L_p-Sobolev spaces

$$(u_1, u_2, u_3) \in C([0,T]; W_p^2(\Omega)) \times C([0,T]; W_p^2(\Omega)) \times W_p^{1,2}((0,T) \times \Omega)$$

under natural assumptions on the data, see Theorem 6.10. The basic idea for obtaining our analytical results for the system (2) is to decouple and linearize the equations and then apply two fixed point principles consecutively. More precisely, we will separate the elliptic part from the parabolic equation by fixing u_3 in an appropriate function space. For an arbitrary, fixed time $t \in [0,T]$ we will first prove the well-posedness of the semilinear elliptic system

$$(3) \qquad \begin{cases} -\nabla \cdot (\kappa_1(u_3(t,x))\nabla u_1(x)) &= \quad F((u_2 - u_1)(x), u_3(t,x), x) \quad \text{in } \Omega, \\ -\nabla \cdot (\kappa_2(u_3(t,x))\nabla u_2(x)) &= -F((u_2 - u_1)(x), u_3(t,x), x) \quad \text{in } \Omega, \end{cases}$$

which will be achieved by linearizing the system (3) and applying the Leray-Schauder principle (see Theorem 4.14). Afterwards, the time dependence and the parabolic equation will be reincluded by using a Schauder fixed point argument in Chapter 6.

The W_p^2-regularity with respect to the space-variable will be obtained by paying particular attention to the situation near the surfaces Σ_i: We will regard this situation as a problem with additional transmission conditions on the interfaces Σ_i.

Generally speaking, transmission conditions on a surface Γ, separating two domains Ω_+ and Ω_- from each other, are a special form of boundary conditions that act on both unknown functions u_+ and u_- which are solutions of the related partial differential equations in Ω_+ and Ω_-, respectively. If these equations are of order $2m$, one also imposes $2m$ transmission conditions of the form

$$C_j^+(x,\mathcal{D})u_+ + C_j^-(x,\mathcal{D})u_- = \sum_{|\gamma|\le m_j^+} c_{j\gamma}^+(x)\mathcal{D}^\gamma u_+ + \sum_{|\gamma|\le m_j^-} c_{j\gamma}^-(x)\mathcal{D}^\gamma u_- = h_j(x) \quad (x\in\Gamma),$$

for $j = 1,\dots,2m$. One important aspect of transmission conditions is that they can be used to include physical properties into a mathematical model. For instance, in equations which describe an electrical potential ϕ, a common assumption is the continuity of ϕ across certain interfaces, e.g. a membrane or an electrode surface, as well as the continuity of a corresponding flux J. A well-known example for this is the Nernst-Planck flux for a species i of charge $z_i \in \mathbb{Z}$, with diffusivity D_i, and of the concentration c_i:

$$J_i = -D_i\nabla c_i - \tfrac{F}{RT} D_i z_i c_i \nabla\phi + c_i \boldsymbol{v}$$

which consists of diffusion, electromigration, and convection in a velocity field $\boldsymbol{v}$. The continuity of ϕ means $\phi_+ = \phi_-$ on Γ and if we regard the continuity of J_i on Γ as a condition on the corresponding derivative of ϕ on Γ, we infer additional transmission conditions for ϕ:

$$\tfrac{F}{RT} z_i \left(D_i c_i^+ \nabla\phi_+ - D_i c_i^- \nabla\phi_-\right) = \left(-D_i\nabla c_i^+ + c_i^+\boldsymbol{v}\right) + \left(D_i\nabla c_i^- - c_i^-\boldsymbol{v}\right).$$

Another example can be formulated for the equations of thermoelasticity: Consider two domains Ω_1 and Ω_2 with C^2- boundaries $\partial\Omega_1 = \Gamma_1\cup\Gamma_2$ and $\partial\Omega_2 = \Gamma_2$, respectively, such that $\Gamma_1\cap\Gamma_2 = \emptyset$. If Ω_1 and Ω_2 consist of different materials, this results in different positive Lamé constants λ_i and μ_i, mass densities m_i, thermal conductivities κ_i, specific heat capacities c_i, and coupling parameters γ_i. On Γ_2 the following transmission conditions are imposed for the displacements u_i and the temperature differences θ_i (each with respect to a given reference state):

$$u_1 = u_2, \quad \theta_1 = \theta_2, \quad \mu_1\nabla u_1 + (\mu_1+\lambda_1)\nabla\cdot u_1 + \gamma_1\theta_2 = \mu_2\nabla u_2 + (\mu_2+\lambda_2)\nabla\cdot u_2 + \gamma_2\theta_1.$$

This represents the continuity of u and θ across Γ_2 as well as the equality of the forces on the interface. For the mathematical treatment of this example, we refer to the article [AMSV].

As mentioned above, a second important aspect of transmission conditions is that despite possible discontinuities of the coefficients in the equations, a certain regularity of the solution can be obtained if transmission conditions of an according order are satisfied. We will pay particular attention to the so-called canonical transmission conditions which read as

$$\partial_n^k u_+ = \partial_n^k u_- \quad \text{on } \mathbb{R}^{n-1} \quad (k = 0,\dots,m-1)$$

for $u_\pm \in W_p^m(\mathbb{R}^n_\pm)$ in the case of the half spaces $\mathbb{R}^n_\pm$. In Chapter 5, we will present general results for parabolic systems with transmission conditions. The results for the corresponding resolvent problems, i.e., for parameter-elliptic equations, will be derived in Chapter 1 for the case of a domain with sufficiently smooth boundary and in Chapter 2 for the case of a rectangular geometry, respectively. In the context of the L_p-theory for a scalar parameter-elliptic equation with transmission conditions in a C^{2m}-domain, solvability and regularity results have been obtained by M. S. Agranovich, R. Denk, and M. Faierman in [ADF97]. Besides the consideration of general transmission conditions, another central aspect of the present thesis is the treatment of parameter-elliptic systems with data in function spaces of higher regularity, namely for data in the Besov space W_p^s with $1 < p < \infty$ and $s < 1/p$.

The L_p-theory for parameter-elliptic boundary value problems of order $2m$ with the general structure

$$\left\{ \begin{array}{rcll} (\lambda - A(x,\mathcal{D}))u & = & f & \text{in } \Omega, \\ B_j(x,\mathcal{D})u & = & g_j & \text{on } \partial\Omega,\ j = 1,\dots,m, \end{array} \right. \tag{4}$$

with the data $f \in L_p(\Omega)$ and $g_j \in W_p^{2m-m_j-1/p}(\partial\Omega)$ was developed quite a long time ago: see [AV63] for the case $p = 2$, [ADF97] for scalar boundary value problem and the case $1 < p < \infty$, and [DFM02] for systems in L_p. Under suitable assumptions on the domain and the coefficients, the L_p-realization of (4) with homogeneous boundary conditions which is given by $D(A_{\mathcal{B},p,0}) := \big\{u \in W_p^{2m}(\Omega) : B_j u = 0,\ j = 1,\dots,m\big\}$, $A_{\mathcal{B},p,0}u := Au$ defines a sectorial operator. Moreover, for a wide class of operators the stronger property of $\mathcal{R}$-sectoriality for $A_{\mathcal{B},p,0}$, which implies the unique solvability of the corresponding time dependent problem, was proved in [DHP03], [KW04], and [DHP07].

However, the situation essentially changes if the data provide higher regularity, say, $f \in W_p^s(\Omega)$ and $g_j \in W_p^{2m+s-m_j-1/p}(\partial\Omega)$ for sufficiently large $s > 0$, and we ask for resolvent estimates in the ground space $W_p^s(\Omega)$. In fact, for $s > \frac{1}{p}$, the underlying operator $A_{\mathcal{B},p,s}$ with the domain $D(A_{\mathcal{B},p,s}) := \big\{u \in W_p^{2m+s}(\Omega) : B_j(x,\mathcal{D})u = 0\big\}$ is no longer sectorial, because the corresponding resolvent $(\lambda - A_{\mathcal{B},p,s})^{-1}$ does not decay in the W_p^s-norm at the rate $|\lambda|^{-1}$, unless additional compatibility conditions of the form $B_j A^k u = 0$ for $2mk + m_j < s - \frac{1}{p}$ are incorporated into $D(A_{\mathcal{B},p,s})$. Similar assertions hold for the case of inhomogeneous boundary conditions. We refer to the thesis [Ne09] for an apposite example and the article [DD11] for general resolvent estimates. In Chapter 1, we will apply the functional analytic concept of Banach scales from [Ama95] and interpolation theory to obtain natural resolvent estimates and the sectoriality for $A_{\mathcal{B},p,s}$ in the ground space $W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$, which contains compatibility conditions as indicated above. For the case of inhomogeneous boundary values, we will derive an explicit estimate in certain parameter-dependent norms for the case $s = 1$ in Theorem 1.39. Eventually, the general result, Corollary 1.44, for the case $s \in (0, \frac{1}{p})$ will be obtained by interpolation.

One motivation to consider the equations (4) in function spaces of higher regularity lies in the compactness of the embedding $B^t_{pq}(\Omega) \overset{c}{\hookrightarrow} B^s_{pq}(\Omega)$ of Besov spaces, for $t > s$, which is satisfied for bounded domains Ω with a Lipschitz boundary, see [Tri02]. This compactness can be exploited to acquire the compactness of the solution operator to the linearized equations which makes a fixed point argument of Schauder type accessible. In this thesis, we will pursue this approach in order to prove the well-posedness of the nonlinear elliptic equations (3) in the weak sense in Chapter 3 and in the strong sense in Chapter 4.

Taking into account the monotonicity properties of the nonlinear function F, one could also try to establish a monotone method for the corresponding elliptic system (3) in order to obtain a classical solution under suitable simplifications. However, such a method is not directly applicable to the system (3) which will be discussed in Chapter 7. To the author's knowledge, a monotone method for elliptic-parabolic systems of the present structure has not been developed yet. Therefore, solvability results for the whole system with a monotone method seem to be out of range.

The importance of elliptic-parabolic systems of partial differential equations becomes evident through their occurrence in various physical and mathematical contexts. In connection with free boundary value problems, in [Esc04] a tumor growth model was treated by J. Escher. In this article, local in time existence is proved for a system which couples an elliptic equation for the internal pressure with a parabolic equation for a nutrient-concentration.
Other elliptic-parabolic systems naturally arise when one considers purely parabolic systems or hyperbolic-parabolic systems and passes to a quasi-stationary problem in one of the equations or in the hyperbolic equation, respectively. A related example for this is the macroscopic chemotaxis model by E. F. Keller and L. A. Segel.

In its original version, this model leads to a fully parabolic system which describes the dynamics of a species density subject to the concentration of a chemoattractant which is produced by the species itself at a certain rate. By rescaling and passing to the limit, an elliptic-parabolic version of this system was derived and considered in [JL92]. Existence, uniqueness, and the qualitative behaviour of a solution to the chemotaxis system have been studied by many authors in various frameworks. We refer to the review article [Hor03] for further references on this topic.

Whereas the previous systems have been studied in great detail and multifarious settings, this has not yet been the case for the elliptic-parabolic equations developed by Newman et. al. (cf. the articles [DFN93], [DFN94], and [DNT94]) which describe the processes during charge or discharge of a lithium-ion battery. For simplicity, the domain Ω is supposed to be an interval ($n = 1$), a paraxial rectangle ($n = 2$), or a cuboid ($n = 3$) which is divided by two parallel $(n-1)$-dimensional surfaces Σ_i into the electrodes Ω_1 and Ω_3, and the separator Ω_2. The situation is illustrated for the case $n = 2$ below:

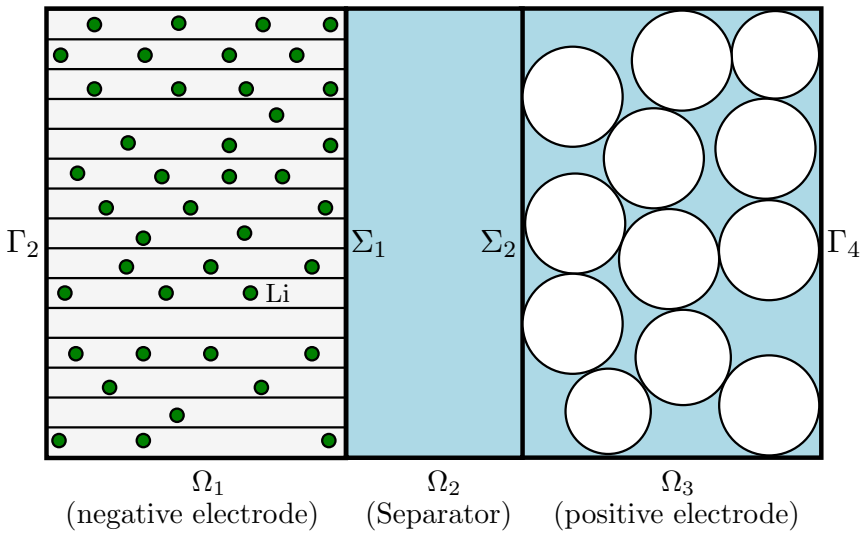

Figure 1: Schematic design of a charged lithium-ion battery.

With the unknown potentials ϕ_1 in the liquid phase, being the whole of Ω, and ϕ_2 in the solid electrodes Ω_1 and Ω_3 as well as the concentration c of electrolyte, the model equations read as

$$
(5)\quad \begin{cases}
-\nabla\cdot\left(\kappa_i(c)\left(\nabla\phi_1 - \frac{2RT}{F}\nabla\ln(c)\right)\right) &=& j(\phi_2-\phi_1,c) & \text{in } J\times\Omega_i,\ (i=1,2,3),\\
-\nabla\cdot(\sigma_i\nabla\phi_2) &=& -j(\phi_2-\phi_1,c) & \text{in } J\times\Omega_i,\ (i=1,3),\\
\partial_t c - \nabla\cdot(D_i\nabla c) &=& j(\phi_2-\phi_1,c) & \text{in } J\times\Omega_i,\ (i=1,2,3),\\
\partial_\nu\phi_1 - \frac{2RT}{F}\frac{1}{c}\partial_\nu c &=& 0 & \text{on } J\times\partial\Omega,\\
\sigma\partial_\nu\phi_2 &=& I & \text{on } J\times\Gamma_k,\ k\in\{n,2n\},\\
\partial_\nu\phi_2 &=& 0 & \text{on } J\times((\partial\Omega_1\backslash\Gamma_n)\cup(\partial\Omega_3\backslash\Gamma_{2n})),\\
\partial_\nu c &=& 0 & \text{on } J\times\partial\Omega,\\
c(0,x) &=& c_0(x) & (x\in\Omega).
\end{cases}
$$

The voltage I is deflected or supplied at two parallel surfaces Γ_n and Γ_{2n} of the boundary, while isolation and non-flux conditions are imposed on the remaining parts of $\partial\Omega$. The nonlinear function j is given by an expression of the form

$$
(6)\quad j(\phi_2-\phi_1,c) = kc^\alpha\left[\exp\left(\tfrac{\alpha F}{RT}(\phi_2-\phi_1-U)\right) - \exp\left(\tfrac{-(1-\alpha)F}{RT}(\phi_2-\phi_1-U)\right)\right] \text{ in } \Omega_1 \text{ and } \Omega_3
$$

and represents the flux of charges across the electrode surfaces Σ_i which is proportional to the reaction rate of oxidation at the anode or the rate of reduction at the cathode, respectively. Such kinetic expressions of Butler-Volmer form are typical for (electro-) chemical reactions.

In the present model, the electrodes are regarded as superposition of liquid and solid phase at every point and the properties are averaged, such that the source term j appears as a right-hand side in each equation instead as a boundary condition. In the separator Ω_2, one formally sets $j := 0$. For a brief discussion and a derivation of the model equations, we refer to Section 7.1 of this thesis.

Note that the conductivities $\kappa_i : [0,\infty) \to [0,\infty)$ in (5) depend continuously on c only in each of the subdomains Ω_i. Even for the case of a smooth concentration c, one cannot expect the continuity of the coefficient $\kappa(c)$ on all of Ω, because of possible jumps in the dependence on c at the electrode surfaces. Similarly, if we define the effective diffusion coefficient $D := D_i$ in Ω_i for $i = 1,2,3$, D is only sufficiently smooth in each subdomain, but not across the electrode surfaces Σ_i.

In order to handle the logarithmic term in the first equation, substituting ϕ_1 with the function $u_1 := \phi_1 - \frac{2RT}{F}\ln(c)$ is helpful. Consequently, the nonlinearity j takes the form

$$N(\phi_2 - u_1, c) := \begin{cases} c^{-\alpha} e^{\frac{\alpha F}{RT}(\phi_2 - u_1 - U)} - c^{2-\alpha} e^{\frac{-(1-\alpha)F}{RT}(\phi_2 - u_1 - U)}, & \text{in } \Omega_1 \cup \Omega_3, \\ 0, & \text{in } \Omega_2. \end{cases} \tag{7}$$

This function becomes singular if the concentration c approaches zero and it has exponential growth with respect to the difference of the potentials. After this modification, the system (5) structurally matches the general equations (2) to a wide extent and adapted variants of the fundamental assumptions (i)-(iii) can be verified for the nonlinearity N. Note that the consideration of the system (5) in two or three dimensions combines all the mathematical difficulties we outlined at the beginning.

A first theoretical treatment for the system (5) with the particular nonlinearity (6) was presented in the article [WXZ06] in the framework of weak solutions. The authors already followed the approach of decoupling the equations and treating the elliptic system and the parabolic equation separately. The essential compactness properties for the fixed point arguments were obtained from the Hölder continuity of weak solution to the linearized equations.

This thesis is structured as follows.

Chapter 1 contains the results on general parameter-elliptic systems in the ground spaces L_p, W_p^1, and W_p^s with $s < \frac{1}{p}$. We follow a standard approach in elliptic theory and first develop the assertions for model problems in the whole and the half space. The sectoriality of the corresponding realization $A_{\mathcal{B},p,s}$ of the model problem in the ground space $W^s_{p;(A,\mathcal{B})}(\Omega)$, where additional compatibility conditions are included, is proved in Theorem 1.33 for general s. Corresponding results for inhomogeneous boundary conditions are obtained in Theorems 1.39 (for the case $s = 1$) and Corollary 1.44 (for $0 < s < \frac{1}{p}$), respectively. Subsequently, the results are generalized for a bounded domain with sufficiently smooth boundary and varying coefficients of suitable smoothness via perturbation theory and a localization procedure. The main result on parameter-elliptic systems in the space $W_p^s(\Omega)$ with $s < \frac{1}{p}$ is Theorem 1.49. Besides the results for standard boundary value problems, transmission problems are briefly addressed. In particular, we give a proof for the absolute ellipticity of the canonical transmission conditions in Theorem 1.9.

Chapter 2 transfers the results from Chapter 1 to the case of a rectangular domain with additional transmission conditions across an interface. In order to accomplish this, we restrict ourselves to differential operators with even principal part and boundary conditions of even or odd order on each surface, to make reflection techniques applicable. The main result on the model problem in a divided k-corner $\mathbb{K}^n_{k,\pm}$ with additional transmission conditions is stated in Theorem 2.12, see also Corollary 2.15 for the case of inhomogeneous transmission conditions. After a localization procedure, we obtain the unique solvability of a general parameter-elliptic boundary value problem in a cuboid Ω with transmission conditions across certain interfaces for the ground space $W_p^s(\Omega)$, see Corollary 2.26 for the precise statement.

Chapter 3 deals with the existence and uniqueness of a weak solution to a nonlinear elliptic system of the form (3) in a Lipschitz domain. Regarding the nonlinear function F merely condition (i), which was introduced at the beginning, is assumed to be satisfied. The problem with homogeneous Neumann boundary conditions and the particular nonlinearity (6) has been the scope of the article [WXZ06]. By virtue of the Leray-Schauder principle in $(L_\infty(\Omega))^2$, we generalize the corresponding well-posedness result in the space $H^1_*(\Omega)\cap L_\infty(\Omega)$ in Theorem 3.18. We also refer to Corollary 3.20 for the case of inhomogeneous Neumann boundary conditions. Beyond that, we prove analogous results in $H^1_\Gamma(\Omega) \cap L_\infty(\Omega)$ for Dirichlet and mixed boundary conditions under some restrictions on the geometry of Ω in Theorem 3.34.

Chapter 4 considers the nonlinear elliptic system (3) in $L_p(\Omega)$ for the case of a rectangular domain. Here, we ask for a solution in the space $W^2_p(\Omega)$ and impose the assumptions (i) and (ii) on the nonlinearity F. In comparison with the previous chapter, our conditions on the coefficients are strengthened, too, but we still permit discontinuities at interfaces which divide the domain Ω. In order to allow for the discontinuous coefficients, we impose additional canonical transmission conditions of order two across these interfaces and then apply a general result from Chapter 2 to a linearized version of the equations. Making use of the L_∞-bounds we proved in the previous chapter, we can again establish a Leray-Schauder fixed point argument to prove our main result (Theorem 4.14): the strong well-posedness for the system in $W^2_p(\Omega)$.

Chapter 5 focuses on general parabolic equations with transmission conditions in the case of two domains Ω_1 and Ω_2 which are separated by a C^{2m}-surface and for the situation of a rectangular domain which is divided by an interface, respectively. First, we briefly introduce the notion of $\mathcal{R}$-boundedness of an operator family and quote two operator-valued variants of Michlin's multiplier theorem (see Theorem 5.10). We further introduce the functional analytic concepts of $\mathcal{R}$-sectoriality and of maximal regularity for an operator as well as a connection between them which is given by a theorem due to L. Weis (Theorem 5.18). By deriving a resolvent representation and estimates of the $\mathcal{R}$-bounds of the resolvent, the property of $\mathcal{R}$-sectoriality is then established for the realization of a parabolic initial boundary value problem with transmission conditions in the space $L_p((0,T);L_p(\Omega_1))\times L_p((0,T);L_p(\Omega_2))$. The key results in this context are Theorem 5.28 for the case of a sufficiently smooth domain and Theorem 5.33 for the case of a rectangular domain.

Chapter 6 connects the results on the nonlinear elliptic system from Chapter 4 and linear parabolic equations with transmission conditions from Chapter 5. In Theorem 6.10, we prove local in time existence of a solution which possesses W^2_p-regularity with respect to the space variables for an elliptic-parabolic system of the structure (2) in a general setting. In order to establish a Schauder fixed point argument, the main tasks are to obtain compactness and continuity properties when plugging functions into the nonlinearity G.

Chapter 7 starts with a short derivation of the lithium-ion battery model equations (5) which were a principal motivation for this thesis. Afterwards, we transfer our general results from Chapter 6 to this concrete situation and obtain the existence of a local in time solution

$$(\phi_1,\phi_2,c) \in C([0,T];W^2_p(\Omega)) \times C([0,T];W^2_p(\Omega)) \times W^{1,2}_{p,+}((0,T)\times\Omega)$$

in Theorem 7.5. In particular, the nonlinearity (6) will serve as an applied example which satisfies the assumptions (i)-(iii) we imposed on F and G in Chapters 4 and 6. We eventually conclude with a brief discussion of a more recent battery model.

Basic notations and definitions as well as some important facts and results on particular topics are summarized in Appendix A.

Konstanz, April 2013 Tim Seger

Chapter 1

Parameter-elliptic systems

In this chapter, we present well-posedness results for parameter-elliptic boundary value problems. We briefly introduce the notion of parameter-ellipticity and summarize some well-known results on a model problem in the space $L_p(\mathbb{R}^n_+)$ which go back to M. S. Agranovich and M. I. Vishik, see [AV63]. For a survey and the generalization to systems we refer to the articles [ADF97] and [DFM02].

In the subsequent sections, we then focus on two new situations: First, we derive solvability results and a priori estimates in the space $W^s_p(\mathbb{R}^n_+)$, to obtain higher regularity of the solution to the boundary value problem if the data permit this. By perturbation theory and a localization procedure, the results are proved for a bounded $C^{m,1}$-domain Ω and for differential operators with varying coefficients. Secondly, besides the results for standard boundary value problems, first results on systems with transmission conditions will be obtained as well.

1.1 Boundary value problems in L_p

We start with the definition of a parameter-elliptic system with a complex parameter $\lambda \in \mathbb{C}$ varying in a sector

$$\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \varphi\} \quad (\varphi \in [0,\pi)).$$

Given $\lambda \in \Sigma_\varphi$, we consider the boundary value problem $(A,\mathcal{B}) := (A, B_1, \dots, B_{mN/2})$

$$\left\{ \begin{array}{rcll} (\lambda - A(x,\mathcal{D}))u & = & f & \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n} B_j(x,\mathcal{D})u & = & g_j & \text{on } \mathbb{R}^{n-1},\ (j = 1,\dots,mN/2), \end{array} \right. \tag{1.1}$$

where $\gamma_{0,n}$ denotes the trace on the boundary $\partial\mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$ (cf. Theorem A.2). The following list of assumptions may also be regarded as a definition of the postulated properties.

Assumption 1.1

(i) ***Differential Operators****: We consider operators of the structure*

$$A(x,\mathcal{D}) := \sum_{|\alpha|\le m} a_\alpha(x)\mathcal{D}^\alpha \quad \textit{and} \quad B_j(x,\mathcal{D}) := \sum_{|\beta|\le m_j} b_{j\beta}(x)\mathcal{D}^\beta \quad (j = 1,\dots,mN/2) \tag{1.2}$$

acting on $\mathbb{R}^n_+$ and $\mathbb{R}^{n-1}$, respectively, with $m_j := \operatorname{ord}(B_j) \le m-1$. For the coefficients of the differential operator A we assume

$$a_\alpha \in \left\{ \begin{array}{ll} BUC(\mathbb{R}^n_+;\mathbb{C}^{N\times N}), \quad a_\alpha(\infty) := \lim\limits_{|x|\to\infty} a_\alpha(x) \textit{ exists}, & (|\alpha| = m), \\ L_\infty(\mathbb{R}^n_+;\mathbb{C}^{N\times N}), & (|\alpha| < m). \end{array} \right.$$

For the coefficients of the boundary operators we assume that $b_{j\beta} \in C^{m+1-m_j}(\mathbb{R}^{n-1};\mathbb{C}^{1\times N})$ for all $j = 1,\dots,mN/2$ and $|\beta| \le m_j$). For $|\beta| = m_j$ it is additionally required that the limits $b_{j\beta}(\infty) := \lim\limits_{\substack{|x'|\to\infty \\ x'\in\mathbb{R}^{n-1}}} b_{j\beta}(x',0) \in \mathbb{C}^{1\times N}$ exist.

The principal parts A_0 of A and B_{j0} of B_j are defined by

$$A_0(x,\xi) := \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \quad \textit{and} \quad B_{j0}(x,\xi) := \sum_{|\beta|=m_j} b_{j\beta}(x)\xi^\beta \quad (j = 1,\dots,mN/2).$$

(ii) ***Parameter-ellipticity****: The differential operator A is parameter-elliptic in the sector $\overline{\Sigma}_\varphi$, i.e.*

$$\det(\lambda - A_0(x,\xi)) \neq 0 \quad (x \in \overline{\mathbb{R}^n_+},\ (\xi,\lambda) \in \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\}).$$

(iii) ***Shapiro-Lopatinskii condition****: Let $x_0 \in \mathbb{R}^{n-1}$ be fixed and assume that (1.1) is rewritten in local coordinates with respect to x_0, i.e., $x_0 \mapsto 0$ and the inner normal vector is given by e_n. Then we assume that for all $(\xi',\lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$ the system of ordinary differential equations*

$$\left\{ \begin{array}{rcll} (\lambda - A_0(x_0,\xi',\mathcal{D}_n))u & = & 0 & (x_n > 0), \\ B_{j0}(x_0,\xi',\mathcal{D}_n)u|_{x_n=0} & = & h_j & (j = 1,\dots,mN/2), \end{array} \right. \tag{1.3}$$

admits a unique, stable, continuous solution for all $h_j \in \mathbb{C}$. Furthermore, we assume that the limit problem, where the highest order coefficients $a_\alpha(x_0)$ and $b_{j\beta}(x_0)$ are replaced by their limits $a_\alpha(\infty)$ and $b_{j\beta}(\infty)$

$$\left\{ \begin{array}{rcll} (\lambda - A_0(\infty,\xi',\mathcal{D}_n))u & = & 0 & (x_n > 0), \\ B_{j0}(\infty,\xi',\mathcal{D}_n)u|_{x_n=0} & = & h_j & (j = 1,\dots,mN/2), \end{array} \right. \tag{1.4}$$

has a unique, stable, continuous solution for all $h_j \in \mathbb{C}$.

(iv) ***Data****: We assume $f \in L_p(\mathbb{R}^n_+;\mathbb{C}^N)$ and $g_j \in W_p^{m-m_j-1/p}(\mathbb{R}^{n-1})$ for $j = 1,\dots,mN/2$.*

Definition 1.2
The boundary value problem $(A,\mathcal{B})$ in the half space is called parameter-elliptic if all assumptions on the operators from 1.1 are satisfied, i.e., the differential operator A is parameter-elliptic and the Shapiro-Lopatinskii condition is satisfied. We refer to $(A,\mathcal{B})$ as a model problem if all differential operators coincide with their principal parts and have constant coefficients.

Remark 1.3

(i) Note that for $n > 1$ the number mN is even if A is a parameter-elliptic differential operator (see [AV63], Section 6). For $n = 1$, we additionally impose the condition of proper ellipticity: For all $x_0 \in \overline{\mathbb{R}^n_+}$ and all $(\lambda,\xi') \in \overline{\Sigma}_\varphi \times \mathbb{R}^{n-1} \setminus \{0\}$, the polynomial $\tau \mapsto \det(\lambda - A(x_0,\xi',\tau))$ possesses the same number of roots with positive and negative imaginary part.

(ii) The principal symbol A_0 is positively homogeneous in ξ of degree m, hence we have

$$\rho^m\lambda - A_0(x,\rho\xi) = \rho^m(\lambda - A_0(x,\xi)) \quad (\rho > 0,\ x \in \mathbb{R}^n_+, (\lambda,\xi) \in \overline{\Sigma}_\varphi \times \mathbb{R}^n \setminus \{0\}).$$

We say that $(x,\xi,\lambda) \mapsto \lambda - A_0(x,\xi)$ is quasi-homogeneous in (λ,ξ) of degree m. Consequently, $(x,\xi,\lambda) \mapsto \det(\lambda - A_0(x,\xi))$ is quasi-homogeneous in (λ,ξ) of degree mN. Since $\{(\lambda,\xi) \in \overline{\Sigma}_\varphi \times \mathbb{R}^n : |\lambda| + |\xi|^m = 1\}$ is a compact set, the assumptions on parameter-ellipticity and the continuity imply

$$C_P := \inf\left\{ |\det(\lambda - A_0(x,\xi))| :\ x \in \mathbb{R}^n_+,\ \lambda \in \overline{\Sigma}_\varphi,\ \xi \in \mathbb{R}^n, |\lambda| + |\xi|^m = 1 \right\} > 0,$$

Hence, scaling with $\rho := |\lambda| + |\xi|^m$ yields an estimate of the form

$$|\det(\lambda - A_0(x,\xi))| \geq C_P(|\lambda| + |\xi|^m)^N. \tag{1.5}$$

The results for the L_p-setting are quite well-known (see [AV63], [DHP03], [ADF97]), so we omit their proofs, but we will partially extend the results to transmission problems. Before we state the main theorem of the L_p-theory, we introduce the important property of sectoriality of an operator.

Definition 1.4
A closed linear operator $A : X \supset D(A) \to X$ in a complex Banach space X is called sectorial if the following conditions are satisfied:

(i) $\overline{D(A)} = X$ *and* $\overline{R(A)} = X$.

(ii) There exists an angle $\varphi \in (0, \pi)$ such that $\rho(A) \supset \Sigma_\varphi$ and the set $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_\varphi\}$ is bounded in $L(X)$.

The supremum over all angles such that condition (ii) is satisfied is called the spectral angle φ_A of A.

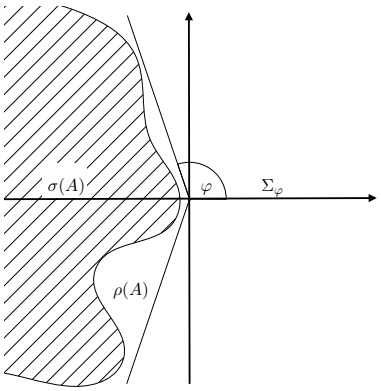

Figure 1.1: The sector Σ_φ, the spectrum and the resolvent set of an operator A.

Remark 1.5
(i) Recall that every sectorial operator with a spectral angle $\varphi_A > \frac{\pi}{2}$ generates a bounded analytic semigroup on the Banach space X, see [KW04], Theorem 1.1.

(ii) If the Banach space X is reflexive, condition (ii) implies that $\overline{D(A)} = X$ and in this situation $\overline{R(A)} = X$ is equivalent to $N(A) = \{0\}$. For further properties of sectorial operators, we refer to the textbook [Haa06].

At this point, it is convenient to introduce certain parameter-dependent norms for a priori estimates of a solution.

Definition 1.6
Let $\Omega \subset \mathbb{R}^n$ be a domain. For $\lambda \in \mathbb{C}$ we set

$$\begin{aligned} |\!|\!| u |\!|\!|_{k,p,\Omega} &:= \|u\|_{k,p,\Omega} + |\lambda|^{\frac{k}{m}} \|u\|_{0,p,\Omega} && (1 \le k \le m,\ u \in W_p^k(\Omega)), \\ |\!|\!| u |\!|\!|_{k-1/p,p,\partial\Omega} &:= \|u\|_{k-1/p,p,\partial\Omega} + |\lambda|^{\frac{k-1/p}{m}} \|u\|_{0,p,\partial\Omega} && (1 \le k \le m,\ u \in W_p^{k-1/p}(\partial\Omega)). \end{aligned}$$

The basic theorem of the L_p-theory for the problem (1.1) in the half space reads as follows:

Theorem 1.7
Let the model problem $(A, \mathcal{B})$ be parameter-elliptic in the sense of Definition 1.2. Then for all $f \in L_p(\mathbb{R}^n_+; \mathbb{C}^N)$, $g_j \in W_p^{m-m_j-1/p}(\mathbb{R}^{n-1})$, and $\lambda \in \overline{\Sigma}_\varphi$, the problem (1.1) with constant coefficients admits a unique solution $u \in W_p^m(\mathbb{R}^n_+; \mathbb{C}^N)$. In addition, for $|\lambda| \ge \lambda_0 > 0$ the following a priori estimate is satisfied:

$$|\!|\!| u |\!|\!|_{m,p,\mathbb{R}^n_+} \le C \left(\|f\|_{0,p,\mathbb{R}^n_+} + \sum_{j=1}^{mN/2} |\!|\!| g_j |\!|\!|_{m-m_j-1/p,p,\mathbb{R}^{n-1}} \right), \tag{1.6}$$

with C independent of f, g_j, and λ.

In particular, the L_p-realization $A_{\mathcal{B},p}$ of $(A,\mathcal{B})$, defined by

$$\begin{aligned} D(A_{\mathcal{B},p}) &:= \{u \in W_p^m(\mathbb{R}^n_+) :\ \gamma_{0,n} B_j(\mathcal{D})u = 0,\ j = 1,\dots,mN/2\}, \\ A_{\mathcal{B},p} &: L_p(\mathbb{R}^n_+) \supset D(A_{\mathcal{B},p}) \to L_p(\mathbb{R}^n_+),\ u \mapsto A(\mathcal{D})u, \end{aligned}$$

is a sectorial operator of angle φ.

Proof. A proof can be found in [ADF97], Theorem 2.1 for the scalar case and in [DFM02], Theorem 2.4 for the matrix-valued case. □

1.1.1 Transmission problems in the half spaces $\mathbb{R}^n_\pm$

Theorem 1.7 has a straightforward consequence for transmission problems in the half spaces $\mathbb{R}^n_\pm$. Here, we give a brief introduction to transmission problems and present an application which will be generalized in a subsequent chapter. For simplicity, we consider a scalar boundary value problem in the half spaces with parameter-elliptic operators $\widetilde{A_1}$ acting in $\mathbb{R}^n_-$, and A_2 acting in $\mathbb{R}^n_+$ of the same order $2m$:

$$\begin{cases} (\lambda - \widetilde{A}_1(\mathcal{D}))\tilde{u}_1 &= \tilde{f}_1 \quad \text{in } \mathbb{R}^n_-, \\ (\lambda - A_2(\mathcal{D}))u_2 &= f_2 \quad \text{in } \mathbb{R}^n_+. \end{cases} \tag{1.7}$$

Generally speaking, transmission conditions are boundary conditions imposed on the surface $\Gamma_n := \{x \in \mathbb{R}^n : x_n = 0\} \cong \mathbb{R}^{n-1}$ which include both solutions $\tilde{u}_1$ and u_2:

$$\gamma_{0,n} B_j^{(1)}(x,\mathcal{D})\tilde{u}_1 + \gamma_{0,n} B_j^{(2)}(x,\mathcal{D})u_2 = g_j \quad \text{on } \Gamma_n \quad (j = 0,\dots,2m-1),$$

with differential operators $B_j^{(1)}(x,\mathcal{D})$ of order m_{1j} acting on $\mathbb{R}^n_-$ and $B_j^{(2)}(x,\mathcal{D})$ of order m_{2j} acting on $\mathbb{R}^n_+$ and given boundary values g_j in an appropriate trace space. Note that one usually imposes $2m$ of such conditions, whereas for a standard scalar boundary value problem of order $2m$, only m boundary conditions are imposed. For our applications in the present chapter, we restrict ourselves to the important case of canonical transmission conditions which for $\tilde{u}_1 \in W_p^{2m}(\mathbb{R}^n_-)$ and $u_2 \in W_p^{2m}(\mathbb{R}^n_+)$ read as

$$\gamma_{l,n} u_2 - \gamma_{l,n}\tilde{u}_1 := \gamma_{0,n}\mathcal{D}_n^l u_2 - \gamma_{0,n}\mathcal{D}_n^l \tilde{u}_1 = g_l \quad \text{on } \Gamma_n, \quad (l = 0,\dots,2m-1), \tag{1.8}$$

with $g_j \in W_p^{2m-l-1/p}(\Gamma_n)$. If $g_j = 0$ in (1.8) we have the following 'patching' theorem.

Lemma 1.8
Let $1 < p < \infty, k \in \mathbb{N}$, $u_+ \in W_p^k(\mathbb{R}^n_+)$, and $u_- \in W_p^k(\mathbb{R}^n_-)$. Then

$$u(x) := \begin{cases} u_+(x), & (x \in \mathbb{R}^n_+), \\ u_-(x), & (x \in \mathbb{R}^n_-), \end{cases}$$

is an element of $W_p^k(\mathbb{R}^n)$ if and only if

$$\gamma_0 \partial_n^l u_+ = \gamma_0 \partial_n^l u_- = (-1)^l \gamma_0 \partial_{-n} u_- \quad (l = 0,\dots,k-1).$$

The same assertions hold for the case of the Besov spaces $B_{pq}^{k+s}(\mathbb{R}^n)$ for $(1 \le p,q < \infty)$ and the Bessel potential spaces $H_p^{k+s}(\mathbb{R}^n)$ with $0 \le s < 1/p$.

Proof. See [Ama00], Theorem 4.7.3. □

The next theorem states the well-posedness of a model transmission problem with these conditions. We also refer to Section 7 of [ADF97] for a comparable result.

Theorem 1.9
Let $\widetilde{A}_1$ and A_2 be differential operators of order $2m$ in the half spaces $\mathbb{R}^n_-$ and $\mathbb{R}^n_+$, respectively. Assume that both operators have constant coefficients and are parameter-elliptic in a sector $\overline{\Sigma}_\varphi$. Then for all $(\tilde{f}_1, f_2) \in L_p(\mathbb{R}^n_-) \times L_p(\mathbb{R}^n_+)$ and $g_l \in W_p^{2m-l-1/p}(\mathbb{R}^{n-1})$, the transmission problem

$$\left\{ \begin{array}{rcll} (\lambda - \widetilde{A}_1(\mathcal{D}))\tilde{u}_1 & = & \tilde{f}_1 & \text{in } \mathbb{R}^n_-, \\ (\lambda - A_2(\mathcal{D}))u_2 & = & f_2 & \text{in } \mathbb{R}^n_+, \\ \gamma_{l,n}u_2 - \gamma_{l,n}\tilde{u}_1 & = & g_l & \text{on } \mathbb{R}^{n-1},\ (l = 0, \dots, 2m-1) \end{array} \right. \tag{1.9}$$

admits a unique solution $(\tilde{u}_1, u_2) \in W_p^{2m}(\mathbb{R}^n_-) \times W_p^{2m}(\mathbb{R}^n_+)$. In addition, there exists a constant $\lambda_0 > 0$ such that for all $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| > \lambda_0$ the a priori estimate

$$\|\tilde{u}_1\|_{2m,p,\mathbb{R}^n_-} + \|u_2\|_{2m,p,\mathbb{R}^n_+} \leq C \left(\|\tilde{f}_1\|_{0,p,\mathbb{R}^n_-} + \|f_2\|_{0,p,\mathbb{R}^n_+} + \sum_{l=0}^{2m-1} \|g_l\|_{2m-l-1/p,p,\mathbb{R}^{n-1}} \right)$$

holds. Furthermore, the function $u(x) := \begin{cases} \tilde{u}_1(x), & (x \in \mathbb{R}^n_-), \\ u_2(x), & (x \in \mathbb{R}^n_+) \end{cases}$ belongs to $W_p^{2m}(\mathbb{R}^n)$ if and only if $g_l = 0$ for $l = 0, \dots, 2m-1$.

Proof. By means of the transformation $x_n \mapsto -x_n$, we rewrite the problem (1.9) as a diagonal system in the half space $\mathbb{R}^n_+$ by setting $A_1(\mathcal{D}) := \widetilde{A}_1(\mathcal{D}', -\mathcal{D}_n)$, $f_1(x) := \tilde{f}_1(x', -x_n) \in L_p(\mathbb{R}^n_+)$, and $u_1(x) := \tilde{u}_1(x', -x_n)$ for $x_n > 0$. Note that A_1 is parameter-elliptic in $\overline{\Sigma}_\varphi$ by the properties of the principal symbol under change of coordinates (see, e.g., [Kum81], Chapter 2, Theorem 6.3). The transformed problem reads as

$$\left\{ \begin{array}{rcll} (\lambda - A_1(\mathcal{D}))u_1 & = & f_1 & \text{in } \mathbb{R}^n_+, \\ (\lambda - A_2(\mathcal{D}))u_2 & = & f_2 & \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n}\mathcal{D}_n^l u_2 + (-1)^{l+1}\gamma_{0,n}\mathcal{D}_n^l u_1 & = & g_l & \text{on } \mathbb{R}^{n-1},\ (l = 0, \dots, 2m-1), \end{array} \right. \tag{1.10}$$

and is equivalent to the system (1.1) if we set $A(\mathcal{D}) = \operatorname{diag}(A_1(\mathcal{D}), A_2(\mathcal{D}))$, $u := (u_1, u_2)^T$, $f := (f_1, f_2)^T$, and $B_l(\mathcal{D})u := (1, (-1)^{l+1})u$ for $l = 0, \dots, 2m-1$. A is parameter-elliptic because the parameter-ellipticity of A_1 and A_2 implies that for all $(\lambda, \xi) \in \overline{\Sigma}_\varphi \times \mathbb{R}^n \setminus \{0\}$:

$$|\det(\lambda - A(\xi))| = |(\lambda - A_1(\xi))(\lambda - A_2(\xi))| \geq C_1 C_2 (|\lambda| + |\xi|^{2m})^2.$$

It remains to verify the Shapiro-Lopatinskii condition for the boundary operators B_l, i.e., we have to show that for all $(\lambda, \xi') \in \overline{\Sigma}_\varphi \times \mathbb{R}^{n-1} \setminus \{0\}$, the system of ordinary differential equations of order $2m$

$$\left\{ \begin{array}{rcll} (\lambda - A_1(\xi', \mathcal{D}_n))v_1 & = & 0 & (x_n > 0), \\ (\lambda - A_2(\xi', \mathcal{D}_n))v_2 & = & 0 & (x_n > 0), \\ [\mathcal{D}_n^l v_2 + (-1)^{l+1}\mathcal{D}_n^l v_1](\xi', x_n)\big|_{x_n=0} & = & g_l & (l = 0, \dots, 2m-1) \end{array} \right.$$

admits a unique, stable solution for any choice of $g_l \in \mathbb{C}$, or, equivalently,

$$\left\{ \begin{array}{rcll} (\lambda - \widetilde{A}_1(\xi', \mathcal{D}_n))\tilde{v}_1 & = & 0 & (x_n < 0), \\ (\lambda - A_2(\xi', \mathcal{D}_n))v_2 & = & 0 & (x_n > 0), \\ \mathcal{D}_n^l v_2(\xi', x_n)\big|_{x_n=0} - \mathcal{D}_n^l \tilde{v}_1(\xi', x_n)\big|_{x_n=0} & = & g_l & (l = 0, \dots, 2m-1) \end{array} \right.$$

admits a unique, stable solution for arbitrary $g_l \in \mathbb{C}$. Suppose $\mathcal{M}_1 := \{\widetilde{w}_1, \dots, \widetilde{w}_m\}$ is a fundamental system of stable solutions for the first equation, i.e., $(\lambda - \widetilde{A}_1(\xi', \mathcal{D}_n))\widetilde{w}_i = 0$ and there holds $\lim_{x_n \to -\infty} \widetilde{w}_i(x_n) = 0$. Similarly, let $\mathcal{M}_2 := \{w_1, \dots, w_m\}$ be a fundamental system of stable solutions for the second equation, i.e. $(\lambda - A_2(\xi', \mathcal{D}_n))w_i = 0$ and $\lim_{x_n \to +\infty} w_i(x_n) = 0$. We define complex polynomials of degree $2m$ by

$$\widetilde{a}_1(\tau, \xi', \lambda) := (\lambda - \widetilde{A}_1(\xi', \tau)) \quad \text{and} \quad a_2(\tau, \xi', \lambda) := (\lambda - A_2(\xi', \tau)).$$

By the assumption on the proper ellipticity of $\widetilde{A}_1$ and A_2, each of these polynomials has exactly m roots in the upper and in the lower complex half plane $\mathbb{C}_\pm$, respectively. Let $\widetilde{\tau}_k^{(1)} := \widetilde{\tau}_k^{(1)}(\xi', \lambda)$ $(k = 1, \dots, m)$ denote the m roots of $\widetilde{a}_1(\xi', \tau, \lambda)$ in $\mathbb{C}_-$ and $\tau_k^{(2)} := \tau_k^{(2)}(\xi', \lambda)$, $(k = 1, \dots, m)$ denote the m roots of $a_2(\xi', \tau, \lambda)$ in $\mathbb{C}_+$. Then we define two complex polynomials of degree m by

$$\widetilde{a}_{1-}(\tau) := \widetilde{a}_{1-}(\tau, \xi', \lambda) := \prod_{k=1}^{m} (\tau - \widetilde{\tau}_k^{(1)}) \quad \text{and} \quad a_{2+}(\tau) := a_{2+}(\tau, \xi', \lambda) := \prod_{k=1}^{m} (\tau - \tau_k^{(2)})$$

and consider the following differential equation on the whole of $\mathbb{R}$:

$$a_{12}(\mathcal{D}_n, \xi', \lambda) z(x_n) := a_{1-}(\mathcal{D}_n, \xi', \lambda) a_{2+}(\mathcal{D}_n, \xi', \lambda) z(x_n) = 0 \quad \text{in } \mathbb{R}. \tag{1.11}$$

Note that by definition $a_{12}(\tau, \xi', \lambda)$ has exactly m roots in the upper and in the lower complex half plane. Define $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2$. Clearly, each element of $\mathcal{M}$ solves (1.11) and each of the subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ is linearly independent. If we assume $\mathcal{M}$ to be linearly dependent, then we have

$$\widetilde{w}_i(x_n) = \sum_{k=1}^{m} \alpha_k w_k(x_n) \quad (x_n \in \mathbb{R}),$$

for some index $i \in \{1, \dots, m\}$ and coefficients $\alpha_k \in \mathbb{C}$ that cannot vanish simultaneously. By the definition of $\mathcal{M}_1$ it follows $\lim_{x_n \to -\infty} \widetilde{w}_i(x_n) = 0$ and on the other hand there holds $\lim_{x_n \to +\infty} \widetilde{w}_i(x_n) = \lim_{x_n \to +\infty} \sum_{k=1}^{m} \alpha_k w_k(x_n) = 0$ by the definition of $\mathcal{M}_2$. Consequently, $\widetilde{w}_i(x_n)$ is a solution of (1.11) which is bounded on the entire real line and hence, $\widetilde{w}_i(x_n) = 0$ which contradicts the linear independence of $\mathcal{M}_2$. Therefore, $\mathcal{M}$ is a fundamental system of (1.11) and for all $x_n \in \mathbb{R}$ the Wronskian of this system does not vanish:

$$\det \begin{pmatrix} \widetilde{w}_1(x_n) & \cdots & w_m(x_n) \\ \mathcal{D}_n \widetilde{w}_1(x_n) & \cdots & \mathcal{D}_n w_m(x_n) \\ \vdots & & \vdots \\ \mathcal{D}_n^{2m-1} \widetilde{w}_1(x_n) & \cdots & \mathcal{D}_n^{2m-1} w_m(x_n) \end{pmatrix} \neq 0 \quad (x_n \in \mathbb{R}).$$

Now we insert the approach $\widetilde{v}_1 = \sum_{k=1}^{m} \alpha_k \widetilde{w}_k$ and $v_2 = \sum_{k=1}^{m} \beta_k w_k$ with unknown coefficients $\alpha_k, \beta_k \in \mathbb{C}$ into the transmission conditions to obtain the following system of linear equations:

$$\begin{pmatrix} \widetilde{w}_1(x_n) & \cdots & w_m(x_n) \\ \mathcal{D}_n \widetilde{w}_1(x_n) & \cdots & \mathcal{D}_n w_m(x_n) \\ \vdots & & \vdots \\ \mathcal{D}_n^{2m-1} \widetilde{w}_1(x_n) & \cdots & \mathcal{D}_n^{2m-1} w_m(x_n) \end{pmatrix} \Bigg|_{x_n = 0} \begin{pmatrix} -\alpha_1 \\ \vdots \\ -\alpha_m \\ \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} g_0 \\ \vdots \\ g_{2m-1} \end{pmatrix}$$

as well as its unique solvability. We are now able to apply Theorem 1.7 to the system (1.10) and this yields a unique solution $(u_1, u_2) \in W_p^{2m}(\mathbb{R}^n_+)^2$ that satisfies the desired a priori estimate for parameters $\lambda \in \overline{\Sigma}_\varphi$ with a sufficiently large modulus. The claim of the theorem follows, since the transformation $x_n \mapsto -x_n$ does not affect the norms of the functions. The last statement is a direct consequence of Lemma 1.8. □

Some concluding comments on the preceding results are in order.

Remark 1.10

(i) An application of Theorem 1.7 in its general version permits the treatment of lower order terms as well as varying coefficients of the differential operators $\tilde{A}_1$ and A_2. The theorem easily extends to this situation, since the main part of the proof, i.e., the verification of the

Shapiro-Lopatinskii condition for the canonical transmission conditions, is then performed for fixed $(x_0, \xi', \lambda) \in \Gamma_n \times \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi$ *with* $(\xi, \lambda) \neq (0,0)$*. However, in this introductory chapter we restrict ourselves to the constant coefficient result because we will carry out the generalizations to lower order terms and varying coefficients via perturbation theory and a localization procedure for a more complex problem in a subsequent chapter.*

(ii) The last proof contains an important property of the scalar canonical transmission conditions: in combination with general properly (parameter-) elliptic operators they satisfy the Shapiro-Lopatinskii condition. This property of a set of boundary conditions is sometimes called absolute ellipticity.

(iii) General transmission conditions of the form (1.7) *can be treated similarly. In this case, a substitute for the Shapiro-Lopatinskii condition has to be imposed such that the corresponding system in* $\mathbb{R}^n_+$ *then satisfies the classical Shapiro-Lopatinskii condition from Assumption 1.1. We postpone details to Chapter 5 where we prove these results in view of parabolic transmission problems.*

(iv) The operators $\tilde{A}_1$ *and* A_2 *do not have to be parameter-elliptic with the same angle* φ*. If* $\tilde{A}_1$ *is parameter-elliptic in the sector* $\overline{\Sigma}_{\varphi_1}$ *and* A_2 *in a sector* $\overline{\Sigma}_{\varphi_2}$*, the assertion still holds for all angles* $0 < \varphi \leq \min(\varphi_1, \varphi_2)$*.*

(v) A localization procedure makes the results accessible to the situation of a bounded C^{2m}*-domain which is divided by a closed* C^{2m}*-surface into two subdomains* Ω_1 *and* Ω_2*. See [ADF97], Sections 1.7 and 7 for further information.*

1.2 Boundary value problems in W_p^s

In contrast to the preceding section, we now change our ground space from L_p to W_p^1 in order to obtain analogous results in this space of higher regularity. In analogy to the well-known L_p-case, we would like to prove sectoriality of the underlying operator $A_{\mathcal{B}}$, but it turns out that we have to handle this new situation more carefully: It was demonstrated in [Ne09] and [DD11] that the necessary resolvent estimate

$$\left\|\lambda(\lambda - A_{\mathcal{B}})^{-1}\right\|_{L(W_p^1(\mathbb{R}^n_+))} \leq C$$

fails, unless additional compatibility conditions are included into the ground space. More precisely, the boundary values of the right-hand side f have to vanish up to a certain order and this is the reason why we merely consider an appropriate subspace of $W_p^1(\mathbb{R}^n_+)$, which includes these homogeneous boundary conditions as soon as boundary conditions of order 0 appear.

1.2.1 Parameter-elliptic operators in $W_p^k(\mathbb{R}^n)$

We first summarize some properties of a parameter-elliptic operator in the whole space where the ground space is a Sobolev space of higher order. Therefore, we briefly recall the results from [KS12] for the case of a model problem where even $\mathcal{R}$-sectoriality of the W_p^k-realization of a parameter-elliptic operator is proved. Keeping in mind that $\mathcal{R}$-boundedness implies uniform boundedness and $\mathcal{R}$-sectoriality of an operator is a stronger property than sectoriality, the relevance of the presented results will be clear without the precise definitions of $\mathcal{R}$-boundedness and $\mathcal{R}$-sectoriality, which will be given in Chapter 5.

Let

$$A(\mathcal{D}) := \sum_{|\alpha|=m} a_\alpha \mathcal{D}^\alpha$$

be a differential operator of order m without lower order terms and with constant coefficients $a_\alpha \in \mathbb{C}^{N\times N}$.

We assume that A is parameter-elliptic in a complex sector $\overline{\Sigma}_\varphi$ with constant C_P, i.e.,

$$|\det(\lambda - A(\xi))| \geq C_P(|\xi|^m + |\lambda|)^N \quad ((\xi,\lambda) \in \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\}).$$

Definition 1.11
Let the differential operator $A(\mathcal{D})$ be given as described above and let $k \in \mathbb{N}_0$. Then the W_p^k-realization $A_{k,p}$ of A is defined by $D(A_{k,p}) := W_p^{m+k}(\mathbb{R}^n;\mathbb{C}^N)$ and

$$A_{k,p} : W_p^k(\mathbb{R}^n;\mathbb{C}^N) \supset D(A_{k,p}) \to W_p^k(\mathbb{R}^n;\mathbb{C}^N), \quad u \mapsto A_{k,p}u := A(\mathcal{D})u.$$

First, we state some results which will turn out to be useful when proving certain mapping properties of Fourier multipliers between Sobolev spaces of higher order.

Remark 1.12 (Order reductions)
For $k \in \mathbb{N}$, consider the symbols

$$m_{\pm k} : \mathbb{R}^n \to \mathbb{C}^{N\times N}, \quad \xi \mapsto \mathrm{id}_{\mathbb{C}^N}(1+|\xi|^2)^{\pm\frac{k}{2}}.$$

Then the operators

$$\begin{aligned} \Lambda_{-k} : L_p(\mathbb{R}^n;\mathbb{C}^N) &\to W_p^k(\mathbb{R}^n;\mathbb{C}^N), \quad f \mapsto T_{m_{-k}}f, \\ \Lambda_k : W_p^k(\mathbb{R}^n;\mathbb{C}^N) &\to L_p(\mathbb{R}^n;\mathbb{C}^N), \quad f \mapsto T_{m_{+k}}f \end{aligned}$$

are well-defined and $(\Lambda_{-k})^{-1} = \Lambda_k$. In addition, for all $l \in \mathbb{N}$:

$$\Lambda_{-k}|_{W_p^l(\mathbb{R}^n;\mathbb{C}^N)} \in L(W_p^l(\mathbb{R}^n;\mathbb{C}^N), W_p^{l+k}(\mathbb{R}^n;\mathbb{C}^N)),$$

and for all $l \geq k$:

$$\Lambda_k|_{W_p^l(\mathbb{R}^n;\mathbb{C}^N)} \in L(W_p^l(\mathbb{R}^n;\mathbb{C}^N), W_p^{l-k}(\mathbb{R}^n;\mathbb{C}^N)).$$

Lemma 1.13
Let $A(\mathcal{D})$ be a differential operator which is parameter-elliptic in a sector $\overline{\Sigma}_\varphi$ with constant C_P and coefficients $a_\alpha \in \mathbb{C}^{N\times N}$. For $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m$, consider the matrix-valued symbols

$$m_\beta : \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\} \to \mathbb{C}^{N\times N}, \quad (\xi,\lambda) \mapsto \xi^\beta \lambda^{\frac{m-|\beta|}{m}}(\lambda - A(\xi))^{-1}.$$

Then the following assertions hold:

(i) For $\beta \in \mathbb{N}_0^n$ with $|\beta| \leq m$ and $\lambda \in \overline{\Sigma}_\varphi \setminus \{0\}$, $m_\beta(\cdot,\lambda)$ defines a Fourier multiplier. The corresponding linear operator is denoted by $T_{m_\beta(\cdot,\lambda)} \in L(L_p(\mathbb{R}^n;\mathbb{C}^N))$.

(ii) The family of operators

$$\mathcal{T}_\beta := \left\{ T_{m_\beta(\cdot,\lambda)},\ \lambda \in \overline{\Sigma}_\varphi \setminus \{0\} \right\} \subset L(L_p(\mathbb{R}^n;\mathbb{C}^N))$$

is $\mathcal{R}$-bounded and the $\mathcal{R}$-bound only depends on m, n, p, N, and C_P.

(iii) For every $k \in \mathbb{N}$, the family of operators

$$\mathcal{T}_{k,\beta} := \left\{ T_{m_\beta(\cdot,\lambda)}\Big|_{W_p^k(\mathbb{R}^n;\mathbb{C}^N)},\ \lambda \in \overline{\Sigma}_\varphi \setminus \{0\} \right\} \subset L(W_p^k(\mathbb{R}^n;\mathbb{C}^N))$$

is $\mathcal{R}$-bounded as well and the $\mathcal{R}$-bound only depends on k, m, n, p, N, and C_P.

Proof. See [KS12], Lemma 5.11. □

Theorem 1.14 ($\mathcal{R}$-sectoriality of parameter-elliptic operators in $W_p^k(\mathbb{R}^n)$)
Let $k \in \mathbb{N}_0$ and $A(\mathcal{D})$ be a differential operator which is parameter-elliptic in a sector $\overline{\Sigma}_\varphi$. Then the following assertions for the W_p^k-realization of A hold true:

(i) $\rho(A_{k,p}) \supset \overline{\Sigma}_\varphi \setminus \{0\}$.

(ii) For $\beta \in \mathbb{N}_0^n$ with $|\beta| \le m$, the set

$$\left\{ |\lambda|^{\frac{m-|\beta|}{m}} \mathcal{D}^\beta (\lambda - A_{k,p})^{-1} : \lambda \in \overline{\Sigma}_\varphi \setminus \{0\} \right\} \subset L(W_p^k(\mathbb{R}^n; \mathbb{C}^N))$$

is $\mathcal{R}$-bounded and the $\mathcal{R}$-bound only depends on k, m, n, p, N, and the constant C_P of parameter-ellipticity.

In particular, $A_{k,p}$ is $\mathcal{R}$-sectorial with angle $\varphi_{\mathcal{R}} \ge \varphi$.

(iii) For all $\lambda \in \overline{\Sigma}_\varphi \setminus \{0\}$ there holds $\lambda^{\frac{m-|\beta|}{m}} \mathcal{D}^\beta (\lambda - A_{k,p})^{-1} = T_{m_\beta(\cdot,\lambda)}$ in $L(L_p(\mathbb{R}^n; \mathbb{C}^N))$.

Proof. See [KS12], Proposition 5.12. □

In addition, it was proved in [KS12] that this result remains true under interpolation which yields $\mathcal{R}$-sectoriality for the realizations of A in the whole Besov- and the Bessel potential scale, see [KS12], Corollary 5.31.

Corollary 1.15
Let $s \ge 0$ and $A(\mathcal{D})$ be a parameter-elliptic differential operator of order m without lower order terms and constant coefficients. Then for all $f \in W_p^s(\mathbb{R}^n)$, the equation

$$(\lambda - A(\mathcal{D}))u = f \quad \text{in } \mathbb{R}^n$$

has a unique solution $u \in W_p^{m+s}(\mathbb{R}^n)$ and there exists a $\lambda_0 > 0$ such that for all $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| \ge \lambda_0$ the solution satisfies the following a priori estimate with a constant C which only depends on k, m, n, p, N, and C_P:

$$|\!|\!| u |\!|\!|_{m+s,p,\mathbb{R}^n} \le C \|f\|_{s,p,\mathbb{R}^n}.$$

Remark 1.16
The last Theorem shows that in the situation of a parameter-elliptic problem in $\mathbb{R}^n$, the regularity of the right-hand side is inherited by the solution. The statements presented above, also hold in full strength for operators including lower order terms and with varying coefficients (under appropriate smoothness assumptions). For the proofs we refer to the article [KS12].

1.2.2 Smoothness assumptions and continuity in higher order Sobolev spaces

The goal of the next subsections is to obtain additional regularity for the solution of a boundary value problem, provided by the corresponding data. In a first step, we derive the continuity of the differential operators in certain parameter-dependent norms which will be convenient for the corresponding a priori estimates. In view of later applications, this will be discussed in full generality for varying coefficients here.

Let $\Omega \subset \mathbb{R}^n$ be an open set with boundary $\partial\Omega$ and $s \in [0, \infty)$. Let a function $u \in W_p^{m+s}(\Omega)$, a differential operator $A(x, \mathcal{D})$, and boundary operators $B_j(x, \mathcal{D})$ of the structure (1.2) be given. Furthermore, let $\lambda \in \overline{\Sigma}_\varphi \setminus \{0\}$ and set $f := (\lambda - A(x, \mathcal{D}))u$ as well as $g_j := \gamma_{\partial\Omega} B_j(x, \mathcal{D})u$. We consider the equations

$$\left\{ \begin{array}{rcll} (\lambda - A(x, \mathcal{D}))u & = & f & \text{in } \Omega, \\ \gamma_{\partial\Omega} B_j(x, \mathcal{D}) & = & g_j & \text{on } \partial\Omega, \ (j = 1, \dots, mN/2), \end{array} \right. \tag{1.12}$$

and ask for, preferably weak, sufficient conditions on the boundary and the coefficients of the operators, such that the natural regularities $f \in W_p^s(\Omega)$ and $g_j \in W_p^{m+s-m_j-1/p}(\partial\Omega)$ are fulfilled. Conditions which will turn out to be adequate for our applications are contained in the following theorem.

Theorem 1.17 (Multiplication properties in Besov spaces)

(i) Let $\Omega \subset \mathbb{R}^n$ be an open set. Let $0 < s_1 \le s_2$ and $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$. Assume that the following constraints are satisfied:

$$\text{(1.13a)} \qquad \frac{1}{p} \quad \le \quad \frac{1}{p_1} + \frac{1}{p_2},$$

$$\text{(1.13b)} \qquad \frac{n}{p} - s_1 \quad > \quad \begin{cases} (\frac{n}{p_1} - s_1)_+ + (\frac{n}{p_2} - s_2)_+, & \text{if } \max\limits_{i=1,2}(\frac{n}{p_i} - s_i) > 0, \\ \max\limits_{i=1,2}(\frac{n}{p_i} - s_i), & \text{otherwise,} \end{cases}$$

$$\text{(1.13c)} \qquad s_1 + s_2 \quad > \quad n\left(\frac{1}{p_1} + \frac{1}{p_2} - 1\right),$$

$$\text{(1.13d)} \quad \text{if } s_1 < s_2, \quad \text{then} \quad q \ge q_1, \text{ and if } s_1 = s_2, \text{ then } q \ge \max(q_1, q_2).$$

Then we have the following continuous pointwise multiplication: For all $u_1 \in B^{s_1}_{p_1,q_1}(\Omega)$ and $u_2 \in B^{s_2}_{p_2,q_2}(\Omega)$ it holds $u_1 \cdot u_2 \in B^{s_1}_{p,q}(\Omega)$ and there is a constant C (independent of u_1 and u_2) such that

$$\text{(1.14)} \qquad \|u_1 \cdot u_2\|_{B^{s_1}_{p,q}(\Omega)} \le C \|u_1\|_{B^{s_1}_{p_1,q_1}(\Omega)} \|u_2\|_{B^{s_2}_{p_2,q_2}(\Omega)} \quad (u_i \in B^{s_i}_{p_i,q_i}(\Omega),\ i = 1, 2).$$

(ii) Suppose that (1.13a) *and* (1.13c) *are satisfied and equality holds in* (1.13b). *Assume in addition that $q \ge \max(q_1, q_2)$ if $s_1 - \frac{n}{p} = s_2 - \frac{n}{p_2}$. Then* (1.14) *remains valid if*

$$\left\{ i : s_i = \frac{n}{p_i} \text{ and } q_i > 1 \right\} \cup \left\{ i : s_i \ge \frac{n}{p_i} \text{ and } q_i > \frac{n}{\frac{n}{p_i} - s_i} \right\} = \emptyset.$$

(iii) In the case $p_2 = \infty$ and $p = p_1 > 1$, assume $0 < s_1 \le s_2$, and (1.13d). *Then the multiplication property* (1.14) *holds as well.*

Proof. The results are stated in [RS96], Theorems 1 and 2 of Chapter 4.4.4 and Theorem 4.7.1. □

Up to some extent, the presented multiplier results include those of H. Triebel (see [Tri83], Theorem 3.3.2, p.198) where a sufficient pointwise multiplier condition in $B^s_{p,q}(\Omega)$ is fulfilled for functions which belong to the Hölder-Zygmund space $\mathscr{C}^t(\Omega) = B^t_{\infty,\infty}(\Omega)$ with $t > s$. Taking into account that $\mathscr{C}^t(\Omega)$ coincides with the classical Hölder space $C^{k,\alpha}(\Omega)$ for $t = k + \alpha \notin \mathbb{N}$, the above assertions match with those of P. Grisvard in [Gri85] and H. Amann in [Ama91]. We also briefly outline these results here.

Theorem 1.18
Let Ω be a Lipschitz domain.

(i) Let $k \in \mathbb{N}$, $\alpha \in [0, 1]$, and $s \in \mathbb{R}$. Then for $k + \alpha \ge |s|$ if $s \in \mathbb{N}$ and for $k + \alpha > |s|$ if $s \in \mathbb{R} \setminus \mathbb{N}$, the functions $\varphi \in BUC^{k,\alpha}(\overline{\Omega})$ are pointwise multipliers in $W^s_p(\Omega)$. In particular, $\varphi u \in W^s_p(\Omega)$ for $\varphi \in BUC^{k,\alpha}(\overline{\Omega})$ and $u \in W^s_p(\Omega)$ and there is a constant K, independent of u and φ, such that

$$\|\varphi u\|_{s,p,\Omega} \le K \|\varphi\|_{C^{k,\alpha}(\overline{\Omega})} \|u\|_{s,p,\Omega}.$$

Under the same preliminaries, similar results hold in the spaces $W^s_p(\overline{\Omega})$ and $W^s_{p,0}(\Omega)$.

(ii) *Let $s_1, s_2 \geq s$ such that one of the following conditions is fulfilled:*

$$s_1 + s_2 - s \geq n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0 \text{ and } s_i - s > n\left(\frac{1}{p_i} - \frac{1}{p}\right) \quad (i = 1, 2), \tag{1.15a}$$

$$s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right) \geq 0 \text{ and } s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right) \quad (i = 1, 2). \tag{1.15b}$$

Then the pointwise multiplication $W_{p_1}^{s_1}(\Omega) \times W_{p_2}^{s_2}(\Omega) \to W_p^s(\Omega)$ is continuous.

Proof. See [Ama91], Section 4. □

For the consideration of (1.12) in $W_p^1(\Omega)$, we make the following assumptions on the occurring coefficients:

Assumption 1.19
Let Ω be a bounded domain. Then we impose the following assumptions on the variable coefficients in equation (1.12):

(i) *For $|\alpha| = m$, we assume $a_\alpha \in C^1(\overline{\Omega}; \mathbb{C}^{N\times N})$.*

(ii) *For $|\alpha| = k < m$, we assume $a_\alpha \in L_\infty(\Omega; \mathbb{C}^{N\times N}) + L_{r_k}(\Omega; \mathbb{C}^{N\times N})$ with $r_k \geq p$ and $m - k > \frac{n}{r_k}$ such that for all $j = 1, \dots n$ we have $\partial_j a_\alpha \in L_\infty(\Omega; \mathbb{C}^{N\times N}) + L_{r_k'}(\Omega; \mathbb{C}^{N\times N})$ with $r_k' \geq p$ and $m - k + 1 > \frac{n}{r_k'}$.*

(iii) *For $j = 1, \dots, mN/2$ and $|\beta| \leq m_j$, we assume $b_{j\beta} \in C^{m-m_j+1,\alpha_j}(\overline{\Omega}; \mathbb{C}^{1\times N})$ and $\alpha_j \in (0, 1]$.*

Keeping in mind that we will consider (1.12) the spaces $W_p^s(\Omega)$ with $s \in (0, 1)$, too, we state the corresponding smoothness assumptions for this situation.

Assumption 1.20
Let Ω be a bounded domain and let $s \in (0, 1)$. Then we impose the following assumptions on the variable coefficients in equation (1.12) considered as an equation in $W_p^s(\Omega)$:

(i) *For $|\alpha| \leq m$, we assume $a_\alpha \in C^t(\overline{\Omega}; \mathbb{C}^{N\times N})$ with $t > s$.*

(ii) *For $j = 1, \dots, mN/2$ and $|\beta| \leq m_j$, we assume $b_{j\beta} \in C^{k_j,\alpha_j}(\overline{\Omega}; \mathbb{C}^{1\times N})$ with $k_j \in \mathbb{N}$ and $\alpha_j \in [0, 1]$ such that $k_j + \alpha_j > m + s - m_j$.*

Besides the smoothness assumptions, we will impose the following ellipticity conditions:

Definition 1.21
Let Ω be a bounded $C^{m,1}$-domain. Then we say that a boundary value problem $(A, \mathcal{B})$ of the form (1.12) in Ω is parameter-elliptic if the following conditions on the differential operators $(A, B_1, \dots, B_{mN/2})$ are satisfied:

(i) *The smoothness assumptions 1.19 or 1.20 for the coefficients are satisfied if (1.12) is considered in the ground space $W_p^1(\Omega)$ or in $W_p^s(\Omega)$, respectively.*

(ii) *The differential operator A is parameter-elliptic in a closed sector $\overline{\Sigma}_\varphi$, i.e., for all $x \in \overline{\Omega}$, $(\xi, \lambda) \in \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\}$ it holds $\det(\lambda - A_0(x, \xi)) \neq 0$.*

(iii) *The Shapiro-Lopatinskii condition is satisfied, i.e., for all $x_0 \in \partial\Omega$, $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$, the ordinary differential equation on the half line, rewritten in local coordinates with respect to x_0 (cf. Assumption 1.1)*

$$\begin{cases} (\lambda - A_0(x_0, \xi', \mathcal{D}_n))u &= 0 \quad (x_n > 0), \\ B_{j0}(x_0, \xi', \mathcal{D}_n)u|_{x_n=0} &= h_j \quad (j = 1, \dots, mN/2), \\ u(x_n) &\to 0 \quad (x_n \to \infty), \end{cases}$$

has a unique solution for all $h_j \in \mathbb{C}$.

As a first step, we will merely consider problems in the half space $\mathbb{R}^n_+$. For this situation, our assumptions have to be modified as follows.

Definition 1.22
A boundary value problem $(A, \mathcal{B})$ of the form (1.12) in the half space $\mathbb{R}^n_+$ will be called parameter-elliptic in the sector $\overline{\Sigma}_\varphi$ if the following conditions are satisfied:

(i) The smoothness assumptions 1.19 or 1.20 for the coefficients are satisfied if (1.12) is considered in the ground space $W_p^1(\mathbb{R}^n_+)$ or in $W_p^s(\mathbb{R}^n_+)$, respectively, where C^k is replaced by BUC^k. In addition, we assume that the limits of the highest order coefficients

$$a_\alpha(\infty) := \lim_{|x|\to\infty} a_\alpha(x), \quad (|\alpha| = m) \quad \text{and} \quad b_{j\beta}(\infty) := \lim_{|x|\to\infty} b_{j\beta}(x), \quad (|\beta| = m_j)$$

exist in $\mathbb{C}^{N\times N}$ and $\mathbb{C}^{1\times N}$, respectively.

(ii) The condition $\det(\lambda - A_0(x,\xi)) \neq 0$ of parameter-ellipticity of the operator A is satisfied for all $x \in \mathbb{R}^n_+ \cup \{\infty\}$ and $(\xi,\lambda) \in \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\}$.

(iii) The Shapiro-Lopatinskii condition is satisfied for all $x_0 \in \mathbb{R}^{n-1} \cup \{\infty\}$.

In the preceding definition, the assumptions at infinity have to be understood in the sense that the operators $A_0(\infty, \mathcal{D}) := \sum_{|\alpha|=2m} a_\alpha(\infty)\mathcal{D}^\alpha$ and $B_{j0}(\infty,\mathcal{D}) := \sum_{|\beta|=m_j} b_{j\beta}(\infty)\mathcal{D}^\beta$ with the limits of the coefficients satisfy the respective conditions in (ii) and (iii), too.

Remark 1.23

(i) In view of Theorem 1.17, the assumptions on the lower-order coefficients can somewhat be weakened. Since under multiplication in Besov spaces less smoothness is preserved, one can not reduce the requirement of $t \geq s$ and the additional smoothness of the lower order derivatives $\mathcal{D}^\alpha u$ ($|\alpha| < m$) cannot be exploited, unlike in the L_p-case where Hölder's and Gagliardo-Nirenberg's estimates permit a reduction. However, a sufficient condition on a_α which ensures that the term $a_\alpha \mathcal{D}^\alpha u$ belongs to $W_p^s(\Omega)$ for $u \in W_p^{m+s}(\Omega)$ and $|\alpha| < m$ is: $a_\alpha \in B^s_{p_1,p}(\Omega)$ with p_1 sufficiently large. Note that most of the conditions in Theorem 1.17 become redundant in this case; essentially only (1.13b) has to be met.

(ii) In the localization procedure, we will see that for the accessibility of results on bounded domains it would be sufficient to consider half space problems where the highest order coefficients are of the form $a_\alpha(x) = a^0_\alpha + a^1_\alpha(x)$ where $a^1_\alpha \in C^t(\mathbb{R}^n_+; \mathbb{C}^{N\times N})$ has a compact support and $a^0_\alpha \in \mathbb{C}^{N\times N}$ is fixed. Likewise, it is sufficient to consider highest order coefficients $b_{j\beta}(x) = b^0_{j\beta} + b^1_{j\beta}(x)$ with fixed $b^0_{j\beta} \in \mathbb{C}^{1\times N}$ and compactly supported $b^1_{j\beta} \in C^{m-m_j,\alpha_j}(\mathbb{R}^n_+; \mathbb{C}^{1\times N})$ for the boundary operators.

(iii) Under the assumptions stated in Definition 1.22, a corresponding result for parameter-elliptic problems with varying coefficients in the half space as well as for unbounded domains with compact $C^{m,1}$-boundary can be established, cf. [DHP03], Sections 7.4 and 8.2.

Using the smoothness assumptions stated above, we can prove the continuity of the corresponding operators in the half space and for domains with sufficiently smooth boundary.

Lemma 1.24
Let Ω denote either the half space or a bounded $C^{m,1}$-domain. Let $s \in (0,1]$ and let a boundary value problem $(A, B_1, \ldots, B_{mN/2})$ in Ω of the structure (1.12) be given. Let the coefficients satisfy Assumption 1.19 for the case $s = 1$ or Assumption 1.20 for $0 < s < 1$. In the case $\Omega = \mathbb{R}^n_+$, assume additionally that the Assumption (i) from Definition 1.22 is satisfied, too.

Then the operator

$$\begin{aligned}(A,\mathcal{B}):\quad W_p^{m+s}(\Omega) &\to W_p^s(\Omega)\times\prod_{j=1}^{mN/2} W_p^{m+s-m_j-1/p}(\partial\Omega)\\ u &\mapsto (A(\cdot,\mathcal{D})u,\gamma_{\partial\Omega}B_1(\cdot,\mathcal{D})u,\dots,\gamma_{\partial\Omega}B_{mN/2}(\cdot,\mathcal{D})u)\end{aligned}$$

is continuous.

Proof. For $s=1$, note that Gagliardo-Nirenberg's estimate (Theorem A.5) for given $p\in(1,\infty)$, $\tilde m\in\mathbb{N}$, and $p<p_1<\frac{pn}{n-\tilde m p}$ yields

$$\|u\|_{0,p_1,\Omega}\le C\|u\|_{\tilde m,p,\Omega}^{\tau}\|u\|_{0,p,\Omega}^{1-\tau}\le C\|u\|_{\tilde m,p,\Omega}\quad(u\in W_p^{\tilde m}(\Omega)) \tag{1.16}$$

with $\tau:=\frac{n}{\tilde m}\left(\frac1p-\frac{1}{p_1}\right)\in(0,1)$. Now, for $|\alpha|=k<m$ we choose $\tilde m:=m-k\ge1$ and use Assumption 1.19 (ii) on a_α to employ Hölder's inequality and obtain for $j=1,\dots,n$:

$$\|a_\alpha\mathcal{D}^{\alpha+e_j}u\|_{0,p,\Omega}\le C\|a_\alpha\|_{0,r_k,\Omega}\|\mathcal{D}^{\alpha+e_j}u\|_{m-k,p,\Omega}\le C\|a_\alpha\|_{0,r_k,\Omega}\|u\|_{m+1,p,\Omega}.$$

Again, for $|\alpha|=k<m$ and $j=1,\dots,n$, we can choose $\tilde m:=m-k+1$ to estimate

$$\|\mathcal{D}_j a_\alpha\mathcal{D}^\alpha u\|_{0,p,\Omega}\le C\|\mathcal{D}_j a_\alpha\|_{0,r_k',\Omega}\|\mathcal{D}^\alpha u\|_{m-k+1,p,\Omega}\le C\|\mathcal{D}_j a_\alpha\|_{0,r_k',\Omega}\|u\|_{m+1,p,\Omega}.$$

By the product rule, we have for all $|\alpha|<m$:

$$\begin{aligned}\|a_\alpha\mathcal{D}^\alpha u\|_{1,p,\Omega} &\le \|a_\alpha\mathcal{D}^\alpha u\|_{0,p,\Omega}+\sum_{j=1}^n\|\mathcal{D}_j a_\alpha\mathcal{D}^\alpha u\|_{0,p,\Omega}+\|a_\alpha\mathcal{D}^{\alpha+e_j}u\|_{0,p,\Omega}\\ &\le C\left(2\|a_\alpha\|_{0,r_k,\Omega}+\max_{j=1,\dots,n}\|\mathcal{D}_j a_\alpha\|_{0,r_k',\Omega}\right)\|u\|_{m+1,p,\Omega}.\end{aligned}$$

For $|\alpha|=m$, we apply Hölder's inequality to find

$$\|a_\alpha\mathcal{D}^\alpha u\|_{1,p,\Omega}\le C\left(2\|a_\alpha\|_{0,\infty,\Omega}+\max_{j=1,\dots,n}\|\mathcal{D}_j a_\alpha\|_{0,\infty,\Omega}\right)\|u\|_{m+1,p,\Omega}.$$

For the boundary operators, we find with the trace theorem (Theorem A.2) and Theorem 1.18

$$\begin{aligned}\|\gamma_{\partial\Omega}B_j(x,\mathcal{D})u\|_{m+s-m_j-1/p,p,\partial\Omega} &\le C\sum_{|\beta|\le m_j}\|b_{j\beta}\mathcal{D}^\beta u\|_{m+s-m_j,p,\Omega}\\ &\le C\max_{|\beta|\le m_j}\|b_{j_\beta}\|_{C^{k_j,\alpha_j}(\overline\Omega)}\sum_{|\beta|\le m_j}\|\mathcal{D}^\beta u\|_{m+s-m_j,p,\Omega}\\ &\le C\max_{|\beta|\le m_j}\|b_{j_\beta}\|_{C^{k_j,\alpha_j}(\overline\Omega)}\|u\|_{m+s,p,\Omega}.\end{aligned}$$

For $s\in(0,1)$, the assertion follows even more directly via the pointwise multiplier theorems. □

So far we have not taken into account the parameter-dependence in problem (1.12). In order to do this, we define new appropriate parameter-dependent norms.

Definition 1.25
Let $s\in(0,1]$ and a parameter-dependent boundary value problem $(A,B_1,\dots,B_{mN/2})$ of the structure (1.12) with $\lambda\in\overline{\Sigma}_\varphi$ be given. Denote by $0\le m_j\le m-1$ the orders of the boundary operators B_j. We define the following norm expressions:

$$|\!|\!|u|\!|\!|_{m+s,p,\Omega} := \|u\|_{m+s,p,\Omega}+|\lambda|\|u\|_{s,p,\Omega}\quad(u\in W_p^{m+s}(\Omega)), \tag{1.17}$$

$$|\!|\!|g|\!|\!|_{m+s-m_j,p,\partial\Omega} := \|g\|_{m+s-m_j-1/p,p,\partial\Omega}+|\lambda|^{\frac{m+\min(0,s-m_j-1/p)}{m}}\|g\|_{\max(0,s-m_j-1/p),p,\partial\Omega}, \tag{1.18}$$

where the latter norm expression is defined for all functions $g\in W_p^{m+s-m_j-1/p}(\partial\Omega)$.

Note that these norms completely coincide with the parameter-dependent norms of the L_p-case for $s = 0$. We first give an equivalent expression for $\|\!|\cdot|\!\|_{k+s,p,\Omega}$.

Lemma 1.26
Let $k \in \mathbb{N}$ and $s > 0$ with $s - 1/p \notin \mathbb{N}$. Then there exist positive constants $C_1, C_2 > 0$ such that for all $u \in W_p^{k+s}(\Omega)$ and $\lambda \in \overline{\Sigma}_\varphi$:

$$C_1 \, \|\!|u|\!\|_{k+s,p,\Omega} \le \sum_{|\alpha|\le k} \left\| \lambda^{1-\frac{|\alpha|}{k}} \mathcal{D}^\alpha u \right\|_{s,p,\Omega} \le C_2 \, \|\!|u|\!\|_{k+s,p,\Omega} \, .$$

Proof. Using the well-known equivalences $\|\cdot\|_{k,p,\Omega} + |\lambda| \|\cdot\|_{0,p,\Omega} \approx \sum_{|\alpha|\le k} \left\| \lambda^{1-\frac{|\alpha|}{k}} \mathcal{D}^\alpha \cdot \right\|_{0,p,\Omega}$ and $\|\cdot\|_{s,p,\Omega} \approx \|\cdot\|_{0,p,\Omega} + [\,\cdot\,]_{s,p,\Omega}$, we find

$$\sum_{|\alpha|\le k} \left\| \lambda^{1-\frac{|\alpha|}{k}} \mathcal{D}^\alpha u \right\|_{s,p,\Omega} \approx \|u\|_{k,p,\Omega} + |\lambda| \|u\|_{0,p,\Omega} + \sum_{|\alpha|\le k} \left[\lambda^{1-\frac{|\alpha|}{k}} \mathcal{D}^\alpha u \right]_{s,p,\Omega} . \tag{1.19}$$

It is easy to see that $\|\!|u|\!\|_{k+s,p,\Omega} \approx \|u\|_{k,p,\Omega} + \sum_{|\alpha|=k} [\mathcal{D}^\alpha u]_{s,p,\Omega} + |\lambda| \left(\|u\|_{0,p,\Omega} + [u]_{s,p,\Omega} \right)$ is contained in the right-hand side, which proves the first inequality. For the second one, we have to estimate the terms $\left\| \lambda^{1-\frac{|\alpha|}{k}} \mathcal{D}^\alpha u \right\|_{s,p,\Omega}$ against $\|\!|u|\!\|_{k+s,p,\Omega}$ for $0 < |\alpha| = l < k$. The continuity of $\mathcal{D}^\alpha : W_p^{s+|\alpha|}(\Omega) \to W_p^s(\Omega)$ (cf. [Gri85], Theorem 1.4.4.6) and interpolation between $W_p^{k+s}(\Omega)$ and $W_p^s(\Omega)$ then yield

$$\left\| \lambda^{1-\frac{|\alpha|}{k}} \mathcal{D}^\alpha u \right\|_{s,p,\Omega} \le C_l |\lambda|^{1-\frac{l}{k}} \|u\|_{l+s,p,\Omega} \le C_l \left(\|u\|_{k+s,p,\Omega} + |\lambda| \|u\|_{s,p,\Omega} \right) .$$

□

Next, we prove the continuity of the operator $(A, \mathcal{B})$ with respect to the parameter-dependent norms.

Lemma 1.27
Let Ω denote either the half space or a bounded domain with $C^{m,1}$-boundary. Let $s \in (0, 1]$ and let a boundary value problem $(A, B_1, \ldots, B_{mN/2})$ in Ω of the structure (1.12) be given. Let the coefficients fulfill Assumption 1.19 in the case $s = 1$ or Assumption 1.20 in the case $s < 1$ with C^k replaced by BUC^k for $\Omega = \mathbb{R}^n_+$. Then the operator

$$\begin{aligned} (A, \mathcal{B}) : \left(W_p^{m+s}, \|\!|\cdot|\!\|_{m+s,p,\Omega} \right) &\to \left(W_p^s, \|\cdot\|_{s,p,\Omega} \right) \times \prod_{j=1}^{mN/2} \left(W_p^{m+s-m_j-1/p}, \|\!|\cdot|\!\|_{m+s-m_j-1/p,p,\partial\Omega} \right) \\ u &\mapsto \left(A(x,\mathcal{D})u, \gamma_0 B_1(x,\mathcal{D})u, \ldots, \gamma_0 B_{mN/2}(x,\mathcal{D})u \right) \end{aligned}$$

is continuous. More precisely, the following estimate is satisfied:

$$\|Au\|_{s,p,\Omega} + \sum_{j=1}^{mN/2} \|\!|\gamma_0 B_j u|\!\|_{m+s-m_j-1/p,p,\partial\Omega} \le C \, \|\!|u|\!\|_{m+s,p,\Omega} \quad (u \in W_p^{m+s}(\Omega)). \tag{1.20}$$

Proof. In Lemma 1.24, we already proved the inequalities $\|A(x,\mathcal{D})u\|_{s,p,\Omega} \le C\|u\|_{m+s,p,\Omega}$ and $\|\gamma_0 B_j(x,\mathcal{D})u\|_{m+s-m_j-1/p,p,\partial\Omega} \le C\|u\|_{m+s,p,\Omega}$ for all $u \in W_p^{m+s}(\Omega)$. Now, first assume that $s - m_j - 1/p \ge 0$. Then, by the multiplication properties of the coefficients $b_{j\beta}$, we find

$$|\lambda| \|\!|\gamma_0 B_j(x,\mathcal{D})u|\!\|_{s-m_j-1/p,p,\partial\Omega} \le C|\lambda| \max_{|\beta|\le m_j} \|b_{j\beta}\|_{C^{k_j,\alpha_j}(\overline{\Omega})} \sum_{|\beta|\le m_j} \|\mathcal{D}^\beta u\|_{s-m_j,p,\Omega} \le C|\lambda| \|\!|u|\!\|_{s,p,\Omega}.$$

In the case $s - m_j - 1/p < 0$, we first assume $m_j \geq s$ and use Lemma A.6 (iii) with $\sigma_0 := 0$, $\sigma_1 := m + s$, $\theta := \frac{1}{p(m+s-m_j)}$, and $\rho := |\lambda|^{\frac{m+s-m_j}{m}}$ to infer

$$\begin{aligned} |\lambda|^{\frac{m+s-m_j-1/p}{m}} \|\gamma_0 B_j(x,\mathcal{D})u\|_{0,p,\partial\Omega} &\leq C\left(\|B_j(x,\mathcal{D})u\|_{m+s,p,\Omega} + \rho\|B_j(x,\mathcal{D})u\|_{0,p,\Omega}\right) \\ &\leq C\left(\|u\|_{m+s-m_j,p,\Omega} + |\lambda|^{\frac{m+s-m_j}{m}}\|u\|_{m_j,p,\Omega}\right). \end{aligned}$$

By interpolation (note that $s - m_j \leq 0$), we find

$$|\lambda|^{\frac{m+s-m_j}{m}}\|u\|_{m_j,p,\Omega} \leq C\left(\|u\|_{m+s,p,\Omega} + |\lambda|\|u\|_{s,p,\Omega}\right) = C\,|\!|\!|u|\!|\!|_{m+s,p,\Omega},$$

which completes the proof in the case $m_j \geq s$. For the remaining case $m_j < s < m_j + 1/p$, we use Lemma A.6 (iii) with $\sigma_1 := m + s - m_j$, $\sigma_0 := s - m_j$, $\theta := -(s - m_j - 1/p)/m \in (0,1)$, and $\rho := |\lambda|$ to obtain

$$\begin{aligned} |\lambda|^{\frac{m+s-m_j-1/p}{m}} \|\gamma_0 B_j(x,\mathcal{D})u\|_{0,p,\partial\Omega} &\leq C\left(\|B_j(x,\mathcal{D})u\|_{m+s-m_j,p,\Omega} + |\lambda|\|B_j(x,\mathcal{D})u\|_{s-m_j,p,\Omega}\right) \\ &\leq C\left(\|u\|_{m+s,p,\Omega} + |\lambda|\|u\|_{s,p,\Omega}\right) = C\,|\!|\!|u|\!|\!|_{m+s,p,\Omega}. \end{aligned}$$

□

The previous Lemma shows that the parameter-dependent norms are a natural choice for estimates of solutions to problems of the structure (1.12) in the ground space $W_p^s(\Omega)$. We now pursue the questions of existence of a solution to (1.12) in the case $s = 1$ as well as the validity of an a priori estimate in terms of the parameter-dependent norms, i.e., if the estimate (1.20) is two-sided. To clarify our assumptions on the ground space, we first cite the following result:

Theorem 1.28 ([DD11], Theorem 3.2)
Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and a parameter-elliptic boundary value problem $(A, \mathcal{B})$ with data $f \in W_p^s(\Omega)$ and $g_j \in W_p^{m+s-m_j-1/p}(\partial\Omega)$ be given. Suppose that for all solutions $u \in W_p^{m+s}(\Omega)$ of (1.12) and $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus the estimate

$$|\!|\!|u|\!|\!|_{m+s,p,\Omega} \leq C\left(\|f\|_{s,p,\Omega} + \sum_{j=1}^{mN/2} |\!|\!|g_j|\!|\!|_{m+s-m_j-1/p,p,\partial\Omega}\right)$$

is satisfied. Then $g_j = 0$ for all j with $m_j < s - 1/p$.

In our situation for $s = 1$ and $1 < p < \infty$, the preceding theorem indicates the necessity of additional compatibility conditions in order to prove an estimate in the parameter-dependent norms in the case $m_j = 0$ for at least one j. Furthermore, since the trace of $f \in W_p^1(\Omega)$ is now well defined, we have the compatibility condition that $\gamma_0 B_j(x,\mathcal{D})f = \gamma_0 B_j(x,\mathcal{D})A(x,\mathcal{D})u$ for all j with $m_j = 0$.

1.2.3 Homogeneous boundary conditions

We start with the treatment of the constant coefficient problem with homogeneous boundary values

$$\left\{\begin{array}{rcll} (\lambda - A(\mathcal{D}))u &=& f & \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n} B_j(\mathcal{D})u &=& 0 & \text{on } \mathbb{R}^{n-1},\ (j = 1,\ldots,mN/2). \end{array}\right. \tag{1.21}$$

We will employ the functional analytic concept of Banach scales (see [Ama95], Chapter V) and find a suitable space for the right-hand side f in the case of homogeneous boundary values as a byproduct. So far all the subsequent definitions are only required for the constant coefficient case here, but if we assume that the coefficients of the operators are pointwise multipliers in the respective spaces, the definitions and assertions carry over to a more general setting. First, we briefly describe the construction of a Banach scale which fits to our given situation.

Definition 1.29
Let X be a Banach space, $\{\cdot,\cdot\}_\theta$ denote an exact interpolation functor, and $A : X \supset D(A) \to X$ be the generator of a C_0-semigroup. We make the following definitions:

(i) For $k \in \mathbb{N}_0$, we inductively define the spaces $E_k := D(A^k)$ (with the convention $A^0 = \mathrm{id}_X$, hence $E_0 = X$), the operators $A_0 := A$, and A_k as the E_k-realization of A_{k-1}, i.e.,

$$D(A_k) := \{u \in E_k \cap D(A_{k-1}) : A_{k-1}u \in E_k\}, \quad A_k u := A_{k-1}u \ (u \in D(A_k)).$$

(ii) For $\alpha \in (0,\infty) \setminus \mathbb{N}$, with $\alpha = k + \theta$ for some $k \in \mathbb{N}_0$ and $\theta \in (0,1)$, define the space $E_\alpha := \{E_k, E_{k+1}\}_\theta$ and the operator A_α as the E_α-realization of A_k, i.e.,

$$D(A_\alpha) := \{u \in E_\alpha \cap D(A_k) : A_k u \in E_\alpha\}, \quad A_\alpha u := A_k u \ (u \in D(A_\alpha)).$$

Theorem 1.30
Let X be a Banach space and $A : X \supset D(A) \to X$ be the generator of a C_0-semigroup.

(i) $[(E_\alpha, A_\alpha); \alpha \geq 0]$ defines a scale of Banach spaces in the sense of [Ama95], Definition V.1.1. More precisely, $[(E_\alpha, A_\alpha); \alpha \geq 0]$ is the interpolation-extrapolation scale of order 0, generated by (X, A).

(ii) The operator $A_\alpha : E_\alpha \supset D(A_\alpha) \to E_\alpha$ is again the generator of a C_0-semigroup in E_α.

(iii) For $\alpha \geq 0$ the equality $\rho(A_\alpha) = \rho(A)$ holds.

Proof. By the Hille-Yosida theorem, we can choose $B := \omega - A$ with $\omega > 0$ sufficiently large such that $0 \in \rho(B)$ in Corollary V.2.1.4 of [Ama95]. Since $D(B) = D(A)$, it follows that the Definition 1.29 is meaningful and the first assertion follows from [Ama95], Theorem V.1.5.1. Assertions (ii) and (iii) are immediate consequences of [Ama95] Theorem V.2.1.3 and Corollary V.2.1.4. □

The scale with integer indices $\alpha \in \mathbb{N}_0$ is also called a discrete Banach scale or the power scale generated by (X, A). All assertions remain valid if one replaces the term of 'semigroup' by 'analytic semigroup'. For a detailed discussion on Banach scales and their properties, we refer to Chapter V of the textbook [Ama95]. Now, we apply the general definition to the present situation of the L_p-realization of a parameter-elliptic boundary value problem $A_{\mathcal{B},p}$.

Definition 1.31
For $1 < p < \infty$ and $s \geq 0$, let $\mathcal{K}_p^s$ denote one of the spaces H_p^s or W_p^s. Let $(A, \mathcal{B})$ be a parameter-elliptic boundary value problem in $\mathbb{R}^n_+$ with $\mathrm{ord}(A) = m$ and $\mathrm{ord}(B_j) = m_j \leq m - 1$ for $j = 1, \ldots, mN/2$ such that its L_p-realization $A_{\mathcal{B},p} : L_p(\mathbb{R}^n_+) \supset D(A_{\mathcal{B},p}) \to L_p(\mathbb{R}^n_+)$ generates a semigroup. For $s \in [0,\infty)$, we write $s = mk + t$ with $k \in \mathbb{N}_0$ and $t \in [0, m)$, where $t \neq m_j + 1/p$ for $j = 1, \ldots, mN/2$, and define the spaces

$$\mathcal{K}^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) := \left\{ u \in \mathcal{K}^s_p(\mathbb{R}^n_+) : \gamma_0 B_j A^k u = 0, \text{ for } k \in \mathbb{N}_0, \ j = 1, \ldots, mN/2 \text{ with } s - mk - m_j > \frac{1}{p} \right\}.$$

Lemma 1.32
In the situation of Definition 1.31, the following assertions hold true:

(i) $\mathcal{K}^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$ equipped with the norm $\|\cdot\|_{s,p,\mathbb{R}^n_+}$ (see Appendix A.1) is a closed subspace of $\mathcal{K}^s_p(\mathbb{R}^n_+)$ and therefore a Banach space.

(ii) For $s_0 < s_1$ with $s_i \neq km + m_j + 1/p$ ($k \in \mathbb{N}_0$), the embedding $\mathcal{K}^{s_1}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) \hookrightarrow \mathcal{K}^{s_0}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$ is satisfied.

(iii) The space $W^{km}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$ coincides with $D(A^k_{\mathcal{B},p})$ (where $D(A^0_{\mathcal{B},p}) := L_p(\mathbb{R}^n_+)$).

(iv) The following interpolation properties are satisfied: Let $k \in \mathbb{N}_0$ and $\theta \in (0,1)$ such that $m(\theta+k) \notin \{km + m_j + \frac{1}{p},\ k \in \mathbb{N}_0,\ j = 1,\dots,mN/2\}$. Then

$$\left(W^{km}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+), W^{(k+1)m}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)\right)_{\theta,p} = W^{m(k+\theta)}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+), \tag{1.22}$$

$$\left[H^{km}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+), H^{(k+1)m}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)\right]_{\theta} = H^{m(k+\theta)}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+). \tag{1.23}$$

Proof. Part (i) and (ii) are immediate consequences of the definition. In (iii), the inclusion $W^{km}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) \subset D(A^k_{\mathcal{B},p})$ is obvious, but for the converse inclusion we have to prove that

$$D(A^k_{\mathcal{B},p}) = \{u \in D(A_{\mathcal{B},p}) : A^l u \in D(A_{\mathcal{B},p}),\ l = 0,\dots,k\} \subset W^{km}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+).$$

The definitions ensure that all boundary conditions of the form $\gamma_0 B_j A^l u$, which are well-defined, vanish and there remains to prove that $u \in W^{km}_p(\mathbb{R}^n_+)$. This has to be done iteratively. We start with the facts $u \in W^m_p(\mathbb{R}^n_+)$, $Au \in W^m_p(\mathbb{R}^n_+)$ and $\gamma_0 B_j u = 0 \in W^{2m-m_j-1/p}_p(\mathbb{R}^{n-1})$ and pick $\lambda_0 \in \overline{\Sigma}_\varphi$ with sufficiently large modulus. Then, by definition, $(\lambda_0 - A, \mathcal{B})$ is a regular elliptic boundary value problem in the sense of [Tri95], Definition 5.2.1/4, (p.363). With minor changes in the proof, the elliptic regularity result from [Tri95], Theorem 5.4.1 carries over to the situation of a system in $\mathbb{R}^n_+$ and we may conclude $u \in W^{2m}_p(\mathbb{R}^n_+)$. Replacing u by Au, we can now prove $Au \in W^{2m}_p(\mathbb{R}^n_+)$ using $A^2 u \in W^m_p(\mathbb{R}^n_+)$ and the corresponding boundary conditions. Iterating this procedure, we find after k steps that $u \in W^{km}_p(\mathbb{R}^n_+)$.
For the proof of the interpolation properties (iv), we use an idea from [Ne09]: the case $k = 0$ and $s_0 = 0$ is contained in [Ama09], Theorem 4.9.1. The general case is proved by induction over $k \in \mathbb{N}_0$. Using (iii), we can consider the Banach scale from Definition 1.29 which is generated by $(L_p(\mathbb{R}^n_+), A_{\mathcal{B},p})$ with $D(A_{\mathcal{B},p}) \subset L_p(\mathbb{R}^n_+)$ (cf. Theorem 1.7). Theorem 1.30 implies that both realizations of $A_{\mathcal{B},p}$ in $W^{mk}_{p;(A,\mathcal{B})}$ and $W^{m(k+1)}_{p;(A,\mathcal{B})}$, respectively, are generators of semigroups and that their resolvent sets coincide with $\rho(A_{\mathcal{B},p})$. Hence, by the Hille-Yosida theorem, there exists $\omega > 0$ such that the following mappings are topological isomorphisms:

$$\begin{aligned} \omega - A_{\mathcal{B},p} :&\quad W^{mk}_{p;(A,\mathcal{B})} \supset W^{m(k+1)}_{p;(A,\mathcal{B})} \to W^{mk}_{p;(A,\mathcal{B})}, \\ \omega - A_{\mathcal{B},p} :&\quad W^{m(k+1)}_{p;(A,\mathcal{B})} \supset W^{m(k+2)}_{p;(A,\mathcal{B})} \to W^{m(k+1)}_{p;(A,\mathcal{B})}. \end{aligned}$$

By real interpolation we deduce an isomorphism

$$\omega - A_{\mathcal{B},p} : \left(W^{m(k+1)}_{p;(A,\mathcal{B})}, W^{m(k+2)}_{p;(A,\mathcal{B})}\right)_{\theta,p} \to \left(W^{km}_{p;(A,\mathcal{B})}, W^{m(k+1)}_{p;(A,\mathcal{B})}\right)_{\theta,p} = W^{m(k+\theta)}_{p;(A,\mathcal{B})},$$

where we employed the induction hypothesis (1.22) for k on the right-hand side. Hence, for $u \in W^{m(k+1)}_{p;(A,\mathcal{B})}$, $u \in \left(W^{m(k+1)}_{p;(A,\mathcal{B})}, W^{m(k+2)}_{p;(A,\mathcal{B})}\right)_{\theta,p}$ holds true if and only if $(\omega - A_{\mathcal{B},p})u \in W^{m(k+\theta)}_{p;(A,\mathcal{B})}$ and the last statement is equivalent to $u \in W^{m(k+1+\theta)}_{p;(A,\mathcal{B})}$ by the definitions. The case of complex interpolation can be treated analogously. □

Note that for the interpolation properties of the spaces $W^s_{p;(A,\mathcal{B})}$ we had to leave out the gaps $s = mk + m_j + 1/p$ where the boundary values do not belong to the scale of Besov spaces.

Theorem 1.33
Assume that the model boundary value problem $(A, \mathcal{B})$ be parameter-elliptic in a sector $\overline{\Sigma}_\varphi$ and let $s \in [0,m] \setminus \{m_j + 1/p,\ j = 1,\dots,mN/2\}$. Then for all $f \in W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$, the equation (1.21) with homogeneous boundary values admits a unique solution $u \in W^{m+s}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$ which satisfies

$$|\!|\!| u |\!|\!|_{m+s,p,\mathbb{R}^n_+} = \|u\|_{m+s,p,\mathbb{R}^n_+} + |\lambda| \|u\|_{s,p,\mathbb{R}^n_+} \le C \|f\|_{s,p,\mathbb{R}^n_+} \tag{1.24}$$

for parameters $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus. In particular, the realization of $A_{\mathcal{B},p}$ in the space $W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$ defined by

$$D(A_{\mathcal{B},p,s}) := \{u \in W^m_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) : Au \in W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)\}, \quad A_{\mathcal{B},p,s}u := Au, \quad (u \in D(A_{\mathcal{B},p,s}))$$

is sectorial with angle φ and for the resolvent $R_s(\lambda) := (\lambda - A_{\mathcal{B},p,s})^{-1}$ we have the estimate

$$\left\|\lambda^{1-\frac{|\alpha|}{m}}\mathcal{D}^\alpha R_s(\lambda)f\right\|_{s,p,\mathbb{R}^n_+} \leq C\|f\|_{s,p,\mathbb{R}^n_+} \quad (f \in W^s_p(\mathbb{R}^n_+)), \tag{1.25}$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m$.

Proof. In case $s = 0$, this result is well-known, see Theorem 1.7. Now suppose that $f \in D(A_{\mathcal{B},p}) = \left\{u \in W^m_p(\mathbb{R}^n_+) : \gamma_0 B_j(\mathcal{D})u = 0,\ j = 1,\dots,mN/2\right\}$. Fix $\lambda_0 \in \rho(A_{\mathcal{B},p})$ and consider the right-hand side $g := (\lambda_0 - A(\mathcal{D}))f \in L_p(\mathbb{R}^n_+)$. The sectoriality of $A_{\mathcal{B},p}$ in $L_p(\mathbb{R}^n_+)$ yields a unique solution $v = (\lambda - A_{\mathcal{B},p})^{-1}g \in D(A_{\mathcal{B},p})$ of the equation

$$\left\{\begin{array}{rcl} (\lambda - A(\mathcal{D}))v & = & g \quad \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n}B_j(\mathcal{D})v & = & 0 \quad \text{on } \mathbb{R}^{n-1},\ (j = 1,\dots,mN/2) \end{array}\right.$$

and by the continuity of the differential operator (see Lemma 1.24), the a priori estimate

$$|\!|\!|v|\!|\!|_{m,p,\mathbb{R}^n_+} \leq C\|g\|_{0,p,\mathbb{R}^n_+} \leq C(|\lambda_0|\|f\|_{0,p,\mathbb{R}^n_+} + \|Af\|_{0,p,\mathbb{R}^n_+}) \leq C(|\lambda_0|)\|f\|_{m,p,\mathbb{R}^n_+}. \tag{1.26}$$

is satisfied. Now, we define $u := (\lambda_0 - A_{\mathcal{B},p})^{-1}v$ and exploit the fact that $(\lambda - A_{\mathcal{B},p})^{-1}$ and $(\lambda_0 - A_{\mathcal{B},p})^{-1}$ commute:

$$\begin{array}{rcl} (\lambda - A(\mathcal{D}))u & = & (\lambda_0 - A_{\mathcal{B},p})^{-1}(\lambda - A(\mathcal{D}))v = (\lambda_0 - A_{\mathcal{B},p})^{-1}g = f, \\ \gamma_0 B_j(\mathcal{D})u & = & \gamma_0 B_j(\mathcal{D})(\lambda_0 - A_{\mathcal{B},p})^{-1}v = 0, \\ \gamma_0 B_j(\mathcal{D})A(\mathcal{D})u & = & \gamma_0 B_j(\mathcal{D})(\lambda u - f) = 0, \end{array}$$

where we used the second line and the fact $f \in D(A_{\mathcal{B},p})$ for the last equality. Clearly, we have $u \in W^m_p(\mathbb{R}^n_+)$, hence $Au = f - \lambda u \in W^m_p(\mathbb{R}^n_+)$ and therefore $u \in D(A^2_{\mathcal{B},p}) = W^{2m}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$ by Lemma 1.32 (iii). Note that the construction of u does not depend on the choice of λ_0. It remains to verify the a priori estimate. This can be proved by employing the estimates (1.26) for v as well as estimate

$$\|u\|_{m+k,p,\mathbb{R}^n_+} \leq C\left(\|(\lambda_0 - A)u\|_{k,p,\mathbb{R}^n_+} + \|u\|_{m+k-1,p,\mathbb{R}^n_+}\right), \tag{1.27}$$

for u with $k = 0$ and $k = m$. The latter estimate is stated in [Tri95], Section 5.3.3 for a regular elliptic boundary value problem which is here given by $(\lambda_0 - A, \mathcal{B})$. Furthermore, due to uniqueness, the term $\|u\|_{m+k-1,p,\mathbb{R}^n_+}$ can be omitted. See also [Agr97], Theorem 2.2.1 and Remark 2.2.2 for this fact. We obtain

$$\begin{array}{rcl} \|u\|_{2m,p,\mathbb{R}^n_+} + |\lambda|\|u\|_{m,p,\mathbb{R}^n_+} & \leq & C\left(\|(\lambda_0 - A)u\|_{m,p,\mathbb{R}^n_+} + |\lambda|\|(\lambda_0 - A)u\|_{0,p,\mathbb{R}^n_+}\right) \\ & \leq & C(|\lambda_0|)\left(\|(\lambda_0 - A)f\|_{0,p,\mathbb{R}^n_+} + |\lambda_0|\|f\|_{0,p,\mathbb{R}^n_+}\right) \leq C(|\lambda_0|)\|f\|_{m,p,\mathbb{R}^n_+}, \end{array}$$

and hence, for $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus we have linear topological isomorphisms

$$\begin{array}{lccccc} (\lambda - A_{\mathcal{B},p}): & L_p(\mathbb{R}^n_+) & \supset & W^m_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) & \to & L_p(\mathbb{R}^n_+), \\ (\lambda - A_{\mathcal{B},p}): & W^m_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) & \supset & W^{2m}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) & \to & W^m_{p;(A,\mathcal{B})}(\mathbb{R}^n_+). \end{array}$$

By real interpolation and Lemma 1.32, we deduce the isomorphism

$$(\lambda - A_{\mathcal{B},p}) : W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) \supset W^{m+s}_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) \to W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$$

for all $s \in (0,m) \setminus \{m_j + 1/p,\ j = 1,\dots,mN/2\}$. The exactness of interpolation yields the estimate (1.24), which can be rewritten in terms of the operator $(\lambda - A_{\mathcal{B},p,s})$ as

$$\|\lambda(\lambda - A_{\mathcal{B},p,s})^{-1}\|_{L(W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+))} \le C(\lambda_0).$$

Furthermore, Theorem 2.1.3 of [Ama95] shows that the resolvent sets of the $W^s_{p;(A,\mathcal{B})}$-realizations of $A_{\mathcal{B},p}$ coincide, hence, they all contain the sector $\overline{\Sigma}_\varphi$. The estimate (1.25) is an immediate consequence of (1.24) and Lemma 1.26. □

1.2.4 Inhomogeneous boundary conditions

For the case of general boundary data $g_j \in W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1})$, we first state a solvability result and an a priori estimate in certain norms, which (in case $s > 1/p$ and $m_j = 0$ for some j) somewhat differ form the 'natural' norms from Definition 1.25. We start with an extension operator to the half space that incorporates the parameter λ and derive some continuity properties for this operator.

Lemma 1.34
Let $k \in \mathbb{N}$ and $1 < p < \infty$. For $g \in W_p^{k-\frac{1}{p}}(\mathbb{R}^{n-1})$ and $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| > \lambda_0 > 0$, define the extension $(E_\lambda g)(x', x_n) := h(x', x_n) := \mathcal{F}'^{-1}\exp(-x_n(|\xi'|+|\lambda|^{1/m}))(\mathcal{F}'g)(\xi')$. Then the following assertions hold true:

(i) $E_\lambda g \in W_p^k(\mathbb{R}^n_+)$ with $\gamma_{0,n}E_\lambda g = g$ and there is a constant $C = C(\lambda_0, n, p, k) > 0$ such that

$$\|E_\lambda g\|_{k,p,\mathbb{R}^n_+} + |\lambda|^{\frac{k}{m}}\|E_\lambda g\|_{0,p,\mathbb{R}^n_+} \le C\left(\|g\|_{k-\frac{1}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{k-1/p}{m}}\|g\|_{0,p,\mathbb{R}^{n-1}}\right). \tag{1.28}$$

(ii) For $0 < s < \frac{1}{p}$ and $0 \le m_j \le m-1$, the following equivalence of norms holds:

$$\|E_\lambda g\|_{m+s-m_j,p,\mathbb{R}^n_+} + |\lambda|^{\frac{m-m_j}{m}}\|E_\lambda g\|_{s,p,\mathbb{R}^n_+} \approx \left(\|g\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|g\|_{0,p,\mathbb{R}^{n-1}}\right).$$

Proof. (i) is proved in Proposition 2.3 from [ADF97]. To prove the second assertion, we first interpolate between (1.28) for $k = m+1-m_j$ and $k = m-m_j$ to see

$$\|E_\lambda g\|_{m+s-m_j,p,\mathbb{R}^n_+} \le C\left(\|g\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|g\|_{0,p,\mathbb{R}^{n-1}}\right).$$

Next, we use the continuity property of the extension operator E_λ in the L_p-norm, which is stated in [ADF97], (A.18): $|\lambda|^{\frac{1}{m}}\|E_\lambda g\|_{0,p,\mathbb{R}^n_+} \le C|\lambda|^{\frac{1-1/p}{m}}\|g\|_{0,p,\mathbb{R}^{n-1}}$ and interpolate with (1.28) for $k = 1$ to deduce that

$$\|E_\lambda g\|_{s,p,\mathbb{R}^n_+} + |\lambda|^{s/m}\|E_\lambda g\|_{0,p,\mathbb{R}^n_+} \le C\left(|\lambda|^{\frac{s-1}{mp}}\|g\|_{s-\frac{s}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{s-1/p}{m}}\|g\|_{0,p,\mathbb{R}^{n-1}}\right).$$

This leads to

$$\begin{aligned}
|\lambda|^{\frac{m-m_j}{m}}\|E_\lambda g\|_{s,p,\mathbb{R}^n_+} &\le C\left(|\lambda|^{\frac{m-m_j+(s-1)/p}{m}}\|g\|_{s-\frac{s}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|g\|_{0,p,\mathbb{R}^{n-1}}\right)\\
&\le C\left(\|g\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|g\|_{0,p,\mathbb{R}^{n-1}}\right),
\end{aligned}$$

where the last step follows by interpolation with $s_1 := m+s-m_j-1/p$, $s_0 = 0$, $s_\theta := s - s/p$, and $\rho := |\lambda|^{\frac{m+s-m_j-1/p}{m}}$.

For the converse inequality, we use the relation $g = \gamma_{0,n} E_\lambda g$ and Lemma A.6 with $s_1 := m+s-m_j$ and $s_0 := s$ to see

$$|\lambda|^{\frac{m+s-m_j-1/p}{m}} \|g\|_{0,p,\mathbb{R}^{n-1}} \le C \left(\|E_\lambda g\|_{m+s-m_j,p,\mathbb{R}^n_+} + |\lambda|^{\frac{m+s-m_j-1/p}{m}} \|E_\lambda g\|_{s,p,\mathbb{R}^n_+} \right).$$

Furthermore, the continuity of the trace operator $\gamma_{0,n} : W_p^{m+s-m_j}(\mathbb{R}^n_+) \to W_p^{m+s-m_j-1/p}(\mathbb{R}^{n-1})$ (see Theorem A.2) implies that

$$\|g\|_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}} \le C \|E_\lambda g\|_{m+s-m_j,p,\mathbb{R}^n_+},$$

which proves the assertion. □

In a first step we derive a representation of a solution to the boundary value problem with $f = 0$ and inhomogeneous boundary values.

Lemma 1.35 (Basic solutions)
Let the model problem $(A, \mathcal{B})$ in the half space be parameter-elliptic in a sector $\overline{\Sigma}_\varphi$. Denote by M the set $M := \{(\xi', t, \lambda) \in \mathbb{R}^{n-1} \times [0,\infty) \times \overline{\Sigma}_\varphi : |\xi'| + |\lambda| \neq 0\}$. Then for $j = 1, \dots, mN/2$, there exist functions $W_j : M \to \mathbb{C}^N$ with the following properties:

(i) W_j is smooth with respect to ξ' and t and all derivatives are jointly continuous in (ξ', t, λ).

(ii) $W_j(\rho\xi', \rho^{-1}t, \rho^m\lambda) = \rho^{-m_j}$ for $\rho > 0$ and $j = 1, \dots, mN/2$.

(iii) W_j admits a representation of the form

$$W_j(\xi', x_n, \lambda) = \int_{\gamma_+} e^{ix_n\tau} (\lambda - A(\xi', \tau))^{-1} N_j(\xi', \tau, \lambda)\, d\tau, \tag{1.29}$$

where $\gamma_+ = \gamma_+(\xi', \lambda)$ is a smooth contour in the upper complex half-plane which encloses all roots of the polynomial $\tau \mapsto \det(\lambda - A(\xi', \tau))$ with positive imaginary part. The functions $N_j : \{(\xi', \tau, \lambda) \in \mathbb{R}^{n-1} \times \gamma_+ \times \overline{\Sigma}_\varphi \ : \ |\xi'| + |\lambda| \neq 0\} \to \mathbb{C}^N$ are smooth with respect to ξ' and all derivatives are jointly continuous with respect to all variables. Furthermore, for $\rho > 0$ we have the quasi-homogeneity relation $N_j(\rho\xi', \rho\tau, \rho^m\lambda) = \rho^{m-m_j-1} N_j(\xi', \tau, \lambda)$.

(iv) For $k = 1, \dots, mN/2$ and all $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$, the function W_k solves the system of ordinary differential equations

$$\begin{cases} (\lambda - A(\xi', \mathcal{D}_n)) W_k(\xi', x_n, \lambda) &= 0 \quad (x_n > 0), \\ B_j(\xi', \mathcal{D}_n) W_k(\xi', x_n, \lambda)|_{x_n=0} &= \delta_{jk} \quad (j = 1, \dots, mN/2). \end{cases} \tag{1.30}$$

Proof. The assertions are stated in Lemma 2.5 of [DFM02]. See also [Vol68] and Remark 5.20 below for an explicit construction of W_j. □

Corollary 1.36
Let the model problem $(A, \mathcal{B})$ in the half space be parameter-elliptic in a sector $\overline{\Sigma}_\varphi$ and assume $g_j \in W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1})$ for $j = 1, \dots, mN/2$. Let $u \in W_p^{m+1}(\mathbb{R}^n_+)$ be a solution to

$$\begin{cases} (\lambda - A(\mathcal{D}))u &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \gamma_0 B_j(\mathcal{D}) u &= g_j \quad \text{on } \mathbb{R}^{n-1}, \ (j = 1, \dots, mN/2). \end{cases} \tag{1.31}$$

Then the partial Fourier transform $\mathcal{F}'u$ of u admits the representation

$$(\mathcal{F}'u)(\xi', x_n, \lambda) = -\sum_{j=1}^{mN/2} \int_0^\infty (\partial_n W_j)(\xi', x_n + y, \lambda) \mathcal{F}'(E_\lambda g_j)(\xi', y)\, dy \\ - \sum_{j=1}^{mN/2} \int_0^\infty W_j(\xi', x_n + y, \lambda) \mathcal{F}'(\partial_n E_\lambda g_j)(\xi', y)\, dy,$$

where E_λ denotes the parameter-dependent extension operator defined in Lemma 1.34.

Proof. Applying the partial Fourier transform to (1.31), we see that $v(\xi', x_n, \lambda) := (\mathcal{F}'u)(\xi', x_n)$ solves the following system of ordinary differential equations of order m for all $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$:

$$\left\{ \begin{array}{rcll} (\lambda - A(\xi', \mathcal{D}_n))v(\xi', x_n, \lambda) & = & 0 & (x_n > 0), \\ B_j(\xi', \mathcal{D}_n)v(\xi', x_n, \lambda)|_{x_n=0} & = & (\mathcal{F}'g_j)(\xi') & (j = 1, \dots, mN/2). \end{array} \right. \tag{1.32}$$

Consequently, using an idea from Volevich (see [Vol68]) $v(\xi', x_n, \lambda)$ admits the representation

$$v(\xi', x_n, \lambda) = \sum_{j=1}^{mN/2} W_j(\xi', x_n, \lambda)\mathcal{F}'g_j(\xi') = \sum_{j=1}^{mN/2} W_j(\xi', x_n, \lambda)\mathcal{F}'(E_\lambda g_j)(\xi', 0)$$

$$= -\sum_{j=1}^{mN/2} \int_0^\infty \frac{\partial}{\partial y} \left[W_j(\xi', x_n + y, \lambda)\mathcal{F}'(E_\lambda g_j)(\xi', y) \right] dy$$

$$= -\sum_{j=1}^{mN/2} \int_0^\infty (\partial_n W_j)(\xi', x_n+y, \lambda)\mathcal{F}'(E_\lambda g_j)(\xi', y)\, dy + \int_0^\infty W_j(\xi', x_n+y, \lambda)\mathcal{F}'(\partial_n E_\lambda g_j)(\xi', y)\, dy,$$

where we used $(E_\lambda g_j)(x', 0) = g_j(x')$ with the parameter-dependent extension operator E_λ from Lemma 1.34. □

The representation for $\mathcal{F}'u$ from the preceding lemma motivates the following

Definition 1.37 (Solution operators for inhomogeneous boundary conditions)
For $\lambda \in \overline{\Sigma}_\varphi$ and $x_n > 0$, define the solution operators corresponding to the boundary value problem with $f = 0$ and inhomogeneous boundary values $g_j \in W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1})$ by

$$\left(T_j^{(1)}(\lambda)E_\lambda g_j\right)(x) := -\int_0^\infty \mathcal{F}'^{-1}\left(\partial_n W_j(\xi', x_n + y, \lambda)(\mathcal{F}'E_\lambda g_j)(\xi', y)\right) dy, \tag{1.33a}$$

$$\left(T_j^{(2)}(\lambda)\partial_n E_\lambda g_j\right)(x) := -\int_0^\infty \mathcal{F}'^{-1}\left(W_j(\xi', x_n + y, \lambda)(\mathcal{F}'\partial_n E_\lambda g_j)(\xi', y)\right) dy, \tag{1.33b}$$

where E_λ denotes the parameter-dependent extension operator to the half space, defined in Lemma 1.34.

Lemma 1.38
The following assertions hold for $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| > \lambda_0$:
(i) The linear operator $\left(T_j^{(1)}E_\lambda\right) : W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1}) \to W_p^{m+1}(\mathbb{R}^n_+; \mathbb{C}^N)$ is continuous. More precisely, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m + 1$:

$$\left\| \lambda^{\frac{m+1-|\alpha|}{m}} \mathcal{D}^\alpha T_j^{(1)}(\lambda)E_\lambda g_j \right\|_{0,p,\mathbb{R}^n_+} \le C \left(\|g_j\|_{m+1-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+1-m_j-\frac{1}{p}}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}} \right). \tag{1.34}$$

(ii) The linear operator $\left(T_j^{(2)}(\lambda)\partial_n E_\lambda\right) : W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1}) \to W_p^{m+1}(\mathbb{R}^n_+; \mathbb{C}^N)$ is continuous. More precisely, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m + 1$:

$$\left\| \lambda^{\frac{m+1-|\alpha|}{m}} \mathcal{D}^\alpha T_j^{(2)}(\lambda)\partial_n E_\lambda g_j \right\|_{0,p,\mathbb{R}^n_+} \le C \left(\|g_j\|_{m+1-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+1-m_j-\frac{1}{p}}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}} \right). \tag{1.35}$$

Proof. Let $\alpha \in N_0^n$ with $|\alpha| = k \leq m+1$ and $g_j \in W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1})$ with $m_j \leq m-1$. Then, by Lemma 1.34 (i), $h_j := E_\lambda g_j = \mathcal{F}'^{-1} e^{-\rho x_n} \mathcal{F}' g_j \in W_p^{m+1-m_j}(\mathbb{R}^n_+)$, and we have to estimate the expressions

$$\begin{aligned}
&\left\| \lambda^{\frac{m+1-k}{m}} \mathcal{D}^\alpha T_j^{(1)}(\lambda) E_\lambda g_j \right\|^p_{0,p,\mathbb{R}^n_+} \\
&= \sum_{l=0}^{k} \sum_{|\alpha'|=l} \int_0^\infty \left\| \int_0^\infty \mathcal{F}'^{-1} \lambda^{\frac{m+1-k}{m}} \xi'^{\alpha'} \mathcal{D}_n^{k-l+1} W_j(\xi', x_n+y, \lambda) (\mathcal{F}' h_j)(\cdot, y)\, dy \right\|^p_{0,p,\mathbb{R}^{n-1}} dx_n \\
&\leq \sum_{\substack{l=0 \\ |\alpha'|=l}}^{k} \int_0^\infty \left(\int_0^\infty \left\| \mathcal{F}'^{-1} (|\lambda|^{\frac{1}{m}} + |\xi'|)^{m_j-m-1} \lambda^{\frac{m+1-k}{m}} \xi'^{\alpha'} \mathcal{D}_n^{k-l+1} W_j(\xi', x_n+y, \lambda) \mathcal{F}' \widetilde{h}_j(\cdot, y) \right\|_{0,p,\mathbb{R}^{n-1}} dy \right)^p dx_n,
\end{aligned}$$

where we have set $\widetilde{h}_j := \mathcal{F}'^{-1}(|\lambda|^{\frac{1}{m}} + |\xi'|)^{m+1-m_j} (\mathcal{F}' h_j)$. We now prove that for fixed $\lambda \in \overline{\Sigma}_\varphi$, all $x_n, y > 0$ and $0 \leq k \leq m+1$, $0 \leq l \leq k$ as well as $\alpha' \in \mathbb{N}_0^{n-1}$ with $|\alpha'| = l$, the symbols

$$\begin{aligned}
M_{jk}^{(1)}(\xi', x_n+y, \lambda) &:= (|\lambda|^{\frac{1}{m}} + |\xi'|)^{m_j-m-1} \lambda^{\frac{m+1-k}{m}} \xi'^{\alpha'} (\mathcal{D}_n^{k-l+1} W_j)(\xi', x_n+y, \lambda) \\
\text{and} \quad M_{jk}^{(2)}(\xi', x_n+y, \lambda) &:= (|\lambda|^{\frac{1}{m}} + |\xi'|)^{m_j-m} \quad \lambda^{\frac{m+1-k}{m}} \xi'^{\alpha'} (\mathcal{D}_n^{k-l} W_j)(\xi', x_n+y, \lambda)
\end{aligned}$$

define Fourier multipliers in $L_p(\mathbb{R}^{n-1}; \mathbb{C}^N)$. In order to verify Michlin's condition, let $\beta' \in \mathbb{N}_0^{n-1}$ with $|\beta'| \leq n-1$. In the sequel, we abbreviate $\rho := \rho(\xi', \lambda) := (|\lambda|^{1/m} + |\xi'|)$. We now employ the homogeneity and continuity properties of $(\lambda - A(\xi', \tau))^{-1} N_j$ and use the facts, that the contour γ_+ in the representation (1.29) of W_j can be chosen (locally) independent of ξ' as well as the inequality

$$|\mathrm{Im}(\tau)| \geq C_\gamma (|\lambda|^{\frac{1}{m}} + |\xi'|) = C_\gamma \rho \quad (\tau \in R(\gamma_+)).$$

Then we can estimate

$$\begin{aligned}
&\left| \xi'^{\beta'} \mathcal{D}_{\xi'}^{\beta'} M_{jk}^{(1)}(\xi', x_n+y, \lambda) \right| \\
&= \left| \xi'^{\beta'} \lambda^{\frac{m+1-k}{m}} \int_{\gamma_+(\xi', \lambda)} \tau^{k-l+1} e^{i\tau(x_n+y)} \mathcal{D}_{\xi'}^{\beta'} \left(\rho^{m_j-m-1} \xi'^{\alpha'} (\lambda - A(\xi', \tau))^{-1} N_j(\xi', \tau, \lambda) \right) d\tau \right| \\
&= \left| \left(\frac{\xi'}{\rho} \right)^{\beta'} \left(\frac{\lambda}{\rho^m} \right)^{\frac{m+1-k}{m}} \int_{\gamma_+(\xi', \lambda)} \left(\frac{\tau}{\rho} \right)^{k-l+1} e^{i\tau(x_n+y)} \left[\mathcal{D}_{\xi'}^{\beta'} \left(\rho^{m_j-m-1} (\cdot)^{\alpha'} (\lambda - A)^{-1} N_j \right) \right] \left(\frac{\xi'}{\rho}, \frac{\tau}{\rho}, \frac{\lambda}{\rho^m} \right) d\tau \right| \\
&\leq C \left| \int_{\gamma_+ \left(\frac{\xi'}{\rho}, \frac{\lambda}{\rho^m} \right)} \rho\, \widetilde{\tau}^{k-l+1} e^{i\rho\widetilde{\tau}(x_n+y)} \left[\mathcal{D}_{\xi'}^{\beta'} \left(\rho^{m_j-m-1} (\cdot)^{\alpha'} (\lambda - A)^{-1} N_j \right) \right] \left(\frac{\xi'}{\rho}, \widetilde{\tau}, \frac{\lambda}{\rho^m} \right) d\widetilde{\tau} \right| \\
&\leq C \rho e^{-C_\gamma \rho (x_n+y)} \leq \frac{C}{x_n+y}.
\end{aligned}$$

For the second equality, note that $\mathcal{D}_{\xi'}^{\beta'} \left(\rho^{-m+m_j-1} \xi'^{\alpha'} (\lambda - A(\xi', \tau))^{-1} N_j(\xi', \tau, \lambda) \right)$ is quasi-homogeneous in (ξ', τ, λ) of degree $l - m - 2 - |\beta'|$ and the continuity of this function on the compact set $\left\{ (\xi', \widetilde{\tau}, \lambda) : |\xi'| + |\lambda| \leq 1, \widetilde{\tau} \in \gamma_+ \left(\frac{\xi'}{|\xi'|+|\lambda|^{1/m}}, \frac{\lambda}{(|\xi'|+|\lambda|^{1/m})^m} \right) \right\}$ (see Lemma 1.35) was used to obtain the second to last estimate. Furthermore, we employed the inequality $te^{-t} \leq 1$ for $t \geq 0$ in the last step. The calculation for $M_{jk}^{(2)}$ follows the same steps. Michlin's multiplier theorem ([Tri95], Section 2.2) yields the continuity of $T_{M_{jk}^{(i)}} \in L(L_p(\mathbb{R}^{n-1}; \mathbb{C}^N))$ for $i = 1, 2$ and for all $f \in L_p(\mathbb{R}^{n-1}; \mathbb{C}^N)$, we have the estimates

$$\| T_{M_{jk}^{(i)}} f \|_{0,p,\mathbb{R}^{n-1}} = \| \mathcal{F}'^{-1} M_{jk}^{(i)}(\xi', x_n+y, \lambda) \mathcal{F}' f \|_{0,p,\mathbb{R}^{n-1}} \leq \frac{C_i}{x_n+y} \| f \|_{0,p,\mathbb{R}^{n-1}} \quad (i = 1, 2).$$

Based on the functions $\widetilde{h}_j$, we define $\overline{h}_j \in L_p(\mathbb{R})$ by

$$\overline{h}_j(y) := \begin{cases} \|\widetilde{h}_j(\cdot,-y)\|_{0,p,\mathbb{R}^{n-1}} & (y \le 0), \\ 0, & (y>0) \end{cases}.$$

Using the fact that the Hilbert transform is a continuous operator in $L_p(\mathbb{R})$ (i.e., $L_p(\mathbb{R})$ is a Banach space of class $\mathcal{HT}$, cf. Definition 5.6 and Remark 5.8 as well as the characterization from Remark 5.7) and rewriting the inner integral as a Hilbert transform of $\overline{h}_j$, we can continue the estimate for $T_j^{(1)}E_\lambda$ from the beginning as follows:

$$\begin{aligned}
&\left\|\lambda^{\frac{m+1-k}{m}}\mathcal{D}^\alpha T_j^{(1)}(\lambda)E_\lambda g_j\right\|_{0,p,\mathbb{R}^n_+} \\
&\le \sum_{l=0}^{k}\sum_{|\alpha'|=l}\left(\int_0^\infty\left(\int_0^\infty\left\|\mathcal{F}'^{-1}M_{jk}^{(1)}(\xi',x_n+y,\lambda)\mathcal{F}'\widetilde{h}_j(\cdot,y)\right\|_{0,p,\mathbb{R}^{n-1}}dy\right)^p dx_n\right)^{\frac{1}{p}} \\
&\le C\left(\int_0^\infty\left(\int_0^\infty\frac{\|\widetilde{h}_j(\cdot,y)\|_{0,p,\mathbb{R}^{n-1}}}{x_n+y}dy\right)^p dx_n\right)^{\frac{1}{p}} = C\left(\int_0^\infty\left(\int_{-\infty}^0\frac{\|\widetilde{h}_j(\cdot,-y)\|_{0,p,\mathbb{R}^{n-1}}}{x_n-y}dy\right)^p dx_n\right)^{\frac{1}{p}} \\
&\le C\left(\int_{\mathbb{R}}|(\mathcal{H}\overline{h}_j)(x_n)|^p dx_n\right)^{\frac{1}{p}} \le C\left(\int_{\mathbb{R}}|\overline{h}_j(x_n)|^p dx_n\right)^{\frac{1}{p}} = C\|\widetilde{h}_j\|_{0,p,\mathbb{R}^n_+}.
\end{aligned}$$

In order to obtain the estimate in terms of g_j, we rewrite

$$\begin{aligned}
\|\widetilde{h}_j\|_{0,p,\mathbb{R}^n_+} &= \left\|\mathcal{F}'^{-1}(|\lambda|^{\frac{1}{m}}+|\xi'|)^{m+1-m_j}\mathcal{F}'h_j\right\|_{0,p,\mathbb{R}^n_+} \\
&\le C\left(\|h_j\|_{m+1-m_j,p,\mathbb{R}^n_+}+|\lambda|^{\frac{m+1-m_j}{m}}\|h_j\|_{0,p,\mathbb{R}^n_+}\right) \\
&\overset{(1.28)}{\le} C\left(\|g_j\|_{m+1-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}+|\lambda|^{\frac{m+1-m_j-1/p}{m}}\|g_j\|_{0,p,\mathbb{R}^{n-1}}\right) = C\,|\!|\!|g_j|\!|\!|_{m+1-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}}.
\end{aligned}$$

Exploiting the structure of the extension operator E_λ, the operator $T_j^{(2)}\partial_n E_\lambda$ is treated in the same way since $T_j^{(2)}\partial_n E_\lambda g_j = -T_j^{(2)}\mathcal{F}'^{-1}(|\lambda|^{\frac{1}{m}}+|\xi'|)\mathcal{F}'E_\lambda g_j$ and, replacing $M_{jk}^{(1)}$ by $M_{jk}^{(2)}$, the above estimates follow analogously. □

With the above results at hand, we can now prove the following

Theorem 1.39
Let the model problem $(A,\mathcal{B})$ be elliptic with parameter $\lambda \in \overline{\Sigma}_\varphi$. Then for all $f \in W_p^1(\mathbb{R}^n_+;\mathbb{C}^N)$ and all $g_j \in W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1})$, the system

$$\left\{\begin{array}{rcll}(\lambda - A(\mathcal{D}))u &=& f & \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n}B_j(\mathcal{D})u &=& g_j & \text{on } \mathbb{R}^{n-1},\ (j=1,\dots,mN/2)\end{array}\right. \tag{1.36}$$

admits a unique solution $u \in W_p^{m+1}(\mathbb{R}^n_+;\mathbb{C}^N)$. Additionally, for $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus the following estimate is valid:

$$\begin{aligned}
|\!|\!|u|\!|\!|_{m+1,p,\mathbb{R}^n_+} &= \|u\|_{m+1,p,\mathbb{R}^n_+}+|\lambda|\|u\|_{1,p,\mathbb{R}^n_+} \\
&\le C\Big(\|f\|_{1,p,\mathbb{R}^n_+}+\max_{1\le j\le mN/2}(0,1-m_j)|\lambda|^{\frac{1}{m}}\|f\|_{0,p,\mathbb{R}^n_+} \\
&\qquad +\sum_{j=1}^{mN/2}\|g_j\|_{m+1-m_j-1/p,p,\mathbb{R}^{n-1}}+|\lambda|^{\frac{m+1-m_j-1/p}{m}}\|g_j\|_{0,p,\mathbb{R}^{n-1}}\Big).
\end{aligned} \tag{1.37}$$

Proof. **Step 1**. Assume $f = 0$ and suppose we are given a solution $u \in W_p^{m+1}(\mathbb{R}^n_+;\mathbb{C}^N)$ of (1.31). By Lemma 1.36 u exhibits a representation of the form

$$u = \sum_{j=1}^{mN/2} T_j^{(1)}(\lambda)E_\lambda g_j + T_j^{(2)}(\lambda)\partial_n E_\lambda g_j.$$

Then $|\!|\!| u |\!|\!|_{m+1,p,\mathbb{R}^n_+} = \|u\|_{m+1,p,\mathbb{R}^n_+} + |\lambda||u|_{1,p,\mathbb{R}^n_+}$ can be estimated by the sum of the expressions (1.34) and (1.35) for $|\alpha| = m+1$ and $|\alpha| = 1$. Lemma 1.38 then implies

$$|\!|\!| u |\!|\!|_{m+1,p,\mathbb{R}^n_+} \le C \sum_{j=1}^{mN/2} \left(\|g_j\|_{m+1-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+1-m_j-1/p}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}} \right),$$

yielding (1.38) for the case $f = 0$.
Step 2. Let now $f \neq 0$. We choose a suitable extension $Ef \in W_p^1(\mathbb{R}^n)$ of f to the whole space $\mathbb{R}^n$ with $\|Ef\|_{1,p,\mathbb{R}^n} \le C\|f\|_{1,p,\mathbb{R}^n_+}$ (for instance, we may choose the even reflection $E_{1,n}$, see Definition 2.7 below). According to Theorem 1.14, we can uniquely solve the problem

$$(\lambda - A(\mathcal{D}))\widetilde{u}_1 = Ef \quad \text{in } \mathbb{R}^n$$

with $\widetilde{u}_1 \in W_p^{m+1}(\mathbb{R}^n;\mathbb{C}^N)$ and we decompose $u = u_1 + u_2$ with $u_1 := \widetilde{u}_1|_{\mathbb{R}^n_+}$ and u_2 is the unique solution to

$$(1.38) \qquad \begin{cases} (\lambda - A(\mathcal{D}))u_2 &= 0 & \text{in } \mathbb{R}^n_+, \\ \gamma_0 B_j(\mathcal{D})u_2 &= g_j - \gamma_0 B_j(\mathcal{D})u_1 & \text{on } \mathbb{R}^{n-1},\ (j = 1,\dots,mN/2). \end{cases}$$

From Corollary 1.15, we deduce an estimate for u_1:

$$\|u_1\|_{m+1,p,\mathbb{R}^n_+} + |\lambda|\|u_1\|_{1,p,\mathbb{R}^n_+} \le \|\widetilde{u}_1\|_{m+1,p,\mathbb{R}^n} + |\lambda|\|\widetilde{u}_1\|_{1,p,\mathbb{R}^n} \le C\|Ef\|_{1,p,\mathbb{R}^n} \le C\|f\|_{1,p,\mathbb{R}^n_+},$$

and Step 1 allows us to estimate u_2 for sufficiently large $|\lambda|$:

$$\begin{aligned} & \|u_2\|_{m+1,p,\mathbb{R}^n_+} + |\lambda|\|u_2\|_{1,p,\mathbb{R}^n_+} \\ \le\ & C \sum_{j=1}^{mN/2} \|g_j - \gamma_0 B_j(\mathcal{D})u_1\|_{m+1-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+1-m_j-1/p}{m}} \|g_j - \gamma_0 B_j(\mathcal{D})u_1\|_{0,p,\mathbb{R}^{n-1}} \\ \le\ & C \sum_{j=1}^{mN/2} \Big(\|g_j\|_{m+1-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+1-m_j-1/p}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}} \\ & \qquad + \|f\|_{1,p,\mathbb{R}^n_+} + (1 - \min(1, m_j))|\lambda|^{1/m}\|f\|_{0,p,\mathbb{R}^n_+} \Big), \end{aligned}$$

where we applied the inequality (1.20) to the terms that stem from the solution in the whole space $\|\gamma_0 B_j(\mathcal{D})u_1\|_{m+1-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+1-m_j-1/p}{m}} \|\gamma_0 B_j(\mathcal{D})u_1\|_{0,p,\mathbb{R}^{n-1}}$ in the case $m_j \ge 1$, and in case $m_j = 0$ we used Lemma (A.6) (iii) with $\sigma_0 := 0, \sigma_1 := m, \theta := \frac{1}{pm}$, and $\rho := |\lambda|^{\frac{m+1}{m}}$ as well as the a priori estimate (1.6) from the L_p-setting:

$$|\lambda|^{\frac{m+1-1/p}{m}} \|\gamma_0 B_j(\mathcal{D})u_1\|_{0,p,\mathbb{R}^{n-1}} \le C \left(\|u_1\|_{m,p,\mathbb{R}^n_+} + \lambda^{\frac{m+1}{m}} \|u_1\|_{0,p,\mathbb{R}^n_+} \right) \le C|\lambda|^{1/m}\|f\|_{0,p,\mathbb{R}^n_+}.$$

This completes the proof of the a priori estimate in the general case.
Step 3. Uniqueness: Suppose there exist two solutions $u, v \in W_p^{m+1}(\mathbb{R}^n_+)$. Then $w := u - v$ solves the equations with $f = g_j = 0$ and the a priori estimate implies $|\!|\!| w |\!|\!|_{m+1,p,\mathbb{R}^n_+} = 0$, thus $u = v$.

Step 4. Existence: We abbreviate the vector $(\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_{mN/2})$ by $(\tilde{f}, \tilde{\boldsymbol{g}})$ and define the operator

$$Z : \mathscr{S}(\mathbb{R}^n) \times \prod_{j=1}^{mN/2} \mathscr{S}(\mathbb{R}^n_+) \to \mathscr{S}'(\mathbb{R}^n_+), \quad (\tilde{f}, \tilde{\boldsymbol{g}}) \mapsto Z(\tilde{f}, \tilde{\boldsymbol{g}}),$$

with

$$Z(\tilde{f}, \tilde{\boldsymbol{g}}) := r_+ \mathcal{F}^{-1}(\lambda - A(\xi))^{-1} \mathcal{F} \tilde{f} + \sum_{j=1}^{mN/2} \mathcal{F}'^{-1} W_j(\xi', x_n, \lambda) \mathcal{F}' \left(\tilde{g}_j - B_j(\mathcal{D}) r_+ \mathcal{F}^{-1}(\lambda - A(\xi))^{-1} \mathcal{F} \tilde{f} \right),$$

where $r_+ : W_p^k(\mathbb{R}^n; \mathbb{C}^N) \to W_p^k(\mathbb{R}^n_+; \mathbb{C}^N)$ denotes the restriction to the half space. We denote the differential operator $(\lambda - A(\mathcal{D}), \gamma_{0,n} B_1(\mathcal{D}), \dots, \gamma_{0,n} B_{mN/2}(\mathcal{D}))$ by $(\lambda - A(\mathcal{D}), \gamma_{0,n} \mathcal{B}(\mathcal{D}))$. Then the properties of the basis functions W_j (cf. Lemma 1.35) and the fact that the partial Fourier transform commutes with the trace operator $\gamma_{0,n}$ and with ∂_n yields the following equality:

$$(\lambda - A(\mathcal{D}), \gamma_{0,n} \mathcal{B}(\mathcal{D})) Z(\tilde{f}, \tilde{\boldsymbol{g}}) = (r_+ \tilde{f}, \gamma_{0,n} \tilde{\boldsymbol{g}}) \quad ((\tilde{f}, \tilde{\boldsymbol{g}}) \in \mathscr{S}(\mathbb{R}^n) \times \prod_{j=1}^{mN/2} \mathscr{S}(\mathbb{R}^n_+)).$$

On the other hand, the Fourier multiplier results in Theorem 1.14 and Lemma 1.38 show that Z extends to a continuous operator $\widetilde{Z} : W_p^1(\mathbb{R}^n) \times \prod_{j=1}^{mN/2} W_p^{m+1-m_j}(\mathbb{R}^n_+) \to W_p^{m+1}(\mathbb{R}^n_+)$. Consequently, the composition

$$\overline{Z} := \widetilde{Z} \circ (E, E_\lambda) : W_p^1(\mathbb{R}^n_+) \times \mathbb{Y} \to W_p^{m+1}(\mathbb{R}^n_+)$$

is continuous, too, where we introduced the abbreviation $\mathbb{Y} := \prod_{j=1}^{mN/2} W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1})$. Now, Lemma 1.24, i.e., the continuity of

$$(\lambda - A(\mathcal{D}), \gamma_{0,n} \mathcal{B}(\mathcal{D})) : W_p^{m+1}(\mathbb{R}^n_+) \to W_p^1(\mathbb{R}^n_+) \times \mathbb{Y}$$

and $(\lambda - A(\mathcal{D}), \mathcal{B}(\mathcal{D}))\overline{Z} = \mathrm{id}_{W_p^1(\mathbb{R}^n_+) \times \mathbb{Y}}$ on a dense subset show that for given $(f, \boldsymbol{g}) \in W_p^1(\mathbb{R}^n_+) \times \mathbb{Y}$, $u := \overline{Z}(f, \boldsymbol{g}) \in W_p^{m+1}(\mathbb{R}^n_+)$ defines a solution to the boundary value problem. □

Remark 1.40

(i) In case $m_j \geq 1$ for all $j = 1, \dots, mN/2$, the preceding theorem shows that (1.12) *is well-posed in $W_p^1(\mathbb{R}^n_+)$ and the solution satisfies the natural two-sided a priori estimate in the parameter-dependent norms: $|\!|\!| u |\!|\!|_{m+1,p,\mathbb{R}^n_+} \approx \|f\|_{1,p,\mathbb{R}^n_+} + \sum_{j=1}^{mN/2} |\!|\!| g_j |\!|\!|_{m+1-m_j-1/p,p,\mathbb{R}^{n-1}}$.*

(ii) If $m_j = 0$ for at least one $j \in \{1, \dots, mN/2\}$, the expression on the right-hand side contains an additional term $|\lambda|^{\frac{1}{m}} \|f\|_{0,p,\mathbb{R}^n_+}$. Furthermore, the norm of the corresponding boundary value is not given by $\|g_j\|_{1-\frac{1}{p},p,\mathbb{R}^{n-1}}$ (cf. Definition 1.25 and Lemma 1.27), but $|\lambda|^{\frac{m+1-m_j-1/p}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}}$. However, Theorem 1.28 already indicates that additional compatibility conditions on f and g_j are necessary to derive the converse of the estimate (1.20) *in this case.*

(iii) Note that we obtain an estimate with respect to the parameter-dependent norms from Definition 1.25 in the special case $g_j - \gamma_0 B_j(\mathcal{D})(\lambda - A_{1,p})^{-1} E f = 0$ for j with $m_j = 0$.

Corollary 1.41 (Solution operator for the inhomogeneous boundary value problem)
Let $s \in [0,1]$. Based on the Definition 1.37, we define the operator

$$\begin{aligned} \mathbb{T} : \prod_{j=1}^{mN/2} W_p^{m+s-m_j-1/p}(\mathbb{R}^{n-1}) &\to W_p^{m+s}(\mathbb{R}^n_+;\mathbb{C}^N), \\ g := (g_1,\dots,g_{mN/2}) &\mapsto \sum_{j=1}^{mN/2} (T_j^{(1)}(\lambda)E_\lambda g_j + T_j^{(2)}(\lambda)\partial_n E_\lambda g_j). \end{aligned} \tag{1.39}$$

Then $\mathbb{T}$ is well-defined, $\mathbb{T}g$ solves the boundary value problem (1.31), and the estimate

$$\|\!|\mathbb{T}g|\!\|_{m+s,p,\mathbb{R}^n_+} \le C \sum_{j=1}^{mN/2} \left(\|g_j\|_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}} \right) \tag{1.40}$$

holds for $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda|$ sufficiently large.

Proof. Following the scheme from the first step in the proof of Lemma 1.38, one can prove the L_p-estimate

$$\|\mathbb{T}g\|_{m,p,\mathbb{R}^n_+} + |\lambda|\|\mathbb{T}g\|_{0,p,\mathbb{R}^n_+} \le C \sum_{j=1}^{mN/2} \left(\|g_j\|_{m-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m-m_j-1/p}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}} \right),$$

which proves the assertion for $s = 0$, see also Theorem 1.7 and the references for its proof. The desired estimate in the case $s = 1$ is contained in Theorem 1.39 for $f = 0$. The general case follows now by real interpolation. □

Our main result for the setting of a model problem in $W_p^1(\mathbb{R}^n_+)$ is now a simple modification of Theorem 1.39.

Theorem 1.42
Assume that the model boundary value problem $(A,\mathcal{B})$ is parameter-elliptic in a sector $\overline{\Sigma}_\varphi$. Then for all $\lambda \in \overline{\Sigma}_\varphi$, all $f \in W^1_{p;(A,\mathcal{B})}(\mathbb{R}^n_+;\mathbb{C}^N) = \{f \in W_p^1(\mathbb{R}^n_+) : \gamma_0 B_j(\mathcal{D})f = 0, \text{ if } m_j = 0\}$ and all boundary values $g_j \in W_p^{m+1-m_j-1/p}(\mathbb{R}^{n-1})$ $(j = 1,\dots,mN/2)$, the problem (1.12) admits a unique solution $u \in W_p^{m+1}(\mathbb{R}^n_+;\mathbb{C}^N)$. For $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus, the following a priori estimate is satisfied:

$$\|\!|u|\!\|_{m+1,p,\Omega} \le C\left(\|f\|_{1,p,\Omega} + \sum_{j=1}^{mN/2} \|g_j\|_{m+1-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+1-m_j-1/p}{m}} \|g_j\|_{0,p,\mathbb{R}^{n-1}} \right).$$

Proof. The major difference to the proof of Theorem 1.39 consists in defining the function $u_1 := (\lambda - A_{\mathcal{B},p,s})^{-1} f$ for $f \in W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$ in Step 2. Following the proof of Theorem 1.39 and using Theorem 1.33, the claim is proved. □

Remark 1.43

(i) The approach presented here differs from the standard L_p-approach in one major point: In L_p one also proves an estimate for the case $f = 0$ and then, to handle a general right-hand side, one extends $f \in L_p(\mathbb{R}^n_+)$ from the half space to $Ef \in L_p(\mathbb{R}^n)$ (e.g., by setting $f = 0$ in $\mathbb{R}^n_-$), solves the corresponding problem in the whole space with $\tilde{u}_1 \in W_p^m(\mathbb{R}^n)$, and the restriction $u_1 := \tilde{u}_1|_{\mathbb{R}^n_+} \in W_p^m(\mathbb{R}^n_+)$ then solves the equation with right-hand side f. However, this reduction to inhomogeneous boundary values changes the original boundary values by $-\gamma_0 B_j(\mathcal{D})u_1$, and there remains to solve the problem with $f = 0$ and the boundary values $\widetilde{g_j} := g_j - \gamma_0 B_j(\mathcal{D})u_1$ to correct this (cf. Step 2 of the proof of Theorem 1.39).

In the W_p^1 setting, Theorem 1.28 tells us that in this situation $\widetilde{g}_j = 0$ for $m_j = 0$ is necessary for a natural estimate of the solution. However, for a general $f \in W_p^1(\mathbb{R}^n_+)$, it is not clear how to extend f to $\mathbb{R}^n$ such that $\gamma_0 B_j(\mathcal{D})u_1 = 0$ for $m_j = 0$ and hence it is not clear how to obtain the natural estimates with this approach. Therefore, we considered the completely subdivided problems (1.31) *and* (1.21).

(ii) The proof technique sketched in (i) works for an operator A with even principal part since then for $f \in W^1_{p,0}(\mathbb{R}^n_+)$ one can choose the odd extension $E_{0,n}$ (cf. Lemma 2.7 below), and then the proof of Theorem 2.10 yields that the restriction $u_1 := \tilde{u}_1|_{\mathbb{R}^n_+}$ of the solution to $(\lambda - A(\mathcal{D}))\tilde{u}_1 = E_{0,n}f$ in $\mathbb{R}^n_+$ satisfies $\gamma_0 u_1 = 0$, and one can follow the lines of the proof of Theorem 1.39 again. We postpone details of this case to the subsequent chapter.

(iii) A proof of corresponding results in W_p^k with $2 \le k \le m$ can be carried out with the same techniques: Theorem 1.39 generalizes to higher regularity without essential changes of the proof, and if the boundary values of order $m_j < m + k - 1/p$ (where the 'unnatural' norms of the boundary values would appear) vanish, the results follow directly with Theorem 1.33.

The situation significantly improves when we consider the case $0 < s < \frac{1}{p}$, which will be crucial for our applications in Chapter 4. For such s the compatibility conditions vanish, i.e., we have $W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+) = W^s_p(\mathbb{R}^n_+)$, and the natural parameter-dependent norms on the boundary are given by $|||\cdot|||_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}} := \|\cdot\|_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}}\|\cdot\|_{0,p,\mathbb{R}^{n-1}}$. A simple combination of Corollary 1.41 and Theorem 1.33 then yields

Corollary 1.44
Let $s < \frac{1}{p}$ and let a model boundary value problem $(A, \mathcal{B})$, being parameter-elliptic in $\overline{\Sigma}_\varphi$ be given. Then for all $f \in W^s_p(\mathbb{R}^n_+;\mathbb{C}^N)$ and all $g_j \in W_p^{m+s-m_j-1/p}(\mathbb{R}^{n-1})$, the boundary value problem (1.12) *admits a unique solution $u \in W_p^{m+s}(\mathbb{R}^n_+)$ that satisfies*

$$\|u\|_{m+s,p,\mathbb{R}^n_+} + |\lambda|\|u\|_{s,p,\mathbb{R}^n_+} \le C\left(\|f\|_{s,p,\mathbb{R}^n_+} + \sum_{j=1}^{mN/2} |||g_j|||_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}}\right)$$

for $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus. The solution admits the representation

$$u = R_s(\lambda)f + \mathbb{T}g$$

with the resolvent $R_s(\lambda) := (\lambda - A_{\mathcal{B},p,s})^{-1}$ for the model problem with homogeneous boundary conditions constructed in Theorem 1.33 and the solution operator $\mathbb{T}$ for the inhomogeneous boundary conditions defined in Corollary 1.41.

1.2.5 Perturbation theory

After presenting the results for the case of a model problem, we now treat small perturbations of the principal parts of the differential operators in order to consider varying coefficients. From now we restrict ourselves to the case $s < \frac{1}{p}$. Our method follows the strategy which is carried out in [DHP03], Section 7.3 for the L_p-case.

Lemma 1.45
Let the model boundary value problem $(A, \mathcal{B})$ be elliptic with parameter $\lambda \in \overline{\Sigma}_\varphi$. Then there exist $\varepsilon > 0$ and $\lambda_0 > 0$ such that for all differential operators of the form

$$S_0(x,\mathcal{D}) = \sum_{|\alpha|=m} s_{0\alpha}(x)\mathcal{D}^\alpha, \quad S_j(x,\mathcal{D}) = \sum_{|\beta|=m_j} s_{j\beta}(x)\mathcal{D}^\beta \quad (j = 1,\dots,mN/2) \tag{1.41}$$

with coefficients that satisfy the smoothness assumptions 1.20 and

$$\|s_{0\alpha}\|_{C^t(\overline{\mathbb{R}^n_+};\mathbb{C}^{N\times N})} < \varepsilon, \quad \text{and} \quad \|s_{j\beta}\|_{C^t(\overline{\mathbb{R}^n_+};\mathbb{C}^{1\times N})} < \varepsilon \quad (j = 1,\dots,mN/2),$$

the perturbed boundary value problem $(A+S_0, \mathcal{B}+\mathcal{S}) := (A+S_0, B_1+S_1, \ldots, B_{mN/2}+S_{mN/2})$ *given by*

$$\left\{ \begin{array}{rcll} (\lambda - (A(\mathcal{D}) + S_0(x,\mathcal{D})))u & = & f & \text{in } \mathbb{R}^n_+, \\ \gamma_0(B_j(\mathcal{D}) + S_j(x,\mathcal{D}))u & = & 0 & \text{on } \mathbb{R}^{n-1},\ (j=1,\ldots,mN/2), \end{array} \right. \tag{1.42}$$

admits a unique solution $u \in W_p^{m+s}(\mathbb{R}^n_+;\mathbb{C}^N)$ *for all* $f \in W_p^s(\mathbb{R}^n_+;\mathbb{C}^N)$*, provided* $|\lambda| > \lambda_0$*. In this situation the solution again satisfies an a priori estimate of the form* (1.24)*. In particular, the* W_p^s*-realization of the shifted boundary value problem* $(A-\lambda_0+S_0, \mathcal{B}+\mathcal{S})$ *defines a sectorial operator of angle* φ*.*

Proof. Let $f \in W_p^s(\mathbb{R}^n_+;\mathbb{C}^N)$. Then $u \in W_p^{m+s}(\mathbb{R}^n_+;\mathbb{C}^N)$ is a solution to (1.42) if and only if u satisfies

$$\left\{ \begin{array}{rcll} (\lambda - A(\mathcal{D}))u & = & f + S_0(x,\mathcal{D})u & \text{in } \mathbb{R}^n_+, \\ \gamma_0 B_j(\mathcal{D})u & = & -\gamma_0 S_j(x,\mathcal{D})u & \text{on } \mathbb{R}^{n-1},\ (j=1,\ldots,mN/2). \end{array} \right. \tag{1.43}$$

A solution of this problem satisfies the representation stated in Corollary 1.44 which leads to the following expression:

$$\left(I - \left[R_s(\lambda) S_0(x,\mathcal{D}) + \sum_{j=1}^{mN/2} T_j^{(1)}(\lambda) E_\lambda \gamma_0 S_j(x,\mathcal{D}) + \sum_{j=1}^{mN/2} T_j^{(2)}(\lambda) \gamma_0 \partial_n S_j(x,\mathcal{D}) \right] \right) u = R_s(\lambda) f,$$

where $R_s(\lambda) := (\lambda - A_{\mathcal{B},p,s})^{-1}$ again denotes the resolvent for the W_p^s-realization of the model boundary value problem with homogeneous boundary conditions (see Theorem 1.33) and I denotes the identity in $W_p^{m+s}(\mathbb{R}^n_+;\mathbb{C}^N)$. In the sequel, the operator on the left-hand side will be denoted by $I - S(\lambda)$. Towards a Neumann series argument in $L((W_p^{m+s}(\mathbb{R}^n_+;\mathbb{C}^N), \|\!|\cdot|\!\|_{m+s,p,\mathbb{R}^n_+}))$, we will estimate the parameter-dependent norms of the operators in the squared brackets. By the results of Lemma 1.33,

$$R_s(\lambda) : \left(W_p^s(\mathbb{R}^n_+;\mathbb{C}^N), \|\cdot\|_{s,p,\mathbb{R}^n_+} \right) \to \left(W_p^{m+s}(\mathbb{R}^n_+;\mathbb{C}^N), \|\!|\cdot|\!\|_{m+s,p,\mathbb{R}^n_+} \right)$$

is a continuous isomorphism and by the continuity of $(\lambda - A(\mathcal{D}))$ (cf. Lemma 1.27), the desired estimate $\|\!|S(\lambda)w|\!\|_{m+s,p,\mathbb{R}^n_+} \le C \|\!|w|\!\|_{m+s,p,\mathbb{R}^n_+}$ with $C < 1$ is equivalent to

$$\|\!|S(\lambda)R_s(\lambda)v|\!\|_{m+s,p,\mathbb{R}^n_+} \le C\|v\|_{s,p,\mathbb{R}^n_+} = C\|(\lambda - A(\mathcal{D}))w\|_{s,p,\mathbb{R}^n_+} \le CC_0 \|\!|w|\!\|_{m+s,p,\mathbb{R}^n_+},$$

with $v \in W_p^s(\mathbb{R}^n_+)$ and $R_s(\lambda)v = w$.

(i) Estimate of $\|\!|R_s(\lambda)S_0(x,\mathcal{D})w|\!\|_{m+s,p,\mathbb{R}^n_+}$: For all $w \in W_p^{m+s}(\mathbb{R}^n_+)$ and $0 < \varepsilon < \varepsilon_0$, we have

$$\begin{aligned} \|\!|R_s(\lambda)S_0(x,\mathcal{D})w|\!\|_{m+s,p,\mathbb{R}^n_+} &\le C\|S_0(x,\mathcal{D})w\|_{s,p,\mathbb{R}^n_+} \le C \max_{|\alpha|=m} \|s_{0\alpha}\|_{C^t(\mathbb{R}^n_+)} \|w\|_{m+s,p,\mathbb{R}^n_+} \\ &\le C\varepsilon \|\!|w|\!\|_{m+s,p,\mathbb{R}^n_+} < \frac{1}{4} \|\!|w|\!\|_{m+s,p,\mathbb{R}^n_+}. \end{aligned}$$

(ii) Estimate of $\left\| T_j^{(1)}(\lambda) E_\lambda \gamma_0 S_j(x,\mathcal{D}) w \right\|_{m+s,p,\mathbb{R}^n_+}$: We write $w = R_s(\lambda)v \in W_p^{m+s}(\mathbb{R}^n_+)$ and use Corollary 1.41 and Lemma A.6 (ii) to estimate:

$$
\begin{aligned}
&\left\| T_j^{(1)}(\lambda) E_\lambda \gamma_0 S_j(x,\mathcal{D}) R_s(\lambda) v \right\|_{m+s,p,\mathbb{R}^n_+} \\
\leq\ & C\left(\|\gamma_0 S_j(x,\mathcal{D}) R_s(\lambda) v\|_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}} + |\lambda|^{\frac{m+s-m_j-1/p}{m}} \|\gamma_0 S_j(x,\mathcal{D}) R_s(\lambda) v\|_{0,p,\mathbb{R}^{n-1}} \right) \\
\overset{(A.8)}{\leq}\ & C\left(2\|S_j(x,\mathcal{D}) R_s(\lambda) v\|_{m+s-m_j,p,\mathbb{R}^n_+} + |\lambda|^{\frac{m-m_j}{m}} \|S_j(x,\mathcal{D}) R_s(\lambda) v\|_{s,p,\mathbb{R}^n_+} \right) \\
\leq\ & C\Bigg(\sum_{|\alpha|=m-m_j} \|\mathcal{D}^\alpha S_j(x,\mathcal{D}) R_s(\lambda) v\|_{s,p,\mathbb{R}^n_+} + |\lambda|^{\frac{m-m_j}{m}} \|S_j(x,\mathcal{D}) R_s(\lambda) v\|_{s,p,\mathbb{R}^n_+} \Bigg) \\
\leq\ & C\Bigg(\sum_{|\alpha|=m-m_j} \|S_j(x,\mathcal{D}) \mathcal{D}^\alpha R_s(\lambda) v\|_{s,p,\mathbb{R}^n_+} + \sum_{|\alpha|=m-m_j} \| \left[\mathcal{D}^\alpha, S_j(x,\mathcal{D})\right] R_s(\lambda) v\|_{s,p,\mathbb{R}^n_+} \\
&\qquad + |\lambda|^{\frac{m-m_j}{m}} \|S_j(x,\mathcal{D}) R_s(\lambda) v\|_{s,p,\mathbb{R}^n_+} \Bigg) \\
\leq\ & C_1\varepsilon \left(\|R_s(\lambda) v\|_{m+s,p,\mathbb{R}^n_+} + |\lambda|^{\frac{m-m_j}{m}} \|R_s(\lambda) v\|_{m_j+s,p,\mathbb{R}^n_+} \right) + C_2 |\lambda|^{-1/m} \|v\|_{s,p,\mathbb{R}^n_+} \\
\leq\ & C_1\varepsilon \left\| R_s(\lambda) v \right\|_{m+s,p,\mathbb{R}^n_+} + C_2 |\lambda|^{-1/m} \|v\|_{s,p,\mathbb{R}^n_+} \leq \frac{1}{4C_0 m N} \|v\|_{s,p,\mathbb{R}^n_+}
\end{aligned}
$$

for $|\lambda| \geq \lambda_1 > 1$ and a sufficiently small choice of ε. In the second inequality, we used the fact that $\gamma_0 : W_p^{m+s-m_j}(\mathbb{R}^n_+) \to W_p^{m+s-m_j-1/p}(\mathbb{R}^{n-1})$ is continuous and (A.8) with $\sigma_1 := m+s-m_j$, $\sigma_0 := s < \frac{1}{p}$, $\rho := |\lambda|^{\frac{m-m_j}{m}}$. Furthermore, the fact that the commutators $[\mathcal{D}^\alpha, S_j(x,\mathcal{D})]$ are differential operators of order not higher than $m-1$ and the estimate (1.25) have been employed in the argumentation.
The estimate of $\left\| T_j^{(2)}(\lambda) \partial_n E_\lambda \gamma_0 S_j(x,\mathcal{D}) w \right\|_{m+s,p,\mathbb{R}^n_+}$ is almost the same. For a suitable choice of $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$ and $|\lambda| \geq \lambda_1 \geq 1$ we obtain the estimate

$$
\left\| R_s(\lambda) S_0(x,\mathcal{D}) + \sum_{j=1}^{mN/2} T_j^{(1)}(\lambda) E_\lambda \gamma_0 S_j(x,\mathcal{D}) + \sum_{j=1}^{mN/2} T_j^{(2)}(\lambda) \partial_n E_\lambda \gamma_0 S_j(x,\mathcal{D}) \right\|_{L(W_p^{m+s}(\mathbb{R}^n_+))} \leq \frac{1}{2},
$$

where $W_p^{m+s}(\mathbb{R}^n_+)$ is equipped with the parameter-dependent norm. Hence, there exists a unique $u \in W_p^{m+s}(\mathbb{R}^n_+)$, that solves (1.42). By Neumann series this u admits the representation

$$
u = (I - S(\lambda))^{-1} R_s(\lambda) f = \sum_{l=0}^{\infty} S(\lambda)^l R_s(\lambda) f
$$

and with $\left\| (I - S(\lambda))^{-1} R_s f \right\|_{m+s,p,\mathbb{R}^n_+} \leq 2 \left\| R_s f \right\|_{m+s,p,\mathbb{R}^n_+} \leq 2C \|f\|_{s,p,\mathbb{R}^n_+}$, the claimed a priori estimate follows. □

Lemma 1.46
Let the assumptions of Lemma 1.45 be satisfied. Then for $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus, the perturbed boundary value problem

$$
\left\{ \begin{array}{rcll} (\lambda - (A(\mathcal{D}) + S_0(x,\mathcal{D})))u & = & 0 & \text{in } \mathbb{R}^n_+, \\ \gamma_0 (B_j(\mathcal{D}) + S_j(x,\mathcal{D}))u & = & g_j & \text{on } \mathbb{R}^{n-1}, \ (j = 1, \ldots, mN/2) \end{array} \right. \tag{1.44}
$$

admits a unique solution $u \in W_p^{m+s}(\mathbb{R}^n_+)$ for all $g_j \in W_p^{m+s-m_j-1/p}(\mathbb{R}^{n-1})$. This solution can be represented as

$$
u = \sum_{j=1}^{mN/2} \mathbb{T}_j(\lambda) g_j \tag{1.45}
$$

and again satisfies an a priori estimate of the form (1.40).

Proof. Adopting the notations from the preceding lemma, we write (1.44) in the form

$$\left\{\begin{array}{rcll}(\lambda - A(\mathcal{D}))u & = & S_0(x,\mathcal{D})u & \text{in } \mathbb{R}^n_+,\\ \gamma_0 B_j(\mathcal{D})u & = & g_j - \gamma_0 S_j(x,\mathcal{D})u & \text{on } \mathbb{R}^{n-1},\ (j=1,\dots,mN/2).\end{array}\right. \tag{1.46}$$

Corollary 1.44 shows then that any solution $u \in W_p^{m+s}(\mathbb{R}^n_+)$ satisfies

$$\left(I - \left[R_s(\lambda)S_0(x,\mathcal{D}) + \sum_{j=1}^{mN/2} T_j^{(1)}(\lambda)E_\lambda\gamma_0 S_j(x,\mathcal{D}) + \sum_{j=1}^{mN/2} T_j^{(2)}(\lambda)\partial_n E_\lambda\gamma_0 S_j(x,\mathcal{D})\right]\right)u = \mathbb{T}g.$$

The estimates we derived in the proof of Lemma 1.45 show that for an appropriate choice of $\varepsilon < \varepsilon_0$ and for parameters λ with a sufficiently large modulus the operator on the left-hand side is invertible. Hence, the solution $u \in W_p^{m+s}(\mathbb{R}^n_+)$ of (1.44) is given by

$$u = (I - S(\lambda))^{-1}\mathbb{T}g = \sum_{j=1}^{mN/2} (I - S(\lambda))^{-1}\left(T_j^{(1)}(\lambda)E_\lambda + T_j^{(2)}(\lambda)\partial_n E_\lambda\right) g_j.$$

The a priori estimate follows from $\|(I - S(\lambda))^{-1}\|_{L(W_p^{m+s}(\mathbb{R}^n_+),\|\cdot\|_{m+s,p,\mathbb{R}^n_+})} \leq 2$ and Corollary 1.41. □

Remark 1.47
Note that the smallness condition on the coefficients is only required with respect to the C^t-norm, since the multiplication property of the coefficients is only required in the space W_p^s. Higher norms can be rewritten as a combination of terms of lower order, which can be made small by using the estimate (1.25) *for sufficiently large modulus of λ, and an expression of the form $\|S_j(x,\mathcal{D})\mathcal{D}^\alpha u\|_{s,p,\mathbb{R}^n_+}$, which is then estimated against $C\varepsilon\|u\|_{|\alpha|+s,p,\mathbb{R}^n_+}$.*

1.2.6 Bounded $C^{m,1}$-domains

We now extend the results for the half space and (possibly perturbed) principle parts to the case of a bounded domain with sufficiently smooth boundary and operators with varying coefficients including lower order terms. This will be obtained via a classical localization procedure. Nevertheless at some points this procedure differs from the typical approach in L_p (see [DHP03], pp. 92 ff.) due to the higher smoothness of the solution.

Let Ω be a bounded domain with a $C^{m,1}$-boundary and let differential operators

$$A(x,\mathcal{D}) := \sum_{|\alpha|\leq m} a_\alpha(x)\mathcal{D}^\alpha \quad \text{and} \quad B_j(x,\mathcal{D}) := \sum_{|\beta|\leq m_j} b_{j\beta}(x)\mathcal{D}^\beta \quad (j=1,\dots,mN) \tag{1.47}$$

with coefficients satisfying Assumption 1.20 be given. Furthermore, we assume parameter-ellipticity of the boundary value problem $(A, B_1, \dots, B_{mN/2})$ in the sense of Definition 1.21. For data

$$(f,g) := (f, g_1, \dots, g_{mN/2}) \in \mathbb{Y} := W_p^s(\Omega) \times \prod_{j=1}^{mN/2} W_p^{m+s-m_j-1/p}(\partial\Omega),$$

we consider the boundary value problem

$$\left\{\begin{array}{rcll}(\lambda - A(x,\mathcal{D}))u & = & f & \text{in } \Omega,\\ \gamma_{\partial\Omega} B_j(x,\mathcal{D})u & = & g_j & \text{on } \partial\Omega,\ (j=1,\dots,mN/2).\end{array}\right. \tag{1.48}$$

Step 1: Transformation and extension to the half space.
Fix an arbitrary point $x_0 \in \partial\Omega$. By the definition of a $C^{m,1}$-boundary, there is an open neighbourhood $U_{x_0} := U_1 \times U_2 \subset \mathbb{R}^{n-1} \times \mathbb{R}$ of x_0 as well as a $C^{m,1}$-diffeomorphism $J_{x_0} : U_{x_0} \to V_{x_0}$, such that $J_{x_0}(U_{x_0} \cap \Omega) = V_{x_0} \cap \mathbb{R}^n_+$ and $J_{x_0}(U_{x_0} \cap \partial\Omega) = V_{x_0} \cap \partial\mathbb{R}^n_+$. Without loss of generality, we may assume $(DJ_{x_0})(x_0) = I_n$ (see [DHP03], p. 93). Furthermore, $J_{x_0} \in C^{m,1}(U_{x_0}, V_{x_0})$ can be extended to

$$\widetilde{J}_{x_0} : \widetilde{U}_{x_0} := \{x \in \mathbb{R}^n_+ : x_n > \widetilde{h}(x')\} \to \mathbb{R}^n_+,$$

where $\widetilde{h} \in C^{m,1}(\mathbb{R}^{n-1})$ is an extension of the function $h \in C^{m,1}(\overline{U_1}, \mathbb{R}^{n-1})$ which describes the boundary in U_{x_0}. For the whole section, we make the convention to denote the coordinates in Ω by $x = (x', x_n)$ and the coordinates in the half space by $y = (y', y_n)$. For the image of x_0 under J_{x_0}, we write $y_0 := J_{x_0}(x_0)$.

Remark 1.48

(i) For $k = 0, \dots, m+1$, the mappings J_{x_0} and $\widetilde{J}_{x_0}$ induce continuous linear isomorphisms

$$\mathcal{J}_{x_0} : W_p^k(U_{x_0} \cap \Omega) \to W_p^k(V_{x_0} \cap \mathbb{R}^n_+), \quad u \mapsto u \circ J_{x_0}^{-1}, \qquad \text{and} \qquad \widetilde{\mathcal{J}}_{x_0} : W_p^k(\widetilde{U}_{x_0}) \to W_p^k(\mathbb{R}^n_+), \quad u \mapsto u \circ \widetilde{J}_{x_0}^{-1}.$$

For the validity of a corresponding chain rule, we refer to [Maz85], Theorem 1.1.7. Using real interpolation, we can obtain the continuity for the spaces W_p^{m+s} as well.

(ii) In [Wlo87], Theorem 4.1, the continuity properties of the mappings $\mathcal{J}_{x_0}$ and $\widetilde{\mathcal{J}}_{x_0}$, which we indicated in (i), are proved for the case of a $C^{k,\kappa}$-diffeomorphism in a convex domain for the Slobodeckii spaces W_2^l with $0 < l < k + \kappa$. For our case $l = m + s$, a $C^{m,t}$-boundary with $t > s$ would then be sufficient.

(iii) Due to the fact $J_{x_0}(U_{x_0} \cap \partial\Omega) = V_{x_0} \cap \partial\mathbb{R}^n_+$, for functions in W_p^t with $t < m$, we have the mapping properties $\mathcal{J}_{x_0} : W_p^t(U_{x_0} \cap \partial\Omega) \to W_p^t(V_{x_0} \cap \mathbb{R}^{n-1})$ and $\widetilde{\mathcal{J}}_{x_0} : W_p^t(\partial\widetilde{U}_{x_0}) \to W_p^t(\mathbb{R}^{n-1})$, respectively.

We now consider the push-forwards of the operators, i.e., $A^{\mathcal{J}_{x_0}}(y, \mathcal{D}) := \mathcal{J}_{x_0} A(x, \mathcal{D}) \mathcal{J}_{x_0}^{-1}$ and $B_j^{\mathcal{J}_{x_0}}(y, \mathcal{D}) := \mathcal{J}_{x_0} B_j(x, \mathcal{D}) \mathcal{J}_{x_0}^{-1}$ for $y \in V_{x_0} \cap \mathbb{R}^n_+$. Note that the coefficients of the transformed operators still satisfy the smoothness assumptions stated in 1.20. In order to check that the Shapiro-Lopatinskii condition is preserved, we use the following transformation formula for the principal parts:

$$\begin{aligned} A_0^{\mathcal{J}_{x_0}}(y, \xi) &= A_0(J_{x_0}^{-1}(y), \left[DJ_{x_0}(J_{x_0}^{-1}(y))\right]^T \xi) \quad (y \in V_{x_0} \cap \mathbb{R}^n_+, \xi \in \mathbb{R}^n), \\ B_{j0}^{\mathcal{J}_{x_0}}(y, \xi) &= B_{j0}(J_{x_0}^{-1}(y), \left[DJ_{x_0}(J_{x_0}^{-1}(y))\right]^T \xi) \quad (y \in V_{x_0} \cap \mathbb{R}^n_+, \xi \in \mathbb{R}^n). \end{aligned}$$

The proof of the Shapiro-Lopatinskii condition for the transformed operators is given in [Wlo87], Theorem 11.3. Furthermore, with this representation one can prove, that the condition of parameter-ellipticity in the sector $\overline{\Sigma}_\varphi$ holds for $A^{\mathcal{J}_{x_0}}(y, \xi)$ as well and $A^{\mathcal{J}_{x_0}}(y, \xi)$ and $B_j^{\mathcal{J}_{x_0}}(y, \xi)$ are again differential operators of order m and $m_j \le m - 1$, respectively. Accordingly, we may rewrite their symbols in the form

$$(1.49) \quad A^{\mathcal{J}_{x_0}}(y, \xi) := \sum_{|\alpha| \le m} a_\alpha^{\mathcal{J}_{x_0}}(y) \xi^\alpha \quad \text{and} \quad B_j^{\mathcal{J}_{x_0}}(y, \xi) := \sum_{|\beta| \le m_j} b_{j\beta}^{\mathcal{J}_{x_0}}(y) \xi^\beta \quad (j = 1, \dots, mN/2).$$

Next, we extend the coefficients of the operators for $y \in \mathbb{R}^n_+$. To this end, we choose a cutoff function $\chi \in C_0^\infty(\mathbb{R}^n)$ with $0 \le \chi \le 1$, such that $\chi \equiv 1$ for $|y| \le 1$ and $\chi \equiv 0$ for $|y| \ge 2$.

For a sufficiently small radius $0 < r \le r(x_0)$, which will be chosen in Step 2, we define

$$\text{(1.50a)} \quad a_\alpha^{(x_0)}(y) := a_\alpha^{\mathcal{J}_{x_0}}\left(y_0 + \chi\left(\tfrac{y-y_0}{r}\right)(y-y_0)\right), \quad (|\alpha| \le m),$$
$$\text{(1.50b)} \quad b_{j\beta}^{(x_0)}(y) := b_{j\beta}^{\mathcal{J}_{x_0}}\left(y_0 + \chi\left(\tfrac{y-y_0}{r}\right)(y-y_0)\right), \quad (j = 1,\dots,mN/2,\ |\beta| \le m_j).$$

This means, for y sufficiently close to y_0, the coefficients are left unchanged, for larger distances $|y-y_0|$, the extended coefficients become constant and in between one chooses the value at a point $\widetilde{y} \in V_{x_0} \cap \mathbb{R}^n_+$.
Let us verify that the smoothness conditions of Assumption 1.20 still hold: For $a_\alpha^{(x_0)}$, this can be checked by a simple case differentiation using the fact that $a_\alpha^{\mathcal{J}_{x_0}} \in C^t(V_{x_0} \cap \mathbb{R}^n_+)$. For the coefficients of the boundary operators, we have to consider the derivatives up to order $|\alpha| \le k_j$ with $k_j + \alpha_j > m + s - m_j$, too. These derivatives can be written as a finite sum of the form

$$(\mathcal{D}^{\alpha-\gamma} b_{j\beta}^{\mathcal{J}_{x_0}})(\widetilde{y})(\mathcal{D}^\gamma \chi)(y) := (\mathcal{D}^{\alpha-\gamma} b_{j\beta}^{\mathcal{J}_{x_0}})\left(y_0 + \chi\left(\tfrac{y-y_0}{r}\right)(y-y_0)\right)\left(\mathcal{D}^\gamma\left(\chi\left(\tfrac{\cdot-y_0}{r}\right)(\cdot-y_0)\right)\right)(y),$$

with $\gamma \le \alpha$ and the seminorm $[\,\cdot\,]_{\alpha_j}$ of their Hölder norms can be estimated as follows:

$$\begin{aligned}
&\frac{\left|(\mathcal{D}^{\alpha-\gamma} b_{j\beta}^{\mathcal{J}_{x_0}})(\widetilde{y}_1)(\mathcal{D}^\gamma\chi)(y_1) - (\mathcal{D}^{\alpha-\gamma} b_{j\beta}^{\mathcal{J}_{x_0}})(\widetilde{y})(\mathcal{D}^\gamma\chi)(y_2)\right|}{|y_1-y_2|^{\alpha_j}} \\
\le\ & |(\mathcal{D}^\gamma\chi)(y_1)|\,\frac{\left|(\mathcal{D}^{\alpha-\gamma} b_{j\beta}^{\mathcal{J}_{x_0}})(\widetilde{y}_1) - (\mathcal{D}^{\alpha-\gamma} b_{j\beta}^{\mathcal{J}_{x_0}})(\widetilde{y}_2)\right|}{|y_1-y_2|^{\alpha_j}} + \left|(\mathcal{D}^{\alpha-\gamma} b_{j\beta}^{\mathcal{J}_{x_0}})(\widetilde{y}_2)\right|\frac{|(\mathcal{D}^\gamma\chi)(y_1) - (\mathcal{D}^\gamma\chi)(y_2)|}{|y_1-y_2|^{\alpha_j}} \\
\le\ & C_\chi \|b_{j\beta}^{\mathcal{J}_{x_0}}\|_{C^{k,\alpha_j}(V_{x_0}\cap\mathbb{R}^n_+)} + \|b_{j\beta}^{\mathcal{J}_{x_0}}\|_{C^{k,\alpha_j}(V_{x_0}\cap\mathbb{R}^n_+)}\|\chi\|_{C^{|\gamma|,\alpha_j}(\mathbb{R}^n)} \le C(\chi)\|b_{j\beta}^{\mathcal{J}_{x_0}}\|_{C^{k,\alpha_j}(V_{x_0}\cap\mathbb{R}^n_+)}.
\end{aligned}$$

Step 2: Choice of an open covering of $\overline{\Omega}$.
For given $\varepsilon > 0$ and any fixed $x_0 \in \partial\Omega$, we can choose $r(x_0)$ such that for some $s < t' < t$ the following properties are valid:

$$\begin{aligned}
&(i) \quad J_{x_0}^{-1}(B(y_0, 2r(x_0))) \subset U_{x_0}, \\
&(ii) \quad \sum_{|\alpha|=m} \left\| a_\alpha^{(x_0)}(y) - a_\alpha^{(x_0)}(y_0)\right\|_{C^{t'}(\mathbb{R}^n_+;\mathbb{C}^{N\times N})} < \varepsilon, \\
&(iii) \quad \sum_{|\beta|=m_j} \left\| b_{j\beta}^{(x_0)}(y) - b_{j\beta}^{(x_0)}(y_0)\right\|_{C^{t'}(\mathbb{R}^n_+;\mathbb{C}^{1\times N})} < \varepsilon.
\end{aligned}$$

The smallness parameter ε is determined by Lemma 1.45 (applied to the parameter-elliptic problem with frozen coefficients at x_0) and the choice in (ii) and (iii) is possible because of

$$\begin{aligned}
&\left\| a_\alpha^{(x_0)}(y) - a_\alpha^{(x_0)}(y_0)\right\|_{C^{t'}(B(y_0,2r(x_0)))} \\
=\ & \|a_\alpha^{(x_0)}(y) - a_\alpha^{(x_0)}(y_0)\|_{0,\infty,B(y_0,2r(x_0))} + \sup_{x,y\in B(y_0,2r(x_0))} \frac{\left|a_\alpha^{(x_0)}(x) - a_\alpha^{(x_0)}(y)\right|}{|x-y|^{t'+(t-t')}}|x-y|^{t-t'} \\
\le\ & \frac{\varepsilon}{2} + \left[a_\alpha^{(x_0)}\right]_{C^t(B(y_0,2r(x_0)))} (4r(x_0))^{t-t'}.
\end{aligned}$$

Recall that $a_\alpha^{(x_0)}(y) = a_\alpha^{(x_0)}(y_0)$ and $b_{j\beta}^{(x_0)}(y) = b_{j\beta}^{(x_0)}(y_0)$ outside of $B(y_0, 2r(x_0))$. We obtain an open covering of the boundary

$$\partial\Omega \subset \bigcup_{x_0\in\partial\Omega} J_{x_0}^{-1}(B(y_0, r(x_0))),$$

and due to the compactness of $\partial\Omega$, there exist $x_1,\dots,x_{M_1}$ such that

$$\partial\Omega \subset \bigcup_{k=1}^{M_1} J_{x_k}^{-1}(B(J_{x_k}(x_k), r(x_k))).$$

In the sequel, we abbreviate $J_k := J_{x_k}$, $y_k := J_k(x_k)$, $r_k := r(x_k)$, $U_k := J_k^{-1}(B(y_k, r_k))$, and for the coefficients we write $a_\alpha^{(k)} := a_\alpha^{(x_k)}$, $b_{j\beta}^{(k)} := b_{j\beta}^{(x_k)}$. Furthermore, for $k = 1, \dots, M_1$, we define corresponding differential operators with the extended coefficients consisting only of the highest order terms:

$$A^{(k)}(y, \mathcal{D}) := \sum_{|\alpha|=m} a_\alpha^{(k)}(y)\mathcal{D}^\alpha \quad \text{and} \quad B_j^{(k)}(y, \mathcal{D}) := \sum_{|\beta|=m_j} b_{j\beta}^{(k)}(y)\mathcal{D}^\beta \quad (j = 1, \dots, mN). \tag{1.51}$$

By construction, these operators satisfy the assumptions of Lemma 1.45 for each k, and for sufficiently large $|\lambda|$ we obtain the resolvent $R_s^{(k)}(\lambda)$ for the W_p^s-realization of the local problem $\left(A^{(k)}, B_1^{(k)}, \dots, B_{mN/2}^{(k)}\right)$ in the half space.

Note that for $|y - y_k| \le r$ the operator $A^{(k)}(y, \mathcal{D})$ coincides with the principal part of $A^{\mathcal{J}_{x_k}}(y, \mathcal{D})$ defined in (1.49), hence, for $u \in W_p^{m+s}(\mathbb{R}^n_+)$ with $\operatorname{supp}(u) \subset B(y_k, r_k)$, the order of differentiation in the expression $\left(A^{\mathcal{J}_{x_k}}(y, \mathcal{D}) - A^{(k)}(y, \mathcal{D})\right) u$ is at most $m - 1$. The same holds for $\left(B_j^{\mathcal{J}_{x_k}}(y, \mathcal{D}) - B_j^{(k)}(y, \mathcal{D})\right) u$ with order of differentiation not greater than $m_j - 1$.

In order to cover the whole domain $\overline{\Omega}$, we exploit the compactness of $\widetilde{\Omega} := \overline{\Omega} \setminus \bigcup_{k=1}^{M_1} U_k$ to find finitely many points $x_{M_1+1}, \dots, x_M$, and corresponding balls $U_k := B(x_k, r_k)$ such that $U_k \subset \widetilde{\Omega}$, $\widetilde{\Omega} \subset \bigcup_{k=M_1+1}^{M} U_k$, and $\sum_{|\alpha|=m} \|a_\alpha^{(k)}(x) - a_\alpha^{(k)}(x_k)\|_{C^{t'}(B(x_k, r_k))} < \varepsilon$, where the extension procedure for the coefficients is the same as in (1.50a), but now the coefficients are continued to the whole of $\mathbb{R}^n$. As in the case of the neighbourhoods being close to the boundary, we define a corresponding local operator with the extended coefficients and without lower order terms by

$$A^{(k)}(x, \mathcal{D}) := \sum_{|\alpha|=m} a_\alpha^{(k)}(x)\mathcal{D}^\alpha \quad (k = M_1 + 1, \dots, M). \tag{1.52}$$

Note that here no transformation was necessary so that we could extend the coefficients directly and remain in the x-coordinates in this case. Furthermore, $A^{(k)}$ is again elliptic with parameter as a small perturbation of the parameter-elliptic operator with constant coefficients $A_0(x_k, \mathcal{D})$. As an immediate consequence of [KS12], Proposition 5.22, a shift of the W_p^s-realization of $A^{(k)}$ is a sectorial operator, which means that $\lambda - A^{(k)} : W_p^{m+s}(\mathbb{R}^n) \to W_p^s(\mathbb{R}^n)$ is invertible for sufficiently large modulus of λ. The corresponding resolvent will be denoted by $R^{(k)}(\lambda)$ for $k = M_1 + 1, \dots, M$.

Subordinated to the finite covering $\overline{\Omega} \subset \bigcup_{k=1}^{M} U_k$, we choose a smooth partition of unity $(\varphi_k)_{k=1}^M \in C^\infty(\mathbb{R}^n)$ with $0 \le \varphi_k \le 1$, $\operatorname{supp}(\varphi_k) \subset U_k$, and $\sum_{k=1}^M \varphi_k(x) = 1$ for all $x \in \overline{\Omega}$. Then, by multiplying the equation (1.48) with φ_k, its unique solvability is equivalent to the unique solvability of the local problems

$$\begin{cases} \varphi_k(\lambda - A(x, \mathcal{D}))u &= \varphi_k f & \text{in } \Omega \cap U_k, \\ \gamma_{\partial\Omega}\varphi_k B_j(x, \mathcal{D})u &= (\gamma_{\partial\Omega}\varphi_k) \cdot g_j & \text{on } \partial\Omega \cap U_k, \ (j = 1, \dots, mN/2), \end{cases} \quad (k = 1, \dots, M_1) \tag{1.53}$$

and

$$\varphi_k(\lambda - A(x, \mathcal{D}))u = \varphi_k f \quad \text{in } \Omega \cap U_k, \quad (k = M_1 + 1, \dots, M). \tag{1.54}$$

Note that $\gamma_{\partial\Omega}\varphi_k \in C^\infty(\partial\Omega)$ is a pointwise multiplier in the space $W_p^{m+s-m_j-1/p}(\partial\Omega)$.

Step 3a: Local resolvents for interior neighbourhoods.

We write $A(x, \mathcal{D}) = A_0(x, \mathcal{D}) + A_{\text{low}}(x, \mathcal{D})$ and interchange the principal part A_0 with φ_k using $A_0(x, \mathcal{D})(\varphi_k u) = \varphi_k A_0(x, \mathcal{D})u + [A_0, \varphi_k]u$, where the commutator

$$[A_0(x, \mathcal{D}), \varphi_k]u := \sum_{|\alpha|=m} a_\alpha(x) \sum_{\substack{\beta \le \alpha \\ |\beta| < m}} \binom{\alpha}{\beta} \mathcal{D}^{\alpha-\beta}\varphi_k \mathcal{D}^\beta u$$

is a differential operator of order not exceeding $m - 1$.

Then equation (1.54) can be rewritten as

$$(\lambda - A_0(x,\mathcal{D}))(\varphi_k u) = \varphi_k f + \varphi_k A_{\text{low}}(x,\mathcal{D})u + [A_0(x,\mathcal{D}),\varphi_k]u = \varphi_k f + C_k(x,\mathcal{D})u$$

with a differential operator $C_k(x,\mathcal{D})u$ of order at most $m-1$ and coefficients in $C^t(\overline{\Omega}\cap U_k)$. Now we apply the resolvent $R^{(k)}(\lambda)$ to this equation. Note that $\operatorname{supp}(\varphi_k u) \subset U_k$, hence, $A_0(x,\mathcal{D})(\varphi_k u) = A^{(k)}(x,\mathcal{D})(\varphi_k u)$, and we obtain

$$\varphi_k u = R^{(k)}(\lambda)\varphi_k f + R^{(k)}(\lambda)C_k(x,\mathcal{D})u \qquad (k = M_1+1,\dots,M). \tag{1.55}$$

Step 3b: Local resolvents for neighbourhoods intersecting with the boundary.
In (1.53), we first interchange $A(x,\mathcal{D})$ and φ_k as well as $B_j(x,\mathcal{D})$ and φ_k to obtain an equation for $\varphi_k u$:

$$\begin{cases} (\lambda - A(x,\mathcal{D}))(\varphi_k u) &= \varphi_k f + [A(x,\mathcal{D}),\varphi_k]u & \text{in } \Omega\cap U_k,\\ \gamma_{\partial\Omega}B_j(x,\mathcal{D})(\varphi_k u) &= (\gamma_{\partial\Omega}\varphi_k)g_j - \gamma_{\partial\Omega}[B_j(x,\mathcal{D}),\varphi_k]u & \text{on } \partial\Omega\cap U_k,\ (j=1,\dots,mN/2). \end{cases}$$

Assuming that all occurring functions are extended by zero to $\widetilde{U}_{x_k}$ outside $\Omega\cap U_k$, we may apply $\widetilde{\mathcal{J}}_k$ to change the coordinates $x\mapsto y$ and artificially insert the terms from (1.51) $A^{(k)}(y,\mathcal{D})\widetilde{\mathcal{J}}_k(\varphi_k u)$ into the equation and $\gamma_0 B_j^{(k)}(y,\mathcal{D})\widetilde{\mathcal{J}}_k(\varphi_k u)$ into the boundary conditions, respectively. After rearranging the equations, we obtain the following boundary value problem in the half space:

$$\begin{cases} (\lambda - A^{(k)}(y,\mathcal{D}))\widetilde{\mathcal{J}}_k(\varphi_k u) &= \widetilde{\mathcal{J}}_k\varphi_k f + \widetilde{\mathcal{J}}_k[A(x,\mathcal{D}),\varphi_k]u + \left(A^{\widetilde{\mathcal{J}}_k}(y,\mathcal{D}) - A^{(k)}(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k(\varphi_k u),\\ \gamma_0 B_j^{(k)}(y,\mathcal{D})\widetilde{\mathcal{J}}_k(\varphi_k u) &= \widetilde{\mathcal{J}}_k\left((\gamma_{\partial\Omega}\varphi_k)g_j\right) - \gamma_0\widetilde{\mathcal{J}}_k[B_j(x,\mathcal{D}),\varphi_k]u\\ &\qquad + \gamma_0\left(B_j^{(k)}(y,\mathcal{D}) - B_j^{\widetilde{\mathcal{J}}_k}(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k(\varphi_k u). \end{cases}$$

We define the lower order differential operators C_k of order at most $m-1$ and D_{jk} of order at most $\max(m_j-1,0)$ by

$$\begin{aligned} C_k(x,\mathcal{D}) &:= [A(x,\mathcal{D}),\varphi_k] + \widetilde{\mathcal{J}}_k^{-1}\left(A^{\widetilde{\mathcal{J}}_k}(y,\mathcal{D}) - A^{(k)}(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k\varphi_k,\\ D_{jk}(x,\mathcal{D}) &:= -[B_j(x,\mathcal{D}),\varphi_k] + \widetilde{\mathcal{J}}_k^{-1}\left(B_j^{(k)}(y,\mathcal{D}) - B_j^{\widetilde{\mathcal{J}}_k}(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k\varphi_k. \end{aligned}$$

Due to the smoothness properties of $\widetilde{\mathcal{J}}_k$ and by the construction of the extensions for the coefficients, the coefficients of C_k belong to $C^t(\widetilde{U}_{x_k})$ and those of D_{jk} belong to $C^{k_j,\alpha_j}(\widetilde{U}_{x_k})$. Furthermore, $\widetilde{\mathcal{J}}_k((\gamma_{\partial\Omega}\varphi_k)g_j)$ may be regarded as an element of $W_p^{m+s-m_j-1/p}(\mathbb{R}^{n-1})$, where we assumed $(\gamma_{\partial\Omega}\varphi_k)g_j$ to be extended to $\partial\widetilde{U_k}$ by zero.

Now we apply the resolvent $R_s^{(k)}(\lambda)$ for the perturbed boundary value problem in the half space $(A^{(k)}, B_1^{(k)},\dots,B_{mN/2}^{(k)})$ (with homogeneous boundary values) to the equation and, in order to correct the boundary values, we apply the operator $\mathbb{T}_j^{(k)}$ from Lemma 1.46 to the inhomogeneity $\widetilde{\mathcal{J}}_k(\gamma_{\partial\Omega}\varphi_k)g_j + \gamma_0\widetilde{\mathcal{J}}_k D_{jk}(x,\mathcal{D})$. Finally, changing back to the x-coordinates, we obtain the following representation for the local solution

$$\begin{aligned} \varphi_k u = \widetilde{\mathcal{J}}_k^{-1}R_s^{(k)}\widetilde{\mathcal{J}}_k(\varphi_k f) + \widetilde{\mathcal{J}}_k^{-1}\sum_{j=1}^{mN/2}\mathbb{T}_j^{(k)}(\lambda)\widetilde{\mathcal{J}}_k\left((\gamma_{\partial\Omega}\varphi_k)g_j\right)\\ + \left[\widetilde{\mathcal{J}}_k^{-1}R_s^{(k)}\widetilde{\mathcal{J}}_k C_k(x,\mathcal{D}) + \widetilde{\mathcal{J}}_k^{-1}\sum_{j=1}^{mN/2}\mathbb{T}_j^{(k)}(\lambda)\gamma_0\widetilde{\mathcal{J}}_k D_{jk}(x,\mathcal{D})\right]u. \end{aligned} \tag{1.56}$$

Step 4: Proof of the a priori estimate.
In the sequel, we write for short $R^{(k)}(\lambda) := \widetilde{\mathcal{J}}_k^{-1} R_s^{(k)} \widetilde{\mathcal{J}}_k$ for $k = 1, \dots, M_1$. Let $\psi_k \in C^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\psi_k) \subset U_k$ and $\psi_k \equiv 1$ on $\operatorname{supp}(\varphi_k)$ for $k = 1, \dots, M$. Then we multiply each of the equations (1.55) and (1.56) by the corresponding ψ_k and sum them up to obtain

$$
\begin{aligned}
(1.57)\quad u = \sum_{k=1}^{M} \psi_k R^{(k)}(\lambda)\varphi_k f + \sum_{k=1}^{M_1}\sum_{j=1}^{mN/2} \psi_k \widetilde{\mathcal{J}}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \widetilde{\mathcal{J}}_k((\gamma_{\partial\Omega}\varphi_k) g_j) \\
\Big[\sum_{k=1}^{M} \psi_k R^{(k)}(\lambda) C_k(x,\mathcal{D}) + \sum_{k=1}^{M_1}\sum_{j=1}^{mN/2} \psi_k \widetilde{\mathcal{J}}_k^{-1} \mathbb{T}_j^{(k)}(\lambda)\gamma_0 \widetilde{\mathcal{J}}_k D_{jk}(x,\mathcal{D}) \Big] u.
\end{aligned}
$$

At this point, we assume that we are given a solution $u \in W_p^{m+s}(\Omega)$ of (1.48). This solution necessarily satisfies the latter equation, and we will now derive an estimate for u from this fact. Similarly to the proofs of the Lemmas 1.45 and 1.46, we have to ensure that the norm of the operator in the squared brackets, which will again be denoted by $S(\lambda)$, considered as an operator in $L(W_p^{m+s}(\Omega), \|\cdot\|_{m+s,p,\Omega})$, is smaller than $\frac{1}{2}$ for large modulus of $\lambda \in \overline{\Sigma}_\varphi$.
(i) For $k = M_1 + 1, \dots, M$, we use the interpolation inequality (Lemma A.6) with $\varepsilon := \frac{1}{4MC_k}$, where C_k contains the norm C_R of $R^{(k)}(\lambda)$ in $L((W_p^s, \|\cdot\|_{s,p,\mathbb{R}^n}), W_p^{m+s}(\mathbb{R}^n), \|\cdot\|_{m+s,p,\mathbb{R}^n})$ and the norm of the lower order differential operator $C_k(\cdot,\mathcal{D})$ in $L(W_p^{m+s}(\mathbb{R}^n), W_p^s(\mathbb{R}^n))$, which can be estimated in terms of $\max_{|\gamma|\le m-1} \|c_{k\gamma}\|_{C^t(\overline{\Omega}\cap U_k)}$ when $c_{k\gamma}$ denote the coefficients of $C_k(\cdot,\mathcal{D})$. Then, for $|\lambda| > \lambda_1 := 4MC_kC(\varepsilon)$ we can estimate

$$
\begin{aligned}
\left\| \psi_k R^{(k)}(\lambda) C_k(x,\mathcal{D}) u \right\|_{m+s,p,\Omega} &\le \left\| R^{(k)}(\lambda) C_k(x,\mathcal{D}) u \right\|_{m+s,p,\mathbb{R}^n} \le C_R \|C_k(x,\mathcal{D})u\|_{s,p,\Omega\cap U_k} \\
&\le C_k \|u\|_{m+s-1,p,\Omega} \\
&\le \frac{1}{4M}\left(\|u\|_{m+s,p,\Omega} + 4MC_kC(\varepsilon)\|u\|_{s,p,\Omega}\right) \le \frac{1}{4M}\,|\!|\!|u|\!|\!|_{m+s,p,\Omega}\,.
\end{aligned}
$$

Note that $C_k(x,\mathcal{D}) \equiv 0$ for $x \in \mathbb{R}^n \setminus (\overline{\Omega} \cap U_k)$.
(ii) For $k = 1, \dots, M_1$, one has to take the coordinate changes $\widetilde{\mathcal{J}}_k$ into account and obtains for parameters $|\lambda| > \lambda_2 := 4MC_kC_JC_{J^{-1}}$:

$$
\begin{aligned}
\left\| \psi_k R^{(k)}(\lambda) C_k(x,\mathcal{D}) u \right\|_{m+s,p,\Omega} &\le C_{J^{-1}} \left\| R_s^{(k)}(\lambda) \widetilde{\mathcal{J}}_k C_k(x,\mathcal{D}) u \right\|_{m+s,p,\mathbb{R}^n_+} \\
&\le C_{J^{-1}} C_R \left\| \widetilde{\mathcal{J}}_k C_k(x,\mathcal{D}) u \right\|_{s,p,\mathbb{R}^n_+} \\
&\le C_{J^{-1}} C_R C_J \|C_k(x,\mathcal{D})u\|_{s,p,\Omega\cap U_k} \le \frac{1}{4M}\,|\!|\!|u|\!|\!|_{m+s,p,\Omega}\,.
\end{aligned}
$$

(iii) For the solution operators acting on the boundary values, there holds the continuity

$$
\mathbb{T}_j^{(k)}(\lambda) : \left(W_p^{m+s-m_j-1/p}(\mathbb{R}^{n-1}), \|\cdot\|_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}} \right) \to \left(W_p^{m+s}(\mathbb{R}^n_+), \|\cdot\|_{m+s,p,\mathbb{R}^n_+} \right),
$$

for each $k \in \{1, \dots, M_1\}$ with a constant C_{T_j}. The mapping properties of the coordinate transformation then yield the following estimates for $h_j \in W_p^{m+s-m_j}(\widetilde{U}_k)$:

$$
\begin{aligned}
\left\| \widetilde{\mathcal{J}}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \gamma_0 \widetilde{\mathcal{J}}_k h_j \right\|_{m+s,p,\Omega} &\le C_{J^{-1}} \left\| \mathbb{T}_j^{(k)}(\lambda) \gamma_0 \widetilde{\mathcal{J}}_k h_j \right\|_{m+s,p,\mathbb{R}^n_+} \le C_{J^{-1}} C_{T_j} \left\| \gamma_0 \widetilde{\mathcal{J}}_k h_j \right\|_{m+s-m_j-\frac{1}{p},p,\mathbb{R}^{n-1}} \\
&\le C_{J^{-1}} C_{T_j} C_{\gamma_0} \left(\|\widetilde{\mathcal{J}}_k h_j\|_{m+s-m_j,p,\mathbb{R}^n_+} + |\lambda|^{\frac{m-m_j}{m}} \|\widetilde{\mathcal{J}}_k h_j\|_{s,p,\mathbb{R}^n_+} \right) \\
&\le C_{J^{-1}} C_{T_j} C_{\gamma_0} C_J \left(\|h_j\|_{m+s-m_j,p,\widetilde{U}_k} + |\lambda|^{\frac{m-m_j}{m}} \|h_j\|_{s,p,\widetilde{U}_k} \right).
\end{aligned}
$$

Here, $h_j = D_{jk}(x,\mathcal{D})u$ with $u \in W_p^{m+s}(\Omega)$, hence, there holds $\operatorname{supp}(h_j) \subset \Omega \cap U_k$, and the estimate can be continued:

$$\begin{aligned}
&\left\| \psi_k \widetilde{\mathcal{J}}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \gamma_0 \widetilde{\mathcal{J}}_k D_{jk}(x,\mathcal{D})u \right\|_{m+s,p,\Omega} \\
\leq\ & C \left(\|D_{jk}(x,\mathcal{D})u\|_{m+s-m_j,p,\Omega\cap U_k} + |\lambda|^{\frac{m-m_j}{m}} \|D_{jk}(x,\mathcal{D})u\|_{s,p,\Omega\cap U_k} \right) \\
\leq\ & CC_D \left(\|u\|_{m+s-1,p,\Omega\cap U_k} + |\lambda|^{\frac{m-m_j}{m}} \|u\|_{s+m_j-1,p,\Omega\cap U_k} \right).
\end{aligned}$$

Note that in case $m_j = 0$ we have $D_{jk} \equiv 0$. Now, the interpolation inequality yields

$$|\lambda|^{\frac{1}{m}} \|u\|_{m+s-1,p,\Omega\cap U_k} \leq C_1(\|u\|_{m+s,p,\Omega\cap U_k} + |\lambda|\|u\|_{s,p,\Omega\cap U_k}) \leq C_1 \|\!|u|\!\|_{m+s,p,\Omega}$$

and

$$|\lambda|^{\frac{m-m_j+1}{m}} \|u\|_{s+m_j-1,p,\Omega\cap U_k} \leq C_2(\|u\|_{m+s,p,\Omega\cap U_k} + |\lambda|\|u\|_{s,p,\Omega\cap U_k}) \leq C_2 \|\!|u|\!\|_{m+s,p,\Omega}$$

such that for $|\lambda| > \lambda_3 := (2M_1 mNCC_D \max(C_1, C_2))^m$, we have

$$\left\| \psi_k \widetilde{\mathcal{J}}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \gamma_0 \widetilde{\mathcal{J}}_k D_{jk}(x,\mathcal{D})u \right\|_{m+s,p,\Omega} \leq \frac{1}{2mNM_1} \|\!|u|\!\|_{m+s,p,\Omega}.$$

Combining the estimates we developed in (i)-(iii), we see that for $|\lambda| > \lambda_0 := \max(\lambda_1, \lambda_2, \lambda_3)$ the estimate

$$\|\!|S(\lambda)u|\!\|_{m+s,p,\Omega} \leq \tfrac{1}{2} \|\!|u|\!\|_{m+s,p,\Omega}$$

is valid. By a Neumann series argument, there exists $(I - S(\lambda))^{-1} = \left(\sum_{l=0}^{\infty} S(\lambda)^l\right)$ and satisfies $\|(I - S(\lambda))^{-1}\|_{L(W_p^{m+s}(\Omega), \|\!|\cdot|\!\|_{m+s,p,\Omega})} \leq 2$. Using the abbreviation $g := (g_1, \ldots, g_{mN/2})^T$, we deduce the following representation of a left inverse for the operator $(\lambda - A, B_1, \ldots, B_{mN/2}) : W_p^{m+s}(\Omega) \to \mathbb{Y}$ for parameters $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| > \lambda_0$:

$$L_\lambda \begin{pmatrix} f \\ g \end{pmatrix} := (I - S(\lambda))^{-1} \left(\sum_{k=1}^{M} \psi_k R^{(k)}(\lambda) \varphi_k f + \sum_{k=1}^{M_1} \sum_{j=1}^{mN/2} \psi_k \widetilde{\mathcal{J}}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \widetilde{\mathcal{J}}_k((\gamma_{\partial\Omega}\varphi_k) g_j) \right). \tag{1.58}$$

In order to complete the proof of the a priori estimate, we first apply the a priori estimates for the resolvents $R^{(k)}(\lambda)$ and the continuity of $\widetilde{\mathcal{J}}_k$ to obtain

$$\begin{aligned}
\left\| \sum_{k=1}^{M} \psi_k R^{(k)}(\lambda) \varphi_k f \right\|_{m+s,p,\Omega} &\leq \sum_{k=1}^{M_1} \left\| R^{(k)}(\lambda) \varphi_k f \right\|_{m+s,p,\mathbb{R}^n} + \sum_{k=M_1+1}^{M} \left\| \widetilde{\mathcal{J}}_k^{-1} R_s^{(k)}(\lambda) \widetilde{\mathcal{J}}_k \varphi_k f \right\|_{m+s,p,\widetilde{U}_k} \\
&\leq C_1 \sum_{k=1}^{M_1} \|\varphi_k f\|_{s,p,\Omega\cap U_k} + C_2 \sum_{k=M_1+1}^{M} \left\| R_s^{(k)}(\lambda) \widetilde{\mathcal{J}}_k \varphi_k f \right\|_{m+s,p,\mathbb{R}^n_+} \\
&\leq C_1 \sum_{k=1}^{M_1} \|\varphi_k f\|_{s,p,\Omega\cap U_k} + C_2 \sum_{k=M_1+1}^{M} \left\| \widetilde{\mathcal{J}}_k \varphi_k f \right\|_{s,p,\mathbb{R}^n_+} \\
&\leq C \sum_{k=1}^{M} \|\varphi_k f\|_{s,p,\Omega\cap U_k} \leq CM_0 \|f\|_{s,p,\Omega},
\end{aligned}$$

where M_0 denotes the maximal number of neighbourhoods which intersect with each other. For the boundary terms, we proceed analogously to part (iii) of Step 4 and apply the estimates for $\mathbb{T}_j$ from Corollary 1.41 to find

$$
\begin{array}{lll}
& \left\| \psi_k \widetilde{J}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \widetilde{\mathcal{J}}_k \varphi_k g_j \right\|_{m+s,p,U_k} & \\
\leq C & \left\| \widetilde{J}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \widetilde{\mathcal{J}}_k \varphi_k g_j \right\|_{m+s,p,\widetilde{U}_k} & \leq C_{J^{-1}} \left\| \mathbb{T}_j^{(k)}(\lambda) \widetilde{\mathcal{J}}_k \varphi_k g_j \right\|_{m+s,p,\mathbb{R}^n_+} \\
\leq C_{J^{-1}} C_{\mathbb{T}_j} & \left\| \widetilde{\mathcal{J}}_k \varphi_k g_j \right\|_{m+s-m_j-1/p,p,\mathbb{R}^{n-1}} & \leq C_{J^{-1}} C_{\mathbb{T}_j} C_J \left\| \varphi_k g_j \right\|_{m+s-m_j-1/p,p,\partial \widetilde{U}_k} \\
\leq C & \left\| g_j \right\|_{m+s-m_j-1/p,p,\partial\Omega \cap U_k} . &
\end{array}
$$

Using the first estimate of Step 4, we then deduce

$$
\left\| u \right\|_{m+s,p,\Omega} \leq 2C \left(M_0 \| f \|_{s,p,\Omega} + \sum_{j=1}^{mN/2} C_j \left\| g_j \right\|_{m+s-m_j-1/p,p,\partial\Omega} \right).
$$

Step 5: Existence of a right inverse.
There remains to prove that we can in fact solve the boundary value problem (1.48) for all data $(f,g) \in \mathbb{Y}$, meaning the surjectivity of $(\lambda - A, \mathcal{B})$ for sufficiently large modulus of $\lambda \in \overline{\Sigma}_\varphi$. To this end, we first apply the operator $(\lambda - A_0(x,\mathcal{D}))$ to the representation of $L_\lambda f$ stated in (1.57), where we abbreviate $T_j^{(k)}(\lambda) := \widetilde{\mathcal{J}}_k^{-1} \mathbb{T}_j^{(k)}(\lambda) \gamma_0 \widetilde{\mathcal{J}}_k$.

$$
\begin{aligned}
(\lambda - A_0(x,\mathcal{D})) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix} &= \sum_{k=1}^{M} (\lambda - A_0(x,\mathcal{D})) \psi_k R^{(k)}(\lambda) \left(\varphi_k f + C_k(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix} \right) \\
&+ \sum_{k=1}^{M_1} \sum_{j=1}^{mN/2} (\lambda - A_0(x,\mathcal{D})) \psi_k T_j^{(k)}(\lambda) \left[D_{jk}(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix} + (\gamma_{\partial\Omega} \varphi_k) g_j \right].
\end{aligned}
$$

Using the facts that $\operatorname{supp}(\varphi_k f + C_k(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix}) \subset \operatorname{supp}(\varphi_k) \subset U_k$ and $\psi_k \equiv 1$ on $\operatorname{supp}(\varphi_k)$, we can calculate

$$
\begin{aligned}
& (\lambda - A_0(x,\mathcal{D})) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix} \\
= & \sum_{k=1}^{M} \psi_k (\lambda - A_0(x,\mathcal{D})) R^{(k)}(\lambda) (\varphi_k f + C_k(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix}) \\
& + \sum_{k=1}^{M} [-A_0, \psi_k] R^{(k)}(\lambda) (\varphi_k f + C_k(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix}) \\
& + \sum_{k=1}^{M} \sum_{j=1}^{mN/2} (\psi_k (\lambda - A_0(x,\mathcal{D})) + [-A_0, \psi_k]) T_j^{(k)}(\lambda) \left[D_{jk}(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix} + (\gamma_{\partial\Omega} \varphi_k) g_j \right] \\
= & \sum_{k=1}^{M} (\varphi_k f + C_k(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix}) + [-A_0, \psi_k] R^{(k)}(\lambda) (\varphi_k f + C_k(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix}) \\
& + \sum_{k=1}^{M} \sum_{j=1}^{mN/2} [-A_0, \psi_k] T_j^{(k)}(\lambda) \left[D_{jk}(x,\mathcal{D}) L_\lambda \begin{pmatrix} f \\ g \end{pmatrix} + (\gamma_{\partial\Omega} \varphi_k) g_j \right].
\end{aligned}
$$

Note that $(\lambda - A_0(x,\mathcal{D})) T_j^{(k)}(\lambda) = 0$ and $(\lambda - A_0(x,\mathcal{D})) R^{(k)}(\lambda) g = g$ for $g \in W_p^s(\Omega)$ with $\operatorname{supp}(g) \subset U_k \cap \Omega$. Making use of the equalities $\sum_{j=1}^{M} \varphi_k = 1$ and $\sum_{j=1}^{M} [A_0(x,\mathcal{D}), \varphi_k] = 0$ as well as recalling the definitions of $C_k(x,\mathcal{D})$ (cf. Step 3a and 3b)

$$
C_k(x,\mathcal{D}) = \begin{cases} [A(x,\mathcal{D}), \varphi_k] + \widetilde{\mathcal{J}}_k^{-1} \left(A^{\widetilde{J}_k}(y,\mathcal{D}) - A^{(k)}(y,\mathcal{D}) \right) \widetilde{\mathcal{J}}_k \varphi_k, & (1 \leq k \leq M_1), \\ \varphi_k (A - A_0)(x,\mathcal{D}) + [A_0(x,\mathcal{D}), \varphi_k], & (M_1 < k \leq M), \end{cases}
$$

the first sum can be rewritten as

$$\begin{aligned}
&\sum_{k=1}^{M}\left(\varphi_k f + C_k(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}\right)\\
=\;& f+\sum_{k=1}^{M_1}([A_0,\varphi_k]+[A-A_0,\varphi_k])L_\lambda\begin{pmatrix}f\\g\end{pmatrix}+\widetilde{\mathcal{J}}_k^{-1}\left((A^{\widetilde{J}_k}-A^{(k)})(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k\varphi_k L_\lambda\begin{pmatrix}f\\g\end{pmatrix}\\
&+\sum_{k=M_1+1}^{M}\varphi_k(A-A_0)(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}+[A_0(x,\mathcal{D}),\varphi_k]L_\lambda\begin{pmatrix}f\\g\end{pmatrix}\\
=\;& f+(A-A_0)(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}+\sum_{k=1}^{M_1}(A-A_0)(x,\mathcal{D})\varphi_k L_\lambda\begin{pmatrix}f\\g\end{pmatrix}\\
&+\widetilde{\mathcal{J}}_k^{-1}\left((A^{\widetilde{J}_k}-A^{(k)})(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k\varphi_k L_\lambda\begin{pmatrix}f\\g\end{pmatrix}+\sum_{k=M_1+1}^{M}[A_0(x,\mathcal{D}),\varphi_k]L_\lambda\begin{pmatrix}f\\g\end{pmatrix}.
\end{aligned}$$

We have proved

$$(\lambda-A(x,\mathcal{D}))L_\lambda\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}f\\0\end{pmatrix}+R_1(x,\lambda)\begin{pmatrix}f\\g\end{pmatrix},$$

with an operator R_1 which consists of combinations of lower order differential operators and solution operators. Hence, its norm in $L(\mathbb{Y})$ (with the parameter-dependent norms for the boundary values) can be estimated by virtue of the interpolation inequality and the same techniques as in Step 4 to see that $\|R_1\|_{L(\mathbb{Y})}<\frac{1}{2}$ for $|\lambda|\geq\lambda_1$.

Regarding the boundary conditions, we first apply the principal parts $\gamma_{\partial\Omega}B_{l,0}(x,\mathcal{D})$ to $L_\lambda\begin{pmatrix}f\\g\end{pmatrix}$. Note that $\gamma_{\partial\Omega}\mathcal{D}^\alpha\psi_k\equiv 0$ for the interior neighbourhoods, i.e., for $k=M_1+1,\dots,M$. Furthermore, we make use of the facts that $\widetilde{\mathcal{J}}_kR^{(k)}(\lambda)$ solves the homogeneous boundary value problem $\left(\lambda-A^{(k)},B_1^{(k)},\dots,B_{mN/2}^{(k)}\right)$ in the half space, which coincides with $\left(\lambda-A_0^{\widetilde{\mathcal{J}}_k},B_{1,0}^{\widetilde{\mathcal{J}}_k},\dots,B_{mN/2,0}^{\widetilde{\mathcal{J}}_k}\right)$ on $V_k\cap\mathbb{R}^n_+$, and we have $\gamma_{\partial\Omega}B_{l,0}\widetilde{\mathcal{J}}_k^{-1}\mathbb{T}_j^{(k)}(\lambda)\gamma_0\widetilde{\mathcal{J}}_k g=\delta_{lj}g$ for $g\in W_p^{m+s-m_j-1/p}(\partial\Omega)$ with $\operatorname{supp}(g)\subset\partial\Omega\cap U_k$:

$$\begin{aligned}
\gamma_{\partial\Omega}B_{l,0}(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}&=\sum_{k=1}^{M_1}(\gamma_{\partial\Omega}\varphi_k)g_l+\gamma_{\partial\Omega}\sum_{k=1}^{M_1}\psi_k D_{lk}(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}+\\
+\gamma_{\partial\Omega}\sum_{k=1}^{M_1}[B_{l,0},\psi_k]&\left(R^{(k)}(\lambda)(\varphi_k f+C_k(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix})+\sum_{j=1}^{mN/2}\mathbb{T}_j^{(k)}(\lambda)\left[D_{jk}(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}+(\gamma_{\partial\Omega}\varphi_k)g_j\right]\right).
\end{aligned}$$

The first sum equals to g_l and we consider the second sum. We apply the definition of $D_{lk}(x,\mathcal{D})$ (cf. Step 3b) to write

$$\begin{aligned}
\sum_{k=1}^{M_1}\psi_k D_{lk}(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}&=\sum_{k=1}^{M_1}[-B_l(x,\mathcal{D}),\varphi_k]L_\lambda\begin{pmatrix}f\\g\end{pmatrix}\\
&\quad+\widetilde{\mathcal{J}}_k^{-1}\left(B_{lk}(y,\mathcal{D})-B_l^{\widetilde{\mathcal{J}}_k}(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k\varphi_k L_\lambda\begin{pmatrix}f\\g\end{pmatrix}.
\end{aligned}$$

Note that $\psi_k\equiv 1$ over the set, where $D_{lk}\neq 0$ and that for $x\in\partial\Omega$ we have $\sum_{k=1}^{M_1}\varphi_k(x)=1$, thus $\gamma_0\sum_{k=1}^{M_1}[B_l,\varphi_k]\equiv 0$. Furthermore, by construction, the latter operators can be rewritten in the form

$$\widetilde{\mathcal{J}}_k^{-1}\left(B_{lk}(y,\mathcal{D})-B_l^{\widetilde{\mathcal{J}}_k}(y,\mathcal{D})\right)\widetilde{\mathcal{J}}_k\varphi_k=-(B-B_{l,0})(x,\mathcal{D})\varphi_k+\widetilde{C_l}(x,\mathcal{D})\varphi_k,$$

where $\widetilde{C_l}(x,\mathcal{D})$ is a differential operator of order at most m_l-1, because the principal parts of B_{lk} and $B_l^{\widetilde{\mathcal{J}}_k}$ coincide on $V_k\cap\mathbb{R}^n_+$ and $B_l^{\widetilde{\mathcal{J}}_k}$ also contains the transformed lower order terms of B_l. Using again $\gamma_{\partial\Omega}\sum_{k=1}^{M_1}\varphi_k=1$, we obtain the equality

$$\gamma_{\partial\Omega}\mathcal{B}(x,\mathcal{D})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}0\\g\end{pmatrix}+R_2(x,\lambda)\begin{pmatrix}f\\g\end{pmatrix},$$

where $R_2(x,\lambda)$ is again a combination of solution operators and differential operators of lower order. In virtue the interpolation inequality, the norm of R_2 in $L(\mathbb{Y})$ becomes arbitrarily small for sufficiently large $|\lambda|\geq\lambda_2$. Summarizing the above considerations and putting the lower order terms into R_1+R_2, we can write

$$(\lambda-A,B_1,\ldots,B_{mN/2})L_\lambda\begin{pmatrix}f\\g\end{pmatrix}=\begin{pmatrix}f\\g\end{pmatrix}+(R_1+R_2)\begin{pmatrix}f\\g\end{pmatrix},$$

thus, for $|\lambda|>\lambda_0:=\max(\lambda_1,\lambda_2)$, the operator $L_\lambda(I+(R_1+R_2))^{-1}:\mathbb{Y}\to W_p^{m+s}(\Omega)$ is the desired right inverse. We can now formulate our main result for parameter-elliptic systems in bounded domains with sufficiently smooth boundary.

Theorem 1.49
Let $\Omega\subset\mathbb{R}^n$ be a bounded $C^{m,1}$-domain, $1<p<\infty$, and $0<s<\frac{1}{p}$. Let the boundary value problem $(A,B_1,\ldots,B_{mN/2})$, with operators of the form (1.47), whose coefficients satisfy Assumptions 1.20, be parameter-elliptic in $\overline{\Sigma}_\varphi$. Then there exists a $\lambda_0>0$ such that for all $\lambda\in\overline{\Sigma}_\varphi$ with $|\lambda|>\lambda_0$, all $f\in W_p^s(\Omega)$, and $g_j\in W_p^{m+s-m_j-1/p}(\partial\Omega)$ $(j=1,\ldots,mN/2)$, the boundary value problem (1.48) admits a unique solution $u\in W_p^{m+s}(\Omega)$ and the a priori estimate

$$|\!|\!|u|\!|\!|_{m+s,p,\Omega}\leq C\left(\|f\|_{s,p,\Omega}+\sum_{j=1}^{mN/2}|\!|\!|g_j|\!|\!|_{m+s-m_j-1/p,p,\partial\Omega}\right)$$

is valid. In particular, the shift $A_{\mathcal{B},p,s}-\lambda_0$ of the W_p^s-realization $A_{\mathcal{B},p,s}$, which is given by

$$\begin{aligned}D(A_{\mathcal{B},p,s})&:=\left\{u\in W_p^{m+s}(\Omega):\gamma_{\partial\Omega}B_j(x,\mathcal{D})u=0,\ j=1,\ldots,mN/2\right\},\\ A_{\mathcal{B},p,s}u&:=A(x,\mathcal{D})u,\quad(u\in D(A_{\mathcal{B},p,s}))\end{aligned}$$

defines a sectorial operator in $L(W_p^s(\Omega))$ with $\left\|\lambda(\lambda-(A_{\mathcal{B},p,s}-\lambda_0))^{-1}\right\|_{L(W_p^s(\Omega))}\leq C$.

1.2.7 Fredholm property

For general elliptic boundary value problems in bounded smooth domains without parameter, the Fredholm property of the underlying operator $(A,\mathcal{B})$ is known in the scales H^s (see [Agr97], Section 2.4 and the references therein) and W_p^s (see [Tri95], Sections 5.4.3 and 5.4.4 and the references therein). For the basic definitions and properties of Fredholm operators we refer to Section A.6 of the Appendix. In the situation of a parameter-dependent boundary value problem in a bounded domain, we will now employ Theorem 1.49, which implies the Fredholm property for $(\lambda-A,\mathcal{B})$ with large $|\lambda|$, and compactness results to deduce unique solvability for parameters with smaller modulus. It will be convenient to abbreviate the function spaces and their norms by

$$\begin{aligned}(X_1,\|\cdot\|_{X_1})&:=\left(W_p^{m+s}(\Omega),\|\cdot\|_{m+s,p,\Omega}\right),\quad\text{and}\\ (X_2,\|\cdot\|_{X_2})&:=\left(W_p^s(\Omega)\times\prod_{j=1}^{mN/2}W_p^{m+s-m_j-1/p}(\partial\Omega),\|\cdot\|_{s,p,\Omega}+\sum_{j=1}^{mN/2}\|\cdot\|_{m+s-m_j-1/p,p,\partial\Omega}\right).\end{aligned}$$

Theorem 1.50
Let $(A, \mathcal{B})$ be a parameter-elliptic boundary value problem in the situation of Theorem 1.49. Then the following assertions hold true:

(i) For $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| > \lambda_0$, the operator corresponding to the boundary value problem

$$(\lambda - A, \mathcal{B}) : X_1 \to X_2, \quad u \mapsto \big(\lambda - A(\cdot, \mathcal{D})u, \gamma_{\partial\Omega} B_1(\cdot, \mathcal{D})u, \dots, \gamma_{\partial\Omega} B_{mN/2}(\cdot, \mathcal{D})u\big)$$

is a Fredholm operator of index 0.

(ii) Suppose that the operator $(\mu - A, \mathcal{B}) : X_1 \to X_2$ is injective for a given $\mu \in \mathbb{C}$. Then for all $f \in W_p^s(\Omega)$ and $g_j \in W_p^{m+s-m_j-1/p}(\partial\Omega)$ $(j = 1, \dots, mN/2)$, the boundary value problem

$$\left\{ \begin{array}{rcll} (\mu - A(x, \mathcal{D}))u & = & f & \text{in } \Omega, \\ \gamma_0 B_j(x, \mathcal{D})u & = & g_j & \text{on } \partial\Omega, \ (j = 1, \dots, mN/2) \end{array} \right. \tag{1.59}$$

admits a unique solution $u \in W^{m+s}(\Omega)$. This solution satisfies the a priori estimate

$$\|u\|_{m+s,p,\Omega} \le C \left(\|f\|_{s,p,\Omega} + \sum_{j=1}^{mN/2} \|g_j\|_{m+s-m_j-1/p,p,\partial\Omega} \right). \tag{1.60}$$

Proof. (i) We obtain $\dim(\ker((\lambda - A, \mathcal{B}))) = \operatorname{codim}(R(\lambda - A, \mathcal{B})) = 0$, since $(\lambda - A, \mathcal{B})$ is an isomorphism for $\lambda \in \overline{\Sigma}_\varphi$ with a large modulus.
(ii) Fix $\lambda \in \overline{\Sigma}_\varphi$ such that $[(\lambda - A), \mathcal{B}]$ is invertible. We write

$$[(\mu - A), \mathcal{B}] = [(\mu - \lambda)\,\mathrm{id}_{(W_p^{m+s}, W_p^s)}, 0, \dots, 0] + [(\lambda - A), \mathcal{B}],$$

where $\mathrm{id}_{(W_p^{m+s}, W_p^s)}$ denotes the embedding $W_p^{m+s}(\Omega) \hookrightarrow W_p^s(\Omega)$, which is compact by Theorem A.1 (ii). Together with (i) and Lemma A.19 (iii), we infer that $[(\mu - A), \mathcal{B}]$ is a Fredholm operator of index 0, too. Now, the injectivity and Lemma A.19 (i) show existence and uniqueness for the problem (1.60). Regarding the a priori estimate, first note that $[(\mu - A), \mathcal{B}] : X_1 \to X_2$ is a continuous linear operator between X_1 and X_2. The proof for this assertion is literally the same as in Lemma 1.24. Then, [Agr94], Theorem 2.3.4 (and the subsequent Remark) shows that the finiteness of $\dim(\ker([(\mu - A), \mathcal{B}]))$ and $\operatorname{codim}([(\mu - A), \mathcal{B}])$ (which implies that $R([(\mu - A), \mathcal{B}])$ is closed, see Lemma A.19 (ii)) is sufficient for an estimate of the form

$$\|u\|_{X_1} = \|u\|_{m+s,p,\Omega} \le C \, \|[(\mu - A), \mathcal{B}]u\|_{X_2} = C \left(\|f\|_{s,p,\Omega} + \sum_{j=1}^{mN/2} \|g_j\|_{m+s-m_j-1/p,p,\partial\Omega} \right)$$

for all $u \in X_1$. This completes the proof. □

Chapter 2

Boundary value problems in rectangular domains

In this chapter, we extend our results to the case of rectangular domains. Whereas we studied problems in the half space to obtain the results for a domain with smooth boundary in the previous chapter, our starting point is now the so-called k-corner $\mathbb{K}^n_k := \mathbb{R}^{n-k} \times (\mathbb{R}_+)^k \subset \mathbb{R}^n$. We will employ reflection techniques in order to extend the given boundary value problem to a problem in the half space. This method leads to several restrictions concerning the principal part of the differential operator as well as the feasible boundary operators. Namely, we will require an even principal part of the differential operator and we are merely restricted to boundary conditions which are of even or odd order.

The boundary $\partial\mathbb{K}^n_k$ of the k-corner splits into k parts. More precisely, $\partial\mathbb{K}^n_k$ is the union of the surfaces

$$\Gamma_j := \left\{x \in \overline{\mathbb{K}^n_k} : x_j = 0\right\} \quad (j = n-k+1, \dots, n). \tag{2.1}$$

We identify $\Gamma_j \cong \overline{\mathbb{K}^{n-1}_{k-1}}$ by skipping the j-th component and $\partial\mathbb{K}^n_k = \bigcup_{j=1}^{k} \Gamma_j$.

2.1 Symmetric boundary value problems

The subsequent definitions are required to specify the class of boundary value problems under consideration. We give them for the general case of non-constant coefficients.

Definition 2.1
Let $\Omega \subset \mathbb{R}^n$ be a domain. A differential operator $A(x, \mathcal{D}) = \sum_{|\alpha|\le 2m} a_\alpha(x)\mathcal{D}^\alpha$ with coefficients $a_\alpha : \Omega \to \mathbb{C}^{N\times N}$ is called even if the implication $a_\alpha \neq 0 \Rightarrow \alpha \in 2\mathbb{N}_0^n$ holds, i.e., all orders of derivatives are even. We say that A has even principal part if A_0 is even.

Regarding the boundary conditions, we restrict ourselves to conditions of even or odd type. For second order differential equations, these types of boundary conditions correspond to Dirichlet and Neumann boundary conditions, respectively. Note that we now have to impose boundary conditions on each part Γ_j of $\partial\mathbb{K}^n_k$ separately.

Definition 2.2
Let $j \in \{n-k+1, \dots, n\}$ and let $\mathcal{B}_j(x, \mathcal{D}) := \left(B_1^{(j)}(x, \mathcal{D}), \dots, B_{mN}^{(j)}(x, \mathcal{D})\right)$ be a tuple of mN boundary operators of the structure (1.2).

(i) We refer to $\mathcal{B}_j$ as an even boundary condition in direction j if

$$\begin{aligned} \mathcal{B}_j &= \left(B^{(j)}_{1,0}, \dots B^{(j)}_{N,0}, B^{(j)}_{1,2}, \dots, B^{(j)}_{N,2}, \dots, B^{(j)}_{1,2m-2}, \dots B^{(j)}_{N,2m-2}\right) \\ \text{with } B^{(j)}_{s,t}(x,\mathcal{D})u &= b^{(j)}_{st}(x)\mathcal{D}_j^t \quad (s = 1, \dots, N;\ t = 0, 2, \dots 2m-2). \end{aligned} \tag{2.2}$$

(ii) We refer to $\mathcal{B}_j$ as an odd boundary condition in direction j if

$$\begin{aligned} \mathcal{B}_j &= \left(B^{(j)}_{1,1}, \dots B^{(j)}_{N,1}, B^{(j)}_{1,3}, \dots, B^{(j)}_{N,3}, \dots, B^{(j)}_{1,2m-1}, \dots B^{(j)}_{N,2m-1}\right) \\ \text{with } B^{(j)}_{s,t}(x,\mathcal{D})u &= b^{(j)}_{st}(x)\mathcal{D}_j^t \quad (s = 1, \dots, N;\ t = 1, 3, \dots 2m-1). \end{aligned} \tag{2.3}$$

Notation 2.3

(i) We introduce a vector $\boldsymbol{\nu} \in \{0,1\}^k$ *which describes the type of the boundary condition on the surface* Γ_{n-k+j} *for* $j = 1, \dots, k$*:*

$$\nu_j := \begin{cases} 0, & \text{if } \mathcal{B}_{n-k+j} \text{ is of even type,} \\ 1, & \text{if } \mathcal{B}_{n-k+j} \text{ is of odd type.} \end{cases}$$

(ii) We write for short $\gamma_{0,j}\mathcal{B}_j(x,\mathcal{D})u = 0$ *on* Γ_j *if and only if* $\gamma_{0,j}B_{st}^{(j)}(x,\mathcal{D}_j)u = 0$ *on* Γ_j *holds for all* $t = \begin{cases} 2k & \text{with } k = 0, \dots, m-1, \text{ if } \nu_{j+k-n} = 0 \\ 2k+1 & \text{with } k = 0, \dots, m-1, \text{ if } \nu_{j+k-n} = 1 \end{cases}$*, and* $s = 1, \dots, N$*. Here,* $\gamma_{0,j}$ *denotes the trace in direction* x_j *for* $j = n-k+1, \dots, n$*.*

Note that in both cases for all $x \in \partial\mathbb{K}_k^n$*, each of the* m *admissible orders* t *appears* N *times with* N *different row vectors* $b_{st}(x)$*, which are necessarily linearly independent for the Shapiro-Lopatinskii condition.*

We first consider the problem $(A, \mathcal{B}) := (A, \mathcal{B}_{n-k+1}, \dots, \mathcal{B}_n)$ in the k-corner $\mathbb{K}_k^n$

$$\left\{ \begin{array}{rcll} (\lambda - A(x,\mathcal{D}))u & = & f & \text{in } \mathbb{K}_k^n, \\ \gamma_{0,j}\mathcal{B}_j(x,\mathcal{D})u & = & 0 & \text{on } \Gamma_j, \ (j = n-k+1, \dots, n), \end{array} \right. \tag{2.4}$$

in the ground space $L_p(\mathbb{K}_k^n)$ under the following

Assumption 2.4

(i) In case of variable coefficients, the smoothness assumptions from Assumption 1.1 (i) are satisfied, with $\mathbb{R}_+^n$ *replaced by* $\mathbb{K}_k^n$*.*

(ii) The differential operator A *has even principal part in the sense of Definition 2.1.*

(iii) A *is parameter-elliptic in a closed sector* $\overline{\Sigma}_\varphi$*, i.e.,* $\det(\lambda - A_0(\xi)) \neq 0$ *for all* $x \in \overline{\mathbb{K}_k^n}$ *and* $(\xi, \lambda) \in \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\}$*.*

(iv) For $j = n-k+1, \dots, n$*, the boundary operators* $\mathcal{B}_j = \left(B_1^{(j)}(\mathcal{D}_j), \dots, B_{mN}^{(j)}(\mathcal{D}_j)\right)$ *acting on* Γ_j *are of even or odd type in the sense of Definition 2.2.*

(v) For $j = n-k+1, \dots, n$ *and all* $x_0 \in \Gamma_j$ *the problem* $(A_0(x_0,\mathcal{D}), \mathcal{B}_j(x_0,\mathcal{D}_j))$ *in local coordinates with respect to* x_0 *satisfies the Shapiro-Lopatinskii condition.*

Definition 2.5
We call the boundary value problem (2.4) *symmetric if Assumption 2.4 is satisfied. We refer to* $(A, \mathcal{B})$ *as a symmetric model problem if* A *and* $\mathcal{B}$ *have constant coefficients and if* A *coincides with its principal part.*

Using Assumption 2.4 (i), a continuity result for the differential operator $(A, \mathcal{B})$ can be proved analogously to Lemmas 1.24 and 1.27. The well-posedness of the symmetric boundary value problem (2.4) is closely related to the study of the resolvent of the underlying operator, to be defined next.

Definition 2.6
Set $D(A_\mathcal{B}) := \left\{u \in W_p^{2m}(\mathbb{K}_k^n; \mathbb{C}^N) : \gamma_{0,j}\mathcal{B}_j u = 0, \ j = n-k+1, \dots, n\right\}$*. Then the* L_p*-realization of* $(A, \mathcal{B})$ *is defined as*

$$A_\mathcal{B} : L_p(\mathbb{K}_k^n; \mathbb{C}^N) \supset D(A_\mathcal{B}) \to L_p(\mathbb{K}_k^n; \mathbb{C}^N), \quad A_\mathcal{B}u := A(x,\mathcal{D})u \quad (u \in D(A_\mathcal{B})).$$

In the sequel, we describe the construction of an appropriate extension operator for the symmetric boundary value problem (2.4). We first define odd and even extension operators that will act in each component x_j for $j = n-k+1, \dots, n$, separately.

Definition and Lemma 2.7
Let $f \in L_p(\mathbb{K}^n_k; \mathbb{C}^N)$. For fixed $j \in \{n-k+1, \dots, n\}$, we then define extensions of f in the direction x_j onto $\mathbb{R}^{n-k} \times (\mathbb{R}_+)^{n-j+k-1} \times \mathbb{R} \times (\mathbb{R}_+)^{j-n}$ by means of

(i) $E_{0,j} : L_p(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^k; \mathbb{C}^N) \to L_p(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^{n-j+k-1} \times \mathbb{R} \times (\mathbb{R}_+)^{j-n}; \mathbb{C}^N)$

$$(E_{0,j}f)(x) := \begin{cases} f(x), & (x_j > 0), \\ -f(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n), & (x_j < 0). \end{cases}$$

(ii) $E_{1,j} : L_p(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^k; \mathbb{C}^N) \to L_p(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^{n-j+k-1} \times \mathbb{R} \times (\mathbb{R}_+)^{j-n}; \mathbb{C}^N)$

$$(E_{1,j}f)(x) := \begin{cases} f(x), & (x_j > 0), \\ f(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n), & (x_j < 0). \end{cases}$$

The following estimates are satisfied:

$$\|E_{i,j}f\|_{0,p,\mathbb{R}^{n-k}\times(\mathbb{R}_+)^{n-j+k-1}\times\mathbb{R}\times(\mathbb{R}_+)^{j-n}} \le 2\|f\|_{0,p,\mathbb{R}^{n-k}\times(\mathbb{R}_+)^k} \quad (i = 0, 1).$$

Remark 2.8
The extension operators from the preceding definition may also be considered in the Sobolev spaces $W^1_{p,0}(\mathbb{K}^n_k; \mathbb{C}^N)$ or $W^1_p(\mathbb{K}^n_k; \mathbb{C}^N)$. Note that the additional zero boundary condition in case of an odd extension is necessary to preserve the regularity by Lemma 1.8. In this case, we obtain continuous extensions

$$\begin{aligned} E_{0,j} : W^1_{p,0}(\mathbb{K}^n_k; \mathbb{C}^N) &\to W^1_{p,0}(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^{n-j+k-1} \times \mathbb{R} \times (\mathbb{R}_+)^{j-n}; \mathbb{C}^N) \\ E_{1,j} : W^1_p(\mathbb{K}^n_k; \mathbb{C}^N) &\to W^1_p(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^{n-j+k-1} \times \mathbb{R} \times (\mathbb{R}_+)^{j-n}; \mathbb{C}^N) \end{aligned}$$

with

$$\|E_{i,j}f\|_{1,p,\mathbb{R}^{n-k}\times(\mathbb{R}_+)^{n-j+k-1}\times\mathbb{R}\times(\mathbb{R}_+)^{j-n}} \le 2\|f\|_{1,p,\mathbb{R}^{n-k}\times(\mathbb{R}_+)^k} \quad (i = 0, 1).$$

In order to construct an extension operator which corresponds to the boundary value problem (2.4) in case of a model problem with $g^{(j)} = 0$, we start with an extension of f in the variable x_{n-k+1} and choose the extension operator $E_{\nu_1,n-k+1}$ where ν_l denotes the l-th entry of the vector $\boldsymbol{\nu}$ from Notation 2.3. We end up with a boundary value problem in $\mathbb{K}^n_{k-1}$ with the right-hand side $E_{i,n-k+1}f \in L_p(\mathbb{K}^n_{k-1})$ and $k-1$ remaining boundary operators $\mathcal{B}_j$ with constant coefficients, which trivially extend to $\{x \in \mathbb{K}^n_{k-1} : x_j = 0\}$. Iterating this procedure for $x_{n-k+2}, \dots, x_n$, we obtain a problem in $\mathbb{R}^n$ with the right-hand side $E_{\nu_k,n}E_{\nu_{k-1},n-1} \cdots E_{\nu_1,n-k+1}f \in L_p(\mathbb{R}^n)$. This procedure defines an extension operator subject to the boundary conditions:

$$E_{\boldsymbol{\nu}} : L_p(\mathbb{K}^n_k; \mathbb{C}^N) \to L_p(\mathbb{R}^n; \mathbb{C}^N), \quad f \mapsto E_{\boldsymbol{\nu}}f := E_{\nu_k,n}E_{\nu_{k-1},n-1} \cdots E_{\nu_1,n-k+1}f \tag{2.5}$$

Clearly, we have $\|E_{\boldsymbol{\nu}}f\|_{0,p,\mathbb{R}^n} \le 2^k\|f\|_{0,p,\mathbb{K}^n_k}$ for all $f \in L_p(\mathbb{K}^n_k)$.

Remark 2.9
In view of the ground spaces $W^1_p(\mathbb{R}_+)$ in directions with Neumann boundary conditions and $W^1_{p,0}(\mathbb{R}_+)$ in directions with Dirichlet boundary conditions, we obtain a continuous extension operator by the procedure described above in the space

$$\mathbb{W}^1_{p,\boldsymbol{\nu}}(\mathbb{K}^n_k) := W^1_p(\mathbb{R}^{n-k}) \times \prod_{j=n-k+1}^{n} W^1_{p,\nu_{j-n+k}}(\mathbb{R}_+),$$

if we read $W^1_{p,1}(\mathbb{R}_+) := W^1_p(\mathbb{R}_+)$. Then the operator $E_{\boldsymbol{\nu}} : \mathbb{W}^1_{p,\boldsymbol{\nu}}(\mathbb{K}^n_k) \to W^1_p(\mathbb{R}^n)$ is continuous with $\|E_{\boldsymbol{\nu}}f\|_{1,p,\mathbb{R}^n} \le 2^k\|f\|_{1,p,\mathbb{K}^n_k}$ for $f \in \mathbb{W}^1_{p,\boldsymbol{\nu}}(\mathbb{K}^n_k)$.

We can now prove the basic theorem for a boundary value problem in a k-corner which is an analogue of Theorem 1.7.

Theorem 2.10
Let $k \leq n$ and let the constant coefficient boundary value problem $(A, \mathcal{B})$ be symmetric in the sense of Definition 2.4. Then for all $f \in L_p(\mathbb{K}_k^n; \mathbb{C}^N)$ and $\lambda \in \overline{\Sigma}_\varphi \setminus \{0\}$, the problem (2.4) with homogeneous boundary conditions admits a unique solution $u \in W_p^{2m}(\mathbb{K}_k^n; \mathbb{C}^N)$. In addition, for $|\lambda| \geq \lambda_0 > 0$ the following a priori estimate holds:

$$|\!|\!| u |\!|\!|_{2m,p,\mathbb{K}_k^n} \leq C \| f \|_{0,p,\mathbb{K}_k^n},$$

with C independent of u, f, and λ.

Proof. Let $\boldsymbol{\nu}$ describe the types of boundary conditions and extend f with the corresponding extension operator $E_{\boldsymbol{\nu}}$ to the whole space $\mathbb{R}^n$. According to Corollary 1.15, for all $\lambda \in \overline{\Sigma}_\varphi \setminus \{0\}$ the problem

$$(\lambda - A(\mathcal{D}))v = E_{\boldsymbol{\nu}} f \quad \text{in } \mathbb{R}^n$$

admits a unique solution $v \in W_p^{2m}(\mathbb{R}^n)$ such that $|\!|\!| v |\!|\!|_{2m,p,\mathbb{R}^n} \leq C \| E_{\boldsymbol{\nu}} f \|_{0,p,\mathbb{R}^n} \leq 2^k C \| f \|_{0,p,\mathbb{K}_k^n}$. We define $u := v|_{\mathbb{K}_k^n}$ by means of restriction. Then $|\!|\!| u |\!|\!|_{2m,p,\mathbb{K}_k^n} \leq 2^k C \| f \|_{0,p,\mathbb{K}_k^n}$, and there remains to check that the boundary conditions are satisfied.

First, assume that $\mathcal{B}_j$ is of even type, i.e., $\nu_{j-n+k} = 0$, and $\mathcal{B}_j$ reads as $\gamma_{0,j} b_{s,t} \mathcal{D}_j^t u = 0$ for $s = 1, \ldots, N$ and $t = 0, 2, \ldots, 2m-2$. We define

$$w(x) := -v(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) \quad (x \in \mathbb{R}^n).$$

Due to Assumption 2.4 (i) and the odd extension of f in direction x_j, we have:

$$(\lambda - A(\mathcal{D}))w(x) = -\left(\lambda - A(\mathcal{D})\right) v(\ldots, -x_j, \ldots) = -(E_{\boldsymbol{\nu}} f)(\ldots, -x_j, \ldots) = E_{\boldsymbol{\nu}} f(x) \quad \text{in } \mathbb{R}^n.$$

Hence, $v = w$ by uniqueness, i.e., v is odd with respect to x_j:

$$v(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = -v(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n).$$

By differentiating this equation with respect to x_j, we obtain for even $t = 0, 2, \ldots, 2m-2$:

$$\mathcal{D}_j^t v(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = -\mathcal{D}_j^t v(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n).$$

Hence, $\gamma_{0,j}(\mathcal{D}_j^t v)(x) = 0$ and the linearity the boundary condition yields $\gamma_{0,j} b_{st}(\mathcal{D}_j^t v)(x) = 0$, whence $\gamma_{0,j} \mathcal{B}_j u = 0$. For the case of an odd boundary condition in direction x_j, i.e., $\nu_{j-n+k} = 1$, we define

$$w(x) := v(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) \quad (x \in \mathbb{R}^n).$$

Again by Assumption 2.4(i) and the even extension of f in this case we find

$$(\lambda - A(\mathcal{D}))w(x) = \left(\lambda - A(\mathcal{D})\right) v(\ldots, -x_j, \ldots) = (E_{\boldsymbol{\nu}} f)(\ldots, -x_j, \ldots) = E_{\boldsymbol{\nu}} f(x)$$

and once more $v = w$ by uniqueness, i.e., v is even with respect to x_j:

$$v(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = v(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n).$$

By differentiating this equation with respect to x_j, we obtain for odd $t = 1, 3, \ldots, 2m-1$:

$$\mathcal{D}_j^t v(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n) = -\mathcal{D}_j^t v(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n).$$

From this $\gamma_{0,j}(\mathcal{D}_j^t v)(x) = 0$ easily follows, whence $\gamma_{0,j} \mathcal{B}_j u = 0$.
On the other hand, assume we are given a solution $v \in W_p^{2m+s}(\mathbb{K}_k^n)$ of the boundary value problem (2.4) with homogeneous boundary conditions.

Then, by the linear independence of the set of vectors b^j_{st} for $s = 1, \dots, N$ and any fixed order of differentiation t, we obtain from $\gamma_{0,j} b^j_{st} \mathcal{D}^t_j v = 0$, that $\gamma_{0,j} \mathcal{D}^t_j v = 0$ either for $t = 1, 3, \dots, 2m-1$ (odd boundary condition) or $t = 0, 2, \dots, 2m-2$ (even boundary condition). If we set $w_j(x) := v(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n)$ for $(x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_n) \in \mathbb{K}^n_k$, this is equivalent to

$$\gamma_{0,j} \mathcal{D}^t_j v = \gamma_{0,j} \mathcal{D}^t_j w_j = (-1)^t \gamma_{0,j} (\mathcal{D}^t_j v)$$

for odd $t \leq 2m-1$, and for even $t \leq 2m-2$ the condition $\gamma_{0,j} \mathcal{D}^t_j v = \gamma_{0,j} \mathcal{D}^t_j w_j$ is trivially satisfied. According to Lemma 1.8, these conditions yield that the even extension $E_{1,j} v$ in direction x_j is an element of $W^{2m+s}_p(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^{n-j+k-1} \times \mathbb{R} \times (\mathbb{R}_+)^{j-n})$. Replacing w_j by $-w_j$ in case of even boundary conditions, we see that the odd extension $E_{0,j} v$ in direction x_j is an element of $W^{2m+s}_p(\mathbb{R}^{n-k} \times (\mathbb{R}_+)^{n-j+k-1} \times \mathbb{R} \times (\mathbb{R}_+)^{j-n})$. Performing the extension procedure successively for $j = n-k+1, \dots, n$, we obtain that $V := E_{\boldsymbol{\nu}} v \in W^{2m+s}_p(\mathbb{R}^n)$. Since A is even, it is easy to see that V solves $(\lambda - A(\mathcal{D}))V = E_\nu f$ in $\mathbb{R}^n$. Thus, the unique solvability of this problem yields uniqueness for the problem in the corner. □

2.2 Transmission problems in a k-corner

For later applications it is desirable to impose transmission conditions across a certain interface which is given here by Γ_n w.l.o.g. In the case of a model problem in an $(n-1)$-dimensional k-corner with $k \leq n-1$, we impose additional transmission conditions across the surface Γ_n. We abbreviate $\mathbb{K}^n_{k+} := \mathbb{K}^{n-1}_k \times (0, \infty)$ and $\mathbb{K}^n_{k-} := \mathbb{K}^{n-1}_k \times (-\infty, 0)$. By the definitions, there holds $\mathbb{K}^n_{k\pm} \cong \mathbb{K}^n_{k+1}$.

Assumption 2.11

(i) Let $\widetilde{A}_1$ be a differential operator of order $2m$ of the form (1.2) acting in $\mathbb{K}^n_{k-}$ and A_2 be a differential operator of the same order acting in $\mathbb{K}^n_{k+}$. We assume that both operators have constant coefficients, coincide with their principal parts, are even in the sense of Definition 2.1 and parameter-elliptic in a sector $\overline{\Sigma}_\varphi$.

(ii) On each part Γ_j of $\partial \mathbb{K}^n_k \cong \mathbb{K}^{n-1}_k$ $(j = n-k+1, \dots, n-1)$, let a set of mN boundary operators $\mathcal{B}_j(\mathcal{D}_j) = (B^{(j)}_1(\mathcal{D}_j), \dots, B^{(j)}_{mN}(\mathcal{D}_j))$ of even or odd type be given which act on the function $u(x) := \begin{cases} \widetilde{u}_1(x), & (x \in \mathbb{K}^n_{k-}) \\ u_2(x), & (x \in \mathbb{K}^n_{k+}) \end{cases}$.

(iii) On the surface Γ_n, which separates $\mathbb{K}^n_{k+}$ and $\mathbb{K}^n_{k-}$, we impose $2mN$ transmission conditions of order at most $2m-1$ of the structure

$$\gamma_{0,n} \left[\widetilde{C}^{(1)}_l(\mathcal{D}) \widetilde{u}_1 - C^{(2)}_l(\mathcal{D}) u_2 \right] = h_l \quad \text{on } \Gamma_n, \quad (l = 1, \dots, 2mN) \tag{2.6}$$

with $C^{(i)}_l(\mathcal{D}) = C^{(i)}_l(\mathcal{D}_1, \dots, \mathcal{D}_{n-k}, \mathcal{D}_n) = \sum_{|\gamma| \leq m_l} c^{(i)}_{l\gamma} \mathcal{D}^\gamma$ and constant coefficients. Recall that we assume $\operatorname{ord}(C^{(1)}_l) = \operatorname{ord}(C^{(2)}_l) = m_l$ here.

(iv) Assume that the following substitute for the Shapiro-Lopatinskii condition is satisfied for the transmission conditions: For all $h_l \in \mathbb{C}$ and $(\lambda, \xi') \in \overline{\Sigma}_\varphi \times \mathbb{R}^{n-1} \setminus \{0\}$, the system of ordinary differential equations

$$\begin{cases} (\lambda - \widetilde{A}_1(\xi', \mathcal{D}_n)) \widetilde{v}_1(x_n) = 0 \quad (x_n < 0), \\ (\lambda - A_2(\xi', \mathcal{D}_n)) v_2(x_n) = 0 \quad (x_n > 0), \\ \left[\widetilde{C}^{(1)}_l(\xi_1, \dots, \xi_{n-k}, \mathcal{D}_n) \widetilde{v}_1 - C^{(2)}_l(\xi_1, \dots, \xi_{n-k}, \mathcal{D}_n) v_2 \right]\Big|_{x_n=0} = h_l \quad (l = 1, \dots, 2mN), \\ \widetilde{v}_1(x_n) \to 0 \quad (x_n \to -\infty), \qquad v_2(x_n) \to 0 \quad (x_n \to +\infty), \end{cases}$$

admits a unique solution.

We now consider the boundary value problem with the transmission conditions in x_n-direction:

$$(2.7)\qquad \begin{cases} (\lambda - \tilde{A}_1(\mathcal{D}))\tilde{u}_1 &= \tilde{f}_1 \quad \text{in } \mathbb{K}^n_{k-}, \\ (\lambda - A_2(\mathcal{D}))u_2 &= f_2 \quad \text{in } \mathbb{K}^n_{k+}, \\ \gamma_{0,j}\mathcal{B}_j(\mathcal{D})u &= 0 \quad \text{on } \Gamma_j,\ (j = n-k+1,\dots,n-1), \\ \gamma_{0,n}\Big[\widetilde{C}^{(1)}_l(\mathcal{D})\widetilde{u}_1 - C^{(2)}_l(\mathcal{D})u_2\Big] &= h_l \quad \text{on } \Gamma_n,\ (l = 1,\dots,2mN). \end{cases}$$

Theorem 2.12
Let Assumption 2.11 be satisfied. Then for all $(\widetilde{f}_1, f_2) \in L_p(\mathbb{K}^n_{k-}) \times L_p(\mathbb{K}^n_{k+})$ and all $\lambda \in \overline{\Sigma}_\varphi$, the problem (2.7) with homogeneous transmission conditions ($h_l = 0$) admits a unique solution $(\tilde{u}_1, u_2) \in W^{2m}_p(\mathbb{K}^n_{k-}) \times W^{2m}_p(\mathbb{K}^n_{k+})$. For $|\lambda| \geq \lambda_0 > 0$ the following a priori estimate holds:

$$\|\tilde{u}_1\|_{2m,p,\mathbb{K}^n_{k-}} + \|u_2\|_{2m,p,\mathbb{K}^n_{k+}} \leq C\left(\|\tilde{f}_1\|_{0,p,\mathbb{K}^n_{k-}} + \|f_2\|_{0,p,\mathbb{K}^n_{k+}}\right).$$

Proof. Let $\boldsymbol{\nu}$ be the vector describing the types of the boundary conditions and $E_{\boldsymbol{\nu}}$ denote the corresponding extension operator. We extend (2.7) to a transmission problem in the half spaces $\mathbb{R}^n_\pm$, where the differential and boundary (or transmission) operators extend trivially because of their constant coefficients:

$$\begin{cases} (\lambda - \widetilde{A}_1(\mathcal{D}))\widetilde{U}_1 &= E_{\boldsymbol{\nu}}\widetilde{f}_1 \quad \text{in } \mathbb{R}^n_-, \\ (\lambda - A_2(\mathcal{D}))U_2 &= E_{\boldsymbol{\nu}}f_2 \quad \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n}\Big[\widetilde{C}^{(1)}_l(\mathcal{D})\widetilde{U}_1 - C^{(2)}_l(\mathcal{D})U_2\Big] &= 0 \quad \text{on } \mathbb{R}^{n-1},\ (l = 1,\dots,2mN). \end{cases}$$

Here, Assumption 2.11 (iii), i.e., the independence of the transmission conditions of the variables $x_{n-k+1},\dots,x_{n-1}$, where the boundary conditions act on, was exploited as well. Applying Theorem 1.9 and Remark 1.10 (iii), we obtain the existence of unique $\widetilde{U_1} \in W^{2m}_p(\mathbb{R}^n_-)$ and $U_2 \in W^{2m}_p(\mathbb{R}^n_+)$, which enjoy the desired estimates in $\mathbb{R}^n_\pm$ for sufficiently large $|\lambda|$. With the same steps as in the proof of Theorem 2.10, we can show that $\gamma_{0,j}\mathcal{B}_j(\mathcal{D})\widetilde{U}_1 = 0$ on $\Gamma_j \cap \mathbb{R}^n_-$ and $\gamma_{0,j}\mathcal{B}_j(\mathcal{D})U_2 = 0$ on $\Gamma_j \cap \mathbb{R}^n_+$. Hence, the restrictions $\widetilde{u}_1 := \widetilde{U}_1\big|_{\mathbb{K}^n_{k-}} \in W^{2m}_p(\mathbb{K}^n_{k-})$ and $u_2 := U_2|_{\mathbb{K}^n_{k+}} \in W^{2m}_p(\mathbb{K}^n_{k+})$ uniquely solve (2.7) and we have

$$\begin{aligned} \|\widetilde{u}_1\|_{2m,p,\mathbb{K}^n_{k-}} + \|u_2\|_{2m,p,\mathbb{K}^n_{k+}} &\leq \|\widetilde{v}_1\|_{2m,p,\mathbb{R}^n_-} + \|v_2\|_{2m,p,\mathbb{R}^n_+} \leq C\left(\|E_{\boldsymbol{\nu}}\widetilde{f}_1\|_{0,p,\mathbb{R}^n_-} + \|E_{\boldsymbol{\nu}}f_2\|_{0,p,\mathbb{R}^n_+}\right) \\ &\leq 2^k C\left(\|\widetilde{f}_1\|_{0,p,\mathbb{K}^n_{k-}} + \|f_2\|_{0,p,\mathbb{K}^n_{k+}}\right). \end{aligned}$$

□

Remark 2.13 (Inhomogeneous transmission conditions)
In order to consider inhomogeneous transmission conditions of the form (2.6) with boundary data $h_l \in W^{2m-m_l-1/p}_p(\Gamma_n)$, additional compatibility conditions on the boundary data have to be imposed. As we wish to argue with the unique solvability of the corresponding transmission problem in $\mathbb{R}^n_\pm$, in order to obtain the even or odd symmetry in the variables x_j for $j = n-k+1,\dots,n-1$, we require that in addition to $U = (\widetilde{U}_1, U_2)$ also $V_+ := U(\dots,-x_j,\dots)$ or $V_- := -U(\dots,-x_j,\dots)$, respectively, defines a solution to the transmission problem, too. Inserting this into the transmission conditions and exploiting that these conditions only act on $x_1,\dots,x_{n-k}$, and x_n, we find the following condition, where $\overline{h}_l \in W^{2m-m_l-1/p}_p(\mathbb{R}^{n-1})$ denotes an extension of h_j:

$$\gamma_{0,n}\Big[\widetilde{C}^{(1)}_l(\mathcal{D})\widetilde{V}_{1,\pm} - C^{(2)}_l(\mathcal{D})V_{2,\pm}\Big] = \pm\overline{h}_l(\dots,-x_j,\dots) \quad \text{on } \mathbb{R}^{n-1},\ (l = 1,\dots,2mN).$$

Accordingly, the compatibility conditions read as $\overline{h}_l(\dots,-x_j,\dots) = \overline{h}_l(\dots,x_j,\dots)$ if $\mathcal{B}_j$ is an odd boundary condition (i.e. $\nu_{j-n+k} = 1$), and $-\overline{h}_l(\dots,-x_j,\dots) = \overline{h}_l(\dots,x_j,\dots)$ if $\mathcal{B}_j$ is an even boundary condition (i.e. $\nu_{j-n+k} = 0$). This means, the extension $\overline{h}_j$ has to be even in case of odd boundary conditions on Γ_j and odd in case of even boundary conditions on Γ_j as well. Hence, the only possible choice of an extension is $\overline{h}_l := E_{\boldsymbol{\nu}}h_l$.

In order to preserve the regularity of a boundary value h_l when applying $E_{\boldsymbol{\nu}}$, conditions of the form $\gamma_{0,j}\mathcal{D}_j^r h_l = 0$ on $\Gamma_j \cap \Gamma_n$ become necessary and sufficient:

Lemma 2.14
Let $h \in W_p^{k-1/p}(\Gamma_n)$ for some $k \in \mathbb{N}$. Then the extension $E_{\boldsymbol{\nu}}h$ is an element of $W_p^{k-1/p}(\mathbb{R}^{n-1})$ if and only if the following compatibility conditions are satisfied for $j = n-k+1, \dots, n-1$:

$$\text{(2.8a)} \quad \gamma_{0,j}\partial_j^r h = 0, \quad \text{on } \Gamma_n \cap \Gamma_j \quad \text{for } r = 1, 3, \dots, 2\lceil \tfrac{k-1}{2} \rceil - 1, \text{ if } \nu_{j-n+k} = 1,$$
$$\text{(2.8b)} \quad \gamma_{0,j}\partial_j^r h = 0, \quad \text{on } \Gamma_n \cap \Gamma_j \quad \text{for } r = 0, 2, \dots, 2\lfloor \tfrac{k-1}{2} \rfloor, \quad \text{if } \nu_{j-n+k} = 0.$$

Proof. We use $\Gamma_n = \{x \in \overline{\mathbb{K}_k^n} : x_n = 0\} \cong \mathbb{R}^{n-k} \times (\mathbb{R}_+)^{k-1}$ and prove the case $\nu_1 = 1$. The remaining extension procedures (for $\nu_2, \dots, \nu_{k-1}$) follow iteratively and the case $\nu_{j-n+k} = 0$ only leads to a change of the signs. With $h_+(x_1, \dots, x_{n-k+1}, \dots, x_{n-1}) := h(x_1, \dots, x_{n-k+1}, \dots, x_{n-1})$ for $x_{n-k+1} > 0$, and $h_-(x_1, \dots, x_{n-k+1}, \dots, x_{n-1}) := h(x_1, \dots, -x_{n-k+1}, \dots, x_{n-1})$ in the case $x_{n-k+1} < 0$, the even extension $E_{1,n-k+1}h : \mathbb{R}^{n-k+1} \times (\mathbb{R}_+)^{k-2} \to \mathbb{C}$ is given by

$$E_{1,n-k+1}h(x) := \begin{cases} h_+(x), & (x_{n-k+1} > 0), \\ h_-(x), & (x_{n-k+1} < 0). \end{cases}$$

Lemma 1.8 yields $E_{1,n-k+1}h(x) \in W_p^{k-1/p}(\mathbb{R}^{n-k+1} \times \mathbb{R}_+^{k-2})$ if and only if for $l = 0, \dots, k-1$

$$0 = \gamma_{0,n-k+1}\left(\partial_{n-k+1}^l h_+ - \partial_{n-k+1}^l h_-\right) = \gamma_{0,n-k+1}\left((\partial_{n-k+1}^l h) + (-1)^{l+1}(\partial_{n-k+1}^l h)\right).$$

This condition is trivially satisfied for even $l \leq k-1$, while for odd $l \leq k-1$ we obtain the condition $2\gamma_{0,n-k+1}\partial_{n-k+1}^l h = 0$, which is (2.8a). □

Roughly speaking, the inhomogeneous transmission values h_l have to satisfy the adjacent boundary conditions on the intersection $\Gamma_j \cap \Gamma_n$, as far as they are well-defined for h_l, too. Note that for an adjacent boundary condition of maximal order $2m-1$ no additional compatibility conditions appear. As a direct consequence of the previous lemma and Theorem 2.12, we obtain

Corollary 2.15
Let Assumption 2.11 be satisfied and let $h_l \in W_p^{2m-m_l-1/p}(\Gamma_n)$ be given for $l = 1, \dots, 2mN$, subject to the compatibility conditions for all $j = n-k+1, \dots, n-1$:

$$\text{(2.9)} \quad \gamma_{0,j}\partial_j^r h_l = 0, \quad \text{on } \Gamma_n \cap \Gamma_j \quad \text{for } r = 1, 3, \dots, 2\lceil \tfrac{2m-m_l-1}{2} \rceil - 1, \text{ if } \nu_{j-n+k} = 1,$$
$$\text{(2.10)} \quad \gamma_{0,j}\partial_j^r h_l = 0, \quad \text{on } \Gamma_n \cap \Gamma_j \quad \text{for } r = 0, 2, \dots, 2\lfloor \tfrac{2m-m_l-1}{2} \rfloor, \quad \text{if } \nu_{j-n+k} = 0.$$

Then for all $(\widetilde{f}_1, f_2) \in L_p(\mathbb{K}_{k-}^n) \times L_p(\mathbb{K}_{k+}^n)$ and all $\lambda \in \overline{\Sigma}_\varphi$, the system (2.7) with inhomogeneous transmission conditions admits a unique solution $(\tilde{u}_1, u_2) \in W_p^{2m}(\mathbb{K}_{k-}^n) \times W_p^{2m}(\mathbb{K}_{k+}^n)$ and for $|\lambda| \geq \lambda_0 > 0$ the following a priori estimate is satisfied:

$$\|\tilde{u}_1\|_{2m,p,\mathbb{K}_{k-}^n} + \|u_2\|_{2m,p,\mathbb{K}_{k+}^n} \leq C\left(\|\tilde{f}_1\|_{0,p,\mathbb{K}_{k-}^n} + \|f_2\|_{0,p,\mathbb{K}_{k+}^n} + \sum_{l=1}^{2mN} \|h_l\|_{2m-m_l-1/p,p,\Gamma_n}\right).$$

Remark 2.16
By using Theorem 1.7 instead of Theorem 1.9, a similar result holds for a general parameter-elliptic system with an even differential operator A of order $2m$, even or odd boundary conditions on $k-1$ surfaces $\Gamma_{n-k+1}, \dots, \Gamma_{n-1}$, and a general set of mN boundary operators $B_j^{(n)}(\mathcal{D}_k)$ $(j = 1, \dots, mN)$ acting on Γ_n such that $\left(A, B_1^{(n)}, \dots, B_{mN}^{(n)}\right)$, considered as a boundary value problem in $\mathbb{R}_+^n$, fulfills the Shapiro-Lopatinskii condition. Here, the boundary operator $B^{(n)}$ may act on the variables $x_1, \dots, x_{n-k}$, and x_n, but not on $x_{n-k+1}, \dots, x_{n-1}$.

2.3 Higher regularity: the ground space $W_p^s(\Omega)$

In this section, we extend the previous results to the case of the ground space $W_p^s(\mathbb{K}_k^n)$ with $s < \frac{1}{p}$ and varying coefficients. We first state our assumptions on the structure and the smoothness of the perturbed boundary value problem under consideration.

Assumption 2.17

(i) The constant coefficient operator of order $2m$

$$A(\mathcal{D}) = \sum_{|\alpha|=2m} a_\alpha \mathcal{D}^\alpha$$

is even in the sense of Definition 2.1, i.e. $a_\alpha \in \mathbb{C}^{N\times N}$ *equals* 0 *if* $\alpha \notin 2\mathbb{N}_0^n$. *In addition,* A *is parameter-elliptic in a closed sector* $\overline{\Sigma}_\varphi$ *(cf. Assumption 1.1).*

(ii) Let $S(x,\mathcal{D})$ *be an operator of the structure*

$$S(x,\mathcal{D}) = \sum_{|\alpha|=2m} s_\alpha(x)\mathcal{D}^\alpha$$

with coefficients $s_\alpha \in BC^t(\overline{\mathbb{K}_k^n};\mathbb{C}^{N\times N})$ *for some* $t \in (s,1)$. *We assume that* S *is a small perturbation of* A *in the sense that* $s_\alpha \equiv 0$ *for* $\alpha \notin 2\mathbb{N}_0^n$ *and* $\sum_{|\alpha|=2m} \|s_\alpha\|_{C^t(\overline{\mathbb{K}_k^n})} < \varepsilon$, *where* ε *will be determined explicitly later.*

(iii) For $j = n-k+1,\dots,n-1$, *the boundary conditions* $\mathcal{B}_j(x,\mathcal{D}_j) = \left(B_1^{(j)}(x,\mathcal{D}),\dots,B_{mN}^{(j)}(x,\mathcal{D})\right)$ *on the surfaces* Γ_j *are of even or odd type in the sense of Definition 2.2, with coefficients* $b_{rt}^{(j)} \in BUC^{2m-t,\alpha_j}(\overline{\mathbb{K}_k^n};\mathbb{C}^{1\times N})$, *where* $\alpha_j \in (s,1)$. *For fixed* j *and every fixed order of differentiation* t *in direction* x_j, *we assume that for all* $x_0 \in \Gamma_j$ *the corresponding* N *vectors* $b_{1t}^{(j)}(x_0),\dots,b_{Nt}^{(j)}(x_0)$ *are linearly independent.*

(iv) The boundary operator $\mathcal{B}_n(\mathcal{D}) = \left(B_1^{(n)}(\mathcal{D}),\dots,B_{mN}^{(n)}(\mathcal{D})\right)$ *acting on* Γ_n *is given by a set of constant coefficient differential operators, which only act on the variables* $x_1,\dots,x_{n-k}$, *and* x_n *and coincide with their principal parts. In addition, the corresponding constant coefficient problem* $\left(A, B_1^{(n)},\dots,B_{mN}^{(n)}\right)$ *in* $\mathbb{R}_+^n$ *satisfies the Shapiro-Lopatinskii condition. The order of* $B_l^{(n)}$ *will be denoted by* $m_{nl} := \operatorname{ord}(B_l^{(n)})$ $(l = 1,\dots,mN)$.

Theorem 2.18
Let $0 < s < \frac{1}{p}$ *and let Assumption 2.17 be satisfied with a sufficiently small* $\varepsilon < \varepsilon_0$ *in (ii). Then for all* $\lambda \in \overline{\Sigma}_\varphi$ *with* $|\lambda| > \lambda_0$ *and all* $f \in W_p^s(\mathbb{K}_k^n;\mathbb{C}^N)$, *there exists a unique solution* $u \in W_p^{2m+s}(\mathbb{K}_k^n;\mathbb{C}^N)$ *to the boundary value problem*

$$\left\{\begin{array}{rcll} (\lambda - (A(\mathcal{D}) + S(x,\mathcal{D})))u &=& f & \text{in } \mathbb{K}_k^n, \\ \gamma_{0,j}\mathcal{B}_j(x,\mathcal{D})u &=& 0 & \text{on } \Gamma_j,\ (j = n-k+1,\dots,n-1), \\ \gamma_{0,n}\mathcal{B}_n(\mathcal{D})u &=& 0 & \text{on } \Gamma_n. \end{array}\right. \tag{2.11}$$

In this situation the solution satisfies the estimate

$$|\!|\!| u |\!|\!|_{2m+s,p,\mathbb{K}_k^n} \le C\|f\|_{s,p,\mathbb{K}_k^n}.$$

Proof. Let $\boldsymbol{\nu} \in \{0,1\}^{k-1}$ denote the vector which describes the types of the boundary conditions for $j = n-k+1,\dots,n-1$. We extend the right-hand side f with $E_{\boldsymbol{\nu}}$ to an element of $W_p^s(\mathbb{R}_+^n;\mathbb{C}^N)$, while we extend all coefficients of S in the variables $x_{n-k+1},\dots,x_{n-1}$

with the even extension $E_{(1,\dots,1)}$. Note that the Hölder continuity of s_α is preserved with $\|E_{(1,\dots,1)}s_\alpha\|_{C^t(\mathbb{R}^n_+;\mathbb{C}^{N\times N})} \le \|s_\alpha\|_{C^t(\overline{\mathbb{K}^n_k};\mathbb{C}^{N\times N})}$ in this extension process, since for $x \neq y$ we have

$$\frac{|E_{(1,\dots,1)}s_\alpha(x) - E_{(1,\dots,1)}s_\alpha(y)|}{|x-y|^t} = \frac{|s_\alpha(\widetilde{x}) - s_\alpha(\widetilde{y})|}{|\widetilde{x}-\widetilde{y}|^t}\left(\frac{|\widetilde{x}-\widetilde{y}|}{|x-y|}\right)^t \le \|s_\alpha\|_{C^t(\overline{\mathbb{K}^n_k};\mathbb{C}^{N\times N})},$$

where $\widetilde{x} := (x_1,\dots,x_{n-k},|x_{n-k+1}|,\dots,|x_{n-1}|,x_n)$ denotes the vector in $\mathbb{K}^n_k$, which corresponds to x and is obtained from x by at most $k-1$ reflections. Hence, the extended differential operator

$$S_1(x,\mathcal{D}) := (E_{(1,\dots,1)}S)(x,\mathcal{D}) := \sum_{|\alpha|=2m}(E_{(1,\dots,1)}s_\alpha)(x)\mathcal{D}^\alpha$$

defines a continuous linear map $S_1 : W_p^{2m+s}(\mathbb{R}^n_+;\mathbb{C}^N) \to W_p^s(\mathbb{R}^n_+;\mathbb{C}^N)$ (cf. the proof of Theorem 1.24) and $\widetilde{f} := E_{\boldsymbol{\nu}}f + S_1(x,\mathcal{D})U \in W_p^s(\mathbb{R}^n_+;\mathbb{C}^N)$ for $U \in W_p^{2m+s}(\mathbb{R}^n_+;\mathbb{C}^N)$. Since A and $\mathcal{B}_n$ trivially extend to $\mathbb{R}^n_+$, we arrive at the following constant coefficient problem in the half space

$$\left\{\begin{array}{rcll}(\lambda - A(\mathcal{D}))U &=& E_{\boldsymbol{\nu}}f + S_1(x,\mathcal{D})U & \text{in } \mathbb{R}^n_+,\\ \gamma_{0,n}\mathcal{B}_n(\mathcal{D})U &=& 0 & \text{on } \mathbb{R}^{n-1}.\end{array}\right. \tag{2.12}$$

By Theorem 1.33, a solution $U \in W_p^{2m+s}(\mathbb{R}^n_+;\mathbb{C}^N)$ of (2.12) satisfies

$$[I - R_s(\lambda)S_1(x,\mathcal{D})]U = E_{\boldsymbol{\nu}}f,$$

where $R_s(\lambda)$ denotes the resolvent of the W_p^s-realization of the constant coefficient problem $(A,\mathcal{B}_n)$. The estimate

$$\begin{aligned}\|\!| R_s(\lambda)S_1(x,\mathcal{D})U |\!\|_{2m+s,p,\mathbb{R}^n_+} &\le C\|S_1(x,\mathcal{D})U\|_{s,p,\mathbb{R}^n_+}\\ &\le C\left(\sum_{|\alpha|=2m}\|s_\alpha\|_{C^t(\overline{\mathbb{K}^n_k})}\right)\|U\|_{2m+s,p,\mathbb{R}^n_+} \le C\varepsilon\,\|\!| U |\!\|_{2m+s,p,\mathbb{R}^n_+}\end{aligned}$$

ensures the existence of $U = [I - R_s(\lambda)S_1(x,\mathcal{D})]^{-1}E_{\boldsymbol{\nu}}f$ for sufficiently small $\varepsilon < \frac{1}{2C}$ and the a priori estimate for $u := U|_{\mathbb{K}^n_k}$ reads as

$$\|\!| u |\!\|_{2m+s,p,\mathbb{K}^n_+} \le \|\!| U |\!\|_{2m+s,p,\mathbb{R}^n_+} \le 2\|E_{\boldsymbol{\nu}}f\|_{s,p,\mathbb{R}^n_+} \le 2^k\|f\|_{s,p,\mathbb{K}^n_k}.$$

Using the symmetry of A and S, respectively, as well as the fact that the boundary operator $\mathcal{B}_n$ does not act on the variables $x_{n-k+1},\dots,x_{n-1}$, we see that

$$w_j(x) := \begin{cases} U(x_1,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_n), & \text{if } \nu_{j-n+k}=1,\\ -U(x_1,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_n), & \text{if } \nu_{j-n+k}=0,\end{cases}$$

defines a solution to (2.12) as well:

$$\begin{aligned}(\lambda - A(D))w_j(x) &= \pm\left(\lambda U(\dots,-x_j,\dots) - (A(D)U)(\dots,-x_j,\dots)\right)\\ &= \pm(E_{\boldsymbol{\nu}}f)(\dots,-x_j,\dots) \pm (S_1((\dots,-x_j,\dots),\mathcal{D})U)(\dots,-x_j,\dots)\\ &= (E_{\boldsymbol{\nu}}f)(x) + S_1(x,\mathcal{D})w_j(x).\end{aligned}$$

Hence, U is even with respect to x_j in case $\nu_{j-n+k}=1$, which implies that the odd boundary conditions are satisfied for U on $\{x \in \mathbb{R}^n_+ : x_j = 0\}$ and for u on Γ_j, respectively. Likewise, U is odd with respect to x_j in case $\nu_{j-n+k}=0$, which yields the corresponding even boundary conditions.

Again, uniqueness follows from the fact that any solution $v \in W_p^{2m+s}(\mathbb{K}^n_k)$ already satisfies $\gamma_{0,j}\mathcal{D}_j^t v = 0$ on Γ_j for $j = n-k+1,\dots,n-1$ and $t = 1,3,\dots,2m-1$ in case of odd, and

for $t = 0, 2, \dots, 2m-2$ in case of even boundary conditions, respectively. In fact, this follows from the linear independence of the vectors $b_{st}^{(j)}(x)$ in the boundary conditions $\mathcal{B}_j$. Hence, the extension operator $E_{\boldsymbol{\nu}}$ leads to a function $V := E_{\boldsymbol{\nu}} v \in W_p^{2m+s}(\mathbb{R}^n_+)$, which solves the corresponding boundary value problem in the half space (2.12). Due to the parameter-ellipticity of the problem $(A, \mathcal{B}_n)$, the solution is unique, and thus the uniqueness for the problem in the corner follows from Corollary 1.44. □

Remark 2.19 *(i) Note that the coefficients of $\mathcal{B}_{n-k+1}, \dots, \mathcal{B}_{n-1}$ could be chosen variable since the method of extension already ensures $\gamma_{0,j}\mathcal{D}^t u = 0$ on Γ_j for $t = 1, 3, \dots, 2m-1$, and the multiplication with $b_{st} \in BUC^{2m-t,\alpha_j}(\overline{\mathbb{K}^n_k}; \mathbb{C}^{1\times N})$ does not change this. The linear independence of every set of N vectors $b_{1t}^{(j)}(x), \dots, b_{Nt}^{(j)}(x)$ for all $x \in \Gamma_j$ which belong to the same order of differentiation in direction x_j is necessary for the uniqueness here.*

(ii) The coefficients of $\mathcal{B}_n$ were chosen to be constant since it is not clear that it is possible to extend them in a way such that their smoothness and the Shapiro-Lopatinskii condition are preserved at the same time, which is necessary to solve the occurring half space problem.

(iii) As indicated in Theorem 2.12, homogeneous transmission conditions of the form (2.6) on Γ_n can be treated as well if both unknown functions satisfy the same boundary conditions on each of the other corresponding parts of $\partial\mathbb{K}^n_{k\pm}$. Indeed, after the change of variables $x_n \mapsto -x_n$ for u_-, we deduce a system (of size $2N$) for $v(x) = (u_-(x', -x_n), u_+(x))^T$ in $\mathbb{K}^n_k$ with even or odd boundary conditions on Γ_j $(j = n-k+1, \dots, n-1)$ and the constant coefficient boundary conditions

$$\gamma_{0,n} B_l^{(n)} v = \gamma_{0,n} \begin{pmatrix} \widetilde{C}_l^{(1)}(\mathcal{D}', -\mathcal{D}_n) & 0 \\ 0 & C_l^{(2)}(\mathcal{D}) \end{pmatrix} \begin{pmatrix} u_-(x', -x_n) \\ u_+(x) \end{pmatrix} = 0 \quad \text{on } \Gamma_n, \ (l = 1, \dots, 2mN).$$

For the particular case of homogeneous canonical transmission conditions, i.e.,

$$B_{rt}^{(n)} v := \begin{pmatrix} (-1)^{l+1} e_r^T \partial_n^t & 0 \\ 0 & e_r^T \partial_n^t \end{pmatrix} \begin{pmatrix} u_-(x', -x_n) \\ u_+(x) \end{pmatrix} = 0, \quad (r = 1, \dots, N; \ t = 0, \dots, 2m-1),$$

we obtain solutions $u_\pm$ in $W_p^{2m+s}(\mathbb{K}^n_{k\pm})$, which together with the transmission conditions define a function $u \in W_p^{2m+s}(\mathbb{K}^n_k)$. This is only possible for $s < 1/p$, because for $1/p < s < 1 + 1/p$ the additional transmission conditions $\gamma_{0,n}\mathcal{D}_n^{2m} e_r^T (u_+ - u_-) = 0$ on Γ_n become necessary to have $u \in W_p^{2m+s}(\mathbb{K}^n_k)$. But this condition is not satisfied in general, since it can no longer be included into the boundary conditions on Γ_n.

As in the L_p-case, some comments and results on inhomogeneous boundary values and lower oder terms in the boundary operators can be stated.

Corollary 2.20 (Inhomogeneous boundary conditions on Γ_n)
Let Assumption 2.17 be satisfied with sufficiently small $\varepsilon < \varepsilon_0$ in (ii). Let $\boldsymbol{\nu} \in \{0,1\}^{k-1}$ be the vector describing the type of the boundary conditions. Let $f \in W_p^s(\mathbb{K}^n_k; \mathbb{C}^N)$ and let boundary values $g_n := (g_1^{(n)}, \dots, g_{mN}^{(n)})^T \in \prod_{l=1}^{mN} W_p^{2m+s-m_{nl}-1/p}(\Gamma_n)$ be given subject to the following compatibility conditions: For all $l = 1, \dots, mN$, there holds

$$\begin{aligned} &\gamma_{0,j}\mathcal{D}_j^r g_l^{(n)} = 0 \quad \text{on } \Gamma_n \cap \Gamma_j, \qquad r = 1, 3, \dots, 2m-1-2\lfloor \tfrac{m_{nl}}{2} \rfloor, \qquad \text{if } \nu_{j-n+k} = 1, \\ &\gamma_{0,j}\mathcal{D}_j^r g_l^{(n)} = 0 \quad \text{on } \Gamma_n \cap \Gamma_j, \qquad r = 0, 2, \dots, 2m-m_{nl}-2, \qquad \text{if } \nu_{j-n+k} = 0. \end{aligned}$$

Then there exists $\lambda_1 > 0$, independent of f and g_n, such that for all $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| \geq \lambda_1$, there exists a unique solution $u \in W_p^{2m+s}(\mathbb{K}^n_k; \mathbb{C}^N)$ to the boundary value problem

$$\left\{ \begin{aligned} (\lambda - (A(\mathcal{D}) + S(x, \mathcal{D})))u &= f \quad \text{in } \mathbb{K}^n_k, \\ \gamma_{0,j}\mathcal{B}_j(x, \mathcal{D})u &= 0 \quad \text{on } \Gamma_j, \ (j = n-k+1, \dots, n-1), \\ \gamma_{0,n}\mathcal{B}_n(\mathcal{D})u &= g_n \quad \text{on } \Gamma_n. \end{aligned} \right.$$

In addition, the following a priori estimate is satisfied:

$$|||u|||_{2m+s,p,\mathbb{K}_k^n} \le C\left(\|f\|_{s,p,\mathbb{K}_k^n} + \sum_{l=1}^{mN} \left|\left|\left| g_l^{(n)} \right|\right|\right|_{2m+s-m_{nl}-1/p,p,\Gamma_n}\right).$$

Proof. Under the given assumptions and according to Lemma 2.14, the extension operator $E_{\boldsymbol{\nu}}$ can be applied to the data f and $g_l^{(n)}$ without loss of any regularity. The coefficients of the even differential operator S are again extended evenly, hence, S extends to the differential operator $S_1 \in L(W_p^{2m+s}(\mathbb{R}_+^n;\mathbb{C}^N), W_p^s(\mathbb{R}_+^n;\mathbb{C}^N))$ (see the proof of Theorem 2.18). We obtain an inhomogeneous boundary value problem in the half space

$$\begin{cases} (\lambda - A(\mathcal{D}))U &= E_{\boldsymbol{\nu}} f + S_1(x,\mathcal{D})U & \text{in } \mathbb{R}_+^n, \\ \gamma_{0,n}\mathcal{B}_l^{(n)}(\mathcal{D})U &= E_{\boldsymbol{\nu}} g_l^{(n)} & \text{on } \mathbb{R}^{n-1},\ (l=1,\dots,mN), \end{cases}$$

which is uniquely solvable for $|\lambda| > \lambda_1$, with $U \in W_p^{2m+s}(\mathbb{R}_+^n)$ (combine the estimates derived in the proofs of the Lemmas 1.45, 1.46, and the preceding Theorem for the existence of an inverse). U admits the representation

$$U = R_s(\lambda)S_1(x,\mathcal{D})U + R_s(\lambda)E_{\boldsymbol{\nu}} f + \sum_{l=1}^{mN} \mathbb{T}_l(\lambda)(E_{\boldsymbol{\nu}} g_l^{(n)}),$$

where R_s denotes the resolvent of the W_p^s-realization of the constant coefficient problem $(A,\mathcal{B}_n)$ (see Theorem 1.33). Furthermore, the desired a priori estimate holds for U in the form

$$|||U|||_{2m+s,p,\mathbb{R}_+^n} \le C\left(\|E_{\boldsymbol{\nu}} f\|_{s,p,\mathbb{R}_+^n} + \sum_{l=1}^{mN} \left|\left|\left| E_{\boldsymbol{\nu}} g_l^{(n)} \right|\right|\right|_{2m+s-m_{nl}-1/p,p,\mathbb{R}^{n-1}}\right).$$

Now, all assertions for $u := U|_{\mathbb{K}_k^n}$ follow analogously to the proof of Theorem 2.18. For the boundary conditions on Γ_j, note that in the case $\nu_{j-n+k} = 0$, there holds

$$\begin{aligned} \gamma_{0,n}B_l^{(n)}(\mathcal{D}_n)\left(-U(\dots,-x_j,\dots)\right) &= -\gamma_{0,n}B_l^{(n)}(\mathcal{D}_n)U(\dots,-x_j,\dots) \\ &= -(E_{\boldsymbol{\nu}} g_l^{(n)})(x_1,\dots,x_{j-1},-x_j,x_{j+1},\dots,x_{n-1}) \\ &= E_{\boldsymbol{\nu}} g_l^{(n)}(x'), \end{aligned}$$

and then uniqueness for the problem in the half space yields that U is odd with respect to x_j. The case of odd boundary conditions is treated in the same way. □

Remark 2.21 (Lower order terms in $\mathcal{B}_n$)
Under the same assumptions as in Corollary 2.20, we can also allow lower order terms in the operators $B_l^{(n)}(\mathcal{D}) = B_{l,0}^{(n)}(\mathcal{D}) + B_{l,low}^{(n)}(\mathcal{D})$ which have constant coefficients and only act on the variables $x_1,\dots,x_{n-k}$, and x_n. Rewriting the boundary condition $\gamma_{0,n}B_l^{(n)}u = g_l$ as $\gamma_{0,n}B_{l,0}^{(n)}u = g_l - \gamma_{0,n}B_{l,low}^{(n)}u = \widetilde{g}_l$, we have to ensure that $\widetilde{g}_l$ satisfies the compatibility conditions. Since g_l is compatible, $\widetilde{g}_l$ satisfies these conditions if and only if for all $j = n-k+1,\dots,n-1$:

$$\gamma_{0,j}\mathcal{D}_j^t\left[\gamma_{0,n}B_{l,low}^{(n)}u\right] = 0 \quad \text{on } \Gamma_j \cap \Gamma_n \quad \text{for } t = \begin{cases} 1,3,\dots,2m-1-2\lfloor\frac{m_l}{2}\rfloor, & \text{if } \nu_{j-n+k}=1, \\ 0,2,\dots,2m-2\lfloor\frac{m_l}{2}\rfloor, & \text{if } \nu_{j-n+k}=0. \end{cases}$$

On the other hand, the boundary condition for u on Γ_j and the linear independence of the vectors $b_{rt}(x)$ ensures that

$$\gamma_{0,j}\mathcal{D}_j^t u = 0 \quad \text{on } \Gamma_j \quad \text{for } t = \begin{cases} 1,3,\dots,2m-1-2\lfloor\frac{m_l}{2}\rfloor, & \text{if } \nu_{j-n+k}=1, \\ 0,2,\dots,2m-2\lfloor\frac{m_l}{2}\rfloor, & \text{if } \nu_{j-n+k}=0. \end{cases}$$

Since $\gamma_{0,j}\mathcal{D}_j^t\left[\gamma_{0,n}B_{l,low}^{(n)}u\right] = \gamma_{0,n}B_{l,low}^{(n)}\left[\gamma_{0,j}\mathcal{D}_j^t u\right]$, which holds because $B_{l,low}^{(n)}$ has constant coefficients and does not act on x_j, the compatibility conditions are satisfied. A similar argument applies for the transmission conditions (cf. Corollary 2.15).

2.4 Localization procedure for a cuboid

In view of applications to elliptic systems (with parameter) which appear in several battery models, it is desirable to take a simplified geometry of a battery into account. So far, our preparations of this section gear towards a rectangular domain $\Omega = \prod_{k=1}^{n}(0, L_k)$ with $L_k > 0$ lying parallel to the coordinate axes. In addition, we now assume the domain to be divided into three rectangular subdomains $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$ by two disjoint, paraxial interfaces Σ_1 and Σ_2, which are given by $\Sigma_i := \{x \in \overline{\Omega} : x_n = \sigma_i\}$, with $0 < \sigma_1 < \sigma_2 < L_n$. In a battery, for instance, these interfaces may represent the surfaces of the electrodes which separate a solid electrode from the liquid electrolyte. Let

$$\Gamma_i := \{x \in \overline{\Omega} : x_i = 0\}, \quad (1 \le i \le n) \quad \text{and} \quad \Gamma_i := \{x \in \overline{\Omega} : x_{i-n} = L_{i-n}\}, \quad (n+1 \le i \le 2n)$$

denote the surfaces of Ω. Note that Γ_i and Γ_{i+n} are parallel to each other and have no points in common. First, we give the precise assumptions on the structure and the smoothness of the boundary value problems under consideration:

Assumption 2.22
Let $1 < p < \infty$, $m \in \mathbb{N}$, $s < 1/p$, and $t \in (s, 1)$.

(i) For $i = 1, 2, 3$, let A_i be differential operators of order $2m$

$$A_i(x, \mathcal{D}) := \sum_{|\alpha| \le 2m} a_{i,\alpha}(x)\mathcal{D}^\alpha \quad (x \in \Omega_i)$$

with coefficients $a_{i,\alpha} \in C^t(\overline{\Omega}_i; \mathbb{C}^{N\times N})$ and even principal part, i.e., $a_{i,\alpha} \equiv 0$ for all α with $|\alpha| = 2m$ and $\alpha \notin 2\mathbb{N}_0$. We define the corresponding differential operator on the whole domain by

$$A(x, \mathcal{D}) = A_i(x, \mathcal{D}) \quad (x \in \Omega_i).$$

(ii) Each operator A_i is parameter-elliptic in a complex sector $\overline{\Sigma}_{\varphi_i}$. We set $\varphi := \min_{1\le i\le 3} \varphi_i$ such that A is parameter-elliptic in $\overline{\Sigma}_\varphi$.

(iii) For each of the surfaces Γ_j, let a corresponding set of boundary operators of the following form be given:

(1) If $j \notin \{n, 2n\}$, the set of boundary conditions $\mathcal{B}_j = \left(B_1^{(j)}(\mathcal{D}_j), \dots, B_{mN}^{(j)}(\mathcal{D}_j)\right)$ on Γ_j is of even or odd type in the sense of Definition 2.2 with constant coefficients. Here, we denote $\mathcal{D}_j = \mathcal{D}_j$ if $i \le n$, and $\mathcal{D}_j := -\mathcal{D}_{j-n}$ for $n < j \le 2n$.

(2) If $j \in \{n, 2n\}$, the boundary conditions $\mathcal{B}_j = \left(B_1^{(j)}(\mathcal{D}_j), \dots, B_{mN}^{(j)}(\mathcal{D}_j)\right)$ have constant coefficients, and on each point $x_0 \in \Gamma_j$, the Shapiro-Lopatinskii condition holds for the constant coefficient problem $\left(A_0(x_0, \cdot), B_{1,0}^{(j)}, \dots, B_{mN,0}^{(j)}\right)$ considered as a problem in the half space $\mathbb{R}^n_+$, i.e., for all $\lambda \in \overline{\Sigma}_\varphi$, $\xi' \in \mathbb{R}^{n-1}$ with $|\lambda| + |\xi'| \ne 0$ and all $g_l \in \mathbb{C}$, the system of ordinary differential equations

$$\text{(2.13)} \qquad \begin{cases} (\lambda - A_0(x_0, \xi', \mathcal{D}_n))v(x_n) &= 0 \quad (x_n > 0), \\ B_{l,0}^{(j)}(\xi', \mathcal{D}_n)v(0) &= g_l \quad (l = 1, \dots, mN/2), \end{cases}$$

admits a unique, stable solution $v \in C([0, \infty); \mathbb{C}^N)$.

(iv) On each of the surfaces Σ_i, we impose $2mN$ general transmission conditions with constant coefficients which only act on the x_n-variable:

$$\gamma_{\Sigma_i}\left[C_l^{(i)}(\mathcal{D}_n)u_i + C_l^{(i+1)}(\mathcal{D}_n)u_{i+1}\right] = 0 \quad \text{on } \Sigma_i \quad (i = 1, 2;\ l = 1, \dots, 2mN).$$

We assume that $\operatorname{ord}(C_l^{(i)}) = \operatorname{ord}(C_l^{(i+1)}) = m_{il} \leq 2m-1$. *In addition, we demand that for all* $x_0 \in \Sigma_i$ *the constant coefficient problems for* $A_i(x_0, \mathcal{D})$ *and* $A_{i+1}(x_0, \mathcal{D})$, *which are coupled through the transmission conditions* $\left(C_1^{(i)}, \ldots, C_{2mN}^{(i+1)}\right)$, *satisfy the following substitute for the Shapiro-Lopatinskii condition: For all* $x_0 \in \Sigma_i, (\lambda, \xi') \in \overline{\Sigma}_\varphi \times \mathbb{R}^{n-1} \setminus \{0\}$, *and all* $g_l \in \mathbb{C}$, *the system of ordinary differential equations*

$$
\left\{
\begin{array}{rcll}
(\lambda - A_{i,0}(x_0, \xi', \mathcal{D}_n))v_i(x_n) & = & 0 & (x_n < 0), \\
(\lambda - A_{i+1,0}(x_0, \xi', \mathcal{D}_n))v_{i+1}(x_n) & = & 0 & (x_n > 0), \\
\left[C_{l,0}^{(i)}(\mathcal{D}_n)v_i + C_{l,0}^{(i+1)}(\mathcal{D}_n)v_{i+1}\right]\Big|_{x_n=0} & = & g_l & (l = 1, \ldots, 2mN), \\
v_i(x_n) \to 0 \quad (x_n \to -\infty), \qquad v_{i+1}(x_n) & \to & 0 & (x_n \to +\infty),
\end{array}
\right. \tag{2.14}
$$

which is rewritten in local coordinates with respect to x_0, *admits a unique solution* (v_i, v_{i+1}).

Under the above conditions, we now consider the following system for given $f \in W_p^s(\Omega)$:

$$
\left\{
\begin{array}{rcll}
(\lambda - A(x, \mathcal{D}))u & = & f & \text{in } \Omega, \\
\gamma_{\partial\Omega} B_l^{(j)}(\mathcal{D}_j)u & = & 0 & \text{on } \Gamma_j, \ (j = 1, \ldots, 2n; \ l = 1, \ldots, mN), \\
\gamma_{\Sigma_i}\left[C_l^{(i)}(\mathcal{D}_n)u_i + C_l^{(i+1)}(\mathcal{D}_n)u_{i+1}\right] & = & 0 & \text{on } \Sigma_i, \ (i = 1, 2; \ l = 1, \ldots, 2mN).
\end{array}
\right. \tag{2.15}
$$

Similar to the approach for a smooth domain, we first have to find a suitable neighbourhood U_{x_0} for a point $x_0 \in \partial\Omega$ where we can define a corresponding local problem which satisfies the assumptions of Theorem 2.18. The localization at different points of $\partial\Omega$ will lead to several different types of boundary value problems in an n-corner or on the edges (lower dimensional corners), which have already been treated in the previous sections.
The rectangular geometry somewhat simplifies the situation because the mapping from a neighbourhood U_{x_0} of a boundary point to a neighbourhood V_{x_0} of the corresponding k-corner is trivial in contrast to the mapping $\mathcal{J}_{x_0}$ (cf. Remark 1.48). Instead of spheres with respect to $|\cdot|_2$ in $\mathbb{R}^n$, which we used for the covering of the smooth domain, cubes, i.e., spheres with respect to $|\cdot|_\infty$, seem to be more appropriate this time.

Step 1 - Choice of grid and neighbourhoods. Fix a point $x_0 \in \partial\Omega$. Then there exists a $\delta_0 > 0$ such that, up to a rotation of the coordinate axes, in the intersection of the cube $Q(x_0, \delta_0) := \{x \in \mathbb{R}^n : |x - x_0|_\infty < \delta_0\}$ with $\overline{\Omega}$ we are in exactly one of the following situations:

Ⓘ x_0 is one of the $2n$ corners of $\overline{\Omega}$. This will lead to a boundary value problem in the n-corner $\mathbb{K}_n^n = (0, \infty)^n$ with $n-1$ sets of even or odd boundary conditions (cf. Assumption 2.22 (iii), (1)) and one set of general boundary conditions (cf. Assumption 2.22 (iii), (2)).

Ⓘⓘ x_0 lies on one of the edges of $\overline{\Omega}$, namely $x_0 \in \bigcap_{l=1}^k \Gamma_{i_l}$ for some $2 \leq k < n$, but $x_0 \notin \Sigma_i$. This leads to a boundary value problem in a k-corner with k sets of even or odd boundary conditions.

Ⓘⓘⓘ x_0 lies inside a surface, i.e., $x_0 \in \Gamma_i$ for exactly one $i \in \{1, \ldots, 2n\}$, but $x_0 \notin \Sigma_i$. This situation leads to a local boundary value problem in the half space.

Ⓘⓥ $x_0 \in \Sigma_i$ for one $i \in \{1, 2\}$ and x_0 lies on an edge of $\partial\Omega$, i.e. $x_0 \in \bigcap_{l=1}^k \Gamma_{i_l}$ for some $2 \leq k < n$. Then we are in the situation of a divided k-corner $\mathbb{K}_{k,\pm}^n$ with even or odd boundary conditions on k surfaces and additional transmission conditions in x_n-direction.

Ⓥ $x_0 \in \Sigma_i$ for one $i \in \{1, 2\}$ and $x_0 \in \Gamma_j$ for exactly one $j \in \{1, \ldots, 2n\}$. Then we have the case of a divided corner $\mathbb{K}_{1\pm}^n$ with transmission conditions in x_n-direction and one set of even or odd boundary conditions on $\{x \in \overline{\mathbb{K}_1^n} : x_{n-1} = 0\}$.

Ⓥⓘ x_0 lies inside of an interface, but not on the boundary, i.e., $x_0 \in \Sigma_i$ for one $i \in \{1, 2\}$ and $x_0 \notin \Gamma_j$. This leads to a transmission problem in the half spaces $\mathbb{R}_\pm^n$.

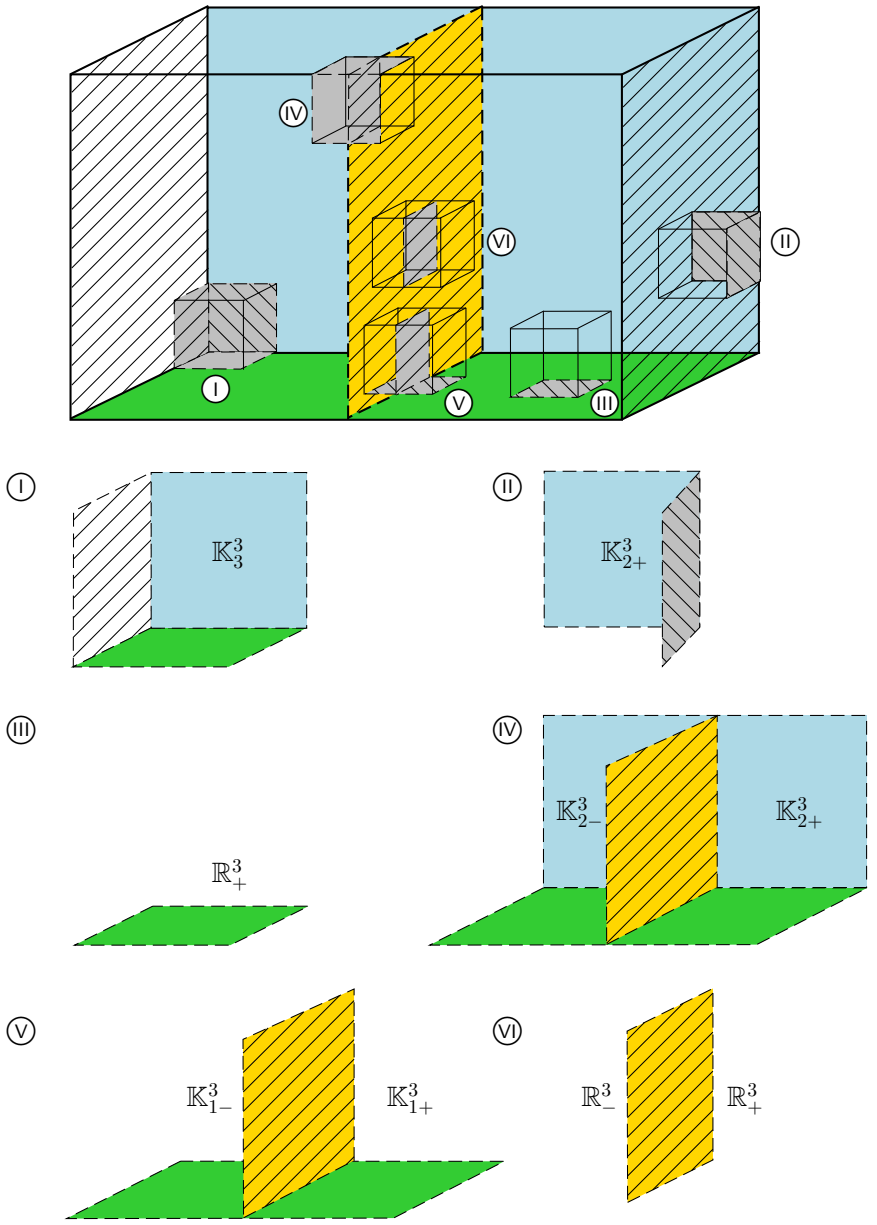

Figure 2.1: A divided cuboid and the corresponding local problems for the case $n = 3$.

Next, we extend the coefficients of A_i from $Q(x_0, \delta_0)$ to the whole unbounded corner $\mathbb{K}^n_k$ in the cases Ⅰ-Ⅲ and in the cases Ⅳ-Ⅵ, we extend the coefficients of both participating operators from $W_-(x_0, \delta_0) := Q(x_0, \delta_0) \cap \Omega_i$ and $W_+(x_0, \delta_0) := Q(x_0, \delta_0) \cap \Omega_{i+1}$ to their corresponding half-corner $\mathbb{K}^n_{k\pm}$. The extension procedure is the same as that for smooth domains (cf. page 40), i.e., we define

$$a^{(x_0)}_{\alpha,i}(x) := a_{\alpha,i}\left(x_0 + \chi\left(\frac{x - x_0}{r}\right)(x - x_0)\right),$$

where $\chi \in C^\infty_0(\mathbb{R}^n)$ with $0 \le \chi \le 1$ such that $\chi \equiv 1$ for $|x|_\infty \le 1$ and $\chi \equiv 0$ for $|x|_\infty \ge 2$ and the radius $r \in (0, \delta_0)$ can still be chosen sufficiently small. It was demonstrated above that we obtain coefficients $a^{(x_0)}_{\alpha,i} \in C^t(\overline{\mathbb{K}^n_k}; \mathbb{C}^{N\times N})$ and $a^{(x_0)}_{\alpha,i} \in C^t(\overline{\mathbb{K}^n_{k\pm}}; \mathbb{C}^{N\times N})$ by this extension procedure, respectively. Given $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon) \le \delta_0$ such that we have the following estimate for the variable top order coefficients of A_i:

$$\sum_{|\alpha|=2m} \|a^{(x_0)}_{\alpha,i}(x) - a^{(x_0)}_{\alpha,i}(x_0)\|_{C^{t'}(\overline{\mathbb{K}^n_k};\mathbb{C}^{N\times N})} < \varepsilon.$$

This holds with a Hölder index $t' \in (s,t)$, as we have already seen in the estimate on page 40. Note that $a_{\alpha,i}^{(x_0)}(x) - a_{\alpha,i}^{(x_0)}(x_0) = 0$ for $|x - x_0| > 2r$. In the cases Ⓘ-Ⓜ, where a cube intersects with two of the subdomains Ω_i and Ω_{i+1}, these estimates hold for each of the operators separately but remain valid for the $\mathbb{C}^{2N\times 2N}$ valued coefficients

$$a_{\alpha}^{(x_0)}(x) := \begin{pmatrix} a_{\alpha,i}^{(x_0)}(x',-x_n) & 0_{N\times N} \\ 0_{N\times N} & a_{\alpha,i+1}^{(x_0)}(x) \end{pmatrix} \quad (x \in Q(x_0,\delta_1) \cap \Omega_{i+1})$$

of the corresponding system in the $(k+1)$-corner (or the 2-corner or the half space, respectively).

We will now choose grid points on $\partial\Omega$ to construct a finite covering of the boundary. To this end, we start with all $2n$ corners $x_1 := 0, \dots, x_{2n}$, and all points $x_{2n+1}, \dots, x_{M_1}$, where a one-dimensional edge (i.e., the intersection of $n-1$ of the surfaces Γ_j) intersects with one of the $(n-1)$-dimensional interfaces Σ_i. In the first case, we will obtain a local problem in an n-corner (case Ⓘ), and in the second case, we will have to deal with a problem in an $(n-1)$-corner with additional transmission conditions (case Ⓜ). By the compactness of $\Sigma_i \cap \partial\Omega$, we can choose finitely many points $x_{M_1+1}, \dots, x_{M_2} \in \Sigma_i \cap \partial\Omega$ such that $\Sigma_i \cap \partial\Omega \subset \bigcup_{k=2n+1}^{M_2} Q(x_k,\delta_k)$ and for $k = M_1+1, \dots, M_2$, in each part $Q(x_k,\delta_k)\cap\overline{\Omega}$ we have the situations Ⓜ (with $k < n-1$) or Ⓥ. To cover the interfaces Σ_i, we choose additional points $x_{M_2+1}, \dots, x_{M_3} \in \Sigma_i$ such that $x_k \notin Q(x_l,\delta_l)$ if $k \neq l$ for $k,l \in \{1,\dots,M_3\}$ and the corresponding local problems are transmission problems in $\mathbb{R}^n_\pm$ (case Ⓥ). Finally, we add finitely many points $x_{M_3+1}, \dots, x_{M_4}$ which lie on the edges or surfaces of $\partial\Omega$ with $x_k \notin Q(x_l,\delta_l)$ if $k \neq l$ for $k,l \in \{1,\dots,M_4\}$ such that we have the situations Ⓘ or Ⓘ in these cases. Then we have $\partial\Omega \subset \bigcup_{k=1}^{M_3} Q(x_k,\delta_k)$. By adding finitely many points and making the radius smaller than $\min_{1\le k\le M_4}\delta_k$, we may assume w.l.o.g. that all cubes have the same diameter δ. Using the compactness of $\overline{\Omega} \setminus \bigcup_{k=1}^{M_4} Q(x_k,\delta)$, we can add interior points $x_{M_4+1} \dots, x_M \in \Omega$ such that (w.l.o.g.) $\{x_1,\dots,x_M\}$ forms a grid on $\overline{\Omega}$ and

$$\overline{\Omega} = \bigcup_{k=1}^{M} Q(x_k,\delta) \cap \overline{\Omega}.$$

Step 2 - Cutoff functions, local operators, and resolvents. In the sequel, we will write $Q_k := Q(x_k,\delta)$ for short. Note that $Q_k = (x_k - x_l) + Q_l$, and by construction at most $2n$ of the cubes intersect at one point. Subordinated to the open covering of $\overline{\Omega}$ with cubes, we choose a partition of unity φ_k $(k = 1,\dots,M)$ with the following properties:

(i) $\varphi_k \in C^\infty(\mathbb{R}^n)$, with $\operatorname{supp}(\varphi_k) \subset Q_k$,

(ii) $\varphi_k \equiv 1$ on $\widetilde{Q}_k := \{x \in Q_k : |x - x_k|_\infty < \frac{1}{3}\delta\}$,

(iii) $\sum_{k=1}^{M} \varphi_k(x) = 1$ for all $x \in \overline{\Omega}$.

Note that, if $x_k \in \Gamma_i$, the corresponding cutoff function φ_k is independent of x_i (or x_{i-n} if $i > n$) in a neighbourhood of x_k, namely in $\{x \in \Omega : \operatorname{dist}(x,\Gamma_i) < \frac{1}{3}\delta\}$. This situation is illustrated for the two-dimensional case in the Figures 2.2 and 2.3 below and will become important when we consider commutators for the boundary operators.

Multiplying equations (2.13) by φ_k, we obtain a boundary value problem in a corner without transmission conditions in the cases $k = 1,\dots,2n, M_1+1,\dots,M_2$, and including transmission conditions in cases $k = 2n+1,\dots,M_3$. We find a local transmission problem in the half spaces $\mathbb{R}^n_\pm$ if $M_2+1 \le k \le M_3$ and, as in the localization procedure for the smooth domain, we have a local problem in $\mathbb{R}^n$ if $M_4+1 \le k \le M$. In order to construct a representation of a solution to (2.13) we will employ the solutions of corresponding prolonged local problems and a partition of unity.

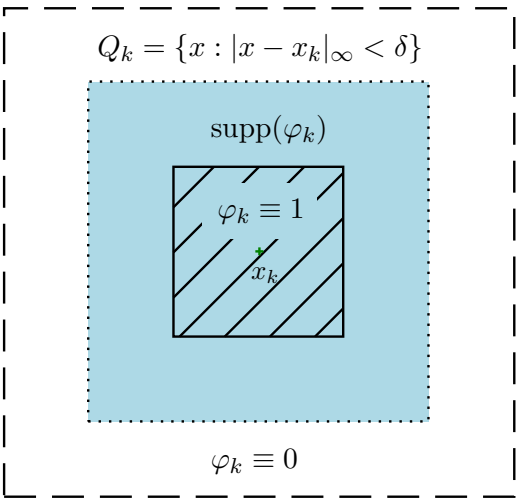

Figure 2.2: The cube Q_k and areas with values of the cutoff function φ_k.

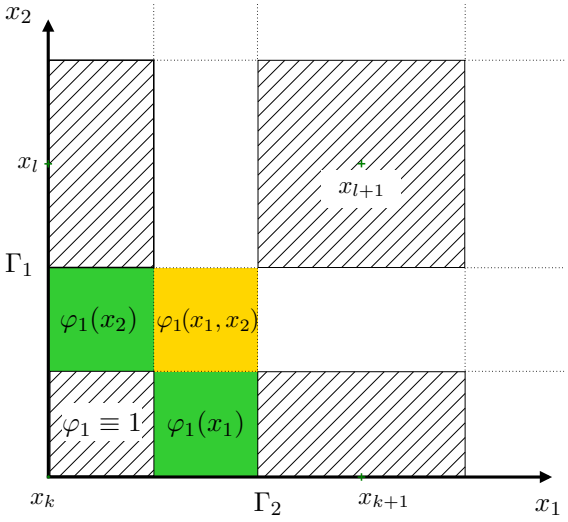

Figure 2.3: Areas of dependencies of the cutoff function φ_1.

Definition 2.23 (Local operators)
We define local differential operators $A^{(k)}$ as follows:

(i) For $k = 1, \dots, 2n$ and $k = M_3 + 1, \dots, M_4$, we have $Q_k \cap \partial\Omega \cap \overline{\Omega}_i \neq \emptyset$ for exactly one $i \in \{1, 2, 3\}$. For this situation, we define

$$A^{(k)}(x, \mathcal{D})u := \sum_{|\alpha|=2m} a_{\alpha,i}^{(x_k)}(x)\mathcal{D}^\alpha u \quad (u \in W_p^{2m+s}(G_k; \mathbb{C}^N)),$$

where G_k denotes the corresponding unbounded corner $\mathbb{K}_l^n$ with $1 \leq l \leq n$.

(ii) In case $Q_k \subset \mathring{\Omega}_i$ for exactly one $i \in \{1, 2, 3\}$ (i.e., for $k = M_4 + 1, \dots, M$) we define the operator $A^{(k)}$ in $\mathbb{R}^n$ in the same way.

(iii) For $k = 2n + 1, \dots, M_3$, we have additional transmission conditions across Σ_i, i.e., Q_k intersects with Ω_i and Ω_{i+1} for some $i \in \{1, 2\}$. Then we define

$$\begin{aligned} A_-^{(k)}(x, \mathcal{D})u_- &:= \sum_{|\alpha|=2m} a_{\alpha,i}^{(x_k)}(x)\mathcal{D}^\alpha u_- \quad (u_- \in W_p^{2m+s}(\mathbb{K}_{l-}^n; \mathbb{C}^N)), \\ A_+^{(k)}(x, \mathcal{D})u_+ &:= \sum_{|\alpha|=2m} a_{\alpha,i+1}^{(x_k)}(x)\mathcal{D}^\alpha u_+ \quad (u_+ \in W_p^{2m+s}(\mathbb{K}_{l+}^n; \mathbb{C}^N)). \end{aligned}$$

We also define the differential operator for the corresponding system in $\mathbb{K}_{l+}^n \cong \mathbb{K}_{l+1}^n$ by

$$A^{(k)}(x, \mathcal{D})v := \sum_{|\alpha|=2m} a_\alpha^{(x_k)}(x)\mathcal{D}^\alpha v \quad (v \in W_p^{2m+s}(\mathbb{K}_{l+}^n; \mathbb{C}^{2N}))$$

with coefficients

$$a_\alpha^{(x_k)}(x) := \begin{pmatrix} a_{\alpha,i}^{(x_k)}(x', -x_n) & 0 \\ 0 & a_{\alpha,i+1}^{(x_k)}(x', x_n) \end{pmatrix} \in C^t(\overline{\mathbb{K}_{l,+}^n}; \mathbb{C}^{2N \times 2N}), \quad v := \begin{pmatrix} u_-(x', -x_n) \\ u_+(x) \end{pmatrix}.$$

Note that, for a sufficiently small choice of δ, in (iii) we still may assume

$$\sum_{|\alpha|=2m} \|a_\alpha^{(x_k)}(x) - a_\alpha^{(x_k)}(x_k)\|_{C^{t'}(\overline{\mathbb{K}_{l+}^n}; \mathbb{C}^{2N \times 2N})} < \varepsilon.$$

Lemma 2.24 (Local resolvents)
Let Assumptions 2.22 be satisfied. Then for all $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus, the following local problems admit a unique solution:

(i) For $k = 1, \dots, 2n$, the boundary value problem in the n-corner $\mathbb{K}_n^n$ $(1 \le l \le n)$ with $n-1$ even or odd and one general boundary condition as well as for $k = M_3+1, \dots, M_4$, the boundary value problem in the l-corner $\mathbb{K}_l^n$ with $(1 \le l < n)$, for given $f \in W_p^s(\mathbb{K}_l^n; \mathbb{C}^N)$:

$$\left\{ \begin{array}{rcll} (\lambda - A^{(k)}(x,\mathcal{D}))u & = & f & \text{in } \mathbb{K}_l^n, \\ \gamma_{0,i_r} B_j^{(i_r)}(\mathcal{D}_{i_r})u & = & 0 & \text{on } \widetilde{\Gamma}_{i_r},\ (r = 1, \dots, l;\ j = 1, \dots, mN). \end{array} \right.$$

Here, i_r denotes the indices of the boundary conditions which are imposed on the surfaces Γ_{i_r} that form the original corner of Ω, and the surfaces $\widetilde{\Gamma}_{i_r} := \{x \in \mathbb{K}_l^n : x_{i_r} = 0\}$ are the natural extensions of these surfaces to the unbounded corner $\mathbb{K}_l^n$.

(ii) For $k = 2n+1, \dots, M_2$, the boundary value problem in the divided l-corners $\mathbb{K}_{l\pm}^n$ where $1 \le l < n$, with additional transmission conditions in x_n-direction which reads as

$$\left\{ \begin{array}{rcll} (\lambda - A_-^{(k)}(x,\mathcal{D}))u_- & = & f_- & \text{in } \mathbb{K}_{l-}^n, \\ (\lambda - A_+^{(k)}(x,\mathcal{D}))u_+ & = & f_+ & \text{in } \mathbb{K}_{l+}^n, \\ \gamma_{0,i_r} B_j^{(i_r)}(\mathcal{D}_{i_r})u & = & 0 & \text{on } \partial\mathbb{K}_{l\pm}^n,\ (r = 1, \dots, l;\ j = 1, \dots, mN), \\ \gamma_{0,n}\left[C_l^{(-)}(\mathcal{D}_n)u_- + C_l^{(+)}(\mathcal{D}_n)u_+ \right] & = & 0 & \text{on } \widetilde{\Gamma}_n, \quad (l = 1, \dots, 2mN). \end{array} \right.$$

Here, of course, $u(x) := \begin{cases} u_+(x), & (x \in \mathbb{K}_{l+}^n) \\ u_-(x), & (x \in \mathbb{K}_{l-}^n) \end{cases}$, and $f_\pm \in W_p^s(\mathbb{K}_{l\pm}^n; \mathbb{C}^N)$. The assertion remains also valid for $k = M_2+1, \dots, M_3$, i.e., for a transmission problem in the half spaces $\mathbb{R}_\pm^n$.

(iii) For $k = M_4+1, \dots, M$, the problem in the full space $(\lambda - A^{(k)}(x,\mathcal{D}))u = f$ in $\mathbb{R}^n$ with $f \in W_p^s(\mathbb{R}^n; \mathbb{C}^N)$.

This gives rise to the local resolvents $R^{(k)}(\lambda) : W_p^s \to W_p^{2m+s}$ and for $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| \ge \lambda_0 > 0$ we have the estimate

$$\left\| R^{(k)}(\lambda) f \right\|_{m+s,p} \le C \|f\|_{s,p}.$$

Proof. We apply Theorem 2.18 with $A(\mathcal{D}) := A^{(k)}(x_k, \mathcal{D})$ and $S(x,\mathcal{D}) := A^{(k)}(x,\mathcal{D}) - A^{(k)}(x_k,\mathcal{D})$. By construction and an appropriate choice of the grid parameter δ, we see that Assumption 2.17 (ii) is satisfied for S. Hence, part (i) immediately follows. For (ii) we can consider the corresponding diagonal system in $\mathbb{K}_{l+}^n$, for $v(x) := (u_-(x', -x_n), u_+(x))^T$ and

$$A(x,\mathcal{D}) := \begin{pmatrix} A_-^{(k)}(x', -x_n, \mathcal{D}', -\mathcal{D}_n) & 0_{N\times N} \\ 0_{N\times N} & A_+^{(k)}(x,\mathcal{D}) \end{pmatrix}, \quad f(x) := \begin{pmatrix} f_-(x', -x_n) \\ f_+(x) \end{pmatrix} \quad (x \in \mathbb{K}_{l+}^n),$$

which then reads as

$$\left\{ \begin{array}{rcll} (\lambda - A(x,\mathcal{D}))v & = & f & \text{in } \mathbb{K}_{l+}^n, \\ \gamma_{0,i_r}(B_j^{(i_r)}(\mathcal{D}_{i_r}), B_j^{(i_r)}(\mathcal{D}_{i_r}))v & = & 0 & \text{on } \partial\mathbb{K}_{l+}^n, \quad (r = 1, \dots, l;\ j = 1, \dots, mN), \\ \gamma_{0,n}\left((-1)^{m_l} C_l^{(-)}(\mathcal{D}_n), C_l^{(+)}(\mathcal{D}_n) \right) v & = & 0 & \text{on } \widetilde{\Gamma}_n, \quad (l = 1, \dots, 2mN). \end{array} \right.$$

Note that the form of the boundary conditions for u_- remains unchanged here, since they are independent of x_n. The new boundary condition on Γ_n satisfies the Shapiro-Lopatinskii condition due to Assumption 2.22 (iv), whence (ii) is now a consequence of (i). Part (iii) follows from [KS12], Corollary 5.31. □

Step 3 - Construction of the resolvent. Here, we merely consider the structural most complicated type of local problems which is definitely the case when additional transmission conditions are considered for a problem in a divided l-corner. We denote the corresponding sets of boundary operators by $B^{(j)}(\mathcal{D}_j)$, $(j = 1, \dots, l)$ instead of $B^{(i_r)}(\mathcal{D}_{i_r})$, $(r = 1, \dots, l)$ and use the same notation for the surfaces Γ_{i_r}. Namely, the multiplication of (2.15) with φ_k for some $k \in \{2n+1, \dots, M_2\}$ leads to a problem of the following structure:

$$\begin{cases} \varphi_k(\lambda - A_i(x,\mathcal{D}))u_i &=& \varphi_k f_i & \text{in } \Omega_i \cap Q_k, \\ \gamma_{0,j}\varphi_k B_r^{(j)}(\mathcal{D}_j)u_i &=& 0 & \text{on } \Gamma_j \cap \partial\Omega_i \cap Q_k,\ (r = 1, \dots, mN), \\ \varphi_k(\lambda - A_{i+1}(x,\mathcal{D}))u_{i+1} &=& \varphi_k f_{i+1} & \text{in } \Omega_{i+1} \cap Q_k, \\ \gamma_{0,j}\varphi_k B_r^{(j)}(\mathcal{D}_j)u_{i+1} &=& 0 & \text{on } \Gamma_j \cap \partial\Omega_{i+1} \cap Q_k,\ (r = 1, \dots, mN), \\ \gamma_{0,n}\varphi_k\Big[C_r^{(i)}(\mathcal{D}_n)u_i + C_r^{(i+1)}(\mathcal{D}_n)u_{i+1}\Big] &=& 0 & \text{on } \Sigma_i \cap Q_k, \quad (r = 1, \dots, 2mN). \end{cases}$$

Note that the boundary conditions on Γ_j are the same for u_i and u_{i+1} (only on different parts of the surfaces Γ_j). To obtain equations for $\varphi_k u_i$ and $\varphi_k u_{i+1}$, we wish to interchange φ_k with the differential operators. For the equation, we thus obtain a commutator $[A_i(x,\mathcal{D}), \varphi_k]$ and $[A_{i+1}(x,\mathcal{D}), \varphi_k]$ on the right-hand side, but for the boundary and transmission conditions we can exploit the special structure of the cutoff function: Since φ_k is independent of x_j in a neighbourhood of Γ_j the corresponding commutators vanish: $[B_r^{(j)}(\mathcal{D}_j), \varphi_k] = 0$ for $j = 1, \dots, l$. Furthermore, φ_k can be chosen constant in the x_n-direction across the interface, i.e., in an neighbourhood of $Q_k \cap \Sigma$ and consequently $\gamma_\Sigma \mathcal{D}_n^j \varphi_k = 0$ so that φ_k commutes with the transmission conditions which only act on the x_n-variable. These changes lead to the system

$$\begin{cases} (\lambda - A_i(x,\mathcal{D}))(\varphi_k u_i) &=& \varphi_k f_i + [A_i(x,\mathcal{D}), \varphi_k]u_i & \text{in } \Omega_i \cap Q_k, \\ (\lambda - A_{i+1}(x,\mathcal{D}))(\varphi_k u_{i+1}) &=& \varphi_k f_{i+1} + [A_{i+1}(x,\mathcal{D}), \varphi_k]u_{i+1} & \text{in } \Omega_{i+1} \cap Q_k, \\ \gamma_{0,j} B_r^{(j)}(\mathcal{D}_j)(\varphi_k u) &=& 0 \quad \text{on } \Gamma_j \cap \overline{Q}_k,\ (j = 1, \dots, l;\ r = 1, \dots, mN), \\ \gamma_{0,n}\Big[C_r^{(i)}(\mathcal{D}_n)\varphi_k u_i + C_r^{(i+1)}(\mathcal{D}_n)\varphi_k u_{i+1}\Big] &=& 0 & \text{on } \Sigma_i \cap Q_k,\ (r = 1, \dots, 2mN). \end{cases}$$

Finally, we replace $A_- := A_i(x,\mathcal{D})$ and $A_+ := A_{i+1}(x,\mathcal{D})$ by the local operators $A_\pm^{(k)}(x,\mathcal{D})$ from Definition 2.23 (iii). We abbreviate the differential operators on the right-hand side of the equations by

$$C_{k,\pm}(x,\mathcal{D}) := (A_\pm - A_\pm^{(k)})(x,\mathcal{D})\varphi_k + [A_\pm(x,\mathcal{D}), \varphi_k]$$

and point out that $C_{k,\pm}$ is of order at most $2m-1$ for functions $v \in W_p^{2m+s}(\mathbb{K}_{l\pm}^n)$. Extending the right-hand sides by 0 outside of $\mathrm{supp}(\varphi_k)$ to the whole of $\mathbb{K}_{l\pm}^n$, we obtain a transmission boundary value problem in the divided l-corners $\mathbb{K}_{l\pm}^n$ for $u_- := u_i$ and $u_+ := u_{i+1}$:

(2.16)
$$\begin{cases} (\lambda - A_-^{(k)}(x,\mathcal{D}))(\varphi_k u_-) &=& \varphi_k f_- + C_{k,-}(x,\mathcal{D})u_- & \text{in } \mathbb{K}_{l-}^n, \\ (\lambda - A_+^{(k)}(x,\mathcal{D}))(\varphi_k u_+) &=& \varphi_k f_+ + C_{k,+}(x,\mathcal{D})u_+ & \text{in } \mathbb{K}_{l+}^n, \\ \gamma_{0,j} B_j^{(i)}(\mathcal{D}_i)(\varphi_k u) &=& 0 \ \text{ on } \partial\mathbb{K}_{l\pm}^n,\ (i = 1, \dots, l;\ j = 1, \dots, mN/2), \\ \gamma_{0,n}\big[C_r^{(-)}(\mathcal{D}_n)\varphi_k u_- + C_r^{(+)}(\mathcal{D}_n)\varphi_k u_+\big] &=& 0 & \text{on } \Gamma_n,\ (r = 1, \dots, 2m). \end{cases}$$

Now, we apply the resolvent $R^{(k)}(\lambda)$ from Lemma 2.24 (ii) to the local problem in the divided l-corner to obtain the representation

$$\begin{pmatrix} \varphi_k u_- \\ \varphi_k u_+ \end{pmatrix} = R^{(k)}(\lambda)\begin{pmatrix} \tilde{f}_- \\ \tilde{f}_+ \end{pmatrix},$$

which can be rewritten in terms of $u := (u_-, u_+)^T$ and $\tilde{f} := (\tilde{f}_-, \tilde{f}_+)^T$ as

$$\varphi_k u = R^{(k)}(\lambda)\left[\varphi_k f + C_k(x,\mathcal{D})u\right].$$

The first component of $\varphi_k u$ will contribute to the solution u_i in Ω_i and the second component contributes to u_{i+1} in Ω_{i+1}. For the remaining cubes Q_k, where no transmission conditions

occur, the representation of $\varphi_k u_i$ is similar, but does not depend on the solution u_{i+1} in the adjacent subdomain. The case of a local problem in $\mathbb{R}^n$, which corresponds to an interior point x_k has already been treated in Step 3a for the smooth domain. Now, let $\psi_k \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}(\psi_k) \subset Q_k$ and $\psi_k = 1$ on $\operatorname{supp}(\varphi_k)$. Then we have the representation

$$u_i = \sum_{k=1}^{M} \psi_k R^{(k)}(\lambda)\,[\varphi_k f + C_k(x,\mathcal{D})u] \quad (i = 1,2,3),$$

and the existence of a left inverse as well as the a priori estimate can be proved in the same way as in Step 4 of Section 1.2.6. The proof for the existence of a right inverse (Step 5) also carries over to the situation here. We have proved

Theorem 2.25
Under the Assumptions 2.22 and for $\lambda \in \overline{\Sigma}_\varphi$ with sufficiently large modulus, the boundary value problem (2.15) *with homogeneous boundary and transmission conditions admits a unique solution $u = u_i$ in Ω_i with $u_i \in W_p^{2m+s}(\Omega_i;\mathbb{C}^N)$ for every given $f \in W_p^s(\Omega;\mathbb{C}^N)$. In each subdomain Ω_i the solution satisfies the a priori estimate*

$$|\!|\!| u_i |\!|\!|_{2m+s,p,\Omega_i} \le C \|f\|_{s,p,\Omega}.$$

Corollary 2.26
Under the assumptions of Theorem 2.25, let boundary values $g_i = \left(g_1^{(i)},\dots,g_{mN}^{(i)}\right)$ on Γ_r with $g_l^{(i)} \in W_p^{2m+s-m_{rl}-1/p}(\Gamma_i)$ be given for $i \in \{n,2n\}$. Furthermore, let $h_i = \left(h_1^{(i)},\dots,h_{2mN}^{(i)}\right)$ on Σ_i with $h_l^{(i)} \in W_p^{2m+s-m_{il}-1/p}(\Gamma_r)$ be given for $i \in \{1,2\}$, where $m_{il} = \operatorname{ord}(C_l^{(i)}) = \operatorname{ord}(C_l^{(i+1)})$ denotes the order of the corresponding transmission condition. Let $\boldsymbol{\nu} \in \{0,1\}^{2n-2}$ be the vector describing the type of the boundary conditions on Γ_j for $j = 1,\dots,n-1,n+1,\dots,2n-1$. Assume that the following compatibility conditions are satisfied for all $j = 1,\dots,n-1,n+1,\dots,2n-1$:

$$\begin{aligned}
&\gamma_{0,j}\mathcal{D}_j^r u_l^{(i)} = 0 \quad \text{on } \Gamma_i \cap \Gamma_j, \qquad r = 1,3,\dots,2m-1-2\lfloor \tfrac{m_{il}}{2} \rfloor, \qquad \text{if } \nu_j = 1,\\
&\gamma_{0,j}\mathcal{D}_j^r u_l^{(i)} = 0 \quad \text{on } \Gamma_i \cap \Gamma_j, \qquad r = 0,2,\dots,2m-m_{il}-2, \qquad \text{if } \nu_j = 0,
\end{aligned}$$

where either $u = g$, $i \in \{n,2n\}$ and $l = 1,\dots,mN$, or $u = h$, $i \in \{1,2\}, l = 1,\dots,2mN$ and Γ_i is replaced by Σ_i.

Then there exists $\lambda_1 > 0$ such that for all $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| > \lambda_1$, the boundary value problem

$$(2.17)\quad \begin{cases}
(\lambda - A(x,\mathcal{D}))u_i &=& f_i & \text{in } \Omega_i,\ (i=1,2,3),\\
\gamma_{\partial\Omega} B_l^{(j)}(\mathcal{D}_j)u &=& 0 \quad \text{on } \Gamma_j,\ (j = 1,\dots,n-1,n+1,\dots,2n-1;\ l = 1,\dots,mN),\\
\gamma_{\partial\Omega} B_l^{(j)}(\mathcal{D}_j)u &=& g_l^{(j)} & \text{on } \Gamma_j,\ (j \in \{n,2n\};\ l = 1,\dots,mN),\\
\gamma_{\Sigma_i}\left[C_l^{(i)}(\mathcal{D}_n)u_i \ +\ C_l^{(i+1)}(\mathcal{D}_n)u_{i+1}\right] = h_l^{(i)} & & & \text{on } \Sigma_i,\ (i=1,2,\ l=1,\dots,2mN),
\end{cases}$$

admits a unique solution $u_i \in W_p^{2m+s}(\Omega_i)$ for $i = 1,2,3$, and the following a priori estimate is satisfied:

$$(2.18)\quad |\!|\!| u_i |\!|\!|_{2m+s,p,\Omega_i}$$
$$\le C\left(\|f\|_{s,p,\Omega} + \sum_{j\in\{n,2n\}} \sum_{l=1}^{mN} \left|\!\left|\!\left| g_l^{(j)} \right|\!\right|\!\right|_{2m+s-m_{jl}-1/p,p,\Gamma_j} + \sum_{j\in\{1,2\}} \sum_{l=1}^{2mN} \left|\!\left|\!\left| h_l^{(j)} \right|\!\right|\!\right|_{2m+s-m_{jl}-1/p,p,\Sigma_j} \right).$$

Proof. The main difference to the method of the above proof is to use Corollary 2.20 for the local problems in the l corner $\mathbb{K}_l^n$ of the form given in Lemma 2.24 (i) whenever a boundary condition on Γ_n or Γ_{2n} participates in this problem. For these problems, the local solution $\varphi_k u$ is given by the restriction of

$$R^{(k)}(\lambda)E_{(1,\dots,1)}(A^{(k)}(x,\mathcal{D}) - A^{(k)}(x_k,\mathcal{D}))E_{\boldsymbol{\nu}}(\varphi_k u) + R^{(k)}(\lambda)E_{\boldsymbol{\nu}} f + \sum_{l=1}^{mN} \mathbb{T}_l(\lambda)(E_{\boldsymbol{\nu}}\varphi_k g_l^{(j)})$$

to the corner $\mathbb{K}_l^n$, where $R^{(k)}(\lambda)$ denotes the resolvent of the homogeneous constant coefficient problem $(A^{(k)}(x_k,\mathcal{D}),\mathcal{B}_i)$ with $i=n$ or $i=2n$ in the half space. Note that the trivial extension of $\varphi_k g_l^{(i)}$ to $\mathbb{K}_l^n$ again satisfies the compatibility conditions because of the special construction of φ_k.
For the local problems which include inhomogeneous transmission conditions, we proceed as in Step 3 of the proof of Theorem 2.25, but apply Corollary 2.20 to the system of size $2N$ with inhomogeneous transmission conditions (see (2.16)). This yields a representation of the solution to the local problem of the form

$$\varphi_k\begin{pmatrix}u_-(x',-x_n)\\ u_+(x)\end{pmatrix} \;=\; R^{(k)}(\lambda)\left[\varphi_k\begin{pmatrix}f_-\\ f_+\end{pmatrix}+C_k(x,\mathcal{D})\begin{pmatrix}u_-(x',-x_n)\\ u_+(x)\end{pmatrix}\right] \;+\; \sum_{l=1}^{2mN}\mathbb{T}_l(\lambda)\varphi_k h_l^{(j)}.$$

To verify the a priori estimate and the existence of a right inverse for sufficiently large modulus of λ, one can again follow the structure of the proof given for the case of a smooth domain. □

Remark 2.27
If we combine the localization techniques we employed for $C^{m,1}$-domains on pages 38 ff., and for the rectangular domain with two interfaces on pages 61 ff., we can prove analogous results for domains with piecewise $C^{2m,1}$-boundary which is divided by a $C^{2m,1}$-surface Σ such that for every point on an edge or a corner $x_0\in\bigcap_j\Gamma_j$ and for each point $x_0\in\Sigma\cap\partial\Omega$, there is a neighbourhood $U(x_0)$ which is $2m$-diffeomorphic to a corresponding neighbourhood of the reference cuboid treated in the present chapter. More illustrative, each of the surfaces Γ_j which form the boundary of Ω as well as the interface Σ may be a $C^{2m,1}$-surface, but each intersection of k of the surfaces Γ_j must be locally diffeomorphic to a k-corner and the intersection of Σ with $k-1$ surfaces Γ is locally diffeomorphic to a divided corner $\mathbb{K}^n_{k-1,\pm}$.

We draw a final conclusion for the case of canonical transmission conditions.

Corollary 2.28 (Canonical transmission conditions)
Under the assumptions of Corollary 2.26, let homogeneous transmission conditions be given by

$$[\![\partial_n^l u]\!] := e_r^T\left[\gamma_{\Sigma_i}(\partial_n^l u_i)-\gamma_{\Sigma_i}(\partial_n^l u_{i+1})\right]=0\quad \text{on } \Sigma_i,\ (i=1,2;\ l=0,\dots,2m-1;\ r=1,\dots,N),$$

where e_r denotes the r-th unit vector. Then for $\lambda\in\overline{\Sigma}_\varphi$ with $|\lambda|>\lambda_1$, we obtain a unique solution $u\in W_p^{2m+s}(\Omega;\mathbb{C}^N)$ to the system

$$(2.19)\quad \begin{cases}(\lambda-A(x,\mathcal{D}))u_i &=& f_i & \text{in } \Omega_i,\ (i=1,2,3)\\ \gamma_{\partial\Omega}B_l^{(j)}(\mathcal{D}_j)u &=& 0 & \text{on } \Gamma_j,\ (j=1,\dots,n-1,n+1,\dots,2n-1;\ l=1,\dots,mN),\\ \gamma_{\partial\Omega}B_l^{(j)}(\mathcal{D}_j)u &=& g_l^{(j)} & \text{on } \Gamma_j,\ (j\in\{n,2n\};\ l=1,\dots,mN),\\ [\![\partial_n^l u]\!] &=& 0 & \text{on } \Sigma_i,\ (i=1,2;\ l=0,\dots,2m-1),\end{cases}$$

and u satisfies the a priori estimate

$$|\!|\!|u_i|\!|\!|_{2m+s,p,\Omega_i}\le C\left(\|f\|_{s,p,\Omega}+\sum_{j\in\{n,2n\}}\sum_{l=1}^{mN}\left|\!\left|\!\left|g_l^{(j)}\right|\!\right|\!\right|_{2m+s-m_{jl}-1/p,p,\Gamma_j}\right).$$

In addition, if $N=1$, the Assumption 2.22 (iv) becomes redundant.

Proof. This is a direct consequence of Corollary 2.26 and Lemma 1.8. The claim for $N=1$ follows from the proof of Theorem 1.9. □

Remark 2.29
The condition (ii) from Assumption 2.22 can be relaxed. In fact, the principal part of A only has to be even near corner points and even with respect to certain variables near the intersections of the surfaces Γ_j. Similar assertions hold for the Assumption 2.22 (iii) regarding the dependence of the boundary operators $\mathcal{B}_j$ which were assumed to act only on the x_j-variable. Note that the parameter-ellipticity condition A, as well as the Shapiro-Lopatinskii condition for the local boundary value (or transmission) problems still have to be satisfied.

2.5 Fredholm property

To conclude this chapter, we state an analogous result on the Fredholm property for a parameter-elliptic system in a rectangular domain. For this purpose, we consider the differential operator from Assumption 2.22

$$(\lambda - A, \mathbb{B}) := \left(\lambda - A(\cdot, \mathcal{D}), \gamma_{\Gamma_1} B_1^{(1)}(\mathcal{D}_1), \dots, \gamma_{\Gamma_1} B_{mN}^{(1)}(\mathcal{D}_1), \gamma_{\Gamma_2} B_1^{(2)}(\mathcal{D}_2), \dots, \gamma_{\Gamma_{2n}} B_{mN}^{(2n)}(\mathcal{D}_{2n})\right)$$

which acts between the following Banach spaces:

$$X_1 := \Big\{ u \in W_p^{2m+s}(\Omega) : \gamma_{0,i} B_j^{(i)}(\mathcal{D}_i) u = 0 \quad \text{on } \Gamma_i, \\ (i = 1, \dots, n-1, n+1, \dots, 2n-1;\ j = 1, \dots, mN) \Big\}$$

and

$$X_2 := W_p^s(\Omega) \times \{0\}^{(n-1)mN} \times \prod_{j=1}^{mN} W_{p,\mathbb{B}}^{2m+s-m_{n,j}-1/p}(\Gamma_n) \times \{0\}^{(n-1)mN} \times \prod_{j=1}^{mN} W_{p,\mathbb{B}}^{2m+s-m_{2n,j}-1/p}(\Gamma_{2n}) \\ \subset W_p^s(\Omega) \times \prod_{i=1}^{2n} \prod_{j=1}^{mN} W_p^{2m+s-m_{i,j}-1/p}(\Gamma_i).$$

Here, the subscript $\mathbb{B}$ indicates that the compatibility conditions from Corollary 2.26 are included into the space for the boundary value. This means for $g \in W_p^{2m+s-m_{l,j}-1/p}(\Gamma_l)$ $(l \in \{n, 2n\})$ there holds $g \in W_{p,\mathbb{B}}^{2m+s-m_{l,j}-1/p}(\Gamma_l)$ if and only if for all $i = 1, \dots, n-1, n+1, \dots, 2n-1$ the conditions

$$\gamma_{0,i} \partial_i^r g = 0 \quad \text{on } \Gamma_l \cap \Gamma_i, \quad r = \begin{cases} 1, 3, \dots, 2m - 1 - 2\lfloor \frac{m_{l,j}}{2} \rfloor, & \text{if } \nu_i = 1 \\ 0, 2, \dots, 2m - 2 - m_{l,j}, & \text{if } \nu_i = 0 \end{cases}$$

are satisfied. Note that for $u \in X_1$, the compatibility conditions for its boundary values are satisfied because for $l \in \{n, 2n\}$ we may interchange the boundary operators $\gamma_{0,l} B_j^{(l)}(\mathcal{D}_l)$, and the linear independence of the vectors $b_{sr}^{(i)}$ for $s = 1, \dots, N$ ensures $\gamma_{0,i} \mathcal{D}_i^r u = 0$:

$$\gamma_{0,i} \mathcal{D}_i^r \left(\gamma_{0,l} B_j^{(l)}(\mathcal{D}_l) u \right) = \gamma_{0,l} B_j^{(l)}(\mathcal{D}_l) \left(\gamma_{0,i} \mathcal{D}_i^r u \right) = 0 \quad (i = 1, \dots, n-1, n+1, \dots, 2n-1),$$

for even (odd) $r \in \mathbb{N}_0$ such that $2m + s - 1/p - m_{l,j} - r > 1/p$ in case of an even (odd) boundary condition on the surface Γ_i.

Theorem 2.30
Let $(A, \mathbb{B})$ be a symmetric, parameter-elliptic boundary value problem in the setting of Corollary 2.26. Then the following assertions hold true:

(i) For $\lambda \in \overline{\Sigma}_\varphi$ with $|\lambda| > \lambda_0$, the operator corresponding to the boundary value problem

$$\begin{aligned} (\lambda - A, \mathbb{B}) : (X_1, \|\cdot\|_{2m+s,p,\Omega}) &\to \left(X_2, \sum_{i=1}^{2n} \sum_{j=1}^{mN} \|\cdot\|_{2m+s-m_{i,j}-1/p,p,\Gamma_i} \right), \\ u &\mapsto (\lambda - A(\cdot, \mathcal{D})u, \mathbb{B}(\mathcal{D})u) \end{aligned}$$

is a Fredholm operator of index 0.

(ii) *Let $\mu \in \mathbb{C}$ such that the operator $(\mu - A, \mathbb{B}) : X_1 \to X_2$ is injective. Then for all $f \in W_p^s(\Omega)$ and $g_j^{(l)} \in W_{p,\mathbb{B}}^{2m+s-m_{l,j}-1/p}(\Gamma_l)$ $(l \in \{n, 2n\},\ j = 1, \dots, mN)$, the boundary value problem (2.17) (where $h_l^{(i)} = 0$) with $\lambda = \mu$ admits a unique solution $u \in X_1$. The solution satisfies the a priori estimate*

$$\|u\|_{2m+s,p,\Omega} \le C(\mu) \left(\|f\|_{s,p,\Omega} + \sum_{l \in \{n,2n\}} \sum_{j=1}^{mN} \|g_j^{(l)}\|_{2m+s-m_j-1/p,p,\partial\Omega} \right). \tag{2.20}$$

Proof. Up to some minor changes of the notations, the proof is the same as the one given for Theorem 1.50. □

Remark 2.31
The following generalizations can be deduced for the case $1 \le n \le 3$ from Theorem A.3.

(i) *Theorem 2.30 can be extended to inhomogeneous boundary values on all smooth parts of $\partial\Omega$. Assuming that all boundary values $g_{m_{j,l}}^{(l)} \in W_p^{2m+s-m_{l,j}-1/p}(\Gamma_l)$ $(l = 1, \dots, 2n,\ j = 1, \dots, m)$ combined with the coefficients of the corresponding boundary operators satisfy the compatibility conditions of Theorem A.3, there exists $w \in W_p^{2m+s}(\Omega)$ such that $\gamma_{0,l} B_j^{(l)}(\mathcal{D}_l) w = g_{m_{j,l}}^{(l)}$ on Γ_l and*

$$\|w\|_{2m+s,p,\Omega} \le C \sum_{l=1}^{2n} \sum_{j=1}^{m} \|g_{m_{l,j}}^{(l)}\|_{2m+s-m_{l,j}-1/p,p,\Gamma_l}.$$

The approach $u = w + v$ then yields an equation with entirely homogeneous boundary conditions and right-hand side $f - (\mu - A(\mathcal{D}))w$ and Theorem 2.30 (ii) applies with $g_j^{(l)} = 0$.

(ii) *Theorem A.3 can be applied to prove a corresponding trace theorem for $\mathbb{K}_k^n$. For the two-dimensional case the idea is the following: Let $r > \frac{1}{p}$ and $0 \le j \le [r]$. Assume that $g^{(l)} \in W_p^{r-1/p}(\Gamma_l)$ for $l = 1, \dots, k$ satisfies the compatibility conditions of Theorem A.3. Here, $\Gamma_l \cong [0, \infty)$. Using a cutoff function χ, which is equal to one near $\Gamma_{l_-} \cap \Gamma_{l_+}$, we can decompose the boundary values $g^{(l_\pm)} = \chi g^{(l_\pm)} + (1 - \chi) g^{(l_\pm)}$. Then we use Theorem A.3 for $\chi g^{(l_\pm)}$ in a finite rectangle $Q \supset supp(\chi)$ and the trace theorem in $\mathbb{R}^n$ for the functions $(1 - \chi) g^{(l_\pm)} \in W_p^{r-1/p}(\mathbb{R})$ separately. This yields functions $w_0 \in W_p^r(Q)$ with $\gamma_{\partial Q \cap \Gamma_\pm} = \chi g^{(l_\pm)}$ and $w_\pm \in W_p^r(\mathbb{R}^2)$ with $\gamma_{\Gamma_\pm} w_\pm = (1 - \chi) g^{(l_\pm)}$. Choosing a smooth function $\psi \in C_c^\infty(Q)$ with $\psi \equiv 1$ on $supp(\chi)$, we can define $w := \psi w_0 + w_+|_{\mathbb{K}_k^2} + w_-|_{\mathbb{K}_k^2} \in W_p^r(\mathbb{K}_k^2)$. Since $w_\pm = 0$ near $\Gamma_\mp$, we have $\gamma_{\Gamma_\pm} w = g^{(l_\pm)}$.*

(iii) *A generalization of Theorem A.3 makes it possible to carry over the assertions of Section 2.3 to the case of entirely inhomogeneous boundary values including lower order terms in the boundary operators. For this one would have to prove the continuity of the right inverse of the trace operator with respect to the parameter-dependent norms. For the case $r = 2m + s$ with $s < \frac{1}{p}$ this means that an estimate of the form*

$$\|u\|_{r,p,\Omega} + |\lambda| \|u\|_{s,p,\Omega} \le C \sum_{l=1}^{2n} \sum_{j=0}^{m-1} \|g_j^{(l)}\|_{r-j-1/p,p,\Gamma_l} + |\lambda|^{\frac{r-j-1/p}{2m}} \|g_j^{(l)}\|_{0,p,\Gamma_l}$$

is satisfied. Here, we will not further pursue this issue.

Chapter 3

Weak well-posedness of a nonlinear elliptic system

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary. In this chapter we consider a nonlinear elliptic system of the following structure:

$$\begin{cases} -\nabla \cdot (\kappa_1(x)\nabla u_1(x)) &= F(u_2(x) - u_1(x), x) + f_1(x) \quad \text{in } \Omega, \\ -\nabla \cdot (\kappa_2(x)\nabla u_2(x)) &= -F(u_2(x) - u_1(x), x) + f_2(x) \quad \text{in } \Omega, \end{cases}$$

when Neumann, Dirichlet, or mixed boundary conditions on $\partial\Omega$ are imposed. For $i = 1, 2$, we assume that $\kappa_i : \Omega \to \mathbb{R}$ belongs to $L_\infty(\Omega)$ with $0 < k_i^2 \le \kappa_i \le K_i$ as well as $f_i \in L_q(\Omega)$ with $q > \frac{n}{2}$ if $n \ge 2$, and $q = 2$ if $n = 1$. We will prove that this system admits a unique weak solution $(u_1, u_2) \in H^1(\Omega)^2$, or in an appropriate subspace of $H^1(\Omega)^2$ under the following assumptions on the nonlinearity F:

Assumption 3.1
Let $F : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $s \mapsto F(s, \cdot)$ is continuous from $\mathbb{R}$ into $L_\infty(\Omega)$, i.e., for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $s_1, s_2 \in \mathbb{R}$ with $|s_1 - s_2| < \delta$, there holds $\|F(s_1, \cdot) - F(s_2, \cdot)\|_{0,\infty,\Omega} < \varepsilon$. We further assume that the following estimates are satisfied:
There is a constant $C_0 \ge 0$ such that

$$(F(s, x) - F(t, x))(s - t) \ge C_0(s - t)^2 \quad (s, t \in \mathbb{R},\ \text{a.e. } x \in \Omega). \tag{3.1}$$

There are constants $C_1 > 0$ and $C_2 \ge 0$ such that

$$F(s, x)s \ge C_1 s^2 - C_2 \quad (s \in \mathbb{R},\ \text{a.e. } x \in \Omega). \tag{3.2}$$

Clearly, the first condition is met with $C_0 = 0$ if $F(\cdot, x)$ is monotonically nondecreasing for all $x \in \Omega$. For later applications, we will sometimes impose the more restrictive

Assumption 3.2
$F : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ is piecewise continuous on $\overline{\Omega}$, F is differentiable with respect to the first variable, and there is a positive constant $C > 0$ such that for almost all $x \in \Omega$ and all $w \in \mathbb{R}$:

$$\frac{\partial}{\partial w} F(w, x) \ge C > 0.$$

We first prove some technical but useful conclusions from the proposed properties of F.

Lemma 3.3
Let $F : \mathbb{R} \times \Omega \to \mathbb{R}$ such that $s \mapsto F(s, \cdot)$ is continuous from $\mathbb{R}$ into $L_\infty(\Omega)$.

(i) Let (3.2) be satisfied. Then $F(s, x)\ \operatorname{sgn}(s) \ge -C_3$ for all $s \in \mathbb{R}$ and almost all $x \in \Omega$.

(ii) Let Assumption 3.2 be satisfied. Then Assumption 3.1 is satisfied as well.

Proof. (i) For $|s| \le 1$, the mapping property of F yields the boundedness of the image $\{F(s, \cdot) : s \in [-1, 1]\} \subset L_\infty(\Omega)$, i.e. $|F(s, x)\operatorname{sgn}(s)| = |F(s, x)| \le K$ for a.e. $x \in \Omega$. Hence, $F(s, x)\operatorname{sgn}(s) \ge -K$. For $|s| \ge 1$, we divide (3.2) by $|s|$ and obtain the estimate $F(s, x)\ \operatorname{sgn}(s) \ge C_1 s \operatorname{sgn}(s) - \frac{C_2}{|s|} \ge -C_2$. Consequently, we can set $C_3 := \max(K, C_2)$.

(ii) For almost every fixed $x \in \Omega$, $F(\cdot, x)$ is strictly monotonically increasing, hence (3.1) holds with $C_0 = C$. For $|s| \le R$, the estimate (3.2) follows by the boundedness of $(s, x) \mapsto F(s, x)s - s^2$. For $|s| > R$, we use the mean value theorem: for some $t \in (0, s)$ (resp. $(s, 0)$) we have:

$$Cs^2 \le s^2(\partial_1 F)(t, x) = s\,(F(s, x) - F(0, x)), \text{ thus } F(s, x)s \ge Cs^2 + F(0, x)s \ge C_1 s^2 - C_2,$$

where the last inequality is trivial for $F(0, x) \ge 0$ (or $F(0, x) \le 0$) and for $F(0, x) < 0$ (or $F(0, x) > 0$), one can choose R sufficiently large such that $\frac{C}{2}s^2 + F(0, x)s \ge -C_2$.

□

3.1 Homogeneous Neumann boundary conditions

We first consider the elliptic system with homogeneous Neumann boundary conditions

$$\left\{\begin{array}{rcll} -\nabla\cdot(\kappa_1\nabla u_1) &=& F(u_2-u_1)+f_1 & \text{in } \Omega,\\ -\nabla\cdot(\kappa_2\nabla u_2) &=& -F(u_2-u_1)+f_2 & \text{in } \Omega,\\ \frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu} &=& 0 & \text{on } \partial\Omega. \end{array}\right. \tag{3.3}$$

When we consider the equations with pure Neumann boundary conditions, the following spaces which include a vanishing integral mean value will be required in view of uniqueness.

Definition and Lemma 3.4
Let Ω be a bounded domain. Denote by $C^\infty_*(\Omega)$ the space of smooth functions in Ω with vanishing integral.

(i) For $1 \le p \le \infty$, we define the space $L_{p,*}(\Omega) := \left\{u \in L_p(\Omega) \;:\; \int_\Omega u\,dx = 0\right\}$. Then $L_{p,*}(\Omega)$ is a closed subspace of $L_p(\Omega)$ and for $1 < p < \infty$, $C^\infty_*(\Omega)$ is dense in $L_{p,*}(\Omega)$.

(ii) For $1 < p < \infty$, we define the space $H^1_{p,*}(\Omega) := \left\{u \in H^1_p(\Omega) \;:\; \int_\Omega u\,dx = 0\right\}$. Then $H^1_{p,*}(\Omega)$ is a closed subspace of $H^1_p(\Omega)$ and $C^\infty_*(\Omega)$ is dense in $H^1_{p,*}(\Omega)$.

Definition 3.5
A pair of functions $(u_1, u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty := (H^1_*(\Omega) \times H^1(\Omega)) \cap (L_{\infty,*}(\Omega) \times L_\infty(\Omega))$ is called a (bounded) weak solution to the system (3.3) if for all $\phi \in H^1_*(\Omega)$ and all $\psi \in H^1(\Omega)$ the following equalities are satisfied:

$$\int_\Omega \kappa_1(x)\nabla u_1(x)\nabla\phi(x)\,dx - \int_\Omega F(u_2(x)-u_1(x),x)\phi(x)\,dx - \int_\Omega f_1(x)\phi(x)\,dx = 0, \tag{3.4a}$$

$$\int_\Omega \kappa_2(x)\nabla u_2(x)\nabla\psi(x)\,dx + \int_\Omega F(u_2(x)-u_1(x),x)\psi(x)\,dx - \int_\Omega f_2(x)\psi(x)\,dx = 0. \tag{3.4b}$$

Lemma 3.6 (Uniqueness)
Let the condition (3.1) on F be satisfied. If $C_0 > 0$ then (3.3) admits at most one weak solution $(u_1, u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$. If $C_0 = 0$, then a weak solution $(u_1, u_2) \in (H^1_*(\Omega) \cap L_\infty(\Omega))^2$ is unique.

Proof. Assume there are two solutions $(u_1, u_2), (v_1, v_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$. Taking the difference of the equations (3.4a) for u_1 and v_1, and choosing $\phi := u_1 - v_1 \in H^1_*(\Omega) \cap L_{\infty,*}(\Omega)$ as an admissible test function shows

$$\int_\Omega \kappa_1|\nabla(u_1-v_1)|^2\,dx - \int_\Omega \left(F(u_2-u_1) - F(v_2-v_1)\right)(u_1-v_1)\,dx = 0.$$

Similarly, by testing the difference of the equations (3.4b) for u_2 and v_2 with $\psi := u_2 - v_2$, we obtain

$$\int_\Omega \kappa_2|\nabla(u_2-v_2)|^2\,dx + \int_\Omega \left(F(u_2-u_1) - F(v_2-v_1)\right)(u_2-v_2)\,dx = 0.$$

Taking the sum, we end up with

$$\int_\Omega \kappa_1|\nabla(u_1-v_1)|^2\,dx+\int_\Omega \kappa_2|\nabla(u_2-v_2)|^2\,dx \\ +\int_\Omega \left(F(u_2-u_1)-F(v_2-v_1)\right)\left((u_2-u_1)-(v_2-v_1)\right)\,dx=0.$$

We now employ the assumptions on κ_i and (3.1) to find

$$k_1^2\|\nabla(u_1-v_1)\|_{0,2,\Omega}^2+k_2^2\|\nabla(u_2-v_2)\|_{0,2,\Omega}^2+C_0\|(u_1-u_2)-(v_1-v_2)\|_{0,2,\Omega}^2\le 0.$$

Hence, u_1-v_1 is constant and together with $u_1-v_1\in H^1_*(\Omega)$ this implies $u_1=v_1$. For $C_0>0$, the last term now equals $\|u_2-v_2\|_{0,2,\Omega}$ and we also have $u_2=v_2$. In case of $C_0=0$, we conclude similarly as for u_1-v_1 with the vanishing mean value that $u_2=v_2$. □

Remark 3.7
The vanishing mean value condition is rather technical but necessary. If we drop this condition without replacement, we would only deduce uniqueness of the functions u_i up to constants. A possible replacement for this condition is to impose homogeneous Dirichlet boundary conditions on a part Γ_D of $\partial\Omega$ with $\lambda_{n-1}(\Gamma_D)>0$ for one of the functions (provided $C_0>0$). In this situation, Poincaré's inequality (see Lemma A.7) holds and ensures the coerciveness of a corresponding bilinear form as well as uniqueness for this problem. For the treatment of a structural similar equation with such techniques, we refer to the subsequent sections and [Hol94]. The case $C_0=0$ will not further be treated in the sequel to avoid technicalities.

In order to prove existence, we consider a modified, linearized problem with an additional parameter $\delta\in[0,1]$: Let $(w_1,w_2)\in\mathbb{L}_\infty$ be fixed. Then we investigate the weak form of the equations

$$\left\{\begin{array}{rcll} u_1-\nabla\cdot(\kappa_1\nabla u_1) &=& \delta(\ \ F(w_2-w_1)+f_1+w_1) & \text{in } \Omega,\\ u_2-\nabla\cdot(\kappa_2\nabla u_2) &=& \delta(-F(w_2-w_1)+f_2+w_2) & \text{in } \Omega,\\ \frac{\partial u_1}{\partial\nu}=\frac{\partial u_2}{\partial\nu} &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{3.5}$$

which reads as follows: Find $(u_1,u_2)\in\mathbb{H}^1\cap\mathbb{L}_\infty$ such that for all $\phi\in H^1_*(\Omega)$ and $\psi\in H^1(\Omega)$

$$\int_\Omega u_1\phi\,dx+\int_\Omega \kappa_1\nabla u_1\nabla\phi\,dx-\delta\int_\Omega (F(w_2-w_1)+f_1+w_1)\phi\,dx=0, \tag{3.6a}$$

$$\int_\Omega u_2\psi\,dx+\int_\Omega \kappa_2\nabla u_2\nabla\psi\,dx+\delta\int_\Omega (F(w_2-w_1)-f_2-w_2)\psi\,dx=0. \tag{3.6b}$$

Note that these changes lead to a decoupling of the equations (3.4a)-(3.4b).

Lemma 3.8 (Uniqueness)
For all $\delta\in[0,1]$, the decoupled system (3.6a)-(3.6b) *admits at most one solution (u_1,u_2) in the space $H^1(\Omega)\times H^1(\Omega)$.*

Proof. Suppose there are solutions $(u_1,u_2),(v_1,v_2)\in H^1(\Omega)\times H^1(\Omega)$. Then we proceed in the same way as in the proof of Lemma 3.6 to obtain the inequality

$$\|u_1-v_1\|_{0,2,\Omega}^2+\|u_2-v_2\|_{0,2,\Omega}^2+k_1^2\|\nabla(u_1-v_1)\|_{0,2,\Omega}^2+k_2^2\|\nabla(v_1-v_2)\|_{0,2,\Omega}^2\le 0.$$

This time we can immediately conclude that $(u_1,u_2)=(v_1,v_2)$ in $(H^1(\Omega))^2$. □

Lemma 3.9 (Existence)
For all $\delta \in [0,1]$, $\varepsilon > 0$, $f_1, f_2 \in L_q(\Omega)$ with $q \geq \max\left(2, \frac{n}{2} + \varepsilon\right)$, and $w_1, w_2 \in L_\infty(\Omega)$, the equations (3.6a)-(3.6b) *admit a unique solution $(u_1, u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$. In addition, there are a constant $C > 0$ and Hölder exponents $\alpha_i \in (0,1)$ $(i = 1,2)$, depending on n, δ, k_i^2, K_i, Ω, and $\|w_i\|_{0,\infty,\Omega}$ such that the a priori estimates*

$$\begin{aligned}
(3.7) \quad \|u_i\|_{1,2,\Omega} &\leq C\left(\|F(w_2 - w_1, \cdot) + w_i\|_{0,2,\Omega} + \|f_i\|_{0,2,\Omega}\right) \quad (i = 1,2),\\
(3.8) \quad \|u_1\|_{C^{0,\alpha_1}(\overline{\Omega})} &\leq C_1\left(\|u_1\|_{0,2,\Omega} + (1 + |\Omega|)\left(\|F(w_2 - w_1, \cdot) + w_1\|_{0,\infty,\Omega} + \|f_1\|_{0,q,\Omega}\right)\right),\\
(3.9) \quad \|u_2\|_{C^{0,\alpha_2}(\overline{\Omega})} &\leq C_2\left(\|u_2\|_{0,2,\Omega} + \left(\|F(w_2 - w_1, \cdot) + w_2\|_{0,\infty,\Omega} + \|f_2\|_{0,q,\Omega}\right)\right),
\end{aligned}$$

where C_i has the same dependencies as C, but depends on α_i, too.

Proof. For $i = 1,2$, we write $g_i := \delta(f_i + w_i + (-1)^{i+1}F(w_2 - w_1, \cdot)) \in L_q(\Omega)$. Then the existence of a solution $u_2 \in H^1(\Omega)$ to

$$\int_\Omega u_2\psi \, dx + \int_\Omega \kappa_2 \nabla u_2 \nabla\psi \, dx - \int_\Omega g_2\psi \, dx = 0 \quad (\psi \in H^1(\Omega)),$$

follows immediately by the Lax-Milgram theorem (Theorem A.11). In order to prove the existence of $u_1 \in H^1_*(\Omega)$, we define the bilinear form

$$B : H^1_*(\Omega) \times H^1_*(\Omega) \to \mathbb{R}, \quad (u, \phi) \mapsto \int_\Omega u\phi \, dx + \int_\Omega \kappa_1 \nabla u \nabla\phi \, dx.$$

Using the assumptions on κ_1 and Hölder's inequality it is easy to see that B is continuous and coercive:

$$\begin{aligned}
|B(u,\phi)| \leq K_1\|\nabla u\|_{0,2,\Omega}\,\|\nabla\phi\|_{0,2,\Omega} + \|u\|_{0,2,\Omega}\|\phi\|_{0,2,\Omega} &\leq \max(1, K_1)\left(\|u\|_{1,2,\Omega}\,\|\phi\|_{1,2,\Omega}\right),\\
B(u,u) \geq \|u\|^2_{0,2,\Omega} + k_1^2\|\nabla u\|^2_{0,2,\Omega} &\geq \min(1, k_1^2)\|u\|^2_{1,2,\Omega}.
\end{aligned}$$

Furthermore, $G : H^1_*(\Omega) \to \mathbb{R}$, $\phi \mapsto \int_\Omega g_1\phi \, dx$ defines a continuous functional:

$$|G(\phi)| \leq \|g_1\|_{0,2,\Omega}\|\phi\|_{0,2,\Omega} \leq C\|g_1\|_{0,q,\Omega}\|\phi\|_{1,2,\Omega},$$

i.e., $\|G\|_{(H^1_*)'} \leq C\|g_1\|_{0,q,\Omega}$. Thus the Lax-Milgram theorem (Theorem A.11) for the Hilbert space $H^1_*(\Omega)$ yields the existence of a unique $u_1 \in H^1_*(\Omega)$ such that for all $\phi \in H^1_*(\Omega)$

$$\int_\Omega g_1\phi \, dx = G(\phi) = B(u_1, \phi) = \int_\Omega u_1\phi \, dx + \int_\Omega \kappa_1 \nabla u_1 \nabla\phi \, dx.$$

In addition, we obtain the estimate

$$\|u_1\|_{1,2,\Omega} \leq \min(1, k_1^2)^{-1}\|g_1\|_{0,2,\Omega}.$$

There remains to prove that $u := (u_1, u_2) \in C^{0,\alpha_1}(\overline{\Omega}) \times C^{0,\alpha_2}(\overline{\Omega})$ for some $\alpha_1, \alpha_2 \in (0,1)$. We are going to apply the general result [Nit10], Theorem 3.1.5 for weak solutions in $H^1(\Omega)$, which is summarized in the Appendix of this thesis, see Theorem A.14. First, observe that u_1 also solves a weak problem in $H^1(\Omega)$: Let $\psi \in H^1(\Omega)$ and set $\alpha_\psi := \frac{1}{|\Omega|}\int_\Omega \psi \, dx$. Then $\phi := \psi - \alpha_\psi \in H^1_*(\Omega)$ and hence

$$\begin{aligned}
0 &= \int_\Omega (u_1 - g_1)(\psi - \alpha_\psi) \, dx + \int_\Omega \kappa_1 \nabla u_1 \nabla(\psi - \alpha_\psi) \, dx\\
&= \int_\Omega (u_1 - g_1)\psi \, dx + \int_\Omega \kappa_1 \nabla u_1 \nabla\psi \, dx + \frac{1}{|\Omega|}\int_\Omega g_1 \, dx \int_\Omega \psi \, dx\\
&= \int_\Omega (u_1 - \underbrace{\left(g_1 - \frac{1}{|\Omega|}\int_\Omega g_1 \, dx\right)}_{=:h_1 \in L_{q,*}(\Omega)})\psi \, dx + \int_\Omega \kappa_1 \nabla u_1 \nabla\psi \, dx.
\end{aligned}$$

Hence, if we define $A(x,p) := \kappa_1(x)p$ and $a(x,u) := (u - h_1)(x)$, then u_1 (now considered as an element of $H^1(\Omega)$) solves

$$\int_\Omega A(x, \nabla u_1)\nabla\psi \, dx + \int_\Omega a(x, u_1)\psi \, dx = 0 \quad (\psi \in H^1(\Omega)).$$

Since $p \cdot A(x,p) \geq k_1^2|p|^2$, $|A(x,p)| \leq K_1|p|$, and $|a(x,u)| \leq |u(x)| + |h_1(x)|$, Theorem A.14 applies with $m = 2$ and $\psi_i = 0$ $(1 \leq i \leq 5)$, $\psi_6 = 1 \in L_\infty(\Omega)$, and $\psi_7 = |h_1| \in L_q(\Omega)$. Note that $|\psi_7|_{0,q,\Omega} \leq (1 + |\Omega|)\|g_1\|_{0,q,\Omega}$. We obtain that u_1 is Hölder continuous up to the boundary with some exponent $\alpha_1 \in (0,1)$ in Ω and

$$\|u_1\|_{C^{0,\alpha_1}(\overline{\Omega})} \leq C\left(\|u_1\|_{0,2,\Omega} + (1 + |\Omega|)\|g_1\|_{0,q,\Omega}\right).$$

For u_2 the proof of existence in $H^1(\Omega)$ and of the Hölder estimate follows the same lines. □

Remark 3.10

(i) As an immediate consequence of the two preceding lemmas, the operator

$$T : [0,1] \times \mathbb{L}_\infty \to \mathbb{H}^1 \cap \mathbb{L}_\infty \subset \mathbb{L}_\infty, \quad (\delta, (w_1, w_2)) \mapsto (u_1, u_2) \tag{3.10}$$

is well-defined and enjoys the estimate

$$\begin{aligned} \|T(\delta,(w_1,w_2))\|_{0,\infty,\Omega} = \|u_1\|_{0,\infty,\Omega} + \|u_2\|_{0,\infty,\Omega} &\leq C\left(\|u_1\|_{C^{0,\alpha_1}(\overline{\Omega})} + \|u_2\|_{C^{0,\alpha_2}(\overline{\Omega})}\right) \\ &\leq C_T\left(2\|F((w_2 - w_1)(\cdot),\cdot)\|_{0,\infty,\Omega} + \|(f_1,f_2)\|_{0,q,\Omega} + \|(w_1,w_2)\|_{0,\infty,\Omega}\right), \end{aligned} \tag{3.11}$$

where C_T unites the constants of the estimates (3.7)-(3.9)*. In the sequel, we will consider T as an operator from the space $\mathbb{L}_\infty$ into itself. The estimate* (3.11) *and the continuity of $s \mapsto F(s,\cdot)$ yield that T maps bounded subsets of $\mathbb{L}_\infty$ into bounded subsets of $\mathbb{L}_\infty$.*

(ii) The assumption $f_i \in L_q(\Omega)$ for $q \geq \max\left(2, \frac{n}{2} + \varepsilon\right)$ was chosen because on the one hand, we require the right-hand sides g_i to define a functional on $H^1(\Omega)$ for the Lax-Milgram Theorem and on the other hand the preliminaries of Theorem A.14, i.e., $g_i \in L^q$ for some $q = \frac{n}{2} + \varepsilon$ (with $n \geq 2$ and $\varepsilon > 0$ or $q = 1$ if $n = 1$) have to be fulfilled to obtain the Hölder estimates.

(iii) The bilinear form B from the preceding proof is in fact continuous and coercive as a mapping $B : H^1 \times H^1 \to \mathbb{R}$. Similarly, the functional G of course belongs to $H^1(\Omega)'$ as well. Since we are interested to find a solution $u_1 \in H^1_*(\Omega)$ we have to consider their restrictions to this closed subspace of $H^1(\Omega)$.
Of course, this procedure works for u_2 as well and existence of a weak solution $(u_1, u_2) \in (H^1_*(\Omega) \cap L_\infty(\Omega))^2$ (now for both unknown functions in the sense of equation (3.6a)*) can be shown analogously. This fact is relevant for uniqueness if one would like to consider the nonlinear problem and the nonlinearity F satisfying* (3.1) *with $C_0 = 0$, see Lemma 3.6.*

Remark 3.11
As a strategy to prove the existence of a solution to the nonlinear problem, we will work towards an application of the Leray-Schauder principle (Theorem A.9) in order to obtain $w_1 = u_1$, $w_2 = u_2$, and $\delta = 1$ in the linearized problem, i.e., a fixed point for the mapping $T(1,\cdot) : \mathbb{L}_\infty \to \mathbb{L}_\infty$. Let us briefly check that $T(0,w) = 0$ for all $w \in \mathbb{L}_\infty$. This follows from the fact that $(0,0)$ is a solution to (3.6a)-(3.6b) *and this solution is unique according to Lemma 3.8.*

The remaining preliminaries of Theorem A.9 are contained in the subsequent lemmas.

Lemma 3.12
Let $\delta \in [0,1]$ be fixed. Then $T(\delta,\cdot) : \mathbb{L}_\infty \to \mathbb{L}_\infty$ is continuous and compact.

Proof. Let $w := (w_1, w_2)$, $\widetilde{w} := (\widetilde{w}_1, \widetilde{w}_2) \in \mathbb{L}_\infty$ and define $u := (u_1, u_2) := T(\delta, w)$ as well as $\widetilde{u} := (\widetilde{u}_1, \widetilde{u}_2) := T(\delta, \widetilde{w})$. Furthermore, we abbreviate $g_i := \delta((-1)^{i+1}F(w_2 - w_1, \cdot) + w_i + f_i)$ for $i = 1, 2$ and set $h_1 := g_1 - \frac{1}{|\Omega|}\int_\Omega g_1 \, dx$. The functions $\widetilde{g}_i$ and $\widetilde{h}_1$ are defined similarly. Then $u_1 - \widetilde{u}_1$ fulfills

$$\int_\Omega (u_1 - \widetilde{u}_1)\phi \, dx + \int_\Omega \kappa_1 \nabla(u_1 - \widetilde{u}_1)\nabla\phi \, dx - \int_\Omega (h_1 - \widetilde{h}_1)\phi \, dx = 0 \quad (\phi \in H^1(\Omega)).$$

The Lax-Milgram theorem particularly yields an estimate of $\|u_1 - \widetilde{u}_1\|_{0,2,\Omega}$ (cf.(3.7)) and hence, from Theorem A.14 we deduce $\|u_1 - \widetilde{u}_1\|_{C^{0,\alpha_1}(\overline{\Omega})} \le C\|h_1 - \widetilde{h}_1\|_{0,q,\Omega}$. Using the embedding $C^{0,\alpha_1}(\overline{\Omega}) \hookrightarrow L_\infty(\Omega)$, there remains to show that $\|h_1 - \widetilde{h}_1\|_{0,q,\Omega} < \varepsilon$ if $\|w - \widetilde{w}\|_{0,\infty,\Omega} < \eta$. This follows from the continuity of $s \mapsto F(s,\cdot)$ from $\mathbb{R}$ into $L_\infty(\Omega)$:

$$\|h_1 - \widetilde{h}_1\|_{0,\infty,\Omega} \le (1 + |\Omega|)\|F(w_2 - w_1) - F(\widetilde{w}_2 - \widetilde{w}_1)\|_{0,\infty,\Omega} + \|w_1 - \widetilde{w}_1\|_{0,\infty,\Omega},$$

since $\|F((w_2 - w_1)(\cdot),\cdot) - F((\widetilde{w}_2 - \widetilde{w}_1)(\cdot),\cdot)\|_{0,\infty,\Omega} < \varepsilon$ for $\|w - \widetilde{w}\|_{0,\infty,\Omega} < \eta$. The proof for the estimate of $\|u_2 - \widetilde{u}_2\|_{C^{0,\alpha_2}(\overline{\Omega})}$ follows the same lines.

For the compactness, let a bounded sequence $(w^{(k)})_{k\in\mathbb{N}} \subset \mathbb{L}_\infty$ be given. Lemma 3.9 yields a bounded sequence $u^{(k)} := T(\delta, w^{(k)}) \subset \mathbb{L}_\infty \cap C^{0,\alpha_1^{(k)}}(\overline{\Omega}) \times C^{0,\alpha_2^{(k)}}(\overline{\Omega})$ and since the Hölder indices only depend on $\|w^{(k)}\|_{0,\infty,\Omega}$ as well as δ, n, and Ω, there are α_1 and α_2, such that $\alpha_i^{(k)} \ge \alpha_i$ for all $k \in \mathbb{N}$. Hence, $u^{(k)}$ is bounded in $C^{0,\alpha_1}(\overline{\Omega}) \times C^{0,\alpha_2}(\overline{\Omega}) \overset{c}{\hookrightarrow} C^{0,\alpha_1/2}(\overline{\Omega}) \times C^{0,\alpha_2/2}(\overline{\Omega}) \hookrightarrow L_\infty(\Omega)^2$, yielding the desired convergent subsequence. □

Lemma 3.13
For every bounded set $X_0 \subset \mathbb{L}_\infty$ and all $\varepsilon > 0$, there exists $\eta > 0$ such that for all $w \in X_0$ and all $\delta, \widetilde{\delta} \in [0,1]$ with $|\delta_1 - \widetilde{\delta}| < \eta$, there holds $\|T(\delta_1, w) - T(\widetilde{\delta}, w)\|_{0,\infty,\Omega} < \varepsilon$.

Proof. Let $\delta, \widetilde{\delta} \in [0,1]$. For arbitrary $w = (w_1, w_2) \in X_0$ set $u := (u_1, u_2) := T(\delta, w)$ and $\widetilde{u} := (\widetilde{u}_1, \widetilde{u}_2) := T(\widetilde{\delta}, w)$. It is easy to see that $u_1 - \widetilde{u}_1$ then satisfies

$$\int_\Omega (u_1 - \widetilde{u}_1)\phi \, dx + \int_\Omega \kappa_1 \nabla(u_1 - \widetilde{u}_1)\nabla\phi \, dx + (\widetilde{\delta} - \delta)\int_\Omega h_1\phi \, dx = 0 \quad (\phi \in H^1(\Omega)),$$

where we abbreviated $h_1 := F(w_2 - w_1, \cdot) + w_1 + f_1 - \frac{1}{|\Omega|}\int_\Omega (F((w_2 - w_1)(y), y) + f_1(y)) \, dy$. Recall that $\int_\Omega w_1 \, dx = 0$. Due to the continuity of $s \mapsto F(s,\cdot)$ and the boundedness $\|w\|_{0,\infty,\Omega} \le K_0$, we infer that $\|F((w_2 - w_1)(\cdot),\cdot)\|_{0,\infty,\Omega} \le K$ and hence, $\|h_1\|_{0,q,\Omega} \le (1 + |\Omega|)(K + \|f_1\|_{0,q,\Omega}) + K_0$, independent on the choice of $w \in X_0$. Using the Lax-Milgram theorem and Theorem A.14, we obtain

$$\|u_1 - \widetilde{u}_1\|_{0,\infty,\Omega} \le \|u_1 - \widetilde{u}_1\|_{C^{0,\alpha}(\Omega)} \le C|\widetilde{\delta} - \delta|\|h_1\|_{0,q,\Omega}.$$

For $u_2 - \widetilde{u}_2$ the calculation is even simpler, so that we have proved that $T(\cdot, w) : [0,1] \to \mathbb{L}_\infty$ is Lipschitz continuous with a Lipschitz constant independent from the particular choice of $w \in X_0$. The lemma is proved. □

The remaining condition in Theorem A.9 is the a priori boundedness of possible fixed points in $\mathbb{L}_\infty$: there is a constant $K_T \ge 0$ (independent of u and δ) such that, if $T(\delta, (u_1, u_2)) = (u_1, u_2)$ is satisfied for some $\delta \in [0,1]$, then already $\|(u_1, u_2)\|_{0,\infty,\Omega} \le K_T$ holds. This a priori boundedness in $\mathbb{L}_\infty$ will be obtained via an iteration scheme, which is comparable to the well-known method of Moser iteration, which was successfully employed to deduce L_∞-estimates of weak solutions, see [GT77], Section 8.5 for an introduction to this method.

For the remaining part of the present subsection, we suppose that $T(\delta, (u_1, u_2)) = (u_1, u_2)$ is satisfied for some $\delta \in [0,1]$ and an element $u = (u_1, u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$. We start with a uniform L_2-estimate for such u. This can be regarded as the start of an induction argument.

Lemma 3.14
There are positive constants $M_i = M_i(n, \Omega, f_1, f_2, C_1, C_2, k_1, k_2)$ such that

(i) $\|u_i\|_{0,2,\Omega} \le M_i \quad (i = 1, 2)$,

(ii) $\delta \left| \int_\Omega F((u_2 - u_1)(x), x)\ dx \right| \le M_3$.

The constants C_i and k_i stem from Assumption 3.1 on the nonlinearity and the coefficients κ_i.

Proof. We test (3.6a) with $\phi := u_1$, (3.6b) with $\psi := u_2$ and sum up the resulting equalities:

$$\int_\Omega \kappa_1 |\nabla u_1|^2\ dx + \int_\Omega \kappa_2 |\nabla u_2|^2\ dx + (1-\delta)\|u_1\|^2_{0,2,\Omega} + (1-\delta)\|u_2\|^2_{0,2,\Omega} + \delta \int_\Omega F((u_2 - u_1)(x), x)(u_2 - u_1)(x)\ dx = \delta \left(\int_\Omega f_1 u_1\ dx + \int_\Omega f_2 u_2\ dx \right).$$

With the assumption on the coefficients, the property (3.2) of F, and by applying Hölder's and Young's inequalities to the first integral on the right-hand side, we can estimate

$$k_1^2 \|\nabla u_1\|^2_{0,2,\Omega} + k_2^2 \|\nabla u_2\|^2_{0,2,\Omega} + (1-\delta)\|u_1\|^2_{0,2,\Omega} + (1-\delta)\|u_2\|^2_{0,2,\Omega} + \delta C_1 \|u_2 - u_1\|^2_{0,2,\Omega} \le \frac{\varepsilon}{2}\|u_1\|^2_{0,2,\Omega} + \delta \left(C_2 + \frac{\delta}{2\varepsilon}\|f_1\|^2_{0,2,\Omega} + \int_\Omega f_2 u_2\ dx \right).$$

For u_1 we have the second Poincaré inequality (A.10) which leads to

$$\left(\frac{k_1^2}{C_P} - \frac{\varepsilon}{2} + 1 - \delta \right) \|u_1\|^2_{0,2,\Omega} + k_2^2 \|\nabla u_2\|^2_{0,2,\Omega} + (1-\delta)\|u_2\|^2_{0,2,\Omega} + \delta C_1 \|u_2 - u_1\|^2_{0,2,\Omega} \le \delta \left(C_2 + \frac{\delta}{2\varepsilon}\|f_1\|^2_{0,2,\Omega} + \int_\Omega f_2 u_2\ dx \right).$$

Hence, if we choose $\varepsilon := \frac{k_1^2}{C_P}$, we obtain the estimates

$$\|u_1\|^2_{0,2,\Omega} \le 2\frac{\delta C_P C_2}{k_1^2} + \left(\frac{\delta C_P}{k_1^2} \right)^2 \|f_1\|^2_{0,2,\Omega} + 2\frac{\delta C_P}{k_1^2}\|u_2\|_{0,2,\Omega}\|f_2\|_{0,2,\Omega},$$
$$(1-\delta)\|u_2\|^2_{0,2,\Omega} + \delta C_1 \|u_2 - u_1\|^2_{0,2,\Omega} \le 2\delta \left(C_2 + \frac{\delta C_P}{2k_1^2}\|f_1\|^2_{0,2,\Omega} + \|u_2\|_{0,2,\Omega}\|f_2\|_{0,2,\Omega} \right).$$

Now, we write $u_2 = \delta(u_2 - u_1) + (1-\delta)u_2 + \delta u_1$ and obtain with the above estimates, $\delta \le 1$, and Young's inequality

$$\begin{aligned} \frac{1}{3}\|u_2\|^2_{0,2,\Omega} &\le \delta\|u_2 - u_1\|^2_{0,2,\Omega} + (1-\delta)\|u_2\|^2_{0,2,\Omega} + \delta\|u_1\|^2_{0,2,\Omega} \\ &\le C_1' \left(C_2 + \frac{C_P}{2k_1^2}\|f_1\|^2_{0,2,\Omega} + \|u_2\|_{0,2,\Omega}\|f_2\|_{0,2,\Omega} \right) \\ &\le C_1' C_2 + \frac{C_1' C_P}{2k_1^2}\|f_1\|^2_{0,2,\Omega} + \frac{C_1'^2}{2\varepsilon}\|f_2\|^2_{0,2,\Omega} + \frac{\varepsilon}{2}\|u_2\|_{0,2,\Omega}, \end{aligned}$$

with $C_1' := 2\left(1 + \frac{1}{C_1} + \frac{C_P}{k_1^2}\right)$. Choosing $\varepsilon := \frac{1}{3}$, we obtain

$$\|u_2\|^2_{0,2,\Omega} \le 6 \left(C_1' C_2 + \frac{C_1' C_P}{2k_1^2}\|f_1\|^2_{0,2,\Omega} + \frac{3C_1'^2}{2}\|f_2\|^2_{0,2,\Omega} \right) =: M_2^2.$$

Now, going back to u_1, we find

$$\|u_1\|^2_{0,2,\Omega} \le 2\frac{C_P C_2}{k_1} + \left(\frac{C_P}{k_1^2} \right)^2 \|f_1\|^2_{0,2,\Omega} + 2\frac{C_P}{k_1^2} M_2 \|f_2\|_{0,2,\Omega} =: M_1^2.$$

To prove the second assertion, we test equation (3.6b) with $1 \in H^1(\Omega)$ and rearrange the expressions which leads to

$$\delta \int_\Omega F((u_2-u_1)(x),x)\ dx = \delta \int_\Omega f_2\ dx - (1-\delta)\int_\Omega u_2\ dx,$$

and with the results from the first part, we have

$$\delta \left| \int_\Omega F((u_2-u_1)(x),x)\ dx \right| \le \|f_2\|_{0,1,\Omega} + (1-\delta)\|u_2\|_{0,1,\Omega} \le M_3.$$

□

Remark 3.15
Note that the latter method of estimating $\left|\int_\Omega F((u_2-u_1)(x),x)\ dx\right|$ fails in the case $C_0 = 0$ (see (3.1) of Assumption 3.1): in the weak form of the equations for u_2 only test functions $\psi \in H^1_*(\Omega)$ are admissible because we then also seek $u_2 \in H^1_*(\Omega)$ in order to have uniqueness for the nonlinear system (see Remark 3.6). Hence, the choice $\psi = 1$ is no longer possible.

From now, we impose the stronger assumption on the right-hand sides

$$f_1, f_2 \in L_\infty(\Omega),$$

which will be required to deduce the uniform L_∞-bound for functions $(u_1,u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$ with $T(\delta,(u_1,u_2)) = (u_1,u_2)$. The next lemma contains the essential estimate, which will enable us to carry out the iteration scheme. Note that we already know from Lemma 3.9 that any solution of (3.6a)-(3.6b) belongs to $\mathbb{L}_\infty(\Omega)$, but we need a uniform L_∞-bound for possible solutions of the corresponding nonlinear problem and up to now we only proved a uniform L_2-bound in the preceding lemma.

Lemma 3.16
Suppose there is $(u_1,u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$ with $T(\delta,(u_1,u_2)) = (u_1,u_2)$ for some $\delta \in [0,1]$. Let $p \in \mathbb{N}$ be an even integer. Then there exists a positive constant $C = C(n,\Omega,f_1,f_2,k_1,k_2)$, independent of u_i, such that

$$\left\|\nabla(u_1^{p/2})\right\|^2_{0,2,\Omega} + \left\|\nabla(u_2^{p/2})\right\|^2_{0,2,\Omega} \le C\ p\left(\|u_1\|^{p-1}_{0,p-1,\Omega} + \|u_2\|^{p-1}_{0,p-1,\Omega}\right). \tag{3.12}$$

Proof. We have $u_i^{p-1} \in H^1(\Omega)$ by the chain rule and Hölder's inequality:

$$\left\|\nabla\left(u_i^{p-1}\right)\right\|_{0,2,\Omega} = \left\|(p-1)u_i^{p-2}\nabla u_i\right\|_{0,2,\Omega} \le (p-1)\|u_i\|^{p-2}_{0,\infty,\Omega}\ \|\nabla u_i\|_{0,2,\Omega} < \infty.$$

First, we test (3.6a) (where now $(w_1,w_2) = (u_1,u_2)$) with $\phi := u_1^{p-1} - \frac{1}{|\Omega|}\int_\Omega u_1^{p-1}\ dx\ \in H^1_*(\Omega)$ and (3.6b) (again with $(w_1,w_2) = (u_1,u_2)$) with $\psi := u_2^{p-1} \in H^1(\Omega)$. Note that the occurring term $(1-\delta)\left(\frac{1}{|\Omega|}\int_\Omega u_1^{p-1}\ dx\right)\int_\Omega u_1\ dx$ vanishes because of $u_1 \in H^1_*(\Omega)$. After a rearrangement of the terms, the sum of the resulting equations reads

$$\begin{aligned}
&\int_\Omega \kappa_1 \nabla u_1 \nabla(u_1^{p-1})\ dx + \int_\Omega \kappa_2 \nabla u_2 \nabla(u_2^{p-1})\ dx + (1-\delta)\int_\Omega (u_1^p + u_2^p)\ dx \\
&= \delta\left(\int_\Omega f_1 u_1^{p-1}\ dx + \int_\Omega f_2 u_2^{p-1}\ dx - \int_\Omega F((u_2-u_1)(x),x)(u_2^{p-1}-u_1^{p-1})(x)\ dx\right) \\
&\quad - \delta\left(\frac{1}{|\Omega|}\int_\Omega u_1^{p-1}\ dx\right)\left(\int_\Omega f_1\ dx\right) - \delta\left(\frac{1}{|\Omega|}\int_\Omega u_1^{p-1}\ dx\right)\left(\int_\Omega F((u_2-u_1)(x),x)\ dx\right)
\end{aligned} \tag{3.13}$$

We will now estimate each term of the right-hand side from above:

(i) Condition (3.2) on F and Lemma 3.3 (i) show that

$$\begin{aligned} & -\int_\Omega F((u_2-u_1)(x),x)(u_2^{p-1}-u_1^{p-1})(x)\ dx \\ = & \int_\Omega -F((u_2-u_1)(x),x)\operatorname{sgn}(u_2-u_1)\left|(u_2^{p-1}-u_1^{p-1})(x)\right|\ dx \\ \le & \ C_3\int_\Omega \left|(u_2^{p-1}-u_1^{p-1})(x)\right|\ dx \le C_3\int_\Omega \left(|u_2(x)|^{p-1}+|u_1(x)|^{p-1}\right)\ dx. \end{aligned}$$

Recall that p is even, hence $(u_2^{p-1}-u_1^{p-1})$ and (u_2-u_1) have the same sign.

(ii)
$$\int_\Omega f_i u_i^{p-1}\ dx \le \|f_i\|_{0,\infty,\Omega}\int_\Omega |u_i|^{p-1}\ dx \quad (i=1,2).$$

(iii)
$$\delta\left|\frac{1}{|\Omega|}\int_\Omega u_1^{p-1}\ dx\ \int_\Omega f_1\ dx\right| \le \frac{\|f_1\|_{0,1,\Omega}}{|\Omega|}\int_\Omega |u_1|^{p-1}\ dx.$$

(iv) By Lemma 3.14 (ii), we can estimate

$$\delta\left|\frac{1}{|\Omega|}\int_\Omega u_1^{p-1}\ dx\ \int_\Omega F((u_2-u_1)(x),x)\ dx\right| \le \frac{M_3}{|\Omega|}\int_\Omega |u_1|^{p-1}\ dx.$$

We use (i)-(iv) in (3.13) and sum up the constants to $C=C(|\Omega|,M_3,f_1,f_2)$ to obtain

$$\begin{aligned} (3.14)\quad \int_\Omega \kappa_1\nabla u_1\nabla(u_1^{p-1})\ dx+\int_\Omega \kappa_2\nabla u_2\nabla(u_2^{p-1})\ dx+(1-\delta)\int_\Omega (u_1^p+u_2^p)\ dx \\ \le C\int_\Omega |u_1|^{p-1}+|u_2|^{p-1}\ dx. \end{aligned}$$

Since $p-2$ is even, the first two integrals can be estimated from below:

$$\int_\Omega \kappa_i\nabla u_i\nabla(u_i^{p-1})\ dx \ge (p-1)k_i^2\int_\Omega |u_i|^{p-2}|\nabla u_i|^2\ dx = \frac{4(p-1)k_i^2}{p^2}\int_\Omega \left|\nabla(u_i^{p/2})\right|^2\ dx.$$

Combining this and (3.14), the claim easily follows. □

We can now prove the desired L_∞-estimate:

Theorem 3.17
Let $1\le n\le 4$ and assume $T(\delta,(u_1,u_2))=(u_1,u_2)$ holds for some $(u_1,u_2)\in \mathbb{H}^1\cap\mathbb{L}_\infty$ and $\delta\in[0,1]$. Then there is a constant $C=C(n,\Omega,k_i,f_i)$ independent of u such that

$$(3.15)\qquad \|(u_1,u_2)\|_{0,\infty,\Omega}\le C.$$

Proof. **Step 1.** Let us briefly consider the case $n=1$: Since $H^1(\Omega)\hookrightarrow C^{0,\alpha}(\overline{\Omega})$ with $\alpha\in(0,\frac12]$, Lemma 3.16 for $p=2$ and Lemma 3.14 yield

$$\|u_i\|_{0,\infty,\Omega}\le C(\|u_i\|_{0,2,\Omega}+\|\nabla u_i\|_{0,2,\Omega})\le C(\|u_1\|_{0,2,\Omega}+\|u_2\|_{0,2,\Omega})\le C.$$

Step 2. Let $p\in\mathbb{N}$ be an even integer. We can assume $p/2<n$, because else we can argue as in Step 1. By Lemma 3.16 we have $u_i^{p/2}\in H^1(\Omega)$ and may apply Sobolev's embedding theorem ([AF03], Theorem 4.12) which reads as

$$(3.16)\qquad H^1(\Omega)\hookrightarrow L_{2\chi}(\Omega),\quad \text{i.e.,}\quad \|v\|_{0,2\chi,\Omega}\le C\|v\|_{1,2,\Omega}\quad (v\in H^1(\Omega)).$$

Here, $\chi := \frac{n}{n-2}$ for $n \geq 3$ and χ can be chosen arbitrarily large for $n = 2$. For $v = u_i^{p/2}$, this yields

$$\|u_i\|_{0,p\chi,\Omega}^p = \left\|u_i^{p/2}\right\|_{0,2\chi,\Omega}^2 \leq C\left\|u_i^{p/2}\right\|_{1,2,\Omega}^2 = C\left(\|u_i\|_{0,p,\Omega}^p + \left\|\nabla\left(u_i^{p/2}\right)\right\|_{0,2,\Omega}^2\right)$$
$$\leq C\sum_{i=1}^{2}\left(\|u_i\|_{0,p,\Omega}^p + p\|u_i\|_{0,p-1,\Omega}^{p-1}\right) \leq C\sum_{i=1}^{2}\left(\|u_i\|_{0,p,\Omega}^p + p|\Omega|^{1/p}\|u_i\|_{0,p,\Omega}^{p-1}\right),$$

where we used Hölder's inequality in the last step. We now want to consider the case when p approaches infinity and may assume $\|u_i\|_{0,p,\Omega} \geq 1$. Then the above estimate simplifies to

$$\|(u_1,u_2)\|_{0,p\chi,\Omega} \leq (Cp)^{1/p}\|(u_1,u_2)\|_{0,p,\Omega}. \tag{3.17}$$

We consider the sequence $p_k := 2\chi^k$ for $k \in \mathbb{N}_0$. Since the fundamental estimate (3.12) in our proof is only valid for even $p \in \mathbb{N}$ and we have to start with $p_0 = 2$, because we have a uniform estimate for this case, we have to ensure that in each iteration step, the improved index of integrability is greater or equal to the next even number: $p_{k+1} = p_k\chi \geq p_k + 2$. This leads to the condition $n \leq p_0 + 2$ and hence, the argument works for $n \leq 4$. In particular, $2\chi^k$ is always an even number in this case, since $\chi = 3$ for $n = 3$ and $\chi = 2$ for $n = 4$. In the case $n > p_0 + 2$ one has to prove a uniform estimate in $L_{n-2}(\Omega)$ before one can argue with the presented iteration scheme again. With these choices, an iterated application of (3.17) leads to

$$\|(u_1,u_2)\|_{0,p_{k+1},\Omega} \leq (Cp_k)^{1/p_k}\|(u_1,u_2)\|_{0,p_k,\Omega}$$
$$\leq (C\chi)^{k/p_k}\|(u_1,u_2)\|_{0,p_k,\Omega} \leq (C\chi)^{\sum_{j=0}^{k}\frac{j}{\chi^j}}\|(u_1,u_2)\|_{0,2,\Omega}$$

and $\displaystyle\sum_{j=0}^{\infty}\frac{j}{\chi^j}$ is finite, since $\frac{j+1}{\chi^{j+1}}\frac{\chi^j}{j} = \frac{j+1}{j\chi} < 1$ because of $\chi \geq 2$ and thus the right-hand side is independent of k. Passing to the limit $k \to \infty$ yields

$$\|(u_1,u_2)\|_{0,\infty,\Omega} \leq C\|(u_1,u_2)\|_{0,2,\Omega} \leq C.$$

□

We finally state our main result of this section.

Theorem 3.18
Let $1 \leq n \leq 4$. Let $\kappa_i \in L_\infty(\Omega)$ such that $\kappa_i \geq k_i^2 > 0$ and assume that $F : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ satisfies Assumption 3.1. Then given $f_1, f_2 \in L_\infty(\Omega)$, the nonlinear system (3.3) with homogeneous Neumann boundary conditions admits a unique weak solution $(u_1,u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$ in the sense that (3.4a)-(3.4b) is satisfied. There is a constant $K = K(F, f_1, f_2, \kappa_1, \kappa_2, \Omega, n)$ such that

$$\|u\|_{1,2,\Omega} + \|u\|_{0,\infty,\Omega} \leq K.$$

In addition, there are $\alpha_i \in (0,1)$ and a constant $C > 0$ with the same dependencies as K such that $u \in C^{0,\alpha_1}(\overline{\Omega}) \times C^{0,\alpha_2}(\overline{\Omega})$ with

$$\|u_1\|_{C^{0,\alpha_1}(\overline{\Omega})} + \|u_2\|_{C^{0,\alpha_2}(\overline{\Omega})} \leq C.$$

Proof. The preliminaries of the Leray-Schauder fixed point theorem (Theorem A.9) for the operator $T : [0,1] \times \mathbb{L}_\infty \to \mathbb{L}_\infty$ are contained in Remarks 3.10 and 3.11, Lemmas 3.12 and 3.13, and Theorem 3.17. Thus, there exists $u \in \mathbb{L}_\infty$ such that $T(1,u) = u$, i.e., a solution to (3.3) in the sense that (3.4a)-(3.4b) is satisfied and $\|(u_1,u_2)\|_{0,\infty,\Omega} \leq C$. Taking the functions $\widetilde{f_1} := f_1 + u_1 + F(u_2 - u_1, \cdot) \in L_\infty(\Omega)$ and $\widetilde{f_2} := f_2 + u_2 - F(u_2 - u_1, \cdot) \in L_\infty(\Omega)$ as data for the equations (3.6a)-(3.6b), we deduce from Lemma 3.9 that $u \in \mathbb{H}^1$. The solution u is unique according to Lemma 3.6 and using the estimates (3.8) and (3.9) as well as the continuity of F, we deduce the bound for $\|u_i\|_{C^{0,\alpha_i}}$. □

3.2 Generalizations

3.2.1 Inhomogeneous Neumann boundary conditions

Lemma 3.19
Let $F : \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ satisfy Assumption 3.1. For given $U \in L_\infty(\Omega)$, consider the function $G : \mathbb{R} \times \Omega \to \mathbb{R}$, $(s, x) \mapsto F(s - U(x), x)$. Then G satisfies Assumption 3.1 as well.

Proof. Obviously $s \mapsto G(s, \cdot)$ belongs to $C(\mathbb{R}; L_\infty(\Omega))$, too. For a.e. $x \in \Omega$, the estimate (3.1) follows from

$$\begin{aligned}(G(s,x) - G(t,x))(s-t) &= (F(s-U(x),x) - F(t-U(x),x))(s-U(x)-(t-U(x))) \\ &\geq C_0(s-U(x)-(t-U(x)))^2 = C_0(s-t)^2,\end{aligned}$$

and for the second estimate we first employ $\|U\|_{0,\infty,\Omega} \leq K$ which together with (3.1) yields $F(s-K,x) \leq G(s,x) \leq F(s+K,x)$. For $s > 0$, we find

$$\begin{aligned}G(s,x)s &\geq F(s-K,x)s \geq C_1(s-K)^2 - C_2 + KF(s-K,x) \\ &\geq C_1 s^2 - 2C_1 sK + C_1K^2 - C_2 + KF(-K,x) \\ &\geq \tfrac{1}{2}C_1 s^2 - C_1K^2 - C_2 - KC_3 = \tfrac{1}{2}C_1 s^2 - C_4.\end{aligned}$$

Here, we used the fact that $F(-K, \cdot) \in L_\infty(\Omega)$. For negative s the inequality follows from

$$\begin{aligned}G(s,x)s &\geq F(s+K,x)s \geq C_1(s+K)^2 - C_2 - KF(s+K,x) \\ &\geq C_1 s^2 + 2C_1 sK + C_1K^2 - C_2 - KF(K,x) \\ &\geq \frac{1}{2}C_1 s^2 - C_1K^2 - C_2 - KC_3 = \tfrac{1}{2}C_1 s^2 - C_4.\end{aligned}$$

□

On the one hand, the preceding lemma allows us to include additional physical functions into the nonlinearity (e.g. the open circuit potential for a battery, see Chapter 7). From a mathematical point of view, we are now in the position to treat inhomogeneous Neumann boundary conditions as well. We consider the system

$$\left\{\begin{array}{rcll} -\nabla\cdot(\kappa_1\nabla u_1) &=& F(u_2-u_1,x)+f_1 & \text{in } \Omega, \\ -\nabla\cdot(\kappa_2\nabla u_2) &=& -F(u_2-u_1,x)+f_2 & \text{in } \Omega, \\ \kappa_1\frac{\partial u_1}{\partial\nu} &=& g_1, \quad \kappa_2\frac{\partial u_2}{\partial\nu} = g_2 & \text{on } \partial\Omega, \end{array}\right. \tag{3.18}$$

with boundary values $g_i \in L_\infty(\partial\Omega)$. We reduce to the case of homogeneous boundary conditions by considering the linear Neumann problems

$$\left\{\begin{array}{rcll} -\nabla\cdot(\kappa_1\nabla w_1) &=& h_1 & \text{in } \Omega, \\ \kappa_1\frac{\partial w_1}{\partial\nu} &=& g_1 & \text{on } \partial\Omega, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rcll} -\nabla\cdot(\kappa_2\nabla w_2) &=& h_2 & \text{in } \Omega, \\ \kappa_2\frac{\partial w_2}{\partial\nu} &=& g_2 & \text{on } \partial\Omega, \end{array}\right.$$

with functions $h_1, h_2 \in L_\infty(\Omega)$ such that $\int_\Omega h_i \, dx = \int_{\partial\Omega} g_i \, dS(x)$ and $\|h_i\|_{0,\infty,\Omega} \leq C\|g_i\|_{0,\infty,\partial\Omega}$ for $i = 1, 2$. For instance, we can explicitly define $h_i := \chi_\Omega \frac{1}{|\Omega|}\left(\int_{\partial\Omega} g_i \, dS(x)\right)$. These conditions are necessary and sufficient for the existence of unique weak solutions $w_1, w_2 \in H^1_*(\Omega)$ to the above problems and [Nit10], Theorem 3.1.5 again yields $w_i \in C^{0,\alpha_i}(\overline{\Omega})$, which ensures Hölder continuity and an estimate of $\|u_i\|_{C^{\alpha_i}(\overline{\Omega})}$ against the norms of the data in their respective spaces. There remains to find a solution $v \in \mathbb{H}^1 \cap \mathbb{L}_\infty$ to the system

$$\left\{\begin{array}{rcll} -\nabla\cdot(\kappa_1\nabla v_1) &=& F(v_2+w_2-(v_1+w_1),x)+f_1-h_1 & \text{in } \Omega, \\ -\nabla\cdot(\kappa_2\nabla v_2) &=& -F(v_2+w_2-(v_1+w_1),x)+f_2-h_2 & \text{in } \Omega, \\ \frac{\partial v_1}{\partial\nu} = \frac{\partial v_2}{\partial\nu} &=& 0 & \text{on } \partial\Omega, \end{array}\right.$$

which again admits the structure and the same properties as the system (3.3) but the nonlinearity is now given by $G(v_2 - v_1, x) := F(v_2 - v_1 + (w_2 - w_1), x)$. Theorem 3.18 yields a unique solution $v \in \mathbb{H}^1 \cap \mathbb{L}_\infty$ and then $u := (v_1 + w_1, v_2 + w_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$ solves (3.18).

Corollary 3.20
Under the preliminaries stated in Theorem 3.18 and the Assumption 3.1 on F with $C_0 > 0$, let boundary values $g_1, g_2 \in L_\infty(\partial\Omega)$ be given. Then the system (3.18) possesses a unique weak solution $u = (u_1, u_2) \in \mathbb{H}^1 \cap \mathbb{L}_\infty$, i.e.:

$$\text{(3.19a)} \quad \int_\Omega \kappa_1 \nabla u_1 \nabla \phi \, dx - \int_\Omega (F(u_2 - u_1) + f_1)\, \phi \, dx - \int_{\partial\Omega} g_1 \phi \, dS(x) = 0, \quad (\phi \in H^1_*(\Omega)),$$

$$\text{(3.19b)} \quad \int_\Omega \kappa_2 \nabla u_2 \nabla \psi \, dx + \int_\Omega (F(u_2 - u_1, x) - f_2)\psi \, dx - \int_{\partial\Omega} g_2 \psi \, dS(x) = 0, \quad (\psi \in H^1(\Omega)).$$

Furthermore, the solution is bounded in $\mathbb{H}^1$ by a constant C_1 which only depends on the quantities $n, \Omega, \|f_i\|_{0,\infty,\Omega}, \|g_i\|_{0,\infty,\partial\Omega}, k_i$, and K_i and there are $\alpha_i \in (0,1)$ and a positive constant C_2, which additionally depends on α_i such that

$$\|u_1\|_{C^{0,\alpha_1}(\overline{\Omega})} + \|u_2\|_{C^{0,\alpha_2}(\overline{\Omega})} \leq C_2.$$

Proof. The procedure we sketched above yields a solution in the claimed spaces. The uniqueness is proved in the same way as in Lemma 3.6. □

3.2.2 Dirichlet and mixed boundary conditions

The case when homogeneous or inhomogeneous Dirichlet conditions are imposed on some relatively open set $\Gamma_D \subset \partial\Omega$ is also of mathematical and physical interest and we are going to generalize the results we have obtained for pure Neumann boundary conditions so far to this setting. We consider the system

$$\text{(3.20)} \qquad \begin{cases} -\nabla \cdot (\kappa_1 \nabla u_1) = F(u_2 - u_1, x) + f_1 & \text{in } \Omega, \\ -\nabla \cdot (\kappa_2 \nabla u_2) = -F(u_2 - u_1, x) + f_2 & \text{in } \Omega, \\ u_1 = g_1, \quad u_2 = g_2 & \text{on } \Gamma_D, \\ \kappa_1 \frac{\partial u_1}{\partial \nu} = h_1, \quad \kappa_2 \frac{\partial u_2}{\partial \nu} = h_2 & \text{on } \Gamma_N \end{cases}$$

under the following

Assumption 3.21 *(i) $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and the boundary admits a splitting of the form $\partial\Omega = \Gamma_D \cup \Gamma_N$ such that $\lambda_{n-1}(\Gamma_D \cap \Gamma_N) = 0$ and $\lambda_{n-1}(\Gamma_D) > 0$. In addition, we assume that the boundary $\partial\Omega$ is regular in the sense of Definition A.15.*

(ii) The data satisfy $f_1, f_2 \in L_\infty(\Omega)$, $g_1, g_2 \in W_m^{1-1/m}(\partial\Omega)$ for some $m > n$ as well as $h_1, h_2 \in L_\infty(\partial\Omega)$. By the surjectivity of the trace operator $\gamma_{\partial\Omega} : W_m^1(\Omega) \to W_m^{1-1/m}(\partial\Omega)$ (cf. Theorem A.2 (ii)), there exists $w_i \in W_m^1(\Omega)$ such that $\gamma_{\partial\Omega} w_i = g_i$ and with Sobolev's embedding we also have $w_i \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0, 1 - n/m)$.

(iii) For the coefficients $\kappa_1, \kappa_2 \in L_\infty(\Omega)$, there holds $\kappa_i \geq k_i^2 > 0$ which yields ellipticity with constant k_i^2 for the corresponding bilinear form.

(iv) The nonlinearity F satisfies Assumption 3.2.

We only require that the different types of boundary conditions are well-separated in the sense of Assumption 3.21 (i) in order to have a result on the Hölder continuity for weak solutions to mixed boundary value problems (Theorem A.17) at hand. The Dirichlet conditions on Γ_D have to be included into the definition of an appropriate Hilbert space. The following definition generalizes the idea that a function vanishes on the part Γ_D of $\partial\Omega$ and is adopted from [Grö89] (Definition 3) and [GM00] (Remark 4.3.1).

Definition 3.22
Let Ω be a bounded Lipschitz domain and let $\partial\Omega$ be regular in the sense of Definition A.15.

(i) Let $C^\infty_{\Gamma_D}(\Omega) := \{ v|_\Omega : v \in C^\infty_c(\mathbb{R}^n),\ \mathrm{supp}(v) \cap (\partial\Omega \setminus \Gamma_N) = \emptyset\}$. For the case of pure Dirichlet conditions, we formally set $C^\infty_{\partial\Omega}(\Omega) = C^\infty_c(\Omega)$ and $C^\infty_\emptyset(\Omega) = C^\infty(\Omega)$ in the case of pure Neumann conditions.

(ii) Define $H^1_{\Gamma_D}(\Omega)$ as the closure of $C^\infty_{\Gamma_D}(\Omega)$ in $H^1(\Omega)$. Then $H^1_{\Gamma_D}(\Omega)$ is a Hilbert space with respect to the standard scalar product in $H^1(\Omega)$. In the case $\Gamma_D = \partial\Omega$, we formally have $H^1_{\Gamma_D}(\Omega) = H^1_0(\Omega)$ and in case of pure Neumann conditions, there holds $H^1_\emptyset(\Omega) = H^1(\Omega)$.

Conversely, in this situation it can be proved that $C^\infty_{\Gamma_D}(\Omega)$, defined as above, is dense in the closed subspace $\{u \in H^1(\Omega) : u|_{\Gamma_D} = 0\}$ of $H^1(\Omega)$. Note that for more general Lipschitz domains and $\Gamma_D \subset \partial\Omega$ with $\lambda_{n-1}(\Gamma_D) > 0$, the density of the restrictions of smooth functions to Ω, which vanish in a neighbourhood of Γ_D, in the space $W^1_{p,\Gamma_D}(\Omega) := \{u \in W^1_p(\Omega) : u|_{\Gamma_D} = 0\}$ has only recently been proved, see the article [Ber11].

Definition 3.23 (Weak solution of the problem with mixed boundary conditions)
A pair $(u_1, u_2) \in H^1(\Omega)^2$ is called a weak solution of the nonlinear system (3.20) if (u_1, u_2) admits a decomposition $u_i = v_i + w_i$ with an element $w_i \in H^1(\Omega)$ that satisfies $\gamma_{\partial\Omega} w_i = g_i$ and $v_i \in H^1_{\Gamma_D}$ such that for all $\varphi \in H^1_{\Gamma_D}(\Omega)$ the following equalities are valid:

$$\begin{aligned} \int_\Omega \kappa_i \nabla v_i \nabla\varphi \, dx - \int_{\Gamma_N} h_1 \varphi \, dS &= \int_\Omega (-1)^{i+1} F(v_2 + w_2 - v_1 - w_1, x)\varphi \, dx \\ &\quad + \int_\Omega f_i \varphi \, dx - \int_\Omega \kappa_i \nabla w_i \nabla\varphi \, dx \quad (i = 1, 2). \end{aligned} \tag{3.21}$$

Lemma 3.24 (Uniqueness)
The system (3.20) admits at most one weak solution $u = (u_1, u_2) \in H^1(\Omega) \times H^1(\Omega)$ in the sense of Definition 3.23.

Proof. Given two solutions $u = (u_1, u_2),\ z = (z_1, z_2) \in H^1(\Omega) \times H^1(\Omega)$, their difference satisfies

$$\begin{aligned} \int_\Omega \kappa_1 \nabla(u_1 - z_1)\nabla\varphi \, dx &= \int_\Omega \left(F(u_2 - u_1, x) - F(z_2 - z_1, x)\right)\varphi \, dx \quad (\varphi \in H^1_{\Gamma_D}(\Omega)), \\ \int_\Omega \kappa_2 \nabla(u_2 - z_2)\nabla\psi \, dx &= -\int_\Omega \left(F(u_2 - u_1, x) - F(z_2 - z_1, x)\right)\psi \, dx \quad (\psi \in H^1_{\Gamma_D}(\Omega)). \end{aligned}$$

Taking $\varphi := u_1 - z_1 \in H^1_{\Gamma_D}(\Omega)$ and $\psi := u_2 - z_2 \in H^1_{\Gamma_D}(\Omega)$, summing up the equations, and using the estimates for κ_i, we arrive at

$$\begin{aligned} & \|\nabla(u_1 - z_1)\|^2_{0,2,\Omega} + \|\nabla(u_2 - z_2)\|^2_{0,2,\Omega} \\ \le\ & -C \int_\Omega \left[F((u_2 - u_1)(x), x) - F((z_2 - z_1)(x), x)\right](u_2 - u_1 - (z_2 - z_1))(x) \, dx \\ \le\ & -C\|u_2 - u_1 - (z_2 - z_1)\|^2_{0,2,\Omega} \le 0. \end{aligned}$$

Now, the first Poincaré inequality (Theorem A.7 (i)), which holds for functions in $H^1_{\Gamma_D}(\Omega)$, yields $u = z$. □

Remark 3.25
The regularity of the Dirichlet boundary values $g_i \in W^{1-1/m}_m(\partial\Omega)$ and the surjectivity of the trace operator yield the existence of $w_i \in W^1_m(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$ $(\alpha \in (0, 1 - \frac{n}{m}))$ with $\gamma_{\partial\Omega} w_i = g_i$. Since Ω is bounded and $m > \max(n, 2)$, we have $W^1_m(\Omega) \hookrightarrow H^1_2(\Omega)$. The assumption on g_i is stated here, because we intend to have a Hölder continuous solution and the correction functions w_i should be bounded, since they occur in the nonlinearity F. For the linearized equations, the proceeding will also work for $g_i \in H^{1/2}(\partial\Omega)$, see [Dro00], but will then not necessarily lead to a $C^{0,\alpha}$-solution.

Example 3.26
One may ask, how a strong solution $(u_1, u_2) \in W_p^2(\Omega)$ of (3.20) leads to a solution of (3.21). We first assume that Ω is a $C^{1,1}$-domain. In this situation the trace theorem (Theorem A.2) yields $g_i := \gamma_{\partial\Omega} u_i \in W_p^{2-1/p}(\partial\Omega)$ and $h_i := \gamma_{\partial\Omega} \frac{\partial u_i}{\partial \nu} \in W_p^{1-1/p}(\partial\Omega)$. In order to correct the Dirichlet boundary value, we use the fact that the mapping

$$W_p^2(\Omega) \to W_p^{2-1/p}(\partial\Omega) \times W_p^{1-1/p}(\partial\Omega),\ w \mapsto \left(\gamma_{\partial\Omega} w\ ,\ \gamma_{\partial\Omega} \frac{\partial w}{\partial \nu}\right) \tag{3.22}$$

is a retraction, where ν denotes the outer normal. We choose an element $w_i \in W_p^2(\Omega)$ with $(\gamma_{\partial\Omega} w_i, \gamma_{\partial\Omega} \frac{\partial w_i}{\partial \nu}) = (g_i, 0)$. Then $v_i := u_i - w_i \in W_p^2(\Omega)$ satisfies $v_i = 0$ on Γ_D and solves the equation

$$\left\{\begin{array}{rcll} -\nabla\cdot(\kappa_1 \nabla v_1) &=& F(u_2 - u_1, x) + f_1 + \nabla\cdot(\kappa_1 \nabla w_1) & \text{in } \Omega, \\ -\nabla\cdot(\kappa_2 \nabla v_2) &=& -F(u_2 - u_1, x) + f_2 + \nabla\cdot(\kappa_2 \nabla w_2) & \text{in } \Omega, \\ v_1 &=& 0, \quad v_2 = 0 & \text{on } \Gamma_D, \\ \kappa_1 \frac{\partial v_1}{\partial \nu} &=& h_1, \quad \kappa_2 \frac{\partial v_2}{\partial \nu} = h_2 & \text{on } \Gamma_N. \end{array}\right. \tag{3.23}$$

Now, multiplication of the differential equations with $\varphi \in C^\infty_{\Gamma_D}(\Omega)$ and integration by parts leads to equations of the form (3.21). For the case of a Lipschitz domain, additional conditions on the Dirichlet boundary values have to be imposed, since the mapping in (3.22) is in general no longer surjective, unless certain compatibility conditions are satisfied by the boundary values. A precise statement on this is presented in Theorem A.3 of the Appendix.

We henceforth impose the additional

Assumption 3.27 (Inhomogeneous Dirichlet data)
In the case of an inhomogeneous Dirichlet condition $\gamma_{\partial\Omega} u_i = g_i$ on Γ_D, we assume that there exists $w_i \in H^1(\Omega) \cap L_\infty(\Omega)$ with $\nabla w_i \in L_\infty(\Omega)$ such that $\gamma_{\partial\Omega} w_i = g_i$ on Γ_D.

The boundedness of ∇w_i will only be required to prove a uniform L_∞-estimate for a solution. In order to obtain a linear ground space for the system (3.20) with inhomogeneous Dirichlet boundary values, we first reduce to the case of homogeneous Dirichlet boundary conditions, i.e., the Dirichlet conditions are understood in the sense $u_i - g_i \in H^1_{\Gamma_D}(\Omega)$. To this end, we fix $w_1, w_2 \in W_m^1(\Omega) \cap L_\infty(\Omega)$ such that $\gamma_{\partial\Omega} w_i = g_i$ is satisfied and seek the solution of the form $u_i = v_i + w_i$, where $v_i \in H^1_{\Gamma_D}(\Omega)$ is a solution to the nonlinear system (3.21) for $i = 1, 2$.

Remark 3.28
Note that there may exist several functions w_i with $\gamma_{\partial\Omega} w_i = g_i$ (e.g., one can add a function $\varphi \in C_0^\infty(\Omega)$ to w_i without changing its regularity and its boundary value) and each of them will uniquely lead to a function v_i such that $u_i := v_i + w_i$ solves (3.20). However, Lemma 3.24 shows that these are only different decompositions of the same function.

We now follow the same scheme which we employed for the case of homogeneous Neumann boundary conditions. We henceforth consider the linearized and decoupled equations, i.e., we fix $z_1, z_2 \in L_\infty(\Omega)$ and introduce an additional parameter $\delta \in [0, 1]$ so that the resulting equations read as

$$\begin{aligned} \int_\Omega \kappa_1 \nabla v_1 \nabla \varphi\ dx = \delta\Big(&\int_\Omega f_1 \varphi\ dx + \int_\Omega F(z_2 - z_1 + w_2 - w_1, x)\varphi\ dx \\ &- \int_\Omega \kappa_1 \nabla w_1 \nabla \varphi\ dx + \int_{\Gamma_n} h_1 \varphi\ dS \Big) \quad (\varphi \in H^1_{\Gamma_D}(\Omega)), \end{aligned} \tag{3.24a}$$

$$\begin{aligned} \int_\Omega \kappa_2 \nabla v_2 \nabla \psi\ dx = \delta\Big(&\int_\Omega f_2 \psi\ dx - \int_\Omega F(z_2 - z_1 + w_2 - w_1, x)\psi\ dx \\ &- \int_\Omega \kappa_2 \nabla w_2 \nabla \psi\ dx + \int_{\Gamma_n} h_2 \psi\ dS \Big) \quad (\psi \in H^1_{\Gamma_D}(\Omega)). \end{aligned} \tag{3.24b}$$

Lemma 3.29
For every fixed $z_1, z_2 \in L_\infty(\Omega)$ and $\delta \in [0,1]$, the equations (3.24a) *and* (3.24b) *admit unique solutions $v_1, v_2 \in H^1_{\Gamma_D}(\Omega)$. Furthermore, there exist Hölder exponents $\alpha_i \in (0, 1-n/m)$ and constants $C_i(\alpha_i)$, depending on $\Omega, \Gamma_D, n, m, k_i, \|\kappa_i\|_{0,\infty,\Omega}$, and the norms $\|g_i\|_{1-1/m,m,\partial\Omega}$, $\|h_i\|_{0,\infty,\partial\Omega}$, and $\|f_i + F(z_2 - z_1 + w_2 - w_1, \cdot)\|_{0,\infty,\Omega}$ $(i = 1, 2)$ such that the estimate*

$$\|v_i\|_{C^{0,\alpha_i}(\overline{\Omega})} \leq C_i, \quad (i = 1, 2)$$

is satisfied.

Proof. Clearly, the bilinear forms

$$A_i : H^1_{\Gamma_D}(\Omega) \times H^1_{\Gamma_D}(\Omega) \to \mathbb{R}, \quad (u, v) \mapsto \int_\Omega \kappa_i \nabla u \nabla v \, dx$$

are continuous and coercive, where the latter follows from the first Poincaré inequality. For the continuity of the right-hand sides of the equations (3.24a) and (3.24b) as functionals on $H^1_{\Gamma_D}(\Omega)$, we only note that

$$\left| \int_\Omega \kappa_i \nabla w_i \nabla \varphi \, dx \right| \leq \|\kappa_i\|_{0,\infty,\Omega} \|\nabla w_i\|_{0,2,\Omega} \|\nabla \varphi\|_{0,2,\Omega} \leq C_i(g_i, \kappa_i) \|\varphi\|_{1,2,\Omega}.$$

The continuity of the other components of the functionals has already been proved in Lemma 3.9. The existence in the Hilbert space $H^1_{\Gamma_D}(\Omega)$ now follows by the Lax-Milgram theorem (Theorem A.11). Regarding the Hölder continuity, we note that for $m > \max(n, 2)$ and with $\frac{1}{m} + \frac{1}{q} = 1$ the embeddings $L_\infty(\partial\Omega) \hookrightarrow L_{\frac{(n-1)p}{n}}(\partial\Omega) \hookrightarrow (W_q^{1-1/q}(\partial\Omega))'$ hold true. In addition, it is easy to check that $\varphi \mapsto \int_\Omega (f_i \pm F(z_2 - z_1 + w_2 - w_1, x))\varphi \, dx$ defines a continuous functional on $W^1_q(\Omega)$:

$$\begin{aligned} & \left| \int \delta(\pm F((z_2 - z_1 + w_2 - w_1)(x), x) + f_i(x))\varphi(x) \, dx \right| \\ \leq \; & \| \pm F((z_2 - z_1 + w_2 - w_1)(\cdot), \cdot) + f_i\|_{0,m,\Omega} \|\varphi\|_{0,q,\Omega} \\ \leq \; & (\|F((z_2 - z_1 + w_2 - w_1)(\cdot), \cdot)\|_{0,\infty,\Omega} + \|f_i\|_{0,\infty,\Omega}) \|\varphi\|_{0,q,\Omega}. \end{aligned}$$

Hence, Theorem A.17 applies with $b = \lambda = 0$ as well as $\boldsymbol{v} = 0$ and yields the desired Hölder continuity as well as the claimed dependencies of α_i and the constants C_i. □

As a direct consequence, for fixed data $f_i \in L_\infty(\Omega)$, $h_i \in L_\infty(\partial\Omega)$, and $w_i \in W^1_m(\Omega) \cap L_\infty(\Omega)$ with $\gamma_{\partial\Omega} w_i = g_i \in W_m^{1-1/m}(\partial\Omega)$, the corresponding solution operator

$$T : [0, 1] \times L_\infty(\Omega)^2 \to L_\infty(\Omega)^2, \quad (\delta, z_1, z_2) \mapsto (v_1, v_2)$$

is well-defined.

Remark 3.30

(i) The continuity conditions (ii) and (iii) of the Leray-Schauder fixed point theorem can be verified analogously to the proofs of the Lemmas 3.12 and 3.13, respectively.

(ii) With the same arguments as in the proof of Lemma 3.12, we can show that $T(\delta, \cdot)$ maps bounded sets of $L_\infty(\Omega)^2$ into bounded sets of $L_\infty(\Omega)^2$.

(iii) The Hölder exponents in Lemma 3.29 only depend on Ω and the fixed quantities n, $\|f_i\|_{0,\infty,\Omega}$, $\|F((z_2 - z_1 - (w_2 - w_1))(\cdot), \cdot)\|_{0,\infty,\Omega}$, k_i^2, and $\|g_i\|_{1-1/m,m,\partial\Omega}$. So they do not depend on the particular choice of $z_1, z_2 \in L_\infty(\Omega)$ but only on the norm of $\|z_2 - z_1\|_{0,\infty,\Omega}$. Consequently, for a bounded sequence $(z_k)_{k\in\mathbb{N}} = (z_1^{(k)}, z_2^{(k)})_{k\in\mathbb{N}} \subset L_\infty(\Omega)^2$, there are α_1, $\alpha_2 \in (0, 1)$ such that for all $k \in \mathbb{N}$, the Hölder exponents $\alpha_i^{(k)}$ of $v_k := T(\delta, z_k) \in C^{0,\alpha_1^{(k)}}(\overline{\Omega}) \times C^{0,\alpha_2^{(k)}}(\overline{\Omega})$ satisfy $\alpha_i^{(n)} \geq \alpha_i$ for $i = 1, 2$. Hence, a subsequence of $(v_k)_{k\in\mathbb{N}}$ converges in $C^{0,\alpha_1/2}(\overline{\Omega}) \times C^{0,\alpha_2/2}(\overline{\Omega})$. In particular, this subsequence converges in $L_\infty(\Omega)^2$. For all $\delta \in [0, 1]$, this proves that the mapping $T(\delta, \cdot) : L_\infty(\Omega)^2 \to L_\infty(\Omega)^2$ is compact.

(iv) Obviously, there holds $T(0, z_1, z_2) = 0$ for all $z \in L_\infty(\Omega)^2$, by the coerciveness of the bilinear forms A_1 and A_2 (cf. the proof of Lemma 3.29).

It remains to prove the a-priori boundedness in $L_\infty(\Omega)$ of functions $(v_1, v_2) \in (H^1_{\Gamma_D}(\Omega) \cap L_\infty(\Omega))^2$ which satisfy $T(\delta, v_1, v_2) = (v_1, v_2)$ for some $\delta \in [0, 1]$.

Lemma 3.31
Suppose that Assumptions 3.21 and 3.27 are satisfied and $1 \le n \le 4$. Then there exists a constant $M > 0$, depending on Ω, Γ_D, n, p, k_i, $\|\kappa_i\|_{0,\infty,\Omega}$, and the norms $\|f_i\|_{0,\infty,\Omega}$, $\|g_i\|_{1-1/m,m,\partial\Omega}$, and $\|h_i\|_{0,\infty,\partial\Omega}$, such that $\|v_1\|_{0,\infty,\Omega} + \|v_2\|_{0,\infty,\Omega} \le M$ holds for all $(v_1, v_2) \in (H^1_{\Gamma_D} \cap L_\infty(\Omega))^2$ which satisfy $v = (v_1, v_2) = T(\delta, v_1, v_2)$ for some $\delta \in [0, 1]$.

Proof. We first prove uniform boundedness in $H^1(\Omega)$. We test (3.24a) (where now $v_i = z_i$) with $v_1 \in H^1_{\Gamma_D}(\Omega)$, (3.24b) with $v_2 \in H^1_{\Gamma_D}(\Omega)$, and take the sum of the resulting equations:

$$\int_\Omega \kappa_1 |\nabla v_1|^2 \, dx + \int_\Omega \kappa_2 |\nabla v_2|^2 \, dx = \delta \bigg(\int_\Omega f_1 v_1 + f_2 v_2 \, dx + \int_\Omega \kappa_1 \nabla w_1 \nabla v_1 + \kappa_2 \nabla w_2 \nabla v_2 \, dx \\ - \int_\Omega F((v_2 - v_1 + (w_2 - w_1))(x), x)(v_2 - v_1) \, dx + \int_{\Gamma_N} h_1 v_1 + h_2 v_2 \, dS(x) \bigg).$$

Using the ellipticity of κ_i, Hölder's inequality, $\delta \le 1$, and the Assumption (3.2) on the nonlinearity, we find

$$\|\nabla v_1\|^2_{0,2,\Omega} + \|\nabla v_2\|^2_{0,2,\Omega} \; + C_1 \delta \|v_2 - v_1\|^2_{0,2,\Omega} \\ \le \varepsilon \left(\|v_1\|^2_{1,2,\Omega} + \|v_2\|^2_{1,2,\Omega} \right) + C(\kappa_1, \kappa_2, \varepsilon) \bigg(C_2 + \sum_{i=1}^{2} \left(\|f_i\|^2_{0,\infty,\Omega} + \|\nabla w_i\|^2_{0,2,\Omega} + \|h_i\|^2_{0,\infty,\partial\Omega} \right) \bigg).$$

Together with Poincaré's inequality and an appropriate choice of ε, the claim follows. Note that in case $n = 1$, we are finished by now, since $H^1_2(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega}) \hookrightarrow L_\infty(\Omega)$ for $\alpha < \frac{1}{2}$. To obtain uniform estimates in higher L_p-norms, we set $\varphi := v_1^{p-1}$ and $\psi := v_2^{p-1}$ for an even integer p. Then $\varphi, \psi \in H^1_{\Gamma_D}(\Omega) \cap L_\infty(\Omega)$. Now, we can literally follow the proof of Lemma 3.16: The sum of the resulting equations reads

$$\int_\Omega \kappa_1 \nabla v_1 \nabla (v_1^{p-1}) \, dx + \int_\Omega \kappa_2 \nabla v_2 \nabla (v_2^{p-1}) \, dx \\ = \delta \bigg(\int_\Omega f_1 v_1^{p-1} + f_2 v_2^{p-1} \, dx - \int_\Omega F((v_2 - v_1 + (w_2 - w_1))(x), x)(v_2^{p-1} - v_1^{p-1})(x) \, dx \\ + \int_\Omega \kappa_1 \nabla w_1 \nabla (v_1^{p-1}) \, dx + \int_\Omega \kappa_2 \nabla w_2 \nabla (v_2^{p-1}) \, dx + \int_{\Gamma_N} h_1 v_1^{p-1} + h_2 v_2^{p-1} \, dS(x) \bigg).$$

As in the proof of Lemma 3.16, the integrals on the left-hand side can be estimated from below against $\frac{C(p-1)}{p^2} \|\nabla(v_i^{p/2})\|^2_{0,2,\Omega}$ and the first two integrals on the right-hand side can be estimated from above. For the remaining integrals, we use the higher integrability of $\nabla w_i \in L_\infty(\Omega)$:

$$\int_\Omega \kappa_i \nabla w_i \nabla (v_i^{p-1}) \, dx = (p-1) \int_\Omega \kappa_i v_i^{p-2} \nabla w_i \nabla v_i \, dx \\ \le C_i (p-1) \left(\tfrac{2\varepsilon}{p^2} \|\nabla(v_i^{p/2})\|^2_{0,2,\Omega} + \tfrac{1}{2\varepsilon} \|v_i^{p-2}\|_{0,1,\Omega} \right) \\ \le \varepsilon \frac{C_i (p-1)}{p^2} \|\nabla(v_i^{p/2})\|^2_{0,2,\Omega} + \frac{C_i (p-1)}{2\varepsilon} \|v_i\|^{p-2}_{0,p-2,\Omega},$$

where we denoted $C_i := \|\kappa_i \nabla w_i\|_{0,\infty,\Omega}$. Here, we used $\nabla w_i \in L_\infty(\Omega)$, because the weaker condition $\nabla w_i \in L_m(\Omega)$ would only allow an estimate against $\|v_i^{p-2}\|_{0,\frac{m}{m-2},\Omega}$ and this term

would become dominant in the iteration scheme for $p \geq 4$, such that an estimate of the form $\|v\|_{0,p_{k+1},\Omega} \leq (Cp_k)^{1/p_k}\|v\|_{0,p_k,\Omega}$ with a sequence p_k, which satisfies $p_k \to \infty$, $p_{k+1} \geq p_k + 2$, and $\prod_{j=0}^{\infty}(Cp_j)^{1/p_j} < \infty$ seems no longer possible. For the boundary integrals, we use Hölder's inequality, the continuity of the trace operator as well as Poincaré's inequality for some $t \in (0,1)$:

$$\begin{aligned}\int_{\Gamma_N} h_i v_i^{p-1}\, dS(x) &\leq \|h_i\|_{0,\infty,\Gamma_N}\|v_i^{p-1}\|_{0,1,\Gamma_N} \leq C\|v_i^{p-1}\|_{0,1+t,\partial\Omega} \leq C\|v_i^{p-1}\|_{1,1+t,\Omega}\\ &\leq C\|\nabla(v_i^{p-1})\|_{0,1+t,\Omega}.\end{aligned}$$

Using Young's inequality (with $q = \frac{2}{1+t} > 1$ and $q' = \frac{2}{1-t}$) for the norm of the gradient, we find

$$\begin{aligned}\|\nabla(v_i^{p-1})\|_{0,1+t,\Omega} &= (p-1)\|v_i^{p-2}\nabla v_i\|_{0,1+t,\Omega}\\ &= (p-1)\left(\int_\Omega \left(\tfrac{4}{p^2}|\nabla(v_i^{p/2})|^2\right)^{\frac{1+t}{2}} v_i^{\frac{(p-2)(1+t)}{2}}\, dx\right)^{\frac{1}{1+t}}\\ &\leq (p-1)\left(\int_\Omega 2\varepsilon\tfrac{1+t}{p^2}|\nabla(v_i^{p/2})|^2 + \tfrac{1-t}{2\varepsilon} v_i^{\frac{(p-2)(1+t)}{1-t}}\, dx\right)^{\frac{1}{1+t}}\\ &\leq \varepsilon\tfrac{C(p-1)}{p^2}\left\|\nabla(v_i^{p/2})\right\|^2_{0,2,\Omega} + \tfrac{C(p-1)}{\varepsilon}\|v_i\|^{p-1}_{0,p,\Omega}.\end{aligned}$$

For the latter estimates, we could assume that $\|\nabla(v_i)^{p/2}\|^2_{0,2,\Omega} > 1$ and $\|v_i\|_{0,p,\Omega} > 1$. Furthermore, for even $p \geq 4$, we chose $\frac{1+t}{1-t} := \frac{p}{p-2}$, i.e., $t = \frac{1}{(p-1)}$ and in the case $p = 2$, we have set $t := \frac{1}{2}$. With an appropriate choice of ε, it is no longer difficult to establish the following gradient estimate

$$\left\|\nabla({v_1}^{p/2})\right\|^2_{0,2,\Omega} + \left\|\nabla({v_2}^{p/2})\right\|^2_{0,2,\Omega} \leq Cp\left(\sum_{i=1}^{2}\|v_i\|^{p-1}_{0,p-1,\Omega} + \|v_i\|^{p-1}_{0,p,\Omega} + \|v_i\|^{p-2}_{0,p-2,\Omega}\right). \tag{3.25}$$

Following the arguments in the proof of Lemma 3.17 on page 79 and assuming w.l.o.g. that the occurring norms are larger than 1 (else we can consider a larger index p), we can now take again the inequality

$$\|(v_1,v_2)\|_{0,p\chi,\Omega} \leq (Cp)^{1/p}\|(v_1,v_2)\|_{0,p,\Omega}$$

with $\chi := \frac{n}{n-2}$ if $n = 3$ or $n = 4$, and χ arbitrarily large if $n = 2$, as a starting point for the iteration scheme and the uniform boundedness of $\|v\|_{0,\infty,\Omega}$ follows. □

Theorem 3.32 (Existence for Dirichlet and mixed boundary conditions)
Let $1 \leq n \leq 4$. Under the hypotheses stated in Assumptions 3.21 and 3.27, the nonlinear problem (3.21) *admits a unique solution $(v_1,v_2) \in (H^1_{\Gamma_D}(\Omega) \cap L_\infty(\Omega))^2$. For $i = 1,2$, there are constants $M_i > 0$, which depend on the quantities Ω, n, $\|f_i\|_{0,\infty,\Omega}$, $\|g_i\|_{1-1/m,m,\partial\Omega}$, $\|h_i\|_{0,\infty,\partial\Omega}$, $\|\nabla w_i\|_{0,\infty,\Omega}$, $\|\kappa_i\|_\infty$, and the constants of ellipticity k_i, such that*

$$\|v_1\|_{1,2,\Omega} + \|v_1\|_{0,\infty,\Omega} \leq M_1 \qquad \text{and} \qquad \|v_2\|_{1,2,\Omega} + \|v_2\|_{0,\infty,\Omega} \leq M_2.$$

In addition, there exist $\alpha_1, \alpha_2 \in (0,1)$ and a constant $M_3 > 0$, which depends on α_i and the same quantities as M_i, such that

$$\|v_1\|_{C^{0,\alpha_1}(\overline{\Omega})} + \|v_2\|_{C^{0,\alpha_2}(\overline{\Omega})} \leq M_3.$$

Proof. Consider the mapping $T : [0,1] \times L_\infty(\Omega)^2 \to L_\infty(\Omega)^2$ defined on page 85. As a consequence of Remark 3.30 and Lemmas 3.29 and 3.31, the Leray-Schauder fixed point theorem (Theorem A.9) applies and yields the asserted existence and boundedness in $L_\infty(\Omega)^2$. It follows from the continuity of $s \mapsto F(s,\cdot)$, that $f_i \pm F((v_2 - v_1 + w_2 - w_1)(\cdot),\cdot) \in L_\infty(\Omega)$ is bounded by a constant that depends on $\|f_i\|_{0,\infty,\Omega}, \|g_i\|_{1-1/m,m,\Omega}$, and M_i. Then, considering (3.24a)-(3.24b) with $\delta = 1$ and using the unique solvability of the equations, the Lax-Milgram theorem yields the boundedness of the H^1-norms and Lemma 3.29 contains the boundedness of the Hölder-norms. □

Corollary 3.33
Under the hypotheses stated in Assumptions 3.21 and 3.27, the system (3.21) *admits a unique weak solution* $(u_1, u_2) \in H^1(\Omega)^2$ *in the sense of Definition 3.22. The solution is bounded in the norms of* $H^1(\Omega)^2$ *and* $C^{0,\alpha_1}(\overline{\Omega}) \times C^{0,\alpha_2}(\overline{\Omega})$ *by positive constants which depend on the norms of the given data* f_i, g_i, *and* h_i *in their respective spaces as well as on* $\|w_i\|_{1,\infty,\Omega}$, $\|\kappa_i\|_{0,\infty,\Omega}$, k_i, Ω, n, *and additionally on* α_i *in the second case.*

3.2.3 Different types of boundary conditions on the same surface

There remains to consider the case when each component of (u_1, u_2) satisfies mixed boundary conditions on $\partial\Omega$, but the types of boundary conditions do not coincide on the same parts of the boundary, i.e., we have two decompositions $\overline{\Gamma}_{N,1} \cup \overline{\Gamma}_{D,1} = \partial\Omega = \overline{\Gamma}_{N,2} \cup \overline{\Gamma}_{D,2}$ and impose the boundary conditions

$$\gamma_{\partial\Omega} u_i = g_i \text{ on } \Gamma_{D,i}, \quad \gamma_{\partial\Omega} \kappa_i \frac{\partial u_i}{\partial \nu} = h_i \text{ on } \Gamma_{N,i} \quad (i = 1, 2).$$

We assume $\lambda_{n-1}(\Gamma_{D,i}) > 0$ in order to have a Poincaré inequality and all hypotheses from Assumption 3.21 to be valid, where the geometric condition 3.21 (i) for the boundary parts is now to be satisfied for both decompositions separately. In case of $g_i \neq 0$, we additionally impose Assumption 3.27. In this situation, the following steps can be carried over from the preceding section where we considered the case $\Gamma_{N,1} = \Gamma_{N_2} = \Gamma_N$:

(i) The reduction to homogeneous Dirichlet boundary conditions by subtracting $w_i \in H^1(\Omega)$ with $\nabla w_i \in L_\infty(\Omega)$ and $\gamma_{\partial\Omega} w_i = g_i$. This leads to a nonlinear problem of the form (3.21), but now we seek the solution u_i in the Hilbert space $H^1_{\Gamma_{D,i}}(\Omega)$.

(ii) The linearization process which leads to completely decoupled equations of the form (3.24a) in $H^1_{\Gamma_{D,i}}(\Omega)$.

(iii) The proof of existence and uniqueness for the decoupled linearized equations as well as the Hölder estimate (see Lemma 3.29). Again we have a well-defined solution operator $T : [0,1] \times L_\infty(\Omega)^2 \to L_\infty(\Omega)^2$. This operator maps bounded sets of $L_\infty(\Omega)^2$ into bounded sets of $L_\infty(\Omega)^2$ and satisfies $T(0, z_1, z_2) = 0$. Furthermore, the continuity and compactness of $T(\delta, \cdot) : L_\infty(\Omega)^2 \to L_\infty(\Omega)^2$ and the uniform continuity of $T(\cdot, z) : [0,1] \to L_\infty(\Omega)^2$ with respect to δ, if $z = (z_1, z_2)$ is taken from a bounded subset of $L_\infty(\Omega)^2$ are proved in the same way as the Lemmas 3.12 and 3.13.

(iv) The only difference when we intend to prove the uniform L_∞-estimate for $v \in L_\infty(\Omega)^2$ with $T(\delta, v) = v$ for some $\delta \in [0,1]$, is to estimate the boundary terms on $\Gamma_{N,i}$ separately.

We can finally state the main result on the weak well-posedness of the nonlinear elliptic system in its most general form:

Theorem 3.34
Let $\Omega \subset \mathbb{R}^n$ $(1 \leq n \leq 4)$ *be a Lipschitz domain with two decompositions of the boundary* $\partial\Omega = \overline{\Gamma}_{N,1} \cup \overline{\Gamma}_{D,1} = \overline{\Gamma}_{N,2} \cup \overline{\Gamma}_{D,2}$ *which both satisfy the Assumption 3.21 (i). Assume that the conditions (ii)-(iv) from Assumption 3.21 hold for the data* f_i, g_i, *and* h_i, *the coefficients* κ_i, *and the nonlinearity* F. *In case of* $g_i \neq 0$ *assume that Assumption 3.27 is satisfied, too. Then the nonlinear coupled system*

$$\left\{ \begin{array}{rcll} -\nabla \cdot (\kappa_1 \nabla u_1) &=& F(u_2 - u_1, x) + f_1 & \text{in } \Omega, \\ -\nabla \cdot (\kappa_2 \nabla u_2) &=& -F(u_2 - u_1, x) + f_2 & \text{in } \Omega, \\ u_1 &=& g_1 \quad \text{on } \Gamma_{D,1}, \quad \kappa_1 \frac{\partial u_1}{\partial \nu} = h_1 & \text{on } \Gamma_{N,1}, \\ u_2 &=& g_2 \quad \text{on } \Gamma_{D,2}, \quad \kappa_2 \frac{\partial u_2}{\partial \nu} = h_2 & \text{on } \Gamma_{N,2}, \end{array} \right. \tag{3.26}$$

admits a unique weak solution $u = (u_1, u_2) \in H^1(\Omega)^2$*, i.e.,* $u_i = v_i + w_i$ *with* $v_i \in H^1_{\Gamma_{D,i}}(\Omega)$*,* $w_i \in H^1(\Omega) \cap L_\infty(\Omega)$ *such that* $\gamma_{\partial\Omega} w_i = g_i$ *and*

$$\int_\Omega \kappa_1 \nabla v_1 \nabla \varphi \, dx = \int_\Omega (F(v_2 - v_1 + w_2 - w_1) + f_1)\, \varphi \, dx - \int_\Omega \kappa_1 \nabla w_1 \nabla \varphi \, dx + \int_{\Gamma_{N,1}} h_1 \varphi \, dx, \tag{3.27a}$$

$$\int_\Omega \kappa_2 \nabla v_2 \nabla \psi \, dx = \int_\Omega (-F(v_2 - v_1 + w_2 - w_1) + f_2)\, \psi \, dx - \int_\Omega \kappa_2 \nabla w_2 \nabla \psi \, dx + \int_{\Gamma_{N,2}} h_2 \psi \, dx, \tag{3.27b}$$

holds for all $\varphi \in H^1_{\Gamma_{D,1}}(\Omega)$ *and* $\psi \in H^1_{\Gamma_{D,2}}(\Omega)$*. In addition,* $u \in C^{0,\alpha_1}(\overline{\Omega}) \times C^{0,\alpha_2}(\overline{\Omega})$ *for some* $\alpha_1, \alpha_2 \in (0,1)$ *and there is a constant* $C(\alpha_i) > 0$*, depending on all the data in their respective norms,* $\|\kappa_i\|_{0,\infty,\Omega}$*, the constants of ellipticity* k_i *as well as on* Ω *and* n*, such that*

$$\|u_1\|_{1,2,\Omega} + \|u_2\|_{1,2,\Omega} + \|u_1\|_{C^{0,\alpha_1}(\overline{\Omega})} + \|u_2\|_{C^{0,\alpha_2}(\overline{\Omega})} \leq C.$$

Chapter 4

Strong well-posedness of a nonlinear elliptic system

In the preceding chapter, we proved the unique solvability of the nonlinear elliptic system

$$\left\{ \begin{array}{rcll} -\nabla \cdot (\kappa_1(x)\nabla u_1(x)) & = & F(u_2(x)-u_1(x),x)+f_1(x) & \text{in } \Omega, \\ -\nabla \cdot (\kappa_2(x)\nabla u_2(x)) & = & -F(u_2(x)-u_1(x),x)+f_2(x) & \text{in } \Omega, \end{array} \right. \tag{4.1}$$

with different types of boundary conditions in appropriate subspaces of $H^1(\Omega)$. Here, we ask for sufficient conditions on the nonlinearity F and the coefficients, under which a unique solution with higher regularity exists in the Sobolev space $W_p^2(\Omega)$. In view of later applications, we restrict ourselves to the case of a rectangular domain which is divided by two interfaces. Still, the stated results easily carry over to the situation when Ω is a bounded $C^{1,1}$-domain which is divided by disjoint, closed $C^{1,1}$-surfaces Γ_i $(i=1,\dots,M)$ which do not intersect with each other or with the boundary, into $M+1$ subdomains $\Omega_0,\dots,\Omega_M$ such that $\partial\Omega_i=\Gamma_i$ for $i=1,\dots,M$. In order to obtain a W_p^2-solution, we have to impose some stronger assumptions on the occurring data and coefficients which will be specified here for the case of a rectangular domain. Preserving the notation from Chapter 3, we consider the nonlinear system (4.1) with boundary conditions of Neumann, Dirichlet or mixed type which will be made precise in the subsequent paragraphs.

Assumption 4.1
Let $1 \le n \le 4$, $p \in (n,\infty)$, $s < \frac{1}{p}$, and $t \in (s,1)$.

(i) ***Geometry**: Let $\Omega = \prod_{k=1}^n (0,L_k) \subset \mathbb{R}^n$, with $0 < L_k < \infty$ be a rectangular domain whose boundary is the union of the $(n-1)$-dimensional surfaces $\Gamma_k := \{x \in \overline{\Omega} : x_k = 0\}$ for $k=1,\dots,n$, and $\Gamma_k := \{x \in \overline{\Omega} : x_{k-n} = L_{k-n}\}$ for $k=n+1,\dots,2n$. We assume that Ω is divided by two $(n-1)$-dimensional surfaces $\Sigma_i := \{x \in \overline{\Omega} :\ x_n = \sigma_i\}$ $(i=1,2)$, with $0 < \sigma_1 < \sigma_2 < L_n$, into three subdomains: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3$.*

(ii) ***Coefficients**: Let $\kappa_i \in C^{1,t}(\overline{\Omega}_j)$ for $i=1,2$, and $j=1,2,3$, i.e., we allow possible discontinuities at the interfaces Σ_i. We will frequently use $\|\nabla\kappa_i\|_{0,\infty,\overline{\Omega}} \le K_i$ and further assume $\kappa_i \ge k_i^2 > 0$ which implies the ellipticity for the principal parts of the corresponding differential operators.*

(iii) ***Nonlinearity**: For $j=1,2,3$, the function $F|_{\mathbb{R}\times\overline{\Omega}_j} : \mathbb{R}\times\overline{\Omega}_j \to \mathbb{R}$ is continuous, differentiable with respect to the first variable, $\partial_1 F|_{\mathbb{R}\times\overline{\Omega}_j} : \mathbb{R}\times\overline{\Omega}_j \to \mathbb{R}$ is continuous and there is a constant $C>0$ such that*

$$\frac{\partial}{\partial w} F(w,x) \ge C_F > 0 \quad ((w,x) \in \mathbb{R}\times\Omega).$$

In addition, we assume that F acts between the spaces $W_p^2(\Omega)$ and $W_p^s(\Omega)$ in the following sense: for all $u \in W_p^2(\Omega)$ we have $F(u(\cdot),\cdot) \in W_p^s(\Omega)$ and F maps bounded subsets of $W_p^2(\Omega)$ into bounded subsets of $W_p^s(\Omega)$, i.e. for all $u \in W_p^2(\Omega)$ with $\|u\|_{2,p,\Omega} \le K_1$, there exists $K_2 = K_2(K_1) > 0$ such that $\|F(u(\cdot),\cdot)\|_{s,p,\Omega} \le K_2$.

(iv) ***Boundary conditions****: For $k = 1, \dots, n-1, n+1, \dots, 2n-1$, the boundary condition on Γ_k is of the form*

$$\gamma_{0,k} u_i = 0 \quad \text{or} \quad \gamma_{0,k} \kappa_i \partial_k u_i = 0 \quad \text{on } \Gamma_k \ (i = 1, 2),$$

i.e., we either impose homogeneous Dirichlet or homogeneous Neumann boundary conditions. Again we agree to write $\partial_{n+k} = -\partial_k$, for $k = 1, \dots, n$. Regarding the boundary conditions on the parallel surfaces Γ_n and Γ_{2n}, we suppose to be given one of the following cases

$$\gamma_{0,k} u_i = g_{0,i}^{(k)}, \quad \text{or} \quad \gamma_{0,k} \kappa_i \partial_n u_i = g_{1,i}^{(k)} \quad (k \in \{n, 2n\},\ i = 1, 2),$$

with a boundary value $g_{m_k,i}^{(k)} \in W_p^{2+s-m_k-1/p}(\Gamma_k)$ which is subject to the compatibility conditions

$$\text{(4.2a)} \qquad \gamma_{0,j} g_{0,i}^{(k)} = \gamma_{0,j} \partial_j g_{0,i}^{(k)} = 0 \quad \text{on } \Gamma_j \cap \Gamma_k \quad \text{if } m_k = 0,$$

$$\text{(4.2b)} \qquad \gamma_{0,j} g_{0,i}^{(k)} = 0 \quad \text{on } \Gamma_j \cap \Gamma_k \quad \text{if } m_k = 1,$$

for $j \in \{1, \dots, n-1, n+1, \dots, 2n-1\}$. See the preliminaries of Corollary 2.26.

(v) ***Transmission conditions****: On the surfaces Σ_j $(j = 1, 2)$, we impose homogeneous canonical transmission conditions in x_n-direction for both functions u_i:*

$$\text{(4.3a)} \quad \gamma_{\Sigma_1} u_i|_{\Omega_1} - \gamma_{\Sigma_1} u_i|_{\Omega_2} = 0, \quad \gamma_{\Sigma_1}(\partial_n u_i)|_{\Omega_1} - \gamma_{\Sigma_1}(\partial_n u_i)|_{\Omega_2} = 0 \quad \text{on } \Sigma_1,\ (i = 1, 2),$$

$$\text{(4.3b)} \quad \gamma_{\Sigma_2} u_i|_{\Omega_2} - \gamma_{\Sigma_2} u_i|_{\Omega_3} = 0, \quad \gamma_{\Sigma_2}(\partial_n u_i)|_{\Omega_2} - \gamma_{\Sigma_2}(\partial_n u_i)|_{\Omega_3} = 0 \quad \text{on } \Sigma_2,\ (i = 1, 2).$$

We will write for short $[\![u_i]\!] = [\![\partial_n u_i]\!] = 0$ on Σ_j for $i, j \in \{1, 2\}$, where $[\![u_i]\!]$ denotes the jump of u_i across Σ_j. The latter conditions ensure that $u_i \in W_p^{2+s}(\Omega)$ if and only if $u_i|_{\Omega_j} \in W_p^{2+s}(\Omega_j)$ for $j = 1, 2, 3$ (cf. Lemma 1.8).

(vi) ***Data****: We assume $f_1, f_2 \in W_p^s(\Omega) \cap L_\infty(\Omega)$.*

Note that for $k = 1, \dots, 2n$, the boundary conditions for u_1 and u_2 on each surface Γ_k may be of different type.

Notation 4.2
For $k = 1, \dots, 2n$, the boundary operator which acts on u_i on the surface Γ_k will be denoted by $\gamma_{0,k} \mathcal{B}_i^{(k)}$. The corresponding boundary values will be denoted by $g_{m_k,i}^{(k)}$, where we set $g_{m_k,i}^{(k)} := 0$, if $k = 1, \dots, n-1, n+1, \dots, 2n-1$, and the index $m_k \in \{0, 1\}$ corresponds to Dirichlet- or Neumann boundary conditions, respectively. We will use the symbolic notation

$$\gamma_{\partial\Omega} \mathcal{B}_i u_i = g_i \quad \text{on } \partial\Omega, \quad \text{if and only if} \quad \gamma_{0,k} \mathcal{B}_i^{(k)} u_i = g_{m_k,i}^{(k)} \quad \text{on } \Gamma_k \quad (k = 1, \dots, 2n).$$

Remark 4.3 (Acting properties of nonlinear functions)
The question, under which conditions $F : \mathbb{R} \times \Omega \to \mathbb{R}$ acts between the spaces $W_p^2(\Omega)$ and $W_p^s(\Omega)$ is a question on the properties of the corresponding superposition (or Nemytskij) operator

$$(T_F(u))(x) = F(u(x), x) \quad (x \in \Omega).$$

T_F can be considered for a Carathéodory function F, i.e. $F(y, \cdot) : \Omega \to \mathbb{R}$ is measurable for $y \in \mathbb{R}$ and $F(\cdot, x) : \mathbb{R} \to \mathbb{R}$ is continuous for almost every $x \in \Omega$. The mapping properties as well as boundedness and continuity for this operator are known for the L_p-spaces, the Sobolev spaces $W_p^k(\Omega)$ with $k \in \mathbb{N}$, and the Hölder spaces $C^{0,\alpha}(\overline{\Omega})$ (see [AZ90], Chapters 3, 7, and 9). For the Besov spaces, the question is still not answered for the whole scale, but at least some sufficient conditions are known and the autonomous case, i.e. F is independent of x, is studied in much more detail, see [RS96], Chapter 5 and [AZ90], Chapter 9.

First of all, let us mention that it suffices to have $T_F(u)|_{\Omega_j} \in W_p^s(\Omega_j)$ *for* $j = 1, 2, 3$, *because for the present case* $s < 1/p$, *one can approximate functions in* $W_p^s(\Omega_j)$ *by smooth functions with compact support in* Ω_j. *More precisely,* $W_p^s(\Omega_j)$ *coincides with the closure of* $C_c^\infty(\Omega_j)$ *with respect to* $\|\cdot\|_{s,p,\Omega_j}$ *for* $s < 1/p$ *(cf. [Tri02], Proposition 3.1). An approximation of* $T_F(u)|_{\Omega_j}$ *in each* Ω_j *with functions in* $C_c^\infty(\Omega_j)$ *will then yield an approximation of* $T_F(u)$ *on all of* Ω *with functions in* $C_c^\infty(\Omega)$, *hence* $T_F(u) \in W_p^s(\Omega)$.

In the following, we list some sufficient properties for the function F *such that* T_F *admits the mapping property from Assumption 4.1 (iii).*

(i) Assume that F *is real analytic with respect to* y *and independent of* x. *In this case, we can use the well-known fact that for* $p > \frac{n}{2}$, *the space* $W_p^2(\Omega)$ *is a Banach algebra with an equivalent norm* $\|\cdot\|_{2,p,\Omega}^* := K_*\|\cdot\|_{2,p,\Omega}$ *(cf. [AF03], pp.106 ff.), with a constant* $K_* = K_*(n, p, \Omega)$, *such that* $F : W_p^2(\Omega) \to W_p^2(\Omega)$ *is bounded because of the absolute convergence of the series*

$$\|F(u)\|_{2,p,\Omega}^* \le \sum_{k=0}^{\infty} |c_k| \ \|u^k\|_{2,p,\Omega}^* \le \sum_{k=0}^{\infty} |c_k| \ (\|u\|_{2,p,\Omega}^*)^k.$$

(ii) Let $F \in C^2(\mathbb{R})$ *with bounded derivatives* $F', F'' \in L_\infty(\mathbb{R})$. *Then we have by the chain and product rule (see [GT77], Lemma 7.5) as well as Hölder's inequality that for all* $u \in W_p^2(\Omega)$ *with* $p > n$, *the composition* $F \circ u \in W_p^2(\Omega)$ *is well-defined with*

$$\|F \circ u\|_{2,p,\Omega} = \|F \circ u\|_{0,p,\Omega} + \sum_{i,j=1}^{n} \|F'(u)\partial_j u\|_{0,p,\Omega} + \|F'(u)\partial_i\partial_j u + F''(u)\partial_i u \partial_j u\|_{0,p,\Omega} \le C(F)\|u\|_{2,p,\Omega},$$

where we used the embedding $W_p^1(\Omega) \hookrightarrow L_\infty(\Omega)$.

(iii) From [Val85], we cite the following sufficient condition: Assume that $\Omega \subset \mathbb{R}^n$ *is bounded and satisfies the cone condition. Let* $k \in \mathbb{N}$ *and assume* $F \in C^k(\mathbb{R} \times \overline{\Omega})$. *Then the operator* $T_F : W_p^k(\Omega) \to W_p^k(\Omega), \ u \mapsto F(u(\cdot), \cdot)$ *is continuous. See also [AZ90], Theorem 9.7. Of course, the operator* $T_F : W_p^k(\Omega) \to W_p^s(\Omega)$ *is even compact in this case.*

(iv) We can exploit that there are sharp conditions on $F : \mathbb{R} \times \Omega \to \mathbb{R}$ *to act in the spaces* $C^\alpha(\overline{\Omega})$ *([AZ90], Theorem 7.1): Let* $0 < \beta \le \alpha \le 1$. *Then* T_F *maps* $C^{0,\alpha}(\overline{\Omega})$ *to* $C^{0,\beta}(\overline{\Omega})$ *and is bounded if and only if for all* $r > 0$, *there exists* $m = m(r) > 0$, *such that*

$$|F(u, x) - F(v, y)| \le m \left(|x - y|^\beta + (r^{-1}|u - v|)^{\beta/\alpha} \right)$$

holds for all $|u|, |v| \le r$ *and* $x, y \in \overline{\Omega}$. *The latter condition is equivalent to the local Lipschitz continuity of* $F(\cdot, x)$ *uniformly in* $x \in \overline{\Omega}$ *and to the Hölder continuity with exponent* β/α *of* $F(u, \cdot)$ *uniformly with respect to* u *in compact subsets of* $\mathbb{R}$, *see also [Nug93].*
Assuming these properties for F *with* $\alpha = 1$ *and* $0 < s < \beta \le 1$ *and setting* $\varepsilon := \beta - s > 0$, *we can conclude as follows: By embedding we have* $u \in W_p^2(\Omega) \hookrightarrow C^{1,\lambda}(\overline{\Omega}) \overset{c}{\hookrightarrow} C^{0,1}(\overline{\Omega})$. *Now,* F *is bounded from* $C^{0,1}(\overline{\Omega})$ *to*

$$C^\beta(\overline{\Omega}) \overset{c}{\hookrightarrow} C^{\beta-\varepsilon/2}(\overline{\Omega}) = B_{\infty,\infty}^{\beta-\varepsilon/2}(\Omega) \hookrightarrow B_{\infty,p}^s(\Omega) \hookrightarrow B_{p,p}^s(\Omega) = W_p^s(\Omega).$$

Overall, the operator $T_F : W_p^2(\Omega) \to W_p^s(\Omega), u \mapsto F(u, \cdot)$ *is compact. Note that in this situation continuity of* T_F *holds for any space* $W_q^t(\Omega)$, *which continuously embeds into* $C^{0,1}(\overline{\Omega})$.

Example 4.4
As an easy consequence of Remark 4.3(i), we can consider a nonlinearity of the structure $F(y,x) = m(x)\cdot\exp(y)$, *where* m *is an arbitrary pointwise multiplier function in the space* $W_p^s(\Omega)$ *with* $0<s<1$, *which holds for example in case* $m\in C^t(\overline{\Omega})$ *with* $t>s$. *In this case, we find for* $u\in W_p^2(\Omega)$ *with* $p>\frac{n}{2}$*:*

$$\|F(u,x)\|_{s,p,\Omega}\le\|m\|_{C^t(\overline{\Omega})}\|\exp(u)\|_{s,p,\Omega}\le\frac{1}{K_*}\|m\|_{C^t(\overline{\Omega})}\exp(K_*\|u\|_{1,p,\Omega}).$$

In analogy to the preceding chapter, we linearize and decouple the system by considering the equations

$$\text{(4.4a)}\qquad \begin{cases} u_1-\nabla\cdot(\kappa_1\nabla u_1) &=& \delta\,(F(z_2-z_1)+f_1+z_1) & \text{in } \Omega,\\ \gamma_{\partial\Omega}\mathcal{B}_1u_1 &=& \delta g_1 & \text{on } \partial\Omega,\\ [\![u_1]\!]=[\![\partial_n u_1]\!] &=& 0 & \text{on } \Sigma_j,\ (j\in\{1,2\}),\end{cases}$$

$$\text{(4.4b)}\qquad \begin{cases} u_2-\nabla\cdot(\kappa_2\nabla u_2) &=& \delta\,(-F(z_2-z_1)+f_2+z_2) & \text{in } \Omega,\\ \gamma_{\partial\Omega}\mathcal{B}_2u_2 &=& \delta g_2 & \text{on } \partial\Omega,\\ [\![u_2]\!]=[\![\partial_n u_2]\!] &=& 0 & \text{on } \Sigma_j,\ (j\in\{1,2\}),\end{cases}$$

for given $z_1,z_2\in W_p^2(\Omega)$ and with an additional parameter $\delta\in[0,1]$.

Remark 4.5 ('Strong implies weak')
Here, we outline a connection between a solution $(u_1,u_2)\in W_p^2(\Omega)^2$ *to the nonlinear system* (4.1) *and a weak solution in the sense of Definition 3.23 to the equations* (3.27a)-(3.27b). *We write* J_D *and* $J_N:=\{1,\dots,2n\}\setminus J_D$ *for the index sets of the surfaces with a Dirichlet and a Neumann boundary condition, respectively. Let* $(u_1,u_2)\in W_p^2(\Omega)$ *satisfy* (4.1) *with boundary conditions* $\gamma_{\partial\Omega}\mathcal{B}_iu_i=g_i$ *on* $\partial\Omega$, *which satisfy Assumption 4.1. Here, we only consider the most complicated case in detail which is given by the nonlinear system with an inhomogeneous Dirichlet condition on* Γ_n *or* Γ_{2n}.
In this case, we apply Theorem A.3 with $m=2$ *and* $r=2+s$ *which is possible to the compatibility conditions from* (4.2a) *(iv). Hence, we can choose* $w_i\in W_p^{2+s}(\Omega)$ *such that*

$$(\gamma_{0,1}w_i,\gamma_{0,1}\partial_1w_i,\dots,\gamma_{0,n}w_i,\gamma_{0,n}\partial_nw_i,\dots,\gamma_{0,2n}w_i,\gamma_{0,2n}\partial_{2n}w_i)=(0,0,\dots,g_{0,i}^{(n)},0,\dots,g_{0,i}^{(2n)},0).$$

We henceforth consider the equations for $v_i:=u_i-w_i$, *which now satisfies a homogeneous Dirichlet condition on* $\Gamma_D:=\bigcup_{j\in J_D}\Gamma_j$*:*

$$\text{(4.5)}\qquad \begin{cases} -\nabla\cdot(\kappa_i\nabla v_i) &=& (-1)^{i+1}F(v_2-v_1+w_2-w_1)+f_i+\nabla\cdot(\kappa_i\nabla w_i) & \text{in } \Omega,\\ \gamma_{0,j}v_i &=& 0 & \text{on } \Gamma_j\quad (j\in J_D),\\ \gamma_{0,j}\kappa_i\partial_jv_i &=& 0 & \text{on } \Gamma_j\quad (j\in J_N\setminus\{n,2n\}),\\ \gamma_{0,j}\kappa_i\partial_jv_i &=& g_{1,i}^{(j)} & \text{on } \Gamma_j\quad (j\in J_N\cap\{n,2n\}).\end{cases}$$

Let $\varphi\in C^\infty_{\Gamma_D}(\Omega)$ *(see Definition 3.22). Then we can multiply the first line of* (4.5) *with* φ *in* $L_2(\Omega)$ *and use integration by parts to find that* $v_i\in W_p^2(\Omega)\cap H^1_{\Gamma_D}(\Omega)$ *satisfies*

$$\begin{aligned}\int_\Omega\left((-1)^{i+1}F(v_2-v_1+w_2-w_1)+f_i\right)\varphi\,dx &= -\int_\Omega\nabla\cdot(\kappa_i\nabla(v_i+w_i))\,\varphi\,dx\\ =\int_\Omega\kappa_i\nabla(v_i+w_i)\nabla\varphi\,dx-\sum_{j\in J_N}\int_{\Gamma_j}\kappa_i\underbrace{\left(\tfrac{\partial v_i}{\partial\nu}+\tfrac{\partial w_i}{\partial\nu}\right)}_{=\gamma_{0,j}\partial_jv_i+\gamma_{0,j}\partial_jw_i}\varphi\,dS(x)-\sum_{j\in J_D}\int_{\Gamma_j}\kappa_i\left(\tfrac{\partial v_i}{\partial\nu}+\tfrac{\partial w_i}{\partial\nu}\right)\underbrace{\varphi}_{=0}dS(x)&\\ =\int_\Omega\kappa_i\nabla v_i\nabla\varphi\,dx+\int_\Omega\kappa_i\nabla w_i\nabla\varphi\,dx-\int_{\Gamma_l}g_{1,i}^{(l)}\varphi\,dS(x).&\end{aligned}$$

Note that in the situation of an inhomogeneous Dirichlet boundary value at most one inhomogeneous Neumann boundary term (on the opposite surface) occurs. Hence, we have a decomposition $u_i = v_i + w_i \in H^1_{\Gamma_D}(\Omega) + H^1(\Omega)$ *and* v_i *satisfies an equation of the form* (3.27a)-(3.27b). *The treatment of the cases when no inhomogeneous Dirichlet boundary condition occurs is much easier. For the situation of pure Neumann boundary conditions, one has to consider test functions with vanishing mean value* $\varphi \in C^\infty_*(\Omega)$ *(cf. Definition 3.4).*

Corollary 4.6
Suppose that Assumption 4.1 is satisfied. Then the following systems admit at most one solution $u = (u_1, u_2)$ *in the indicated spaces:*

(i) The linear equations (4.4a) *and* (4.4b) *in the space* $W^2_p(\Omega)$.

(ii) The nonlinear system (4.1) *in the space* $W^2_p(\Omega)^2$ *with two sets of boundary conditions* $\mathcal{B}_1$ *for* u_1 *and* $\mathcal{B}_2$ *for* u_2, *which contain at least one Dirichlet condition.*

(iii) The nonlinear system (4.1) *with pure Neumann boundary conditions when the solution is considered in the space* $W^2_{p,*}(\Omega) \times W^2_p(\Omega)$ *(cf. Definition 3.4).*

Proof. (i) The difference of two solutions $u_1 - \widetilde{u}_1$ to (4.4a) satisfies this equation with homogeneous boundary values and right-hand side, which only admits the trivial solution due to Lemma 3.29 with $w_i = 0$, i.e. $u_1 = \widetilde{u}_1$ holds in $L_2(\Omega)$. (ii) is a direct consequence of Lemma 3.24 and (iii) follows from Lemma 3.6. □

We emphasize that no additional compatibility conditions were necessary to prove uniqueness for the aforementioned systems, since their boundary conditions are linear and hence the difference of two solutions always satisfies homogeneous boundary values. An application of Corollary 2.26 and the Fredholm property (Theorem 2.30) now yield:

Lemma 4.7
Let Assumption 4.1 be satisfied. Then for every $\delta \in [0,1]$ *and* $z_1, z_2 \in W^2_p(\Omega)$, *each of the equations* (4.4a) *and* (4.4b) *admits a unique solution* $u_i \in W^{2+s}_p(\Omega)$. *This solution satisfies the a priori estimate*

$$\|u_i\|_{2+s,p,\Omega} \leq C\left(\|(-1)^{i+1}\ F(z_2 - z_1) + f_i + z_i\|_{s,p,\Omega} + \sum_{l \in \{n,2n\}} \|g^{(l)}_{m_l,i}\|_{2+s-m_l-1/p,p,\Gamma_l} \right). \tag{4.6}$$

Proof. Instead of (4.4a) consider the equation

$$\left\{ \begin{array}{rcll} \lambda u_1 - \nabla \cdot (\kappa_1 \nabla u_1) & = & \delta\,(F(z_2 - z_1) + f_1 + z_1) & \text{in } \Omega, \\ \gamma_{\partial\Omega} \mathcal{B}_1 u_1 & = & \delta g_1 & \text{on } \partial\Omega, \\ [\![u_1]\!] = [\![\partial_n u_1]\!] & = & 0 & \text{on } \Sigma_i,\ (i \in \{1,2\}), \end{array} \right. \tag{4.7}$$

with a parameter $\lambda \in \overline{\Sigma}_\varphi$ for some $0 < \varphi < \frac{\pi}{2}$. The principal part of this operator is $\kappa_1 \Delta$ and this operator with the boundary conditions from Assumption 4.1 (iv) leads to a boundary value problem, which is symmetric in the sense of Definition 2.5, in particular this problem is parameter-elliptic in $\overline{\Sigma}_\varphi$. According to Corollary 2.26 for $|\lambda| > \lambda_1$ this problem admits a unique solution $u_1 \in W^{2+s}_p(\Omega)$, which satisfies the a priori estimate from 2.26 in the parameter dependent norms. Due to Corollary 4.6 (i), the operator

$$(1 - \nabla \cdot (\kappa_1 \nabla), \mathcal{B}_1) \in L\left(W^{2+s}_p(\Omega), W^s_p(\Omega) \times \prod_{j=1}^{2n} W^{2+s-m_j-1/p}_p(\Gamma_j) \right)$$

is injective. By Theorem 2.30, equation (4.4a) admits a unique solution which satisfies the a priori estimate (4.6) for $i = 1$. The existence of a solution to (4.4b) follows the same lines. □

The preceding lemma gives rise to the solution operator to the equations (4.4a)-(4.4b):

Definition 4.8
Let Assumption 4.1 be satisfied.

(i) We abbreviate $X := (W_p^2(\Omega))^2$ *and* $X_* := W_{p,*}^2(\Omega) \times W_p^2(\Omega)$. *The norm on* X *and* X_* *will be denoted by*

$$\|u\|_X := \|u_1\|_{2,p,\Omega} + \|u_2\|_{2,p,\Omega} \quad (u = (u_1, u_2) \in X).$$

(ii) Let $\mathrm{ord}(B_i^{(j)}) = 0$ *for at least one pair* (i,j) $(i = 1,2;\ j = 1,\dots,2n)$, *i.e., at least one of the boundary conditions is of Dirichlet type. Then for given data* $f_i \in W_p^s(\Omega)$ *and* $g_{m_{i,j},i}^j \in W_p^{2+s-m_{i,j}-1/p}(\Gamma_j)$ $(i = 1,2;\ j \in \{n, 2n\})$, *which are subject to the compatibility conditions* (4.2a)-(4.2b), *we define the solution operator*

$$T : [0,1] \times X \to (W_p^{2+s}(\Omega))^2 \subset X,\ (\delta, z_1, z_2) \mapsto T(\delta, z_1, z_2) := (u_1, u_2), \tag{4.8}$$

where (u_1, u_2) *denotes the solution of the equations* (4.4a)-(4.4b) *which uniquely exists due to Lemma 4.7.*

(iii) In the case of pure Neumann boundary conditions, i.e., $\mathrm{ord}(B_i^{(j)}) = 1$ *for all* $i = 1,2$, *and* $j = 1,\dots,2n$, *we additionally project the first component of the solution into the space* $W_{p,*}^{2+s}(\Omega)$, *i.e. we define the operator*

$$\begin{aligned} T : [0,1] \times X_* &\to W_{p,*}^{2+s}(\Omega) \times W_p^{2+s}(\Omega) \subset X_*, \\ (\delta, z_1, z_2) &\mapsto T(\delta, z_1, z_2) := \left(u_1 - \frac{1}{|\Omega|} \int_\Omega u_1 \, dx \ ,\ u_2 \right). \end{aligned} \tag{4.9}$$

Remark 4.9
Clearly, T *satisfies* $T(0, z_1, z_2) = 0$, *since the problem with homogeneous data admits the unique solution* 0, *see Corollary 4.6.*

In the subsequent lemmas, we prove continuity properties of the operator T with respect to z and δ.

Lemma 4.10
For fixed $\delta \in [0,1]$, *the operator* $T(\delta, \cdot) : X \to X$ *is continuous, i.e. for all* $y = (y_1, y_2) \in X$ *and all* $\varepsilon > 0$, *there exists* $\eta > 0$ *such that for all* $z = (z_1, z_2) \in X$ *with* $\|y - z\|_X < \eta$, *there holds* $\|T(\delta, y) - T(\delta, z)\|_X < \varepsilon$.

Proof. Let $u := (u_1, u_2) := T(\delta, y)$ and $v := (v_1, v_2) := T(\delta, z)$ be the unique solutions to (4.4a) and (4.4b) with the data $(-1)^{i+1} F(y_2 - y_1) + f_i + y_i$ and $(-1)^{i+1} F(z_2 - z_1) + f_i + z_i$, respectively. Then $w_i := u_i - v_i$ satisfies

$$\begin{cases} w_i - \nabla \cdot (\kappa_i \nabla w_i) &= \delta \left((-1)^{i+1} (F(y_2 - y_1) - F(z_2 - z_1)) + y_i - z_i \right) & \text{in } \Omega, \\ \gamma_{\partial\Omega} \mathcal{B}_i(\mathcal{D}) w_i &= 0 & \text{on } \partial\Omega. \end{cases} \tag{4.10}$$

The a priori estimate (4.6) for this elliptic problem and $\delta \le 1$ yield

$$\|u_i - v_i\|_{2,p,\Omega} \le C \left(\|F(y_2 - y_1) - F(z_2 - z_1)\|_{0,p,\Omega} + \|y_i - z_i\|_{0,p,\Omega} \right).$$

For our choice of p we have $W_p^2(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega})$. In particular, we have the boundedness of $\max_{x \in \overline{\Omega}} |y_2(x) - y_1(x)| \le C \|y_2 - y_1\|_{2,p,\Omega} =: K_1 < \infty$. For $\|y - z\|_X < \eta$ with $0 < \eta < 1$, this implies

$$\max_{x \in \overline{\Omega}} |(z_2 - z_1)(x)| \le C \left(\|(z_2 - y_2) + (y_1 - z_1) + (y_2 - y_1)\|_{2,p,\Omega} \right) \le 2C\eta + K_1 < 2C + K_1 =: K_2.$$

We now apply the mean value theorem to obtain

$$\begin{aligned}\int_\Omega |F(y_2-y_1,x)-F(z_2-z_1,x)|^p\ dx &= \int_\Omega |(\partial_1 F)(\xi,x)|^p\,|(y_2-y_1)-(z_2-z_1)|^p\ dx\\ &\le K_3^p\left(\|y_1-z_1\|_{0,p,\Omega}^p+\|y_2-z_2\|_{0,p,\Omega}^p\right)\end{aligned}$$

with some $\xi\in[-K_2,K_2]$ and $K_3:=\max_{(\xi,x)\in[-K_2,K_2]\times\overline{\Omega}}|(\partial_1F)(\xi,x)|$. This yields the desired continuity of T with $\eta:=\min\left(\frac{\varepsilon}{2C(K_3+1)},\frac{1}{2}\right)$. In case of pure Neumann boundary conditions, we additionally estimate

$$\left\|u_1-v_1-\tfrac{1}{|\Omega|}\int_\Omega v_1-u_1\ dx\right\|_{2,p,\Omega}\le\|u_1-v_1\|_{2,p,\Omega}+|\Omega|^{1/p-1}\|u_1-v_1\|_{0,1,\Omega}\le\varepsilon+\|u_1-v_1\|_{0,p,\Omega}<2\varepsilon.$$

□

Lemma 4.11
The mapping T is equicontinuous on bounded subsets of X uniformly in δ, i.e. for any bounded set $X_0\subset X$ and for all $\varepsilon>0$, there exists $\eta>0$ such that for all $\delta_1,\delta_2\in[0,1]$ with $|\delta_2-\delta_1|<\eta$ and for all $z\in X_0$, there holds $\|T(\delta_1,z)-T(\delta_2,z)\|_X<\varepsilon$.

Proof. Let $z=(z_1,z_2)\in X$ such that $\|z\|_X\le K_1$, hence $\|z_2-z_1\|_{2,p,\Omega}\le 2K_1$. By the Assumption 4.1 (iii), F maps bounded sets of $W_p^2(\Omega)$ into bounded sets of $W_p^s(\Omega)\hookrightarrow L_p(\Omega)$ and we infer $\|F(z_2-z_1)\|_{0,p,\Omega}\le C_F(K_1)$. For $\delta_1,\delta_2\in[0,1]$, we define $u:=(u_1,u_2):=T(\delta_1,z)$ as well as $v:=(v_1,v_2):=T(\delta_2,z)$. Then the difference $w_i:=u_i-v_i$ solves

$$\left\{\begin{array}{rcll} w_i-\nabla\cdot(\kappa_i\nabla w_i) &=& (\delta_2-\delta_1)\left((-1)^{i+1}F(z_2-z_1)+f_i+z_i\right) & \text{in }\Omega,\\ \gamma_{\partial\Omega}\mathcal{B}_i w_i &=& (\delta_2-\delta_1)g & \text{on }\partial\Omega.\end{array}\right. \tag{4.11}$$

For a solution to this problem the a priori estimate (4.6) yields

$$\begin{aligned}\|w_i\|_{2,p,\Omega}&\le C|\delta_2-\delta_1|\left(\|F(z_2-z_1)\|_{0,p,\Omega}+\|f_i\|_{0,p,\Omega}+\|z_i\|_{0,p,\Omega}+\sum_{j\in\{n,2n\}}\|g_i^{(j)}\|_{2-m_{j,i}-1/p,p,\Gamma_j}\right)\\ &\le|\delta_2-\delta_1|\left(C_F(K_1)+C\|f_i\|_{0,p,\Omega}+C\sum_{j\in\{n,2n\}}\|g_i^{(j)}\|_{2-m_{j,i}-1/p,p,\Gamma_j}\right)\le|\delta_2-\delta_1|\,C(F,K_1,f_i,g_i^{(j)}).\end{aligned}$$

This shows that $T(\cdot,z)$ is Lipschitz continuous with a Lipschitz constant which only depends on K_1 and the fixed data, but not on the particular choice of $z\in X$. Hence, the claimed continuity is proved. In case of pure Neumann conditions we proceed similarly as in the proof of the preceding lemma. □

Next, we prove a uniform a priori bound for elements $u\in X$, which satisfy $T(\delta,u)=u$ for some $\delta\in[0,1]$.

Lemma 4.12
There is a constant $M>0$ such that if $u=(u_1,u_2)\in X$ satisfies $T(\delta,u)=u$ for some $\delta\in[0,1]$, it follows $\|u\|_X\le M$.

Proof. The equality $T(\delta,u)=u$ implies that (u_1,u_2) is a solution to the nonlinear system (4.4a)-(4.4b) with $z_1=u_1$ and $z_2=u_2$. Remark 4.5 shows that u_i is also a weak solution to the nonlinear system. In case of an inhomogeneous Dirichlet condition, note that the boundary correction term $w_i\in W_p^2(\Omega)$ only depends on the Dirichlet data $g_{0,i}^{(j)}$ on Γ_j for $j\in\{n,2n\}$ and satisfies an estimate of the form (cf. (A.2))

$$\|w_i\|_{2,p,\Omega}\le C(\|g_{0,i}^{(n)}\|_{2-1/p,p,\Gamma_n}+\|g_{0,i}^{(2n)}\|_{2-1/p,p,\Gamma_{2n}})=:M_{1,i}.$$

In order to obtain a bound for v_i, we deduce from Lemma 3.31 (or from Theorem 3.17 in case of pure Neumann boundary conditions, respectively) that in particular $\|v_i\|_{0,\infty,\Omega} \leq M_{2,i}$ for $i = 1, 2$. By embedding, we have $\|u_i\|_{0,\infty,\Omega} \leq M_i := M_{1,i} + M_{2,i}$ and set $M_3 := M_1 + M_2$. Using the a priori estimate in $W_p^2(\Omega)$ for the linear equations and the piecewise continuity of F on the compact sets $[-M_3, M_3] \times \overline{\Omega}_j$ for $j = 1, 2, 3$, we infer for $i = 1, 2$ that

$$\|u_i\|_{2,p,\Omega} \leq C\delta \left(\|F(u_2 - u_1)\|_{0,p,\Omega} + \|f_i\|_{0,p,\Omega} + \|u_i\|_{0,p,\Omega} \right) \leq C_F(M_3) + C(\|f_i\|_{0,p,\Omega} + M_i).$$

Defining M as the sum of these constants finishes the proof. □

Lemma 4.13
For every fixed $\delta \in [0, 1]$, the operator $T_\delta := T(\delta, \cdot)$ from Definition 4.8 (ii) [resp. (iii)] has the following properties:

(i) T_δ maps bounded subsets of X [resp. of X_*] into bounded subsets of $(W_p^{2+s}(\Omega))^2$ [resp. of $W_{p,*}^{2+s}(\Omega) \times W_p^{2+s}(\Omega)$].

(ii) $T_\delta : X \to X$ [resp. $T_\delta : X_* \to X_*$] is a compact operator.

Proof. (i) Let $z \in X$ with $\|z\|_X \leq K_1$. Then the boundedness of the mapping $z \mapsto F(z, \cdot)$ as a mapping from $W_p^2(\Omega)$ into $W_p^s(\Omega)$ (cf. Assumption 4.1, (iii)) yields $\|F(z(\cdot), \cdot)\|_{s,p,\Omega} \leq C(F, K_1)$. The a priori estimate (4.6) then implies that $\|T(\delta, z)\|_{2+s,p,\Omega}$ is bounded by a constant which depends on the fixed data f_i and $g_{m_j,i}^{(j)}$ in their respective norms as well as on $C(F, K_1)$.

(ii) The continuity of T_δ has been proved in Lemma 4.10. Part (i) shows that any bounded sequence $(z_n)_{n\in\mathbb{N}}$ in X yields a bounded sequence $(T_\delta(z_n))_{n\in\mathbb{N}}$ in $(W_p^{2+s}(\Omega))^2$ and by the compactness of the embedding $W_p^{2+s}(\Omega) \overset{c}{\hookrightarrow} W_p^2(\Omega)$, there exists a subsequence of $(T_\delta(z_n))_{n\in\mathbb{N}}$ which converges in X. In case of pure Neumann boundary conditions note that

$$\frac{1}{|\Omega|} \left\| \int_\Omega u_1 \, dx \right\|_{2+s,p,\Omega} = \frac{1}{|\Omega|} \left\| \int_\Omega u_1 \, dx \right\|_{0,p,\Omega} \leq \|u_1\|_{0,1,\Omega},$$

hence, all assertions remain valid in this case, too. □

Theorem 4.14
Let Assumption 4.1 be satisfied. Then the nonlinear elliptic system

$$\left\{ \begin{array}{rcll} -\nabla \cdot (\kappa_1 \nabla u_1) & = & F(u_2 - u_1) + f_1 & \text{in } \Omega, \\ -\nabla \cdot (\kappa_2 \nabla u_2) & = & -F(u_2 - u_1) + f_2 & \text{in } \Omega, \\ \gamma_{\partial\Omega} \mathcal{B}_1 u_1 & = & g_1, \quad \gamma_{\partial\Omega} \mathcal{B}_2 u_2 = g_2 & \text{on } \partial\Omega, \\ [\![u_i]\!] = [\![\partial_n u_i]\!] & = & 0 & \text{on } \Sigma_j, \ (i, j \in \{1, 2\}) \end{array} \right. \tag{4.12}$$

admits a unique solution $u = (u_1, u_2) \in (W_p^{2+s}(\Omega))^2$. In case of pure Neumann boundary conditions, uniqueness holds in the space $W_{p,*}^{2+s}(\Omega) \times W_p^{2+s}(\Omega)$. In addition, there exists a constant $M = M(n, p, s, \Omega, \kappa_i, f_i, g_i)$ such that

$$\|u_1\|_{2+s,p,\Omega} + \|u_2\|_{2+s,p,\Omega} \leq M.$$

Proof. The Leray-Schauder principle (Theorem A.9), whose preliminaries are contained in Definition 4.8, Remark 4.5, and Lemmas 4.10-4.13, gives rise to a fixed point $u = (u_1, u_2) \in X$ of the operator $T(1, \cdot) : X \to X$. By the construction of the problems (4.4a)-(4.4b), u is a solution to the nonlinear system (4.12) and $\|u\|_X \leq M$ by Lemma 4.12. Taking the functions $(-1)^{i+1} F(u_2 - u_1) + f_i \in W_p^s(\Omega) \cap L_\infty(\Omega)$, which are also bounded by a constant $C(F, M, f_i)$, as data for the equations (4.4a)-(4.4b) and using their unique solvability, Lemma 4.7 yields $u_i \in W_p^{2+s}(\Omega)$ and the bound for the norm in $W_p^{2+s}(\Omega)$. □

Remark 4.15

(i) The Fredholm property of the elliptic boundary value problems is not essential for the proof of the unique solvability of the linearized equations (4.4a) *and* (4.4b) *in Lemma 4.7. In fact, we could consider equations of the structure* (4.7) *with a fixed, sufficiently large $\lambda > \lambda_0$ and a term λz_i instead of z_i on the right-hand side. Then Corollary 2.28 applies and yields the corresponding estimate. The remaining part of the proof strategy remains the same but the estimates will additionally depend on λ.*

(ii) In the present chapter, the homogeneous transmission conditions were imposed to obtain a strong solution $u \in W_p^2(\Omega)^2$, despite the possible discontinuities of the coefficients across the interfaces Σ_i. Using Corollary 2.26, one can consider general transmission conditions on the interfaces Σ_i, but then one will obtain a solution u which satisfies $u|_{\Omega_j} \in W_p^{2+s}(\Omega_j)$ for $j = 1, 2, 3$, but in general one cannot expect a higher global regularity than $u \in W_p^s(\Omega)$ with $s < 1/p$ on the whole of Ω.

Chapter 5

Parabolic transmission problems

The detailed study of parameter-elliptic boundary value problems in different settings in the first chapter has direct consequences for the corresponding parabolic equations. Here, we briefly introduce the abstract concepts of $\mathcal{R}$-boundedness and maximal regularity as well as their implications for the solvability of parabolic equations. Afterwards, we focus on parabolic equations with general transmission conditions.

5.1 Maximal regularity and $\mathcal{R}$-boundedness

Throughout this section let X, Y, and Z denote Banach spaces.

5.1.1 $\mathcal{R}$-bounded operator families

Definition 5.1
A family $\mathcal{T} \subset L(X,Y)$ of linear operators is called $\mathcal{R}$-bounded if there exists a constant $C > 0$ and $p \in [1,\infty)$ such that for any choice of $n \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$, and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega, \mathcal{M}, \mu)$ it holds that

$$\left\| \sum_{j=1}^{n} \varepsilon_j T_j x_j \right\|_{L_p(\Omega;Y)} \leq C \left\| \sum_{j=1}^{n} \varepsilon_j x_j \right\|_{L_p(\Omega;X)}. \tag{5.1}$$

The smallest possible constant C such that (5.1) is satisfied is called the $\mathcal{R}$-bound of the family $\mathcal{T}$ and will be denoted by $\mathcal{R}_p(\mathcal{T})$.

$\mathcal{R}$-boundedness implies uniform boundedness and if X and Y are Hilbert spaces, the concepts are equivalent. An equivalent characterization of $\mathcal{R}$-boundedness via Rademacher functions as well as Kahanes inequality (see [KW04], Definition 1.9 and Theorem 2.4) shows that if $\mathcal{T}$ is $\mathcal{R}$-bounded for one $p \in [1,\infty)$, then for any $q \in [1,\infty)$. Still the $\mathcal{R}$-bound may depend on p, but we do not further indicate this and agree to write $\mathcal{R}(\mathcal{T})$.

Lemma 5.2

(i) Let $\mathcal{T}$ and $\mathcal{S} \subset L(X,Y)$ as well as $\mathcal{U} \subset L(Y,Z)$ be $\mathcal{R}$-bounded. Then the operator families $\mathcal{T} \cup \mathcal{S} \subset L(X,Y)$, $\mathcal{T} + \mathcal{S} \subset L(X,Y)$, and $\mathcal{U}\mathcal{T} \subset L(X,Z)$ are $\mathcal{R}$-bounded as well. For the corresponding $\mathcal{R}$-bounds the estimates

$$\mathcal{R}(\mathcal{T} \cup \mathcal{S}), \mathcal{R}(T+S) \leq \mathcal{R}(\mathcal{S}) + \mathcal{R}(\mathcal{T}) \quad \text{and} \quad \mathcal{R}(\mathcal{U}\mathcal{T}) \leq \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T})$$

are valid. Furthermore, if $\overline{\mathcal{T}}$ denotes the closure of $\mathcal{T}$ with respect to the strong operator topology, it holds $\mathcal{R}(\overline{\mathcal{T}}) = \mathcal{R}(\mathcal{T})$.

(ii) (Kahane's contraction principle) Let $n \in \mathbb{N}$, $x_j \in X$, ε_j be independent, symmetric, $\{-1,1\}$-valued random variables on a probability space $(\Omega, \mathcal{M}, \mu)$. Assume that $\alpha_j, \beta_j \in \mathbb{C}$ with $|\alpha_j| \leq |\beta_j|$ for $j = 1, \dots, n$. Then

$$\left\| \sum_{j=1}^{n} \alpha_j \varepsilon_j x_j \right\|_{L_p(\Omega;X)} \leq 2 \left\| \sum_{j=1}^{n} \beta_j \varepsilon_j x_j \right\|_{L_p(\Omega;X)}. \tag{5.2}$$

The constant 2 can be omitted in the case $\alpha_j, \beta_j \in \mathbb{R}$.

Proof. See e.g. [DHP03], Propositions 3.4 and 3.5. □

A fundamental ingredient when proving $\mathcal{R}$-boundedness of the resolvents corresponding to certain parameter-elliptic problems will be the $\mathcal{R}$-boundedness of kernel functions and of their corresponding integral operators which is contained in the next lemma.

Lemma 5.3
Let (Ω, μ) be a σ-finite measure space and Λ be an index set. Let $K = \{k_\lambda : \lambda \in \Lambda\}$ be a set of measurable, operator-valued kernels $k_\lambda : \Omega \times \Omega \to L(X, Y)$ such that

$$\mathcal{R}\left(\{k_\lambda(x,y) : \lambda \in \Lambda\}\right) \leq k_0(x,y) \quad (x, y \in \Omega)$$

for $k_0 : \Omega \times \Omega \to \mathbb{C}$ and assume that the scalar integral operator

$$(K_0 f)(x) := \int_\Omega k_0(x,y) f(y)\, dy \quad (f \in L_p(\Omega))$$

belongs to $L(L_p(\Omega))$. Then the family

$$(K_\lambda g)(x) := \int_\Omega k_\lambda(x,y) g(y)\, dy \quad (g \in L_p(\Omega; X),\ \lambda \in \Lambda)$$

of vector valued integral operators belongs to $L(L_p(\Omega; X), L_p(\Omega; Y))$ and is $\mathcal{R}$-bounded with $\mathcal{R}(\{K_\lambda : \lambda \in \Lambda\}) \leq \|K_0\|_{L(L_p(\Omega))}$.

Proof. Note that $g \in L_p(\Omega; X)$ means $\|g(\cdot)\|_X \in L_p(\Omega)$. Then the estimate

$$\begin{aligned}
\int_\Omega \|(K_\lambda g)(x)\|_Y^p\ dx &= \int_\Omega \left\| \int_\Omega k_\lambda(x,y) g(y)\, dy \right\|_Y^p\ dx \leq \int_\Omega \left(\int_\Omega \|k_\lambda(x,y) g(y)\|_Y\, dy \right)^p\ dx \\
&\leq \int_\Omega \left(\int_\Omega k_0(x,y) \|g(y)\|_X\, dy \right)^p\ dx \leq \int_\Omega |(K_0 \|g(\cdot)\|_X)(x)|^p\ dx = \left\| K_0 \|g(\cdot)\|_X \right\|_{L_p(\Omega)}^p \\
&\leq \left(\|K_0\|_{L(L_p(\Omega))} \left\| \|g(\cdot)\|_X \right\|_{L_p(\Omega)} \right)^p = \left(\|K_0\|_{L(L_p(\Omega))} \|g\|_{L_p(\Omega;X)} \right)^p
\end{aligned}$$

shows that $K_\lambda \in L(L_p(\Omega; X), L_p(\Omega; Y))$. The $\mathcal{R}$-boundedness is proved in [DHP03], Proposition 4.12. □

The second ingredient is to prove the $\mathcal{R}$-boundedness of certain kernel functions similarly as we proved the norm-boundedness of the solution operators in Chapter 1, using Michlin's multiplier theorem. The necessary generalizations are provided in the next subsection.

5.1.2 Operator valued Fourier multipliers

Let X, Y be Banach spaces and $m \in L_\infty(\mathbb{R}^n \setminus \{0\}; L(X, Y))$. Then, since $m\mathcal{F}f \in L_\infty(\mathbb{R}^n; Y)$ for $f \in \mathscr{S}(\mathbb{R}^n; X)$, the operator

$$T_m f := \mathcal{F}^{-1} m \mathcal{F} f \quad (f \in \mathscr{S}(\mathbb{R}^n; X))$$

is well-defined as a mapping from $\mathscr{S}(\mathbb{R}^n; X)$ into $\mathscr{S}'(\mathbb{R}^n; Y)$.

Definition 5.4
Let $1 < p < \infty$ and $m \in L_\infty(\mathbb{R}^n; L(X, Y))$. Then m is called an (operator-valued) (L_p-)Fourier multiplier if the operator $T_m f \in L_p(\mathbb{R}^n; Y)$ for all $f \in \mathscr{S}(\mathbb{R}^n; X)$ and if there exists $C > 0$ such that

$$\|T_m f\|_{L_p(\mathbb{R}^n;Y)} \leq C \|f\|_{L_p(\mathbb{R}^n;X)} \quad (f \in \mathscr{S}(\mathbb{R}^n; X)).$$

Since $\mathscr{S}(\mathbb{R}^n; X)$ is dense in $L_p(\mathbb{R}^n; X)$, in this situation T_m extends to a continuous linear operator $T_m \in L(L_p(\mathbb{R}^n; X), L_p(\mathbb{R}^n; Y))$.

Remark 5.5 (Necessity of $\mathcal{R}$-boundedness)
If one first restricts to functions $f \in \mathcal{F}^{-1}\mathscr{D}(\mathbb{R}^n;X)$, it suffices to require $m \in L_{1,loc}(\mathbb{R}^n;L(X,Y))$ (see [DHP03], Section 3.3) such that $T_m : \mathcal{F}^{-1}\mathscr{D}(\mathbb{R}^n;X) \to \mathscr{S}'(\mathbb{R}^n;Y)$ is well-defined. However, given $m \in L_{1,loc}(\mathbb{R}^n;L(X,Y))$ such that $T_m \in L(L_p(\mathbb{R}^n;X),L_p(\mathbb{R}^n;Y))$, it is known that $\{m(\xi),\xi \in \mathcal{L}(m)\} \subset L(X,Y)$ is $\mathcal{R}$-bounded, where $\mathcal{L}(m)$ denotes the set of Lebesgue points of m (see [DHP03], Proposition 3.17).

Before we can state sufficient conditions on $m \in L_\infty(\mathbb{R}^n;L(X,Y))$ to be a Fourier multiplier, i.e., an analogue of Michlin's multiplier theorem, two more definitions of properties of the underlying Banach spaces are required.

Definition 5.6 *(i) We define the Hilbert transform*

$$\mathcal{H} : \mathscr{S}(\mathbb{R};X) \to \mathscr{S}'(\mathbb{R};X), \quad f \mapsto \mathcal{H}f := \mathcal{F}^{-1}m_{\mathcal{H}}\mathcal{F}f,$$

with the symbol $m_{\mathcal{H}} := i\frac{\xi}{|\xi|}\,\mathrm{id}_X$.

(ii) The Banach space X is said to be of class $\mathcal{HT}$ if $\mathcal{H}$ extends to a bounded linear operator $\mathcal{H} \in L(L_p(\mathbb{R};X))$ for some $p \in [1,\infty)$.

Remark 5.7
In [Ama95], Section 4.3 the following equivalent definition of the Hilbert transform is justified:

$$(\mathcal{H}u)(t) := \frac{1}{\pi}PV(1/t) * u = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\tau\geq\varepsilon} u(t-\tau)\ \tfrac{d\tau}{\tau}.$$

Remark 5.8
In the literature, Banach spaces of class $\mathcal{HT}$ are often referred to as spaces which admit the unconditional martingale difference property (UMD-property), which is an equivalent description in probability theory. In the textbook [Ama95], Theorem 4.5.2, one can find several properties of spaces of class $\mathcal{HT}$ such as

(i) Finite-dimensional spaces are of class $\mathcal{HT}$.

(ii) Hilbert spaces are of class $\mathcal{HT}$.

(iii) If X is of class $\mathcal{HT}$, so is $L_p(\Omega;X)$ for $1 < p < \infty$.

(iv) If X is of class $\mathcal{HT}$ and $Y \subset X$ is a closed linear subspace, then Y is of class $\mathcal{HT}$, too.

(v) The property of class $\mathcal{HT}$ is stable interpolation, i.e., if (X_0,X_1) is an interpolation couple of Banach spaces of class $\mathcal{HT}$, then for $\theta \in (0,1)$ and $1 < p < \infty$, the spaces $(X_0,X_1)_{\theta,p}$ and $[X_0,X_1]_\theta$ are of class $\mathcal{HT}$, too.

In particular, all function spaces we will be dealing with are of class $\mathcal{HT}$.

Definition 5.9
A Banach space X is said to have property (α) if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $\alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$, all $x_{ij} \in X$, and all independent, symmetric, $\{-1,1\}$-valued random variables $\varepsilon_i^{(1)}$ on a probability space $(\Omega_1,\mathcal{M}_1,\mu_1)$ and $\varepsilon_j^{(2)}$ on a probability space $(\Omega_2,\mathcal{M}_2,\mu_2)$ for $i,j = 1,\dots,n$, there holds

$$\int_{\Omega_1}\int_{\Omega_2}\Big\|\sum_{i,j=1}^n \varepsilon_i^{(1)}(x)\varepsilon_j^{(2)}(x)\alpha_{ij}x_{ij}\Big\|_X\,dx\,dy \leq C\int_{\Omega_1}\int_{\Omega_2}\Big\|\sum_{i,j=1}^n \varepsilon_i^{(1)}(x)\varepsilon_j^{(2)}(x)x_{ij}\Big\|_X\,dx\,dy.$$

We do not discuss this property in detail but mention that if a Banach space X has property (α), so has $L_p(\Omega;X)$ for $p \in (1,\infty)$ and that Hilbert spaces possess property (α). For more details and proofs we refer to [KW04], Section 4. The following theorem contains the vector valued version of Michlin's multiplier theorem.

Theorem 5.10
Let $1 < p < \infty$ and X, Y be Banach spaces of class $\mathcal{HT}$. Assume that $\mathcal{T} \subset L(X,Y)$ is an $\mathcal{R}$-bounded set.

(i) If $m \in C^n(\mathbb{R}^n \setminus \{0\}; L(X,Y))$ satisfies

$$\left\{ |\xi|^{|\gamma|} \mathcal{D}^\gamma m(\xi) : \xi \in \mathbb{R}^n,\ \gamma \in \{0,1\}^n \right\} \subset \mathcal{T}, \tag{5.3}$$

then m is a Fourier multiplier in the sense of Definition 5.4. In addition, the estimate $\|T_m\|_{L(L_p(\mathbb{R}^n;X),L_p(\mathbb{R}^n;Y))} \leq C\mathcal{R}(\mathcal{T})$ is satisfied with a constant $C = C(X,Y,n,p)$.

(ii) If in addition X and Y enjoy property (α) and $m \in C^n(\mathbb{R}^n \setminus \{0\}; L(X,Y))$ satisfies the condition

$$\{ \xi^\gamma \mathcal{D}^\gamma m(\xi) : \xi \in \mathbb{R}^n,\ \gamma \in \{0,1\}^n \} \subset \mathcal{T}, \tag{5.4}$$

then m is a Fourier multiplier in the sense of Definition 5.4 and the set of operators

$$\{ T_m : m \text{ satisfies condition (5.4)} \} \subset L(L_p(\mathbb{R}^n;X), L_p(\mathbb{R}^n;Y))$$

is $\mathcal{R}$-bounded.

Proof. For (i) we refer to [DHP03], Theorem 3.25, [KW04], Theorem 4.13 or [HHN02]. The $\mathcal{R}$-boundedness of a family of Fourier multiplier operators in (ii) was proved by Girardi and Weis, see [GW03] and also [KW04], Theorem 4.6. □

We draw a simple conclusion for the case $X = \mathbb{C}^N$ and $Y = \mathbb{C}^M$ which will be useful for later applications to parabolic systems.

Corollary 5.11
Let $M, N \in \mathbb{N}$, Λ be an index set, and $\{m_\lambda, \lambda \in \Lambda\}$ be a set of $(M \times N)$-matrix valued symbols $m_\lambda \in C^n(\mathbb{R}^n \setminus \{0\}; L(\mathbb{C}^N, \mathbb{C}^M))$ such that

$$|\xi^\gamma \mathcal{D}^\gamma m_\lambda(\xi)| \leq C_0 \quad (\xi \in \mathbb{R}^n \setminus \{0\},\ \gamma \in \{0,1\}^n,\ \lambda \in \Lambda).$$

Then the family of operators $\{T_{m_\lambda} : \lambda \in \Lambda\} \subset L(L_p(\mathbb{R}^n;\mathbb{C}^N), L_p(\mathbb{R}^n;\mathbb{C}^M))$ is $\mathcal{R}$-bounded with $\mathcal{R}(\{T_{m_\lambda} : \lambda \in \Lambda\}) \leq CC_0$, where C only depends on n, N, M, and p.

Proof. Since $\mathbb{C}^N$ and $\mathbb{C}^M$ with their standard inner products are Hilbert spaces, they are of class $\mathcal{HT}$ and possess property (α). The assumption implies that the set

$$\{ \xi^\gamma \mathcal{D}^\gamma m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\},\ \gamma \in \{0,1\}^n, \lambda \in \Lambda \} \subset L(\mathbb{C}^N, \mathbb{C}^M)$$

is uniformly bounded in $L(\mathbb{C}^N, \mathbb{C}^M)$, hence this set is $\mathcal{R}$-bounded. Note that we choose the euclidean norm on $\mathbb{C}^N$ and $\mathbb{C}^M$ here and that only the $\mathcal{R}$-bound, but not the property of $\mathcal{R}$-boundedness in $L(X,Y)$ is affected by passing to an equivalent norm on X or Y. In order to apply Theorem 5.10 (ii), we set $\mathcal{T} := \{A \in L(\mathbb{C}^N, \mathbb{C}^M) : |A| \leq C_0\}$, which is uniformly bounded and again $\mathcal{R}$-bounded, because the operators act between Hilbert spaces. Theorem 5.10 (ii) yields the $\mathcal{R}$-boundedness of $\{T_{m_\lambda} : \lambda \in \Lambda\} \subset L(L_p(\mathbb{R}^n;\mathbb{C}^N), L_p(\mathbb{R}^n;\mathbb{C}^M))$ with the claimed $\mathcal{R}$-bound. □

5.1.3 Maximal regularity and parabolic equations

This subsection connects the notion of $\mathcal{R}$-boundedness to the well-posedness of parabolic equations and the regularity of the solution. We discuss the central condition of $\mathcal{R}$-sectoriality of an operator A in the context of abstract Cauchy problems of the form

$$\begin{cases} u_t - Au &= f \quad \text{in } (0,T), \\ u(0) &= u_0, \end{cases} \tag{5.5}$$

with a closed and densely defined linear operator $A : X \supset D(A) \to X$ on a Banach space X. As a generalization of the sectoriality of an operator (cf. Definition 1.4), we have the following

Definition 5.12
A closed linear operator A on a complex Banach space X is called $\mathcal{R}$-sectorial if the following conditions are satisfied:

(1) $\overline{D(A)} = X$ and $\overline{R(A)} = X$.

(2) There exists an angle $\varphi \in (0, \pi)$ such that $\rho(A) \supset \overline{\Sigma}_\varphi$ and the set $\{\lambda(\lambda - A)^{-1} : \lambda \in \overline{\Sigma}_\varphi\}$ is $\mathcal{R}$-bounded in $L(X)$.

The supremum of all angles such that condition (2) is satisfied is called the $\mathcal{R}$-angle $\varphi_A^{\mathcal{R}}$ of A.

Remark 5.13
Every $\mathcal{R}$-sectorial operator is in particular sectorial and it holds $\varphi_A^{\mathcal{R}} \leq \varphi_A \leq \pi$. In Hilbert spaces the two definitions are equivalent.

When dealing with inhomogeneous initial conditions, we need an appropriate space for the initial value u_0.

Definition 5.14
Let $J := (0, T)$, $1 < p < \infty$, and $A : X \supset D(A) \to X$ be a closed linear operator in X. We define the space for the solution of (5.5) as $\mathbb{E} := W_p^1(J; X) \cap L_p(J; D(A))$ with the natural norm $\|u\|_{\mathbb{E}} := \|u\|_{W_p^1(J;X)} + \|u\|_{L_p(J;D(A))}$. Furthermore, let $I_p(A)$ denote the trace space

$$I_p(A) := \{u_0 : \exists u \in \mathbb{E} : u(0) = u_0\}, \quad \text{with} \quad \|u_0\|_{I_p(A)} := \inf_{u \in \mathbb{E}, u(0)=u_0} \|u\|_{\mathbb{E}},$$

and $\mathbb{F} := L_p(J; X) \times I_p(A)$ the space for the data to the equations (5.5).

Remark 5.15 (Characterization of $I_p(A)$)
It can be shown that $I_p(A)$ is a Banach space with $D(A) \hookrightarrow I_p(A) \hookrightarrow X$. If A is the generator of a strongly continuous semigroup, $I_p(A)$ can be characterized as a real interpolation space: $I_p(A) = (X, D(A))_{1-1/p,p}$, see e.g. [Lun95], Proposition 2.2.2. By means of the trace method (see [Tri95], Section 1.8) one can prove that this is already satisfied for a closed operator A: The equalities

$$V_1(p, \tfrac{1}{p}, D(A), p, \tfrac{1}{p}, X) = \mathbb{E} \quad \text{and} \quad T_0^1(p, \tfrac{1}{p}, D(A), p, \tfrac{1}{p}, X) = I_p(A)$$

hold with equivalent norms. The equivalence Theorem ([Tri95], Theorem 1.8.2) then implies

$$I_p(A) = T_0^1(p, \tfrac{1}{p}, D(A), p, \tfrac{1}{p}, X) = (D(A), X)_{\frac{1}{p},p} = (X, D(A))_{1-1/p,p}$$

with equivalent norms.
In case of our applications we will consider $X := L_p(\Omega; \mathbb{C}^N)$ and for the realization of a differential operator A of order $2m$ on X, the domain is a subspace of $W_p^{2m}(\Omega; \mathbb{C}^N)$. The right choice for the space of the initial conditions will therefore be the Besov space

$$I_p(A) = \big(L_p(\Omega; \mathbb{C}^N), W_p^{2m}(\Omega; \mathbb{C}^N)\big)_{1-1/p,p} = B_{pp}^{2m(1-1/p)}(\Omega; \mathbb{C}^N).$$

Definition 5.16
Let $T > 0$, $J := (0, T)$, $1 < p < \infty$, X a Banach space, and let $A : X \supset D(A) \to X$ be a closed, densely defined operator in X. Then A is said to have maximal L_p-regularity on X, if for all $(f, u_0) \in \mathbb{F}$ there exists a unique solution $u : J \to D(A)$ of the Cauchy problem (5.5) which satisfies the estimate

$$\|u_t\|_{L_p(J;X)} + \|Au\|_{L_p(J;X)} \leq C_p \left(\|f\|_{L_p(J;X)} + \|u_0\|_{(X,D(A))_{1-1/p,p}} \right) \tag{5.6}$$

with a constant $C_p > 0$ independent of $f \in L_p(J; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$.

It was proved by Sobolevskii in [Sob64] that if A has maximal regularity for one $p \in (1,\infty)$, then A enjoys this property for all $p \in (1,\infty)$. In the sequel, we do not further indicate the index p and only speak of maximal regularity.

Remark 5.17

(i) Observe that the definition does not require $u \in L_p(J;X)$, hence $\|u_t\|_{L_p(J;X)}$ cannot be replaced by $\|u\|_{W^1_p(J;X)}$. However, this is possible if $0 \in \rho(A)$ or if the time interval J is finite, i.e., $T < \infty$, see [KW04], for instance. Then A admits maximal regularity if and only if the mapping

$$\begin{pmatrix} \partial_t - A \\ \gamma_0 \end{pmatrix} : W^1_p(J;L_p(X)) \cap L_p(J;D(A)) \to L_p(J;X) \times I_p(A)$$

is a continuous isomorphism of Banach spaces. Here, γ_0 denotes the trace with respect to time.

(ii) The definition of $I_p(A)$ allows to reduce (5.5) to the case $u_0 = 0$, since we may subtract $v \in \mathbb{E}$ with $v(0) = u_0$ and $\|v\|_{\mathbb{E}} \le 2\|u_0\|_{I_p(A)}$. Hence, our definition of maximal regularity is equivalent to the definition in [KW04], Section 1.3, where merely $u(0) = 0$ is considered.

The following theorem was proved by L. Weis in 2001 and connects the concepts of $\mathcal{R}$-sectoriality and maximal regularity.

Theorem 5.18
Let A be the generator of a bounded analytic semigroup on a Banach space X of class $\mathcal{HT}$. Then A has maximal L_p-regularity for one (all) $p \in (1,\infty)$ on $\mathbb{R}_+$ if and only if A is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle $\varphi^{\mathcal{R}}_A > \frac{\pi}{2}$.

Proof. The result is Theorem 1.11 in [KW04]. □

Results on maximal L_p-regularity have been established for general linear parabolic equations under natural assumptions (see for example [DHP03] and [DHP07]) and have been applied in fixed point arguments to several nonlinear equations by numerous authors, e.g. in [EPS03] to the Stefan-problem with Gibbs-Thomson correction, or in [DGHSS] to a free boundary value problem, which is a model for the spin-coating process, to mention only a few. However, the notion of transmission problems, which corresponds to a system in $\mathbb{R}^n_+$ with $2m$ boundary operators, does not fit into the scheme which was developed in [DHP03] or [DHP07] and we present a slightly different approach in the next section.

5.2 Transmission problems in the half space

We consider a general parabolic transmission problem in the half spaces $\mathbb{R}^n_\pm$ of the following structure:

$$(5.7)\quad \begin{cases} \partial_t \tilde{u}_1 - \widetilde{A}_1(x,\mathcal{D})\tilde{u}_1 = \tilde{f}_1 & \text{in } J\times\mathbb{R}^n_-, \\ \partial_t u_2 - A_2(x,\mathcal{D})u_2 = f_2 & \text{in } J\times\mathbb{R}^n_+, \\ \gamma_{0,n}\left(\widetilde{B}^{(1)}_j(x,\mathcal{D})\tilde{u}_1 - B^{(2)}_j(x,\mathcal{D})u_2\right) = g_j & \text{in } J\times\mathbb{R}^{n-1},\ (j=1,\dots,2m), \\ \tilde{u}_1(0,x) = \tilde{u}_{1,0}(x) \quad \text{in } \mathbb{R}^n_-, \quad u_2(0,x) = u_{2,0}(x) & \text{in } \mathbb{R}^n_+. \end{cases}$$

and impose the following

Assumption 5.19
Let Ω_1 and Ω_2 be domains in $\mathbb{R}^n$ which are separated by a surface Γ.

(i) The differential operator $\widetilde{A}_1(x,\mathcal{D}) = \sum_{|\alpha|\le 2m} \tilde{a}_\alpha^{(1)}(x)\mathcal{D}^\alpha$ is of order $2m$ with continuous coefficients $\tilde{a}_\alpha^{(1)} \in BUC(\overline{\Omega}_1)$ if $|\alpha| = 2m$, and $\tilde{a}_\alpha^{(1)} \in L_\infty(\Omega_1)$ if $|\alpha| < 2m$. If Ω_1 is unbounded, assume in addition that $\tilde{a}_\alpha^{(1)}(\infty) := \lim_{|x|\to\infty} \tilde{a}_\alpha^{(1)}$ exists. We further assume parabolicity of the operator $\widetilde{A}_1$, i.e., there exists an angle $\varphi_1 \ge \pi/2$ such that

$$\lambda - \widetilde{A}_{1,0}(x,\xi) := \lambda - \sum_{|\alpha|=2m} \tilde{a}_\alpha^{(1)}(x)\xi^\alpha \neq 0 \quad \text{for all } x \in \overline{\Omega}_1,\ (\xi,\lambda) \in \mathbb{R}^n \times \overline{\Sigma}_{\varphi_1} \setminus \{0\}. \tag{5.8}$$

Here, $\widetilde{A}_{1,0}$ again denotes the principal symbol of $\widetilde{A}_1$. In other words, $\widetilde{A}_1$ is parameter-elliptic in a sector $\overline{\Sigma}_{\varphi_1}$ which contains the closed right halfplane.

(ii) $A_{2,0}(x,\mathcal{D}) = \sum_{|\alpha|\le 2m} a_\alpha^{(2)}(x)\mathcal{D}^\alpha$ is a differential operator of order $2m$, whose coefficients satisfy the same assumptions in Ω_2 as those of $\widetilde{A}_1$ in Ω_1 and A_2 is also parabolic, i.e., there exists an angle $\varphi_2 \ge \pi/2$ such that

$$\lambda - A_{2,0}(x,\xi) \neq 0 \quad \text{for all } x \in \overline{\Omega}_2,\ (\xi,\lambda) \in \mathbb{R}^n \times \overline{\Sigma}_{\varphi_2} \setminus \{0\}. \tag{5.9}$$

(iii) The operators $\widetilde{B}_j^{(1)}$ and $B_j^{(2)}$ in the transmission conditions are of equal order $m_j < 2m$ and take the form $\widetilde{B}_j^{(1)}(x,\mathcal{D}) = \sum_{|\beta|\le m_j} \tilde{b}_{j\beta}^{(1)}(x)\mathcal{D}^\beta$ and $B_j^{(2)}(x,\mathcal{D}) = \sum_{|\beta|\le m_j} b_{j\beta}^{(2)}(x)\mathcal{D}^\beta$ with coefficients $\tilde{b}_{j\beta}^{(1)},\ b_{j\beta}^{(2)} \in C^{2m-m_j}(\Gamma)$. If Γ is unbounded, assume that the limits

$$\tilde{b}_{j\beta}^{(1)}(\infty) := \lim_{|x|\to\infty, x\in\Gamma} \tilde{b}_{j\beta}^{(1)}(x) \quad \text{and} \quad b_{j\beta}^{(2)}(\infty) := \lim_{|x|\to\infty, x\in\Gamma} b_{j\beta}^{(2)}(x)$$

exist for $|\beta| = m_j$ and $j = 1,\dots,2m$.

(iv) The following substitute for the Shapiro-Lopatinskii condition is satisfied: For every $x_0 \in \Gamma$, the system of ordinary differential equations in local coordinates subject to x_0

$$\begin{cases} (\lambda - \widetilde{A}_{1,0}(x_0,\xi',\mathcal{D}_n))\tilde{v}_1(x_n) = 0 & (x_n < 0), \\ (\lambda - A_{2,0}(x_0,\xi',\mathcal{D}_n))v_2(x_n) = 0 & (x_n > 0), \\ \widetilde{B}_{j,0}^{(1)}(x_0,\xi',\mathcal{D}_n)\tilde{v}_1\big|_{x_n=0} - B_{j,0}^{(2)}(x_0,\xi',\mathcal{D}_n)v_2\big|_{x_n=0} = g_j & (j = 1,\dots,2m), \\ \tilde{v}_1(x_n) \to 0 \quad (x_n \to -\infty), \qquad v_2(x_n) \to 0 & (x_n \to +\infty), \end{cases} \tag{5.10}$$

admits a unique solution for all $g_j \in \mathbb{C}$ and all $(\xi',\lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$, where we have set $\varphi := \min(\varphi_1,\varphi_2)$. In addition, for unbounded domains Ω_1 and Ω_2 and unbounded Γ, we impose the condition when the coefficients of the principal parts are replaced by their limits at infinity.

(v) Regarding the data in (5.7), we assume the following regularities: $\tilde{f}_1 \in L_p(J;L_p(\Omega_1))$, $f_2 \in L_p(J;L_p(\Omega_2))$, and $g_j \in W_p^{\kappa_j/2m,\kappa_j}(J\times\Gamma) := W_p^{\kappa_j/2m}(J;L_p(\Gamma)) \cap L_p(J;W_p^{\kappa_j}(\Gamma))$, where we have set $\kappa_j := 2m - m_j - 1/p$ for $j = 1,\dots,2m$. According to Remark 5.15, the initial values satisfy $\tilde{u}_{1,0} \in W_p^{2m-2m/p}(\Omega_1)$ and $u_{2,0} \in W_p^{2m-2m/p}(\Omega_2)$. In addition, if $\kappa_j > 2m/p$, i.e., if the boundary value g_j admits a trace with respect to time, then $\gamma_\Gamma \widetilde{B}_j^{(1)}(x,\mathcal{D})\tilde{u}_{1,0} - \gamma_\Gamma B_j^{(2)}(x,\mathcal{D})u_{2,0} = g_j(0,x)$ for $x \in \Gamma$.

The transmission problem (5.7) will be called parabolic if the assumptions $(i)-(iv)$ are satisfied. The main task will be to treat the transmission problem in the half spaces, i.e. $\Omega_1 := \mathbb{R}^n_-$, $\Omega_2 := \mathbb{R}^n_+$ and $\Gamma \cong \mathbb{R}^{n-1}$. In this situation, we will merely consider the equivalent system for $u = (u_1, u_2)^T := (\tilde{u}_1 \circ \tau_n, u_2)^T$, where $\tau_n : \mathbb{R}^n_+ \to \mathbb{R}^n_-$ denotes the reflection in the x_n-variable. Our assumptions will lead to a system in the half space which is parameter-elliptic in the sector $\overline{\Sigma}_\varphi \supset \{z \in \mathbb{C} : \ \mathrm{Re}(z) \geq 0\}$. We define the reflected right-hand side $f_1 := \tilde{f}_1 \circ \tau_n \in L_p(J; L_p(\mathbb{R}^n_+))$, $f := (f_1, f_2)^T$, and note that the reflected differential operator in $\mathbb{R}^n_+$ with the symbol

$$A_1(x, \xi) := \widetilde{A}_1(x', -x_n, \xi', -\xi_n) \quad (x \in \mathbb{R}^n_+, \ \xi \in \mathbb{R}^n)$$

again satisfies Assumption 5.19 (i) with $\mathbb{R}^n_-$ replaced by $\mathbb{R}^n_+$ and the analogue of (iv), which now reads: for every $x_0 \in \mathbb{R}^{n-1}$, all $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$, and all $g_j \in \mathbb{C}$, the system of ordinary differential equations

$$\left\{ \begin{array}{rcll} (\lambda - A_{1,0}(x_0, \xi', \mathcal{D}_n))v_1(x_n) & = & 0 & (x_n > 0), \\ (\lambda - A_{2,0}(x_0, \xi', \mathcal{D}_n))v_2(x_n) & = & 0 & (x_n > 0), \\ \Big(B^{(1)}_{j,0}(x_0, \xi', \mathcal{D}_n)v_1 - B^{(2)}_{j,0}(x_0, \xi', \mathcal{D}_n)v_2\Big)\Big|_{x_n=0} & = & g_j & (j = 1, \ldots, 2m), \\ v_1(x_n) \to 0, \qquad v_2(x_n) & \to & 0 & (x_n \to +\infty), \end{array} \right. \tag{5.11}$$

is uniquely solvable. Here, we have set $B^{(1)}_j(x_0, \xi', \mathcal{D}_n) := \widetilde{B}^{(1)}_j(x_0, \xi', -\mathcal{D}_n)$ for $x_0 \in \mathbb{R}^{n-1}$. We now follow a standard approach in elliptic theory and first consider the model problem, hence all operators have constant coefficients ($\tilde{a}^{(1)}_\alpha, a^{(2)}_\alpha, \tilde{b}^{(1)}_{j\beta}, b^{(2)}_{j\beta} \in \mathbb{C}$) and coincide with their principal parts. In order to prove maximal regularity, we will derive a resolvent representation for the solution to the problem

$$\left\{ \begin{array}{rcll} (\lambda - \widetilde{A}_1(\mathcal{D}))\tilde{u}_1 & = & \tilde{f}_1 & \text{in } \mathbb{R}^n_-, \\ (\lambda - A_2(\mathcal{D}))u_2 & = & f_2 & \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n}\left(\widetilde{B}^{(1)}_j(\mathcal{D})\tilde{u}_1 - B^{(2)}_j(\mathcal{D})u_2\right) & = & g_j & \text{on } \mathbb{R}^{n-1}, \ (j = 1, \ldots, 2m). \end{array} \right. \tag{5.12}$$

Observe that we can equivalently construct a representation for the solution $u = (u_1, u_2)$ to the corresponding system in the half space and then obtain $(\tilde{u}_1, u_2) = (u_1 \circ \tau_n, u_2)$.

5.2.1 Solution operators

Preserving the notation introduced above and writing $A(\mathcal{D}) := \begin{pmatrix} A_1(\mathcal{D}) & 0 \\ 0 & A_2(\mathcal{D}) \end{pmatrix}$ as well as $B_j(\mathcal{D}) := \Big(B^{(1)}_j(\mathcal{D}), -B^{(2)}_j(\mathcal{D})\Big)$ and $\mathcal{B}(\mathcal{D}) := (B_1(\mathcal{D}), \ldots, B_{2m}(\mathcal{D}))^T$ for the differential operator and the boundary operators, respectively, we construct the resolvent for the system

$$\left\{ \begin{array}{rcll} (\lambda \, \mathrm{id}_{\mathbb{C}^2} - A(\mathcal{D}))u & = & f & \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n}B_j(\mathcal{D})u & = & g_j & \text{on } \mathbb{R}^{n-1}, \ (j = 1, \ldots, 2m) \end{array} \right. \tag{5.13}$$

which is parameter-elliptic in the sector $\overline{\Sigma}_\varphi$. First, we reduce to a homogeneous right-hand side: Let $E_0 : L_p(\mathbb{R}^n_+; \mathbb{C}^2) \to L_p(\mathbb{R}^n; \mathbb{C}^2)$ denote the trivial extension. Then we solve the equation in the whole space (cf. [DHP03], Theorem 5.4) and restrict the solution to $\mathbb{R}^n_+$ again:

$$w_0 := r_+(\lambda - A_{\mathbb{R}^n})^{-1}E_0 f \in W^{2m}_p(\mathbb{R}^n_+; \mathbb{C}^2). \tag{5.14}$$

We choose the approach $u = w_0 + w$, where $w \in W^{2m}_p(\mathbb{R}^n_+; \mathbb{C}^2)$ is a solution to

$$\left\{ \begin{array}{rcll} (\lambda \, \mathrm{id}_{\mathbb{C}^2} - A(\mathcal{D}))w & = & 0 & \text{in } \mathbb{R}^n_+, \\ \gamma_{0,n}B_j(\mathcal{D})w & = & g_j - \gamma_{0,n}B_j(\mathcal{D})w_0 & \text{on } \mathbb{R}^{n-1}, \ (j = 1, \ldots, 2m). \end{array} \right. \tag{5.15}$$

For convenience, we agree to write $\tilde{g}_j := g_j - \gamma_{0,n}B_j(\mathcal{D})w_0 \in W^{2m-m_j-1/p}_p(\mathbb{R}^{n-1})$ for the new inhomogeneous boundary values. Parameter-ellipticity of $(A, B_1, \ldots, B_{2m})$ and Lemma 2.5 of

[DFM02] yield the existence of a $2 \times 2m$ matrix-valued function $W(\xi', x_n, \lambda)$, which is defined on the set $S := \{(\xi', x_n, \lambda) : (\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\},\ x_n \geq 0\}$. W has continuous derivatives with respect to ξ' and x_n of arbitrary order and admits the representation

$$W(\xi', x_n, \lambda) = \int_{\gamma_+(\xi', \lambda)} e^{ix_n\tau}(\lambda - A(\xi', \tau))^{-1}\mathcal{N}(\xi', \tau, \lambda)\ d\tau \tag{5.16}$$

with a $2 \times 2m$ matrix $\mathcal{N}$. This matrix-function has continuous derivatives with respect to ξ' and admits the following quasi-homogeneity for all $\rho > 0$:

$$\mathcal{N}(\rho\xi', \rho\tau, \rho^{2m}\lambda) = \mathcal{N}(\xi', \tau, \lambda)\mathrm{diag}(\rho^{2m-m_1-1}, \ldots, \rho^{2m-m_{2m}-1}),$$

see also Lemma 1.35. The $2m$ columns of the matrix W form a set of basic solutions for the system of ordinary differential equations which is obtained from (5.15) via partial Fourier transform. More precisely, for all $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$, the matrix function $W(\xi', \cdot, \lambda)$ satisfies

$$\left\{ \begin{array}{rcll} (\lambda\,\mathrm{id}_{\mathbb{C}^2} - A(\xi', \mathcal{D}_n))W(\xi', x_n, \lambda) & = & 0_{2m} & (x_n > 0), \\ \mathcal{B}(\xi', \mathcal{D}_n)W(\xi', x_n, \lambda)|_{x_n=0} & = & I_{2m}. & \end{array} \right. \tag{5.17}$$

This leads to the representation $v(\xi', x_n, \lambda) = W(\xi', x_n, \lambda)(\mathcal{F}\tilde{g})(\xi')$ with the vector of boundary values $\tilde{g} := (\tilde{g}_1, \ldots, \tilde{g}_{2m})^T \in \prod_{j=1}^{2m} W_p^{2m-m_j-1/p}(\mathbb{R}^{n-1})$ for every solution v of

$$\left\{ \begin{array}{rcll} (\lambda\,\mathrm{id}_{\mathbb{C}^2} - A(\xi', \mathcal{D}_n))v(\xi', x_n, \lambda) & = & 0 & (x_n > 0), \\ B_j(\xi', \mathcal{D}_n)v(\xi', x_n, \lambda)|_{x_n=0} & = & (\mathcal{F}\tilde{g}_j)(\xi') & (j = 1, \ldots, 2m). \end{array} \right. \tag{5.18}$$

In the sequel, we denote by $h_j := B_j(\mathcal{D})w_0 + \mathcal{F}'^{-1}\exp(-(|\lambda|^{1/2m} + |\xi'|)x_n)\mathcal{F}'g_j \in W_p^{2m-m_j}(\mathbb{R}^n_+)$ a special extension of $\tilde{g}_j$ to the half space to obtain $h := (h_1, \ldots, h_{2m})$ with $\gamma_{0,n}h = \tilde{g}$ (cf. Lemma 1.34 or [DFM02], Lemma 2.6) and rewrite the representation for v as follows:

$$\begin{aligned} v(\xi', x_n, \lambda) = W(\xi', x_n, \lambda)(\mathcal{F}'h)(\xi', 0) &= -\int_0^\infty \partial_n \left[W(\xi', x_n + \sigma, \lambda)(\mathcal{F}'h)(\xi', \sigma)\right]\ d\sigma \\ &= -\int_0^\infty (\partial_n W)(\xi', x_n + \sigma, \lambda)(\mathcal{F}'h)(\xi', \sigma)\ d\sigma - \int_0^\infty W(\xi', x_n + \sigma, \lambda)(\mathcal{F}'\partial_n h)(\xi', \sigma)\ d\sigma. \end{aligned}$$

Finally, we introduce the parameter-dependent order reductions in the tangential variables

$$\Lambda_s(\lambda) : W_p^r(\mathbb{R}^n) \to W_p^{r-s}(\mathbb{R}^n), \quad u \mapsto \Lambda_s(\lambda)u := \mathcal{F}'^{-1}(|\lambda|^{1/2m} + |\xi'|)^s\mathcal{F}'u$$

and use the property $\Lambda_s(\lambda)\Lambda_{-s}(\lambda) = \mathrm{id}_{L_p(\mathbb{R}^n)}$ for $s \in \mathbb{R}$ (cf. Remark 1.12). Abbreviating $\rho := (|\lambda|^{1/2m} + |\xi'|)$, we define the matrices

$$\Psi_1(\rho) := \mathrm{diag}(\rho^{2m-m_1}, \ldots, \rho^{2m-m_{2m}}) \quad \text{and} \quad \Psi_2(\rho) := \rho^{-1}\Psi_1(\rho).$$

Then the operator $T_{\Psi_1} := \mathcal{F}'^{-1}\Psi_1\mathcal{F}'$ maps $\prod_{j=1}^{2m} W_p^{2m-m_j}(\mathbb{R}^n_+)$ continuously into $L_p(\mathbb{R}^n_+)^{2m}$ and $T_{\Psi_2} := \mathcal{F}'^{-1}\Psi_2\mathcal{F}'$ maps $\prod_{j=1}^{2m} W_p^{2m-m_j-1}(\mathbb{R}^n_+)$ continuously into $L_p(\mathbb{R}^n_+)^{2m}$. We obtain the following representation for $w(x', x_n) = (\mathcal{F}'^{-1}v(\cdot, x_n))(x')$:

$$\begin{aligned} w(x', x_n) &= -\int_0^\infty \mathcal{F}'^{-1}(\partial_n W)(\xi', x_n + \sigma, \lambda)\Psi_1^{-1}(\rho)(\mathcal{F}'T_{\Psi_1}h)(\xi', \sigma)\ d\sigma \\ &\quad -\int_0^\infty \mathcal{F}'^{-1}W(\xi', x_n + \sigma, \lambda)\Psi_2^{-1}(\rho)(\mathcal{F}'T_{\Psi_2}\partial_n h)(\xi', \sigma)\ d\sigma \\ &=: \mathbb{T}^{(1)}(\lambda)T_{\Psi_1}h + \mathbb{T}^{(2)}(\lambda)T_{\Psi_2}\partial_n h. \end{aligned} \tag{5.19}$$

Remark 5.20 (Construction and smoothness of the basic solutions)
We take a closer look at the construction of the matrix $W(\xi', x_n, \lambda)$, especially to the $N \times Nm$ matrix $\mathcal{N}(\xi', \tau, \lambda)$. This matrix is constructed in [Vol68] for the case of an elliptic system in $\mathbb{C}^N$ and one can define $\mathcal{N}$ using the pseudo-inverse $\Lambda(\xi', \lambda)^+$ of the $mN \times 2mN$ matrix

$$\Lambda(\xi', \lambda) := \int_{\gamma_+(\xi', \lambda)} \mathcal{B}(\xi', \tau)(\lambda - A(\xi', \tau))^{-1} \left[I_N, \tau I_N, \dots, \tau^{2m-1} I_N \right] \, d\tau$$

which has rank mN for all $(\xi', \lambda) \in \mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi \setminus \{0\}$ if and only if the Shapiro-Lopatinskii condition is satisfied by $(A, \mathcal{B})$ (see [Vol68], Section 1). Explicitly, one defines

$$\mathcal{N}(\xi, \tau, \lambda) := \left[I_N, \tau I_N, \dots, \tau^{2m-1} I_N \right] \Lambda(\xi', \lambda)^T (\Lambda(\xi', \lambda) \Lambda(\xi', \lambda)^T)^{-1},$$

which is an $N \times mN$ matrix, and satisfies $\int_{\gamma_+} \mathcal{B}(\xi', \tau)(\lambda - A(\xi', \tau))^{-1} \mathcal{N}(\xi, \tau, \lambda) \, d\tau = I_{mN}$. From this representation, the smoothness properties can be read off immediately. In addition, since $\lambda \mapsto (\lambda - A(\xi', \tau))^{-1}$ is analytic and this property is preserved under the multiplication and inversion of Λ, the same holds true for $\mathcal{N}$.

5.2.2 $\mathcal{R}$-boundedness for solution operators

We first prove $\mathcal{R}$-boundedness for the integral operators $\mathbb{T}^{(i)}$ in the representation (5.19).

Lemma 5.21
For $i = 1, 2$ and $\lambda_0 > 0$, the sets of operators

$$\left\{ \lambda^{\frac{2m-|\alpha|}{2m}} \mathcal{D}^\alpha \mathbb{T}^{(i)}(\lambda) : \lambda \in \overline{\Sigma}_\varphi, \ |\lambda| \geq \lambda_0, \ \alpha \in \mathbb{N}_0^n, \ |\alpha| \leq 2m \right\}$$

are $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n_+; \mathbb{C}^{2m}), L_p(\mathbb{R}^n_+; \mathbb{C}^2))$.

Proof. We start with the kernel functions that correspond to $\lambda^{\frac{2m-|\alpha|}{2m}} \mathcal{D}^\alpha \mathbb{T}^{(i)}(\lambda)$:

$$k_\alpha^{(1)}(\xi', x_n, \sigma, \lambda) := \lambda^{\frac{2m-|\alpha|}{2m}} \xi'^{\alpha'} \mathcal{D}_n^{\alpha_n} (-\partial_n W)(\xi', x_n + \sigma, \lambda) \Psi_1^{-1}(\rho).$$

Observe that the integrand

$$M : S \to \mathbb{C}^{2 \times 2m}, \quad (\xi', \tau, \lambda) \mapsto (\lambda - A(\xi', \tau))^{-1} \mathcal{N}(\xi', \tau, \lambda) \Psi_1(\rho)$$

is homogeneous in $(\xi', \tau, \lambda^{2m})$ of degree $-2m - 1$ (in fact, the different homogeneities of $\mathcal{N}$ and Ψ_1 in each row cancel out each other), and that differentiation with respect to ξ_i diminishes the degree of homogeneity by 1. Then we can estimate

$$\begin{aligned}
& \left| \xi'^{\gamma'} \mathcal{D}_{\xi'}^{\gamma'} k_\alpha^{(1)}(\xi', x_n, \sigma, \lambda) \right| \\
= \ & \left| \lambda^{\frac{2m-|\alpha|}{2m}} \xi'^{\alpha'+\gamma'} \int_{\gamma_+} \tau^{\alpha_n+1} e^{i(x_n+\sigma)\tau} \mathcal{D}_{\xi'}^{\gamma'} \big((\lambda - A(\xi', \tau))^{-1} N(\xi', \tau, \lambda) \Psi_1^{-1}(\rho) \big) \, d\tau \right| \\
= \ & \left| \lambda^{\frac{2m-|\alpha|}{2m}} \xi'^{\alpha'+\gamma'} \int_{\gamma_+(\xi', \lambda)} \tau^{\alpha_n+1} e^{i(x_n+\sigma)\tau} \rho^{-2m-1-\gamma'} \left(\mathcal{D}_{\xi'}^{\gamma'} M \right) \left(\tfrac{\xi'}{\rho}, \tfrac{\tau}{\rho}, \tfrac{\lambda}{\rho^{2m}} \right) \, d\tau \right| \\
= \ & \left(\tfrac{|\lambda|}{\rho^{2m}} \right)^{\frac{2m-|\alpha|}{2m}} \left(\tfrac{|\xi'|}{\rho} \right)^{\alpha'+\gamma'} \left| \int_{\gamma_+(\xi', \lambda)} \left(\frac{\tau}{\rho} \right)^{\alpha_n+1} e^{i(x_n+\sigma)\tau} \left(\mathcal{D}_{\xi'}^{\gamma'} M \right) \left(\tfrac{\xi'}{\rho}, \tfrac{\tau}{\rho}, \tfrac{\lambda}{\rho^{2m}} \right) \, d\tau \right| \\
\leq \ & C(\alpha, \gamma') \left| \int_{\gamma_+ \left(\frac{\xi'}{\rho}, \frac{\lambda}{\rho^{2m}} \right)} \rho \, \widetilde{\tau}^{\alpha_n+1} e^{i(x_n+\sigma)\rho\widetilde{\tau}} \left(\mathcal{D}_{\xi'}^{\gamma'} M \right) \left(\tfrac{\xi'}{\rho}, \widetilde{\tau}, \tfrac{\lambda}{\rho^{2m}} \right) \, d\widetilde{\tau} \right| \\
\leq \ & \frac{C(\alpha, \gamma')}{x_n + \sigma}.
\end{aligned}$$

For the latter estimates, we used the continuity of $\mathcal{D}_{\xi'}^{\gamma'} M$ on the bounded set

$$\left\{ \left(\tfrac{\xi'}{\rho}, \widetilde{\tau}, \tfrac{\lambda}{\rho^{2m}} \right) : (\xi', \lambda) \in (\mathbb{R}^{n-1} \times \overline{\Sigma}_\varphi) \setminus \{0\},\ \widetilde{\tau} \in R(\gamma_+(\tfrac{\xi'}{\rho}, \tfrac{\lambda}{\rho^{2m}})) \right\}.$$

Furthermore, we used the fact that γ_+ can be chosen locally independent of ξ', so that we do not have to differentiate γ_+, and the inequality $te^t \leq 1$ for $t \geq 0$. For $k_\alpha^{(2)}(\xi', x_n, \sigma, \lambda)$, the calculation only differs in the power of τ and the homogeneity of Ψ. Taking C_0 as the largest of the constants $C(\alpha, \gamma')$ with $\gamma' \in \{0,1\}^n$, Corollary 5.11 applies in $\mathbb{R}^{n-1}$ with $N = 2m$ and $M = 2$. This yields the $\mathcal{R}$-boundedness of the sets

$$\mathcal{K}^{(i)} := \left\{ \mathcal{F}'^{-1} k_\alpha^{(i)}(\cdot, x_n, \sigma, \lambda) \mathcal{F}' : \lambda \in \overline{\Sigma}_\varphi \right\} \subset L(L_p(\mathbb{R}^{n-1}; \mathbb{C}^{2m}), L_p(\mathbb{R}^{n-1}; \mathbb{C}^2)) \quad (i = 1, 2),$$

with an $\mathcal{R}$-bound $\mathcal{R}(\mathcal{K}^{(i)}) \leq \frac{C}{x_n + \sigma}$ for all $x_n, \sigma \in \mathbb{R}_+$. We set $k_0(x_n, \sigma) := \frac{1}{x_n + \sigma}$, then the corresponding integral operator $(K_0 f)(x_n) = \int_0^\infty \frac{f(\sigma)}{x_n + \sigma}\, d\sigma$ is bounded since $L_p(\mathbb{R}_+)$ is of class $\mathcal{HT}$ and we can apply Lemma 5.3 with $X = L_p(\mathbb{R}^{n-1}; \mathbb{C}^{2m})$ and $Y = L_p(\mathbb{R}^{n-1}; \mathbb{C}^2)$. We obtain the $\mathcal{R}$-boundedness of the sets of integral operators

$$(K_{\alpha,\lambda} f)(x_n) := \int_0^\infty \mathcal{F}'^{-1} k_\alpha^{(i)}(\cdot, x_n, \sigma, \lambda)(\mathcal{F}' f)(\xi', \sigma)\, d\sigma \quad (f \in L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; \mathbb{C}^{2m}))),$$

in $L(L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; \mathbb{C}^{2m})), L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}; \mathbb{C}^2))) = L(L_p(\mathbb{R}^n_+; \mathbb{C}^{2m}), L_p(\mathbb{R}^n_+; \mathbb{C}^2))$ which yields the assertion. □

There remains to prove the $\mathcal{R}$-boundedness of the set of operators, where the solution operators $\mathbb{T}^{(i)}(\lambda)$ are applied to. Here, we distinguish between the operators which stem from the reduction to homogeneous right-hand sides and the operators which act on the original inhomogeneous boundary values.

Lemma 5.22
The following sets of operators are $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n_+; \mathbb{C}^2), L_p(\mathbb{R}^n_+; \mathbb{C}^{2m}))$:

$$\left\{ T_{\Psi_1} \mathcal{B}(\mathcal{D}) r_+ (\lambda - A_{\mathbb{R}^n})^{-1} E_0 : \lambda \in \overline{\Sigma}_\varphi \right\} \quad \textit{and} \quad \left\{ T_{\Psi_2} \partial_n \mathcal{B}(\mathcal{D}) r_+ (\lambda - A_{\mathbb{R}^n})^{-1} E_0 : \lambda \in \overline{\Sigma}_\varphi \right\}.$$

Proof. Since the restriction operator commutes with $T_{\Psi_1} \mathcal{B}(\mathcal{D})$ and $T_{\Psi_2} \partial_n \mathcal{B}(\mathcal{D})$ (the constant coefficient operator $\mathcal{B}$ trivially extends to $\mathbb{R}^n$) and the sets with a single continuous operator $\{r_+\} \subset L(L_p(\mathbb{R}^n; \mathbb{C}^{2m}), L_p(\mathbb{R}^n_+; \mathbb{C}^{2m}))$, $\{E_0\} \subset L(L_p(\mathbb{R}^n_+; \mathbb{C}^2), L_p(\mathbb{R}^n; \mathbb{C}^2))$ are $\mathcal{R}$-bounded, it suffices to prove $\mathcal{R}$-boundedness of the sets

$$\left\{ T_{\Psi_1} \mathcal{B}(\mathcal{D})(\lambda - A_{\mathbb{R}^n})^{-1} : \lambda \in \overline{\Sigma}_\varphi \right\} \quad \text{and} \quad \left\{ T_{\Psi_2} \partial_n \mathcal{B}(\mathcal{D})(\lambda - A_{\mathbb{R}^n})^{-1} : \lambda \in \overline{\Sigma}_\varphi \right\}$$

in $L(L_p(\mathbb{R}^n; \mathbb{C}^2), L_p(\mathbb{R}^n; \mathbb{C}^{2m}))$ in view of Lemma 5.2 (i). Up to a factor $(-i)$ for m_2, the corresponding matrix valued symbols are given by

$$m_1(\xi, \lambda) := \Psi_1(|\lambda|^{\frac{1}{2m}} + |\xi'|) \mathcal{B}(\xi)(\lambda - A(\xi))^{-1} \text{ and } m_2(\xi, \lambda) := \Psi_2(|\lambda|^{\frac{1}{2m}} + |\xi'|) \xi_n \mathcal{B}(\xi)(\lambda - A(\xi))^{-1}.$$

Both symbols are quasi-homogeneous in (ξ, λ) of degree 0 and bounded on the compact set $\{(\xi, \lambda) \in \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\} :\ |\lambda|^{1/2m} + |\xi| = 1\}$ by continuity and the same holds for $\xi^\alpha \mathcal{D}_\xi^\alpha m_i(\xi, \lambda)$. Hence,

$$\left| \xi^\alpha \mathcal{D}_\xi^\alpha m_i(\xi, \lambda) \right| \leq C_\alpha \quad ((\xi, \lambda) \in \mathbb{R}^n \times \overline{\Sigma}_\varphi \setminus \{0\},\ \alpha \in \{0,1\}^n)$$

and Corollary 5.11 yields the claimed $\mathcal{R}$-boundedness. □

In order to have a continuous time dependent solution operator for inhomogeneous transmission conditions in the parabolic system, we also need to prove the $\mathcal{R}$-boundedness for the derivative of $\mathbb{T}^{(i)}(\lambda)$ with respect to λ, so that Theorem 5.10 (ii) becomes applicable in one dimension. Taking Fourier transform with respect to $t \in \mathbb{R}$, ∂_t transforms into a multiplication with $i\eta$ where $\eta \in \mathbb{R}$. Hence, we require the Michlin condition for $\mathbb{T}^{(i)}(\lambda)$ with $\lambda \in i\mathbb{R}$.

Lemma 5.23
For $\lambda_0 > 0$, the following sets of operators are $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n_+;\mathbb{C}^{2m}), L_p(\mathbb{R}^n_+;\mathbb{C}^2))$:

$$\Big\{\lambda^{2-\frac{|\alpha|}{2m}}\mathcal{D}^\alpha\partial_\lambda\mathbb{T}^{(j)}(\lambda) : \lambda\in i\mathbb{R},\ |\lambda|\geq\lambda_0,\ \alpha\in\mathbb{N}_0^n,\ |\alpha|\leq 2m\Big\}\quad (j=1,2).$$

Proof. We follow the lines of the proof of Lemma 5.21. This time we have to estimate the kernel functions

$$K_\alpha^{(1)}(\xi',x_n,\sigma,\lambda) := \lambda^{\frac{4m-|\alpha|}{2m}}\xi'^{\alpha'}\partial_\lambda\mathcal{D}_n^{\alpha_n}(-\partial_n W)(\xi',x_n+\sigma,\lambda)\Psi_1^{-1}(\rho).$$

Note that differentiability with respect to λ is ensured by Remark 5.20 and each diagonal element of $\Psi_1^{-1}(\rho)$, given by $(|\lambda|^{1/2m}+|\xi'|)^{-2m+m_j}$, is differentiable with respect to λ for $\lambda\in i\mathbb{R}\setminus\{0\}$. Every differentiation with respect to ξ_i diminishes the degree of homogeneity of the integrand by 1, while a derivative with respect to λ diminishes the degree of homogeneity of the integrand by $2m$, since this is the homogeneity-weight of λ. With these facts at hand, we can estimate

$$\begin{aligned}
&\Big|\xi'^{\gamma'}\mathcal{D}_{\xi'}^{\gamma'}K_\alpha^{(1)}(\xi',x_n+\sigma,\lambda)\Big|\\
=\ &\left|\lambda^{\frac{4m-|\alpha|}{2m}}\xi'^{\alpha'+\gamma'}\int_{\gamma_+}\tau^{\alpha_n+1}e^{i(x_n+\sigma)\tau}\partial_\lambda\mathcal{D}_{\xi'}^{\gamma'}\big((\lambda-A(\xi',\tau))^{-1}N(\xi',\tau,\lambda)\Psi_1^{-1}(\rho)\big)\ d\tau\right|\\
=\ &\left|\lambda^{\frac{4m-|\alpha|}{2m}}\xi'^{\alpha'+\gamma'}\int_{\gamma_+(\xi',\lambda)}\tau^{\alpha_n+1}e^{i(x_n+\sigma)\tau}\rho^{-4m-1-|\gamma'|}\Big(\mathcal{D}_{\xi'}^{\gamma'}M\Big)\Big(\tfrac{\xi'}{\rho},\tfrac{\tau}{\rho},\tfrac{\lambda}{\rho^{2m}}\Big)\ d\tau\right|\\
\leq\ &\frac{C(\alpha,\gamma')}{x_n+\sigma}.
\end{aligned}$$

Note that we do not have to differentiate γ_+ with respect to ξ' and λ, since Cauchy's theorem makes it possible to choose γ_+ locally independent of these variables. With obvious modifications for $K_\alpha^{(2)}$, the rest of the proof is literally the same as in Lemma 5.21. □

5.2.3 Maximal regularity for parabolic transmission problems

Eventually, we can draw the conclusions for parabolic transmission problems in the situation of a model problem. We start with (5.7) when $g_j = 0$ and define $\widetilde{A}(\mathcal{D}) := \begin{pmatrix}\widetilde{A}_1(\mathcal{D}) & 0\\ 0 & A_2(\mathcal{D})\end{pmatrix}$ as well as $\widetilde{B_j}(\mathcal{D}) := (\widetilde{B}_j^{(1)}(\mathcal{D}), -B_j^{(2)}(\mathcal{D}))$ and $\widetilde{\mathcal{B}}(\mathcal{D}) := (\widetilde{B}_1(\mathcal{D}),\dots,\widetilde{B}_{2m}(\mathcal{D}))^T$. Then we define the realization of $(\widetilde{A},\widetilde{\mathcal{B}})$ in $X := L_p(\mathbb{R}^n_-)\times L_p(\mathbb{R}^n_+)$ as

$$D(\widetilde{A}_{\widetilde{\mathcal{B}}}) := \left\{u=\begin{pmatrix}\tilde{u}_1\\ u_2\end{pmatrix}\in W_p^{2m}(\mathbb{R}^n_-)\times W_p^{2m}(\mathbb{R}^n_+) :\ \widetilde{\mathcal{B}}u=0\right\},\quad \widetilde{A}_{\widetilde{\mathcal{B}}}u := A(\mathcal{D})u\quad (u\in D(\widetilde{A}_{\widetilde{\mathcal{B}}})).$$

For $J=(0,T)$ and an open subset $\Omega\subset\mathbb{R}^n$, we agree to write

$$W_p^{1,2m}(J\times\Omega) := W_p^1(J;L_p(\Omega))\cap L_p(J;W_p^{2m}(\Omega)).$$

Theorem 5.24
Let the transmission problem (5.7) be parabolic in the sense of Assumption 5.19. Then the L_p-realization of the model problem is $\mathcal{R}$-sectorial with angle $\varphi := \min(\varphi_1,\varphi_2)\geq\frac{\pi}{2}$, i.e. ,

$$\Big\{\lambda(\lambda-\widetilde{A}_{\widetilde{\mathcal{B}}})^{-1} : \lambda\in\overline{\Sigma}_\varphi,\ |\lambda|\geq\lambda_0\Big\}\subset L(L_p(J;L_p(\mathbb{R}^n_-))\times L_p(J;L_p(\mathbb{R}^n_+)))$$

is $\mathcal{R}$-bounded. In particular, for $\lambda_0>0$ the operator $\widetilde{A}_{\widetilde{\mathcal{B}}}-\lambda_0$ admits maximal regularity and for all $(\tilde{f}_1,f_2)\in L_p(J;L_p(\mathbb{R}^n_-)\times L_p(\mathbb{R}^n_+))$ and $u_0=\Big(\tilde{u}_0^{(1)},u_0^{(2)}\Big)\in W_p^{2m-2m/p}(\mathbb{R}^n_-)\times W_p^{2m-2m/p}(\mathbb{R}^n_+)$, the transmission problem (5.7) with $g_j=0$ and $\widetilde{A}$ replaced by $\widetilde{A}-\lambda_0$ admits a unique solution $(\tilde{u}_1,u_2)$ in the space $\mathbb{E}=W_p^1(J;X)\cap L_p(J;D(\widetilde{A}_{\widetilde{\mathcal{B}}}))$ which enjoys an estimate of the form

$$\begin{aligned}(5.20)\quad &\|\partial_t\tilde{u}_1\|_{L_p(J;L_p(\mathbb{R}^n_-))}+\|\tilde{u}_1\|_{L_p(J;W_p^{2m}(\mathbb{R}^n_-))}+\|\partial_t u_2\|_{L_p(J;L_p(\mathbb{R}^n_+))}+\|u_2\|_{L_p(J;W_p^{2m}(\mathbb{R}^n_+))}\\ &\leq C\left(\|\tilde{f}_1\|_{L_p(J;L_p(\mathbb{R}^n_-))}+\|\tilde{u}_0^{(1)}\|_{W_p^{2m-2m/p}(\mathbb{R}^n_-)}+\|f_2\|_{L_p(J;L_p(\mathbb{R}^n_+))}+\|u_0^{(2)}\|_{W_p^{2m-2m/p}(\mathbb{R}^n_+)}\right).\end{aligned}$$

Proof. By the discussion on page 108, it suffices to prove the properties for the corresponding system in $\mathbb{R}^n_+$, i.e., maximal regularity for the $L_p(\mathbb{R}^n_+)$-realization of $(A,\mathcal{B})$, defined at the beginning of subsection 5.2.1. The solution $w_0\in W_p^{2m}(\mathbb{R}^n_+;\mathbb{C}^2)$ for the constant coefficient problem in $\mathbb{R}^n$ (see (5.14)) is uniquely determined and the part w, given in (5.19), which stems from the boundary terms of w_0 belongs to $W_p^{2m}(\mathbb{R}^n_+;\mathbb{C}^2)$, too, as shown in Lemmas 5.21 and 5.22. Hence, $u:=w_0+w\in W_p^{2m}(\mathbb{R}^n_+;\mathbb{C}^2)$ and direct calculations show that u in fact solves (5.13) with $g_j=0$. Consequently, the resolvent is given by

$$\begin{aligned}(5.21)\quad (\lambda-A_{\mathcal{B}})^{-1}f&=r_+(\lambda-A_{\mathbb{R}^n})^{-1}E_0f\\ &\quad+\mathbb{T}^{(1)}(\lambda)T_{\Psi_1}r_+\mathcal{B}(\lambda-A_{\mathbb{R}^n})^{-1}E_0f+\mathbb{T}^{(2)}(\lambda)T_{\Psi_2}r_+\partial_n\mathcal{B}(\lambda-A_{\mathbb{R}^n})^{-1}E_0f.\end{aligned}$$

Lemma 5.21 and Lemma 5.22 show that the two latter sets of operators are $\mathcal{R}$-bounded and the proof of the $\mathcal{R}$-boundedness of the set $\{\lambda r_+(\lambda-A_{\mathbb{R}^n})^{-1}E_0:\lambda\in\overline{\Sigma}_\varphi\}$ is the same as given for Lemma 5.22, see also [DHP03], Corollary 5.6. This yields the $\mathcal{R}$-boundedness of the set $\{\lambda(\lambda-A_{\mathcal{B}})^{-1},\ \lambda\in\overline{\Sigma}_\varphi,\ |\lambda|\geq\lambda_0\}$, whence $A_{\mathcal{B}}-\lambda_0$ is $\mathcal{R}$-sectorial with angle φ and by Theorem 5.18 all other assertions follow at once. □

The preceding theorem covers an important special case which is of particular interest if the highest order coefficients of a parabolic operator in the half space have discontinuities across the surface $\Gamma_n:=\{x\in\mathbb{R}^n:\ x_n=0\}$.

Corollary 5.25
Let Assumption 5.19 (i), (ii), and (v) be satisfied with $g_j=0$. Assume that $\widetilde{\mathcal{B}}$ represents the canonical transmission conditions, i.e., $\widetilde{B}_j(\mathcal{D})=(\partial_n^{j-1},-\partial_n^{j-1})$ for $j=1,\dots,2m$. Then all assertions of Theorem 5.24 hold true. In addition, the function

$$u(t,x):=\begin{cases}\tilde{u}_1(t,x), & x\in\mathbb{R}^n_-,\\ u_2(t,x), & x\in\mathbb{R}^n_+,\end{cases}$$

is an element of $W_p^{1,2m}(J\times\mathbb{R}^n)$.

We now turn our attention to inhomogeneous transmission conditions and consider (5.7) with $u_{i,0}=0$ and $f_i=0$ for $i=1,2$.

Theorem 5.26
Let the transmission problem (5.7) be parabolic in the sense of Assumption 5.19 with $u_{i,0}=0$ and $f_i=0$. Then for all $\lambda_0>0$, the parabolic transmission model problem with inhomogeneous transmission conditions

$$(5.22)\quad\begin{cases}\partial_t\tilde{u}_1-(\widetilde{A}_1(\mathcal{D})-\lambda_0)\tilde{u}_1 &=& 0 & \text{in } J\times\mathbb{R}^n_-,\\ \partial_t u_2-(A_2(\mathcal{D})-\lambda_0)u_2 &=& 0 & \text{in } J\times\mathbb{R}^n_+,\\ \gamma_{0,n}\left(\widetilde{B}_j^{(1)}(\mathcal{D})\tilde{u}_1-B_j^{(2)}(\mathcal{D})u_2\right) &=& g_j & \text{in } J\times\mathbb{R}^{n-1},\ (j=1,\dots,2m),\\ \tilde{u}_1(0,x)=0,\quad \text{in } \mathbb{R}^n_-,\quad u_2(0,x) &=& 0 & \text{in } \mathbb{R}^n_+\end{cases}$$

admits a unique solution $u=(\tilde{u}_1,u_2)\in\mathbb{E}_1\times\mathbb{E}_2:=W_p^{1,2m}(\mathbb{R}_+\times\mathbb{R}^n_-)\times W_p^{1,2m}(\mathbb{R}_+\times\mathbb{R}^n_+)$, which satisfies the estimate

$$(5.23)\qquad \|\tilde{u}_1\|_{\mathbb{E}_1}+\|u_2\|_{\mathbb{E}_2}\leq C\sum_{j=1}^{2m}\left(\|g_j\|_{W_p^{\kappa_j/2m}(\mathbb{R}_+;L_p(\mathbb{R}^{n-1}))}+\|g_j\|_{L_p(\mathbb{R}_+;W_p^{\kappa_j}(\mathbb{R}^{n-1}))}\right).$$

Proof. Again, we prove the claim for the corresponding system in the half space. The compatibility conditions 5.19 (v) for $u_0 = 0$ read $g_j(0,x') = 0$ if $\frac{\kappa_j}{2m} = \frac{2m-m_j-1/p}{2m} > \frac{1}{p}$. Thus, the trivial extension by zero $E_{0,t}g_j$ of g_j with respect to t yields an element of the space $W_p^{\kappa_j/2m}(\mathbb{R};L_p(\mathbb{R}^{n-1})) \cap L_p(\mathbb{R};W_p^{\kappa_j}(\mathbb{R}^{n-1}))$. We apply the time-dependent extension operator as well as the time dependent order reductions from [DHP07], Section 3, which in include the operator $L := \partial_t + \lambda_0 + (-\Delta_{n-1})^m$ with domain $D(L) := W_p^1(\mathbb{R};L_p(\mathbb{R}^{n-1})) \cap L_p(\mathbb{R};W_p^{2m}(\mathbb{R}^{n-1}))$, to obtain

$$h_j(t,x',x_n) := L^{\frac{2m-m_j}{2m}} e^{-x_n L^{1/2m}} E_{0,t}g_j \in L_p(\mathbb{R};L_p(\mathbb{R}^n_+)).$$

Fourier transform with respect to t applied to the system which corresponds to (5.22) leads to the parameter-dependent problem for $v_i(\eta,x) := (\mathcal{F}_t u_i(\cdot,x))(\eta)$:

$$(5.24)\quad \begin{cases} (i\eta - (A_1(\mathcal{D}) - \lambda_0))v_1(\eta,x) &= 0 & \text{in } \mathbb{R}\times\mathbb{R}^n_+, \\ (i\eta - (A_2(\mathcal{D}) - \lambda_0))v_2(\eta,x) &= 0 & \text{in } \mathbb{R}\times\mathbb{R}^n_+, \\ \gamma_{0,n}\left(B_j^{(1)}(\mathcal{D})v_1 - B_j^{(2)}(\mathcal{D})v_2\right) &= (\mathcal{F}_t E_{0,t}g_j)(\eta,x') & \text{in } \mathbb{R}\times\mathbb{R}^{n-1},\ (j=1,\dots,2m). \end{cases}$$

By the considerations in subsection 5.2.1, we can write the unique solution to this system in the form

$$\begin{aligned} v &= \mathbb{T}^{(1)}(\lambda)T_{\Psi_1}E_\lambda \mathcal{F}_t E_{0,t}g + \mathbb{T}^{(2)}(\lambda)T_{\Psi_2}\partial_{x_n}E_\lambda\mathcal{F}_tE_{0,t}g = \left(\mathbb{T}^{(1)}(\lambda)+\mathbb{T}^{(2)}(\lambda)\right)T_{\Psi_1}E_\lambda\mathcal{F}_tE_{0,t}g \\ &= \left(\mathbb{T}^{(1)}(\lambda)+\mathbb{T}^{(2)}(\lambda)\right)\mathcal{F}_t h =: \mathbb{T}(i\eta)\mathcal{F}_t h \end{aligned}$$

where we have set $\lambda = i\eta$ and $(\mathcal{F}_tE_{0,t}g) := \mathcal{F}_tE_{0,t}(g_1,\dots,g_{2m})^T$, as well as $h := (h_1,\dots,h_{2m})^T$. We further employed the parameter-dependent extension operator

$$E_\lambda : W_p^{\kappa_j}(\mathbb{R}^{n-1}) \to W_p^{2m-m_j}(\mathbb{R}^n_+), \quad g \mapsto e^{-x_n L_\lambda^{1/2m}} g, \quad \text{with} \quad L_\lambda := \lambda + (-\Delta_{n-1})^m$$

and used the property $\partial_n E_\lambda g = -L_\lambda^{1/2m} E_\lambda \partial_n g$ (see [DHP07], Remark 3.6 and Lemma 4.3). The solution to (5.22) is then given by $u = r_{t,+}\mathcal{F}_t^{-1}\mathbb{T}(i\eta)\mathcal{F}_t h$, where $r_{t,+}$ denotes the restriction from $\mathbb{R}$ to the time interval $\mathbb{R}_+$. Now, Lemma 5.21 as well as Lemma 5.23 imply that the sets

$$\left\{\lambda^{1-\frac{|\alpha|}{2m}}\mathcal{D}^\alpha\mathbb{T}(\lambda) : \lambda\in i\mathbb{R}\setminus\{0\},\ |\alpha|\le 2m\right\} \text{ and } \left\{\lambda^{2-\frac{|\alpha|}{2m}}\partial_\lambda\mathcal{D}^\alpha\mathbb{T}(\lambda) : \lambda\in i\mathbb{R}\setminus\{0\},\ |\alpha|\le 2m\right\}$$

are $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n_+;\mathbb{C}^{2m}), L_p(\mathbb{R}^n_+;\mathbb{C}^2))$. Theorem 5.10 (ii) with $n=1$ implies that

$$\{\mathcal{F}_t^{-1}\eta^{1-\frac{|\alpha|}{2m}}\mathcal{D}^\alpha\mathbb{T}(i\eta)\mathcal{F}_t : \eta\in\mathbb{R}\setminus\{0\},\ |\alpha|\le 2m\} \subset L(L_p(\mathbb{R};L_p(\mathbb{R}^n_+;\mathbb{C}^{2m})), L_p(\mathbb{R};L_p(\mathbb{R}^n_+;\mathbb{C}^2)))$$

is $\mathcal{R}$-bounded again, since all spaces enjoy property (α) here. Using the boundedness of these operators for $\alpha = 0$ and $|\alpha| = 2m$, it is not hard to see that

$$\|\partial_t u\|_{L_p(\mathbb{R}_+;L_p(\mathbb{R}^n_+;\mathbb{C}^2))} + \|u\|_{L_p(\mathbb{R}_+;W_p^{2m}(\mathbb{R}^n_+;\mathbb{C}^2))} \le C\|h\|_{L_p(\mathbb{R};L_p(\mathbb{R}^n_+;\mathbb{C}^{2m}))},$$

and by the construction of h we obtain

$$\|h\|_{L_p(\mathbb{R};L_p(\mathbb{R}^n_+;\mathbb{C}^{2m}))} \le C\sum_{j=1}^{2m}\left(\|g_j\|_{W_p^{\kappa_j/2m}(\mathbb{R}_+;L_p(\mathbb{R}^{n-1}))} + \|g_j\|_{L_p(\mathbb{R}_+;W_p^{\kappa_j}(\mathbb{R}^{n-1}))}\right),$$

see also [DHP07]. □

Remark 5.27

(i) The last two theorems for model transmission problems are the analogues of Lemma 4.2 and Proposition 4.5 in [DHP07], see also [DHP03], Corollary 7.5, where even the Banach space valued case and different integrability with respect to time and space are treated. However, in case $m_1 \neq m_2 \neq \ldots \neq m_{2m}$ (which holds in particular for the canonical transmission conditions) our boundary operator $\mathcal{B}$ for the system in $\mathbb{R}^n_+$ does not fit into the setting of the cited papers. In [DHP07] m boundary operators B_j with E-valued coefficients which coincide with their principal parts in case of a model problem are treated and our system cannot be rewritten in this particular form.

(ii) Apart from this difference, the perturbation theory for $\mathcal{R}$-bounded operators as well as the localization procedure which are in order to make the above results accessible for operators with variable coefficients, including lower order terms in a domain with sufficiently smooth, compact boundary, will exactly follow the lines of [DHP03], Sections 7.3-8.2 and [DHP07], Proposition 4.7. Useful perturbation results in this context have been established in the article [DDHPV].

(iii) The aforementioned perturbation results can be deduced for a sufficiently large modulus of the shift λ_0 which already occurred in the above Theorems and they are a parabolic version of the parameter-dependent case, stated in [ADF97], Section 7. If we restrict to a finite time interval $(0,T)$ with $T < \infty$, the shift becomes unnecessary by replacing $f(t,x)$ by $e^{-\lambda_0 t} f(t,x)$, $g_j(t,x')$ by $e^{-\lambda_0 t} g_j(t,x')$, and $u(t,x)$ by $e^{\lambda_0 t} u(t,x)$.

With the techniques mentioned above, the following Theorem for bounded domains with a C^{2m}-boundary which are divided by a closed C^{2m}-surface Γ can be formulated.

Theorem 5.28
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^{2m}-boundary $\partial\Omega$. Let Γ be a closed C^{2m}-surface inside Ω with $\Gamma \cap \partial\Omega = \emptyset$ such that Ω is divided into the subdomains Ω_1 and Ω_2 with $\partial\Omega_2 = \Gamma$. Let parabolic differential operators A_1 and A_2 in Ω_1 and Ω_2, as well as $2m$ transmission conditions on Γ be given such that Assumptions 5.19 are satisfied. For the exterior boundary, assume that m boundary operators $C_j(x,\mathcal{D}) = \sum_{|\gamma|=r_j} c_{j\gamma}(x)\mathcal{D}^\gamma$ are defined on $\partial\Omega$ so that the boundary value problem $(A_1, C_1, \ldots, C_m)$ satisfies the Shapiro-Lopatinskii condition at every point of Γ. Then there exists $\lambda_0 > 0$ such that for all $f_i \in L_p(J; L_p(\Omega_i))$, $g_j \in W_p^{\kappa_l/2m,\kappa_l}(J \times \Gamma)$ $(l = 1, \ldots, 2m)$, $h_j \in W_p^{\kappa_j/2m,\kappa_l}(J \times \partial\Omega)$ $(j = 1, \ldots, m)$ and $u_{i,0} \in W_p^{2m-2m/p}(\Omega_i)$, which are subject to the compatibility conditions

(1) $\gamma_{\partial\Omega} C_j(x,\mathcal{D}) u_{1,0}(x) = h_j(0,x) \quad$ *for* $\quad j = 1, \ldots, m,$

(2) $\gamma_\Gamma \left(B_l^{(1)}(x,\mathcal{D}) u_{1,0} - B_l^{(2)}(x,\mathcal{D}) u_{2,0} \right) = g_l(0,x) \quad$ *for* $\quad l = 1, \ldots, 2m,$

the parabolic transmission boundary value problem

$$
\left\{
\begin{array}{rcll}
(\partial_t - (A_1(x,\mathcal{D}) - \lambda_0))u_1 &=& f_1 & \text{in } J \times \Omega_1, \\
(\partial_t - (A_2(x,\mathcal{D}) - \lambda_0))u_2 &=& f_2 & \text{in } J \times \Omega_2, \\
\gamma_\Gamma \left(B_l^{(1)}(x,\mathcal{D}) u_1 - B_l^{(2)}(x,\mathcal{D}) u_2 \right) &=& g_l & \text{on } J \times \Gamma, \ (l = 1, \ldots, 2m), \\
\gamma_{\partial\Omega} C_j(x,\mathcal{D}) u_1 &=& h_j & \text{on } J \times \partial\Omega, \ (j = 1, \ldots, m), \\
u_1(0,x) = u_{1,0}(x) \quad \text{in } \Omega_1, \quad u_2(0,x) &=& u_{2,0}(x) & \text{in } \Omega_2,
\end{array}
\right. \tag{5.25}
$$

admits a unique solution $(u_1, u_2) \in W_p^{1,2m}(J \times \Omega_1) \times W_p^{1,2m}(J \times \Omega_2)$.

The solution satisfies the estimate

$$
\begin{aligned}
(5.26)\quad & \|\partial_t u_1\|_{L_p(J;L_p(\Omega_1))} + \|u_1\|_{L_p(J;W_p^{2m}(\Omega_1))} + \|\partial_t u_2\|_{L_p(J;L_p(\Omega_2))} + \|u_2\|_{L_p(J;W_p^{2m}(\Omega_2))} \\
& \le C\Big(\|f_1\|_{L_p(J;L_p(\Omega_1))} + \|f_2\|_{L_p(J;L_p(\Omega_2))} + \|u_{1,0}\|_{W_p^{2m-2m/p}(\Omega_1)} + \|u_{2,0}\|_{W_p^{2m-2m/p}(\Omega_2)} \\
& \qquad + \sum_{l=1}^{2m} \|g_l\|_{W_p^{\kappa_l/2m,\kappa_l}(J\times\Gamma)} + \sum_{j=1}^{m} \|h_j\|_{W_p^{\kappa_j/2m,\kappa_j}(J\times\partial\Omega)} \Big).
\end{aligned}
$$

Remark 5.29
The given proofs carry over to parabolic systems with transmission conditions $(\widetilde{A},\widetilde{\mathcal{B}})$ in $\mathbb{R}^n_\pm$ of order $2m$ and dimension N which are coupled by $2mN$ transmission conditions. The reflected system $(A,\mathcal{B})$ of dimension $2N$ in the half space then admits a block-structure and the Shapiro-Lopatinskii condition has to be satisfied for the ODE-system in $\mathbb{R}_+$ with $2mN$ boundary conditions. In this situation, it is no longer guaranteed that the canonical transmission conditions

$$
\gamma_{0,n}\left[\partial_n^{l-1}\tilde{u}_{1,j} - \partial_n^{l-1}u_{2,j}\right] = 0 \quad (l=1,\dots,2m;\ j=1,\dots,N)
$$

satisfy this condition automatically, see [DFM02], Example A.4. The simple argumentation with the roots of the polynomials $\tilde{a}_{1-}$ and a_{2+}, which we presented in the proof of Theorem 1.9 to verify the Shapiro-Lopatinskii condition can no longer be applied, since parabolicity is now characterized by the determinant of $\lambda - A_0(x_0,\xi)$. This property of absolute ellipticity of the canonical transmission conditions remains valid if the matrix $A_0(x_0,\xi)$ is positive definite or admits a triangular structure (see [DFM02], Propositions A.1 and A.3).

5.3 Transmission boundary value problems for a cuboid

In order to solve parabolic boundary value problems in a cube which is divided by an interface, where transmission conditions are imposed, there remains to consider the local problem, where the interface intersects with the boundary.

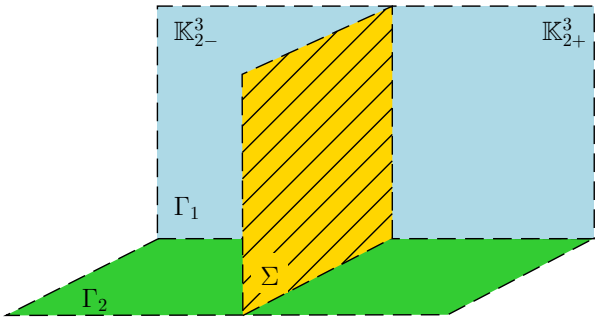

Figure 5.1: Rectangular intersection of a surface Σ with an edge formed by Γ_1 and Γ_2.

The corresponding parameter-dependent problem has already been studied in Section 2.2 under the additional assumption that the principal parts of the operators are even and by restriction to boundary conditions of even or odd type on the lateral surfaces. Here, we preserve the notation which was introduced at the beginning of Chapter 2 for the k-corner and the surfaces Γ_j which form the boundary. We consider the local resolvent problem with constant coefficients an without lower order terms:

$$
(5.27)\quad \begin{cases}
(\lambda - \widetilde{A}_1(\mathcal{D}))\tilde{u}_1 &= \tilde{f}_1 & \text{in } \mathbb{K}^n_{k-}, \\
(\lambda - A_2(\mathcal{D}))u_2 &= f_2 & \text{in } \mathbb{K}^n_{k+}, \\
\gamma_{0,j}\mathcal{B}_j(\mathcal{D})u &= 0 & \text{on } \Gamma_j,\ (j=n-k+1,\dots,n-1), \\
\gamma_{0,n}\left(\widetilde{C}_l^{(1)}(x,\mathcal{D})\tilde{u}_1 - C_l^{(2)}(x,\mathcal{D})u_2\right) &= g_l & \text{on } \Gamma_n,\ (l=1,\dots,2m).
\end{cases}
$$

Here, the boundary condition $\gamma_{0,j}\mathcal{B}_j(\mathcal{D})u = 0$ has to be understood as $\gamma_{0,j}B_l^{(j)}(\mathcal{D}_j)\tilde{u}_1 = 0$ on $\Gamma_{j+} := \{x \in \Gamma_j : \ x_n > 0\}$ and $\gamma_{0,j}B_l^{(j)}(\mathcal{D}_j)u_2 = 0$ on $\Gamma_{j-} := \{x \in \Gamma_j : \ x_n < 0\}$ for $l = 1, \dots, 2m$, respectively, where $\mathcal{B}_j$ is a set of boundary conditions of even or odd type in the sense of Definition 2.2. We still suppose Assumption 5.19 to be satisfied, but in addition, we now assume that the operators $\tilde{A}_1$ and A_2 have even principal parts in the sense of Definition 2.1. We assume that $g_l \in W_p^{2m-m_l-1/p}(\Gamma_n)$ is subject to the compatibility conditions

$$\gamma_{0,j}B_k^{(j)}(\mathcal{D}_j)g_l = 0 \quad \text{on } \Gamma_n \cap \Gamma_j \text{ if } \ 2m - m_l - \operatorname{ord}(B_k^{(j)}) - 2/p > 0 \tag{5.28}$$

for all $j = n-k+1, \dots, n-1$, $k = 1, \dots, m$, and $l = 1, \dots, 2m$. We will again make use of the symbolic notation $\gamma_{0,n}\widetilde{\mathcal{C}}u = 0$ for the transmission conditions. Let $\boldsymbol{\nu} \in \{0,1\}^{k-1}$ be the vector, which describes the type of the boundary conditions $\mathcal{B}_j$ for $j = n-k+1, \dots, n-1$ (cf. Notation 2.3) and $E_{\boldsymbol{\nu}}$ denote the corresponding extension operator (cf. (2.5) on page 51). The L_p-realization of the boundary value problem is then defined by

$$\begin{aligned} D(\widetilde{A}_{\mathcal{B}}) &:= \Big\{u = (\tilde{u}_1, u_2) \in W_p^{2m}(\mathbb{K}_{k-}^n) \times W_p^{2m}(\mathbb{K}_{k+}^n) : \gamma_{0,j}\mathcal{B}_j u = 0 \ \text{ on } \Gamma_j, \ \ \gamma_{0,n}\widetilde{\mathcal{C}}u = 0 \ \text{ on } \Gamma_n\Big\}, \\ \widetilde{A}_{\mathcal{B}}u &:= \Big(\widetilde{A}_1(\mathcal{D})\tilde{u}_1, A_2(\mathcal{D})u_2\Big)^T \quad (u \in D(\widetilde{A}_{\mathcal{B}})). \end{aligned}$$

We first state the main result for a model problem with homogeneous transmission conditions.

Theorem 5.30
Let even differential operators $\widetilde{A}_1$ in $\mathbb{K}_{k-}^n$ and A_2 in $\mathbb{K}_{k+}^n$ as well as transmission conditions $\widetilde{\mathcal{C}}$ with constant coefficients be given such that Assumption 5.19 is satisfied. Assume that the transmission conditions do not act on the variables $x_{n-k+1}, \dots, x_{n-1}$, i.e., $\widetilde{\mathcal{C}}(\mathcal{D}) = \widetilde{\mathcal{C}}(\mathcal{D}_1, \dots, \mathcal{D}_{n-k}, \mathcal{D}_n)$. In addition, on each surface Γ_j $(j = n-k+1, \dots, n-1)$ let a set of m boundary operators $\mathcal{B}_j(\mathcal{D}_j) = \Big(B_1^{(j)}(\mathcal{D}_j), \dots, B_m^{(j)}(\mathcal{D}_j)\Big)$ of even or odd type be given. Then for $\lambda_0 > 0$ we have $\{\lambda \in \overline{\Sigma}_\varphi : |\lambda| > \lambda_0\} \subset \rho(\widetilde{A}_{\mathcal{B}})$ and

$$\Big\{\lambda(\lambda - \widetilde{A}_{\mathcal{B}})^{-1} : \lambda \in \overline{\Sigma}_\varphi, \ |\lambda| > \lambda_0\Big\} \subset L(L_p(\mathbb{K}_{k-}^n) \times L_p(\mathbb{K}_{k+}^n))$$

is $\mathcal{R}$-bounded.

Proof. We consider the corresponding system in $\mathbb{K}_{k+}^n$ and apply the extension operator $E_{\boldsymbol{\nu}}$, so that (5.27) is extended to the following transmission problem in the half spaces

$$\left\{\begin{array}{rcll} (\lambda - A_1(\mathcal{D}))v_1 &=& E_{\boldsymbol{\nu}}f_1 & \text{in } \mathbb{R}_+^n, \\ (\lambda - A_2(\mathcal{D}))v_2 &=& E_{\boldsymbol{\nu}}f_2 & \text{in } \mathbb{R}_+^n, \\ \gamma_{0,n}\Big(C_l^{(1)}(\mathcal{D})v_1 - C_l^{(2)}(\mathcal{D})v_2\Big) &=& 0 & \text{on } \mathbb{R}^{n-1}, \ (l = 1, \dots, 2m). \end{array}\right. \tag{5.29}$$

Theorem 5.24 yields a unique solution to this problem which is given by $v = (\lambda - A_{\mathcal{C}})^{-1}(E_{\boldsymbol{\nu}}f)$, where the explicit form of $(\lambda - A_{\mathcal{C}})^{-1}$ is stated in (5.21). With the same arguments as in the proof of Theorem 2.12, we see that $u := r_{\mathbb{K}_{k+}^n}v$ is a solution to the system in $\mathbb{K}_{k+}^n$. If $u = (u_1, u_2)$ and $v = (v_1, v_2) \in W_p^{2m}(\mathbb{K}_{k+}^n; \mathbb{C}^2)$ are solutions to the system in $\mathbb{K}_{k+}^n$ which corresponds to (5.27), then it is not hard to see that their extensions $U := E_{\boldsymbol{\nu}}u$ and $V := E_{\boldsymbol{\nu}}v \in W_p^{2m}(\mathbb{R}_+^n; \mathbb{C}^2)$ both solve (5.29). Note that the W_p^{2m}-regularity is preserved, because u and v satisfy boundary conditions of even type in case of an odd extension in direction x_j and of odd type in case of an even extension in direction x_j. Uniqueness for the system in $\mathbb{R}_+^n$ yields $U = V$, hence implies uniqueness for the system in $\mathbb{K}_{k+}^n$. There remains to prove the claimed $\mathcal{R}$-boundedness and this follows from the continuity of the mappings $r_{\mathbb{K}_{k+}^n} \in L\left(W_p^{2m}(\mathbb{R}_+^n), W_p^{2m}(\mathbb{K}_{k+}^n)\right)$ and $E_{\boldsymbol{\nu}} \in L\left(L_p(\mathbb{K}_{k+}^n), L_p(\mathbb{R}_+^n)\right)$ and the $\mathcal{R}$-boundedness of the set $\{\lambda(\lambda - A_{\mathcal{C}})^{-1} : \lambda \in \overline{\Sigma}_\varphi, \ |\lambda| > \lambda_0\}$ which is contained in Theorem 5.24. □

We do not emphasize on the consequences for the corresponding parabolic equation, but we state the following analogue of Theorem 5.26 for inhomogeneous transmission conditions in the parabolic version of (5.27) which is given by

$$
(5.30)\qquad \begin{cases} (\partial_t - \widetilde{A}_1(\mathcal{D}))\tilde{u}_1 &= \tilde{f}_1 & \text{in } J\times\mathbb{K}^n_{k-},\\ (\partial_t - A_2(\mathcal{D}))u_2 &= f_2 & \text{in } J\times\mathbb{K}^n_{k+},\\ \gamma_{0,j}\mathcal{B}_j(\mathcal{D}_j)u &= 0 & \text{on } J\times\Gamma_j,\ (j=n-k+1,\dots,n-1),\\ \gamma_{0,n}\left(\widetilde{C}^{(1)}_l(\mathcal{D})\tilde{u}_1 - C^{(2)}_l(\mathcal{D})u_2\right) &= g_l & \text{on } J\times\Gamma_n,\ (l=1,\dots,2m),\\ \tilde{u}_1(0)=\tilde{u}_{1,0}\quad \text{in } \mathbb{K}^n_{k-}, \quad u_2(0) &= u_{2,0} & \text{in } \mathbb{K}^n_{k+}. \end{cases}
$$

Theorem 5.31
Under the assumptions of Theorem 5.30, let values

$$g_l \in W_p^{\kappa_l/2m,\kappa_l}(J\times\Gamma_n) := W_p^{\kappa_l/2m}(J;L_p(\Gamma_n))\cap L_p(J;W_p^{\kappa_l}(\Gamma_n))$$

for the transmission conditions be given and assume $\tilde{f}_1 = f_2 = 0$. Suppose that g_l satisfies the compatibility conditions with the boundary operators $\mathcal{B}_j$ given by (5.28) *for $t\in J$, as well as with the initial conditions $\tilde{u}_{1,0} = u_{2,0} := 0$, i.e., we have $g_j(0,x') = 0$ on Γ_n if $\kappa_l/2m > 1/p$. Then* (5.30) *admits a unique solution $\tilde{u} = (\tilde{u}_1, u_2) \in W_p^{1,2m}(J\times\mathbb{K}^n_{k-})\times W_p^{1,2m}(J\times\mathbb{K}^n_{k+})$ and the estimate* (5.23) *(with $\mathbb{R}^n_\pm$ replaced by $\mathbb{K}^n_{k\pm}$ and $\mathbb{R}^{n-1}$ replaced by Γ_n) is satisfied.*

Proof. The compatibility conditions (5.28) with the boundary operators allow an extension of $g_l \in W_p^{\kappa_l/2m,\kappa_l}(J\times\Gamma_n)$ by $E_{\boldsymbol{\nu}}$ to $E_{\boldsymbol{\nu}}g_l \in W_p^{\kappa_l/2m,\kappa_l}(J\times\mathbb{R}^{n-1})$ (cf. the proof of Corollary 2.20) and after the reflection in the first equation we may consider the inhomogeneous system in the half spaces

$$
(5.31)\qquad \begin{cases} (\partial_t - A_1(\mathcal{D}))v_1 &= 0 & \text{in } J\times\mathbb{R}^n_+,\\ (\partial_t - A_2(\mathcal{D}))v_2 &= 0 & \text{in } J\times\mathbb{R}^n_+,\\ \gamma_{0,n}\left(C^{(1)}_l(\mathcal{D})v_1 - C^{(2)}_l(\mathcal{D})v_2\right) &= E_{\boldsymbol{\nu}}g_l & \text{on } J\times\mathbb{R}^{n-1},\ (l=1,\dots,2m),\\ v_1(0)=v_2(0) &= 0 & \text{in } \mathbb{R}^n_+. \end{cases}
$$

Theorem 5.26 yields a unique solution $v=(v_1,v_2)\in W_p^{1,2m}(J\times\mathbb{R}^n_+)^2$ and as in the proof of Theorem 2.12 by uniqueness, symmetry, and the extension procedure, it is not hard to prove that $u := r_{\mathbb{K}^n_{k+}}v$ satisfies the boundary conditions $\mathcal{B}_j$ on Γ_j for all $t\in J$. The estimate is satisfied for v in $\mathbb{R}^n_+$ and by the continuity of the restriction, it holds for u on $\mathbb{K}^n_{k+}$. If $u=(u_1,u_2)$ and $v=(v_1,v_2)\in W_p^{1,2m}(J\times\mathbb{K}^n_{k+})^2$ are solutions to the reflected version of (5.30) in $\mathbb{K}^n_{k+}$, their extensions $U := E_{\boldsymbol{\nu}}u$ and $V := E_{\boldsymbol{\nu}}v$ belong to $W_p^{1,2m}(J\times\mathbb{R}^n_+)^2$, because they satisfy the appropriate type of homogeneous boundary conditions in direction x_j. U and V both solve (5.31), because the transmission conditions are independent of the variables $x_{n-k+1},\dots,x_{n-1}$, and hence they coincide, which yields uniqueness. □

Remark 5.32
Note that the proofs of the preceding theorems directly carry over to a boundary value problem in $\mathbb{K}^n_{k+}$, where $k-1$ boundary conditions of even or odd type on Γ_j and one set of boundary conditions $\mathcal{B}_n = \left(B^{(n)}_1,\dots,B^{(n)}_m\right)$ on Γ_n of general type (under the restriction, that $\mathcal{B}_n$ only acts on $x_1,\dots,x_{n-k}$, and x_n) such that $(A,\mathcal{B}_n)$ satisfies the Shapiro-Lopatinskii condition at each point of Γ_n are imposed. See also Corollary 2.20 for the parameter-dependent case in the space $W_p^s(\mathbb{R}^n_+)$.

Following the localization procedure, we performed in Section 2.4, but using perturbation results for $\mathcal{R}$-boundedness (see e.g. [KW04], Theorems 6.4 and 6.5) instead of Theorem 2.18 and Corollary 2.20, we can eventually state the parabolic analogue of Corollary 2.26.

Theorem 5.33
Let $1 < p < \infty$ and suppose that the following assumptions are satisfied:

(i) ***Geometry**: Let $0 < L_k < \infty$ and $\Omega := \prod_{k=1}^{n}(0, L_k)$ be a paraxial rectangle. For fixed $0 < \sigma_1 < \sigma_2 < L_n$, let $\Sigma_i := \{x \in \overline{\Omega} : x_n = \sigma_i\}$ be two interfaces which divide Ω into three subdomains Ω_1, Ω_2, and Ω_3.*

(ii) ***Differential operators**: For $i = 1, 2, 3$, let A_i be a differential operator of order $2m$ with even principal part A_i^0 which is parabolic in $\overline{\Omega}_i$. Assume that the coefficients of A_i satisfy the smoothness assumptions from Assumption 5.19 (i).*

(iii) ***Boundary operators I**: On the lateral surfaces Γ_j $(j = 1, \ldots, n-1, n+1, \ldots, 2n-1)$ let homogeneous boundary conditions of even or odd type be given. These conditions will be denoted by $\mathcal{B}_j(x, \mathcal{D}_j) = \left(B_1^{(j)}(x, \mathcal{D}_j), \ldots, B_m^{(j)}(x, \mathcal{D}_j)\right)$. Regarding the smoothness of the coefficients of the boundary operators, assume that $b_{l\beta}^{(j)} \in C^{2m-ord(B_l^{(j)})}(\Gamma_j)$.*

(iv) ***Boundary operators II**: On each of the parallel surfaces $\Gamma_n := \{x \in \overline{\Omega} : x_n = 0\}$ and $\Gamma_{2n} := \{x \in \overline{\Omega} : x_n = L_n\}$ let a set of m constant coefficient boundary operators $\mathcal{B}_j(\mathcal{D}_n) := \left(B_1^{(j)}(\mathcal{D}_n), \ldots, B_m^{(j)}(\mathcal{D}_n)\right)$ $(j \in \{n, 2n\})$ be given which only act on the x_n-variable such that the boundary value problems $(A_1, \mathcal{B}_n)$ and $(A_3, \mathcal{B}_{2n})$ satisfy the Shapiro-Lopatinskii condition at each point $x_0 \in \Gamma_n$ and $x_0 \in \Gamma_{2n}$, respectively.*

(v) ***Transmission conditions**: On each of the surfaces Σ_i let a set C and D of $2m$ transmission conditions of the form*

$$\begin{aligned} \gamma_{\Sigma_1} C_l^{(1)}(\mathcal{D}_n)u_1 - \gamma_{\Sigma_1} C_l^{(2)}(\mathcal{D}_n)u_2 &= h_l^{(1)} \quad \text{on } \Sigma_1, \quad (l = 1, \ldots, 2m), \\ \gamma_{\Sigma_2} D_l^{(2)}(\mathcal{D}_n)u_2 - \gamma_{\Sigma_2} D_l^{(3)}(\mathcal{D}_n)u_3 &= h_l^{(2)} \quad \text{on } \Sigma_2, \quad (l = 1, \ldots, 2m), \end{aligned}$$

be given such that Assumption 5.19 is satisfied for each of the transmission problems (A_1, A_2, C) on Σ_1 and (A_2, A_3, D) on Σ_2. Here, $h_l^{(i)} \in W_p^{\kappa_{il}/2m, \kappa_{il}}(J \times \Sigma_i)$ $(i = 1, 2)$ with $\kappa_{1l} := 2m - ord(C_l^{(1)}) - 1/p$ and $\kappa_{2l} := 2m - \mathrm{ord}(D_l^{(2)}) - 1/p$.

(vi) ***Data and compatibility conditions**: For $i = 1, 2, 3$, let $f_i \in L_p(J; L_p(\Omega_i))$. Further, let initial values $u_{i,0} \in W_p^{2m-2m/p}(\Omega_i)$ and with $\kappa_{jl} := 2m - \mathrm{ord}(B_l^{(j)}) - 1/p$ let boundary values $g_l^{(j)} \in W_p^{\kappa_{jl}/2m, \kappa_{jl}}(J \times \Gamma_j)$ $(j \in \{n, 2n\},\ l = 1, \ldots, m)$ for the conditions on Γ_n and Γ_{2n} be given such that the following compatibility conditions are satisfied:*

(1) $\gamma_{\Sigma_1} C_l^{(1)}(\mathcal{D}_n)u_{1,0} - \gamma_{\Sigma_1} C_l^{(2)}(\mathcal{D}_n)u_{2,0} = h_l^{(1)}(0, x)$ *for $l = 1, \ldots, 2m$, if $\kappa_{1l}/2m > 1/p$.*

(2) $\gamma_{\Sigma_2} D_l^{(2)}(\mathcal{D}_n)u_{2,0} - \gamma_{\Sigma_1} D_l^{(3)}(\mathcal{D}_n)u_{3,0} = h_l^{(2)}(0, x)$ *for $l = 1, \ldots, 2m$, if $\kappa_{2l}/2m > 1/p$.*

(3) $\gamma_{0,j} B_k^{(j)}(x, \mathcal{D}_j)u_{i,0} = 0$ *on Γ_j for $j = 1, \ldots, n-1, n+1, \ldots, 2n-1$, $k = 1, \ldots, m$, and $i = 1, 2, 3$, if $2m - 2m/p - \mathrm{ord}(B_k^{(j)}) > 1/p$.*

(4) $\gamma_{0,j} B_k^{(j)}(\mathcal{D}_j)u_{i,0} = g_k^{(j)}(0, x)$ *on Γ_j for $j \in \{n, 2n\}$, $k = 1, \ldots, m$, and $i = 1, 2, 3$, if $\kappa_{jk}/2m > 1/p$.*

(5) $\gamma_{0,j} B_k^{(j)}(x, \mathcal{D}_j)h_l^{(i)} = 0$ *on $\Sigma_i \cap \Gamma_j$ for all $t \in J$ and $j = 1, \ldots, n-1, n+1, \ldots, 2n-1$, $k = 1, \ldots, 2m$, $l = 1, \ldots, 2m$, and $i = 1, 2$, if $\kappa_{il} - \mathrm{ord}(B_k^{(j)}) > 1/p$.*

(6) $\gamma_{0,j} B_k^{(j)}(x, \mathcal{D}_j)g_l^{(i)} = 0$ *on $\Gamma_i \cap \Gamma_j$ for all $t \in J$ and $j = 1, \ldots, n-1, n+1, \ldots, 2n-1$, $k = 1, \ldots, m$, $l = 1, \ldots, 2m$, and $i \in \{n, 2n\}$, if $\kappa_{il} - \mathrm{ord}(B_k^{(j)}) > 1/p$.*

Then there exists $\lambda_0 > 0$ such that for $i = 1, 2, 3$, the parabolic transmission boundary value problem

$$
(5.32)\quad \begin{cases}
(\partial_t - (A_i(x,\mathcal{D}) - \lambda_0))u_i &=& f_i & \text{in } J\times\Omega_i,\\
\gamma_{0,j}\mathcal{B}_j(x,\mathcal{D}_j)u_i &=& 0 & \text{on } J\times\Gamma_j,\ (j\in\{1,\dots,2n\},\ j\neq n,2n),\\
\gamma_{0,j}B_k^{(j)}(\mathcal{D}_j)u_i &=& g_k^{(j)} & \text{on } J\times\Gamma_j,\ (j=n,2n;\ k=1,\dots,m),\\
\gamma_{\Sigma_1}\Big(C_l^{(1)}(\mathcal{D}_n)u_1 - C_l^{(2)}(\mathcal{D}_n)u_2\Big) &=& h_l^{(1)} & \text{on } \Sigma_1,\ (l=1,\dots,2m),\\
\gamma_{\Sigma_2}\Big(D_l^{(2)}(\mathcal{D}_n)u_2 - D_l^{(3)}(\mathcal{D}_n)u_3\Big) &=& h_l^{(2)} & \text{on } \Sigma_2,\ (l=1,\dots,2m),\\
u_i(0,\cdot) &=& u_{i,0} & \text{in } \Omega_i,
\end{cases}
$$

admits a unique solution $u_i \in W_p^{1,2m}(J\times\Omega_i)$. The solution satisfies the estimate

$$
\begin{aligned}
\sum_{i=1}^{3}\|\partial_t u_i\|_{L_p(J;L_p(\Omega_i))} + \|u_i\|_{L_p(J;W_p^{2m}(\Omega_i))} \le C\Bigg(&\sum_{i=1}^{3}\|f_i\|_{L_p(J;L_p(\Omega_i))} + \|u_{i,0}\|_{W_p^{2m-2m/p}(\Omega_i)}\\
&+ \sum_{j\in\{n,2n\}}\sum_{k=1}^{m}\Big\|g_k^{(j)}\Big\|_{W_p^{\kappa_{jk}/2m,\kappa_{jk}}(J\times\Gamma_j)} + \sum_{j=1}^{2}\sum_{l=1}^{2m}\Big\|h_l^{(j)}\Big\|_{W_p^{\kappa_{jk}/2m,\kappa_{jk}}(J\times\Sigma_j)}\Bigg).
\end{aligned}
$$

Chapter 6

Local existence for a nonlinear elliptic-parabolic system

After the detailed treatment of the coupled elliptic system in Chapters 2 and 3, we introduce a third, parabolic equation which is coupled to the elliptic system via the nonlinearity on the right-hand sides of the equations. Consequently, all unknown functions, as well as the data of the equations will become time dependent: $u_i = u_i(t,x)$, $(t,x) \in (0,T) \times \Omega$ $(i = 1,2,3)$ and initial data have to be imposed.

6.1 Statement of the problem and assumptions

Notation 6.1
For $1 \le p \le \infty$ and $r > 0$ with $rp > n$, we denote by $W^r_{p,+}(\Omega)$ the cone of positive W^r_p-functions:

$$W^r_{p,+}(\Omega) := \{u \in W^r_p(\Omega) \ : \ u(x) > 0 \quad (x \in \Omega)\}.$$

Note that the pointwise condition is meaningful, because of $rp > n$. We will use similar notations for the spaces $C^r_+(\Omega)$ and $W^{1,2}_{p,+}(J \times \Omega)$.

Assumption 6.2
Let $1 \le n \le 4$, $0 < s < \tau < 1$, and $\frac{1}{\tau} + n + 3 < p < \infty$ as well as $s < \frac{1}{p}$. Fix $T \in (0,\infty)$ and abbreviate $J := (0,T)$.

(i) ***Geometry****: Given $L_l \in (0,\infty)$ $(l = 1,\dots,n)$ and $0 < \sigma_1 < \sigma_2 < L_n$, let Ω denote the paraxial cuboid $\Omega := \prod_{l=1}^n (0,L_l)$. The $2n$ surfaces, whose union is the boundary $\partial\Omega$, will be denoted by $\Gamma_k := \{x \in \overline{\Omega} : \ x_k = 0\}$ for $k = 1,\dots,n$ and $\Gamma_k := \{x \in \overline{\Omega} : \ x_{k-n} = L_{k-n}\}$ for $k = n+1,\dots,2n$, respectively. In addition, let Ω be divided by the two parallel surfaces $\Sigma_i := \{x \in \overline{\Omega} : x_n = \sigma_i\}$ $(i = 1,2)$ into three subdomains Ω_j $(j = 1,2,3)$.*

(ii) ***Coefficients****: Let $\kappa_1 : [0,\infty) \to [0,\infty)$ with $\kappa_1(0) = 0$ and $\kappa_1(c) > 0$ for $c > 0$. Regarding the smoothness of κ_1, we assume the following property: for every $c \in C^{1,\tau}_+(\overline{\Omega})$ there holds $\kappa_1(c) \in C^{1,\tau}(\overline{\Omega}_j)$ for $j = 1,2,3$. Likewise, we assume that $\kappa_2 \in C^{1,\tau}(\overline{\Omega}_j)$ for $j = 1,2,3$, with $\kappa_2 \ge k_2^2 > 0$. In addition, let a diffusion coefficient $\kappa_3 \in C(\overline{\Omega}_j)$ with $\kappa_3 \ge k_3^2 > 0$ and $\nabla\kappa_3 \in L_\infty(\Omega_j)$ be given. In each subdomain Ω_j $(j = 1,2,3)$, we consider the following differential operators in divergence form:*

$$\begin{aligned}
A_{1j}(x,\mathcal{D})u_1 &:= \nabla \cdot (\kappa_1(u_3)\nabla u_1) = \kappa_1(u_3)\Delta u_1 + \kappa_1'(u_3)\nabla u_3 \cdot \nabla u_1,\\
A_{2j}(x,\mathcal{D})u_2 &:= \nabla \cdot (\kappa_2 \nabla u_2) = \kappa_2 \Delta u_2 + \nabla\kappa_2 \nabla u_2,\\
A_{3j}(x,\mathcal{D})u_3 &:= \nabla \cdot (\kappa_3 \nabla u_3) = \kappa_3 \Delta u_3 + \nabla\kappa_3 \nabla u_3.
\end{aligned}$$

Note that A_{ij} has even principal part and is parameter-elliptic in every sector $\overline{\Sigma}_\varphi$ with $\varphi < \pi$, since

$$\lambda - A^0_{ij}(x_0,\xi) = \lambda + \kappa_i(x_0)|\xi|^2 \ne 0 \quad ((\xi,\lambda) \in \mathbb{R}^n \times (\mathbb{C} \setminus (-\infty,0)) \setminus \{(0,0)\},\ x_0 \in \overline{\Omega}_j).$$

For the parameter-ellipticity of A_{1j} we assumed $u_3 \in C^1_+(\overline{\Omega})$ here.

(iii) ***Nonlinearity****: Let $F : \mathbb{R} \times (0,\infty) \times \Omega \to \mathbb{R}$ such that for every $j \in \{1,2,3\}$ the function $F_j := F_{\mathbb{R}\times(0,\infty)\times\overline{\Omega}_j}$ is continuous with respect to all variables and continuously differentiable with respect to the first variable. Furthermore, there is a constant $C_j > 0$ such that*

$$(\partial_1 F_j)(v,w,x) \geq C_j > 0, \quad (v,w,x) \in \mathbb{R} \times (0,\infty) \times \overline{\Omega}_j.$$

Additionally, we assume that the mapping $x \mapsto F(u(x),v(x),x)$ is an element of $W_p^s(\Omega)$ for all $u \in W_p^2(\Omega)$ and $v \in W_{p,+}^{2-3/p}(\Omega)$ and the nonlinear operator

$$T_F : W_p^2(\Omega) \times W_{p,+}^{2-3/p}(\Omega) \to W_p^s(\Omega), \quad (u,v) \mapsto F(u(\cdot),v(\cdot),\cdot) \tag{6.1}$$

is bounded. Note that by the embeddings $W_p^2(\Omega_j), W_p^{2-3/p}(\Omega_j) \hookrightarrow C(\overline{\Omega}_j)$ and the piecewise continuity of F, T_F maps bounded sets of $W_p^2(\Omega_j) \times W_{p,+}^{2-3/p}(\Omega_j)$ into bounded sets of $C(\overline{\Omega}_j)$ for $j = 1,2,3$.

(iv) ***Boundary conditions****: The system for (u_1,u_2,u_3) is completed by a homogeneous boundary condition on each surface Γ_k for $k \in \{1,\dots,n-1,n+1,\dots,2n-1\}$:*

$$\mathcal{B}_i(\mathcal{D}) := \left(B_i^{(1)}(\mathcal{D}_1),\dots,B_i^{(n-1)}(\mathcal{D}_{n-1}),B_i^{(n+1)}(\mathcal{D}_{n+1}),\dots,B_i^{(2n-1)}(\mathcal{D}_{2n-1})\right) \quad (i = 1,2,3)$$

with $\partial_{n+k} := -\partial_k$ $(k = 1,\dots,n)$ of even or odd type, i.e.

$$\gamma_{0,k} B_i^{(k)}(\mathcal{D}_k) u_i = \gamma_{0,k} u_i \quad \text{or} \quad \gamma_{0,k} B_i^{(k)}(\mathcal{D}_k) u_i = \gamma_{0,k} \kappa_i \partial_k u_i.$$

Note that these types of boundary conditions satisfy the Shapiro-Lopatinskii condition with the operators A_{ij}, cf. Assumption 4.1 and [Wlo87], Examples 11.1 and 11.2.
Inhomogeneous boundary conditions $\mathcal{B}_i^{(k)}(\mathcal{D}_k)$ of general form can be imposed on the parallel surfaces Γ_n and Γ_{2n} with the constraint that the boundary value problems $(A_{i1},\mathcal{B}_i^{(n)})$ and $(A_{i3},\mathcal{B}_i^{(2n)})$ then satisfy the Shapiro-Lopatinskii condition on Γ_k for $i = 1,2,3$.

(v) ***Transmission Conditions****: On the surfaces Σ_j $(j = 1,2)$, we impose homogeneous canonical transmission conditions of order 2 for all unknown functions:*

$$[\![u_i]\!] = [\![\partial_n u_i]\!] = 0 \quad \text{on } \overline{\Sigma}_j \quad (i = 1,2,3,\ j = 1,2).$$

(vi) ***Data****: For $i = 1,2$, let $f_i \in C(\overline{J}; W_p^s(\Omega) \cap L_\infty(\Omega))$ and assume $f_3 \in L_p(J; L_p(\Omega))$. For possibly inhomogeneous boundary data of u_1 and u_2 on the surfaces Γ_n and Γ_{2n}, we assume the regularity*

$$g_{m_{ik},i}^{(k)} \in C(\overline{J}; W_p^{\kappa_{ik}}(\Gamma_k)),\ (k \in \{n,2n\})$$

with $\kappa_{ik} := 2 + s - m_{ik} - 1/p$ for $i = 1,2$ as well as the compatibility conditions

$$\gamma_{0,j} g_{m_{ik},i}^{(k)} = \gamma_{0,j} \partial_j g_{m_{ik},i}^{(k)} = 0 \quad \text{on } \Gamma_j \cap \Gamma_k \quad (j = 1,\dots,n-1,n+1,\dots,2n-1,\ k \in \{n,2n\})$$

which have to be satisfied for all $t \in J$. Since F is only defined for positive values of r, we assume to be given a positive initial condition $u_{3,0} \in W_{p,+}^{2-2/p}(\Omega)$ with

$$u_3(0,x) = u_{3,0}(x) \geq C_0 > 0 \quad (x \in \Omega).$$

Possibly inhomogeneous boundary values $g_{m_{3k},3}^{(k)} \in W_p^{\frac{\kappa_{3k}}{2},\kappa_{3k}}(J \times \Gamma_k)$ $(k \in \{n,2n\})$ for u_3, where $\kappa_{3k} := 2 - m_{3k} - 1/p$, are assumed to satisfy the compatibility conditions with $u_{3,0}$ and the boundary operators on the adjacent surfaces, which have been stated in Theorem 5.33 (vi).

Preserving the symbolic Notation 4.2 for the boundary conditions, we study the following nonlinear elliptic-parabolic system

$$
(6.2)\quad \begin{cases}
-\nabla\cdot(\kappa_1(u_3)\nabla u_1) &= F(u_2-u_1,u_3,x)+f_1 & \text{in } J\times\Omega,\\
-\nabla\cdot(\kappa_2\nabla u_2) &= -F(u_2-u_1,u_3,x)+f_2 & \text{in } J\times\Omega,\\
\partial_t u_3\ -\nabla\cdot(\kappa_3\nabla u_3) &= F(u_2-u_1,u_3,x)+f_3 & \text{in } J\times\Omega,\\
\gamma_\Omega\mathcal{B}_i u_i &= g_i & \text{on } J\times\partial\Omega,\ (i=1,2,3),\\
[\![u_i]\!]=[\![\partial_n u_i]\!] &= 0 & \text{on } J\times\Sigma_j,\ (i=1,2,3;\ j=1,2),\\
u_3(0,x) &= u_{3,0}(x) & \text{in } \Omega.
\end{cases}
$$

Remark 6.3

(i) The initial conditions for u_1 and u_2 are obtained by solving the elliptic system for $t=0$ with $u_3=u_{3,0}$. This is possible due to Theorem 4.14.

(ii) For u_1 and u_2, we are restricted to Dirichlet or Neumann boundary conditions on all of $\partial\Omega$, because we are going to apply Theorem 4.14. Regarding u_3, we imposed more general boundary conditions which satisfy the Shapiro-Lopatinskii condition on Γ_n and Γ_{2n}, respectively, which makes Theorem 5.33 applicable.

(iii) Regarding the transmission conditions for all three unknown functions, we could also impose conditions of general form under the assumptions of Theorem 5.31. However, in the present chapter, we ask for preferably high regularity of the solution and hence, the homogeneous canonical conditions are advisable.

(iv) The Assumption 6.2 (iii) on F seems somewhat technical, but the application in Chapter 7 will clarify for what kind of nonlinearities this assumption is satisfied.

6.2 Function spaces and continuity results

Here, we introduce certain subsets of the Sobolev and Hölder spaces which will be frequently used when we consider the above equations.

Notation 6.4
Let $\mathfrak{F}$ denote one of the spaces $W_p^s(\Omega)$ or $C^{k,\alpha}(\overline{\Omega})$ for $s>0$, $p\in(1,\infty)$, $k\in\mathbb{N}_0$, and $\alpha\in(0,1)$. For $K>0$ we denote by an additional subscript K the set

$$\mathfrak{F}_K := \left\{ f\in\mathfrak{F}:\ \|f\|_{\mathfrak{F}}\le K,\ f(x)\ge\frac{1}{K}\right\}.$$

The latter condition has to be understood in the L_p-sense in case $sp\le n$ and in the classical sense for $sp>n$ and for the Hölder spaces, respectively. Analogous notations will also be used for spaces of time dependent functions as well as for intersections of these spaces, always equipped with their standard norms.

In order to apply our results from Chapter 4 for the elliptic system, we are going to fix u_3 in an appropriate function space and then fix the time $t\in[0,T)$, to determine $u_i(t,\cdot)$ $(i=1,2)$. We will seek a solution u_3 of the parabolic equation in the set $\mathbb{E}_K$, where $K>0$ will be determined later and $\mathbb{E}$ denotes the space $\mathbb{E}:=W_p^{1,2}(J\times\Omega):=W_p^1(J;L_p(\Omega))\cap L_p(J;W_p^2(\Omega))$. By [Ama00], Theorem 5.2, for $\theta\in(0,1)$ and $0\le\alpha<1-\theta-\frac{1}{p}$, we have the compact embedding

$$
(6.3)\quad \mathbb{E}\overset{c}{\hookrightarrow} C^\alpha(\overline{J};W_p^{2\theta}(\Omega)),
$$

and the latter space is continuously embedded into $C^\alpha(\overline{J};C^{1,\beta}(\overline{\Omega}))$ for $0<\beta\le 2\theta-1-\frac{n}{p}$. Furthermore, from [Ama95], Theorem III.4.10.2, it is known that $\mathbb{E}\hookrightarrow BUC(J;W_p^{2-2/p}(\Omega))$.

Remark 6.5
It is in order to discuss a proper choice of the parameters $\alpha, \theta \in (0,1)$ as well as $p \in (1,\infty)$ for the embedding (6.3). For our purposes, the Hölder index α may become very small, while we wish $\theta \in (0,1)$ to be close to 1 in order to preserve the high regularity. The condition $0 \le \alpha < 1-\theta-\frac{1}{p}$ implies $\theta < 1-\frac{1}{p}-\alpha$, hence $r := 2\theta - 1 < 1 - \frac{2}{p} - 2\alpha$. For the sake of simplicity, we choose $0 < \alpha < \frac{1}{2p}$ in the sequel and set $\theta := 1 - \frac{3}{2p}$, i.e., $r := 1 - \frac{3}{p}$. This explains our assumption on the mapping properties of the nonlinearity (see (6.1)).
For the embedding $W_p^{2\theta}(\Omega) \hookrightarrow C^{1,\beta}(\overline{\Omega})$ we can then choose $\beta = r - \frac{n}{p} = 1 - \frac{n+3}{p} > \frac{1}{tp}$ because of the choice $p > n + 3 + \frac{1}{t}$. This will be important for the coefficient $\kappa_1(u_3)$ in the first equation.

For the time being, let $u_3 \in \mathbb{E}_{K_0}$ and $t \in [0,T)$ be fixed. We use the compact embedding $\mathbb{E} \overset{c}{\hookrightarrow} C^\alpha(\overline{J}; W_p^{1+r}(\Omega))$ with the parameters from Remark 6.5. Then for all $t \in J$, there holds $u_3(t,\cdot) \in W_p^{1+r}(\Omega)$ with $\|u_3(t,\cdot)\|_{1+r,p,\Omega} \le C\|u_3\|_{\mathbb{E}} \le CK_0 =: K_1$. Hence, $u_3(t,\cdot) \in W_{p,K_1}^{1+r}(\Omega)$ and the operator from (6.1) induces a bounded mapping

$$G : W_p^2(\Omega) \to W_p^s(\Omega), \quad u \mapsto F(u(\cdot), u_3(t,\cdot), \cdot).$$

Note that $G : \mathbb{R} \times \overline{\Omega} \to \mathbb{R},\ (w,x) \mapsto F(w, u_3(t,x), x)$ satisfies Assumption 4.1 (iii).

For the coefficient, we use Sobolev's embedding to see $u_3(t,\cdot) \in C^{1,\beta}(\overline{\Omega})$ with $\beta \le r - \frac{n}{p}$ and hence, for $j = 1,2,3$, the coefficient $\kappa_{1,3} := \kappa_1 \circ u_3(t,\cdot)$ is an element of $C^{1,\tau}(\overline{\Omega}_j)$ with $\tau := t\beta > 1/p > s$ (see Remark 6.5). Since $u_3 \in \mathbb{E}_{K_0}$ we have $\kappa_{1,3}(x) \ge k_{1,3} > 0$ for $x \in \Omega$ and the latter estimate is independent of t. Similarly, we have $\|u_3(t,\cdot)\|_{C^{1+\tau}(\overline{\Omega})} \le C_2(K_0)$ for all $t \in \overline{J}$, which yields $\|\kappa_{1,3}\|_{C^{1,\tau}(\overline{\Omega})} \le K_{1,3}(K_0)$ independent of t. By our assumptions, for all $t \in [0,T]$, we can fix data $f_i(t,\cdot) \in W_p^s(\Omega) \cap L_\infty(\Omega)$, $g_{m_{ik},i}^{(k)}(t,\cdot) \in W_p^{2+s-m_{ik}-1/p}(\Gamma_k)$ for $k \in \{n, 2n\}$ and $i = 1,2$, which admit bounds in the norms of the indicated spaces that are independent of t as well. So we are back in the situation of Theorem 4.14 and can consider the elliptic system

$$(6.4) \qquad \begin{cases} -\nabla\cdot(\kappa_{1,3}\nabla u_1) &= G(u_2-u_1,x) + f_1(t,\cdot) & \text{in } \Omega, \\ -\nabla\cdot(\kappa_2\nabla u_2) &= -G(u_2-u_1,x) + f_2(t,\cdot) & \text{in } \Omega, \\ \gamma_\Omega \mathcal{B}_i u_i &= g_i(t,\cdot) & \text{on } \partial\Omega,\ (i=1,2), \\ [\![u_i]\!] = [\![\partial_n u_i]\!] &= 0 & \text{on } J\times\Sigma_j,\ (i,j=1,2). \end{cases}$$

For almost every $t \in [0,T)$, we obtain a unique solution $u_1(t,\cdot), u_2(t,\cdot) \in W_p^{2+s}(\Omega)$ which satisfies the estimate

$$(6.5) \quad \|u_1(t,\cdot)\|_{2+s,p,\Omega} + \|u_2(t,\cdot)\|_{2+s,p,\Omega}$$
$$\le M\Big(\|\kappa_1(u_3(t,\cdot))\|_{C^{1+\tau}(\overline{\Omega})}, \|\kappa_2\|_{C^{1+t}(\overline{\Omega})}, \sum_{\substack{i=1,2\\ j\in\{n,2n\}}} \|f_i(t,\cdot)\|_{W_p^s(\Omega)\cap L_\infty(\Omega)} + \|g_{m_{ij},i}^{(j)}(t,\cdot)\|_{\kappa_{ij},p,\Gamma_j} \Big)$$
$$\le M\Big(K_{1,3}, \|\kappa_2\|_{C^{1+t}(\overline{\Omega})}, \sum_{\substack{i=1,2\\ j\in\{n,2n\}}} \|f_i\|_{L_\infty(J;W_p^s(\Omega)\cap L_\infty(\Omega))} + \|g_{m_{ij},i}^{(j)}\|_{L_\infty(J;W_p^{\kappa_{ij}}(\Gamma_j))} \Big),$$

which is again independent of $t \in J$, hence $(u_1,u_2) \in L_\infty(J; W_p^{2+s}(\Omega)^2)$. This procedure gives rise to a solution operator.

Lemma 6.6 (Time-independent solution operator)
For every $K < \infty$ and $0 \le \tilde{s} < s < \frac{1}{p}$, the operator

$$L : W_{p,K}^{1+r}(\Omega) \to W_p^{2+\tilde{s}}(\Omega)^2, \quad u_3 \mapsto (u_1,u_2),$$

where (u_1,u_2) is the unique solution to the system (6.4), is well-defined and continuous. In addition, L maps bounded sets of $W_{p,+}^{1+r}(\Omega)$ into bounded sets of $W_p^{2+\tilde{s}}(\Omega)^2$, i.e., there is a constant $C = C\left(K, \|f_i\|_{W_p^s(\Omega)\cap L_\infty(\Omega)}, \|g_{m_{ij},i}^{(j)}\|_{2-m_{ij}-1/p,p,\Gamma_j}, \|\kappa_2\|_{C^{1+t}(\overline{\Omega})}, \|\kappa_1\|_{C^{1+t}([0,\infty))}, n, p\right)$ such that

$$\|Lv\|_{2+\tilde{s},p,\Omega} \le C \quad (v \in W_{p,K}^{1+r}(\Omega)).$$

Proof. The above discussion shows that the definition of L is meaningful and the boundedness is contained in (6.5). Let $v_k \to v$ in $W^{1+r}_{p,K}(\Omega)$. Denote by $u_k := (u_{1,k}, u_{2,k}) := Lv_k$ and $u := (u_1, u_2) := Lv$ the corresponding solutions to (6.4). By inequality (6.5), $u_n \subset W^{2+s}_p(\Omega)^2$ is bounded and we may assume that $u_k \to U = (U_1, U_2)$ in $(W^{2+\tilde{s}}_p(\Omega))^2$. We will prove that $u = U$ in $W^1_2(\Omega) \cap L_\infty(\Omega)$. The piecewise continuity of κ_1 and F (see Assumption 6.2 (ii) and (iii)) implies $\kappa_k := \kappa_1 \circ v_k \to \kappa_1 \circ v$ in $L_\infty(\Omega)$ as well as $F(u_{2,k} - u_{1,k}, v_k, \cdot) \to F(U_2 - U_1, v, \cdot)$ in $L_\infty(\Omega)$. Following the proceeding of Remarks 3.26 and 4.5, it is not hard to see that $u_k \in W^1_p(\Omega) \cap L_\infty(\Omega)$ is the unique solution to the corresponding weak problem: In case of an inhomogeneous Dirichlet condition for u_i on Γ_j for $j \in \{n, 2n\}$, first choose $w_i \in W^{2+s}_p(\Omega)$ such that $\gamma_{0,j} w_i = g^{(j)}_{0,i}$ on Γ_j and $\gamma_{0,k} B^{(k)}_i(\mathcal{D}_k) w_i = 0$ on Γ_k for $k \neq j$. The existence of an element $w_i \in W^{2+s}_p(\Omega)$ with these properties is ensured by Theorem A.3 and the compatibility conditions for the Dirichlet data. Then $\tilde{u}_{i,k} := u_{i,k} - w_i \in H^1_{\Gamma_{D,i}}(\Omega)$ solves a problem of the following structure:

$$\begin{aligned}
\int_\Omega \kappa_k \nabla \tilde{u}_{1,k} \nabla \varphi \, dx &= \int_\Omega (F(u_{2,k} - u_{1,k}, v_k, x) + f_1)\, \varphi \, dx - \int_\Omega \kappa_k \nabla w_1 \nabla \varphi \, dx + \int_{\Gamma_{N,1}} h_1 \varphi \, dx, \\
\int_\Omega \kappa_2 \nabla \tilde{u}_{2,k} \nabla \psi \, dx &= \int_\Omega (-F(u_{2,k} - u_{1,k}, v_k, x) + f_2)\, \psi \, dx - \int_\Omega \kappa_2 \nabla w_2 \nabla \psi \, dx + \int_{\Gamma_{N,2}} h_2 \psi \, dx,
\end{aligned}$$

for all $\varphi \in H^1_{\Gamma_{D,1}}(\Omega)$ and $\psi \in H^1_{\Gamma_{D,2}}(\Omega)$. The last term in each equation here only contains a possible remaining inhomogeneous Neumann boundary value on Γ_n or Γ_{2n}. For $k \to \infty$, we find

$$\begin{aligned}
&\left| \int_\Omega (\kappa_k \nabla \tilde{u}_{1,k} - (\kappa_1 \circ v) \nabla (U_1 - w_1)) \nabla \varphi \, dx \right| \\
&\leq \|\kappa_k - (\kappa_1 \circ v)\|_{0,\infty,\Omega} \|\nabla (U_1 - w_1) \nabla \varphi\|_{0,1,\Omega} + \|\kappa_k\|_{0,\infty,\Omega} \|\nabla (\tilde{u}_{1,k} - (U_1 - w_1)) \nabla \varphi\|_{0,1,\Omega} \to 0.
\end{aligned}$$

Analogous convergence properties can be seen for the remaining expressions and this shows that $\tilde{U}_1 := U_1 - w_1$ satisfies

$$\int_\Omega (\kappa_1 \circ v) \nabla \tilde{U}_1 \nabla \varphi \, dx = \int_\Omega (F(U_2 - U_1, v, x) + f_1)\, \varphi \, dx - \int_\Omega (\kappa_1 \circ v) \nabla w_1 \nabla \varphi \, dx + \int_{\Gamma_{N,1}} h_1 \varphi \, dx$$

and a similar equation is satisfied by $\tilde{U}_2 := U_2 - w_2$. By the definition of L and analogous considerations, these equations are also solved by $\binom{u_1 - w_1}{u_2 - w_2} = Lv - \binom{w_1}{w_2}$ and Lemma 3.24 shows $u = U$ in $(W^1_2(\Omega) \cap L_\infty(\Omega))^2$. □

Observe that the mapping $v \mapsto \kappa_1(v)$ is in general not continuous from $C^{1,\tau}(\overline{\Omega}_j)$ to $C^{1,\tau}(\overline{\Omega}_j)$, but our proof only requires uniqueness for the weak solution and continuity in $L_\infty(\Omega)$. Similarly, we only used the piecewise continuity of F and the convergence of u_k and v_k in $C(\overline{\Omega})$, which is guaranteed by embedding. In order to include the time dependence of the data, we make the following

Definition 6.7 (Time-dependent solution operator)

(i) Define the trivial extension S of L to $\mathbb{E}_{K_0} = W^{1,2}_{p,K_0}(J \times \Omega)$ by

$$S : \mathbb{E}_{K_0} \to C(\overline{J}; W^{2+\tilde{s}}_p(\Omega))^2, \quad u \mapsto (S(u))(t, \cdot) := L(u(t, \cdot)) \quad (t \in \overline{J}). \tag{6.6}$$

(ii) We further abbreviate the following set of Hölder continuous functions with values in the Sobolev space $W^{1+r}_p(\Omega)$:

$$\mathbb{E}_{\alpha,1+r,C_1} := \left\{ u \in C^\alpha(\overline{J}; W^{1+r}_p(\Omega)) : \ \|u(t, \cdot)\|_{1+r,p,\Omega} \leq C_1 \ (t \in \overline{J}), \ u(t, \cdot) \geq \tfrac{1}{C_1} \ \text{in } \Omega \right\},$$

with the parameters α and r from Remark 6.5 and where C_1 denotes the constant of the (compact) embedding $\mathbb{E} \hookrightarrow C^\alpha(\overline{J}; W^{1+r}_p(\Omega))$.

Lemma 6.8
For $K_0 < \infty$, the operator $S : \mathbb{E}_{K_0} \to C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2$ is well-defined and compact. In particular, S is continuous and maps $\mathbb{E}_{K_0}$ into a bounded subset of $C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2$, i.e., there exists a constant $K_1 = K_1(K_0) > 0$ such that

$$\|Sv\|_{C(\overline{J};W_p^{2+\tilde{s}}(\Omega))} \leq K_1 \quad (v \in \mathbb{E}_{K_0}).$$

Proof. First, $S : \mathbb{E}_{\alpha,1+r,C_1} \to C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2$ is well-defined because the mapping $t \mapsto u(t,\cdot)$ is continuous from $[0,T]$ into $W^{1+r}_{p,C_1}(\Omega)$ and $L : W^{1+r}_{p,C_1}(\Omega) \to W_p^{2+\tilde{s}}(\Omega)^2$ is continuous by Lemma 6.6 and bounded as well. Thus, the composition $t \mapsto (S(u))(t,\cdot)$ is continuous from $\overline{J}$ to $W_p^{2+\tilde{s}}(\Omega)^2$, i.e., $S(u) \in (C(\overline{J}; W_p^{2+\tilde{s}}(\Omega)^2))_{C_1}$ for all $u \in \mathbb{E}_{\alpha,1+r,C_1}$ and there exists a constant K_1 (see Lemma 6.6 for the dependencies of K_1) such that

$$\|S(u)\|_{C(\overline{J};W_p^{2+\tilde{s}}(\Omega)^2)} \leq K_1, \quad (u \in \mathbb{E}_{\alpha,1+r,C_1}).$$

Assuming $S : \mathbb{E}_{\alpha,1+r,C_1} \to C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2$ is discontinuous, let $v_k \to v$ in $C^\alpha(\overline{J}; W_p^{1+r}(\Omega))$ and assume $\|S(v_k) - S(v)\|_{C(\overline{J};W_p^{2+\tilde{s}}(\Omega))} \geq \varepsilon$ for all $k \in \mathbb{N}$. Then there exists a sequence $(t_k)_{k\in\mathbb{N}} \subset \overline{J}$ such that $\|(S(v_k) - S(v))(t_k,\cdot)\|_{W_p^{2+\tilde{s}}(\Omega)} \geq \varepsilon$ for $k \in \mathbb{N}$. W.l.o.g. let $t_k \to t_0 \in \overline{J}$. Then we have

$$\begin{aligned}\|(S(v_k) - S(v))(t_k,\cdot)\|_{2+\tilde{s},p,\Omega} \\ \leq \|(S(v_k))(t_k,\cdot) - (S(v))(t_0,\cdot)\|_{2+\tilde{s},p,\Omega} + \|(S(v))(t_0,\cdot) - (S(v))(t_k,\cdot)\|_{2+\tilde{s},p,\Omega},\end{aligned}$$

and the second norm is smaller than $\frac{\varepsilon}{2}$ for $k \geq k_1$, because $S(v) \in C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2$. The continuity of L (cf. Lemma 6.6) yields the existence of a positive constant $\delta = \delta(\frac{\varepsilon}{2}, v)$ such that $\|L(v_k(t_k,\cdot)) - L(v(t_0,\cdot))\|_{2+\tilde{s},p,\Omega} < \frac{\varepsilon}{2}$, if $\|(v_k(t_k,\cdot)) - v(t_0,\cdot)\|_{1+r,p,\Omega} < \delta$. We now prove that the latter condition is satisfied for sufficiently large $k \in \mathbb{N}$, to obtain a contradiction to our assumption: First, the convergence in $C^\alpha(\overline{J}; W_p^{1+r}(\Omega))$ yields that there exists $k_2 \in \mathbb{N}$ such that for all $k \geq k_2$ there holds $\|v_k(t_0,\cdot) - v(t_0,\cdot)\|_{1+r,p,\Omega} < \frac{\delta}{2}$. Furthermore, the fact that $v_k \in C^\alpha(\overline{J}; W_p^{1+r}(\Omega))$ is bounded yields the existence of $k_3 \in \mathbb{N}$ with

$$\|v_k(t_k,\cdot) - v_k(t_0,\cdot)\|_{1+r,p,\Omega} \leq C_1|t_k - t_0|^\alpha < \frac{\delta}{2} \quad (k \geq k_3).$$

Hence, for $k \geq \max(k_2, k_3)$ we obtain

$$\|v_k(t_k,\cdot) - v(t_0,\cdot)\|_{1+r,p,\Omega} \leq \|(v_k(t_0,\cdot)) - v(t_0,\cdot)\|_{1+r,p,\Omega} + \|(v_k(t_k,\cdot)) - v_k(t_0,\cdot)\|_{1+r,p,\Omega} < \delta.$$

This proves the continuity of $S : \mathbb{E}_{\alpha,1+r,C_1} \to C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2$ and continuity as well as compactness for $S : \mathbb{E}_{K_0} \to C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2$ follow from the compactness of the embedding $\mathbb{E} \overset{c}{\hookrightarrow} C^\alpha(\overline{J}; W_p^{1+r}(\Omega))$. □

The properties of S lead to a helpful estimate of the nonlinearity which will allow us to prove local existence.

Corollary 6.9
Consider the nonlinear mapping

$$\begin{aligned} N : \mathbb{E}_{K_0} \to \quad & C(\overline{J}; W_p^2(\Omega)) \times \mathbb{E}_{\alpha,1+r,C_2} && \to L_p(J; L_p(\Omega)) \\ u_3 \mapsto \quad & (u_2 - u_1, u_3) && \mapsto F(u_2 - u_1, u_3, \cdot). \end{aligned}$$

Then $N : \mathbb{E}_{K_0} \to L_p(J; L_p(\Omega))$ is compact. In addition, there exists a constant $K_2 = K_2(K_0, p)$ such that

$$\|N(u_3)\|_{L_p(J;L_p(\Omega))} \leq K_2\, T^{1/p} \quad (u_3 \in \mathbb{E}_{K_0}). \tag{6.7}$$

Proof. Lemma 6.8 shows that the first step $u_3 \mapsto (S(u_3), u_3) = (u_1, u_2, u_3)$ is compact from $\mathbb{E}_{K_0}$ into $C(\overline{J}; W_p^{2+\tilde{s}}(\Omega))^2 \times \mathbb{E}_{\alpha,1+r,C_1}$ and obviously, $(u_1, u_2, u_3) \mapsto (u_2 - u_1, u_3)$ is continuous from the latter space into $C(\overline{J}; W_p^{2+\tilde{s}}(\Omega)) \times \mathbb{E}_{\alpha,1+r,C_1}$.

Since $(u_2 - u_1, u_3)(t,\cdot) \in W_p^2(\Omega) \times W_{p,C_1}^{1+r}(\Omega) \hookrightarrow C(\overline{\Omega}) \times C_{C_2}(\overline{\Omega})$ for all $t \in \overline{J}$, the mapping $(u_2-u_1, u_3)(t,\cdot) \mapsto F((u_2-u_1)(t,\cdot), u_3(t,\cdot),\cdot) \in L_\infty(\Omega)$ is well-defined. In addition, the mapping $t \mapsto F((u_2 - u_1)(t,\cdot), u_3(t,\cdot),\cdot)$ is continuous from $[0,T]$ into $L_\infty(\Omega)$, as the composition of the continuous function $t \mapsto (u_2-u_1, u_3)(t,\cdot)$ (from $\overline{J}$ to $C(\overline{\Omega})\times C_{C_2}(\overline{\Omega})$) and the piecewise continuous mapping F. This shows $N(v) \in C(\overline{J}; L_\infty(\Omega))$ for all $v \in \mathbb{E}_{K_0}$.

For the boundedness note that $\|u_2 - u_1\|_{C(\overline{J};W_p^{2+\tilde{s}}(\Omega))} + \|u_3\|_{C^\alpha(\overline{J};W_p^{1+r}(\Omega))} \le 2K_1(K_0) + C_1(K_0)$ by Lemma 6.8 and by embedding $\|(u_2 - u_1)(t,\cdot)\|_{C(\overline{\Omega})} + \|u_3(t,\cdot)\|_{C(\overline{\Omega})} \le 2K_1(K_0) + C_1(K_0)$ for all $t \in \overline{J}$ and the piecewise continuity of F yields the boundedness

$$\|F(u_2-u_1, u_3)\|_{C(\overline{J};L_\infty(\Omega))} \le K_2(K_1(K_0), C_1(K_0)) \quad (u_3 \in \mathbb{E}_{K_0}, (u_1,u_2) = S(u_3) \in C(\overline{J}; W_p^2(\Omega))).$$

There remains to prove that $(u_2 - u_1, u_3) \mapsto F(u_2 - u_1, u_3, \cdot)$ maps continuously from the set $C(\overline{J}; W_p^2(\Omega)) \times C^\alpha(\overline{J}; W_{p,C_1}^{1+r}(\Omega))$ into $L_p(J; L_p(\Omega))$. To this end, let $(u_k, v_k) \to (u, v)$ in $C(\overline{J}; W_p^2(\Omega)) \times C^\alpha(\overline{J}; W_{p,C_1}^{1+r}(\Omega))$. This implies the convergence in $C(\overline{J}; C(\overline{\Omega})) \times C_{C_2}(\overline{J}; C(\overline{\Omega}))$ by continuous embedding. The piecewise continuity of F implies that $F(u_k, v_k) \to F(u, v)$ in $L_\infty(J; L_\infty(\Omega))$. The L_p-estimate follows from the boundedness of N:

$$\|N(u_3)\|_{L_p(J;L_p(\Omega))}^p = \|F(u_2 - u_1, u_3, \cdot)\|_{L_p(J;L_p(\Omega))}^p \le \int_0^T \|T_F(u_2 - u_1, u_3)(t,\cdot)\|_{0,\infty,\Omega}^p \, dt \le T\, K_2^p.$$

□

6.3 Fixed point argument

We are now in the position to establish a Schauder fixed point argument to prove the existence of a solution to (6.2) for a small time interval $(0, T_0)$. To this end, we consider the nonlinear compact mapping $N : \mathbb{E}_{K_0} \to L_p(J; L_p(\Omega))$ from Corollary 6.9 which for given $u \in \mathbb{E}_{K_0}$ provides us the data for the linearized parabolic equation

$$\left\{ \begin{array}{rcll} \partial_t v - \nabla\kappa_3\nabla v &=& N(u) + f_3 & \text{in } J\times\Omega, \\ \gamma_\Omega \mathcal{B}_3 v &=& g_3 & \text{on } J\times\partial\Omega, \\ [\![v]\!] = [\![\partial_n v]\!] &=& 0 & \text{on } \Sigma_j,\ (j=1,2), \\ v(0,x) &=& u_{3,0}(x) & \text{in } \Omega. \end{array} \right. \tag{6.8}$$

The space for the fixed data $(f_3, g_3, u_{3,0})$ to the equation will be denoted by

$$\mathbb{F} := L_p(J; L_p(\Omega)) \times W_p^{\frac{\kappa_{n3}}{2},\kappa_{n3}}(J\times\Gamma_n) \times W_p^{\frac{\kappa_{n3}}{2},\kappa_{2n3}}(J\times\Gamma_{2n}) \times W_{p,+}^{2-2/p}(\Omega).$$

Theorem 5.33 shows that there exists a unique solution $v \in \mathbb{E}$ to (6.8) and v satisfies the a priori estimate

$$\begin{aligned} \|v\|_{\mathbb{E}} &\le C\left(\|f_3 + N(u)\|_{L_p(J;L_p(\Omega))} + \|u_{3,0}\|_{W_p^{2-2/p}(\Omega)} + \sum_{j\in\{n,2n\}} \left\| g_3^{(j)} \right\|_{W_p^{\kappa_{j3}/2,\kappa_{j3}}(J\times\Gamma_j)} \right) \\ &\le C_P\left(K_2 T^{1/p} + \|(f_3, g_3, u_{3,0})\|_{\mathbb{F}} \right). \end{aligned} \tag{6.9}$$

Note that the homogeneous transmission conditions for v ensure that $v(t,\cdot) \in W_p^2(\Omega)$ on the whole domain for almost every $t \in J$ and we omitted the shift λ_0 of the operator A_3, since our considerations are restricted to a finite time interval. The continuity of the solution operator to the linear parabolic equation then allows us to define the compact operator

$$Z : \mathbb{E}_{K_0} \to \mathbb{E}, \quad u \mapsto v.$$

There remains to choose T and K_0 such that $Z : \mathbb{E}_{K_0} \to \mathbb{E}_{K_0}$. The choice will also depend on the condition $u_{3,0} \geq C_0 > 0$. First, we use the embedding $\mathbb{E} \hookrightarrow BUC(J; W_p^{2-2/p}(\Omega))$ and choose $T_1 \in (0,T)$ such that for all $t \in [0,T_1)$ there holds

$$u_{3,0}(x) - v(t,x) \leq |u_{3,0}(x) - v(t,x)| \leq \|u_{3,0}(\cdot) - v(t,\cdot)\|_{0,\infty,\Omega} \leq C\|u_{3,0}(\cdot) - v(t,\cdot)\|_{2-2/p,p,\Omega} \leq \tfrac{C_0}{2}.$$

This implies $v(t,x) \geq u_{3,0}(x) - \frac{C_0}{2} \geq \frac{C_0}{2} > 0$ for all $t \in [0,T_1)$ and $x \in \Omega$. Then we set

$$K_0 := \max\left(C_0^{-1}, C_P\left(\|(f_3,g_3,u_{3,0})\|_{\mathbb{F}} + \tfrac{1}{\|(f_3,g_3,u_{3,0})\|_{\mathbb{F}}}\right)\right)$$

with C_P from (6.9) and define the time $T_2 := \left(C_P^{-1}\ K_2 \min\left(\frac{2}{K_0}, \frac{K_0}{2}\right)\right)^p$, where the constant K_2 stems from the estimate (6.7). The estimate (6.9) then implies

$$\|v\|_{\mathbb{E}} \leq C_P\left(\|N(u)\|_{L_p(J;L_p(\Omega))} + \|(f_3,g_3,u_{3,0})\|_{\mathbb{F}}\right) \leq C_P\left(K_2 T_2^{1/p} + \tfrac{K_0}{2C_P}\right) \leq K_0,$$

i.e., for $0 < T \leq \min(T_1,T_2)$ we have $Zu = v \in \mathbb{E}_{K_0}$ for $u \in \mathbb{E}_{K_0}$. By its definition, $\mathbb{E}_{K_0}$ is a nonempty, closed, convex, bounded set in $\mathbb{E}$, whence Schauder's fixed point theorem (Theorem A.8 (ii)) applies to Z and yields the existence of $u_3 \in \mathbb{E}_{K_0}$ which solves (6.8) with $u = v = u_3$. By construction of the operator S, the corresponding triple (u_1,u_2,u_3) with $(u_1,u_2) := S(u_3)$ is a solution to the nonlinear elliptic-parabolic system (6.2). The main result of this chapter reads as follows:

Theorem 6.10
Let Assumption 6.2 be satisfied. Then there exists a finite time $T_0 > 0$ and a constant $K_0 > 0$ such that the elliptic-parabolic system (6.2) *with $J := (0,T_0)$ admits a solution*

$$(u_1,u_2,u_3) \in C(\overline{J}; W_p^{2+\tilde{s}}(\Omega)) \times C(\overline{J}; W_p^{2+\tilde{s}}(\Omega)) \times W_{p,K_0}^{1,2}(J \times \Omega),$$

where $0 \leq \tilde{s} < s < 1/p$. The norm of (u_1,u_2,u_3) in the latter space is estimated against a constant $K := K_1 + K_0$, where the dependence of K_1 is contained in Lemma 6.8 and in the estimate (6.5) *and K_0 depends on $C_0, n, p, \|\kappa_3\|_{C(\overline{\Omega}_i)}, \|\nabla\kappa_3\|_{L_\infty(\Omega_i)}$, and $\|(f_3,g_3,u_{3,0})\|_{\mathbb{F}}$.*

Remark 6.11
The situation can easily be generalized to the situation when the nonlinearity in the parabolic equation differs from the nonlinear function in the two elliptic equations. In fact, we can consider a nonlinear measurable function

$$G : \mathbb{R}^2 \times (0,\infty) \times \Omega \to \mathbb{R}, \quad (x,y,z) \mapsto G(x,y,z)$$

with continuous restrictions to $\mathbb{R}^2 \times (0,\infty) \times \Omega_j$ ($j = 1,2,3$) and the following property: For a finite time $T \in (0,\infty)$, $J := (0,T)$, and $K > 0$ let $\mathbb{X} := C(\overline{J}; W_p^2(\Omega)) \times C(\overline{J}; W_p^2(\Omega)) \times \mathbb{E}_K$. Then the nonlinear mapping

$$T_G : \mathbb{X} \to L_p(J; L_p(\Omega)), \quad (u_1,u_2,u_3) \mapsto G(u_1,u_2,u_3,\cdot)$$

is well-defined, continuous, and maps bounded subsets of $\mathbb{X}$ into bounded subsets of $L_p(J;L_p(\Omega))$ with an estimate of the form

$$\|G(u_1,u_2,u_3,\cdot)\|_{L_p(J;L_p(\Omega))} \leq C_2(C_1,p)T^\gamma \quad (\|(u_1,u_2,u_3)\|_{\mathbb{X}} \leq C_1)$$

for some $\gamma = \gamma(C_1,p) > 0$. Sufficient conditions on G for these mapping properties have been discussed at the beginning of Chapter 4.

Remark 6.12
A similar proof can be established under the assumptions of Chapter 4 for a weak solution

$$(u_1,u_2,u_3) \in C^\gamma(\overline{J}; H^1(\Omega))^2 \times C(\overline{J}; H^1(\Omega)) \cap C^{\beta/2}(\overline{J}; C^\beta(\Omega)),$$

where the fixed point argument is implemented on the set $(C^{\beta/2}(\overline{J}; C^\beta(\Omega)))_K$ for appropriate $K > 0$ and $T \leq T_0$. See [WXZ06], Section 4 for details.

Chapter 7

Application to a lithium-ion battery model

In this chapter, we apply the general results of the previous chapters to a concrete model for a lithium-ion battery which was developed by J. Newman et al. in [DFN94]. In particular, an application-oriented example of a nonlinearity which admits the properties we required in Chapter 6 will be presented. Before we state and prove our analytical results for the model equations, we give a short derivation of the system. Afterwards, a few generalizations and a different model will be discussed.

7.1 Derivation of the model

Here, we rather want to give an idea where the model equations stem from, than a rigorous derivation of the model. Our presentation closely follows the articles [DNT94] and [DFN94] as well as the textbook [New73]. To begin with, let us consider the components of a lithium-ion battery. We distinguish three different major parts: two solid, negative and positive electrodes which consist of active porous material (e.g. graphite (layered structure) or small particles of $FePO_4$ or MnO_2, where lithium is intercalated) and the liquid electrolyte (e.g. an organic solvent with a high concentration of a dissolved lithium salt; a typical example is $LiPF_6$ in Ethylene carbonate) which occupies the separator region between the electrodes and surrounds (or also permeates) their particles.

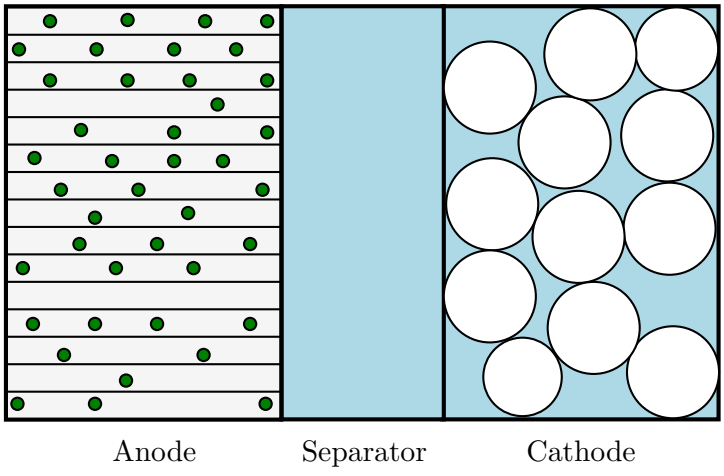

Figure 7.1: Schematic design of a lithium-ion battery (notations for discharge).

Sometimes, metallic lithium is used for the anode (cf. [DFN93]) and the separator is occupied by some conductive polymer (e.g. polyethylene oxide). If an appliance is connected to the battery, neutral lithium in the negative electrode will oxidize to a cation Li^+ which dissolves into the solvent and leaves an electron e^- at the negative electrode which is the anode because of this oxidation process.

$$\mathrm{Li} \quad \longrightarrow \quad \mathrm{Li}^+ + e^- \quad \text{(anodic oxidation)}.$$

This attrition of lithium at the reactive surface of the anode also initiates diffusion processes of lithium-atoms in the solid material. Driven by diffusion and electromigration in the electric field which is formed between the electrodes, the dissolved Li^+-cations move to the positive electrode, the cathode, and at its surface, a reduction process takes place:

$$Li^+ + e^- \quad \longrightarrow \quad Li \quad \text{(cathodic reduction)}.$$

The reduced lithium atoms will also diffuse deeper into the active particles, whose volume increases as a consequence. The discharge process, we sketched above, is illustrated in the following picture.

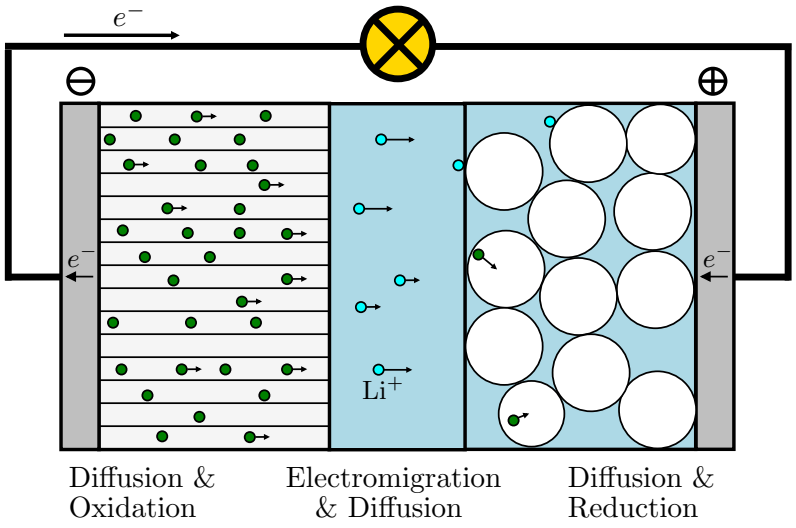

Figure 7.2: Processes during discharge of a lithium-ion battery.

Following Chapter 12 of [SS02], the basic assumptions of the model can be summarized as follows:

- The exact positions and shapes of the particles are not specified.
- Properties are averaged over a test-volume which is large with respect to the porous structure, but which is small with respect to the dimension of the whole battery (thickness parameters suggested in [DFN94]: Anode and Cathode: 200 μm, Separator: 50 μm).
- The electrodes are seen as a superposition of solid active material and liquid electrolyte, and these phases coexist at every point of the electrodes.
- Solid and liquid phase are assumed to be electrically neutral.
- Thermal effects are neglected, i.e., the temperature T is a constant. The volume as well as the geometry of Ω are assumed to be constant, too.

For the sake of simplicity, the geometry of the whole battery Ω, as well as the geometry of the electrodes (Ω_1 and Ω_3) and the separator (Ω_2) will be assumed to be rectangular as indicated in Assumption 6.2 (i). The significant quantities to describe the processes within the battery are

ϕ_1: the potential in the liquid phase (being the whole of Ω by the above assumptions) which is a function $\phi_1 : (0,T) \times \Omega \to \mathbb{R}$.

ϕ_2: the potential in the solid phase ($\Omega' := \Omega_1 \cup \Omega_3$) which is a function $\phi_2 : (0,T) \times \Omega' \to \mathbb{R}$.

c: the concentration of electrolyte which is a function $c : (0,T) \times \Omega \to (0,\infty)$.

In the sequel, we will derive relations between the unknown functions and available data which will eventually lead to a system of partial differential equations.

Potential in the electrolyte

The potential in the electrolyte is usually measured against a reference electrode of metallic lithium. In general, with the species M_i $(i = +, -, 0)$, the half cell reaction can be written as

$$s_- M_-^{z_-} + s_+ M_+^{z_+} + s_0 M_0 \rightleftharpoons n e^-,$$

with the charge numbers $z_\pm \in \mathbb{Z}$, the stoichiometric coefficients $s_i \in \mathbb{Z}$ $(i = +, -, 0)$ and the number of involved electrons $n \in \mathbb{N}_0$. If we assume that the anion $M_- := X^-$ and the solvent M_0 do not participate in this reaction, it simply reads as $-1Li^+ \overset{[Li]}{\rightleftharpoons} 1e^-$ or rewritten: $[Li] \rightleftharpoons Li^+ + e^-$. Note that the metallic lithium of the reference electrode (denoted as $[Li]$) does not explicitly occur in this half cell reaction.
The liquid phase is a highly concentrated solution of a lithium salt in a neutral solvent, whose dissociation reads

$$Li_{\nu_+} X_{\nu_-} \rightleftharpoons \nu_+ Li^+ + \nu_- X^{z_-},$$

with the numbers ν_+ of cations, ν_- of anions and $\nu := \nu_+ + \nu_-$. The electroneutrality condition can then be rewritten as $z_+\nu_+ + z_-\nu_- = 0$. For this situation, the following general relation between the potential, the electrolyte concentration, and the current density vector $\boldsymbol{i}$ was developed:

$$\begin{aligned} \boldsymbol{i} &= -\kappa \nabla \phi_1 - \frac{\kappa}{F}\left(\frac{s_+}{n\nu_+} + \frac{t_+^0}{z_+\nu_+} - \frac{s_0 c}{n c_0}\right)\nabla \mu \\ &= -\kappa \nabla \phi_1 - \frac{\kappa \nu R T}{F}\left(\frac{s_+}{n\nu_+} + \frac{t_+^0}{z_+\nu_+} - \frac{s_0 c}{n c_0}\right)\left(1 + \frac{\partial \ln(f_\pm)}{\partial \ln(c)}\right)\nabla \ln(c) \end{aligned} \tag{7.1}$$

Here, $F = 96485$ C mol^{-1} denotes Faraday's constant, R= 8,3145 J K^{-1}mol^{-1} the ideal gas constant, T the constant temperature [K], μ the chemical potential of the electrolyte [J mol^{-1}], and $t_+^0 \in (0, 1)$ is the transference number of the cation. For the case of a binary electrolyte LiX, we have $\nu_+ = \nu_- = 1$, $\nu = 2$, $z_+ = 1$, $z_- = -1$, $n = 1$, $s_+ = -1$, and $s_0 = s_- = 0$. Then (7.1) simplifies to

$$\boldsymbol{i} = -\kappa \nabla \phi_1 + \frac{\kappa 2RT(1 - t_+^0)}{F}\left(1 + \frac{\partial \ln(f_\pm)}{\partial \ln(c)}\right)\nabla \ln(c) \tag{7.2}$$

This equation is similar to Ohm's law, but the second summand is introduced to include concentration variations. The positive coefficient $\kappa = \kappa(c)$ is called the effective ionic conductivity and depends on the concentration of the electrolyte. As this dependency may change abruptly when we pass from the electrode to the separator, the function κ is only continuous in each of the domains Ω_i. In the sequel, we will omit the term $\left(1 + \frac{\partial \ln(f_\pm)}{\partial \ln(c)}\right)$ which means we assume that the molar activity coefficient $f_\pm$ does not depend on $\ln(c)$ for the sake of simplicity. In order to obtain a current balance in the liquid phase, we use (7.2) in the relation $\nabla \cdot \boldsymbol{i} = j$ (Faraday's law of electrolysis), where j denotes the current flow in normal direction which will be determined later (see reaction rate). The equation for the potential ϕ_1 then reads

$$-\nabla \cdot \left(\kappa(c)\nabla \phi_1 - \kappa(c)\frac{2RT(1 - t_+^0)}{F}\nabla \ln(c)\right) = j \quad \text{in } \Omega_i, \ (i = 1, 2, 3). \tag{7.3}$$

Regarding boundary conditions for ϕ_1, one imposes non flux boundary conditions of the form

$$\kappa(c)\partial_n \phi_1 - \kappa(c)\frac{2RT(1 - t_+^0)}{F}\frac{1}{c}\partial_n c = 0 \quad \text{on } \partial\Omega. \tag{7.4}$$

In this case, the condition $\int_\Omega \phi_1 \, dx = 0$ becomes necessary for uniqueness. Alternatively, ϕ_1 can be set to 0 at one of the current collector surfaces, e.g., at the positive electrode surface Γ_{2n} and the conditions (7.4) are then imposed on $\partial\Omega \setminus \Gamma_{2n}$.

Potential in the solid electrodes

In each of the subdomains Ω_1 and Ω_3 of the solid phase, we directly apply Ohm's law to obtain

$$\boldsymbol{i}_k = -\sigma_k \nabla \phi_2 \quad \text{in } \Omega_k \quad (k \in \{1, 3\}),$$

where the conductivities σ_k may have different values in Ω_1 and Ω_3. Setting $\sigma := \sigma_k$ in Ω_k, we use the relation $\nabla \cdot \boldsymbol{i} = -j$ to obtain the charge balance for the electrodes:

$$-\nabla \cdot (\sigma \nabla \phi_2) = -j \quad \text{in } \Omega_1 \text{ and } \Omega_3. \tag{7.5}$$

Note that the sign of the current flow j is different here, because we changed from the liquid to the solid phase: the outflow of the solid phase equals the inflow of the liquid phase and vice versa. Regarding the boundary conditions for ϕ_2, we impose an inhomogeneous Neumann boundary value I on each of the current collector surfaces Γ_n and Γ_{2n} as well as non-flux boundary conditions on the remaining parts of $\partial\Omega' = \partial\Omega_1 \cup \partial\Omega_3$, i.e.

$$\partial_n \phi_2 = -\frac{I}{\sigma_1} \quad \text{on } \Gamma_n, \quad \partial_n \phi_2 = \frac{I}{\sigma_3} \quad \text{on } \Gamma_{2n}, \quad \partial_\nu \phi_2 = 0 \quad \text{on } \partial\Omega_1 \setminus \Gamma_n \text{ and } \partial\Omega_3 \setminus \Gamma_{2n}. \tag{7.6}$$

I corresponds to the applied current of the lithium-cell.

Transport equations in the electrolyte

While the flux $\boldsymbol{N}_i$ of a species i in a dilute solution takes the simple form

$$\boldsymbol{N}_i \;=\; \underbrace{-z_i u_i F c_i \nabla \phi}_{\text{migration}} \; \underbrace{-D_i \nabla c_i}_{\text{diffusion}} \; \underbrace{+c_i \boldsymbol{v}}_{\text{convection}},$$

where c_i denotes the concentration, u_i the mobility, and D_i a diffusion coefficient, for a concentrated solution, the following expression for the flux was developed in [New73], Section 79:

$$\boldsymbol{N}_i = \frac{t_i^0}{z_i F} \boldsymbol{i} - \nu_i D \nabla c \quad (i = +, -), \tag{7.7}$$

where $c = \frac{c_i}{\nu_i}$ denotes the electrolyte concentration, D is the diffusion coefficient of the electrolyte, and $\boldsymbol{i}$ again denotes the current density in the liquid phase. The convection term and volume changes have already been neglected here. If we use this expression in the general material balance equation $\partial_t c = -\nabla \cdot \boldsymbol{N}$, we obtain (with $\nu_- = \nu_+ = z_+ = 1$, $z_- = -1$, and $t_-^0 = 1 - t_+^0$ which is assumed to be constant)

$$\partial_t c = \nabla \cdot D \nabla c + \frac{(1 - t_+^0)}{F} \nabla \cdot \boldsymbol{i} = \nabla \cdot D \nabla c + \frac{(1 - t_+^0)}{F} j \quad \text{in } \Omega. \tag{7.8}$$

Note that the coefficient D may be discontinuous at the interfaces of the electrodes, because the diffusivity in pure liquid and in the liquid-solid superposition phase is different. The boundary condition for the concentration of the electrolyte is a non-flux condition on the whole boundary and we impose an initial condition:

$$\partial_n c = 0 \quad \text{on } \partial\Omega, \qquad \text{and} \qquad c(0, x) = c_0(x) \quad (x \in \Omega). \tag{7.9}$$

Reaction rate

The current flow j is an expression in ϕ_1, ϕ_2, and c which describes the charge exchange and the intercalation process at the interface of the electrodes. By the superposition assumption, this reaction term enters the differential equations as an inhomogeneous right-hand side rather

than as a boundary condition. For electrochemical reactions, the flow j takes the following Butler-Volmer form (see Section 56 in [New73], for instance):

$$j = j(\phi_2 - \phi_1, c) = i_0 \left(e^{\frac{\alpha_a F}{RT}(\phi_2 - \phi_1 - U)} - e^{\frac{-\alpha_c F}{RT}(\phi_2 - \phi_1 - U)} \right) \tag{7.10}$$

with the so called surface-overpotential $\eta := \phi_2 - \phi_1 - U$ and the exchange current density $i_0 := aFkc^{\alpha_a}$. Here, a is the reactive surface and k denotes a positive kinetic rate constant. The anodic and cathodic transfer coefficients α_a and α_c are assumed to be constants in the interval $(0, 1)$. One often uses the relation $\alpha_c = 1 - \alpha_a$ or sets $\alpha_a = \alpha_c = \frac{1}{2}$ if no data are available. The function U is called the open circuit potential and is defined as the potential of the electrode (measured against a metallic lithium reference electrode), when no current is applied to the cell. This function depends on the concentration of intercalated lithium in the solid phase, which is not further modeled here. In the sequel, we assume U to be a Hölder continuous function of x with a sufficiently large exponent γ: $U \in C^\gamma(\overline{\Omega}_i)$ for $i = 1, 3$. Note that j is only defined in the electrodes Ω_1 and Ω_3, where both of the quantities ϕ_1 and ϕ_2 exist. We formally set $j := 0$ in Ω_2. Lastly, we assume that $(1 - t_+^0) \approx 1$ in the electrodes, i.e., the active material of the electrodes is a good electronic conductor. We summarize the equations of the battery model for the case of pure Neumann boundary conditions:

$$\begin{cases} -\nabla \cdot \left(\kappa_i(c) \left(\nabla\phi_1 - \frac{2RT}{F} \nabla \ln(c) \right) \right) &=& j(\phi_2 - \phi_1, c) & \text{in } J \times \Omega_i,\ (i = 1, 2, 3), \\ -\nabla \cdot (\sigma_i \nabla \phi_2) &=& -j(\phi_2 - \phi_1, c) & \text{in } J \times \Omega_i,\ (i = 1, 3), \\ \partial_t c - \nabla \cdot (D_i \nabla c) &=& j(\phi_2 - \phi_1, c) & \text{in } J \times \Omega_i,\ (i = 1, 2, 3), \\ \partial_\nu \phi_1 - \frac{2RT}{F} \frac{1}{c} \partial_\nu c &=& 0 & \text{on } J \times \partial\Omega, \\ \sigma \partial_\nu \phi_2 &=& I & \text{on } J \times \Gamma_k,\ k \in \{n, 2n\}, \\ \partial_\nu \phi_2 &=& 0 & \text{on } J \times ((\partial\Omega_1 \setminus \Gamma_n) \cup (\partial\Omega_3 \setminus \Gamma_{2n})), \\ \partial_\nu c &=& 0 & \text{on } J \times \partial\Omega, \\ c(0, x) &=& c_0(x) & (x \in \Omega). \end{cases} \tag{7.11}$$

Here, ν denotes the outward normal vector on the respective parts of the boundary, which exists almost everywhere on $\partial\Omega$.

7.2 Local solvability of the model equations

In the sequel, we shall set the physical constants k, a, R, T, and F to 1 for the sake of readability. Before we apply our abstract results to the particular system (7.11), we will rewrite the system in the form of (6.2).

7.2.1 Reformulation of the model equations

We set $u_1 := \phi_1 - 2\ln(c)$. Then we can seek u_1 as a solution of the nonlinear equation

$$-\nabla \cdot (\kappa(c) \nabla u_1) = j(\phi_2 - u_1 - 2\ln(c), c) \quad \text{in } (0, T) \times \Omega_i \ (i = 1, 2, 3),$$

with homogeneous Neumann boundary conditions on all of $\partial\Omega$: $\partial_\nu u_1 = 0$, or an inhomogeneous Dirichlet boundary condition $u_1 = 2\ln(c)$ on $(0, T) \times \Gamma_{2n}$ and homogeneous Neumann conditions on $\partial\Omega \setminus \Gamma_{2n}$. In all three equations the nonlinear function j from (7.10) changes to

$$\begin{aligned} N(\phi_2 - u_1, c) &:= j(\phi_2 - u_1 - 2\ln(c), c) \\ &= \begin{cases} c^\alpha \left(e^{\alpha(\phi_2 - u_1 - 2\ln(c) - U)} - e^{-(1-\alpha)(\phi_2 - u_1 - 2\ln(c) - U)} \right), & \text{in } \Omega_1 \cup \Omega_3 \\ 0, & \text{in } \Omega_2 \end{cases} \\ &= \begin{cases} c^{-\alpha} e^{\alpha(\phi_2 - u_1 - U)} - c^{2-\alpha} e^{-(1-\alpha)(\phi_2 - u_1 - U)}, & \text{in } \Omega_1 \cup \Omega_3 \\ 0, & \text{in } \Omega_2 \end{cases}. \end{aligned} \tag{7.12}$$

Let us verify that the nonlinearity N satisfies Assumption 6.2 (iii) in Ω_1 and Ω_3.

Lemma 7.1
Let $\alpha \in (0,1)$, $T > 0$, $J = (0,T)$, and $p > n+3$. Let $s < \frac{1}{p}$, $\gamma > s$, and assume $U \in C^\gamma(\overline{\Omega}_i)$ for $i = 1,3$. For $K > 0$ let $c \in (C(\overline{J}; C^\gamma(\overline{\Omega}_i)))_K$ (cf. Notation 6.1) and fix an arbitrary $t \in J$.

(i) For $i = 1,3$ consider the mapping

$$F_i : \mathbb{R} \times \overline{\Omega}_i \to \mathbb{R}, (y,x) \mapsto N(y, c(t,x)) = c(t,x)^{-\alpha} e^{\alpha(y-U(x))} - c(t,x)^{2-\alpha} e^{-(1-\alpha)(y-U(x))}.$$

Then F is continuous and continuously differentiable with respect to y and there exists a constant $C_{i,0} > 0$ such that $(\partial_y F)(y,x) \geq C_{i,0}$.

(ii) For every $u \in W_p^2(\Omega)$ there holds $N(u(\cdot), c(t,\cdot)) \in W_p^s(\Omega)$. In addition, the mapping

$$\mathcal{N} : W_p^2(\Omega) \to W_p^s(\Omega), \quad u \mapsto \mathcal{N}(u) := N(u(\cdot), c(t,\cdot))$$

is bounded, i.e., if $\|u\|_{2,p,\Omega} \leq K_1$, there exists a constant $K_2 = K_2(K_1, K)$, independent of u, such that $\|N(u(\cdot), c(t,\cdot))\|_{s,p,\Omega} \leq K_2 \quad (u \in W_p^2(\Omega))$.

Proof. (i) The continuity and the differentiability with respect to y are trivial. Differentiation with respect to y yields

$$(\partial_y N)(y, c(t,x)) = \alpha c(t,x)^{-\alpha} e^{\alpha(y-U(x))} + (1-\alpha) c(t,x)^{2-\alpha} e^{-(1-\alpha)(y-U(x))},$$

and as the continuous functions $\alpha c(t,\cdot)^{-\alpha}$, $(1-\alpha)c(t,\cdot)^{2-\alpha}$ and $e^{\pm U(\cdot)}$ possess a minimum on $\overline{\Omega}$, we can estimate

$$(\partial_y N)(y, c(t,x)) \geq C(K, \alpha, U) \left(e^{\alpha y} + e^{-(1-\alpha)y} \right) \geq C > 0,$$

because the function $y \mapsto \left(e^{\alpha y} + e^{-(1-\alpha)y}\right)$ attains a global positive minimum at the point $y_0 := \ln\left(\frac{1-\alpha}{\alpha}\right)$.

(ii) Let $u \in W_p^2(\Omega)$. We first prove that $v := N(u(\cdot), c(t,\cdot)) \in W_p^s(\Omega_i)$. This is trivial for Ω_2 and for Ω_1 and Ω_3 we first notice that $W_p^2(\Omega_i)$ is a Banach-algebra for $p > \frac{n}{2}$ (cf. [AF03], Theorem 4.39). This yields that $\exp(\pm\alpha u) \in W_p^2(\Omega_i)$ with $\|\exp(\alpha u)\|_{2,p,\Omega_i} \leq K_i(\|u\|_{2,p,\Omega})$. Since U is Hölder continuous, the functions $\exp(-\alpha U)$ and $\exp((1-\alpha)U)$ belong to $C^\gamma(\overline{\Omega}_i)$ as well. Similarly, $c(t,\cdot) \in C^\gamma(\overline{\Omega}_i)$ takes values in the compact interval $[\frac{1}{K}, K]$ and consequently, the functions $c(t,\cdot)^{-\alpha}$ and $c(t,\cdot)^{2-\alpha}$ belong to $C^\gamma(\overline{\Omega}_i)$, too. As a consequence of Theorem 1.18, the functions $c(t,\cdot)^{-\alpha} e^{-\alpha U} \in C^\gamma(\overline{\Omega}_i)$ and $c(t,\cdot)^{2-\alpha} e^{(1-\alpha)U} C^\gamma(\overline{\Omega}_i)$ are pointwise multipliers in the Besov space $W_p^s(\Omega_i)$, which proves $v \in W_p^s(\Omega_i)$. According to [Tri02], Proposition 3.1, we have equality of the spaces $W_p^s(\Omega) = \mathring{W}_p^s(\Omega)$ for $s < \frac{1}{p}$, where the latter space denotes the closure of smooth functions with compact support in Ω with respect to the norm $\|\cdot\|_{s,p,\Omega}$. By approximating v with C_0^∞-functions in Ω_i for $i = 1,3$ and continuing the approximating functions by 0 in Ω_2, we obtain an approximation for v on Ω, hence $v \in W_p^s(\Omega)$.

□

Remark 7.2

(i) We do not further consider the case when the potential ϕ_1 is set to 0 on Γ_{2n}. The corresponding boundary condition for u_1 then reads $u_1 = 2\ln(c)$ on Γ_n and in order to apply Theorem 4.14 and to have a uniform estimate for all $t \in \overline{J}$, we would require the regularity $\ln(c) \in C(\overline{J}; W_p^{2+s-1/p}(\Omega)) \cap W^{\frac{2+s-1/p}{2}}(J; L_p(\Omega))$ with additional compatibility conditions (cf. Assumption 6.2 (vi)). In the fixed point scheme, this regularity cannot be provided by an element $c \in W_p^{1,2}(J \times \Omega)$, in general.

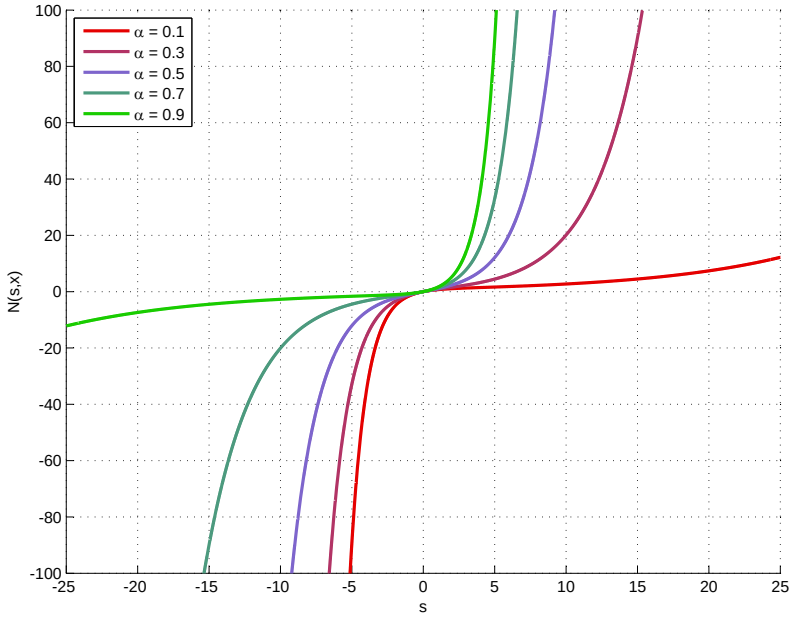

Figure 7.3: Nonlinearity for $U = c = 1$ and different values of α.

(ii) Similarly, in the case of a weak solution, the boundary condition $u_1 = 2\ln(c)$ leads to a regularity problem: in order to fulfill the Assumption 3.27, we require that there exists $w \in H^1(\Omega) \cap L_\infty(\Omega)$ with $\nabla w \in L_\infty(\Omega)$ and $w = 2\ln(c)$ on Γ. A natural space for a weak solution c to the parabolic equation is $C([0,T]; H^1(\Omega)) \cap C^{\beta/2}([0,T]; C^\beta(\overline{\Omega}))$ (cf. [WXZ06], p.13) such that this regularity for $\ln(c)$ is not contained in the equations without further regularity advisements.

(iii) A Dirichlet boundary condition for u_1 which satisfies the conditions of Assumption 6.2 (iv) and (vi) and is independent of the unknown functions can still be treated of course.

Remark 7.3
Assume that $I_k = 0$ for $k \in \{n, 2n\}$. In this case, we have conservation of mass for a strong solution (u_1, ϕ_2, c). To see this, we consider the corresponding weak form of the elliptic equations:

$$\int_\Omega \kappa(c)\nabla u_1 \nabla\varphi \, dx = \int_{\Omega'} N(\phi_2 - u_1, c)\varphi \, dx \quad (\varphi \in H^1_*(\Omega)), \tag{7.13a}$$

$$\int_{\Omega'} \sigma\nabla\phi_2\nabla\psi \, dx = -\int_{\Omega'} N(\phi_2 - u_1, c)\psi \, dx \quad (\psi \in H^1(\Omega')). \tag{7.13b}$$

We test (7.13b) with $\psi := 1$ to obtain that $\int_{\Omega'} N(\phi_2 - u_1, c) \, dx = 0$ in this situation. This yields

$$\frac{d}{dt}\int_\Omega c \, dx = \int_\Omega \nabla\cdot(D\nabla c) \, dx - \int_{\Omega'} N(\phi_2 - u_1, c) \, dx = 0,$$

hence, $\int_\Omega c \, dx$ is constant.

7.2.2 Solvability results

In contrast to the equations we considered in Chapter 6, the second equation is only imposed in Ω_1 and Ω_3. Here, we briefly discuss why this does not affect the existence results we proved for the case that the equation for ϕ_2 is imposed on all of Ω. In order to obtain strong solutions, we will again impose additional homogeneous canonical transmission conditions for u_1 and c on

the interfaces Σ_j.
Given $c \in C^\beta(\overline{J}; C^{1,\gamma}(\overline{\Omega}))$ with $c(t,x) \geq \frac{1}{K}$ and $\|c\|_{C^\beta(\overline{J};C^{1,\gamma}(\overline{\Omega}))} \leq K$, we fix $t \in [0,T]$ and first seek a solution of the nonlinear elliptic system

$$
(7.14) \qquad \begin{cases} -\nabla \cdot (\kappa(c)\nabla u_1) &= N(\phi_2 - u_1 - U, c(t,\cdot)) & \text{in } \Omega_i,\ (i=1,2,3), \\ -\nabla \cdot (\sigma_i \nabla \phi_2) &= -N(\phi_2 - u_1 - U, c(t,\cdot)) & \text{in } \Omega_i,\ (i=1,3), \\ \partial_\nu u_1 &= 0 & \text{on } \partial\Omega, \\ [\![u_1]\!] = [\![\partial_n u_1]\!] &= 0 & \text{on } \Sigma_j,\ (j=1,2), \\ \partial_\nu \phi_2 &= 0 & \text{on } \partial\Omega_1 \setminus \Gamma_n \text{ and } \partial\Omega_1 \setminus \Gamma_{2n}, \\ \sigma \partial_\nu \phi_2 &= I_k & \text{on } \Gamma_k,\ k \in \{n, 2n\}. \end{cases}
$$

We henceforth assume that the functions $I_k : (0,T) \times \Gamma_k \to \mathbb{R}$, which occur as inhomogeneous Neumann boundary values for ϕ_2, belong to the space $C(\overline{J}; W_p^{1+s-1/p}(\Gamma_k))$ for $k \in \{n, 2n\}$ and satisfy the compatibility conditions indicated in Theorem 2.26. In the following, we will use the notation $v \in W_p^2(\Omega')$ if and only if $v|_{\Omega_i} \in W_p^2(\Omega_i)$ for $i = 1, 3$. In order to have uniqueness, we seek the solution in the space $W_{p,*}^2(\Omega) \times W_p^2(\Omega')$.

Consider the linearized, decoupled equations

$$
(7.15a) \qquad \begin{cases} u_1 - \nabla \cdot (\kappa(c)\nabla u_1) &= \delta\,(N(z_2 - z_1 - U, c(t,\cdot)) + z_1) & \text{in } \Omega_i,\ (i=1,2,3), \\ \partial_\nu u_1 &= 0 & \text{on } \partial\Omega, \\ [\![u_1]\!] = [\![\partial_n u_1]\!] &= 0 & \text{on } \Sigma_j,\ (j=1,2), \end{cases}
$$

$$
(7.15b) \qquad \begin{cases} u_2 - \nabla \cdot (\sigma \nabla u_2) &= -\delta\,(N(z_2 - z_1 - U, c(t,\cdot)) - z_1) & \text{in } \Omega_i,\ (i=1,3), \\ \partial_\nu u_2 &= 0 & \text{on } \partial\Omega_1 \setminus \Gamma_n \text{ and } \partial\Omega_3 \setminus \Gamma_{2n}, \\ \partial_\nu u_2 &= \delta \tfrac{I_k}{\sigma} & \text{on } \Gamma_k,\ k \in \{n, 2n\}. \end{cases}
$$

with $\delta \in [0,1]$ and $(z_1, z_2) \in X_* := W_{p,*}^2(\Omega) \times W_p^2(\Omega')$. In this situation, Theorem 2.30 applies and yields a unique solution $u_1 \in W_p^{2+s}(\Omega)$ and $u_2 \in W_p^{2+s}(\Omega')$ and again the operator

$$
T : [0,1] \times X_* \to X_*, \quad (\delta, z_1, z_2) \mapsto \left(u_1 - \frac{1}{|\Omega|} \int_\Omega u_1 \, dx, u_2 \right)
$$

is well-defined. The proofs of the continuity properties (cf. the Lemmas 4.7 and 4.10), the compactness (Lemma 4.13) as well as the a priori boundedness for $z \in X_*$ with $T(\delta, z) = z$ (Lemma 4.12) easily carry over to the present situation. We can again apply the Leray-Schauder principle (Theorem A.9) and have the following

Theorem 7.4
Assume that $1 \leq n \leq 3$, $p > n$, $0 \leq s < \frac{1}{p}$, and $s < \gamma \leq 1$. Let $K > 0$ and $c \in C(\overline{J}; C^{1+\gamma}(\overline{\Omega}))$ with $\|c\|_{C(\overline{J};C^{1+\gamma}(\overline{\Omega}))} \leq K$ and $c(t,x) \geq \frac{1}{K}$. Assume that $\kappa(c(t,\cdot)) \in C^{1+\gamma}(\overline{\Omega}_i)$ for $i = 1,2,3$, $I_k \in C(\overline{J}; W_p^{1+s-1/p}(\Gamma_k))$ for $k \in \{n, 2n\}$, and $U \in C^\gamma(\overline{\Omega}_i)$ for $i = 1, 3$. Then the following assertions hold true:

(i) For every $t \in \overline{J}$, the nonlinear coupled elliptic system (7.14) admits a unique solution $(u_1(t,\cdot), \phi_2(t,\cdot)) \in W_{p,*}^{2+s}(\Omega) \times W_p^{2+s}(\Omega')$. Furthermore, there exists a positive constant $M = M\left(n, p, s, \Omega, K, \kappa, \sigma, \|I_k\|_{C(\overline{J};W_p^{1+s-1/p}(\Gamma_k))}, \|U\|_{C^\gamma(\overline{\Omega}_i)}\right)$, which is independent of t, such that

$$
(7.16) \qquad \|u_1(t,\cdot)\|_{2+s,p,\Omega} + \|\phi_2(t,\cdot)\|_{2+s,p,\Omega'} \leq M.
$$

(ii) Let $0 \leq \tilde{s} < s$ and $r > \gamma + \frac{n}{p}$. Then the mapping

$$
L : W_{p,K}^{1+r}(\Omega) \to W_{p,*}^{2+\tilde{s}}(\Omega) \times W_p^{2+\tilde{s}}(\Omega'),\ c(t,\cdot) \mapsto (u_1(t,\cdot), \phi_2(t,\cdot))
$$

is continuous and bounded.

(iii) The operator

$$S : \mathbb{E}_K := (W_p^{1,2}(J \times \Omega))_K \to C(\overline{J}; W_{p,*}^{2+\tilde{s}}(\Omega)) \times C(\overline{J}; W_p^{2+\tilde{s}}(\Omega')), \quad c \mapsto (u_1, \phi_2)$$

is continuous and compact. Furthermore, this operator is bounded as well, i.e., there exists a positive constant $K_2 = K_2(M)$, such that the estimate

$$\|u_1\|_{C(\overline{J};W_p^{2+\tilde{s}}(\Omega))} + \|\phi_2\|_{C(\overline{J};W_p^{2+\tilde{s}}(\Omega'))} \leq K_2$$

holds for all $(u_1, \phi_2) = S(v)$ with $v \in \mathbb{E}_K$.

Proof. The proofs carry over literally from Theorem 4.14 and from the Lemmas 6.6 and 6.8, respectively. The argumentation applied there never uses that u_1 and u_2 are defined on the same domain. At the points where we choose convergent subsequences for u_2 in function spaces on Ω, we can now first choose a converging subsequence in Ω_1 for ϕ_2 and of this bounded sequence, we can choose another subsequence which then also converges on Ω_3. Regarding the proof of uniqueness, we can copy the one given in Lemma 3.6 with the only change that the nonlinearity is now integrated over Ω', since the integral over Ω_2 vanishes by (7.13). Note that Lemma 7.1 (i) implies that N satisfies the estimate 3.1 in Ω' which is essential for the proof of uniqueness. □

The next step is to prove that

$$N : \mathbb{E}_K \to L_p(J; L_p(\Omega)), v \mapsto N(\phi_2 - u_1, v),$$

with $(u_1, \phi_2) := S(v)$ defines a continuous and compact mapping which satisfies an estimate of the form

$$\|N(v)\|_{L_p(J;L_p(\Omega))} \leq CT^{1/p} \quad (v \in \mathbb{E}_K).$$

This can be proved analogously to Corollary 6.9. In fact, $N : \mathbb{E}_K \to C(J; C(\overline{\Omega}_i))$ is continuous and compact for $i = 1, 2, 3$. Finally, we consider the linearized parabolic equation

$$\left\{ \begin{array}{rcll} \partial_t c - \nabla \cdot (D\nabla c) & = & N(\phi_2 - u_1, c) & \text{in } (0,T) \times \Omega, \\ \partial_\nu c & = & 0 & \text{on } (0,T) \times \partial\Omega, \\ [\![c]\!] = [\![\partial_\nu c]\!] & = & 0 & \text{on } (0,T) \times \Sigma_j, \ (j = 1,2), \\ c(0,x) & = & c_0(x) & \text{in } \Omega. \end{array} \right. \tag{7.17}$$

with a strictly positive initial value $c_0 \in W_{p,+}^{2-2/p}(\Omega)$. According to Theorem 5.33, this equation admits a unique solution $c \in W_p^{1,2}((0,T) \times \Omega)$ which satisfies the estimate

$$\|c\|_{W_p^1(J;L_p(\Omega))} + \|c\|_{L_p(J;W_p^2(\Omega))} \leq C \left(\|c_0\|_{2-2/p,p,\Omega} + \|N(v)\|_{L_p(J;L_p(\Omega))} \right).$$

The whole procedure which we sketched above then defines a continuous and compact operator $Z : \mathbb{E}_K \to \mathbb{E}$ and by choosing the constant $K > 0$ and the time interval $J = (0,T)$ appropriately (cf. the procedure on p. 127), we obtain a continuous and compact mapping from $\mathbb{E}_K$ into itself. Schauder's fixed point theorem (Theorem A.8 (ii)) then yields

Theorem 7.5
Let $1 \leq n \leq 3$, $\tau \in (0,1)$, $p > n + 3 + \frac{1}{\tau}$ and $0 < s < \frac{1}{p}$. Let $c_0 \in W_{p,+}^{2-2/p}(\Omega)$ with $c_0(x) \geq C_0 > 0$ and assume that $I_k \in C(\overline{J}; W_p^{1+s-1/p}(\Gamma_k)) \cap W_p^{\frac{1+s-1/p}{2}}(J; L_p(\Gamma_k))$ satisfies the compatibility conditions

$$\gamma_{\Gamma_j} I_k = \gamma_{\Gamma_j} \partial_j I_k = 0 \quad \text{on } J \times (\Gamma_k \cap \Gamma_j) \quad (k \in \{n, 2n\}).$$

Then there exists a constant $K > 0$ and a time $T_0 > 0$ such that the elliptic-parabolic system

$$
(7.18)\qquad \begin{cases} -\nabla\cdot(\kappa_i(c)\nabla u_1) &= N(\phi_2-u_1,c) & \text{in } J\times\Omega_i,\ (i=1,2,3),\\ -\nabla\cdot(\sigma_i\nabla\phi_2) &= -N(\phi_2-u_1,c) & \text{in } J\times\Omega_i,\ (i=1,3),\\ \partial_t c-\nabla\cdot(D_i\nabla c) &= N(\phi_2-u_1,c) & \text{in } J\times\Omega_i,\ (i=1,2,3),\\ \partial_\nu u_1 &= 0 & \text{on } J\times\partial\Omega,\\ \sigma\partial_\nu\phi_2 &= I & \text{on } J\times\Gamma_k,\ k\in\{n,2n\},\\ \partial_\nu\phi_2 &= 0 & \text{on } J\times((\partial\Omega_1\setminus\Gamma_n)\cup(\partial\Omega_3\setminus\Gamma_{2n})),\\ \partial_\nu c &= 0 & \text{on } J\times\partial\Omega,\\ c(0,x) &= c_0(x) & (x\in\Omega). \end{cases}
$$

admits a solution in the time interval $J=(0,T_0)$ with the following regularity:

$$(u_1,\phi_2,c)\in C(\overline{J};W^2_{p,*}(\Omega))\times C(\overline{J};W^2_p(\Omega'))\times W^{1,2}_{p,K}(J\times\Omega).$$

Corollary 7.6
Under the assumptions of Theorem 7.5 there exists a local solution

$$(\phi_1,\phi_2,c)\in L_p(J;W^2_p(\Omega))\cap C(\overline{J};W^{2-2/p}_p(\Omega))\times C(\overline{J};W^2_p(\Omega'))\times\mathbb{E}_K$$

to the model equations (7.11) *on the time inteval $J=(0,T_0)$.*

Proof. Let $(u_1,\phi_2,c)\in C(\overline{J};W^2_{p,*}(\Omega))\times C(\overline{J};W^2_p(\Omega'))\times\mathbb{E}_K$ denote a solution to (7.18). Then $\ln(c)\in W^1_p(J;L_p(\Omega))\cap L_p(J;W^2_p(\Omega))$, by direct computations (note that $c\in C(\overline{J};C^{1+\beta}(\overline{\Omega}))$ by embedding and $\frac{1}{K_1}\le c(t,x)\le K_1$ for a constant $K_1>0$, hence one can consider an appropriate cutoff of $x\mapsto\ln(x)$ to have chain and product rules available). We define

$$\phi_1 := u_1+2\ln(c)\in C(\overline{J};W^2_p(\Omega))+W^1_p(J;L_p(\Omega))\cap L_p(J;W^2_p(\Omega))\hookrightarrow C(\overline{J};W^{2-2/p}_p(\Omega))$$

and then (ϕ_1,ϕ_2,c) solves (7.11) (when all physical constants are set to 1). □

7.2.3 Applicability of monotone methods

As we have seen in Lemma 7.1, the nonlinear function N from (7.13) possesses a monotonicity with respect to the potentials ϕ_1 and ϕ_2. For elliptic and parabolic systems with a monotone structure, the so called monotone method (or the method of upper and lower solutions) has been applied successfully to several systems of reaction-diffusion equations in view of classical or Hölder continuous solutions. To the author's knowledge, a monotone method for an elliptic-parabolic system of the present structure has not been developed yet, but let us outline the approach for an elliptic system of general form and the difficulties which arise if we try to transfer this method to the concrete system (7.14). We follow the depiction of [Pao92], Sections 3.1, 8.1, and 8.4.

In a bounded domain with $C^{2,\alpha}$-boundary consider a nonlinear system of the form

$$
(7.19)\qquad \begin{cases} -L_1u_1 &= f_1(x,u_1,u_2) & \text{in } \Omega,\\ -L_2u_2 &= f_2(x,u_1,u_2) & \text{in } \Omega,\\ \alpha_1\frac{\partial u_1}{\partial\nu}+\beta_1u_1 &= h_1 & \text{on } \partial\Omega,\\ \alpha_2\frac{\partial u_2}{\partial\nu}+\beta_2u_2 &= h_2 & \text{on } \partial\Omega, \end{cases}
$$

where

$$L_iu := \sum_{j,k=1}^n a^{(i)}_{jk}(x)\partial_j\partial_k u+\sum_{j=1}^n b^{(i)}_j(x)\partial_j u \quad\text{and}\quad B_iu := \alpha_i\frac{\partial u}{\partial\nu}+\beta_iu \quad (i=1,2).$$

We assume that L_i has coefficients $a^{(i)}_{jk},b^{(i)}_j\in C^\alpha(\overline{\Omega})$ and is elliptic, i.e. $\xi^T(a^{(i)}_{jk})_{jk}\xi\ge C|\xi|^2$ for $\xi\in\mathbb{R}^n$. For the boundary operators B_i we assume that $\alpha_i,\beta_i\in C^{1,\alpha}(\partial\Omega)$ with $\alpha_i=0$ and $\beta_i>0$ or $\alpha_i>0$ and $\beta_i\ge0$, as well as $h_i\in C^{1,\alpha}(\partial\Omega)$.

The nonlinear functions (f_1, f_2) are assumed to be Hölder continuous and quasimonotone nondecreasing, i.e. for $i = 1, 2$ and fixed u_i, the function f_i is monotonically nondecreasing in u_j for $j \neq i$. If f_i is differentiable with respect to u_1 and u_2, this property is equivalent to $\partial_{u_2} f_1 \geq 0$ and $\partial_{u_1} f_2 \geq 0$. Evidently, the functions

$$f_1(u_1, u_2) := c^{-\alpha} e^{\alpha(u_2 - u_1 - 2\ln(c) - U)} - c^{2-\alpha} e^{-(1-\alpha)(u_2 - u_1 - 2\ln(c) - U)}$$

and $f_2(u_1, u_2) := -f_1(u_1, u_2)$ satisfy these conditions for $\alpha \in (0, 1)$ and Hölder continuous, bounded functions U and c with $c(x) \geq C_0 > 0$. The following considerations can be modified for the cases when (f_1, f_2) are quasimonotone nonincreasing or mixed quasimonotone, but here we exclusively consider the nondecreasing case.

Definition 7.7
A pair $(\widetilde{u}, \hat{u})$ of ordered upper and lower solutions to (7.19) is a pair of functions $\widetilde{u} = (\widetilde{u}_1, \widetilde{u}_2)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2) \in C(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$, which satisfies $\widetilde{u}_i \geq \hat{u}_i$, $B_i \widetilde{u}_i \geq h_i \geq B_i \hat{u}_i$ on $\partial\Omega$, and

$$\begin{array}{rcccll} L_1\widetilde{u}_1 + f_1(x, \widetilde{u}_1, \widetilde{u}_2) & \leq & 0 & \leq & L_1\hat{u}_1 + f_1(x, \hat{u}_1, \hat{u}_2) & \text{in } \Omega, \\ L_2\widetilde{u}_2 + f_2(x, \widetilde{u}_1, \widetilde{u}_2) & \leq & 0 & \leq & L_2\hat{u}_2 + f_2(x, \hat{u}_1, \hat{u}_2) & \text{in } \Omega. \end{array} \tag{7.20}$$

For the subsequent considerations we assume that $(\widetilde{u}, \hat{u})$ is a pair of ordered upper and lower solutions and that $f = (f_1, f_2)$ satisfies the conditions

$$-c_1(x)(u_1 - v_1) \leq f_1(x, u_1, u_2) - f_1(x, v_1, u_2) \leq C_1(x)(u_1 - v_1) \quad (u_1 \geq v_1) \tag{7.21a}$$

$$-c_2(x)(u_2 - v_2) \leq f_2(x, u_1, u_2) - f_2(x, u_1, v_2) \leq C_2(x)(u_2 - v_2) \quad (u_2 \geq v_2) \tag{7.21b}$$

for all $(u_1, u_2), (v_1, v_2) \in \langle \widetilde{u}, \hat{u} \rangle := \{(u_1, u_2) \in C(\overline{\Omega}) : \widetilde{u}_i \leq u_i \leq \hat{u}_i,\ i = 1, 2\}$ with bounded, nonnegative functions c_i and C_i. Note that then $f_i(x, u_1, u_2) + c_i(x)u_i$ is nondecreasing with respect to u_1 and u_2. Acting on this assumption and given a pair of ordered upper and lower solutions, we can construct sequences $\left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right)$, $\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right)$ of solutions to the linearized problems

$$\left\{ \begin{array}{rcll} -L_i u_i^{(k)} + c_i u_i^{(k)} & = & f_i(x, u_1^{(k-1)}, u_2^{(k-1)}) + c_i(x) u_i^{(k-1)} & \text{in } \Omega,\ (i = 1, 2), \\ B_i u_i^{(k)} & = & h_i & \text{on } \partial\Omega, \end{array} \right. \tag{7.22}$$

where we use the initial functions $\left(\underline{u}_1^{(0)}, \underline{u}_2^{(0)}\right) := (\hat{u}_1, \hat{u}_2)$ for the sequence $\left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right)$ and $\left(\overline{u}_1^{(0)}, \overline{u}_2^{(0)}\right) := (\widetilde{u}_1, \widetilde{u}_2)$ for the sequence $\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right)$. An existence theorem for the linear equation (7.22) can be found in [Pao92], Lemma 3.2.1. Using a variant of the maximum principle (cf. [Pao92], Lemma 3.1.4), we can prove the following monotone property for the sequences:

$$\underline{u}_i^{(k)} \leq \underline{u}_i^{(k+1)} \leq \overline{u}_i^{(k+1)} \leq \overline{u}_i^{(k)} \quad \text{in } \overline{\Omega} \quad (i = 1, 2,\ k \in \mathbb{N}_0).$$

Hence, the pointwise limits $(\underline{u}_1, \underline{u}_2)$ and $(\overline{u}_1, \overline{u}_2)$ exist and the following theorem can be proved:

Theorem 7.8
The sequences $\left(\underline{u}_1^{(k)}, \underline{u}_2^{(k)}\right)$ and $\left(\overline{u}_1^{(k)}, \overline{u}_2^{(k)}\right)$ converge monotonically to their limits $(\underline{u}_1, \underline{u}_2)$ and $(\overline{u}_1, \overline{u}_2) \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ and each pair is a solution to (7.19). Furthermore, any solution $(v_1, v_2) \in C^{2,\alpha}(\Omega) \cap \langle \widetilde{u}, \hat{u} \rangle$ of (7.19) satisfies

$$\underline{u}_i \leq v_i \leq \overline{u}_i \quad \text{in } \overline{\Omega} \quad (i = 1, 2).$$

If additionally one of the conditions $\partial_{u_2} f_1 > 0$ or $\partial_{u_1} f_2 > 0$ is satisfied, then there is a unique solution to (7.19) and $(\underline{u}_1, \underline{u}_2) = (\overline{u}_1, \overline{u}_2)$.

Proof. See [Pao92], Theorems 3.2.1, 8.4.1, and 8.6.1. □

Based on the pointwise convergence, the argumentation in the proof uses a priori estimates for the linear elliptic problems in $W_p^2(\Omega)$, the embedding $W_p^2(\Omega) \hookrightarrow C^{1,\alpha}(\Omega)$, and eventually the Schauder-estimates to obtain a bounded sequence in $C^{2,\alpha}(\overline{\Omega})$. The latter possesses a subsequence, which converges in $C^2(\Omega)$ and necessarily to the pointwise limit. In this argumentation as well as in the classical solvability theorem, which was required to construct the sequences, there lies the main gap between the applicability of the monotone method to the concrete situation, we face with the system (7.14): neither the domain, nor the coefficients or the nonlinearity considered as functions on the whole domain Ω, possess the required smoothness. Beyond that, for the present problem, the task of finding ordered upper and lower solutions such that the inequalities (7.21a)-(7.21b) are satisfied, is by no means easy.

The method extends to equations and systems with nonlinear boundary conditions (cf. [Pao92], Chapters 4 and 9) which - under appropriate simplifications - could be a possible approach for the model (7.25) which will be introduced in the subsequent section. By means of integration by parts, the concept of upper and lower solutions can also be generalized for weak solutions in the spaces $W_p^1(\Omega)$, we refer to the article [Sch07] and the references therein. However, the results stated for this situation again impose growth conditions on the nonlinear functions which cannot be met by a nonlinearity with exponential growth.

7.3 A battery model with nonlinear boundary conditions

In this section, we briefly discuss a lithium-ion battery model which has been developed more recently. The model shares some similarities with the equations we considered in the preceding section, such as the rectangular geometry and the structural properties of the nonlinearity which now occurs as a nonlinear interface condition on the electrode surfaces. The following equations for species and charge transport have been developed by O. Iliev, A. Latz, and J. Zausch at the Fraunhofer ITWM in 2010, see the research report [ILZ10]. In contrast to the preceding section, only two different regions are distinguished: the liquid electrolyte region (Ω_1) and the solid active particles (Ω_2). We do not further specify the geometry, but examples can be found in the article [LZ10]. Two unknown functions describe the concentration c of lithium-ions and the effective potential ϕ and by the subscripts $i = 1, 2$, we will indicate in which domain the respective quantity is considered. The flux of the positive ions is given by

$$\boldsymbol{N}_{1+} = -\frac{\kappa(c)t_+}{F}\nabla\phi - D_1\nabla c - \frac{RTt_+^2}{F^2}\kappa(c)\nabla\ln(c) \quad \text{in } \Omega_1, \tag{7.23a}$$

$$\boldsymbol{N}_{2+} = -D_2\nabla c \quad \text{in } \Omega_2. \tag{7.23b}$$

The transport terms in the flux expression for the liquid part can be regarded as electromigration in the electric field $E = -\nabla\phi$, diffusion, and migration due to a gradient in the chemical potential. In the solid part, however, the ion transport is assumed to be dominated by diffusion. The expressions for the charge transport take the form

$$\boldsymbol{j}_1 = -\kappa(c)\nabla\phi - \frac{RTt_+}{F}\kappa(c)\nabla\ln(c) \quad \text{in } \Omega_1, \qquad \text{and} \qquad \boldsymbol{j}_2 = -\sigma\nabla\phi \quad \text{in } \Omega_2. \tag{7.24}$$

Inserting the fluxes in the equations $\partial_t c + \nabla\cdot\boldsymbol{N}_+ = 0$ for mass balance and $0 = \partial_t q = -\nabla\cdot\boldsymbol{j}$ for the charge balance ($\partial_t q = 0$ due to the assumption of charge neutrality) will then lead to a coupled system for ϕ and c. Regarding interface conditions across the surface Σ, which separates Ω_1 and Ω_2, one imposes the continuity of the fluxes:

$$\boldsymbol{N}_{1+}\cdot\boldsymbol{\nu} = \boldsymbol{N}_{2+}\cdot\boldsymbol{\nu} \qquad \text{and} \qquad \boldsymbol{j}_1\cdot\boldsymbol{\nu} = \boldsymbol{j}_2\cdot\boldsymbol{\nu} \quad \text{on } (0,T)\times\Sigma,$$

where $\boldsymbol{\nu}$ denotes the normal on Σ pointing from the solid into the liquid phase.

In addition, the value of the fluxes is connected to a reaction rate on the surface Σ via the relations $\boldsymbol{j}_k \cdot \boldsymbol{\nu} = i_{12}$ and $\boldsymbol{N}_{k+} \cdot \boldsymbol{\nu} = \frac{i_{12}}{F}$ for $k = 1, 2$. The reaction rate again exhibits the Butler-Volmer form:

$$i_{12} = k(c_1 c_2)^\alpha \left(1 - \tfrac{c_2}{c_{2,\max}}\right)^{1-\alpha} \left(e^{\frac{\alpha F}{RT}(\phi_2 - \phi_1 - U)} - e^{\frac{-(1-\alpha)F}{RT}(\phi_2 - \phi_1 - U)}\right) \quad \text{on } (0,T) \times \Sigma.$$

The boundary conditions on the external parts $\partial\Omega_k \setminus \Sigma$ are similar to the model we discussed in the preceding section, i.e., non-flux conditions are imposed for the concentration c on all of $\partial\Omega_k \setminus \Gamma$ and the potential ϕ is normalized with a Dirichlet condition on one part Γ and non-flux conditions are imposed on the remaining parts of $\partial\Omega$. We can summarize the model equations as follows:

$$\text{(7.25)} \quad \begin{cases} -\nabla \cdot \left(\kappa(c_1)\nabla(\phi_1 + \frac{RTt_+}{F}\ln(c_1))\right) = 0 & \text{in } J \times \Omega_1, \\ -\nabla \cdot (\sigma \nabla \phi_2) = 0 & \text{in } J \times \Omega_2, \\ \partial_t c_1 - \nabla \cdot \left(\frac{\kappa(c_1)t_+}{F}\nabla\phi_1 + D_1\nabla c_1 + \frac{RTt_+^2}{F^2}\kappa(c_1)\nabla\ln(c_1)\right) = 0 & \text{in } J \times \Omega_1, \\ \partial_t c_2 - \nabla \cdot (D_2 \nabla c_2) = 0 & \text{in } J \times \Omega_2, \\ c_k(0,x) = c_{0,k} & \text{in } \Omega, \\ \boldsymbol{N}_{k+} \cdot \boldsymbol{\nu} = 0 & \text{on } J \times \partial\Omega_k \setminus \Sigma, \\ \boldsymbol{j}_1 \cdot \boldsymbol{\nu} = 0 & \text{on } J \times \partial\Omega_1 \setminus \Sigma, \\ \boldsymbol{j}_2 \cdot \boldsymbol{\nu} = 0 & \text{on } J \times \partial\Omega_2 \setminus (\Gamma \cup \Sigma), \\ \phi_2 = 0 & \text{on } J \times \Gamma, \\ \boldsymbol{N}_{1+} \cdot \boldsymbol{\nu} = \boldsymbol{N}_{2+} \cdot \boldsymbol{\nu} = \frac{i_{12}}{F}, \quad \boldsymbol{j}_1 \cdot \boldsymbol{\nu} = \boldsymbol{j}_2 \cdot \boldsymbol{\nu} = i_{12} & \text{on } J \times \Sigma. \end{cases}$$

Similarly as in the preceding section, it is convenient to set all physical constants to 1, to substitute $u_1 := \phi_1 + \frac{RTt_+}{F}\ln(c_1)$, and to rewrite the boundary conditions as Neumann boundary conditions for the unknown functions. Additionally, we assume that the transference number t_+ is a constant in each part which allows to substitute the first into the third equation. We then obtain a system of the form

$$\text{(7.26)} \quad \begin{cases} -\nabla \cdot (\kappa(c_1)\nabla u_1) = 0 & \text{in } J \times \Omega_1, \\ -\nabla \cdot (\sigma \nabla \phi_2) = 0 & \text{in } J \times \Omega_2, \\ \partial_t c_1 - \nabla \cdot (D_1 \nabla c_1) = 0 & \text{in } J \times \Omega_1, \\ \partial_t c_2 - \nabla \cdot (D_2 \nabla c_2) = 0 & \text{in } J \times \Omega_2, \\ c_k(0,x) = c_{0,k}(x) & \text{in } \Omega, \\ D_k \partial_\nu c_k = 0 & \text{on } J \times \partial\Omega_k \setminus \Sigma, \\ \kappa(c_1)\partial_\nu u_1 = 0 & \text{on } J \times \partial\Omega_1 \setminus \Sigma, \\ \sigma \partial_\nu \phi_2 = 0 & \text{on } J \times \partial\Omega_2 \setminus (\Gamma \cup \Sigma), \\ \phi_2 = 0 & \text{on } J \times \Gamma, \\ \kappa(c_1)\partial_\nu u_1 + D_1 \partial_\nu c_1 = D_2 \partial_\nu c_2 = -N(\phi_2 - u_1, c_1, c_2) & \text{on } J \times \Sigma, \\ \kappa(c_1)\partial_\nu u_1 = \sigma \partial_\nu \phi_2 = -N(\phi_2 - u_1, c_1, c_2) & \text{on } J \times \Sigma, \end{cases}$$

with the nonlinearity

$$N(\phi_2 - u_1, c_1, c_2) := (c_1 c_2)^\alpha \left(1 - \tfrac{c_2}{c_{2,\max}}\right)^{1-\alpha} \left(e^{\alpha(\phi_2 - u_1 + \ln(c_1) - U)} - e^{-(1-\alpha)(\phi_2 - u_1 + \ln(c_1) - U)}\right).$$

Despite the structural similarities to the system (7.18), the theoretical treatment of the equations with nonlinear boundary conditions is much more delicate. To the author's knowledge, no analytical results for the above system have been derived yet. If we ask for the existence of a weak solution to the elliptic equations for fixed c_1 and c_2, a fixed point argument in the space $X := L_\infty(\Omega) \cap L_\infty(\Sigma)$ seems to be an equivalent approach to the one we presented in Chapter 3. In fact, under appropriate assumptions on the geometry and the coefficients, a corresponding solution operator $T : [0,1] \times X \to X$ to the linearized equations can be constructed and continuity as well as compactness can be proved analogously to the Lemmas 3.12 and 3.13. However, it is not clear how to prove an a priori bound for elements $x \in X$ with $T(\delta, x) = x$ (cf. Lemma 3.17). Here, we briefly prove uniqueness for the nonlinear elliptic system.

Definition 7.9
Let $c_i \in C([0,T];C^\gamma(\overline{\Omega}_i))$ with $c_i(t,x) \geq C_0 > 0$ and $U \in C([0,T];C^\gamma(\overline{\Omega}))$. Assume that $\kappa \in C^\gamma([0,\infty))$ with $\kappa(0) = 0$ and $\kappa(c) > 0$ for $c > 0$. Fix a time $t \in [0,T]$. By a weak solution to the elliptic equations

$$
\left\{\begin{array}{rcll}
-\nabla\cdot(\kappa(c_1)\nabla u_1) &=& 0 & \text{in } \Omega_1,\\
-\nabla\cdot(\sigma\nabla\phi_2) &=& 0 & \text{in } \Omega_2,\\
\kappa(c_1)\partial_\nu u_1 &=& 0 & \text{on } \partial\Omega_1\setminus\Sigma,\\
\sigma\partial_\nu\phi_2 &=& 0 & \text{on } \partial\Omega_2\setminus(\Gamma\cup\Sigma),\\
\phi_2 &=& 0 & \text{on } \Gamma,\\
\kappa(c_1)\partial_\nu u_1 = \sigma\partial_\nu\phi_2 &=& -N(\phi_2-u_1,c_1,c_2) & \text{on } \Sigma,
\end{array}\right. \tag{7.27}
$$

we mean a pair $(u_1,\phi_2) \in H^1(\Omega_1)\times H^1_\Gamma(\Omega_2)$ which satisfies

$$
\int_{\Omega_1}\kappa(c_1)\nabla u_1\nabla\varphi\,dx = \int_\Sigma N(\phi_2-u_1,c_1,c_2)\varphi\,dS \quad (\varphi\in H^1(\Omega_1)), \tag{7.28a}
$$

$$
\int_{\Omega_2}\sigma\nabla\phi_2\nabla\psi\,dx = -\int_\Sigma N(\phi_2-u_1,c_1,c_2)\psi\,dS \quad (\psi\in H^1_\Gamma(\Omega_2)). \tag{7.28b}
$$

Note that this definition includes that the above integral expressions are well-defined.

The sign of the nonlinearity in the equation for u_1 changes here, because $-\nu$ is the outer normal for Ω_1.

Lemma 7.10
There is at most one weak solution $(u_1,\phi_2) \in H^1(\Omega_1)\times H^1_\Gamma(\Omega_2)$ to the elliptic system (7.27).

Proof. Given two weak solutions (u_1,ϕ_2) and (v_1,ψ_2), take the difference of the equations for u_1 and v_1 and for ϕ_2 and ψ_2, respectively, and set $\varphi := u_1 - v_1$ as well as $\psi := \phi_2 - \psi_2$. The sum of the resulting equations then reads

$$
\begin{aligned}
&\int_{\Omega_1}\kappa(c_1)|\nabla(u_1-v_1)|^2\,dx + \int_{\Omega_2}\sigma|\nabla(\phi_2-\psi_2)|^2\,dx\\
&\qquad = -\int_\Sigma \left(N(\phi_2-u_1,c_1,c_2)-N(\psi_2-v_1,c_1,c_2)\right)\left((\phi_2-u_1)-(\psi_2-v_1)\right)\,dS.
\end{aligned}
$$

Analogously to Lemma 7.1, we can prove that $(N(a,c_1,c_2)-N(b,c_1,c_2))(a-b) \geq C_0(a-b)^2$ for fixed $x\in\Sigma$ and $a,b\in\mathbb{R}$. Together with the ellipticity ($\kappa(c_1)\geq k_1>0$ and $\sigma>0$), we obtain

$$
\|\nabla(u_1-v_1)\|^2_{0,2,\Omega_1} + \|\nabla(\phi_2-\psi_2)\|^2_{0,2,\Omega_2} \leq -C\|(u_1-v_1)-(\phi_2-\psi_2)\|^2_{0,2,\Sigma},
$$

hence, there are constants C_1, C_2, such that $u_1 - v_1 = C_1$ and $\phi_2 - \psi_2 = C_2$ almost everywhere. The condition $\phi_2-\psi_2 \in H^1_\Gamma(\Omega_2)$ then yields $C_2 = 0$, thus $\|(u_1-v_1)-(\phi_2-\psi_2)\|^2_{0,2,\Sigma} = 0$ yields $u_1 - v_1 = 0$ almost everywhere on Σ which implies $C_1 = 0$ as well. □

A first step towards the theoretical treatment of nonlinear boundary conditions of the structure indicated in (7.27) has been achieved by Y. S. Bhat and S. Moskow, see [BM06]. The authors consider a potential-equation on a cylindrical domain Ω when the nonlinear boundary condition is imposed on a part Γ of the lateral boundary:

$$
\left\{\begin{array}{rcll}
\Delta u &=& 0 & \text{in } \Omega,\\
-\frac{\partial u}{\partial\nu} &=& \lambda(y)\left(e^{\alpha(y)(u(y)-V(y))} - e^{-(1-\alpha(y))(u(y)-V(y))}\right) & \text{on } \Gamma,\\
-\frac{\partial u}{\partial\nu} &=& 0 & \text{on } \partial\Omega\setminus\Gamma.
\end{array}\right. \tag{7.29}
$$

Existence and uniqueness in $H^1(\Omega)$ are proved via minimization techniques for the energy functional

$$
E(v) := \int_\Omega |\nabla v|^2\,dx + \int_\Gamma F(x,v)\,d\sigma(x),
$$

where F denotes a primitive for the nonlinear boundary term and satisfies $\frac{\partial^2}{\partial v^2} F(x,v) \geq C > 0$. A second-order Taylor series expansion then yields

$$F(x, v+w) + F(x, v-w) - 2F(x,v) = \frac{\partial^2}{\partial v^2} F(x,\xi) w^2 \geq C w^2$$

and from this the crucial estimate for the energy functional

$$E(v+w) + E(v-w) - 2E(v) \geq \int_\Omega |\nabla w|^2 \, dx + C \int_\Gamma w^2 \, d\sigma(x) \geq C \|w\|^2_{1,2,\Omega}$$

follows from a variant of the Poincaré-Friedrichs inequality.

Appendix

A.1 Notation and function spaces

We write $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_n > 0\}$. For $1 < p < \infty$ and $\theta \in (0,1)$, we denote by $(\cdot,\cdot)_{\theta,p}$ and $[\cdot,\cdot]_\theta$ the real and the complex interpolation functors, respectively. For an introduction to the theory of interpolation of Banach spaces, we refer to Chapter 1 of [Tri95]. The space of bounded linear operators between two Banach spaces X and Y is denoted by $L(X,Y)$ and we abbreviate $L(X) := L(X,X)$. We use the notations $X \overset{d}{\hookrightarrow} Y$ and $X \overset{c}{\hookrightarrow} Y$ for a continuous, dense and a continuous, compact embedding of X into Y, respectively. For a linear and closed operator A, we denote its domain, spectrum, and resolvent set by $D(A)$, $\sigma(A)$, and $\rho(A)$, respectively. For two nonnegative quantities A and B, we write $A \approx B$ if and only if there are positive constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$.

We now briefly introduce the function spaces under consideration. For scalar valued function spaces, we refer to the textbooks of H. Triebel ([Tri95], see also the paper [Tri02]) and P. Grisvard ([Gri85]), and for the case of vector valued function spaces to the papers of H. Amann ([Ama00], [Ama09], and [Ama97]). For the remainder of this section, let $(X, \|\cdot\|_X)$ denote a Banach space of class $\mathcal{HT}$, see the Definition 5.6, and $\Omega \subset \mathbb{R}^n$ be a domain. Further smoothness assumptions on Ω will be made precise when they are required.

Spaces of continuous functions.

Let $B(\Omega;X)$ and $BC(\Omega;X)$ denote the spaces of bounded and bounded, continuous mappings from Ω into X. For $k \in \mathbb{N}$ we denote by $C^k(\Omega;X)$ and $C^k(\overline{\Omega};X)$ the spaces of X-valued, k-times continuously differentiable functions on Ω and $\overline{\Omega}$, respectively. Further, $BUC^k(\overline{\Omega};X)$ denotes the space of the X-valued, k-times bounded, uniformly continuously differentiable functions on $\overline{\Omega}$ with its canonical norm $|\cdot|_{C^k(\overline{\Omega};X)}$.
For non-integer $s = [s] + \tilde{s}$ with integer part $[s] \in \mathbb{N}_0$ and $\tilde{s} \in (0,1)$, we define the Banach spaces of bounded, uniformly Hölder continuous functions $BUC^s(\Omega;X)$ on Ω of order s by

$$BUC^s(\Omega;X) := \left\{ u \in BUC^{[s]}(\Omega;X) \ : [\partial^\alpha u]_{C^{s-[s]}(\Omega;X)} < \infty, |\alpha| = [s] \right\},$$

where the seminorm $[\,\cdot\,]_{C^t(\Omega;X)}$ is defined for $t \in (0,1)$ by

$$[u]_{C^t(\Omega;X)} := \sup_{x,y \in \Omega} \frac{\|u(x) - u(y)\|_X}{|x-y|^t}.$$

We denote the norm on $BUC^s(\Omega;X)$ by

$$\|u\|_{C^s(\Omega;X)} := |u|_{C^{[s]}(\Omega;X)} + \sum_{|\alpha| \leq [s]} [\partial^\alpha u]_{C^{s-[s]}(\Omega;X)}.$$

Replacing Ω by $\overline{\Omega}$ we define the space $BUC^s(\overline{\Omega};X)$ of bounded, uniformly Hölder continuous functions on $\overline{\Omega}$ of order s. For these spaces we will often use the notation $C^{k,\alpha}(\overline{\Omega};X)$ when $[s] = k \in \mathbb{N}_0$ and $s - [s] = \alpha \in (0,1)$. The space of locally Hölder continuous functions $C^s(\Omega;X)$ is defined as the subspace of functions $u \in C^{[s]}(\Omega;X)$, such that for every $x \in \Omega$ there exists a neighbourhood $V \subset \Omega$ with $x \in V$ and $u|_V \in BUC^s(V;X)$.

Analogously, we define the space of locally Hölder continuous functions $C^s(\overline{\Omega};X)$ on $\overline{\Omega}$. Note that if Ω is bounded, there holds $BUC^s(\overline{\Omega};X) = C^s(\overline{\Omega};X)$ for $s \geq 0$. In addition, we set

$$C^\infty(\Omega;X) := \bigcap_{k\in\mathbb{N}_0} C^k(\Omega;X) \quad \text{and} \quad C_c^\infty(\Omega;X) := \{u \in C^\infty(\Omega;X) : \operatorname{supp}(u) \subset \Omega \text{ compact}\}.$$

By $\mathscr{S}(\mathbb{R}^n;X)$ we denote the Schwartz space of X-valued, smooth, rapidly decreasing functions and we define $\mathscr{S}(\mathbb{R}^n_+;X)$ as the space of all $u \in C^\infty(\mathbb{R}^n_+;X)$ such that

$$\sup_{x\in\mathbb{R}^n_+} \|(1+|x|^2)^{k/2}\partial^\alpha u(x)\|_X < \infty \quad (\alpha \in \mathbb{N}_0^n, k \in \mathbb{N}_0).$$

The space of tempered distributions will be denoted by $\mathscr{S}'(\mathbb{R}^n;X) := L(\mathscr{S}(\mathbb{R}^n;X),\mathbb{C})$. Similarly, we set $\mathscr{S}'(\mathbb{R}^n_+;X) := L(\mathscr{S}(\mathbb{R}^n_+;X),\mathbb{C})$.

L_p-spaces and Sobolev spaces of integer order.

For $1 \leq p < \infty$, we define the X-valued L_p-space on Ω by

$$L_p(\Omega;X) := \left\{ u : \Omega \to X \ : \ u \text{ strongly measurable, } \|u\|_{0,p,\Omega}^p := \int_\Omega \|u(x)\|_X^p \, dx < \infty \right\}.$$

For $p = \infty$, one defines $L_\infty(\Omega;X)$ with the usual modification. For $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$, we define the Sobolev space of order m on Ω by

$$W_p^m(\Omega;X) := \{u \in L_p(\Omega;X) \ : \ \partial^\alpha u \in L_p(\Omega;X), \ |\alpha| \leq m\},$$

where the derivative $\partial^\alpha u$ is to be understood in the sense of distributions. Endowed with the canonical norm

$$\|u\|_{m,p,\Omega} := \left(\sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{0,p,\Omega}^p \right)^{1/p},$$

$W_p^m(\Omega;X)$ is a Banach space. In the scalar case $X = \mathbb{R}$ or $X = \mathbb{C}$, we will write $L_p(\Omega)$ and $W_p^m(\Omega)$ instead of $L_p(\Omega;\mathbb{C})$ and $W_p^m(\Omega;\mathbb{C})$. For the following definitions, we restrict ourselves to the cases that Ω denotes either $\mathbb{R}^n$, $\mathbb{R}^n_+$, an open k-corner $\mathbb{K}^n_k := \mathbb{R}^{n-k} \times (0,\infty)^k$, or a bounded Lipschitz-domain. For these classes of domains, standard interpolation properties hold for the L_p-Sobolev spaces.

Sobolev spaces of non-integer order.

Using real and complex interpolation, we define the following scales of X-valued function spaces: for $1 < p < \infty$ and $s > 0$ the Bessel-potential space

$$H_p^s(\Omega;X) := \left[L_p(\Omega;X), W_p^{[s]+1}(\Omega;X) \right]_{\frac{s}{1+[s]}},$$

for $1 \leq p,q < \infty$ and $s > 0$ the Besov space

$$B_{p,q}^s(\Omega;X) := \left(L_p(\Omega;X), W_p^{[s]+1}(\Omega;X) \right)_{\frac{s}{1+[s]},q},$$

and for $1 \leq p < \infty$ and $s > 0$ the Slobodeckii space

$$W_p^s(\Omega;X) := \begin{cases} B_{p,p}^s(\Omega;X), & \text{if } s \notin \mathbb{N}, \\ W_p^m(\Omega;X), & \text{if } s = m \in \mathbb{N}. \end{cases}$$

Commonly, these spaces are first defined on the whole space $\mathbb{R}^n$ using the Fourier-transform in case of the Bessel-potential spaces (see [Tri95], Section 2.3.2) and by a Paley-Littlewood decomposition in case of the Besov spaces (see [Tri95], Section 2.3.2), respectively.

The corresponding spaces on domains are then obtained by restriction. However, for the domains Ω under consideration, these approaches lead to the same spaces with equivalent norms because of the corresponding interpolation properties (cf. [Tri02], Theorem 2.13 and [Ama09], Theorem 4.4.1). We do not make the norms on $H_p^s(\Omega)$ and $B_{pq}^s(\Omega)$ explicit here, but note that $H_p^k(\Omega;X) = W_p^k(\Omega;X)$ for $k \in \mathbb{N}$. For the Slobodeckii spaces, there is a more direct characterization by an equivalent norm (see e.g. [Ama00], Section 2): For $s = [s] + \tilde{s}$ and $1 \le p < \infty$, we define the expressions

$$[u]_{\tilde{s},p,\Omega} := \left(\int_\Omega \int_\Omega \frac{\|u(x) - u(y)\|_X^p}{|x-y|^{n+p\tilde{s}}}\, dx\, dy \right)^{\frac{1}{p}} \text{ and } \|u\|_{s,p,\Omega} := \left(\|u\|_{[s],p,\Omega}^p + \sum_{|\alpha|=[s]} [\partial^\alpha u]_{\tilde{s},p,\Omega} \right)^{\frac{1}{p}}.$$

Then $W_p^s(\Omega;X)$ is characterized as $W_p^s(\Omega;X) = \left\{ u \in W_p^{[s]}(\Omega;X) \ : \ \|u\|_{s,p,\Omega} < \infty \right\}$. We further abbreviate the closure of the X-valued, smooth functions with compact support in $W_p^s(\Omega;X)$:

$$\mathring{W}_p^s(\Omega;X) := \overline{C_c^\infty(\Omega;X)}^{W_p^s(\Omega;X)}.$$

For the case $s = m \in \mathbb{N}$ we will write $W_{p,0}^m(\Omega;X)$ and $H_{p,0}^m(\Omega;X)$ for the closure of $C_c^\infty(\Omega;X)$ in $W_p^m(\Omega;X)$ and $H_p^m(\Omega;X)$, respectively. These spaces generalize the notion that a function vanishes on the boundary of a domain Ω. We conclude this section with a summary of useful embedding and interpolation results.

Theorem A.1
Let $1 < p, q < \infty$ and Ω denote either $\mathbb{R}^n$, $\mathbb{R}_+^n$, $\mathbb{K}_k^n$, or a bounded Lipschitz-domain.

(i) Let $s \le t$, $p \le q$, and $s - \frac{n}{p} \le t - \frac{n}{q}$. Then $W_q^t(\Omega;X) \hookrightarrow W_p^s(\Omega;X)$.

(ii) If $s - \frac{n}{p} \ge t > 0$, then $W_p^s(\Omega;X) \hookrightarrow C^t(\Omega;X)$.

If the above inequalities hold strictly, Ω is bounded, and X is finite-dimensional, then the embeddings are compact.

(iii) Assume in addition that $0 \le s_0 < s_1$, $\theta \in (0,1)$, and set $s_\theta := (1-\theta)s_0 + \theta s_1$. Then

$$\begin{aligned} \left(H_p^{s_0}(\Omega;X), H_p^{s_1}(\Omega;X)\right)_{\theta,p} &= B_{p,p}^{s_\theta}(\Omega;X), \quad \left[H_p^{s_0}(\Omega;X), H_p^{s_1}(\Omega;X)\right]_{\theta,p} = H_p^{s_\theta}(\Omega;X), \\ \left(B_{p,p}^{s_0}(\Omega;X), B_{p,p}^{s_1}(\Omega;X)\right)_{\theta,p} &= B_{p,p}^{s_\theta}(\Omega;X), \quad \left[B_{p,p}^{s_0}(\Omega;X), B_{p,p}^{s_1}(\Omega;X)\right]_{\theta,p} = B_{p,p}^{s_\theta}(\Omega;X), \end{aligned}$$

where $s_0 = 0$ has to be excluded in case of the Besov spaces.

Proof. We refer to [Ama00] and [Ama09] for the cases $\mathbb{R}^n$, $\mathbb{R}_+^n$, and $\mathbb{K}_k^n$. The case of a bounded Lipschitz-domain is considered in [Tri02], Sections 2.3 and 2.5. □

A.2 Traces

Whereas for smooth functions the trace on a hyperplane or on the boundary of a domain is simply the restriction of the function, this notion needs to be extended to Sobolev functions in terms of a trace operator. The following theorem summarizes known results for the scalar case $X = \mathbb{C}$ in this context.

Theorem A.2
Let $1 < p < \infty$, $s = [s] + \tilde{s} \in \left(\frac{1}{p}, \infty\right)$, and assume that $s - \frac{1}{p} \notin \mathbb{N}_0$.

(i) The mapping

$$u \mapsto \left(\gamma_{0,n} u,\ \gamma_{0,n} \partial_n u, \ldots, \gamma_{0,n} \partial_n^{[s]} u \right)$$

which is defined for $u \in C^\infty(\mathbb{R}^n)$ by $(\gamma_{0,n}u)(x_1, \ldots, x_{n-1}) := u(x_1, \ldots, x_{n-1}, 0)$, extends to a continuous linear operator from $W_p^s(\mathbb{R}^n)$ to $\prod_{j=0}^{[s]} W_p^{s-j-1/p}(\mathbb{R}^{n-1})$. This trace operator is surjective and has a continuous right inverse which is independent of p. The same assertion holds true when $\mathbb{R}^n$ is replaced by $\mathbb{R}_+^n$.

(ii) Let $k \in \mathbb{N}_0$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^{k,1}$-boundary Γ. Assume additionally that $\frac{1}{p} < s \leq k+1$. Then the mapping $u \mapsto \left(\gamma_\Gamma u,\ \gamma_\Gamma \partial_\nu u, \ldots, \gamma_\Gamma \partial_\nu^{[s]} u\right)$ which is defined for $u \in C^{k,1}(\overline{\Omega})$ by $\gamma_\Gamma u := u|_\Gamma$, where ∂_ν denotes the derivative in direction of the outward normal vector ν, extends to a continuous linear operator from $W_p^s(\Omega)$ to $\prod_{j=0}^{[s]} W_p^{s-j-1/p}(\Gamma)$. This mapping is surjective and has a continuous right inverse which is independent of p.

Proof. This is a summary of [Gri85], Theorems 1.5.1.1–1.5.1.3. □

An analogous trace theorem is available for the space $W_p^s(\Omega)$ when $s > 1/p$ and Ω is a bounded $C^{k,\alpha}$-domain with $k+\alpha > r$, see [Sch11], Theorem 3.4. Besides the cases of the half space and a bounded domain with sufficiently smooth boundary, we are interested in the cases of rectangular domains which have a piecewise smooth boundary apart from corner points and wedges. Here, for $s > 2/p$, the trace of an element from $W_p^s(\Omega)$ which is defined on a surface Γ_j also possesses a trace on the edge which is formed by Γ_j and an adjacent surface Γ_k. This results in the fact that the trace operator

$$\begin{aligned} \gamma_{\partial\Omega} : \quad W_p^s(\Omega) &\to \prod_{j=1}^{2n} \left(\prod_{k=0}^{m-1} W_p^{s-k-1/p}(\Gamma_j) \right), \\ u &\mapsto \left(\gamma_{0,1} u, \gamma_{0,1} \partial_{\nu_1} u, \ldots, \gamma_{0,1} \partial_{\nu_1}^{m-1} u, \gamma_{0,2} u, \ldots, \gamma_{0,2n} \partial_{\nu_{2n}}^{m-1} u \right), \end{aligned} \tag{A.1}$$

where ν_j denotes the outward normal vector for the surface Γ_j, is no longer surjective, unless compatibility conditions on the edges and corner points are satisfied. Roughly speaking, normal and tangential derivatives of the boundary values have to match each other. However, the continuity of the mapping $\gamma_{\partial\Omega}$ is preserved in the polygonal or polyhedral case, see Corollary 4.3 in [BDM03]. For the special situation of a rectangular domain, we cite the following trace theorem:

Theorem A.3 ([BDM03], Corollary 6.11)
Let $1 < p < \infty$, $r > 1/p$, and $m \in \mathbb{N}$ such that $0 \leq m-1 < r-1/p$. Assume that $r - 2/p \notin \mathbb{N}_0$. Then a tuple $G = (G^1, \ldots, G^6)$ with $G^j = (g_0^j, \ldots, g_{m-1}^j)$ and $g_l^j \in W_p^{r-l-1/p}(\Gamma_j)$ $(j = 1, \ldots, 6)$ is given as a trace

$$G = (\gamma_{0,1} u, \gamma_{0,1} \partial_1 u, \ldots, \gamma_{0,1} \partial_1^{m-1} u, \gamma_{0,2} u, \ldots, \gamma_{0,6} \partial_6^{m-1} u)$$

of an element $u \in W_p^r(\Omega)$ if and only if for all $1 \leq l \leq 12$ and for all integers $0 \leq n \leq 2m-2$ with $n < r - 2/p$ it holds that

$$\partial_{n_{l_-}}^{n-k} g_k^{j(l_-)} = \partial_{n_{l_+}}^{k} g_{n-k}^{j(l_+)} \quad \text{on } \Gamma_{l_+} \cap \Gamma_{l_-} \quad (0 \leq k, n-k \leq m-1),$$

where $n_{l_\pm}$ denotes the outward normal of the surfaces $\Gamma_{l_\pm}$ which form the edge l. In this case an estimate of the form

$$\|u\|_{r,p,\Omega} \leq C \left(\sum_{j=1}^{6} \sum_{k=0}^{m-1} \|g_k^j\|_{r-k-1/p,p,\Gamma_j} \right) \tag{A.2}$$

is satisfied.

Analogous results have been proved for intervals ($n = 1$), rectangles ($n = 2$), and more general polyhedra ($n = 3$), see [BDM03], Theorem 3.6, Corollary 5.9, and Theorem 6.5.

Example A.4
For the particular case of a paraxial three-dimensional cube we have six surfaces Γ_j, where we either prescribe a Dirichlet or a Neumann boundary condition. We wish to find an element $w \in W_p^2(\Omega)$, which corrects all inhomogeneous Dirichlet boundary values $g_0^{(j)} \in W_p^{2-1/p}(\Gamma_j)$ to 0 and leaves the Neumann boundary values unchanged, i.e.,

$$\left(\gamma_{0,1} w, \gamma_{0,1} \frac{\partial w}{\partial \nu}, \ldots, \gamma_{0,6} w, \gamma_{0,6} \frac{\partial w}{\partial \nu} \right) = \left(g_0^{(1)}, 0, g_0^{(2)}, 0, \ldots, g_0^{(6)}, 0 \right).$$

Here, we have set $g_1^{(j)} = 0$ in case of a Neumann condition on Γ_j. Due to [BDM03], Corollary 6.11 (with $r = 2$, $m = 2$, and $p > 2$) such an element exists if and only if the conditions

$$\begin{aligned} \gamma_{0,j} g_0^{(l)} &= \gamma_{0,l} g_0^{(j)} \quad \text{on } \Gamma_j \cap \Gamma_l \quad (0 \le j, l \le 6), \\ \gamma_{0,j} \partial_n g_0^{(l)} &= 0 \qquad\quad \text{on } \Gamma_j \cap \Gamma_l \quad (0 \le j, l \le 6), \end{aligned}$$

are satisfied on the twelve edges of Ω. In this case the estimate (A.2) *reads*

$$\|w\|_{2,p,\Omega} \le C \sum_{j=1}^{6} \|g_0^{(j)}\|_{2-1/p,p,\Gamma_j}.$$

A.3 Calculus inequalities

Theorem A.5 (Gagliardo-Nirenberg estimates)

(i) Let $m \in \mathbb{N}$ and $1 \le p, q, r \le \infty$. Further, let $\beta \in \mathbb{N}_0^n$ with $0 \le |\beta| \le m-1$, $\theta := \frac{|\beta|}{m} \in [0,1)$, and

$$\frac{1}{r} = \frac{|\beta|}{n} + \theta\left(\frac{1}{p} - \frac{m}{n}\right) + (1-\theta)\frac{1}{q}$$

be valid. Then for all $u \in W_p^m(\mathbb{R}^n) \cap L_q(\mathbb{R}^n)$ the following estimate holds:

$$\|\mathcal{D}^\beta u\|_{0,r,\mathbb{R}^n_+} \le C \|u\|^{\theta}_{m,p,\mathbb{R}^n_+} \|u\|^{1-\theta}_{0,q,\mathbb{R}^n_+}. \tag{A.3}$$

In particular, for $p \le r < \frac{np}{n-mp}$ and $\theta := \frac{n}{m}\left(\frac{1}{p} - \frac{1}{r}\right)$, there holds

$$\|u\|_{0,r,\mathbb{R}^n} \le C\|u\|^{\theta}_{m,p,\mathbb{R}^n} \|u\|^{1-\theta}_{0,p,\mathbb{R}^n} \le C\|u\|_{m,p,\mathbb{R}^n} \quad (u \in W_p^m(\mathbb{R}^n)). \tag{A.4}$$

(ii) Let $0 \le \theta \le 1, 1 \le p_0 < \infty, 1 < p_1, r < \infty$, and $0 \le s_0, s_1, t \le m + l$ with $s_0 \ne s_1$. Further, assume that $s_0 = 0$ for $p_0 = 1$, $\theta > 0$ for $s_0 = 0$, and $\theta < 1$ for $s_1 = 0$. If $\frac{1}{r} \le \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and

$$t - \frac{n}{r} < (1-\theta)\left(s_0 - \frac{n}{p_0}\right) + \theta\left(s_1 - \frac{n}{p_1}\right),$$

then for all $u \in W_{p_0}^{s_0}(\mathbb{R}^n) \cap W_{p_1}^{s_1}(\mathbb{R}^n)$, the following estimate is satisfied:

$$\|u\|_{t,r,\mathbb{R}^n} \le C\|u\|^{1-\theta}_{s_0,p_0,\mathbb{R}^n} \|u\|^{\theta}_{s_1,p_1,\mathbb{R}^n}. \tag{A.5}$$

(iii) Let Ω be a Lipschitz domain with minimally smooth boundary. Then the assertions of (i) and (ii) remain valid for $1 < p < \infty$.

Proof. (i) is stated in [Zei90], Appendix 54(i) (p. 1033). (ii) is proved in [Ama84], Proposition 4.1, even in the case for a uniformly regular domain with a $C^{m,l}$-boundary. Especially this includes the case of bounded $C^{m,l}$-domains and the half space $\mathbb{R}^n_+$. (iii) For a domain Ω with minimally smooth boundary $\partial\Omega$, we use the universal extension operator constructed in [Ste70], Theorem 3.5:

$$E : W_p^s(\Omega) \to W_p^s(\mathbb{R}^n) \quad (s > 0, 1 < p < \infty).$$

For an even more general extension operator, including the cases $0 < p \le 1$ and $p = \infty$, see [Ry99], Theorem 4.1 or [Tri02], Theorem 2.11. Then we can again follow the proof of Proposition 4.1 in [Ama84]: Using the continuity of the restriction and extension operator, we find for all $u \in W_{p_0}^{s_0}(\Omega) \cap W_{p_1}^{s_1}(\Omega)$:

$$\|u\|_{t,r,\Omega} \le C\|Eu\|_{t,p,\mathbb{R}^n} \le C\|Eu\|^{1-\theta}_{s_0,p_0,\mathbb{R}^n} \|Eu\|^{\theta}_{s_1,p_1,\mathbb{R}^n} \le C\|u\|^{1-\theta}_{s_0,p_0,\Omega} \|u\|^{\theta}_{s_1,p_1,\Omega}.$$

Note that the embedding $W_r^s(\Omega) \hookrightarrow W_p^t(\Omega)$ for $s - \frac{n}{r} > t - \frac{n}{p}$, which is now required to prove the estimate in full generality, remains valid in the case that Ω is a Lipschitz domain (see [Tri02], Section 2.3). □

Lemma A.6 (Interpolation inequalities)
Let $1 < p < \infty$.

(i) Let Ω be $\mathbb{R}^n$ or a subset of $\mathbb{R}^n$ which admits the continuation property and assume that $0 \le r < s < t$. Then there exists a constant $C = C(\Omega, p, r, s, t)$ such that for all $u \in W_p^t(\Omega)$ and $\varepsilon > 0$ the following inequality is satisfied:

$$\|u\|_{s,p,\Omega} \le \varepsilon \|u\|_{t,p,\Omega} + C\varepsilon^{-\frac{s-r}{t-s}} \|u\|_{r,p,\Omega}. \tag{A.6}$$

(ii) Replacing ε by $\rho^{\theta-1}$ with $\theta = \frac{s-r}{t-r}$ in (A.6), it holds

$$\rho^{1-\theta}\|u\|_{s,p,\Omega} \le \|u\|_{t,p,\Omega} + C\rho\|u\|_{r,p,\Omega} \quad (u \in W_p^t(\Omega),\ \rho \ge 1). \tag{A.7}$$

(iii) Let $1 < p < \infty$, $0 \le \sigma_0 < 1/p < \sigma_1$, and let Ω be $\mathbb{R}^n, \mathbb{R}^n_+$, or a bounded $C^{k,\alpha}$-domain with $k + \alpha > \max(1, \sigma_1)$. Set $\theta := \frac{1/p - \sigma_0}{\sigma_1 - \sigma_0}$ and let $\rho \ge 1$. Then for all $u \in W_p^{\sigma_1}(\Omega)$, there holds

$$\rho^{1-\theta}\|\gamma_0 u\|_{0,p,\partial\Omega} \le C\left(\|u\|_{\sigma_1,p,\Omega} + \rho\|u\|_{\sigma_0,p,\Omega}\right). \tag{A.8}$$

Proof. (i) is stated in [Gri85], Theorem 1.4.3.3 and (ii) is an immediate consequence. For (iii) in the case $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, note that by [Tri95], Theorems 2.9.3 (c) and 4.7.1 (c), respectively, we have the continuity of $\gamma_0 : B_{p,1}^{1/p}(\Omega) \to L_p(\partial\Omega)$. For the remaining case of a $C^{k,\alpha}$-domain, we refer to Theorem 3.4 in [Sch11]. By the choice of θ, we find

$$B_{p,1}^{1/p}(\Omega) = (W_p^{\sigma_0}(\Omega), W_p^{\sigma_1}(\Omega))_{\theta,1},$$

and since real interpolation is exact, we deduce

$$\rho^{1-\theta}\|\gamma_0 u\|_{0,p,\partial\Omega} \le C\left(\rho\|u\|_{\sigma_0,p,\Omega}\right)^{1-\theta} \|u\|^{\theta}_{\sigma_1,p,\Omega}.$$

An application of Young's inequality with $p := \frac{1}{1-\theta}$ and $q := \frac{1}{\theta}$ finishes the proof. □

Lemma A.7 (Poincaré inequalities)
Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain.

(i) For $u \in W_{p,0}^1(\Omega)$ the following inequality is satisfied with a constant $C > 0$ which depends on Ω, p, and n:

$$\|u\|_{0,p,\Omega} \le C\|\nabla u\|_{0,p,\Omega} \tag{A.9}$$

This inequality remains valid if Ω is only bounded in one direction or if Ω is bounded and u belongs to $H_\Gamma^1(\Omega)$, i.e., u vanishes on a subset $\Gamma \subset \partial\Omega$ with $\lambda_{n-1}(\Gamma) > 0$.

(ii) If in addition Ω is a bounded Lipschitz-domain, then the inequality

$$\left\| u - \left(\tfrac{1}{|\Omega|} \int_\Omega u \, dx \right) \right\|_{0,p,\Omega} \le C\|\nabla u\|_{0,p,\Omega} \tag{A.10}$$

is satisfied for $u \in W_p^1(\Omega)$ with a constant $C > 0$ which depends on Ω, p, and n.

Proof. The assertions can be found in [GT77] (p. 164). For the addendum that the inequality remains valid, if $u \in W_{p,\Gamma}^1(\Omega)$, one can closely follow the idea of the proof of (ii): Suppose there is a sequence $(u_k)_{k\in\mathbb{N}} \subset W_{p,\Gamma}^1(\Omega)$ such that $\|u_k\|_{0,p,\Omega} = 1$ and $\|\nabla u_k\|_{0,p,\Omega} \le \frac{1}{k}$. Then u_k is bounded in $W_{p,\Gamma}^1(\Omega)$ and the latter space is reflexive for $1 < p < \infty$. Hence, there exists a subsequence u_{k_l} and $v \in W_{p,\Gamma}^1(\Omega)$, such that $u_{k_l} \rightharpoonup v$ weakly in $W_p^1(\Omega)$ for $l \to \infty$. In particular, $\nabla u_{k_l} \rightharpoonup \nabla v$ weakly in $L_p(\Omega)$. Since $\|\nabla u_{k_l}\|_{0,p,\Omega} \to 0$, we have $\nabla v = 0$. On the other hand, Rellich's Theorem yields that $u_{k_l} \to v$ strongly in $L_p(\Omega)$ and this implies $\|v\|_{0,p,\Omega} = 1$. But $\nabla v = 0$ yields that v is constant on Ω and then $v \in W_{p,\Gamma}^1(\Omega)$ implies $v = 0$, a contradiction. □

A.4 Fixed point theorems

Theorem A.8 (Schauder fixed point theorem)
Let X be a Banach space.

(i) Let $S \subset X$ be a nonempty, compact, convex set and let $T : S \to S$ be a continuous mapping of S into itself. Then T has a fixed point $x \in S$.

(ii) Let $S \subset X$ be a nonempty, convex, closed set and let $T : S \to S$ be a continuous mapping of S into itself. In addition, assume that the image $T(S)$ is precompact, i.e., for every sequence $(x_k)_{k\in\mathbb{N}}$ in the bounded set S, the sequence $(T(x_k))_{k\in\mathbb{N}} \subset S$ possesses a convergent subsequence. Then T has a fixed point $x \in S$.

Proof. See [GT77], Theorem 11.1 and Corollary 11.2. □

Theorem A.9 (Leray-Schauder principle)
Let X be a Banach space, $a, b \in \mathbb{R}$ with $a < b$, and $T : [a, b] \times X \to X$ a mapping with the following properties:

(i) $T(\delta, x)$ is defined for all $\delta \in [a, b]$ and $x \in X$.

(ii) For any fixed $\delta \in [a, b]$, $T(\delta, \cdot) : X \to X$ is continuous, i.e., for any $x_0 \in X$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in X$ with $\|x - x_0\|_X < \delta$: $\|T(\delta, x) - T(\delta, x_0)\|_X < \varepsilon$.

(iii) For x in bounded sets of X, $T(\cdot, x) : [a, b] \to X$ is uniformly continuous in the following sense: for any bounded set $X_0 \subset X$ and for any $\varepsilon > 0$, there exists $\eta > 0$ such that if $x \in X_0$, $\delta_1, \delta_2 \in [a, b]$ with $|\delta_1 - \delta_2| < \eta$, there holds $\|T(\delta_1, x) - T(\delta_2, x)\|_X < \varepsilon$.

(iv) For any fixed $\delta \in [a, b]$, the mapping $T(\delta, \cdot) : X \to X$ is compact, i.e., if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in X, then $(T(\delta, x_n))_{n\in\mathbb{N}}$ possesses a convergent subsequence.

(v) There exists a constant $M > 0$ such that every possible solution $x \in X$ of $T(\delta, x) = x$ for some $\delta \in [a, b]$ satisfies $\|x\|_X \leq M$.

(vi) The equation $T(a, x) = x$ has a unique solution in X.

Then there exists a solution $x \in X$ of the equation $T(b, x) = x$.

Proof. See [Fri84], Theorem 3 (p. 189). □

A.5 Hölder continuity of weak solutions

The question on sufficient conditions for the coefficients and data in the weak formulation of an elliptic equation such that the solution exists and is Hölder continuous have been studied rather extensively. We briefly present the necessary definitions and results which will be applied in this thesis.

Definition A.10
Let H be a real Hilbert space. A bilinear form $B : H \times H \to \mathbb{R}$ is called

(i) continuous if there exists $\Lambda \geq 0$ such that $|B(u, v)| \leq \Lambda \|u\|_H \|v\|_H$ for all $u, v \in H$.

(ii) coercive if there exists $\lambda > 0$ such that $B(u, u) \geq \lambda \|u\|_H^2$ for all $u \in H$.

The main tool to prove the existence of a weak solution is the following theorem.

Theorem A.11 (Lax-Milgram)
Let $B : H \times H \to \mathbb{R}$ be a bounded, coercive bilinear form on a Hilbert space H. Then for every bounded linear functional $F : H \to \mathbb{R}$, there exists a unique element $u \in H$ such that $B(u, v) = F(v)$ holds for all $v \in H$. In addition, there is a constant $C > 0$ such that the estimate

$$\|u\|_H \leq C\|F\|_{L(H,\mathbb{R})}$$

is satisfied.

Proof. See [GT77], Theorem 5.8 (p. 83). □

A.5.1 Neumann boundary conditions

Regarding the boundedness and Hölder continuity of weak solutions, we first restrict ourselves to elliptic equations of second order in divergence form with Neumann boundary conditions on a Lipschitz domain Ω:

$$\text{(A.11)} \qquad \begin{cases} -\operatorname{div}(A(x,u,\nabla u)) + a(x,u,\nabla u) &= f(x) \quad \text{in } \Omega, \\ A(z,u,\nabla u)\cdot\nu(z) &= g(z) \quad \text{on } \partial\Omega. \end{cases}$$

Here, $A : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are measurable functions and the vector $\nu(z)$ denotes the outer normal at a point $z \in \partial\Omega$ which exists almost everywhere for the class of Lipschitz domains. Note that an inhomogeneity $f : \Omega \to \mathbb{R}$ may also be included into a. We call a function $u \in W^1_{1,\mathrm{loc}}(\Omega)$ a weak solution to (A.11) if for all $\varphi \in C^\infty(\overline{\Omega})$

$$\text{(A.12)} \qquad \int_\Omega A(x,u,\nabla u)\nabla\varphi \, dx + \int_\Omega a(x,u,\nabla u)\varphi \, dx = \int_{\partial\Omega} g(z)\varphi \, dS(z) + \int_\Omega f\varphi \, dx.$$

This definition can be justified by means of integration by parts. Note that the definition implicitly includes the existence of the occurring integrals in (A.12). In applications this existence can often be ensured by imposing somewhat stronger assumptions than only $u \in W^1_{1,\mathrm{loc}}(\Omega)$. We call (A.12) the weak form of the equations (A.11). We now state the assumptions on A and a that will be sufficient for the Hölder continuity.

Assumption A.12
Let $m \in (1,\infty)$ and $A : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ as well as $a : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ be measurable functions. Assume that there exist constants ϑ and μ as well as functions ψ_k $(1 \leq k \leq 7)$ such that the following inequalities hold for almost all $x \in \Omega$, all $u \in \mathbb{R}$, and all $p \in \mathbb{R}^n$:

$$\text{(A.13a)} \qquad pA(x,u,p) \geq \vartheta|p|^m - \psi_1(x)|u|^m - \psi_2(x),$$

$$\text{(A.13b)} \qquad |A(x,u,p)| \leq \mu|p|^{m-1} + \psi_3(x)|u|^{m-1} + \psi_4(x),$$

$$\text{(A.13c)} \qquad |a(x,u,p)| \leq \psi_5(x)|p|^{m-1} + \psi_6(x)|u|^{m-1} + \psi_7(x).$$

For the functions ψ_k assume the following integrability conditions:

$$\psi_1,\psi_2,\psi_6,\psi_7 \in L_{\frac{N}{m}+\varepsilon}(\Omega), \quad \psi_3,\psi_4 \in L_{\frac{N}{m-1}}(\Omega), \quad \textit{and} \quad \psi_5 \in L_{N+\varepsilon}(\Omega)$$

for some $\varepsilon > 0$, if $m \leq N$. In the case $m < N$, we may relax these conditions to

$$\psi_1,\psi_2,\psi_6,\psi_7 \in L_1(\Omega), \quad \psi_3,\psi_4 \in L_{\frac{m}{m-1}}(\Omega), \quad \textit{and} \quad \psi_5 \in L_m(\Omega).$$

Boundedness and Hölder continuity in the interior of Ω under the assumptions stated above is known since a rather long period of time.

Theorem A.13 (Interior Hölder continuity)
Let $\Omega \subset \mathbb{R}^n$ be open, $\Omega' \subset \Omega$ be relatively compact in Ω, and assume that Assumption A.12 is satisfied. Then there exist $\alpha \in (0,1)$ and $C \geq 0$ (independent of ψ_2, ψ_4 and ψ_7) such that every weak solution $u \in W^1_m(\Omega)$ of (A.11) (with $f = g = 0$) belongs to $C^{0,\alpha}(\overline{\Omega}')$ with

$$\|u\|_{C^{0,\alpha}(\overline{\Omega}')} \leq C \left(\|u\|_{0,m,\Omega} + (\|\psi_4\| + \|\psi_7\|)^{\frac{1}{m-1}} + \|\psi_2\|^{\frac{1}{m}} \right).$$

Here, the norms of ψ_2, ψ_4, and ψ_7 are taken in the spaces given in Assumption A.12.

Proof. This is a survey of the main results from [Ser64], Sections 1, 4, and 5. □

This result on Hölder continuity was recently improved by R. Nittka in [Nit10]: Under the same assumptions on the data and coefficients one obtains the Hölder continuity up to the boundary of Ω. This assertion was proved using localization and reflection methods along the boundary, similar to those applied in Chapter 2.

Theorem A.14
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let Assumption A.12 be satisfied.

(i) Let $1 \leq q \leq \infty$ with $q > \frac{n-1}{m-1}$ if $m \leq N$. Then there exist $\alpha \in (0,1)$ and $C \geq 0$ (independent of ψ_2, ψ_4, and ψ_7) such that every weak solution $u \in W^1_m(\Omega)$ of

$$\begin{cases} -\operatorname{div}(A(x,u,\nabla u)) + a(x,u,\nabla u) &= 0 \quad \text{in } \Omega, \\ A(z,u,\nabla u)\cdot\nu(z) &= g(z) \quad \text{on } \partial\Omega, \end{cases}$$

with $g \in L^q(\partial\Omega)$ belongs to $C^{0,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \left(\|u\|_{0,m,\Omega} + (\|g\|_{0,q,\partial\Omega} + \|\psi_4\| + \|\psi_7\|)^{\frac{1}{m-1}} + \|\psi_2\|^{\frac{1}{m}} \right), \tag{A.14}$$

where the norms of ψ_2, ψ_4, and ψ_7 are taken in the spaces given in Assumption A.12.

(ii) Let $\varepsilon > 0$ and assume $f \in L_{\frac{n}{m+\varepsilon}}(\Omega)$, $g \in L_{\frac{n-1}{m-1+\varepsilon}}(\partial\Omega)$ if $m \leq N$, or $f \in L_1(\Omega)$ and $g \in L_1(\partial\Omega)$ if $m > N$. Then there exist $\alpha \in (0,1)$ and $C \geq 0$ such that every weak solution $u \in W^1_m(\Omega)$ of the Neumann problem

$$\begin{cases} -\operatorname{div}(A(x,u,\nabla u)) + a(x,u,\nabla u) &= f \quad \text{in } \Omega, \\ A(z,u,\nabla u)\cdot\nu(z) &= g(z) \quad \text{on } \partial\Omega, \end{cases}$$

belongs to $C^{0,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq C \left(\|u\|_{0,m,\Omega} + (\|f\| + \|g\|)^{\frac{1}{m-1}} + 1 \right), \tag{A.15}$$

where the norms of f and g have to be taken in the aforementioned spaces.

Proof. Partially, the result is contained in [Nit11], Proposition 3.6. For details and generalizations, we refer to the PhD-Thesis [Nit10], Theorem 3.1.5 for part (i) and Corollary 3.1.6 for part (ii). □

A.5.2 Dirichlet and mixed boundary conditions

For the important cases of the Hölder continuity of the solution to a linear equation with Dirichlet or mixed boundary conditions, we cite the following definition from [Dro00]

Definition A.15
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain such that the boundary $\partial\Omega$ admits a splitting of the form $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\lambda_{n-1}(\Gamma_D \cap \Gamma_N) = 0$ and $\lambda_{n-1}(\Gamma_D) > 0$. The boundary $\partial\Omega$ is called regular if the following condition is satisfied: For every $x_0 \in \partial\Omega$, there is an open neighbourhood $U_0 = U_0(x_0)$ and a mapping $J_0 : U_0 \to B := B(0,1) \subset \mathbb{R}^n$, which is Lipschitz continuous and bijective with Lipschitz continuous inverse $J_0^{-1} : B \to U_0$ (i.e., J_0 is a so-called bi-Lipschitz mapping) such that exactly one of the following cases holds:

(1) $U_0 \cap \Gamma_N = U_0 \cap \partial\Omega$, and J_0 satisfies

$$J_0(U_0 \cap \Omega) = B_+ := B \cap \mathbb{R}^n_+, \quad \text{and} \quad J_0(U_0 \cap \partial\Omega) = B_{n-1} := \{x \in B : x_n = 0\}, \tag{A.16}$$

(2) $U_0 \cap \Gamma_N = \emptyset$, and J_0 satisfies (A.16),

(3) $J_0(U_0 \cap \Omega) = B_{++} := \{x \in B(0,1) : \ x_n > 0, \ x_{n-1} > 0\}$ and the different types of boundary conditions are 'well-separated', i.e., $J_0(U_0 \cap \Gamma_D) = \{x \in \partial B_{++} : \ x_n = 0\}$ and $J_0(U_0 \cap \Gamma_N) = \{x \in \partial B_{++} : \ x_{n-1} = 0\}$.

Roughly speaking, the condition that $\partial\Omega$ is regular means that the different types of boundary conditions are well-separated. Note that we allow $\Gamma_D = \partial\Omega$ and $\Gamma_N = \emptyset$ and in this case, the geometric condition becomes redundant. The previous definition somewhat strengthens the definition of a regular set given by K. Gröger (see Definition 2 in [Grö89]) which plays an important role in the theory for weak solutions to boundary value problems with mixed boundary conditions. See also [Dro00] and [GM00], where these conditions are used to prove regularity results for weak solutions of problems with different types of boundary conditions. We will consider a linear equation of the following structure:

$$\left\{ \begin{array}{rcll} -\operatorname{div}(A(x)\nabla u) + \operatorname{div}(u\boldsymbol{v}) + bu &=& L & \text{in } \Omega, \\ u &=& g_D & \text{on } \Gamma_N, \\ A(z)\nabla u(z)\cdot\nu(z) + \lambda u(z) &=& g_N & \text{on } \Gamma_N, \end{array} \right. \tag{A.17}$$

where the assumptions on the occurring coefficients, the functional L, and the data will be made precise in Theorem A.17 below.

Definition A.16
Let $u_0 \in H^1(\Omega)$ with $\gamma_{\partial\Omega} u_0 = g_D$ on $\partial\Omega$. Then an element $u \in H^1(\Omega)$ is called a weak solution to (A.17) *if $w := u - u_0 \in H^1_{\Gamma_D}$ and w satisfies*

$$\begin{aligned} &\int_\Omega A(x)\nabla w\nabla\varphi + \int_{\Gamma_n} \lambda w\varphi \, dS - \int_\Omega w\boldsymbol{v}\nabla\varphi \, dx + \int_{\Gamma_n} w\varphi\boldsymbol{v}\nu \, dS + \int_\Omega bw\varphi \, dx \\ &= \langle L, \varphi\rangle_{((H^1(\Omega))',H^1(\Omega))} + \langle g_n, \varphi\rangle_{((H^{1/2}(\partial\Omega))',H^{1/2}(\partial\Omega))} - \int_\Omega A(x)\nabla u_0\nabla\varphi \, dx - \int_{\Gamma_n} \lambda u_0\varphi \, dS \\ &\qquad + \int_\Omega u_0\boldsymbol{v}\nabla\varphi \, dx - \int_{\Gamma_n} u_0\varphi\boldsymbol{v}\nu \, dS - \int_\Omega bu_0\varphi \, dx \quad (\varphi \in H^1_{\Gamma_D}(\Omega)). \end{aligned}$$

The theorem on the Hölder continuity of weak solutions now reads as follows:

Theorem A.17
Let $p > n$, $\frac{1}{p} + \frac{1}{q} = 1$, and consider the linear equation with mixed boundary conditions (A.17) *under the following assumptions:*

(i) $\Omega \subset \mathbb{R}^n$ *is a bounded Lipschitz domain with a regular boundary $\partial\Omega$ in the sense of Definition A.15 such that $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\lambda_{n-1}(\Gamma_D \cap \Gamma_N) = 0$, and $\lambda_{n-1}(\Gamma_D) > 0$.*

(ii) $A \in L_\infty(\Omega; \mathbb{R}^{n\times n})$ *and there exists $\alpha_A > 0$ such that $\xi^T A(x)\xi \geq \alpha_A |\xi|^2$ for almost all $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.*

(iii) $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ *is Lipschitz continuous,* $b \in L_{\frac{np}{n+p}}(\Omega)$, *and* $\frac{1}{2}\operatorname{div}(\boldsymbol{v})(x) + b(x) \geq 0$ *for almost every $x \in \Omega$.*

(iv) $\lambda \in L_{\frac{(n-1)p}{n}}(\Gamma_N)$ *and* $\frac{1}{2}\boldsymbol{v}\nu + \lambda \geq 0$ *a.e. on Γ_N.*

(v) $L \in (W_q^1(\Omega))'$, $g_D \in W_p^{1-1/p}(\partial\Omega)$, *and* $g_N \in \left(W_q^{1-1/q}(\partial\Omega)\right)'$.

Then there exists a unique weak solution $u \in W_p^1(\Omega)$ to (A.17) *in the sense of Definition A.16 and there exists a Hölder exponent $\kappa \in (0, 1 - \frac{n}{p}]$, depending only on Ω, α_A, $\|A\|_{L_\infty(\Omega;\mathbb{R}^{n\times n})}$, and p such that $u \in C^{0,\kappa}(\overline{\Omega})$. Furthermore, the norm $\|u\|_{C^{0,\kappa}(\overline{\Omega})}$ is bounded by a constant which depends on Ω, Γ_D, α_A, $\|A\|_{L_\infty(\Omega;\mathbb{R}^{n\times n})}$, p, the Lipschitz constant of $\boldsymbol{v}$, and the norms of all data and coefficients, taken in the aforementioned spaces.*

Proof. See [Dro00], Theorem 2.1. □

A.6 Fredholm theory

Definition A.18
Let X, Y be $\mathbb{K}$-Banach spaces and $T \in L(X, Y)$ be a continuous, linear operator.

(i) Denote by $N(T) := \{x \in X : Tx = 0\}$ the kernel of T. Obviously, $N(T)$ is a closed linear subspace of X.

(ii) Denote by $R(T) = T(X) := \{Tx : x \in X\}$ the range of T, a linear subspace of Y.

(iii) Let U be a linear subspace of Y. Define the corank of U in Y by $\operatorname{codim}(U) := \dim(Y/U)$, *where Y/U denotes the quotient space.*

(iv) T is called a Fredholm operator if $\dim(N(T))$ *and* $\operatorname{codim}(R(T))$ *are finite numbers. The set of all linear Fredholm operators $T : X \to Y$ is denoted by $F(X, Y)$.*

(v) If $T \in F(X, Y)$ is a Fredholm operator, the number $\operatorname{ind}(T) := \operatorname{codim}(U) - \operatorname{codim}(R(T))$ *is called the index of T.*

(vi) The linear operator T is called compact if for any bounded set $M \subset X$, $T(M) \subset Y$ is precompact, i.e., $\overline{T(M)}$ is compact. Equivalently, for any bounded sequence $(x_n)_{n\in\mathbb{N}} \subset X$ the image sequence $(Tx_n)_{n\in\mathbb{N}} \subset Y$ contains a convergent subsequence. The set of all compact linear operators $T : X \to Y$ is denoted by $K(X, Y)$.

Lemma A.19 (Properties of Fredholm operators)
Let $T \in F(X,Y)$ be a Fredholm operator. Then the following assertions hold true:

(i) *If* $\operatorname{ind}(T) = 0$ *and* $N(T) = \{0\}$*, then for all* $y \in Y$*, the equation* $Tx = y$ *admits a unique solution* $x \in X$*.*

(ii) $R(T)$ *is closed.*

(iii) *Let* $C \in K(X,Y)$ *be a compact linear operator. Then* $T + C$ *is also a Fredholm operator of index* $\operatorname{ind}(T + C) = \operatorname{ind}(T)$*. Hence, the Fredholm property and the index are invariant under compact perturbations.*

(iv) *Let* $S \in L(X,Y)$ *be a continuous linear operator with* $\|S\| < \varepsilon$*, where* $\varepsilon > 0$ *may depend on* T*. Then* $T + S$ *is a Fredholm operator of index* $\operatorname{ind}(T + S) = \operatorname{ind}(T)$*, too. Hence,* $F(X,Y)$ *is an open subset of* $L(X,Y)$*.*

(v) *Let* $T \in K(X,Y)$*. Then for all* $\lambda \in \mathbb{K}$*, the operator* $I - \lambda T$ *is a Fredholm operator of index* 0*.*

(vi) *Let* $T \in L(X,Y)$ *be an isomorphism and* $C \in K(X,Y)$*. Then* $T + C$ *is a Fredholm operator of index* 0*, too.*

(vii) *Let* Z *be another Banach space and* $T \in F(X,Y)$ *as well as* $S \in F(Y,Z)$ *be Fredholm operators. Then* $ST \in F(X,Z)$ *with index* $\operatorname{ind}(ST) = \operatorname{ind}(S) + \operatorname{ind}(T)$*.*

Proof. This is a summary of results of the Chapters 5.4, 5.5, and 5.8 of the textbook [Zei95]. □

Deutsche Zusammenfassung

Ziel der vorliegenden Dissertation ist die Bereitstellung vielseitig einsetzbarer Resultate zur Behandlung elliptisch-parabolischer Systeme, wie sie beispielsweise in Lithium-Ionen Batteriemodellen auftreten. Häufig weisen derartige Modellgleichungen die Struktur eines gekoppelten elliptisch-parabolischen Systems der allgemeinen Form

$$(1) \qquad \begin{cases} -\nabla \cdot (\kappa \nabla \phi + \lambda \nabla c) &= F & \text{in } (0,T)\times\Omega, \\ \frac{\partial c}{\partial t} - \nabla \cdot (\alpha \nabla \phi + \beta \nabla c) &= G & \text{in } (0,T)\times\Omega, \end{cases}$$

in einem Gebiet $\Omega \subset \mathbb{R}^n$ auf, wobei zusätzlich nichtlineare Randbedingungen auftreten können. In (1) können sowohl die Koeffizienten als auch die Funktionen F und G von den gesuchten Größen ϕ und c abhängen, was zu einer nichtlinearen Kopplung der Gleichungen führt.

Bezüglich der Geometrie des Gebiets Ω werden zwei anwendungsorientierte Situationen betrachtet: einerseits der Fall eines beschränkten $C^{1,1}$-Gebiets, welches durch geschlossenen $C^{1,1}$-Flächen Σ_i in $M+1$ Teilgebiete aufgeteilt ist. Andererseits werden entsprechende Resultate für eine achsparalleles rechteckförmiges Gebiet, das durch parallele Flächen Σ_i in mehrere rechteckige Teilgebiete unterteilt ist, erzielt. Die notwendigen Ergebnisse zu linearen Gleichungen im $C^{1,1}$-Gebiet sind durch die Behandlung eines entsprechenden Modellproblems im Halbraum $\mathbb{R}^n_+$ zugänglich. Im Fall des rechteckigen Gebiets wird hingegen eine sogenannte k-Ecke der Form $\mathbb{K}^n_k := \mathbb{R}^{n-k} \times (0,\infty)^k$ mit $1 \leq k \leq n$ der Ausgangspunkt für die Modellproblembetrachtungen sein.

Hauptgegenstand dieser Arbeit wird die folgende Variante des Systems (1) mit einer parabolischen und zwei elliptischen Gleichungen sowie linearen Randbedingungen sein

$$(2) \qquad \begin{cases} -\nabla \cdot (\kappa_1(u_3) \nabla u_1) &= F(u_2 - u_1, u_3) & \text{in } (0,T)\times\Omega, \\ -\nabla \cdot (\kappa_2(u_3) \nabla u_2) &= -F(u_2 - u_1, u_3) & \text{in } (0,T)\times\Omega, \\ \frac{\partial u_3}{\partial t} - \nabla \cdot (\kappa_3 \nabla u_3) &= G(u_1, u_2, u_3) & \text{in } (0,T)\times\Omega. \end{cases}$$

Die nichtlineare Kopplung zwischen den Gleichungen wird hier im Wesentlichen durch zwei nichtlineare Funktionen $F : \mathbb{R}\times(0,\infty)\times\Omega \to \mathbb{R}$ und $G : \mathbb{R}^2\times(0,\infty)\times\Omega \to \mathbb{R}$ hervorgerufen, an die sehr allgemeine Anforderungen gestellt werden. Für eine präzise Formulierung der Annahmen sei hier auf Assumption 6.2 verwiesen. Die Anwendungen in Kapitel 7 zeigen, dass die geforderten Eigenschaften nicht notwendigerweise polynomiale Wachstumsbedingungen an F und G zur Folge haben, wie sie häufig in der Theorie semi- oder quasilinearer elliptischer Gleichungen auftreten. Ein Beispiel für derartige 'growth-conditions' findet sich in Kapitel 8 von [GT77].

Die wesentlichen mathematischen Herausforderungen bei der Betrachtung des Systems (2) lassen sich folgendermaßen zusammenfassen:

- Wir betrachten ein nichtlinear gekoppeltes elliptisch-parabolisches System bei dem die Kopplung sowohl über nichtlineare Funktionen auf der rechten Seite der Gleichungen als auch über die Koeffizienten vermittelt wird und somit insgesamt quasilinear ist.

- Die nichtlinearen Funktionen erfüllen im Allgemeinen keine milden Wachstumsbedingungen. Im Gegenteil: Bezüglich der einzelnen Variablen können sogar exponentielles Wachstum oder singuläres Verhalten (z.B. für $u_3 \to 0$) auftreten. Bekannte Resultate aus der Theorie quasilinearer elliptischer Gleichungen werden aufgrund dessen nicht direkt anwendbar sein.

- Für die Koeffizienten κ_i in (2) wird nur eine $C^{1,\tau}$-Regularität von κ_1 und κ_2 bezüglich c sowie von κ_3 bezüglich x in jedem Teilgebiet Ω_i angenommen. Dies führt zu möglichen Unstetigkeiten an den Grenzflächen Σ_i. Eine wesentliche Annahme in der L_p-Theorie elliptischer und parameter-elliptischer Gleichungen ist jedoch die Stetigkeit der Koeffizienten des Hauptteils des Operators (cf. [AV63] or [ADF97]). Folglich können die Ergebnisse aus der L_p-Theorie ebenfalls nicht sofort auf ganz Ω verwendet werden.

- Ein weiteres Problem aufgrund der geringeren Glattheit des Randes tritt im Fall der rechteckigen Geometrie von Ω auf. In diesem Fall können die Form und die Ordnung der Randbedingungen von Fläche zu Fläche variieren.

Trotz der oben geschilderten Schwierigkeiten wird das Hauptergebnis, Theorem 6.10, die Existenz einer zeitlich lokalen Lösung des Systems (2) in den L_p-Sobolevräumen

$$(u_1, u_2, u_3) \in C([0,T]; W_p^2(\Omega)) \times C([0,T]; W_p^2(\Omega)) \times W_p^{1,2}((0,T)\times\Omega)$$

unter natürlichen Voraussetzungen an die Daten sein. Die grundlegende Idee, um analytische Aussagen über das System (2) zu beweisen, ist, die Gleichungen zu entkoppeln und zu linearisieren sowie die Hintereinanderausführung zweier Fixpunktargumente. Genauer gesagt wird zunächst das semilineare elliptische System

$$\text{(3)} \qquad \begin{cases} -\nabla\cdot(\kappa_1(u_3(t,x))\nabla u_1(x)) &= F((u_2-u_1)(x), u_3(t,x), x) & \text{in } \Omega, \\ -\nabla\cdot(\kappa_2(u_3(t,x))\nabla u_2(x)) &= -F((u_2-u_1)(x), u_3(t,x), x) & \text{in } \Omega, \end{cases}$$

für eine feste Zeit t und mit einer fest gewählten Funktion u_3 getrennt von der parabolischen Gleichung gelöst werden. Dies wird in Kapitel 4 über die Linearisierung des Systems (3) und einer Anwendung Leray-Schauder Prinzips (Theorem 4.14) geschehen. In Kapitel 6 wird die Zeitabhängigkeit von u_1 und u_2 sowie die parabolische Gleichung unter Benutzung des Schauderschen Fixpunktsatzes wieder hinzugefügt. Um die W_p^2-Regularität in den Ortsvariablen zu garantieren, werden dabei zusätzliche Transmissionsbedingungen der Form

$$\partial_n^k u_i = \partial_n^k u_j \quad \text{auf } \Sigma \quad (k=0,1)$$

auf jeder der Flächen Σ, für $u_i \in W_p^2(\Omega_i)$ gefordert.

Eine Anwendung elliptisch-parabolischer Systeme der Struktur (2) ist das Lithium-Ionen Batteriemodell von Newman et. al. (siehe [DFN93], [DFN94] und [DNT94]). Der Einfachheit halber sei hier das Gebiet $\Omega \subset \mathbb{R}^n$ als ein Intervall ($n=1$), als ein achsparalleles Rechteck ($n=2$) oder als ein Quader ($n=3$) angenommen, und Ω sei durch zwei $(n-1)$-dimensionale Flächen Σ_i in die Elektroden Ω_1 und Ω_3 sowie den Separator Ω_2 aufgeteilt. Die gesuchten Größen sind die elektrischen Potentiale u_1 in der flüssigen Phase (in ganz Ω), u_2 in den festen Elektroden Ω_1 und Ω_3 sowie die Elektrolytkonzentration c. Die Modellgleichungen lauten nun

$$\text{(4)} \qquad \begin{cases} -\nabla\cdot\left(\kappa_i(c)\left(\nabla\phi_1 - \frac{2RT}{F}\nabla\ln(c)\right)\right) &= j(\phi_2-\phi_1,c) & \text{in } J\times\Omega_i,\ (i=1,2,3), \\ -\nabla\cdot(\sigma_i\nabla\phi_2) &= -j(\phi_2-\phi_1,c) & \text{in } J\times\Omega_i,\ (i=1,3), \\ \partial_t c - \nabla\cdot(D_i\nabla c) &= j(\phi_2-\phi_1,c) & \text{in } J\times\Omega_i,\ (i=1,2,3), \\ \partial_\nu\phi_1 - \frac{2RT}{F}\frac{1}{c}\partial_\nu c &= 0 & \text{auf } J\times\partial\Omega, \\ \sigma\partial_\nu\phi_2 &= I & \text{auf } J\times\Gamma_k,\ k\in\{n,2n\}, \\ \partial_\nu\phi_2 &= 0 & \text{auf } J\times((\partial\Omega_1\backslash\Gamma_n)\cup(\partial\Omega_3\backslash\Gamma_{2n})), \\ \partial_\nu c &= 0 & \text{auf } J\times\partial\Omega, \\ c(0,x) &= c_0(x) & (x\in\Omega). \end{cases}$$

Der Strom I wird über die beiden parallelen Außenflächen Γ_n und Γ_{2n} ab- bzw. zugeführt. Auf den verbleibenden Teilen des Randes werden Isolations- bzw. Non-flux-Randbedingungen gefordert.

Die Nichtlinearität j ist hierbei durch einen Butler-Volmer Term der Form

$$(5)\quad j(\phi_2-\phi_1,c)=kc^{\alpha}\left[\exp\left(\tfrac{\alpha F}{RT}(\phi_2-\phi_1-U)\right)-\exp\left(\tfrac{-(1-\alpha)F}{RT}(\phi_2-\phi_1-U)\right)\right]\ \text{in } \Omega_1 \text{ and } \Omega_3$$

mit $\alpha\in(0,1)$ gegeben und beschreibt den Fluss von Ladung bzw. von Lithiumionen durch die Flächen Σ_i. Die effektiven Leitfähigkeiten $\kappa_i:[0,\infty)\to[0,\infty)$ und der effektive Diffusionskoeffizient D hängen hier nur in jedem Teilgebiet Ω_i hinreichend glatt von c bzw. von x ab.

Zur Behandlung des logarithmischen Terms in der ersten Gleichung bietet sich eine Substitution der Funktion ϕ_1 durch $u_1:=\phi_1-\frac{2RT}{F}\ln(c)$ an. Infolgedessen ändert sich die Nichtlinearität j zu

$$(6)\qquad N(\phi_2-u_1,c):=\begin{cases} c^{-\alpha}e^{\frac{\alpha F}{RT}(\phi_2-u_1-U)}-c^{2-\alpha}e^{\frac{-(1-\alpha)F}{RT}(\phi_2-u_1-U)}, & \text{in } \Omega_1\cup\Omega_3,\\ 0, & \text{in } \Omega_2.\end{cases}$$

Diese Funktion verhält sich singulär, wenn die Konzentration c gegen 0 geht und bezüglich der Potentialdifferenz zeigt sie exponentielles Wachstum. Nach dieser Substitution weist das System (4) weitgehend die gleiche Struktur wie das allgemeinere elliptisch-parabolische System (2) auf. Bei der Betrachtung der Gleichungen (4) in zwei oder drei Dimensionen werden ferner alle oben aufgezählten mathematischen Problemstellungen kombiniert.

Die vorliegende Dissertation ist folgendermaßen gegliedert:

Kapitel 1 behandelt parameter-elliptische Systeme der Form

$$(7)\qquad \begin{cases} (\lambda-A(x,\mathcal{D}))u &= f & \text{in } \Omega,\\ B_j(x,\mathcal{D})u &= g_j & \text{auf } \partial\Omega,\ j=1,\dots,m,\end{cases}$$

in den Grundräumen $L_p(\Omega)$, $W_p^1(\Omega)$ und $W_p^s(\Omega)$ mit $s<\frac{1}{p}$. Das erste Hauptresultat zur Sektorialität der Realisierung $A_{\mathcal{B},p,s}$ des Halbraum-Modellproblems (7) im Grundraum $W^s_{p;(A,\mathcal{B})}(\mathbb{R}^n_+)$, welcher zusätzliche Kompatibilitätsbedingungen enthält, wird durch Theorem 1.33 für allgemeines s erzielt. Weiterführende Resultate zu inhomogenen Randbedingungen finden sich in Theorem 1.39 im Fall $s=1$, beziehungsweise in Korollar 1.44 für den Fall $0<s<\frac{1}{p}$. Über Störungsresultate sowie ein Lokalisierungsverfahren werden die Ergebnisse für das Modellproblem anschließend auf den Fall eines beschränkten $C^{m,1}$-Gebiets und Operatoren mit variablen Koeffizienten verallgemeinert. Das zentrale Ergebnis zu parameter-elliptischen Systemen in $W_p^s(\Omega)$ mit $s<\frac{1}{p}$ ist schließlich Theorem 1.49.

In **Kapitel 2** werden die Resultate aus dem ersten Kapitel auf den Fall eines rechteckigen Gebiets Ω übertragen. Ferner wird die Gleichung (7) nun mit zusätzlichen Transmissionsbedingungen auf einer Fläche, die das Gebiet aufteilt, betrachtet. Die Klasse der betrachteten Differentialoperatoren wird hierzu auf Operatoren mit geradem Hauptteil (siehe Definition 2.1) eingeschränkt und ebenso werden überwiegend Randbedingungen von geradem oder ungeradem Typ (siehe Definition 2.2) betrachtet. Theorem 2.12 behandelt das zugehörige Modellproblem in einer aufgeteilten k-Ecke $\mathbb{K}^n_{k,\pm}$ mit Transmissionsbedingungen. Über ein angepasstes Lokalisierungsverfahren wird die Wohlgestelltheit eines parameter-elliptischen Randwertproblems in einem rechteckigen Gebiet Ω mit zusätzlichen Transmissionsbedingungen im Grundraum $W_p^s(\Omega)$ erhalten. Dieses Hauptresultat ist in Korollar 2.26 zu finden.

In **Kapitel 3** wird die Existenz und Eindeutigkeit einer schwachen Lösung zu einem nichtlinearen elliptischen System der Form (3) in einem Lipschitzgebiet bewiesen. An die Nichtlinearität wird hierzu lediglich die oben genannte Forderung (i) gestellt. Die Gleichungen (3) mit homogenen Neumann Randbedingungen und der speziellen Nichtlinearität (6) ist bereits in [WXZ06] betrachtet worden. Durch eine Anwendung des Leray-Schauder Prinzips in $(L_\infty(\Omega))^2$ werden die dort erzielten Resultate durch Theorem 3.18 verallgemeinert. Des Weiteren werden auch inhomogene Neumann Randbedingungen in Korollar 3.20 behandelt. Unter geringfügig stärkeren

Bedingungen an das Gebiet können mit Theorem 3.34 schließlich auch analoge Wohlgestelltheitsresultate für Dirichlet Randbedingungen sowie gemischte Randbedingungen erzielt werden.

Kapitel 4 betrachtet das System (3) in $L_p(\Omega)$ für den Fall eines rechteckigen Gebiets Ω, welches durch parallele Flächen Σ_i in Teilgebiete unterteilt ist. Mit den Forderungen (i)-(iii) an die Nichtlinearität F sowie der Annahme einer stückweisen $C^{1,\tau}$-Regularität der Koeffizienten wird hier die eindeutige Lösbarkeit in $(W_p^2(\Omega))^2$ bewiesen. Um mögliche Unstetigkeiten der Koeffizienten zu berücksichtigen, werden auf den Flächen Σ_i zusätzliche kanonische Transmissionsbedingungen der Ordnung zwei gestellt und ein allgemeines Ergebnis aus Kapitel 2 zur Lösung der linearisierten Gleichungen benutzt. Unter Verwendung der im vorigen Kapitel gezeigten L_∞-Schranken kann wiederum mit dem Leray-Schauder Prinzip das Hauptresultat des Kapitels, die eindeutige Lösbarkeit des Systems (3) in $W_p^2(\Omega)$, bewiesen werden.

Im Mittelpunkt von **Kapitel 5** stehen parabolische Gleichungen mit Transmissionsbedingungen für den Fall, dass zwei C^{2m}-Gebiete Ω_1 und Ω_2 durch die C^{2m}-Fläche $\partial\Omega_2$ gegeneinander abgegrenzt sind sowie für den Fall eines rechteckigen Gebiets, welches durch eine Fläche in zwei Teilgebiete unterteilt wird. Zunächst wird der Begriff der $\mathcal{R}$-Beschränktheit einer Operatorfamilie definiert und zwei vektorwertige Versionen des Satzes von Michlin über Fouriermultiplikatoren (Theorem 5.10) vorgestellt. Ferner werden die funktionalanalytischen Begriffe der $\mathcal{R}$-Sektorialität und der maximalen Regularität eines Operators eingeführt sowie der Satz von Weis (Theorem 5.18), der eine wesentliche Verbindung zwischen den beiden Begriffen aufzeigt. In Theorem 5.28 wird die Eigenschaft der $\mathcal{R}$-Sektorialität für die Realisierung eines parabolischen Transmissionsproblems in $L_p((0,T);L_p(\Omega_1)) \times L_p((0,T);L_p(\Omega_2))$ bewiesen, was insbesondere dessen eindeutige Lösbarkeit beinhaltet. Für den Fall eines rechteckigen Gebiets findet sich ein analoges Resultat in Theorem 5.33.

Kapitel 6 verbindet schließlich die Ergebnisse über das nichtlineare elliptische System aus Kapitel 4 sowie die Resultate zu linearen parabolischen Gleichungen mit Transmissionsbedingungen aus Kapitel 5. Mit Theorem 6.10 wird die zeitlich lokale Existenz einer Lösung des elliptisch-parabolischen Systems (2) mit W_p^2-Regularität bezüglich der Raumvariablen gezeigt.

Das **7. Kapitel** beginnt mit den wesentlichen Ideen der Herleitung des Batteriemodells (4). Im Anschluss daran werden die allgemeinen Ergebnisse aus Kapitel 6 auf die konkrete Situation angewendet, um die Existenz einer lokalen Lösung

$$(\phi_1, \phi_2, c) \in C([0,T];W_p^2(\Omega)) \times C([0,T];W_p^2(\Omega)) \times W_{p,+}^{1,2}((0,T)\times\Omega)$$

zu erhalten. Das Hauptergebnis dieses Abschnitts ist Theorem 7.5. Insbesondere stellt hierbei die Nichtlinearität (6) ein anwendungsorientiertes Beispiel mit den oben genannten Eigenschaften (i)-(iii) dar. Den Abschluss bildet eine kurze Diskussion eines Batteriemodells mit nichtlinearen Randbedingungen.

Konstanz, April 2013 Tim Seger

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