Potential Analysis (2025) 63:169–222
https://doi.org/10.1007/s11118-024-10165-1

RESEARCH

Sharp Trace and Korn Inequalities for Differential Operators

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Received: 11 July 2023 / Accepted: 8 September 2024 / Published online: 2 October 2024

Abstract
We establish sharp trace and Korn-type inequalities that involve vectorial differential operators, the focus being on situations where global singular integral estimates are not available. Starting from a novel approach to sharp Besov boundary traces by Riesz potentials and oscillations that equally applies to $p = 1$, a case difficult to be handled by harmonic analysis techniques, we then classify boundary trace- and Korn-type inequalities. For $p = 1$ and so despite the failure of the Calderón-Zygmund theory, we prove that sharp trace estimates can be systematically reduced to full k-th order gradient estimates. Moreover, for $1 < p < \infty$, where sharp trace estimates yield Korn-type inequalities on smooth domains, we show for the basically optimal class of John domains that Korn-type inequalities persist – even though the reduction to global Calderón-Zygmund estimates by extension operators might not be possible.

Keywords Trace theorem · Uspenskiĭ's theorem · L^1-estimates · Korn inequality · $\mathbb{C}$-ellipticity · Riesz potentials · Inverse estimates · NTA domains · John domains · Weighted inequalities

Mathematics Subject Classification (2010) 31B10 · 35B33 · 35C15 · 35G45 · 42B15 · 42B20

1 Introduction

1.1 Calderón-Zygmund and L^1-estimates

A variety of partial differential equations or problems from the calculus of variations require to bound the L^p-norms of full higher order gradients by those of given differential operators. To be more precise, let $k \in \mathbb{N}$, $k \geq 1$ and suppose that $\mathbb{A}$ is a homogeneous, k-th order linear, constant-coefficient differential operator on $\mathbb{R}^n$ with $n \geq 2$ between two finite dimensional

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real vector spaces V and W. Then $\mathbb{A}$ has a representation

$$\mathbb{A} = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ |\alpha|=k}} \mathbb{A}_\alpha \partial^\alpha, \tag{1.1}$$

where for each $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k$, $\mathbb{A}_\alpha\colon V \to W$ is a fixed linear map. Following the foundational work of CALDERÓN & ZYGMUND [15], if $1 < p < \infty$, there exists a constant $c = c(\mathbb{A}, p) > 0$ such that there holds

$$\|D^k u\|_{\mathrm{L}^p(\mathbb{R}^n)} \leq c\|\mathbb{A}u\|_{\mathrm{L}^p(\mathbb{R}^n)} \qquad \text{for all } u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V) \tag{1.2}$$

if and only if $\mathbb{A}$ is elliptic; Eq. 1.2 is also referred to as *Korn-type inequality* [46]. Here, in the most general sense, we call $\mathbb{A}$ *elliptic* provided the Fourier symbol

$$\mathbb{A}[\xi] = \sum_{|\alpha|=k} \xi^\alpha \mathbb{A}_\alpha\colon V \to W \qquad \text{is injective for any } \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

cf. HÖRMANDER [34] and SPENCER [56]. Due to a celebrated counterexample of ORNSTEIN [53] – colloquially termed *Ornstein's Non-Inequality* – inequality Eq. 1.2 does not persist for $p = 1$ in general. Despite the failure of Eq. 1.2 for $p = 1$, one might still hope to find conditions on $\mathbb{A}$ such that lower order quantities depending on u can be controlled in terms of $\|\mathbb{A}u\|_{\mathrm{L}^1}$. In the present paper we shall give a resolution of this matter for the boundary traces in the case of k-th order differential operators and *all* $1 \leq p < \infty$, cf. Theorem 1.1 for $p = 1$ (and Theorem 4.4 for $1 < p < \infty$). We then classify this result within situations where *global* Calderón-Zygmund estimates are not available, might it be because of $p = 1$ (Theorem 1.2) or the irregularity of the underlying domains for $1 < p < \infty$ (Theorem 1.3).

1.2 Limiting L^1-trace Estimates via Riesz Potentials

The general objective of limiting L^1-estimates is to classify those operators $\mathbb{A}$ such that well-known inequalities for the full k-th order gradients hold with D^k replaced by $\mathbb{A}$, too. Starting with BOURGAIN & BREZIS [7, 8], the problem of bounding lower order norms against L^1-norms of differential expressions $\mathbb{A}u$ has been intesively studied from various perspectives in recent years, cf. VAN SCHAFTINGEN et al. [9, 31, 65–67] for a systematic treatment of Sobolev-type inequalities. By ORNSTEIN's Non-Inequality [44, 45, 53], none of these results can be directly inferred from the full k-th order gradient estimates. This equally applies to trace estimates, dealt with in the first order case by BREIT and the authors [11], which in turn give rise to Sobolev embeddings on domains, cf. RAITA and the second author [30].

The instrumental condition making boundary trace inequalities work is that of $\mathbb{C}$-*ellipticity*. Following [11, 30, 55], we say that $\mathbb{A}$ of the form Eq. 1.1 is $\mathbb{C}$-*elliptic* provided

$$\mathbb{A}[\xi]\colon V + \mathrm{i}V \to W + \mathrm{i}W \text{ is injective for each } \xi \in \mathbb{C}^n \setminus \{0\}. \tag{1.4}$$

$\mathbb{C}$-ellipticity is a stronger condition than VAN SCHAFTINGEN's cancellation condition [66]. So for instance the generalisation of GAGLIARDO's trace inequality,

$$\|u\|_{\mathrm{L}^1(\partial\Omega)} \leq c\,(\|u\|_{\mathrm{L}^1(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^1(\Omega)}) \qquad \text{for all } u \in \mathrm{C}^\infty(\overline{\Omega}; V) \tag{1.5}$$

for first order $\mathbb{C}$-elliptic operators holds if and only if $\mathbb{A}$ is $\mathbb{C}$-elliptic, see [11]; here $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain. Since the trace operator $\mathrm{Tr}_{\partial\Omega}\colon\ \mathrm{W}^{1,1}(\Omega; V) \to \mathrm{L}^1(\partial\Omega; V)$ is surjective, Eq. 1.5 is optimal. However, if $k \geq 2$, by a result of USPENSKIĬ [63] there exists $c = c(n, k, V) > 0$ such that for any open halfspace $\mathbb{H} \subset \mathbb{R}^n$ there holds

$$\|u\|_{\dot{\mathrm{B}}^{k-1}_{1,1}(\partial\mathbb{H})} \leq c\|D^k u\|_{\mathrm{L}^1(\mathbb{H})} \qquad \text{for all } u \in \mathrm{C}^\infty_c(\overline{\mathbb{H}}; V) \tag{1.6}$$

with the homogeneous Besov space $\dot{\mathrm{B}}^{k-1}_{1,1}(\partial\mathbb{H}; V)(\subsetneq \dot{\mathrm{W}}^{k-1,1}(\partial\mathbb{H}; V))$ (cf. Section 4.1 for the definition), resulting in a *surjective trace operator*. One may thus wonder whether $\mathbb{C}$-ellipticity is equally necessary and sufficient to yield the corresponding variant of the trace estimate Eq. 1.6, also see [30, Open Problem 4.8]. As our first result, we answer this by

Theorem 1.1 (Higher order traces I) *Let $k \in \mathbb{N}$, $k \geq 2$. Then the following are equivalent for a k-th order differential operator $\mathbb{A}$ of the form Eq. 1.1:*

(a) $\mathbb{A}$ is $\mathbb{C}$-elliptic in the sense of Eq. 1.4.
(b) There exists $c = c(\mathbb{A}) > 0$ such that for any open halfspace $\mathbb{H}$ there holds

$$\|u\|_{\dot{\mathrm{B}}^{k-1}_{1,1}(\partial\mathbb{H})} \leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{H})} \qquad \textit{for all } u \in \mathrm{C}^\infty_c(\overline{\mathbb{H}}; V). \tag{1.7}$$

The previous theorem particularly implies that, for open and bounded domains Ω with smooth boundary $\partial\Omega$ and $\mathbb{C}$-elliptic $\mathbb{A}$, there exists a surjective boundary trace operator $\mathrm{Tr}_{\partial\Omega}\colon \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathrm{B}^{k-1}_{1,1}(\partial\Omega; V)$ for *functions of bounded $\mathbb{A}$-variation* (cf. Theorem 4.7).

This also extends to less regular boundaries, but then requires a more careful definition of the underlying Besov spaces. However, our main focus in the previous theorem is not a possibly involved geometry or low regularity of boundaries, but the estimate Eq. 1.7 based on Riesz potential estimates. More general domains are addressed in Sections 1.3 and 1.4.

Note that a weaker estimate, replacing the $\dot{\mathrm{B}}^{k-1}_{1,1}$-norm in Eq. 1.7 by the $\dot{\mathrm{W}}^{k-1,1}$-norm has been obtained by RAITA, VAN SCHAFTINGEN and the second author in [31]; again note that $\dot{\mathrm{B}}^{k-1}_{1,1} \subsetneq \dot{\mathrm{W}}^{k-1,1}$ (cf. BREZIS & PONCE [13, Rem. A.1, p. 1238]).

As pointed out by LEONI & TICE [47], the only available approaches to the classical USPENSKIĬ trace estimate Eq. 1.6 for $p = 1$ rely on the finite difference characterisation of Besov spaces; also see MIRONESCU & RUSS [52]. By ORNSTEIN's Non-Inequality, controlling the requisite finite differences in terms of $\mathbb{A}u$ and *not* $D^k u$ is far from clear, and so Eq. 1.7 must be approached differently. Toward Theorem 1.1, our line of action instead is to combine an oscillation characterisation of Besov spaces (cf. Lemma 4.1) and a novel sharp Riesz potential representation formula for $\mathbb{C}$-elliptic operators (cf. Proposition 3.3). This approach equally seems to be new in the case $1 < p < \infty$ and even for $\mathbb{A} = D^k$, see Section 4.4.

By ARONSZAJN [4] and SMITH [55] (also see KAŁAMAJSKA [41, 42]), it is well-known that for k-th order $\mathbb{C}$-elliptic operators $\mathbb{A}$ and $1 < p < \infty$ one has the norm equivalence

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} \simeq \|u\|_{\mathrm{W}^{k-1,p}(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)}, \qquad u \in \mathrm{W}^{k,p}(\Omega; V) \tag{1.8}$$

for smoothly bounded $\Omega \subset \mathbb{R}^n$. As asserted in Theorem 4.4, for such domains a corresponding sharp variant of the trace estimate Eq. 1.7 for $1 < p < \infty$ does not only follow from but *is equivalent* to Eq. 1.8, thereby providing a self-contained proof of Eq. 1.8. Here, the respective trace estimates yield the existence of a suitable extension operator and so allow to reduce Eq. 1.8 to *global* singular integral estimates on $\mathbb{R}^n$. In particular, the sharp trace estimates obtained by our Riesz potential approach imply Eq. 1.8 and then vice versa by the sharp trace theorem for $\mathrm{W}^{k,p}$.

In view of the failure of Calderón-Zygmund estimates on L^1, inequality Eq. 1.8 cannot be obtained for $p = 1$ in general and, if $1 < p < \infty$, the above approach only allows to conclude Eq. 1.8 for suitable extension domains. We thus proceed by classifying the underlying mechanisms that allow to arrive at Theorem 1.1 and the Korn-type estimate Eq. 1.8 in situations where the above approaches are not necessarily available.

1.3 Reduction to $\mathrm{W}^{k,1}$-estimates Despite Failure of L^1-CZ Theory

Working from Theorem 1.1, $\mathbb{C}$-ellipticity yields the same trace estimates as known from the full k-th order gradient case. Thus one might wonder whether – *despite the failure of the Calderón-Zygmund estimates for* $p = 1$ – $\mathbb{C}$-ellipticity directly allows to

$$\boxed{\text{reduce limiting } \mathrm{L}^1\text{-trace estimates to those available for } \mathrm{W}^{k,1}.} \tag{1.9}$$

As our second main result, Theorem 1.2 below and, more importantly, its proof, this is the case indeed. It is singled out as its proof seems to be *the first instance in the literature which uses the reduction metaprinciple Eq. 1.9* to arrive at limiting L^1-estimates for differential operators in the face of ORNSTEIN's Non-Inequality. The proof is naturally set up for NTA_{n-1}-domains (cf. Section 2 for this terminology), leading to the following

Theorem 1.2 (Higher order traces II) *Let* $k \in \mathbb{N}_{\geq 1}$. *Then the following are equivalent for an operator* $\mathbb{A}$ *of the form Eq. 1.1:*

(a) $\mathbb{A}$ *is* $\mathbb{C}$*-elliptic in the sense of Eq. 1.4.*
(b) For every open, bounded, NTA_{n-1}*-domain* $\Omega \subset \mathbb{R}^n$*, the space* $\mathrm{W}^{\mathbb{A},1}(\Omega) := \{u \in \mathrm{L}^1(\Omega; V)\colon\ \|u\|_{\mathrm{L}^1(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^1(\Omega)} < \infty\}$ has the same trace space on $\partial\Omega$ as $\mathrm{W}^{k,1}(\Omega; V)$.

Here, given a Banach space $(\mathscr{X}(\partial\Omega), \|\cdot\|_{\mathscr{X}(\partial\Omega)})$ with $\mathscr{X}(\partial\Omega) \subset \mathrm{L}^1_{\mathrm{loc}}(\partial\Omega; V)$, we say that $\mathrm{W}^{k,1}(\Omega; V)$ *has trace space* $\mathscr{X}(\partial\Omega; V)$ provided there exists a linear and bounded surjective trace operator $\mathrm{Tr}_{\partial\Omega}\colon\ \mathrm{W}^{k,1}(\Omega; V) \to \mathscr{X}(\partial\Omega; V)$ (which, by definition, satisfies $\mathrm{Tr}_{\partial\Omega}(u) = u$ $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$ for all $u \in \mathrm{C}(\overline{\Omega}; V) \cap \mathrm{W}^{k,1}(\Omega; V)$). For $\mathrm{W}^{\mathbb{A},1}(\Omega)$, the notion of trace space is then defined analogously. Complementing Theorem 1.1, Theorem 1.2 does not specify the trace space of $\mathrm{W}^{\mathbb{A},1}(\Omega)$ but rather asserts that it *equals that of* $\mathrm{W}^{k,1}(\Omega; V)$. If, for instance, $\Omega \subset \mathbb{R}^n$ is open and bounded with boundary of class C^∞, then this especially implies that the trace space of $\mathrm{W}^{\mathbb{A},1}(\Omega)$ is $\mathrm{B}^{k-1}_{1,1}(\partial\Omega; V)$. Also note that we could replace $\mathrm{W}^{\mathbb{A},1}$ by $\mathrm{BV}^{\mathbb{A}}$ in (b).

Theorem 1.2 is approached by an advancement and strenghtening of a method employed in the first order case by the authors and BREIT [11]. The underlying key novelty of the proof of Theorem 1.2 is that for a $\mathbb{C}$-elliptic operator $\mathbb{A}$, the nullspace of $\mathbb{A}$ consists of polynomials of a fixed degree and so, by the equivalence of all norms on finite dimensional spaces, ORNSTEIN's Non-Inequality becomes invisible on $\ker(\mathbb{A})$. Section 5 is devoted to the implementation of this strategy.

1.4 Korn's Inequality Without Global CZ-estimates

Returning to the Korn-type estimate Eq. 1.8 for $1 < p < \infty$, the approach sketched in Section 1.2 works by extensions and applying global Calderón-Zygmund estimates on $\mathbb{R}^n$. Following the discussions in [23, 39] for the symmetric gradient case $\mathbb{A}u = \frac{1}{2}(Du + Du^\top)$, the natural geometric setup for such inequalities is given by *John domains* (see Section 2 for this terminology). However, John domains need not even be extension domains for $\mathrm{W}^{k,p}$. Hence, in this situation, estimate Eq. 1.8 cannot be established by means of global Calderón-Zygmund estimates on $\mathbb{R}^n$. Interestingly, Eq. 1.8 still persists for John domains and $\mathbb{C}$-elliptic operators:

Theorem 1.3 (Korn for John) *Let $1 < p < \infty$ and $\mathbb{A}$ be a differential operator of the form Eq. 1.1 with $k \in \mathbb{N}_{\geq 1}$. Then the following are equivalent:*

(a) $\mathbb{A}$ is $\mathbb{C}$-elliptic.
(b) For all open and bounded John domains $\Omega \subset \mathbb{R}^n$ we have

$$\|u\|_{\mathrm{W}^{k,p}(\Omega)} \simeq \|u\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)} \qquad \textit{for all } u \in \mathrm{W}^{k,p}(\Omega; V).$$

Theorem 1.3 is established by a generalisation of a decomposition method introduced by Růžička, Schumacher and the first author [23] to the sharp class of operators for which the above Korn-type inequalities can hold at all. Theorem 1.3 is in the spirit of Theorem 1.2, however, note that John domains need not allow for a boundary trace operator; but if they do, Theorem 1.3 immediately implies the equality of the trace space of $\mathrm{W}^{k,p}(\Omega; V)$ and that of the $\mathbb{A}$-Sobolev space $\mathrm{W}^{\mathbb{A},p}(\Omega) := \{u \in \mathrm{L}^p(\Omega; V) \colon \ \|u\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)} < \infty\}$. Motivated by recent interest in Korn-type inequalities on more general space scales (cf. [10, 16, 23]), Theorem 1.3 is obtained in Section 6 as a special case of an A_p-weighted version, cf. Theorem 6.10. By Rubio de Francia extrapolation, the latter implies a variety of Korn-type inequalities, so e.g. on Orlicz- or Lorentz spaces; see Section 6.5.

1.5 Organisation of the Paper

Section 2 collects the different notions of domains considered in the main part of the paper. As a technical novelty, Section 3 provides an improved representation formula a lá Smith and Kaamajska. This results in a family of Poincaré-type inequalities for John domains and Riesz potential-type inequalities which should be of independent interest but also display a crucial ingredient for the sequel. In Section 4, after gathering background facts on Besov spaces, we establish Theorem 1.1. Sections 5 and 6 are devoted to the proofs of Theorems 1.2 and 1.3, respectively. The appendix, Section 7 gathers the proofs of auxiliary results.

General Notation

By $\Omega \subset \mathbb{R}^n$ we understand an open set throughout, and the open ball of radius r centered at x_0 is denoted $B(x_0, r) := \{y \in \mathbb{R}^n \colon \ |x_0 - y| < r\}$. For $B = B(x_0, r)$ and $\sigma > 0$, we set $\sigma B := B(x_0, \sigma r)$. The n-dimensional Lebesgue and $(n-1)$-dimensional Hausdorff measures are denoted $\mathscr{L}^n$ and $\mathscr{H}^{n-1}$, respectively; for brevity, we sometimes use $|A| := \mathscr{L}^n(A)$ and $\mathrm{d}^{n-1} := \mathrm{d}\mathscr{H}^{n-1}$. Given a finite dimensional real vector space E, we denote the finite positive Radon measures on Ω by $\mathscr{M}(\Omega)$, the finite E-valued Radon measures on Ω by $\mathscr{M}(\Omega; E)$ and, for $\mu \in \mathscr{M}(\Omega; E)$, $|\mu|(\Omega)$ its total variation. Given $\mu \in \mathscr{M}(\Omega)$ and a Borel subset U of Ω with $\mu(U) > 0$, we put for a μ-integrable map $f \colon \Omega \to E$

$$⨍_U f \, \mathrm{d}\mu := \frac{1}{\mu(U)} \int_U f \, \mathrm{d}\mu,$$

and the choice of μ will be clear from the context. If $\mu = \mathscr{L}^n$, we also use the shorthand $(f)_U := ⨍_U f \, \mathrm{d}\mathscr{L}^n$. The space of E-valued polynomials on $\mathbb{R}^n$ of degree at most $m \in \mathbb{N}_0$ is denoted $\mathscr{P}_m(\mathbb{R}^n; E)$, and the homogeneous E-valued polynomials of degree m by $\mathscr{P}_m^h(\mathbb{R}^n; E)$; we also set $\mathscr{P}_{-m}(\mathbb{R}^n; E) = \{0\}$ for $m \in \mathbb{N}$ and $\mathscr{P}(\mathbb{R}^n; E) := \bigcup_{m \in \mathbb{N}_0} \mathscr{P}_m(\mathbb{R}^n; E)$. The symbol $\odot^m(\mathbb{R}^n; E)$ denotes the symmetric m-multilinear maps from $\mathbb{R}^n$ to E. By $c, C > 0$ we denote generic constants which might change from line to line and shall only be specified provided their precise value is required. As such, we write $a \simeq b$ if there exist $c, C > 0$ such that $ca \leq b \leq Ca$ and both c, C do not depend in any essential way on a and b. We also use $X \simeq Y$ for normed spaces to indicate that $X = Y$ with equivalence of norms, but no ambiguities will arise from this.

2 Domains

In this section we collect the various geometric assumptions on domains Ω of our functions that arise throughout the main part of the paper.

We say that $\Omega \subset \mathbb{R}^n$ satisfies the *interior corkscrew condition* provided there exist $R > 0$ and $M > 2$ such that for all $x \in \partial\Omega$ and all $0 < r < R$ there exists $y \in \Omega$ such that

$$|x - y| < r \quad \text{and} \quad B\Big(y, \frac{r}{M}\Big) \subset \Omega.$$

Likewise, Ω satisfies the *exterior corkscrew condition* provided $\mathbb{R}^n \setminus \Omega$ satisfies the interior corkskrew condition.

Now suppose that Ω satisfies the interior and exterior corkscrew condition with some $M > 2$. For $x_1, x_2 \in \Omega$, we say that a sequence of balls $B_1, ..., B_K \subset \Omega$ is a *Harnack chain from x_1 to x_2 of length $K \in \mathbb{N}$* provided $x_1 \in B_1$, $x_2 \in B_K$ and for all $1 \leq j \leq K - 1$ there holds $B_j \cap B_{j+1} \neq \emptyset$ and their radii satisfy $M^{-1}r(B_j) < \operatorname{dist}(B_j, \partial\Omega) < Mr(B_j)$ for all $1 \leq j \leq K$. The set Ω is then said to satisfy the *interior Harnack chain condition* if, whenever $\varepsilon > 0$ and $x_1, x_2 \in \Omega \cap B(\xi, \frac{r}{4})$ (for some $\xi \in \partial\Omega$ and $0 < r < R$) satisfy $\operatorname{dist}(x_j, \partial\Omega) > \varepsilon$, $j \in \{1, 2\}$, together with $|x_1 - x_2| < 2^l\varepsilon$ for some $l \in \mathbb{N}$, then there exists a Harnack chain $B_1, ..., B_K$ from x_1 to x_2 of length $K \leq Ml$ such that $\operatorname{diam}(B_j) \geq \frac{1}{M} \min\{\operatorname{dist}(x_1, \partial\Omega), \operatorname{dist}(x_2, \partial\Omega)\}$ for all $j = 1, ..., K$.

Definition 2.1 (NTA and NTA$_{n-1}$-domains) An open and bounded set $\Omega \subset \mathbb{R}^n$ is *non-tangentially accessible* (or *NTA* for brevity) provided Ω satisfies *the interior, the exterior corkscrew and the interior Harnack chain condition*. We refer to $R > 0$ and $M > 1$ from above as the *NTA-parameters* of Ω. Moreover, Ω is said to have $(n-1)$*-Ahlfors regular boundary* $\partial\Omega$ if there exist $R > 0$ and $L \geq 1$ such that for all $x \in \partial\Omega$ and $0 < r < R$ there holds

$$\frac{1}{L} r^{n-1} \leq \mathscr{H}^{n-1}(\partial\Omega \cap B(x, r)) \leq L r^{n-1}.$$

If $\Omega \subset \mathbb{R}^n$ is *NTA* and has $(n-1)$-Ahlfors regular boundary, we say that Ω is NTA$_{n-1}$.

See [2, 33, 38] for more information on NTA domains. Now let $\gamma \subset \mathbb{R}^n$ be a rectifiable path with endpoints a and b and length $|\gamma|$. Assuming that $\gamma : [0, |\gamma|] \to \mathbb{R}^n$ is parametrised by arclength, we define the α*-cigar* with core γ and parameter $\alpha > 0$ by

$$\operatorname{cig}(\gamma, \alpha) := \bigcup_{t \in [0, |\gamma|]} B\Big(\gamma(t), \frac{1}{\alpha} \min\{t, |\gamma| - t\}\Big).$$

Definition 2.2 (John domains) An open and bounded set $\Omega \subset \mathbb{R}^n$ is called an α*-John domain*, $\alpha > 0$, if every pair of distinct points $a, b \in \Omega$ can be joined by a rectifiable path γ such that $\operatorname{cig}(\gamma, \alpha) \subset \Omega$. If the constant α is not important, we just say that Ω is a *John domain*.

Even though this definition of John domains automatically implies their boundedness, we will often write *open and bounded John domain* to highlight the boundedness of such sets in stating our results below. Introduced by John [40] and named after him by Martio & Sarvas [48], John domains include sets with fractal boundary such as the Koch snowflake or slit domains. Moreover, they can be decomposed into a suitable set of balls or cubes that satisfy a certain chain condition. In particular, by [23, Thm. 3.8] every bounded John domain also satisfies the *emanating chain condition* in the sense of the following definition.

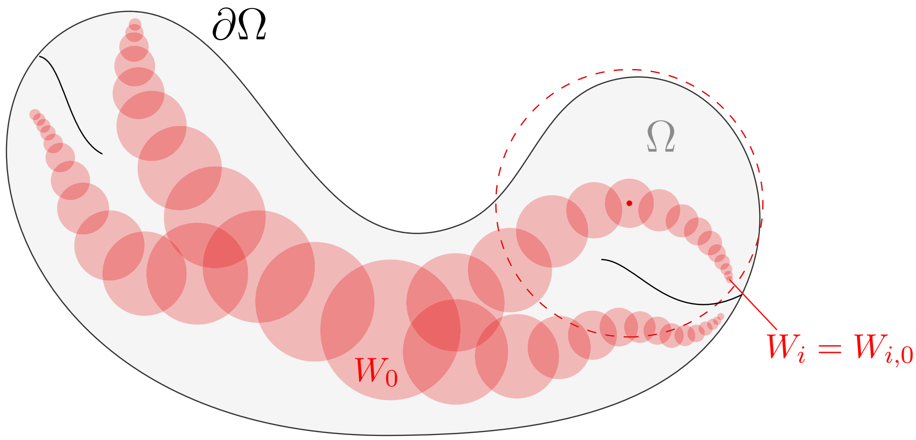

Fig. 1 The emanating chain condition for an exemplary non-extension domain $\Omega \subset \mathbb{R}^2$ with slits, illustrating the key property in 2: In any chain $W_{i,0} = W_i$, ..., $W_{i,m_i} = W_0$, any $W_{i,l}$ might be blown up by σ_2 such that $\sigma_2 W_{i,l}$ contains *all* $W_{i,j}$, $0 \le j \le l$

Definition 2.3 (Emanating chain condition) Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and let $\sigma_1 > 1, \sigma_2 \ge 1$. Then we say that Ω satisfies the *emanating chain condition* with constants σ_1 and σ_2 if there exists a covering $\mathcal{W} = \{ W_i \ : \ i \in \mathbb{N}_0 \mid \}$ of Ω consisting of open balls (or cubes) such that:

(C1) We have $\sigma_1 W \subset \Omega$ for all $W \in \mathcal{W}$ and $\sum_{W \in \mathcal{W}} \mathbb{1}_{\sigma_1 W} \le \sigma_2 \mathbb{1}_\Omega$ on $\mathbb{R}^n$.
(C2) For every $W_i \in \mathcal{W}$ there exists a *chain* of $W_{i,0}, W_{i,1}, \dots, W_{i,m_i}$ (pairwise different) from $\mathcal{W}$ such that $W_{i,0} = W_i$, $W_{i,m_i} = W_0$, and $W_{i,l_1} \subset \sigma_2 W_{i,l_2}$ for $0 \le l_1 \le l_2 \le m_i$. Moreover, $W_{i,l} \cap W_{i,l+1}$, $0 \le l < m_i$, contains a ball $B_{i,l}$ such that $W_{i,l} \cup W_{i,l+1} \subset \sigma_2 B_{i,l}$. The chain $W_{i,0}, \dots, W_{i,m_i}$ is called the *chain emanating from* W_i. The number $m_i \in \mathbb{N}_0$ is called the *length* of this chain.
(C3) The set $\{ i \in \mathbb{N}_0 : \ W_i \cap K \ne \emptyset \mid \}$ is finite for every compact subset $K \subset \Omega$.

The family $\mathcal{W}$ is called the *chain-covering* of Ω. The ball (or cube) W_0 is called the *central ball (or cube)*, since every chain ends in W_0 (Fig 1).

Clearly, all sets satisfying the emanating chain condition are connected. Moreover, if Ω satisfies the emanating chain condition with $\sigma_1 > 1$ and $\sigma_2 \ge 1$, we have

$$\operatorname{diam}(\Omega) \le \sigma_2 \operatorname{diam}(W_0), \tag{2.1}$$

since every $x \in \Omega$ is contained in some $W \in \mathcal{W}$ that can be connected with W_0 by a chain satisfying (C2) and so $W \subset \sigma_2 W_0$. Moreover, note that whenever an open and bounded set satisfies the emanating chain condition with $\sigma_1, \sigma_2 \ge 1$, then by [23, Thm. 3.8] it automatically satisfies the emanating chain condition with $\sigma_1 > 1$ and $\sigma_2 \ge 1$ (for some possibly different chain covering); this is why we directly restrict ourselves to $\sigma_1 > 1$ in Definition 2.3.

Remark 2.4 (Choice of overlap balls) By [23, Rem. 3.15], if $\Omega \subset \mathbb{R}^n$ satisfies the requirements of Definition 2.3, the balls $B_{i,l}$ as in (C2) can be chosen to belong to a family $\mathscr{B}$ of balls such that $\sum_{B \in \mathscr{B}} \mathbb{1}_B \le \sigma_2 \mathbb{1}_\Omega$ as an estimate on $\mathbb{R}^n$.

Following Boman [6] and Iwaniec & Nolder [37], domains merely satisfying (C1) and (C2) are referred to as *Boman (chain) domains*; also see Hurri [35]. By a result due to Buckley, Koskela & Lu [14], an open and bounded domain is a Boman chain domain if and only if it is John; moreover, by [23], any open and bounded John domain satisfies the emanating chain condition. In total, for open and bounded sets $\Omega \subset \mathbb{R}^n$ these notions of sets are connected as follows (see, e.g., [2, Sec. 3.7] and [23, Thm. 3.8]):

$$\boxed{\mathrm{NTA}_{n-1} \Longrightarrow \mathrm{NTA} \Longrightarrow \text{John} = \text{Emanating Chain Condition} = \text{Boman}} \tag{2.2}$$

3 The Sharp Representation Theorem and Poincaré-type Inequalities

In this section we derive a new representation formula in terms of $\mathbb{A}u$. From this we conclude Poincaré-type estimates for balls and John domains which should be of independent interest but also play a pivotal role in the proofs of Theorems 1.1–1.3.

3.1 The Representation Formula

The natural framework for the results of this and the subsequent sections is given by the following variants of Sobolev- and BV-spaces adapted to differential operators $\mathbb{A}$ of the form Eq. 1.1: Given $1 \leq p < \infty$ and an open set $\Omega \subset \mathbb{R}^n$, we define

$$\begin{aligned} \mathrm{W}^{\mathbb{A},p}(\Omega) &:= \{u \in \mathrm{L}^p(\Omega;V)\colon\ \|u\|_{\mathrm{W}^{\mathbb{A},p}(\Omega)} := \|u\|_{\mathrm{L}^p(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)} < \infty\}, \\ \mathrm{BV}^{\mathbb{A}}(\Omega) &:= \{u \in \mathrm{L}^1(\Omega;V)\colon\ \|u\|_{\mathrm{BV}^{\mathbb{A}}(\Omega)} := \|u\|_{\mathrm{L}^1(\Omega)} + |\mathbb{A}u|(\Omega) < \infty\}, \end{aligned} \tag{3.1}$$

where $\|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)} < \infty$ or $|\mathbb{A}u|(\Omega) < \infty$ have to be interpreted in the sense that the distributional differential $\mathbb{A}u$ can be represented by an $\mathrm{L}^p(\Omega;W)$-map or a finite W-valued Radon measure, respectively. Moreover, we set

$$\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega) := \mathrm{W}^{\mathbb{A},p}(\Omega) \cap \mathrm{W}^{k-1,p}(\Omega;V)$$

(in [30] also denoted $\mathrm{V}^{\mathbb{A},p}$) to be endowed with norm $\|u\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)} := \|u\|_{\mathrm{W}^{k-1,p}(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)}$; note that $\mathrm{W}^{k,p}(\Omega;V) \subset \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$. All of these spaces are Banach, and it is routine to check that $\mathrm{C}^\infty(\Omega;V) \cap \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$ is dense in $\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$ and, if $\partial\Omega$ is sufficiently regular, we even have density of $\mathrm{C}^\infty(\overline{\Omega};V) \cap \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$. For future reference, note that such density properties cannot be expected for the norm on $\mathrm{BV}^{\mathbb{A}}$ (cf. Ambrosio [3, Chpt. 3] et al. for $\mathbb{A} = D$), and the correct substitute here is the $\mathbb{A}$-strict metric defined by

$$d_{\mathbb{A}}(u,v) := \|u-v\|_{\mathrm{L}^1(\Omega)} + \big||\mathbb{A}u|(\Omega) - |\mathbb{A}v|(\Omega)\big|, \qquad u,v \in \mathrm{BV}^{\mathbb{A}}(\Omega). \tag{3.2}$$

(Non-)examples of $\mathbb{C}$-elliptic operators are as follows; other relevant instances are gathered in [66, Sec. 6.4]:

Example 3.1 For each $k, n \in \mathbb{N}$ and $N \in \mathbb{N}$, the k-th order gradients D^k acting on $u\colon \mathbb{R}^n \to \mathbb{R}^N$ are $\mathbb{C}$-elliptic. For $n \in \mathbb{N}$ and maps $u\colon \mathbb{R}^n \to \mathbb{R}^n$, the *symmetric gradient operator* $\varepsilon(u) := \frac{1}{2}(Du + Du^\top)$ is $\mathbb{C}$-elliptic for all $n \geq 1$, see [11, Sec. 2.2]. For $n \geq 3$, the *trace-free symmetric gradient* $\varepsilon^D u := \varepsilon(u) - \frac{1}{n}\mathrm{div}(u)E_{n\times n}$ (with the $(n \times n)$-unit matrix $E_{n\times n}$) is equally $\mathbb{C}$-elliptic; again, see [11, Sec. 2.2]. It is easy to see that the composition of two $\mathbb{C}$-elliptic operators is again $\mathbb{C}$-elliptic; an example that occurs frequently (so, e.g., in the regularity theory for elasticity and plasticity, is $\mathbb{A} = D\varepsilon^D$, see Fuchs & Seregin [28]).

Example 3.2 The trace-free symmetric gradient is not $\mathbb{C}$-elliptic for $n = 2$. This can be seen by the fact (following from Proposition 3.3 below) that $\mathbb{C}$-elliptic operators have finite dimensional nullspace on connected sets. Elements of $\ker(\varepsilon^D)$ in $n = 2$ have a one-to-one-correspondence with holomorphic functions on $\mathbb{C}$, hence ε^D is not $\mathbb{C}$-elliptic, see [11, Sec. 2.2]. On the other hand, if $n \geq 2$ and $\mathbb{A}$ is an elliptic operator on $\mathbb{R}^n$ from V to W, then the $\mathbb{A}$-*Laplacian* $\Delta_{\mathbb{A}} := \mathbb{A}^*\mathbb{A}$ is *not* $\mathbb{C}$-*elliptic*. Indeed, in this situation $\Delta_{\mathbb{A}}$ is elliptic as an operator on $\mathbb{R}^n$ from V to V, and hence does not even satisfy VAN SCHAFTINGEN's cancellation condition:

$$\bigcap_{\xi\in\mathbb{R}^n\setminus\{0\}} \Delta_{\mathbb{A}}[\xi](V) = V \neq \{0\}.$$

As $\mathbb{C}$-ellipticity implies ellipticity and cancellation (cf. [30, Lem. 3.2], [31, Prop. 3.1]) for $n \geq 2$, the $\mathbb{A}$-Laplacian $\Delta_{\mathbb{A}}$ is not $\mathbb{C}$-elliptic.

Towards Proposition 3.3, we need to revise a decomposition as established in the pioneering works of SMITH [55] and KAŁAMAJSKA [42], cf. Eq. 3.7 below. Let $\Omega \subset \mathbb{R}^n$ be a domain that is star-shaped with respect to an open ball B and satisfies $\operatorname{diam}(\Omega) \leq c_0 \operatorname{diam}(B)$. The averaged Taylor polynomial $\mathbb{T}_m^B$ of order m then is defined as in [49, Chpt. 1.1.10] and [12, Chpt. 4]: Choose $\omega \in \mathrm{C}_c^\infty(B)$ with $\int_B \omega(y)\,\mathrm{d}y = 1$ and then define for $u \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; V)$:

$$\mathbb{T}_m^B u(x) := \int_B u(y) \sum_{|\alpha|\leq m} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_y^\alpha \big(\omega(y)(x-y)^\alpha\big)\,\mathrm{d}y. \tag{3.3}$$

The averaged Taylor polynomial has good approximation properties and commutes in some sense with derivatives. In particular, we have for $u \in \mathrm{W}^{\mathbb{A},1}_{\mathrm{loc}}(\Omega)$

$$\mathbb{A}\mathbb{T}_{m-1}^B u = \mathbb{T}_{m-k-1}^B \mathbb{A}u. \tag{3.4}$$

Moreover, if $u \in \mathrm{C}^\infty(\overline{\Omega}; V)$, it follows from [49, Chpt. 1.1.10, Thm. 1] that for all $x \in \Omega$ there holds

$$\begin{aligned} u(x) &= \mathbb{T}_{m-1}^B u(x) + \sum_{|\alpha|=m} \int_\Omega K_{\alpha,B}(x,y)\partial^\alpha u(y)\,\mathrm{d}y, \quad \text{where} \\ K_{\alpha,B}(x,y) &= \frac{(-1)^m m}{\alpha!} \frac{(y-x)^\alpha}{|x-y|^n} \int_{|x-y|}^\infty \omega\Big(x + t\frac{y-x}{|y-x|}\Big) t^{n-1}\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

As established in [42], the $\mathbb{C}$-ellipticity in combination with the Hilbert Nullstellensatz [42, Lem. 4] implies the existence of $m \in \mathbb{N}$ and a $\odot^m(\mathbb{R}^n; V)$-valued differential operator $\mathbb{L}$ of order $(m-k)$ such that $D^m = \mathbb{L}\mathbb{A}$. In brevity, the underlying reason for this is that with $\mathbb{A}$ being $\mathbb{C}$-elliptic, for $\xi \in \mathbb{C}^n \setminus \{0\}$ the only solution $v \in V + \mathrm{i}\,V$ of $\mathbb{A}[\xi]v = 0$ is the null vector. Writing this system componentwisely, in this situation (also see [31, Prop. 3.2]) the Hilbert Nullstellensatz yields the existence of such m and $\mathbb{L}$. The constant m is not given explicitly, but at least $\ker(\mathbb{A}) \subset \mathscr{P}_{m-1}(\mathbb{R}^n; V)$; here, $\ker(\mathbb{A})$ is the nullspace of $\mathbb{A}$ relative to $\mathscr{D}'(\Omega; V)$. In particular, the *polynomial degree* $\deg_{\mathscr{P}}(\mathbb{A})$ of $\mathbb{A}$ satisfies

$$m \geq \deg_{\mathscr{P}}(\mathbb{A}) := \min\{\, j \in \mathbb{N}_0 \,:\, \ker(\mathbb{A}) \subset \mathscr{P}_{j-1}(\mathbb{R}^n; V)\ |\geq\}k. \tag{3.6}$$

For future reference, we note that $\deg_{\mathscr{P}}(\mathbb{A}) < \infty$ not only follows but is in fact equivalent to $\mathbb{C}$-ellipticity, cf. [30, Prop. 3.1] and [55]. Now, $D^m = \mathcal{L}\mathbb{A}$ and integrating by parts $(m-k)$-times yields for $u \in \mathrm{C}^\infty(\overline{\Omega}; V)$

$$u(x) = \mathbb{T}_{m-1}^B u(x) + \int_\Omega \widetilde{\mathfrak{K}}_{\mathbb{A}}^B(x,y)\mathbb{A}u(y)\,\mathrm{d}y \qquad \text{for all } x \in \Omega, \tag{3.7}$$

cf. [42, Theorem 4 (v)], where the kernel $\widetilde{\mathfrak{K}}_{\mathbb{A}}^{B}\colon (\Omega\times\Omega)\setminus\Delta\to\mathscr{L}(W;V)$ is smooth off the diagonal Δ, satisfies $\widetilde{\mathfrak{K}}_{\mathbb{A}}^{B}(x,\cdot)=0$ near $\partial\Omega$ for each $x\in\Omega$ and, with $C=C(c_0,\mathbb{A})>0$

$$\left|\partial_x^{\alpha}\partial_y^{\beta}\widetilde{\mathfrak{K}}_{\mathbb{A}}^{B}(x,y)\right|\leq C\,|x-y|^{k-n-|\alpha|-|\beta|}\qquad\text{for all } x,y\in\Omega,\ x\neq y, \tag{3.8}$$

for all $\alpha,\beta\in\mathbb{N}_0^n$ with $|\alpha|+|\beta|\leq k$.

In general, the polynomial $\mathbb{T}_{m-1}^{B}u$ in Eq. 3.7 does not belong to the nullspace of $\mathbb{A}$. As a simple, yet effective improvement, we now modify this representation formula in such a way that we can replace $\mathbb{T}_{m-1}^{B}$ by a projection $\Pi_{\mathbb{A}}^{B}$ to $\ker(\mathbb{A})$, thereby yielding a quick argument for Riesz- and Poincaré-type estimates in Sections 3.2 and 4.

Proposition 3.3 (Representation formula) *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain which is star-shaped with respect to an open ball $B\subset\Omega$ with* $\operatorname{diam}\Omega\leq c_0\operatorname{diam}B$. *Let $\mathbb{A}$ be a k-th order $\mathbb{C}$-elliptic differential operator of the form Eq. 1.1.*

Then there exists an integral kernel $\mathfrak{K}_{\mathbb{A}}^{B}\colon(\Omega\times\Omega)\setminus\Delta\to\mathscr{L}(W;V)$ and a linear projection $\Pi_{\mathbb{A}}^{B}\colon \mathrm{L}^1(\Omega;V)\to\ker(\mathbb{A})$ such that the following hold:

(K1) $\mathfrak{K}_{\mathbb{A}}^{B}$ is smooth off the diagonal, and for $x,y\in\Omega$, $x\neq y$ and $|\alpha|+|\beta|\leq k$ we have

$$|\partial_x^{\alpha}\partial_y^{\beta}\mathfrak{K}_{\mathbb{A}}^{B}(x,y)|\leq C\,|x-y|^{k-n-|\alpha|-|\beta|}$$

with a constant $C=C(\mathbb{A},c_0)>0$.

(K2) $\mathfrak{K}_{\mathbb{A}}^{B}(x,\cdot)$ is zero near $\partial\Omega$ for all $x\in\Omega$, i.e., $\operatorname{spt}(\mathfrak{K}_{\mathbb{A}}^{B}(x,\cdot))\Subset\Omega$.

(K3) For any $u\in \mathrm{C}^{\infty}(\Omega;V)$ and $x\in\Omega$ there holds

$$u(x)=\Pi_{\mathbb{A}}^{B}u(x)+\int_{\Omega}\mathfrak{K}_{\mathbb{A}}^{B}(x,y)\mathbb{A}u(y)\,\mathrm{d}y. \tag{3.9}$$

(K4) $\Pi_{\mathbb{A}}^{B}\colon\mathscr{P}_{\ell}(\mathbb{R}^n;V)\to\mathscr{P}_{\ell}(\mathbb{R}^n;V)\cap\ker(\mathbb{A})$ for every ℓ with $0\leq\ell<\deg_{\mathscr{P}}(\mathbb{A})$.

Especially, $\Pi_{\mathbb{A}}^{B}u$ only depends on B, $\mathbb{A}$ and u, but not on Ω or $u|_{\Omega\setminus B}$.

Proof Using translation and dilation we can assume without loss of generality that B is the unit ball $B(0,1)$.

Let $m\in\mathbb{N}_0$ and $\widetilde{\mathfrak{K}}_{\mathbb{A}}^{B}$ be such that Eq. 3.7 holds. In particular, we have $m\geq\deg_{\mathscr{P}}(\mathbb{A})$ and $\ker(\mathbb{A})\subset\mathscr{P}_{m-1}(\mathbb{R}^n;V)$. Define $\mathcal{Z}_{\ell}:=\ker(\mathbb{A})\cap\mathscr{P}_{\ell}^{h}(\mathbb{R}^n;V)$ for $\ell=0,\dots,m-1$. We claim that

$$\ker(\mathbb{A})=\bigoplus_{\ell=0}^{m-1}\mathcal{Z}_{\ell} \tag{3.10}$$

as a direct sum. For this, let $q\in\ker(\mathbb{A})$. Then $q=\sum_{\ell=0}^{m-1}q_{\ell}$ with $q_{\ell}\in\mathscr{P}_{\ell}^{h}(\mathbb{R}^n;V)$. Since $0=\mathbb{A}q=\sum_{\ell=0}^{m-1}\mathbb{A}q_{\ell}$ and $\mathbb{A}q_{\ell}\in\mathscr{P}_{\ell-k}^{h}(\mathbb{R}^n;W)$ by the homogeneity of $\mathbb{A}$, we obtain $q_{\ell}\in\ker(\mathbb{A})$ for $\ell=0,\dots,m-1$. This proves the decomposition Eq. 3.10.

For $\ell=0,\dots,m-1$, we then choose a linear subspace $\mathcal{W}_{\ell}\subset\mathscr{P}_{\ell}^{h}(\mathbb{R}^n;V)$ such that $\mathscr{P}_{\ell}^{h}(\mathbb{R}^n;V)=\mathcal{Z}_{\ell}\oplus\mathcal{W}_{\ell}$. Then we have

$$\mathscr{P}_{m-1}(\mathbb{R}^n;V)=\bigoplus_{\ell=0}^{m-1}\mathcal{Z}_{\ell}\oplus\bigoplus_{\ell=0}^{m-1}\mathcal{W}_{\ell}=:\ker(\mathbb{A})+\mathcal{W}. \tag{3.11}$$

Let $\psi_1,\dots,\psi_M$ denote a basis of $\mathscr{P}_{m-1}(\mathbb{R}^n;V)$ such that for each $\ell=0,...,m-1$, $\psi_{j_{\ell}},\dots,\psi_{j_{\ell+1}-1}$ is a basis of $\mathcal{Z}_{\ell}$ and $\psi_{j_m},\dots,\psi_M$ is a basis of $\mathcal{W}$; here, $j_0=1$. Now, let $\psi_1^*, ..., \psi_M^*$ be a dual basis with respect to the $\mathrm{L}^2(B;V)$-inner product $\langle\cdot,\cdot\rangle_{\mathrm{L}^2(B)}$, i.e., $\langle\psi_l,\psi_j^*\rangle_{\mathrm{L}^2(B)}=\delta_{l,j}$ for all $j,l=1,...,M$.

Based on Eq. 3.11, we define the projection $\widetilde{\Pi}_{\mathbb{A}}\colon \mathscr{P}_{m-1}(\mathbb{R}^n;V)\to\ker(\mathbb{A})$ by

$$\widetilde{\Pi}_{\mathbb{A}}(q):=\sum_{j=1}^{j_m-1}\langle q,\psi_j^*\rangle_{\mathrm{L}^2(B)}\psi_j.$$

We define $\Pi_{\mathbb{A}}^B:=\widetilde{\Pi}_{\mathbb{A}}\mathbb{T}_{m-1}^B$; since $\mathbb{T}_{m-1}^B$ is a projection to $\mathscr{P}_{m-1}(\mathbb{R}^n;V)$, $\Pi_{\mathbb{A}}^B$ is a projection to $\ker(\mathbb{A})$. Moreover, by construction, $\widetilde{\Pi}_{\mathbb{A}}$ maps $\mathscr{P}_\ell^h(\mathbb{R}^n;V)$ onto $\mathscr{P}_\ell^h(\mathbb{R}^n;V)$ for $\ell=0,\dots,m-1$, thus $\Pi_{\mathbb{A}}^B$ also maps $\mathscr{P}_\ell^h(\mathbb{R}^n;V)$ onto $\mathscr{P}_\ell^h(\mathbb{R}^n;V)$ for $\ell=0,\dots,m-1$. This proves (K4). Moreover, for all $u\in\mathrm{C}^\infty(\overline{\Omega};V)$ we have

$$\begin{aligned}u-\Pi_{\mathbb{A}}^Bu&=u-\widetilde{\Pi}_{\mathbb{A}}\mathbb{T}_{m-1}^Bu\\&=(u-\mathbb{T}_{m-1}^Bu)+(\mathrm{Id}-\widetilde{\Pi}_{\mathbb{A}})\mathbb{T}_{m-1}^Bu.\end{aligned}\tag{3.12}$$

Note that $\langle\mathbb{A}\cdot,\mathbb{A}\cdot\rangle_{\mathrm{L}^2(B)}$ is an inner product on $\mathcal{W}$ as $\mathscr{P}_{m-1}(\mathbb{R}^n;V)=\ker(\mathbb{A})\oplus\mathcal{W}$. Thus, for each ψ_ℓ^* with $j_m\le\ell\le M$ we find $\xi_\ell\in\mathcal{W}$ such that

$$\langle w,\psi_\ell^*\rangle_{\mathrm{L}^2(B)}=\langle\mathbb{A}w,\mathbb{A}\xi_\ell\rangle_{\mathrm{L}^2(B)}\qquad\text{for all } w\in\mathcal{W}\text{ and } j_m\le\ell\le M.$$

If $z\in\ker(\mathbb{A})$, then $\langle z,\psi_\ell^*\rangle_{\mathrm{L}^2(B)}=0=\langle\mathbb{A}z,\mathbb{A}\xi_\ell\rangle_{\mathrm{L}^2(B)}$ for all $j_m\le\ell\le M$. Thus,

$$\langle q,\psi_\ell^*\rangle_{\mathrm{L}^2(B)}=\langle\mathbb{A}q,\mathbb{A}\xi_\ell\rangle_{\mathrm{L}^2(B)}\quad\text{for all } q\in\mathscr{P}_{m-1}(\mathbb{R}^n,V)\text{ and } j_m\le\ell\le M.\tag{3.13}$$

In conclusion,

$$\begin{aligned}(\mathrm{Id}-\widetilde{\Pi}_{\mathbb{A}})\mathbb{T}_{m-1}^Bu&=\sum_{j=j_m}^M\langle\mathbb{T}_{m-1}^Bu,\psi_j^*\rangle_{\mathrm{L}^2(B)}\psi_j\\&=\sum_{j=j_m}^M\langle\mathbb{A}\mathbb{T}_{m-1}^Bu,\mathbb{A}\xi_j\rangle_{\mathrm{L}^2(B)}\psi_j\overset{Eq.3.4}{=}\sum_{j=j_m}^M\langle\mathbb{T}_{m-1-k}^B\mathbb{A}u,\mathbb{A}\xi_j\rangle_{\mathrm{L}^2(B)}\psi_j.\end{aligned}\tag{3.14}$$

Hence, by the definition Eq. 3.3 of $\mathbb{T}_{m-1}^B$, we obtain for all $x\in\Omega$

$$\begin{aligned}&\big((\mathrm{Id}-\widetilde{\Pi}_{\mathbb{A}})\mathbb{T}_{m-1}^Bu\big)(x)\\&\quad=\int_B\int_B\sum_{\substack{|\alpha|\le m-k-1\\ j_m\le j\le M}}\frac{(-1)^{|\alpha|}}{\alpha!}\partial_z^\alpha\big(\omega(z)(y-z)^\alpha\big)\psi_j(x)(\mathbb{A}\xi_j(y))^\top\,\mathrm{d}y\,\mathbb{A}u(z)\,\mathrm{d}z\\&\quad=:\int_B\overline{\mathfrak{K}}_{\mathbb{A}}^B(x,z)\mathbb{A}u(z)\,\mathrm{d}z,\end{aligned}\tag{3.15}$$

where $\overline{\mathfrak{K}}_{\mathbb{A}}^B$ is defined in the obvious manner. Recalling Eq. 3.7, put $\mathfrak{K}_{\mathbb{A}}^B:=\overline{\mathfrak{K}}_{\mathbb{A}}^B+\widetilde{\mathfrak{K}}_{\mathbb{A}}^B$. Then by Eqs. 3.7, 3.12 and 3.15, 3.9 follows. Note that $\overline{\mathfrak{K}}_{\mathbb{A}}^B(x,\cdot)$ is supported on $\operatorname{spt}(\omega)\Subset B\subset\Omega$ and is therefore zero near $\partial\Omega$. On the other hand, by Eq. 3.7ff. we obtain that $\widetilde{\mathfrak{K}}_{\mathbb{A}}^B(x,\cdot)$ is zero near $\partial\Omega$. This proves that $\mathfrak{K}_{\mathbb{A}}^B(x,\cdot)$ is zero near $\partial\Omega$ and so 2 follows. It remains to establish the estimates for $\mathfrak{K}_{\mathbb{A}}^B$ in 1; however, the estimates for $\widetilde{\mathfrak{K}}_{\mathbb{A}}^B$ are much less singular and follow easily using $\operatorname{diam}(\Omega)\le c_0\operatorname{diam}(B)$ and our simplifying assumption $0\in B$. The proof is complete. □

Remark 3.4 It is possible to extend (K4) of Proposition 3.3 to all $0 \leq \ell < m$ for any fixed $m \geq \deg_{\mathscr{P}}(\mathbb{A})$. For this one has to start with Eq. 3.7 with a possible enlarged m. Moreover, if B in Proposition 3.3 is centered at zero or Ω at least contains zero, then it is possible to construct $\Pi_{\mathbb{A}}^{B}$ such $\Pi_{\mathbb{A}}^{B}\colon \mathscr{P}_{\ell}^{h}(\mathbb{R}^n;V) \to \mathscr{P}_{\ell}^{h}(\mathbb{R}^n;V) \cap \ker(\mathbb{A})$ for every ℓ with $0 \leq \ell < \deg_{\mathscr{P}} \mathbb{A}$, where $\mathscr{P}_{\ell}^{h}$ is the linear space of homogeneous polynomials of order ℓ. Note that not increasing the degree of polynomials when projecting *cannot* be improved to the commuting-type relation $\partial^{\alpha}\mathrm{proj}_{\ker(\mathbb{A})}u = \mathrm{proj}_{\partial^{\alpha}\ker(\mathbb{A})}(\partial^{\alpha}u)$ for $u \in \mathscr{P}_{m-1}(\mathbb{R}^n;V)$ for suitable projections onto $\ker(\mathbb{A})$ or $\partial^{\alpha}\ker(\mathbb{A})$, respectively.

This can be seen by means of the symmetric gradient $\mathbb{A} = \varepsilon$ (cf. Example 3.1) in $n = 2$ dimensions. In this case, the nullspace of $\mathbb{A}$ is given by the three-dimensional space of *rigid deformations*

$$\ker(\mathbb{A}) = \left\{(x,y) \mapsto \begin{pmatrix} -ay+b \\ ax+c \end{pmatrix}\colon\; a,b,c \in \mathbb{R}\right\},$$

cf. Reshetnyak [54]. Denote $p(x,y) := (0,x)^{\top}$. Should the commuting relation $\partial^{\alpha}\mathrm{proj}_{\ker(\mathbb{A})}p = \mathrm{proj}_{\partial^{\alpha}\ker(\mathbb{A})}(\partial^{\alpha}p)$ hold, then we have $\partial_x\mathrm{proj}_{\ker(\mathbb{A})}p(x,y) = (0,1)^{\top}$ and $\partial_y\mathrm{proj}_{\ker(\mathbb{A})}p(x,y) = (0,0)^{\top}$. Thus, $\mathrm{proj}_{\ker(\mathbb{A})}p(x,y) = (b,x+c)^{\top}$ for certain $b,c \in \mathbb{R}$, but since $(x,y) \mapsto (b,x+c)^{\top}$ does not belong to $\ker(\mathbb{A})$, such a projection $\mathrm{proj}_{\ker(\mathbb{A})}$ onto $\ker(\mathbb{A})$ cannot exist.

In order to derive bounds for the projctions $\Pi_{\mathbb{A}}^{B}u$ in different function space norms, we will often make use of inverse estimates for elements in the nullspace of $\mathbb{A}$. If $\mathbb{A}$ is $\mathbb{C}$-elliptic, then $\ker(\mathbb{A})$ is a finite dimensional space of V-valued polynomials, and all norms are equivalent on $\ker(\mathbb{A})$. In combination with scaling, this yields that for each $1 \leq p < \infty$ there exists a constant $c > 0$ only depending on $\mathbb{A}$ and p such that

$$\begin{aligned} \|q\|_{\mathrm{L}^{\infty}(B(0,r))} &\leq c\Big(\fint_{B(0,r)} |q|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq c\|q\|_{\mathrm{L}^{\infty}(B(0,r))}, \\ \sum_{|\alpha|\leq k}\Big(r^{|\alpha|p}&\fint_{B(0,r)} |\partial^{\alpha}q|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq c\|q\|_{\mathrm{L}^{\infty}(B(0,r))} \end{aligned} \tag{3.16}$$

hold for all $q \in \ker(\mathbb{A})$ and $r > 0$. Similarly, for all $1 \leq p < \infty$ and all $0 < r' < r < \infty$, there exists a constant $c > 0$ only depending on $\mathbb{A}$, p and $\frac{r}{r'}$ such that

$$\Big(\fint_{B(0,r)} |q|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq c\Big(\fint_{B(0,r')} |q|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \tag{3.17}$$

holds for all $q \in \ker(\mathbb{A})$. Based on the construction of $\Pi_{\mathbb{A}}^{B}$ from above, we obtain the following lemma:

Lemma 3.5 (Inverse estimates for projections) *Under the assumptions of Proposition 3.3, the following holds: For all $1 \leq p < \infty$ and all $u \in \mathrm{L}^1(\Omega;V)$ we have, with $r(B)$ denoting the radius of B,*

$$\begin{aligned} &\Big(\fint_{\Omega} \big|\Pi_{\mathbb{A}}^{B}u\big|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq \big\|\Pi_{\mathbb{A}}^{B}u\big\|_{\mathrm{L}^{\infty}(\Omega)} \leq C\fint_{B} |u|\mathrm{d}x \leq C\Big(\fint_{B} |u|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}, \\ &\frac{1}{C}\Big(\sum_{|\alpha|\leq k} r(B)^{|\alpha|p}\fint_{B} |\partial^{\alpha}\Pi_{\mathbb{A}}^{B}u|^p \,\mathrm{d}x\Big)^{\frac{1}{p}} \leq \fint_{B} |\Pi_{\mathbb{A}}^{B}u| \,\mathrm{d}x \leq C\fint_{B} |u| \,\mathrm{d}x, \end{aligned} \tag{3.18}$$

with a constant $C = C(\mathbb{A},c_0) > 1$.

Proof Since $\Pi_{\mathbb{A}}^{B} = \widetilde{\Pi}_{\mathbb{A}}\mathbb{T}_{m-1}^{B}$ is a projection to a finite dimensional vector space, this follows immediately by inverse estimates for polynomials using $\mathrm{diam}(\Omega) \leq c_0 \,\mathrm{diam}(B)$, translation and scaling. □

By the support and growth properties of the integral kernel in Proposition 3.3, convolving $u \in \mathrm{W}^{\mathbb{A},p}(\Omega)$ or $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$ with smooth bumps and passing to the limit yields

Corollary 3.6 *Under the assumptions of Proposition 3.3, let $u \in \mathrm{W}^{\mathbb{A},p}(\Omega)$ for some $1 \leq p \leq \infty$. Then there holds for all $|\alpha| < k$*

$$\partial^{\alpha} u = \partial^{\alpha}\Pi_{\mathbb{A}}^{B} u + \int_{\Omega} (\partial_x^{\alpha}\mathfrak{K}_{\mathbb{A}}^{B})(\cdot, y)\mathbb{A}u(y)\,\mathrm{d}y \qquad \mathscr{L}^n\text{-almost everywhere.} \tag{3.19}$$

Moreover, for all ℓ with $0 \leq \ell < k$ we have

$$\left|D^{\ell}(u - \Pi_{\mathbb{A}}^{B} u)\right| \leq C \int_{\Omega} |\cdot - y|^{-n+k-\ell}\,|\mathbb{A}u(y)|\,\mathrm{d}y \qquad \mathscr{L}^n\text{-almost everywhere.} \tag{3.20}$$

For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, Eqs. 3.19 and 3.20 remain valid upon replacing $\mathbb{A}u(y)\,\mathrm{d}y$ by $\mathrm{d}\mathbb{A}u(y)$ and $|\mathbb{A}u(y)|\,\mathrm{d}y$ by $\mathrm{d}|\mathbb{A}u|\,(y)$, respectively.

3.2 Poincaré-type Inequalities

Throughout this section, let $\mathbb{A}$ be a k-th order $\mathbb{C}$-elliptic differential operator of the form Eq. 1.1. The present section is devoted to the proof of the following Poincaré-type inequality for the vast class of John domains, announced in [21, Rem. 2.4], which shall turn out a crucial tool for the following sections.

Theorem 3.7 (Poincaré for John) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded α-John domain or, equivalently by Eq. 2.2, an open and bounded set satisfying the emanating chain condition with certain constants σ_1 and σ_2 and central ball B. Then for all $u \in \mathrm{W}^{\mathbb{A},p}(\Omega)$ with $1 \leq p \leq \infty$ and $\ell \in \{0, \ldots, k-1\}$ there holds*

$$\left(\fint_{\Omega} \left|D^{\ell}(u - \Pi_{\mathbb{A}}^{B} u)\right|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}} \leq C\,\mathrm{diam}(\Omega)^{k-\ell}\left(\fint_{\Omega} |\mathbb{A}u|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{3.21}$$

and

$$\begin{aligned}\mathrm{diam}(\Omega)^{\ell}\left(\fint_{\Omega} \left|D^{\ell}u\right|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}} &\leq C\left(\fint_{\Omega} |u|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}} \\ &\quad + C\,\mathrm{diam}(\Omega)^{k}\left(\fint_{\Omega} |\mathbb{A}u|^{p}\,\mathrm{d}x\right)^{\frac{1}{p}},\end{aligned} \tag{3.22}$$

where $C = C(p, \mathbb{A}, \sigma_1, \sigma_2)$. For $p = \infty$, one has to exchange $(\fint_{\Omega} |\cdot|^{p}\,\mathrm{d}x)^{\frac{1}{p}}$ by $\operatorname{ess\,sup}_{\Omega}$. *For $p = 1$ and $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, one replaces $|\mathbb{A}u(y)|\,\mathrm{d}y$ by $\mathrm{d}|\mathbb{A}u|\,(y)$.*

In view of Corollary 3.6, Theorem 3.7 is already available for domains which are star-shaped with respect to a ball; in particular, if Ω is a ball, we may choose $B = \Omega$, giving us back the Poincaré inequalities from [11, 30]. Moreover, the quantity on the left-hand side of Eq. 3.21 enjoys a best approximation property, a fact that we shall return to in slightly higher generality in Section 6.2.

Theorem 3.7 is a direct consequence of the following Riesz-type estimates, improving Corollary 3.6 to John domains. The proof is slightly inspired by [22, Section 8.2].

Proposition 3.8 *In the situation of Theorem 3.7, for all $\ell \in \{0, ..., k-1\}$ and all $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$ there holds with a constant $C = C(\mathbb{A}, \sigma_1, \sigma_2) > 0$*

$$\left|D^{\ell}(u - \Pi_{\mathbb{A}}^{B} u)\right| \leq C \int_{\Omega} |\cdot - y|^{-n+k-\ell}\, |\mathbb{A}u(y)|\, \mathrm{d}y \qquad \mathscr{L}^n\text{-almost everywhere.} \tag{3.23}$$

For $u \in \mathrm{BV}^{\mathbb{A}}(\Omega)$, Eq. 3.23 remains valid upon replacing $|\mathbb{A}u(y)|\,\mathrm{d}y$ by $\mathrm{d}|\mathbb{A}u|\,(y)$.

Proof Let $x \in \Omega$. Slightly abusing notation, let $m \in \mathbb{N}$ and let $(W_j)_{j=0}^m$ be an emanating chain of balls connecting x with the central ball B such that $W_0 = B$ and $x \in W_m$ in the sense of Definition 2.3, and we remark that each $\sigma_1 W_j$ is still contained in Ω. For the following argument, we may assume that $x \notin \sigma_1 W_j$ for all $j \in \{0, ..., m-1\}$ as otherwise we may take the minimal j_0 with $x \in \sigma_1 W_{j_0}$, $x \notin \sigma_1 W_i$ for $0 \leq i < j_0$ and redefine m to be j_0; we will only use that $x \in \sigma_1 W_m$ (and not that $x \in W_m$) in the sequel.

Now consider the chain $\mathbb{W}_0 := W_0, ..., \mathbb{W}_{m-1} := W_{m-1}$ and $\mathbb{W}_m := \sigma_1 W_m$. It is then easy to see that this chain satisfies the following modification of (C2): For any $0 \leq j \leq j' \leq m$ we have $\mathbb{W}_{j'} \subset \sigma_1\sigma_2\mathbb{W}_j$ and each $W_j \cap W_{j+1}$, $j \in \{0, ..., m-1\}$ contains an open ball $\mathbb{B}_j$ with $\mathbb{W}_j \cup \mathbb{W}_{j+1} \subset \sigma_1\sigma_2\mathbb{B}_j$. As a main consequence of this slightly modified construction, every $y \in \mathbb{W}_j$, $j \in \{0, ..., m-1\}$ is at least $(\sigma_1 - 1)r(\mathbb{W}_j)$ distant apart from x, which follows from $x \in \sigma_1 W_m$ and $x \notin \sigma_1 W_j$ for any $0 \leq j \leq m-1$ by the above minimality of m. On the other hand, since $x \in \sigma_1 W_m = \mathbb{W}_m \subset \sigma_1\sigma_2\mathbb{W}_j$ for any $j \in \{0, ..., m\}$, we have $|x - y| \leq 2\sigma_1\sigma_2 r(W_j)$ for all $y \in W_j$. In conclusion, we have

$$(\sigma_1 - 1)r(\mathbb{W}_j) \leq |x - y| \leq 2\sigma_1\sigma_2 r(\mathbb{W}_j) \quad \text{for all } 1 \leq j \leq m-1 \text{ and } y \in \mathbb{W}_j. \tag{3.24}$$

Define $\Pi_{\mathbb{A}}^{\mathbb{W}_j}$ and $\Pi_{\mathbb{A}}^{\mathbb{B}_j}$ as in Proposition 3.3. Then, since $\Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}\Pi_{\mathbb{A}}^{\mathbb{B}_i} = \Pi_{\mathbb{A}}^{\mathbb{B}_i}$, $i \in \{j+1, j\}$,

$$\begin{aligned}
\left|D^{\ell}(u - \Pi_{\mathbb{A}}^{\mathbb{W}_0}u)(x)\right| &\leq \left|D^{\ell}(u - \Pi_{\mathbb{A}}^{\mathbb{B}_{m-1}}u)(x)\right| + \left|D^{\ell}(\Pi_{\mathbb{A}}^{\mathbb{W}_0} - \Pi_{\mathbb{A}}^{\mathbb{B}_0}u)(x)\right| \\
&\quad + \sum_{j=0}^{m-2}\left(\left|D^{\ell}(\Pi_{\mathbb{A}}^{\mathbb{B}_{j+1}}u - \Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}u)(x)\right| + \left|D^{\ell}(\Pi_{\mathbb{A}}^{\mathbb{B}_j}u - \Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}u)(x)\right|\right) \\
&\leq \left|D^{\ell}(u - \Pi_{\mathbb{A}}^{\mathbb{B}_{m-1}}u)(x)\right| + \left|D^{\ell}(\Pi_{\mathbb{A}}^{\mathbb{W}_0} - \Pi_{\mathbb{A}}^{\mathbb{B}_0}u)(x)\right| \\
&\quad + \sum_{j=0}^{m-2}\left(\left|D^{\ell}\Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}(\Pi_{\mathbb{A}}^{\mathbb{B}_{j+1}}u - u)(x)\right| + \left|D^{\ell}\Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}(u - \Pi_{\mathbb{A}}^{\mathbb{B}_j}u)(x)\right|\right) \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Then, since $x \in \mathbb{W}_m \subset \sigma_1\sigma_2\mathbb{W}_{j+1}$ for all $0 \leq j \leq m-2$, we obtain by inverse estimates

$$\begin{aligned}
\mathrm{III} &\leq \sum_{j=0}^{m-2}\sum_{i\in\{j,j+1\}} \|D^{\ell}\Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}(\Pi_{\mathbb{A}}^{\mathbb{B}_i}u - u)\|_{\mathrm{L}^{\infty}(\sigma_1\sigma_2\mathbb{W}_{j+1})} \\
&\overset{Eq.3.16}{\leq} c\sum_{j=0}^{m-2}\sum_{i\in\{j,j+1\}} r(\sigma_1\sigma_2\mathbb{W}_{j+1})^{-\ell}\times \\
&\qquad\qquad \times\left(r(\sigma_1\sigma_2\mathbb{W}_{j+1})^{\ell} \fint_{\sigma_1\sigma_2\mathbb{W}_{j+1}} |D^{\ell}\Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}(\Pi_{\mathbb{A}}^{\mathbb{B}_i}u - u)|\,\mathrm{d}y\right)
\end{aligned}$$

$$
\begin{aligned}
&\overset{Eq.3.18_2}{\leq} c\sum_{j=0}^{m-2}\sum_{i\in\{j,j+1\}} r(\mathbb{W}_{j+1})^{-\ell}\Big(\fint_{\sigma_1\sigma_2\mathbb{W}_{j+1}} |\Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}(\Pi_{\mathbb{A}}^{\mathbb{B}_i}u-u)|\,\mathrm{d}y\Big)\\
&\leq c\sum_{j=0}^{m-2}\sum_{i\in\{j,j+1\}} r(\mathbb{W}_{j+1})^{-\ell}\Big(\fint_{\mathbb{W}_{j+1}} |\Pi_{\mathbb{A}}^{\mathbb{W}_{j+1}}(\Pi_{\mathbb{A}}^{\mathbb{B}_i}u-u)|\,\mathrm{d}y\Big)\\
&\overset{Eq.3.17}{\leq} c\sum_{j=0}^{m-2}\sum_{i\in\{j,j+1\}} r(\mathbb{W}_{j+1})^{-\ell}\Big(\fint_{\mathbb{W}_{j+1}} |u-\Pi_{\mathbb{A}}^{\mathbb{B}_i}u|\,\mathrm{d}y\Big) =: c\sum_{j=0}^{m-2}\mathrm{III}_j,
\end{aligned}
$$

where $c=c(\sigma_1,\sigma_2,\mathbb{A})>0$ is a constant. Since each $\mathbb{W}_{j+1}$, $j\in\{0,\dots,m-2\}$, is starshaped with respect to both $\mathbb{B}_{j+1}$ and $\mathbb{B}_j$, we obtain by use of Corollary 3.6 with a constant $c=c(\sigma_1,\sigma_2,\mathbb{A})>0$ and Fubini's theorem for $i\in\{j,j+1\}$

$$
\begin{aligned}
\mathrm{III}_j &\leq c\sum_{i\in\{j,j+1\}} r(\mathbb{W}_{j+1})^{k-\ell-n}\int_{\mathbb{W}_{j+1}} |\mathbb{A}u(z)|\,\mathrm{d}z\\
&\leq cr(\mathbb{W}_{j+1})^{k-n-\ell}\int_{\mathbb{W}_{j+1}} |\mathbb{A}u(z)|\,\mathrm{d}z \overset{Eq.3.24}{\leq} c\int_{\mathbb{W}_{j+1}} |x-z|^{k-n-\ell}|\mathbb{A}u(z)|\,\mathrm{d}z =: \mathrm{IV}_j.
\end{aligned}
$$

On the other hand, the term I can clearly be bounded against IV_j with $j=m-1$, whereas the term II is estimated along the same lines as III, now using that $\mathbb{W}_0$ is starshaped with respect to $\mathbb{B}_0$. In consequence, by the uniformly finite overlap of the balls $W_0,\dots,W_m$ implied by (C2), we conclude Eq. 3.23. The proof is hereby complete. □

4 Proof of Theorem 1.1: The Trace Estimate via Riesz Potentials

In this section we establish Theorem 1.1. To this end, we recall in Section 4.1 the scale of Besov spaces and prove Theorem 1.1 in Sections 4.2–4.4.

4.1 Besov Spaces

For Theorem 1.1, we briefly recall the scale of Besov spaces and provide an instrumental characterisation by means of oscillations. Besov spaces can be introduced by different means, and we refer to Triebel [60, 62] for comprehensive overviews. In what follows, let $s>0$, $1\leq p<\infty$, $1\leq q<\infty$. Let $\varphi\in \mathrm{C}_c^\infty(B(0,2))$ satisfy $\mathbb{1}_{B(0,1)}\leq\varphi\leq\mathbb{1}_{B(0,2)}$ and put, for $j\in\mathbb{Z}$, $\varphi_j(\xi):=\varphi(2^{-j}\xi)-\varphi(2^{-j+1}\xi)$. We recall that the *Besov space* $\mathrm{B}^s_{p,q}(\mathbb{R}^n;V)$ consists of all $f\in\mathscr{S}'(\mathbb{R}^n;V)$ such that

$$
\|f\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} := \Big(\|(\varphi\widehat{f})^\vee\|^q_{\mathrm{L}^p(\mathbb{R}^n)}+\sum_{j=1}^\infty 2^{jsq}\|(\varphi_j\widehat{f})^\vee\|^q_{\mathrm{L}^p(\mathbb{R}^n)}\Big)^{\frac1q}<\infty.
$$

The *homogeneous Besov space* $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n;V)$ consists of all $f\in(\mathscr{S}'/\mathscr{P})(\mathbb{R}^n;V)$ with

$$
\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} := \Big(\sum_{j=-\infty}^\infty 2^{jsq}\|(\varphi_j\widehat{f})^\vee\|^q_{\mathrm{L}^p(\mathbb{R}^n)}\Big)^{\frac1q}<\infty. \tag{4.1}
$$

Note that, whereas Eq. 4.1 does not give rise to a norm on $\mathscr{S}'(\mathbb{R}^n;V)$ but only on the quotient $(\mathscr{S}'/\mathscr{P})(\mathbb{R}^n;V)$, it does define a norm on $\mathscr{S}(\mathbb{R}^n;V)$. By Bergh & Löfström [5, p. 148] we have for all $s > 0$ and $1 \le p, q < \infty$

$$\mathrm{B}^s_{p,q}(\mathbb{R}^n; V) \simeq (\mathrm{L}^p \cap \dot{\mathrm{B}}^s_{p,q})(\mathbb{R}^n; V). \tag{4.2}$$

Next, for $1 \le p < \infty$ and $M \in \mathbb{N}_0$, we define the (p, M)-*oscillation* of a measurable map $f : \mathbb{R}^n \to V$ at $x_0 \in \mathbb{R}^n$ by

$$\mathrm{osc}^M_p\, f(x_0, r) := \inf\Big\{\Big(\fint_{B(x_0,r)} |f(x) - \pi(x)|^p\, \mathrm{d}x\Big)^{\frac{1}{p}} :\ \pi \in \mathscr{P}_M(\mathbb{R}^n; V)\Big\}.$$

The next lemma is certainly clear to the experts, but we have been unable to find a precise reference; see, however, Dorronsoro [24] for a related result. In our arguments below, we will merely require inequality '$\lesssim$' in Eq. 4.3, and a quick argument for this is sketched in the Appendix, Section 7.2.

Lemma 4.1 *Let $1 \le p, q < \infty$ and $s > 0$. If $M \in \mathbb{N}$ satisfies $M > \lfloor s \rfloor$, then for all $1 \le u \le p$ we have the equivalence of norms*

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \simeq \Big(\int_0^\infty \|\,\mathrm{osc}^M_u\, f(\cdot, t)\|^q_{\mathrm{L}^p(\mathbb{R}^n)} \frac{\mathrm{d}t}{t^{1+sq}}\Big)^{\frac{1}{q}}, \qquad f \in \mathscr{S}(\mathbb{R}^n; V), \tag{4.3}$$

where the constants implicit in '$\simeq$' only depend on n, M, s, p, q, u and V.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^∞-boundary $\partial\Omega$ and outer unit normal $\nu = (\nu_1, ..., \nu_n)$. By definition, there exists $N \in \mathbb{N}$ and open balls $W_1, ..., W_N$ with $\partial\Omega \subset \bigcup_{j=1}^N W_j$ such that the following holds for all $j = 1, ..., N$: We have $\partial\Omega \cap W_j \neq \emptyset$, and there exists $F^{(j)} \in \mathrm{C}^\infty(\overline{W}_j; \mathbb{R}^n)$ with $F^{(j)} : W_j \to \mathbb{R}^n$ injective and having open, bounded image, $F^{(j)}(\partial\Omega \cap W_j) \subset \{x \in \mathbb{R}^n :\ x_n = 0\}$ and $F^{(j)}(\Omega \cap W_j) \subset \mathbb{R}^{n-1} \times (0, \infty)$ is open, simply connected and satisfies $\det(\nabla F^{(j)}) \neq 0$ on $\overline{W}_j$. Let $W_0 \Subset \Omega$ be open and bounded with C^∞-boundary ∂W_0 such that $\overline{\Omega} \subset \bigcup_{j=0}^N W_j$ and $(\psi_j)_{j=0}^N$ be a partition of unity subject to $(W_j)_{j=0}^N$. Following Triebel [60, Chpt. 3.2.2, Def. 2], for $s > 0$, $1 \le p, q < \infty$, we define the Besov norm

$$\|f\|_{\mathrm{B}^s_{p,q}(\partial\Omega)} := \sum_{j=1}^N \|(\psi_j f)(F^{(j)^{-1}}(\cdot, 0))\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^{n-1})}, \qquad f \in \mathscr{D}'(\partial\Omega; V). \tag{4.4}$$

In view of a convenient characterisation of trace spaces for smooth domains, we follow the slightly more general approach of Maz'ya, Mitrea & Shaposhnikova [50] and define for $m \in \mathbb{N}$, $1 < p < \infty$ and $0 < s < 1$

$$\dot{\mathrm{B}}^{m-1+s}_p(\partial\Omega; V) := \overline{\{(\partial^\alpha v|_{\partial\Omega})_{|\alpha| \le m-1} :\ v \in \mathrm{C}^\infty_c(\mathbb{R}^n; V)\}}^{(\mathrm{B}^s_{p,p}(\partial\Omega; V))^{\mathtt{M}}}, \tag{4.5}$$

where $\mathtt{M} := \#\{\alpha \in \mathbb{N}^n_0 :\ |\alpha| \le m - 1\}$. By [50, Prop. 7.7], $(f^\alpha)_{|\alpha| \le m-1}$ belongs to $\dot{\mathrm{B}}^{m-1+s}_p(\partial\Omega; V)$ if and only if

$$\begin{aligned} &f^\alpha \in \mathrm{B}^s_{p,p}(\partial\Omega; V) \qquad \text{for all } |\alpha| \le m-1, \quad \text{and} \\ &(\nu_j \partial_k - \nu_k \partial_j) f^\alpha = \nu_j f^{\alpha+e_k} - \nu_k f^{\alpha+e_j} \quad \text{for all } |\alpha| \le m-2,\ j, k \in \{1, ..., n\}. \end{aligned} \tag{4.6}$$

The natural compatibility condition $Eq.4.6_2$ on the tangential derivatives is a higher order variant of the Whitney array compatibility conditions considered by Adolfsson & Pipher [1]. As in [1, Def. 1.3 ff.], the equations of $Eq.4.6_2$ are understood in the sense of being fulfilled by any extension $\widetilde{f}^\alpha$ of f^α, for which the left-hand side of $Eq.4.6_2$ is well-defined, to a neighbourhood of $\partial\Omega$; since the operator $\nu_j\partial_k - \nu_k\partial_j$ is a tangential derivative on $\partial\Omega$, $(\nu_j\partial_k - \nu_k\partial_j)\widetilde{f}^\alpha$ will be independent of the particular extension and hence is well-defined.

Lemma 4.2 ([50, Prop. 7.3]) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with C^∞-boundary $\partial\Omega$. Let $1 < p < \infty$, $m \in \mathbb{N}$ and define $\mathrm{tr}_{m-1,\partial\Omega}$ on $\mathrm{W}^{m,p}(\Omega; V)$ by*

$$\mathrm{tr}_{m-1,\partial\Omega} f := \big(\mathrm{Tr}_{\partial\Omega}(\partial^\alpha f)\big)_{|\alpha|\leq m-1}, \qquad f \in \mathrm{W}^{m,p}(\Omega; V), \tag{4.7}$$

where $\mathrm{Tr}_{\partial\Omega}\colon \mathrm{W}^{1,p}(\Omega; V) \to \mathrm{B}^{1-1/p}_{p,p}(\partial\Omega; V)$ is the trace operator on $\mathrm{W}^{1,p}(\Omega; V)$ as usual. Then

$$\mathrm{tr}_{m-1,\partial\Omega}\colon \mathrm{W}^{m,p}(\Omega; V) \to \dot{\mathrm{B}}^{m-1/p}_{p}(\partial\Omega; V) \tag{4.8}$$

is a well-defined, linear, bounded operator which is onto. *Moreover, there exists a bounded, linear right-inverse $\mathcal{L}\colon \dot{\mathrm{B}}^{m-1/p}_{p}(\partial\Omega; V) \to \mathrm{W}^{m,p}(\Omega; V)$ of $\mathrm{tr}_{m-1,\partial\Omega}$ in the sense that, whenever $\dot{f} = (f^\alpha)_{|\alpha|\leq m-1} \in \dot{\mathrm{B}}^{m-1/p}_{p}(\partial\Omega; V)$, then*

$$\mathrm{Tr}_{\partial\Omega}(\partial^\alpha(\mathcal{L}\dot{f})) = f^\alpha \qquad \textit{for all } |\alpha| \leq m-1. \tag{4.9}$$

4.2 Geometric Setup and Potential Estimates

In this and the following subsection, we suppose that $\mathbb{A}$ is a $\mathbb{C}$-elliptic differential operator of the form Eq. 1.1. Moreover, we shall tacitly work with the particular halfspace $\mathbb{H} := \mathbb{R}^{n-1} \times (0,\infty)$, but our arguments are easily seen to generalise to any halfspace of $\mathbb{R}^n$.

We start by fixing notation: For some given $z' \in \mathbb{R}^{n-1}$ and $t > 0$ we denote

$$K_{z',t} := \{(y', y_n) \in \mathbb{R}^n : 0 \leq y_n \leq t,\ |y' - z'|^2 < 9y_n^2\} \tag{4.10}$$

the single-sided cone emanating from $(z', 0)$ in direction e_n, of height t and base radius $3t$. To distinguish between n- and $(n-1)$-dimensional balls, we put for $x' \in \mathbb{R}^{n-1}$, $x \in \mathbb{R}^n$ and $r > 0$

$$B_n(x, r) := B(x, r) \quad \text{and} \quad B_{n-1}(x', r) := \partial\mathbb{H} \cap B((x', 0), r).$$

Adopting this notation, let us note that in this situation there holds

$$B_n((x', t), \tfrac{t}{2}) \subset K_{y',2t} \qquad \text{for all } y' \in \mathbb{R}^{n-1} \text{ with } |x' - y'| < t. \tag{4.11}$$

In fact, let $(\xi', \xi_n) \in B_n((x', t), \frac{t}{2})$ so that, firstly, $\frac{1}{2}t \leq \xi_n \leq \frac{3}{2}t$ and hence $\frac{3}{2}t \leq 3\xi_n$. As a consequence,

$$|\xi' - y'| \leq |\xi' - x'| + |x' - y'| < \frac{3}{2}t \leq 3\xi_n$$

and so Eq. 4.11 follows. This situation is depicted in Fig. 2. Combining this setup with Proposition 3.3, we arrive at the following potential estimates on cones:

Lemma 4.3 (Conical Riesz potential estimate) *There exists $c = c(\mathbb{A}) > 0$ with the following property: If $x' \in \mathbb{R}^{n-1}$ and $t > 0$, then for every $y' \in \mathbb{R}^{n-1}$ with $|x' - y'| < t$ there holds for all $u \in \mathrm{C}^\infty(\overline{\mathbb{H}}; V)$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k-1$*

$$\begin{aligned} |(\partial^\alpha u)(y', 0) - (\partial^\alpha \Pi_{\mathbb{A}}^{B_n((x',t),\frac{t}{2})} u)(y', 0)| \\ \leq c \int_{K_{y',2t}} \frac{|\mathbb{A}u(z', z_n)|}{(|z' - y'| + |z_n|)^{n+|\alpha|-k}} \,\mathrm{d}(z', z_n). \end{aligned} \tag{4.12}$$

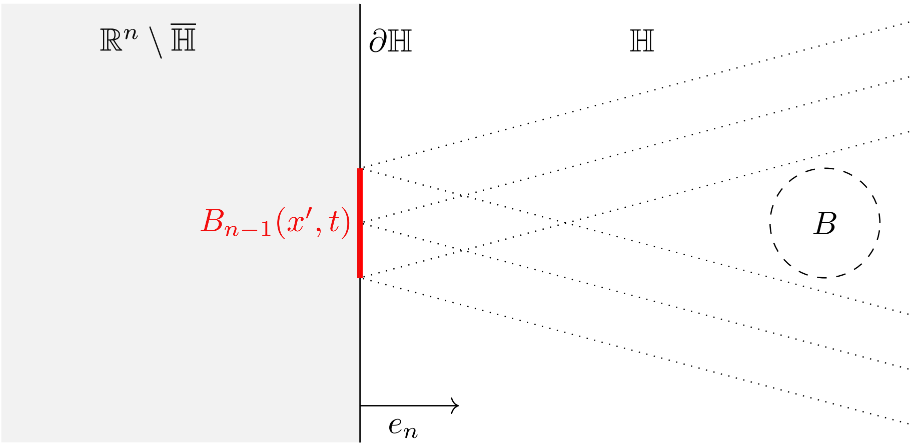

Fig. 2 Not-to-scale construction of the the umbra shadow estimate underlying the proof of Theorem 1.1. To control the oscillation of $u(\cdot, 0)$, the Riesz potential estimate from Lemma 4.3 is applied to shifted cones emanating from points in the red area, where all cones are star-shaped with respect to a joint ball B

Proof Let x', y' and t be as in the lemma. By Eq. 4.11, $B_n((x', t), \frac{t}{2}) \subset K_{y',2t}$ and clearly $K_{y',2t}$ is star-shaped with respect to $B_n((x', t), \frac{t}{2})$. Thus we may apply Proposition 3.3 to $\Omega = K_{y',2t}$ and $B = B_n((x', t), \frac{t}{2})$. Note that $\Pi^B_{\mathbb{A}} u := \widetilde{\Pi}_{\mathbb{A}} \mathbb{T}^B_{m-1} u$ is independent of y' and the diameters of $K_{y',2t}$ and $B_n((x', t), \frac{t}{2})$ are uniformly comparable. Hence Eq. 4.12 follows from (K1) and (K3) via

$$\left| \int_{K_{y',2t}} (\partial^\alpha_y \mathfrak{K}^B_{\mathbb{A}})((y', 0), z)\mathbb{A}u(z)\,\mathrm{d}z \right| \leq C \int_{K_{y',2t}} \frac{|\mathbb{A}u(z', z_n)|}{(|y' - z'| + |z_n|)^{n+|\alpha|-k}}\,\mathrm{d}(z', z_n).$$

The proof is complete. □

4.3 The Trace Estimate for Smooth Maps

After the preparations of the preceding subsection, we can now proceed to show that $\mathbb{C}$-ellipticity suffices for the trace estimate of Theorem 1.1 (b). We provide a slightly stronger variant that is also applicable to the $1 < p < \infty$-scenario of Theorem 4.4 below. More precisely, let $1 \leq p < \infty$ and let $\mathbb{A}$ be a $\mathbb{C}$-elliptic differential operator of the form Eq. 1.1 with $k \geq 2$. Moreover, let $u \in \mathrm{C}^\infty_c(\overline{\mathbb{H}}; V)$.

We distinguish between the cases $|\alpha| \leq k - 2$ and $|\alpha| = k - 1$, referred to as *first* and *second* case in the sequel. In the first case, we will establish that for each $1 \leq p < \infty$ there exists a constant $c = c(\mathbb{A}, p) > 0$ such that we have

$$\|(\partial^\alpha u)(\cdot, 0)\|_{\dot{\mathrm{B}}^{k-|\alpha|-1/p}_{p,p}(\partial\mathbb{H})} \leq c\|\mathbb{A}u\|_{\mathrm{L}^p(\mathbb{H})}, \tag{4.13}$$

whereas in the second case we will only strive for such a bound provided $1 < p < \infty$.

In particular, for the special choice $p = 1$, we will thereby obtain the sufficiency of $\mathbb{C}$-ellipticity for Eq. 1.7.

Proof of Eq. 4.13 in the single cases. For the first case, let $|\alpha| \leq k - 2$ and $1 \leq p < \infty$. Since $\mathbb{A}$ is $\mathbb{C}$-elliptic, we may choose $m := \deg_{\mathscr{P}}(\mathbb{A}) \in \mathbb{N}$ (cf. Eq. 3.6) so that $\partial^\alpha \ker(\mathbb{A}) \subset$

$\mathscr{P}_{m-1}(\mathbb{R}^n; V)(\subset \mathscr{P}_m(\mathbb{R}^n; V))$ and, with $s := k - |\alpha| - 1/p > 0$,

$$m > \deg_{\mathscr{P}}(\mathbb{A}) - 1 \geq k - |\alpha| - 1 = \lfloor k - |\alpha| - 1/p \rfloor = \lfloor s \rfloor.$$

Let $x' \in \partial\mathbb{H}$ and $t > 0$ be arbitrary. Since p and m are fixed, we put for a $\mathscr{H}^{n-1}$-locally integrable map $v \colon \partial\mathbb{H} \to V$

$$\operatorname{osc}^m_{p,\partial\mathbb{H}} v(x', t) := \inf \left\{ \left(\fint_{B_{n-1}(x',t)} |v - \pi|^p \, \mathrm{d}^{n-1} y \right)^{\frac{1}{p}} : \ \pi \in \mathscr{P}_m(\mathbb{R}^n; V) \right\} \tag{4.14}$$

In this situation, Lemma 4.1 implies with $U^\alpha(x') := (\partial^\alpha u)(x', 0)$

$$\|U^\alpha\|^p_{\dot{\mathrm{B}}^{k-|\alpha|-1/p}_{p,p}(\partial\mathbb{H})} \leq c \int_0^\infty \frac{\| \operatorname{osc}^m_{p,\partial\mathbb{H}} U^\alpha(\cdot, t)\|^p_{\mathrm{L}^p(\partial\mathbb{H})}}{t^{(k-|\alpha|)p}} \, \mathrm{d}t. \tag{4.15}$$

To bound the right-hand side, we use Lemma 4.3, $\partial^\alpha \ker(\mathbb{A}) \subset \mathscr{P}_m(\mathbb{R}^n; V)$ and Hölder's inequality to obtain

$$\begin{aligned}
|\operatorname{osc}^m_{p,\partial\mathbb{H}} U^\alpha(x', t)|^p &\leq \fint_{B_{n-1}(x',t)} |U^\alpha(y') - \partial^\alpha \Pi_{\mathbb{A}}^{B_n((x',t),\frac{t}{2})} u(y', 0)|^p \, \mathrm{d}^{n-1} y' \\
&\leq c \fint_{B_{n-1}(x',t)} \left| \int_0^{2t} \int_{\partial\mathbb{H}} \mathbb{1}_{\{|z'-y'|<3\tau\}} \frac{|\mathbb{A}u(z', \tau)|}{(|z'-y'|+\tau)^{n-k+|\alpha|}} \, \mathrm{d}^{n-1} z' \, \mathrm{d}\tau \right|^p \mathrm{d}^{n-1} y' \\
&\leq c \fint_{B_{n-1}(x',t)} t^{p-1} \int_0^{2t} \tau^{(n-1)(p-1)} \times \\
&\times \left(\int_{\partial\mathbb{H}} \mathbb{1}_{\{|z'-y'|<3\tau\}} \frac{|\mathbb{A}u(z', \tau)|^p}{(|z'-y'|+\tau)^{p(n-k+|\alpha|)}} \, \mathrm{d}^{n-1} z' \right) \mathrm{d}\tau \, \mathrm{d}^{n-1} y'.
\end{aligned} \tag{4.16}$$

Let us note that the very first of the previous chain of inequalities is where the $\mathbb{C}$-ellipticity enters as otherwise the nullspace does not consist of polynomials of a fixed maximal degree. In combination with Eq. 4.15, we thus obtain

$$\begin{aligned}
\|U^\alpha\|^p_{\dot{\mathrm{B}}^{k-|\alpha|-1/p}_{p,p}(\partial\mathbb{H})} &\leq c \int_0^\infty \int_{\partial\mathbb{H}} \fint_{B_{n-1}(x',t)} t^{p-1} \int_0^{2t} \tau^{(n-1)(p-1)} \times \\
&\times \left(\int_{\partial\mathbb{H}} \mathbb{1}_{\{|z'-y'|<3\tau\}} \frac{|\mathbb{A}u(z', \tau)|^p}{(|z'-y'|+\tau)^{p(n-k+|\alpha|)}} \, \mathrm{d}^{n-1} z' \right) \mathrm{d}\tau \, \mathrm{d}^{n-1} y' \, \mathrm{d}^{n-1} x' \frac{\mathrm{d}t}{t^{(k-|\alpha|)p}} \\
&\leq c \int_0^\infty \int_{\partial\mathbb{H}} \int_{\partial\mathbb{H}} \int_0^{2t} \tau^{(n-1)(p-1)} \mathbb{1}_{\{|x'-y'|<t\}} \times \\
&\quad \times \left(\int_{\partial\mathbb{H}} \frac{\mathbb{1}_{\{|z'-y'|<3\tau\}} |\mathbb{A}u(z', \tau)|^p}{(|z'-y'|+\tau)^{p(n-k+|\alpha|)}} \, \mathrm{d}^{n-1} z' \right) \mathrm{d}\tau \, \mathrm{d}^{n-1} y' \, \mathrm{d}^{n-1} x' \frac{\mathrm{d}t}{t^{(k-|\alpha|-1)p+n}} = (*)
\end{aligned}$$

Now, if $\mathbb{1}_{\{|z'-y'|<3\tau\}} = 1$, then $\tau \leq |z'-y'| + \tau \leq 4\tau$. Therefore,

$$\begin{aligned}
(*) \leq c \int_0^\infty \int_{\partial\mathbb{H}} \int_{\partial\mathbb{H}} \int_{\partial\mathbb{H}} \int_0^{2t} \tau^{(n-1)(p-1)} \mathbb{1}_{\{|z'-y'|<3\tau\}} \mathbb{1}_{\{|x'-y'|<t\}} \times \\
\times \frac{|\mathbb{A}u(z', \tau)|^p}{\tau^{p(n-k+|\alpha|)}} \, \mathrm{d}\tau \, \mathrm{d}^{n-1} z' \, \mathrm{d}^{n-1} y' \, \mathrm{d}^{n-1} x' \frac{\mathrm{d}t}{t^{(k-|\alpha|-1)p+n}}
\end{aligned}$$

$$
\begin{aligned}
&\leq c\int_0^\infty\int_{\partial\mathbb{H}}\int_{\partial\mathbb{H}}\int_0^{2t}\mathbb{1}_{\{|z'-y'|<3\tau\}}\times \\
&\qquad\qquad\qquad\times\frac{|\mathbb{A}u(z',\tau)|^p}{\tau^{n-1+p(1+|\alpha|-k)}}\,\mathrm{d}\tau\,\mathrm{d}^{n-1}z'\,\mathrm{d}^{n-1}y'\frac{\mathrm{d}t}{t^{(k-|\alpha|-1)p+1}} \\
&\leq c\int_0^\infty\int_{\partial\mathbb{H}}\int_0^{2t}\frac{|\mathbb{A}u(z',\tau)|^p}{\tau^{p(1+|\alpha|-k)}}\,\mathrm{d}\tau\,\mathrm{d}^{n-1}z'\frac{\mathrm{d}t}{t^{(k-|\alpha|-1)p+1}} \\
&= c\int_0^\infty\int_{\partial\mathbb{H}}\int_0^\infty\mathbb{1}_{(0,2t)}(\tau)|\mathbb{A}u(z',\tau)|^p\frac{\mathrm{d}\tau}{\tau^{p(1+|\alpha|-k)}}\,\mathrm{d}^{n-1}z'\frac{\mathrm{d}t}{t^{(k-|\alpha|-1)p+1}} \\
&= c\int_0^\infty\int_{\partial\mathbb{H}}\int_0^\infty\mathbb{1}_{(0,2t)}(\tau)|\mathbb{A}u(z',\tau)|^p\frac{\mathrm{d}t}{t^{(k-1-|\alpha|)p+1}}\,\mathrm{d}^{n-1}z'\frac{\mathrm{d}\tau}{\tau^{p(1+|\alpha|-k)}} \\
&\leq c\int_0^\infty\int_{\partial\mathbb{H}}\Big(\int_{\frac{\tau}{2}}^\infty\frac{\mathrm{d}t}{t^{(k-1-|\alpha|)p+1}}\Big)|\mathbb{A}u(z',\tau)|^p\,\mathrm{d}^{n-1}z'\frac{\mathrm{d}\tau}{\tau^{p(1+|\alpha|-k)}} \\
&\overset{k-|\alpha|-1>0}{\leq} c\int_0^\infty\int_{\partial\mathbb{H}}|\mathbb{A}u(z',\tau)|^p\,\mathrm{d}^{n-1}z'\,\mathrm{d}\tau \\
&= c\|\mathbb{A}u\|^p_{\mathrm{L}^p(\mathbb{H})}.
\end{aligned}
$$

For the second case, the reader will notice that this argument requires refinement for $|\alpha|=k-1$ since in this case, the penultimate estimation cannot be accomplished. Hence let $1<p<\infty$ and $|\alpha|=k-1$; in this case, the reader will notice that we may even allow for $k=1$. Defining $\mathrm{osc}^m_{1,\partial\mathbb{H}}$ in the analogous manner as inEq. 4.14, we imitate Eq. 4.16 to find by changing variables $z'=x'-\xi'$ and $\eta'=x'-y'$ in the third step

$$
\begin{aligned}
|\,\mathrm{osc}^m_{1,\partial\mathbb{H}}U^\alpha(x',t)| &\leq \frac{c}{t^{n-1}}\int_0^{2t}\int_{\partial\mathbb{H}}\int_{\partial\mathbb{H}}\mathbb{1}_{\{|x'-y'|<t\}}\mathbb{1}_{\{|z'-y'|<3\tau\}}\times \\
&\qquad\qquad\times\frac{|\mathbb{A}u(z',\tau)|}{(|z'-y'|+\tau)^{n-1}}\,\mathrm{d}^{n-1}y'\,\mathrm{d}^{n-1}z'\,\mathrm{d}\tau \\
&= \frac{c}{t^{n-1}}\int_0^{2t}\int_{\partial\mathbb{H}}|\mathbb{A}u(z',\tau)|\Big(\int_{\partial\mathbb{H}}\mathbb{1}_{\{|x'-y'|<t\}}\mathbb{1}_{\{|z'-y'|<3\tau\}}\times \\
&\qquad\qquad\times\frac{1}{(|z'-y'|+\tau)^{n-1}}\,\mathrm{d}^{n-1}y'\Big)\,\mathrm{d}^{n-1}z'\,\mathrm{d}\tau \\
&= \frac{c}{t^{n-1}}\int_0^{2t}\int_{\partial\mathbb{H}}|\mathbb{A}u(x'-\xi',\tau)|\mathbb{1}_{\{|\xi'|\leq 7t\}}\Big(\int_{\partial\mathbb{H}}\mathbb{1}_{\{|\eta'|<t\}}\mathbb{1}_{\{|\eta'-\xi'|<3\tau\}}\times \\
&\qquad\qquad\times\frac{1}{(|\eta'-\xi'|+\tau)^{n-1}}\,\mathrm{d}^{n-1}\eta'\Big)\,\mathrm{d}^{n-1}\xi'\,\mathrm{d}\tau=(**),
\end{aligned}
$$

where we have used that if $|\xi'|>7t$, then $|\eta'|<t$ yields

$$3\tau\overset{\tau\leq 2t}{\leq}6t=7t-t<|\xi'|-|\eta'|\leq|\xi'-\eta'|$$

and so $\mathbb{1}_{\{|\eta'|<t\}}\mathbb{1}_{\{|\eta'-\xi'|<3\tau\}}=\mathbb{1}_{\{|\eta'|<t\}}\mathbb{1}_{\{|\eta'-\xi'|<3\tau\}}\mathbb{1}_{\{|\xi'|\leq 7t\}}$. Since we have

$$c\int_{B_{n-1}(0,3\tau)}\frac{\mathrm{d}^{n-1}\theta}{(|\theta|+\tau)^{n-1}}\leq c\frac{\mathscr{H}^{n-1}(B_{n-1}(0,3\tau))}{\tau^{n-1}}\leq c$$

as a bound for the very inner integral in $(**)$, we thus conclude

$$|\,\mathrm{osc}^m_{1,\partial\mathbb{H}}U^\alpha(x',t)|\leq(**)\leq\frac{c}{t^{n-1}}\int_0^{2t}\int_{\partial\mathbb{H}}|\mathbb{A}u(x'-\xi',\tau)|\mathbb{1}_{\{|\xi'|\leq 7t\}}\,\mathrm{d}^{n-1}\xi'\,\mathrm{d}\tau.$$

Integrating the previous inequality with respect to $x' \in \partial\mathbb{H}$, we choose $0 < \delta < p-1$ to find by Hölder's inequality and a change of variables

$$\begin{aligned}
&\|\operatorname{osc}^m_{1,\partial\mathbb{H}} U^\alpha(\cdot,t)\|^p_{\mathrm{L}^p(\partial\mathbb{H})} \\
&\leq \frac{c}{t^{p(n-1)}}\int_{\partial\mathbb{H}}\left|\int_0^{2t}\frac{1}{\tau^{\frac{\delta}{p}}}\int_{\partial\mathbb{H}}\tau^{\frac{\delta}{p}}\mathbb{1}_{\{|\xi'|\leq 7t\}}|\mathbb{A}u(x'-\xi',\tau)|\,\mathrm{d}^{n-1}\xi'\,\mathrm{d}\tau\right|^p\mathrm{d}^{n-1}x' \\
&\leq \frac{ct^{(p-1)(n-1)}}{t^{p(n-1)}}\int_{\partial\mathbb{H}}\Big(\int_0^{2t}\frac{\mathrm{d}\tau}{\tau^{\frac{\delta}{p-1}}}\Big)^{p-1}\times \\
&\quad\times\Big(\int_0^{2t}\tau^\delta\int_{\partial\mathbb{H}}\mathbb{1}_{\{|\xi'|\leq 7t\}}|\mathbb{A}u(x'-\xi',\tau)|^p\,\mathrm{d}^{n-1}\xi'\,\mathrm{d}\tau\Big)\,\mathrm{d}^{n-1}x' \\
&\leq \frac{ct^{(p-1)(n-1)}}{t^{p(n-1)}}t^{p+n-2-\delta}\Big(\int_0^{2t}\tau^\delta\int_{\partial\mathbb{H}}|\mathbb{A}u(\xi',\tau)|^p\,\mathrm{d}^{n-1}\xi'\,\mathrm{d}\tau\Big) \\
&\leq ct^{p-1-\delta}\int_0^{2t}\tau^\delta\|\mathbb{A}u(\cdot,\tau)\|^p_{\mathrm{L}^p(\partial\mathbb{H})}\,\mathrm{d}\tau.
\end{aligned}\tag{4.17}$$

Working from Eq. 4.17, we then obtain

$$\begin{aligned}
\|U^\alpha\|^p_{\dot{\mathrm{B}}^{1-1/p}_{p,p}(\partial\mathbb{H})} &\leq c\int_0^\infty t^{p-1-\delta}\int_0^{2t}\tau^\delta\|\mathbb{A}u(\cdot,\tau)\|^p_{\mathrm{L}^p(\partial\mathbb{H})}\,\mathrm{d}\tau\,\frac{\mathrm{d}t}{t^p} \\
&= c\int_0^\infty\Big(\int_{\frac{\tau}{2}}^\infty\frac{\mathrm{d}t}{t^{1+\delta}}\Big)\tau^\delta\|\mathbb{A}u(\cdot,\tau)\|^p_{\mathrm{L}^p(\partial\mathbb{H})}\,\mathrm{d}\tau \\
&\leq c\|\mathbb{A}u\|^p_{\mathrm{L}^p(\mathbb{H})}.
\end{aligned}$$

This completes the proof. □

Let us note that for $p = 1$ and $|\alpha| = 0$, the above proof works *and is bound to work* for $k \geq 2$ only. Indeed, for $k = 1$, $\mathrm{L}^1(\partial\mathbb{H}; V)$-trace estimates are optimal [11, 29]. However,

$$\dot{\mathrm{B}}^0_{1,1}(\partial\mathbb{H}) \hookrightarrow \dot{\mathrm{F}}^0_{1,2}(\partial\mathbb{H}) = \mathcal{H}^1(\partial\mathbb{H}) \subsetneq \mathrm{L}^1(\partial\mathbb{H}),\tag{4.18}$$

with the homogeneous Hardy space $\mathcal{H}^1$, cf. Triebel [60, Chpt. 5.2.4], so that maps with a finite $\dot{\mathrm{B}}^0_{1,1}(\partial\mathbb{H})$-norm need to have integral zero (cf. [52, Prop. 5.8]). Also note that the oscillation characterisation in the zero order case does not hold, cf. [62, Chpt. 3.5.1].

4.4 Proof of Theorem 1.1 and its Counterpart for $1 < p < \infty$

In this section we prove Theorem 1.1 and state and prove its counterpart for $1 < p < \infty$, Theorem 4.4 below, where stronger statements are available.

Proof of Theorem 1.1 Implication (a)⇒(b) has been established in Section 4.3. On the other hand, since $\dot{\mathrm{B}}^{k-1}_{1,1}(\partial\mathbb{H}; V) \hookrightarrow \dot{\mathrm{W}}^{k-1,1}(\partial\mathbb{H}; V)$ for any open halfspace $\mathbb{H}$, (b) yields the inequality $\|D^{k-1}u\|_{\mathrm{L}^1(\partial\mathbb{H})} \leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{H})}$ for all $u \in \mathrm{C}^\infty_c(\overline{\mathbb{H}}; V)$. By Raita, Van Schaftingen and the second author [31, Thm. 5.2], this already implies that $\mathbb{A}$ is $\mathbb{C}$-elliptic, and the proof is complete. □

The rest of the section is devoted to a self-contained proof of the following theorem in the spirit of [4, 42, 55], where our main focus now is on the equivalence of sharp trace estimates and Korn-type inequalities for smooth domains. Here, sharp trace estimates give access to global Calderón-Zygmund estimates, which is fundamentally different from the non-smooth context of John domains as discussed in Section 6.

Theorem 4.4 (Higher order traces, $1 < p < \infty$) *Let $k \in \mathbb{N}$, $k \geq 1$ and let $1 < p < \infty$. Then the following are equivalent for a k-th order differential operator $\mathbb{A}$ of the form Eq. 1.1:*

(a) $\mathbb{A}$ is $\mathbb{C}$-elliptic.
(b) There exists a constant $c = c(p, \mathbb{A}) > 0$ such that for any open halfspace $\mathbb{H}$ and all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k-1$ there holds

$$\|\partial^\alpha u\|_{\dot{\mathrm{B}}^{k-|\alpha|-1/p}_{p,p}(\partial\mathbb{H})} \leq c\|\mathbb{A}u\|_{\mathrm{L}^p(\mathbb{H})} \qquad \text{for all } u \in \mathrm{C}_c^\infty(\overline{\mathbb{H}}; V). \tag{4.19}$$

(c) For every open and bounded domain Ω with boundary of class C^∞, there exists a constant $c = c(p, \mathbb{A}, \Omega) > 0$ such that for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k-1$ there holds

$$\|\partial^\alpha u\|_{\mathrm{B}^{k-|\alpha|-1/p}_{p,p}(\partial\Omega)} \leq c(\|u\|_{\mathrm{W}^{k-1,p}(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)}) \ \text{ for all } u \in \mathrm{C}^\infty(\overline{\Omega}; V). \tag{4.20}$$

(d) For every open and bounded domain Ω with boundary of class C^∞ and all $1 < p < \infty$, there exists a constant $c = c(p, \mathbb{A}, \Omega) > 0$ such that

$$\sum_{j=0}^{k} \|\nabla^j u\|_{\mathrm{L}^p(\Omega)} \leq c(\|u\|_{\mathrm{W}^{k-1,p}(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)}) \qquad \text{for all } u \in \mathrm{C}^\infty(\overline{\Omega}; V), \tag{4.21}$$

In particular, if any of (a)–(d) holds, then we have (with $\mathrm{W}^{\mathbb{A},p}$ and $\widetilde{\mathrm{W}}^{\mathbb{A},p}$ as in Section 3.1) for any open and bounded domain with smooth boundary $\partial\Omega$

$$\mathrm{W}^{k,p}(\Omega; V) \simeq \mathrm{W}^{\mathbb{A},p}(\Omega) \simeq \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega). \tag{4.22}$$

It will be clear from the proof that some implications also persist for less regular domains (so, e.g., (a)$\Rightarrow$(c) would also follow for domains satisfying a cone condition by a slight variation of the arguments of Section 4.3), but we stick to smooth domains for ease of exposition. For the proof of Theorem 4.4, we require two preparatory lemmas:

Lemma 4.5 (Peetre-Tartar lemma, [59, Lem. 11.1]) *Let $(E_i, \|\cdot\|_{E_i})$, $i \in \{1, 2, 3\}$, be three Banach spaces, $A\colon E_1 \to E_2$ a linear and bounded operator and $B\colon E_1 \to E_3$ be a linear and compact operator. If $x \mapsto \|Ax\|_{E_2} + \|Bx\|_{E_3}$ is a norm on E_1 that is equivalent to $\|\cdot\|_{E_1}$, then $\dim(\ker(A)) < \infty$.*

Lemma 4.6 (Traces and Gauß-Green) *Suppose that assertion (c) of Theorem 4.4 holds. Then for every $1 < p < \infty$ and every open and bounded domain $\Omega \subset \mathbb{R}^n$ with boundary of class C^∞ there exists a linear and bounded* boundary trace operator

$$\mathrm{tr}^{\mathbb{A}}_{k-1,\partial\Omega}\colon \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega) \ni u \mapsto \big(\mathrm{tr}^{\mathbb{A}}_{\alpha,\partial\Omega}(u)\big)_{|\alpha|\leq k-1} \in \dot{\mathrm{B}}^{k-1/p}_{p}(\partial\Omega; V) \tag{4.23}$$

such that, in particular, $\mathrm{tr}^{\mathbb{A}}_{k-1,\partial\Omega}(u) = ((\partial^\alpha u)|_{\partial\Omega})_{|\alpha|\leq k-1}$ holds for all $u \in \mathrm{C}^\infty(\overline{\Omega}; V)$ and so $\mathrm{tr}^{\mathbb{A}}_{k-1,\partial\Omega} = \mathrm{tr}_{k-1,\partial\Omega}$ on $\mathrm{W}^{k,p}(\Omega; V)$ with $\mathrm{tr}_{k-1,\partial\Omega}$ as in Lemma 4.2.

For $\ell \in \{0, \ldots, k-1\}$, put $\mathrm{tr}^{\mathbb{A},\ell}_{\partial\Omega}(u) := (\mathrm{tr}^{\mathbb{A}}_{\alpha,\partial\Omega}(u))_{|\alpha|=\ell}$. For each $l \in \{1, \ldots, n\}$ and each $s \in \{1, \ldots, k\}$ there exist continuous bilinear forms $\mathscr{B}^{\mathbb{A}}_{l,s}\colon \odot^{k-s}(\mathbb{R}^n; V) \times \odot^{s-1}(\mathbb{R}^n; W) \to \mathbb{R}$ such that for all $u \in \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$ and all $\varphi \in \mathrm{C}^\infty(\overline{\Omega}; W)$ we have the Gauß-Green identity *(with $\nu = (\nu_1, \ldots, \nu_n)$ denoting the outer unit normal to $\partial\Omega$)*

$$\begin{aligned}\int_\Omega \mathbb{A}u \cdot \varphi \,\mathrm{d}x &= (-1)^k \int_\Omega u \cdot \mathbb{A}^*\varphi \,\mathrm{d}x \\ &\quad + \sum_{l=1}^n \sum_{s=1}^k \int_{\partial\Omega} \mathscr{B}_{l,s}^{\mathbb{A}}\Big(\mathrm{tr}_{\partial\Omega}^{\mathbb{A},k-s}(u), \nabla^{s-1}\varphi\Big)\nu_l \,\mathrm{d}\mathscr{H}^{n-1},\end{aligned} \tag{4.24}$$

where $\mathbb{A}^* = \sum_{|\alpha|=k} \mathbb{A}_\alpha^\top \partial^\alpha$ is the formal L^2-adjoint of $\mathbb{A}$.

Proof Since Ω has C^∞-boundary, an argument similar to that in, e.g., [26, Chpt. 5.3.3, Thm. 5.3] yields that for every $u \in \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$ there exists $(u_j) \subset \mathrm{C}^\infty(\overline{\Omega}; V)$ such that $\|u - u_j\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)} \to 0$ as $j \to \infty$. Let $\alpha \in \mathbb{N}_0^n$ satisfy $|\alpha| \le k-1$. By Eq. 4.20, $((\partial^\alpha u_j)|_{\partial\Omega})$ is Cauchy in $\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\partial\Omega; V)$ and therefore converges to some $u^\alpha =: \mathrm{tr}_{\alpha,\partial\Omega}^{\mathbb{A}}(u) \in \mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\partial\Omega; V)$, which is clearly independent of the approximating sequence and hereafter well-defined. To conclude Eq. 4.23, we note that $\{u^\alpha : |\alpha| \le k-1\}$ matches the Besov-Whitney array condition $Eq.4.6_2$ $\mathscr{H}^{n-1}$-a.e. on $\partial\Omega$; indeed, for any $|\alpha| \le k-2$ and all $i, l \in \{1, \ldots, n\}$ we have

$$\begin{aligned}&\|\nu_i\partial_l u^\alpha - \nu_l\partial_i u^\alpha - (\nu_i u^{\alpha+e_l} - \nu_l u^{\alpha+e_i})\|_{\mathrm{L}^p(\partial\Omega)} \\ &\quad\le \|(\nu_i\partial_l u^\alpha - \nu_l\partial_i u^\alpha) - (\nu_i\partial_l\partial^\alpha u_j - \nu_l\partial_i\partial^\alpha u_j)\|_{\mathrm{L}^p(\partial\Omega)} \\ &\quad+ \|\nu_i u^{\alpha+e_l} - \nu_i\partial^{\alpha+e_l}u_j\|_{\mathrm{L}^p(\partial\Omega)} + \|\nu_l u^{\alpha+e_i} - \nu_l\partial^{\alpha+e_i}u_j\|_{\mathrm{L}^p(\partial\Omega)} \\ &\quad+ \|\underbrace{\nu_i\partial_l\partial^\alpha u_j - \nu_l\partial_i\partial^\alpha u_j - (\nu_i\partial^{\alpha+e_l}u_j - \nu_l\partial^{\alpha+e_i}u_j)}_{=0}\|_{\mathrm{L}^p(\partial\Omega)} =: \mathrm{I}_j + \ldots + \mathrm{IV}_j.\end{aligned}$$

Since $(\nu_i\partial_l - \nu_l\partial_i)$ is a tangential differential operator on the smooth boundary $\partial\Omega$, we have with the full tangential gradient ∇_τ along $\partial\Omega$

$$\mathrm{I}_j \le \|\nabla_\tau(u^\alpha - \partial^\alpha u_j)\|_{\mathrm{L}^p(\partial\Omega)} \overset{(*)}{\le} c\,\|u^\alpha - \partial^\alpha u\|_{\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\partial\Omega)} \to 0, \qquad j \to \infty,$$

as $k - |\alpha| - 1/p \ge 2 - \frac{1}{p} > 1$ by our assumption on α and p. Up to localisation and flattening, the inequality $(*)$ is a consequence of the embedding $\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1}) \hookrightarrow \mathrm{F}_{p,2}^1(\mathbb{R}^{n-1}) \simeq \mathrm{W}^{1,p}(\mathbb{R}^{n-1})$. Namely, if $1 < p \le 2$, we have that

$$\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1}) \overset{k-|\alpha|\ge 2}{\hookrightarrow} \mathrm{B}_{p,p}^{2-1/p}(\mathbb{R}^{n-1}) \overset{p\le 2}{\hookrightarrow} \mathrm{F}_{p,2}^{2-1/p}(\mathbb{R}^{n-1}) \overset{2-1/p>1}{\hookrightarrow} \mathrm{F}_{p,2}^1(\mathbb{R}^{n-1}),$$

and if $p > 2$, we may choose $\varepsilon > 0$ so small such that $1 < 2 - \frac{1}{p} - \varepsilon$ and obtain

$$\begin{aligned}\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1}) &\overset{k-|\alpha|\ge 2}{\hookrightarrow} \mathrm{B}_{p,2}^{2-1/p-\varepsilon}(\mathbb{R}^{n-1}) \overset{p\ge 2}{\hookrightarrow} \mathrm{F}_{p,2}^{2-1/p-\varepsilon}(\mathbb{R}^{n-1}) \\ &\overset{2-1/p-\varepsilon>1}{\hookrightarrow} \mathrm{F}_{p,2}^1(\mathbb{R}^{n-1}),\end{aligned}$$

from where the underlying inequality follows by virtue of $\mathrm{F}_{p,2}^1(\mathbb{R}^{n-1}) \simeq \mathrm{W}^{1,p}(\mathbb{R}^{n-1})$.

The terms $\mathrm{II}_j, \mathrm{III}_j$ vanish in the limit by construction, and $\mathrm{IV}_j = 0$ for all $j \in \mathbb{N}$; thus, Eq. 4.23 follows. Equation 4.24 follows for maps u and φ which are smooth up to $\partial\Omega$ by repeated use of the usual Gauß-Green theorem, and then inherits to $u \in \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$ by means of smooth approximation. □

***Proof of Theorem* 4.4** Let $1 < p < \infty$. Implication (a)$\Rightarrow$(b) has already been proved in Section 4.3. For implication (b)$\Rightarrow$(c), we assume that Ω is connected, open and bounded with C^{∞}-boundary; if it is not connected, we apply the following to each of the finitely many connected components of Ω. Let $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$. Adopting the notation from Eq. 4.4, for $j \in \{1, \ldots, N\}$, (c) and Eq. 4.2 yield for any $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k-1$ and all $\beta \in \mathbb{N}_0^n$ for the compactly supported, smooth map $\mathrm{I}_{j,\beta} := ((\partial^{\beta}\psi_j)u)(F^{(j)^{-1}}(\cdot, 0))$

$$\begin{aligned}\|\partial^{\alpha}\mathrm{I}_{j,\beta}\|_{\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1})} &\overset{Eq.4.2}{\leq} c\|\partial^{\alpha}\mathrm{I}_{j,\beta}\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} + c\|\partial^{\alpha}\mathrm{I}_{j,\beta}\|_{\dot{\mathrm{B}}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1})} \\ &\leq c\|\partial^{\alpha}\mathrm{I}_{j,\beta}\|_{\dot{\mathrm{B}}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1})} \overset{\text{(b)}}{\leq} c\|\mathbb{A}(((\partial^{\beta}\psi_j)u)(F^{(j)^{-1}}))\|_{\mathrm{L}^p(\mathbb{H})},\end{aligned}$$

where $\mathbb{H} = \mathbb{R}^{n-1} \times (0, \infty)$ and c also depends on $\mathrm{diam}(\mathrm{spt}(\psi_j))$. Thus, by the Leibniz rule and the properties of ψ_j and $F^{(j)}$,

$$\|\partial^{\alpha}\mathrm{I}_{j,\beta}\|_{\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1})} \leq c(\|u\|_{\mathrm{W}^{k-1,p}(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)}) \tag{4.25}$$

where $c = c(\alpha, \beta, \psi_j, F^{(j)}, p, \mathbb{A}, \Omega) > 0$. Based on Eq. 4.25 and again by the Leibniz rule, we conclude inductively for suitable choices of β and by the properties of ψ_j, $F^{(j)}$,

$$\|(\psi_j\partial^{\alpha}u)(F^{(j)^{-1}}(\cdot, 0))\|_{\mathrm{B}_{p,p}^{k-|\alpha|-1/p}(\mathbb{R}^{n-1})} \leq c(\|u\|_{\mathrm{W}^{k-1,p}(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p(\Omega)})$$

and (c) follows.

Ad (c)$\Rightarrow$(d). Let $B \subset \mathbb{R}^n$ be an open ball with $\Omega \Subset B$. Because of (c), we may consider the bounded, linear operator $\mathrm{tr}_{k-1,\partial\Omega}^{\mathbb{A}}\colon \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega) \to \dot{\mathrm{B}}_p^{k-1/p}(\partial\Omega; V)$ from Lemma 4.6. We subsequently pick the bounded and linear operator $\mathcal{L}\colon \dot{\mathrm{B}}_p^{k-1/p}(\partial(B \setminus \overline{\Omega}); V) \to \mathrm{W}^{k,p}(B \setminus \overline{\Omega}; V)$ from Lemma 4.2 and fix some $\rho \in \mathrm{C}_c^{\infty}(\mathbb{R}^n; [0, 1])$ with $\mathbb{1}_{\Omega} \leq \rho \leq \mathbb{1}_B$. Then we define for $u \in \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$

$$\begin{aligned}\mathscr{E}u &:= \begin{cases} u & \text{in } \Omega, \\ \rho\mathcal{L}(\mathbb{1}_{\partial\Omega}\mathrm{tr}_{k-1,\partial\Omega}^{\mathbb{A}}(u)) & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \end{cases} \\ v &:= \begin{cases} \mathbb{A}u & \text{in } \Omega, \\ \mathbb{A}(\rho\mathcal{L}(\mathbb{1}_{\partial\Omega}\mathrm{tr}_{k-1,\partial\Omega}^{\mathbb{A}}(u))) & \text{in } \mathbb{R}^n \setminus \overline{\Omega}. \end{cases}\end{aligned} \tag{4.26}$$

By construction of $\mathscr{E}$, the Gauß-Green formula Eq. 4.24 then implies

$$\int_{\mathbb{R}^n} v \cdot \varphi \,\mathrm{d}x = (-1)^k \int_{\mathbb{R}^n} \mathscr{E}u \cdot \mathbb{A}^*\varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{C}_c^{\infty}(\mathbb{R}^n; W). \tag{4.27}$$

Thus, the distributional expression $\mathbb{A}\mathscr{E}u$ satisfies $\mathbb{A}\mathscr{E}u = v \in \mathrm{L}_{\mathrm{loc}}^1(\mathbb{R}^n; W)$. Moreover, by the usual Gauß-Green formula and Lemma 4.6, we see by similar means that $\mathscr{E}u \in \mathrm{W}_{\mathrm{loc}}^{k-1,1}(\mathbb{R}^n; V)$. Recalling Lemma 4.2 and employing the Leibniz formula, we then have that

$$\begin{aligned}\|\mathscr{E}u\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\mathbb{R}^n\setminus\overline{\Omega})} \leq c\|\mathscr{E}u\|_{\mathrm{W}^{k,p}(\mathbb{R}^n\setminus\overline{\Omega})} &\leq c(\rho)\|\mathcal{L}(\mathbb{1}_{\partial\Omega}\mathrm{tr}_{k-1,\partial\Omega}^{\mathbb{A}}(u))\|_{\mathrm{W}^{k,p}(B\setminus\overline{\Omega})} \\ &\overset{\text{Lem. 4.2}}{\leq} c(\rho, k, p, \Omega, B)\|\mathrm{tr}_{k-1,\partial\Omega}^{\mathbb{A}}(u)\|_{\dot{\mathrm{B}}_p^{k-1/p}(\partial\Omega)} \\ &\overset{Eq.4.23}{\leq} c(\rho, k, p, \mathbb{A}, \Omega, B)\|u\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)}.\end{aligned}$$

Hence, $\mathscr{E}u \in \widetilde{\mathrm{W}}^{\mathbb{A},p}(\mathbb{R}^n)$ together with $\|\mathscr{E}u\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\mathbb{R}^n)} \leq c\|u\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)}$ for all $u \in \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$ with $c > 0$ solely depending on $\mathbb{A}$, p and Ω; note that B and hereafter ρ can be chosen in a way such that they only depend on Ω.

Now note that, if (c) holds, then the differential operator $\mathbb{A}$ is necessarily elliptic in the sense of Eq. 1.3. This can be seen by a modification of a standard construction (see, e.g., [66] or [19]): If it were not elliptic, we would find $\xi \in \mathbb{R}^n \setminus \{0\}$ and $v \in V\setminus\{0\}$ such that $\mathbb{A}[\xi]v = 0$. For any $h \in \mathrm{C}^\infty(\mathbb{R})$ and any open and bounded set $\Omega \subset \mathbb{R}^n$, the function $u_h(x) := h(x\cdot\xi)v$ belongs to $\mathrm{W}^{k-1,p}(\Omega; V)$ and satisfies both $\partial^\alpha u_h(x) = h^{(|\alpha|)}(x\cdot\xi)\xi^\alpha v$ and $\mathbb{A}u_h = 0$. We extend $\{\xi\}$ to an orthonormal basis $\{\xi, \eta_2, ..., \eta_n\}$ and choose an open and bounded set $\Omega \subset \mathbb{R}^n$ with smooth boundary such that $\Gamma := (0,1)\xi \times (0,1)\eta_2 \times ... \times (0,1)\eta_{n-1} \times \{0\} \subset \partial\Omega$ and $\Omega \subset (-2,2)\xi \times (-2,2)\eta_2 \times ... \times (-2,2)\eta_n$. Choose a sequence $(h_j) \subset \mathrm{C}^\infty((-2,2))$ such that $\sup_j \|h_j\|_{\mathrm{W}^{k-1,p}((-2,2))} < \infty$ but $\lim_{j\to\infty} \|h_j^{(k-1)}\|_{\mathrm{L}^q((0,1))} = \infty$ for $q > p$. Let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = k-1$ be such that $\xi^\alpha \neq 0$. By construction and Fubini's theorem, $\sup_j \|u_{h_j}\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)} < \infty$ but, with a constant $c = c(n, \xi, \alpha, v) > 0$, we have for $q > p$

$$\int_{\partial\Omega} |\partial^\alpha u_{h_j}|^q \,\mathrm{d}^{n-1}x \geq \int_\Gamma |\partial^\alpha u_{h_j}|^q \,\mathrm{d}^{n-1}x = c\int_0^1 |h_j^{(k-1)}(t)|^q \,\mathrm{d}t \to \infty. \tag{4.28}$$

Now, if $1 < p < n$, choose $q = \frac{p(n-1)}{n-p}$ and let $q > p$ be arbitrary provided $p \geq n$. Since (c) implies that $(\partial^\alpha u_{h_j}|_{\partial\Omega})$ is bounded in $\mathrm{B}^{1-1/p}_{p,p}(\partial\Omega; V)$ and so, by embedding theorems for Besov spaces, in $\mathrm{L}^q(\partial\Omega; V)$, Eq. 4.28 yields a contradiction. Therefore, $\mathbb{A}$ must be elliptic.

Now, Eq. 4.27ff. and an argument as in [31, Lem. 6.1] or [19, Prop. 4.1] allow to conclude (d): Let $\alpha \in \mathbb{N}_0^n$ satisfy $|\alpha| = k$. Consider for $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^n; W)$ the Fourier multiplication operators $\psi \mapsto \Phi_\alpha(\psi)$, where

$$\Phi_\alpha(\psi)(x) := \mathscr{F}^{-1}[\xi^\alpha(\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}^*[\xi]\widehat{\psi}(\xi)](x), \qquad x \in \mathbb{R}^n. \tag{4.29}$$

Then we have $\Phi_\alpha(\mathbb{A}\varphi) = \partial^\alpha\varphi$ for $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$. Since $\mathbb{A}$ is elliptic, $\mathbb{A}[\xi]\colon V \to W$ is injective and hence $\mathbb{A}^*[\xi]\mathbb{A}[\xi]\colon V \to V$ is bijective for any $\xi \in \mathbb{R}^n \setminus \{0\}$. Thus, $Eq.$4.29 is well-defined. The Fourier multiplier

$$\mathbb{R}^n \setminus \{0\} \ni \xi \mapsto \mathbf{m}_\alpha^{\mathbb{A}}(\xi) := \xi^\alpha(\mathbb{A}^*[\xi]\mathbb{A}[\xi])^{-1}\mathbb{A}^*[\xi] \in \mathscr{L}(W; V) \tag{4.30}$$

is of class $\mathrm{C}^\infty(\mathbb{R}^n \setminus \{0\}; \mathscr{L}(W; V))$, homogeneous of degree zero, and so [25, Thm. 4.13] yields that Φ_α extends to a bounded operator $\Phi_\alpha\colon \mathrm{L}^p(\mathbb{R}^n; W) \to \mathrm{L}^p(\mathbb{R}^n; V)$. Thus,

$$\begin{aligned} \|u\|_{\mathrm{W}^{k,p}(\Omega)} &\leq \|\mathscr{E}u\|_{\mathrm{W}^{k-1,p}(\mathbb{R}^n)} + \|\nabla^k\mathscr{E}u\|_{\mathrm{L}^p(\mathbb{R}^n)} \\ &\leq \|\mathscr{E}u\|_{\mathrm{W}^{k-1,p}(\mathbb{R}^n)} + c\|\mathbb{A}\mathscr{E}u\|_{\mathrm{L}^p(\mathbb{R}^n)} \\ &\leq c\|\mathscr{E}u\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\mathbb{R}^n)} \overset{Eq.4.27\text{ff.}}{\leq} c\|u\|_{\widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)} \end{aligned} \tag{4.31}$$

for all $u \in \mathrm{W}^{\mathbb{A},p}(\Omega)$ with some constant $c = c(\mathbb{A}, p, \rho, \Omega) > 0$. This is (d).

Now, if (d) holds, we apply Lemma 4.5 to $E_1 = \mathrm{W}^{k,p}(\Omega; V)$, $E_2 = \mathrm{L}^p(\Omega; W)$, $E_3 = \mathrm{W}^{k-1,p}(\Omega; V)$, $A = \mathbb{A}$ and B being the compact embedding $\mathrm{W}^{k,p}(\Omega; V) \hookrightarrow \mathrm{W}^{k-1,p}(\Omega; V)$. The equivalence of the $\mathrm{W}^{k,p}$- and $\widetilde{\mathrm{W}}^{\mathbb{A},p}$-norms implied by (d) now yields $\dim(\ker(A)) < \infty$ by virtue of Lemma 4.5, and by Eq. 3.6ff., this implies that $\mathbb{A}$ is $\mathbb{C}$-elliptic. Hence (a) follows, and so all of (a)–(d) are equivalent.

Lastly, by equivalence of (a) and (d), Eq. 4.22 follows if we can establish $\mathrm{W}^{\mathbb{A},p}(\Omega) \hookrightarrow \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$ provided any of the mutually equivalent conditions (a)–(d) holds. We thus assume that $\mathbb{A}$ is $\mathbb{C}$-elliptic and assume without loss of generality that Ω is an open, connected and bounded domain with C^∞-boundary. Since every such domain is John, the Poincaré-type inequality Eq. 3.22 now yields $\mathrm{W}^{\mathbb{A},p}(\Omega) \hookrightarrow \widetilde{\mathrm{W}}^{\mathbb{A},p}(\Omega)$, and the proof is complete. □

By Theorem 4.4 and Lemma 4.6, we obtain the following theorem for $1 < p < \infty$; for $p = 1$ and $\mathrm{BV}^{\mathbb{A}}$, $\mathrm{C}^\infty \cap \mathrm{BV}^{\mathbb{A}}$ in $\mathrm{BV}^{\mathbb{A}}$ is only dense in $\mathrm{BV}^{\mathbb{A}}$ for the $\mathbb{A}$-strict metric and not the norm topology, and so the underlying approximation must be approached differently. For the reader's convenience, the precise argument (hinging on a multiplicative inequality) is given in the Appendix, Section 7.1.

Theorem 4.7 (Trace operator) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary $\partial\Omega$ of class C^∞. Then, for any k-th order $\mathbb{C}$-elliptic differential operator $\mathbb{A}$ of the form Eq. 1.1 and $1 \leq p < \infty$, there exists a bounded and linear trace operator*

$$\mathrm{Tr}_{\partial\Omega}\colon\ \mathrm{W}^{\mathbb{A},p}(\Omega) \to \mathrm{B}^{k-1/p}_{p,p}(\partial\Omega; V)$$

which is onto. Moreover, for $p = 1$, there exists a linear trace operator

$$\mathrm{Tr}_{\partial\Omega}\colon\ \mathrm{BV}^{\mathbb{A}}(\Omega) \to \mathrm{B}^{k-1}_{1,1}(\partial\Omega; V)$$

which is onto and continuous for the $\mathbb{A}$-strict metric defined by Eq. 3.2.

5 Proof of Theorem 1.2: The Trace Estimate via Reduction to $\mathbf{W}^{k,1}(\Omega; V)$

Let $p = 1$. In comparison with the previous section, we now establish the novel reduction principle Eq. 1.9 which leads to the trace estimates underlying Theorem 1.2. This manifests the metaprinciple of $\mathbb{C}$-ellipticity helping to overcome ORNSTEIN's Non-Inequality by passing *to inverse estimates for polynomials* as in Eq. 3.16ff.. Throughout this section, we hereafter suppose that $\mathbb{A}$ is a $\mathbb{C}$-elliptic differential operator of the form Eq. 1.1 with $k \geq 2$.

5.1 Geometric Setup

Throughout this section, we suppose that $\Omega \subset \mathbb{R}^n$ is an open and bounded NTA_{n-1} domain, cf. Definition 2.1. To arrive at the conclusion of Theorem 1.2, we borrow and adapt some technology from the precursor [11] to the present paper. Given $j \in \mathbb{Z}$, pick a countable covering $(B_{j,i})_{i\in\mathbb{N}}$ of $\mathbb{R}^n$ by balls, each having diameter $\mathrm{diam}(B_{j,i}) = 2r(B_{j,i})$ and satisfying each of the following properties:

(C1) For all $j \in \mathbb{Z}$, $i \in \mathbb{N}$, $2^{-j-4} \leq 2r(B_{j,i}) \leq 2^{-j-3}$,
(C2) For each $j \in \mathbb{Z}$, $\bigcup_{i\in\mathbb{N}} \frac{7}{8}B_{j,i} = \mathbb{R}^n$,
(C3) There exists $c > 0$ such that $\sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{N}} \mathbb{1}_{B_{j,i}} \leq c$.

In the sequel, we let $\Omega_j := \{x \in \Omega\colon\ \mathrm{dist}(x, \partial\Omega) < 2^{-j}\}$. Since Ω is a non-tangentially accessible domain and thereby satisfies the interior corkscrew as well as the Harnack chain condition, we may record the following further properties, see [11]:

(C4) There exists an index $j_0 \in \mathbb{Z}$ such that for any ball $B_{j,i}$ with $j \geq j_0$, $i \in \mathbb{N}$ and $B_{j,i} \cap \Omega_j \neq \emptyset$, there exists a *reflected ball* $B^\sharp_{j,i} \subset \Omega$ satisfying

$$\mathrm{diam}(B_{j,i}) \simeq \mathrm{diam}(B^\sharp_{j,i}) \simeq \mathrm{dist}(B^\sharp_{j,i}, \partial\Omega), \quad \mathrm{dist}(B_{j,i}, B^\sharp_{j,i}) \lesssim \mathrm{diam}(B_{j,i}).$$

Here, the constants implicit in '$\simeq$' or '$\lesssim$' are independent of j and i.

(C5) With j_0 as in (C4), if $B_{j,i} \subset \Omega$ and $B_{j,i} \cap \Omega_j \neq \emptyset$ for some $j \geq j_0$, then there exists a chain of balls $\mathbb{B}_1, ..., \mathbb{B}_\gamma \subset \Omega$ with $\gamma \in \mathbb{N}$ independent of j, i such that

(a) $\mathbb{B}_1 = B_{j,i}, \mathbb{B}_\gamma = B^\sharp_{j,i}$,
(b) $\mathscr{L}^n(\mathbb{B}_\beta \cap \mathbb{B}_{\beta+1}) \simeq \mathscr{L}^n(\mathbb{B}_\beta) \simeq \mathscr{L}^n(\mathbb{B}_{\beta+1}) \simeq \mathscr{L}^n(B_{j,i})$ for all $\beta \in \{1, ..., \gamma - 1\}$,
(c) $\mathrm{diam}(\mathbb{B}_\beta) \simeq \mathrm{diam}(B_{j,i})$ for all $\beta \in \{1, ..., \gamma\}$.

Here, the constants implicit in '$\simeq$' are independent of j, i and β. In this situation, we define the *chain set* $\mathrm{ch}(B_{j,i}, B^\sharp_{j,i}) = \bigcup_{\beta=1}^{\gamma} \mathbb{B}_\beta$.

(C6) If j_0 is as in (C4) and $j, l \in \mathbb{Z}$ are such that $j, l \geq j_0$ and $|j - l| \leq 1$, then for all balls $B_{j,i}$ and $B_{l,m}$ with $B_{j,i} \cap B_{l,m} \neq \emptyset$, $B_{j,i} \cap \Omega_j \neq \emptyset$, $B_{l,m} \cap \Omega_l \neq \emptyset$ there exists a chain of balls $\mathbb{B}_1, ..., \mathbb{B}_\gamma \subset \Omega$ with γ independent of j, l, m, i such that

(a) $\mathbb{B}_1 = B^\sharp_{j,i}, \mathbb{B}_\gamma = B^\sharp_{l,m}$,
(b) $\mathscr{L}^n(\mathbb{B}_\beta \cap \mathbb{B}_{\beta+1}) \simeq \mathscr{L}^n(\mathbb{B}_\beta) \simeq \mathrm{diam}(B_{j,i})$ for all $\beta \in \{1, ..., \gamma - 1\}$,
(c) $\mathrm{dist}(\mathbb{B}_\beta, \partial\Omega) \simeq \mathrm{diam}(\mathbb{B}_\beta) \simeq \mathrm{diam}(B_{j,i})$ for all $\beta \in \{1, ..., \gamma\}$.

Here, the constants implicit in '$\simeq$' are independent of j, i and β, and we define similarly as above the respective chain set $\mathrm{ch}(B^\sharp_{j,i}, B^\sharp_{l,m}) := \bigcup_{\beta=1}^{\gamma} \mathbb{B}_\beta$.

Working from (C4)–(C6), we further obtain the existence of an $i_0 \in \mathbb{N}_{\geq 2}$ such that for all $j \geq j_0$ there holds with a universal constant $c > 0$

$$\begin{aligned}
&\sum_{\substack{m:\\ B_{j,m}\cap\Omega_j\neq\emptyset}} \mathbb{1}_{B^\sharp_{j,m}} \leq c\mathbb{1}_{\Omega_{j-i_0}\setminus\Omega_{j+i_0}}, \\
&\sum_{\substack{m:\\ B_{j,m}\cap\Omega_j\neq\emptyset}} \sum_{\substack{i:\\ B_{j+1,i}\cap B_{j,m}\neq\emptyset\\ B_{j+1,i}\cap\Omega_{j+1}\neq\emptyset}} \mathbb{1}_{\mathrm{ch}(B^\sharp_{j,m}, B^\sharp_{j+1,i})} \leq c\mathbb{1}_{\Omega_{j-i_0}\setminus\Omega_{j+i_0}}, \\
&\sum_{\substack{m:\\ B_{j,m}\cap(\Omega_j\setminus\Omega_{j+2})\neq\emptyset}} \mathbb{1}_{\mathrm{ch}(B_{j,m}, B^\sharp_{j,m})} \leq c\mathbb{1}_{\Omega_{j-i_0}\setminus\Omega_{j+i_0}}.
\end{aligned} \tag{5.1}$$

For $Eq.5.1_3$, note that if $B_{j,m} \cap (\Omega_j \setminus \Omega_{j+2}) \neq \emptyset$, then we have $B_{j,m} \subset \Omega \setminus \Omega_{j+3}$ by construction. Given $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$ and $j \geq j_0$, $i \in \mathbb{N}$, we introduce the shorthand notation

$$\Pi_{j,i}u := \Pi_{\mathbb{A}}^{B^\sharp_{j,i}} u, \tag{5.2}$$

with the projection $\Pi_{\mathbb{A}}^{B^\sharp_{j,i}}$ from Proposition 3.3. The proof of the next lemma is accomplished by an argument similar to that underlying Proposition 3.8 and is therefore omitted.

Lemma 5.1 *There exists a constant $c > 0$ that only depends on $\mathbb{A}$ and the* NTA-*parameters of Ω such that the following hold for all $j \geq j_0$ and all $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$:*

(a) For all $B_{j,i}$ with $B_{j,i} \cap \Omega_j \neq \emptyset$ and all $\alpha \in \mathbb{N}_0^n$ there holds

$$\|\partial^\alpha \Pi_{j,i}u\|_{\mathrm{L}^\infty(B_{j,i})} \leq c\,\mathrm{diam}(B_{j,i})^{-|\alpha|} \fint_{B^\sharp_{j,i}} |u|\,\mathrm{d}x.$$

(b) Whenever $B_{j,m} \cap (\Omega_j \setminus \Omega_{j+2}) \neq \emptyset$, then $B_{j,m} \subset \Omega$ and we have, for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq k - 1$,

$$\|\partial^\alpha(u - \Pi_{j,m}u)\|_{\mathrm{L}^1(B_{j,m})} \leq c\,\mathrm{diam}(B_{j,m})^{k-|\alpha|} \int_{\mathrm{ch}(B_{j,m}, B^\sharp_{j,m})} |\mathbb{A}u|\,\mathrm{d}x.$$

(c) Chain control: *Whenever* $B_{j,m} \cap \Omega_j \neq \emptyset$*,* $B_{j+1,i} \cap \Omega_{j+1} \neq \emptyset$ *and* $B_{j+1,i} \cap B_{j,m} \neq \emptyset$*, then we have for all* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \leq k-1$

$$\fint_{B_{j,m}} |\partial^\alpha(\Pi_{j+1,i}u - \Pi_{j,m}u)|\,\mathrm{d}x \leq c\,\mathrm{diam}(B_{j,m})^{k-|\alpha|} \fint_{\mathrm{ch}(B_{j+1,i}^\sharp, B_{j,m}^\sharp)} |\mathbb{A}u|\,\mathrm{d}x.$$

5.2 Proof of Theorem 1.2

In view of the requisite trace estimates, we now introduce a suitable approximation $T_j u$ of a given function $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$. For each $j \in \mathbb{Z}$, pick the covering $(B_{j,i})_i$ from Section 5.1 and let $(\rho_{j,i})_i$ be a smooth partition of unity subject to $(B_{j,i})_i$ such that, with a universal constant $c > 0$,

$$\sum_{|\alpha|\leq k} \mathrm{diam}(B_{j,i})^{|\alpha|} \|\partial^\alpha \rho_{j,i}\|_{\mathrm{L}^\infty(\Omega)} \leq c \qquad \text{for all } i \in \mathbb{N},\ j \in \mathbb{Z}. \tag{5.3}$$

Moreover, let $\rho_j \in \mathrm{C}^\infty(\overline{\Omega}; [0,1])$ be such that $\mathbb{1}_{\Omega_{j+1}} \leq \rho_j \leq \mathbb{1}_{\Omega_j}$ and

$$\|\partial^\alpha \rho_j\|_{\mathrm{L}^\infty(\Omega)} \leq c2^{j|\alpha|} \qquad \text{for all } \alpha \in \mathbb{N}_0^n,\ |\alpha| \leq k. \tag{5.4}$$

We then define the *replacement sequence* $(T_j u)_{j\geq j_0}$ with $j_0 \in \mathbb{Z}$ as in (C4) from above by

$$T_j u := T_j^{(1)}u + T_j^{(2)}u := (1-\rho_j)u + \rho_j \sum_{i\in\mathbb{N}} \rho_{j,i}\Pi_{j,i}u \tag{5.5}$$

in Ω. Note that, as a locally finite sum and by construction, $T_j^{(2)}u \in \mathrm{C}^\infty(\overline{\Omega}; V)$. Thus, in passing from u to $T_j u$ we essentially leave the map u unchanged away from $\partial\Omega$ while close to $\partial\Omega$, we replace it by its projections onto $\ker(\mathbb{A})$ on balls nearby, see Fig. 3. The proof of Theorem 1.2 crucially hinges on the decomposition

$$\begin{aligned} T_{j+1}u - T_j u &= (\rho_j - \rho_{j+1}) \sum_{m\in\mathbb{N}} \rho_{j,m}(u - \Pi_{j,m}u) \\ &\quad + \rho_{j+1} \sum_{i,m\in\mathbb{N}} \rho_{j,m}\rho_{j+1,i}(\Pi_{j+1,i}u - \Pi_{j,m}u) =: \mathrm{I}_j[u] + \mathrm{II}_j[u], \end{aligned} \tag{5.6}$$

which is implied by $\sum_m \rho_{j,m} \equiv 1$ for all j. In comparison with the precursor [11], the key observation is now given by Proposition 5.2 below: The terms $\mathrm{I}_j[u]$ will have trace zero along $\partial\Omega$ and $\mathrm{II}_j[u]$ *uniformly belongs to* $\mathrm{W}^{k,1}(\Omega; V)$. For the reduction argument underlying the proof of Theorem 1.2, it will then suffice to apply the trace estimate for $\mathrm{W}^{k,1}$ to $\mathrm{II}_j[u]$, cf. Corollary 5.3.

Proposition 5.2 *Let* $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$ *and define* $T_j u$, $\mathrm{I}_j[u]$ *and* $\mathrm{II}_j[u]$ *for* $j \geq j_0$ *by Eqs. 5.5 or 5.6, respectively, and let* i_0 *be as in Eq. 5.1. Then the following holds:*

(a) There exists $c > 0$ *such that* $\|\mathrm{I}_j[u]\|_{\mathrm{W}^{\mathbb{A},1}(\Omega)} \leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega_{j-i_0}\setminus\Omega_{j+i_0})}$.
(b) There exists $c > 0$ *such that* $\|\mathrm{II}_j[u]\|_{\mathrm{W}^{k,1}(\Omega)} \leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega_{j-i_0}\setminus\Omega_{j+i_0})}$.
(c) We have $T_j u \to u$ *with respect to the* $\mathrm{W}^{\mathbb{A},1}$*-norm.*
(d) There exists $(u_j) \subset \mathrm{C}^\infty(\overline{\Omega}; V)$ *such that* $u_j \to u$ *with respect to the* $\mathrm{W}^{\mathbb{A},1}$*-norm.*
(e) If $u \in \mathrm{C}(\overline{\Omega}; V) \cap \mathrm{W}^{\mathbb{A},1}(\Omega)$*, then* $T_j u \to u$ *uniformly in* Ω.

In both (a) and (b), the constant $c > 0$ *is independent of* u*.*

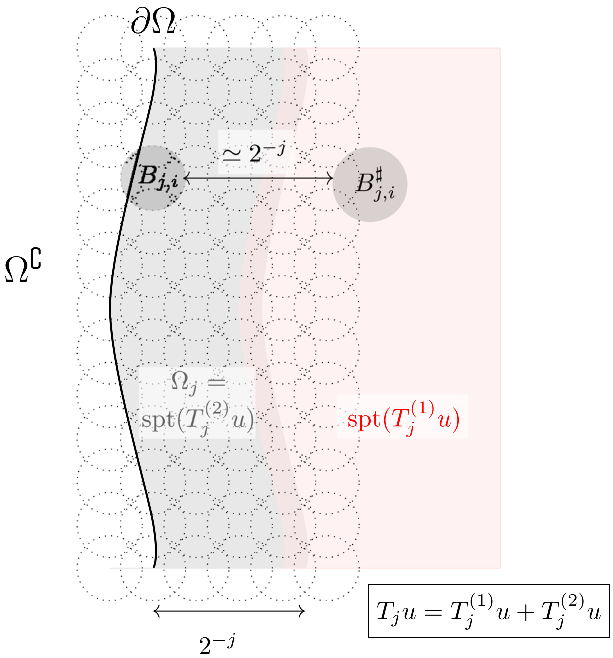

Fig. 3 Construction of T_j and idea of the (trace) estimates of Proposition 5.2 and Corollary 5.3. With the dashed balls indicating the covering $(B_{j,i})_i$, the approximation $T_j u$ given by Eq. 5.5 splits into two parts: At a distance at least 2^{-j} from $\partial\Omega$ (i.e., in the red area), $T_j u$ leaves u unchanged, whereas close to $\partial\Omega$ (i.e., on $\Omega_j := \{\mathrm{dist}(\cdot, \partial\Omega) < 2^{-j}\}$ represented by the grey area), $T_j u$ essentially replaces u by its projections onto the nullspace of $\mathbb{A}$ on the reflected balls $B^{\sharp}_{j,i}$

Proof Let $u \in \mathrm{W}^{\mathbb{A},1}(\Omega)$. Ad (a). For $j \geq j_0$, denote $\mathcal{I}_j$ the set of all indices $m \in \mathbb{N}$ such that $(\rho_j - \rho_{j+1})\rho_{j,m} \not\equiv 0$. Since $\mathrm{spt}(\rho_j - \rho_{j+1}) \subset \Omega_j \setminus \Omega_{j+2}$, $m \in \mathcal{I}_j$ implies that $B_{j,m} \cap (\Omega_j \setminus \Omega_{j+2}) \neq \emptyset$. For such indices, we may use Lemma 5.1 (b). In consequence, we can even estimate the $\widetilde{\mathrm{W}}^{\mathbb{A},1}(\Omega)$-norm of $\mathrm{I}_j[u]$ by

$$
\begin{aligned}
\|\mathrm{I}_j[u]\|_{\widetilde{\mathrm{W}}^{\mathbb{A},1}(\Omega)} &\leq \Big(\sum_{m\in\mathcal{I}_j} \sum_{l=0}^{k-1} \|D^{\ell}((\rho_j - \rho_{j+1})\rho_{j,m}(u - \Pi_{j,m}u))\|_{\mathrm{L}^1(\Omega)} \Big) \\
&\quad + \Big(\sum_{m\in\mathcal{I}_j} \|\mathbb{A}((\rho_j - \rho_{j+1})\rho_{j,m}(u - \Pi_{j,m}u))\|_{\mathrm{L}^1(\Omega)} \Big) \\
&\overset{(*)}{\leq} c\Big(\sum_{m\in\mathcal{I}_j} \sum_{l=0}^{k-1} \sum_{|\alpha|+|\beta|=\ell} \|\partial^{\alpha}((\rho_j - \rho_{j+1})\rho_{j,m})\|_{\mathrm{L}^{\infty}(\Omega)} \|\partial^{\beta}(u - \Pi_{j,m}u)\|_{\mathrm{L}^1(B_{j,m})} \Big) \\
&\quad + c\Big(\sum_{m\in\mathcal{I}_j} \sum_{\substack{|\alpha|+|\beta|=k \\ |\beta|\leq k-1}} \|\partial^{\alpha}((\rho_j - \rho_{j+1})\rho_{j,m})\|_{\mathrm{L}^{\infty}(\Omega)} \|\partial^{\beta}(u - \Pi_{j,m}u)\|_{\mathrm{L}^1(B_{j,m})} \\
&\qquad + \sum_{m\in\mathcal{I}_j} \|(\rho_j - \rho_{j+1})\rho_{j,m}\mathbb{A}u\|_{\mathrm{L}^1(\Omega)} \Big)
\end{aligned}
$$

$$\overset{\text{Lem. 5.1(b), Eqs. 5.3, 5.4}}{\leq} c\Big(\sum_{m\in\mathcal{I}_j}\sum_{l=0}^{k-1}\sum_{|\alpha|+|\beta|=\ell} 2^{j|\alpha|}2^{-j(k-|\beta|)}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B_{j,m},B^{\sharp}_{j,m}))}\Big)$$
$$+c\Big(\sum_{m\in\mathcal{I}_j}\sum_{\substack{|\alpha|+|\beta|=k\\|\beta|\leq k-1}} 2^{j|\alpha|}2^{-j(k-|\beta|)}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B_{j,m},B^{\sharp}_{j,m}))}+\sum_{m\in\mathcal{I}_j}\|\mathbb{A}u\|_{\mathrm{L}^1(B_{j,m})}\Big)$$
$$\leq c\sum_{m\in\mathcal{I}_j}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B_{j,m},B^{\sharp}_{j,m}))}$$
$$\overset{Eq.5.1_3}{\leq} c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega_{j-i_0}\setminus\Omega_{j+i_0})}.$$

Here we used at $(*)$ that, if we expand $\mathbb{A}(\varphi v)$ for sufficiently regular functions $\varphi\colon\,\Omega\to\mathbb{R}$ and $v\colon\,\Omega\to V$ by the Leibniz rule, then the only summand that involves k-th order derivatives of v is of the form $\varphi\mathbb{A}v$. Hence, (b) follows.

Ad (a). Let $\mathcal{J}_j^m$ be the set of all indices $i\in\mathbb{N}$ such that $\rho_{j+1}\rho_{j,m}\rho_{j+1,i}\not\equiv 0$. Then, by (C3), $\#\mathcal{J}_j^m$ is uniformly bounded in j and m. If $i\in\mathcal{J}_j^m$, then $\Omega_{j+1}\cap B_{j,m}\neq\emptyset$, $\Omega_{j+1}\cap B_{j+1,i}\neq\emptyset$ and $B_{j,m}\cap B_{j+1,i}\neq\emptyset$. As such, $Eq.5.1_2$ is available for such indices. Let $|\beta|\leq k$. Since $\ker(\mathbb{A})$ is a finite dimensional space by the $\mathbb{C}$-ellipticity of $\mathbb{A}$, we may invoke the inverse estimates $Eq.3.18_2$ on elements of the nullspace of $\mathbb{A}$ to obtain for all $j\geq j_0$, $m\in\mathbb{N}$ and $i\in\mathcal{J}_j^m$:

$$\|\partial^\beta(\Pi_{j+1,i}u-\Pi_{j,m}u)\|_{\mathrm{L}^1(B_{j,m})}\leq\|D^{|\beta|}(\Pi_{j+1,i}u-\Pi_{j,m}u)\|_{\mathrm{L}^1(B_{j,m})}$$
$$\leq C\,\mathrm{diam}(B_{j,m})^{n-|\beta|}\,\mathrm{diam}(B_{j,m})^{|\beta|}\fint_{B_{j,m}}|D^{|\beta|}(\Pi_{j+1,i}u-\Pi_{j,m}u)|\,\mathrm{d}x$$
$$\leq C\,\mathrm{diam}(B_{j,m})^{n-|\beta|}\times$$
$$\times\Big(\sum_{\ell=0}^{k}\mathrm{diam}(B_{j,m})^{\ell}\fint_{B_{j,m}}|D^{\ell}(\Pi_{j+1,i}u-\Pi_{j,m}u)|\,\mathrm{d}x\Big)$$
$$\overset{Eq.3.18_2}{\leq} C\,\mathrm{diam}(B_{j,m})^{n-|\beta|}\fint_{B_{j,m}}|\Pi_{j+1,i}u-\Pi_{j,m}u|\,\mathrm{d}x$$
$$\overset{\text{Lem. 5.1 (c)}}{\leq} C\,\mathrm{diam}(B_{j,m})^{(k-|\beta|)}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B^{\sharp}_{j+1,i},B^{\sharp}_{j,m}))}$$
$$\leq C2^{-j(k-|\beta|)}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B^{\sharp}_{j+1,i},B^{\sharp}_{j,m}))}.$$

As $\mathcal{J}_j^m$ is non-empty only if $m\in\mathcal{K}_j:=\{l\in\mathbb{N}\colon\;\Omega_{j+1}\cap B_{j,l}\neq\emptyset\}$, we obtain by Eq. 5.3, Eq. 5.4 and the preceding inequality

$$\|\mathrm{II}_j[u]\|_{\mathrm{W}^{k,1}(\Omega)}\leq c\sum_{m\in\mathcal{K}_j}\sum_{i\in\mathcal{J}_j^m}\sum_{\ell=0}^{k}\sum_{|\alpha|+|\beta|=\ell}\Big(\|\partial^\alpha(\rho_{j+1}\rho_{j,m}\rho_{j+1,i})\|_{\mathrm{L}^\infty(\Omega)}\times$$
$$\times\|\partial^\beta(\Pi_{j+1,i}u-\Pi_{j,m}u)\|_{\mathrm{L}^1(B_{j,m})}\Big)$$
$$\leq c\sum_{m\in\mathcal{K}_j}\sum_{i\in\mathcal{J}_j^m}\sum_{\ell=0}^{k}\sum_{|\alpha|+|\beta|=\ell}2^{j|\alpha|}2^{-j(k-|\beta|)}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B^{\sharp}_{j+1,i},B^{\sharp}_{j,m}))}$$

$$\begin{aligned}
&\leq c \sum_{m\in\mathcal{K}_j}\sum_{i\in\mathcal{J}_j^m}\sum_{\ell=0}^{k} 2^{(\ell-k)j}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B^\sharp_{j+1,i},B^\sharp_{j,m}))}\\
&\leq c \sum_{m\in\mathcal{K}_j}\sum_{i\in\mathcal{J}_j^m}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathrm{ch}(B^\sharp_{j+1,i},B^\sharp_{j,m}))}\\
&\overset{Eq. 5.1_2}{\leq} c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega_{j-i_0}\setminus\Omega_{j+i_0})}
\end{aligned}$$

This establishes (b).

Ad (c). By construction of the sets Ω_j, $\sum_{j\geq l}\mathbb{1}_{\Omega_{j-i_0}\setminus\Omega_{j+i_0}}\leq c(i_0)\mathbb{1}_{\Omega_{l-i_0}}$ for all $l\in\mathbb{N}$. Since $\|\cdot\|_{\mathrm{W}^{\mathbb{A},1}}$ always can be dominated by $\|\cdot\|_{\mathrm{W}^{k,1}}$, we obtain by (a) and (b) for all $l'\geq l\geq j_0$ by virtue of a telescope sum argument

$$\|T_{l'}u-T_lu\|_{\mathrm{W}^{\mathbb{A},1}(\Omega)}\leq\sum_{j=l}^{\infty}\|T_{j+1}u-T_ju\|_{\mathrm{W}^{\mathbb{A},1}(\Omega)}\leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega_{l-i_0})}\to 0$$

as $l,l'\to\infty$. Thus (T_lu) is Cauchy in $\mathrm{W}^{\mathbb{A},1}(\Omega)$ and thus converges to some $v\in\mathrm{W}^{\mathbb{A},1}(\Omega)$. On the other hand, since $\mathrm{spt}(u-T_lu)\subset\Omega_l$ holds by construction, we have $\|u-T_lu\|_{\mathrm{W}^{\mathbb{A},1}(\omega)}\to 0$ for any open $\omega\Subset\Omega$ as $l\to\infty$, and so $v=u$. Therefore, $T_ju\to u$ in $\mathrm{W}^{\mathbb{A},1}(\Omega)$ as $j\to\infty$, which is (c).

Ad (d). We recall the decomposition Eq. 5.5. Since $T_j^{(2)}u$ is a locally finite sum of C^∞-maps, $T_j^{(2)}u\in\mathrm{C}^\infty(\overline{\Omega};V)$. Moreover, since $T_j^{(1)}u\in\mathrm{W}^{\mathbb{A},1}(\Omega)$ is compactly supported in Ω, standard mollification yields some $v_j\in\mathrm{C}_c^\infty(\Omega;V)$ such that $\|T_j^{(1)}u-v_j\|_{\mathrm{W}^{\mathbb{A},1}(\Omega)}<\frac{1}{j}$. Put $u_j:=v_j+T_j^{(2)}u$ so that $u_j\in\mathrm{C}^\infty(\overline{\Omega};V)$, and then (c) implies $\|u-u_j\|_{\mathrm{W}^{\mathbb{A},1}(\Omega)}\to 0$ as $j\to\infty$.

To see (e), let $\varepsilon>0$. By uniform continuity of u, we find $\delta>0$ such that $|x-y|<\delta$ implies $|u(x)-u(y)|<\varepsilon$. Let $B_{j,i}$ be such that $B_{j,i}\cap\Omega_j\neq\emptyset$. By (C4) and assuming that $j\geq j_0$, there exists a constant $\mathtt{C}>0$ independent of j and i such that $|x-y|<\mathtt{C}2^{-j}$ holds for all $x\in B_{j,i}$ and $y\in B^\sharp_{j,i}$; moreover, we may assume that $\mathrm{diam}(B^\sharp_{j,i})<\mathtt{C}2^{-j}$ as well. We thus find $J_0\geq j_0$ such that for all $j\geq J_0$ there holds $\mathtt{C}2^{-j}\leq\delta$. Thus, invoking Lemma 5.1(a) and because of $(u)_{B^\sharp_{j,i}}=\Pi_{j,i}(u)_{B^\sharp_{j,i}}$,

$$\begin{aligned}
\|\rho_j\rho_{j,i}(u-\Pi_{j,i}u)\|_{\mathrm{L}^\infty(\Omega)}&\leq\|u-(u)_{B^\sharp_{j,i}}\|_{\mathrm{L}^\infty(B_{j,i}\cap\Omega_j)}+\|\Pi_{j,i}(u-(u)_{B^\sharp_{j,i}})\|_{\mathrm{L}^\infty(B_{j,i}\cap\Omega_j)}\\
&\leq\sup_{x\in B_{j,i}\cap\Omega_j}\fint_{B^\sharp_{j,i}}|u(x)-u(y)|\,\mathrm{d}y+c\fint_{B^\sharp_{j,i}}|u-(u)_{B^\sharp_{j,i}}|\,\mathrm{d}y\\
&\leq(1+c)\varepsilon.
\end{aligned}$$

Since $u-T_ju=\sum_i\rho_j\rho_{j,i}(u-\Pi_{j,i}u)$ and the $B_{j,i}$'s have mutual uniformly finite overlap (in i for each fixed j), we thus obtain $T_ju\to u$ in $\mathrm{L}^\infty(\Omega;V)$ and hence (e) follows. The proof is complete. $\square$

For the following, let $(\mathscr{X}(\partial\Omega;V);\|\cdot\|_{\mathscr{X}(\partial\Omega)})$ be a Banach space with $\mathscr{X}(\partial\Omega;V)\subset\mathrm{L}^1_{\mathrm{loc}}(\partial\Omega;V)$ such that $\mathrm{W}^{k,1}(\Omega;V)$ has trace space $\mathscr{X}(\partial\Omega;V)$ (cf. Theorem 1.2 ff. for this terminology); we denote $\mathrm{Tr}_{\partial\Omega}\colon\mathrm{W}^{k,1}(\Omega;V)\to\mathscr{X}(\partial\Omega;V)$ the underlying trace operator.

Corollary 5.3 *There exists a constant $c = c(\Omega, \mathbb{A}) > 0$ such that the following holds: Whenever $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$ and $j_0 \in \mathbb{Z}$ is as in (C4), then there holds*

$$\sum_{j=j_0}^{\infty} \| \mathrm{Tr}_{\partial\Omega}(T_{j+1}u) - \mathrm{Tr}_{\partial\Omega}(T_j u)\|_{\mathscr{X}(\partial\Omega)} \leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega)}. \tag{5.7}$$

Proof Let $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$ and $j \geq j_0$. Then Eq. 5.6 implies that $T_{j+1}u - T_j u$ is a locally finite sum of $\mathrm{C}^{\infty}(\overline{\Omega}; V)$-maps and so is of class $\mathrm{C}^{\infty}(\overline{\Omega}; V)$, too. Since $\mathrm{spt}(\mathrm{I}_j[u]) \subset \{x \in \Omega\colon\; \mathrm{dist}(x, \partial\Omega) > 2^{-j-2}\}$ because of $\mathrm{spt}(\rho_j - \rho_{j+1}) \subset \Omega_j \setminus \Omega_{j+2}$, we have $\mathrm{Tr}_{\partial\Omega}(\mathrm{I}_j[u]) = 0$. Since $\mathrm{W}^{k,1}(\Omega; V)$ has trace space $\mathscr{X}(\partial\Omega; V)$, there exists a constant $c > 0$ such that $\| \mathrm{Tr}_{\partial\Omega}(v)\|_{\mathscr{X}(\partial\Omega)} \leq c\|v\|_{\mathrm{W}^{k,1}(\Omega)}$ for all $v \in \mathrm{W}^{k,1}(\Omega; V)$. Now, by Eq. 5.6 and Proposition 5.2 (b), this entails

$$\begin{aligned}\sum_{j=j_0}^{\infty} \| \mathrm{Tr}_{\partial\Omega}(T_{j+1}u) - \mathrm{Tr}_{\partial\Omega}(T_j u)\|_{\mathscr{X}(\partial\Omega)} &= \sum_{j=j_0}^{\infty} \| \mathrm{Tr}_{\partial\Omega}(\mathrm{II}_j[u])\|_{\mathscr{X}(\partial\Omega)} \\ &\leq c \sum_{j=j_0}^{\infty} \|\mathrm{II}_j[u]\|_{\mathrm{W}^{k,1}(\Omega)} \\ &\overset{\text{Prop. 5.2 (b)}}{\leq} c \sum_{j=j_0}^{\infty} \|\mathbb{A}u\|_{\mathrm{L}^1(\Omega_{j-i_0}\setminus\Omega_{j+i_0})} \\ &\leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega_{j_0-i_0})},\end{aligned} \tag{5.8}$$

where in the ultimate step we used that the sets $\Omega_{j-i_0} \setminus \Omega_{j+i_0}$ only have a uniformly finite mutual overlap. This is Eq. 5.7, and the proof is complete. □

Before we proceed, we briefly pause to comment on the preceding proof in

Remark 5.4 Based on Proposition 5.2 (b), in the second line of Eq. 5.8 we have employed *full k-th order gradient* L^1*-estimates* to obtain the requisite $\mathscr{X}(\partial\Omega; V)$-estimates for the traces of $T_{j+1}u - T_j u$. We wish to emphasize that a similar strategy *does not* yield any uniform L^1-bounds on the sequence $(D^k(T_{j+1}u - T_j u))_j$, and so this is in line with Ornstein's Non-Inequality. Indeed, even for $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$ the term $\mathrm{I}_j[u]$ in Eq. 5.6, for which Proposition 5.2 only provides $\mathrm{W}^{\mathbb{A},1}$- but not $\mathrm{W}^{k,1}$-bounds, might have diverging $\mathrm{W}^{k,1}$-norm as $j \to \infty$: Precisely by Ornstein's Non-Inequality, when D^k acts on u in the term $\mathrm{I}_j[u]$ in Eq. 5.6, one has $\limsup_{j\to\infty} \|\mathrm{I}_j[u]\|_{\mathrm{W}^{k,1}(\Omega)} = \infty$ in general. By the inverse estimates on polynomials of a fixed degree, this obstruction becomes invisible for $\mathrm{II}_j[u]$, and only $\mathrm{II}_j[u]$ matters for the requisite trace estimate Eq. 5.7.

Towards the proof of Theorem 1.2, let us note that it is only the sufficiency part that requires proof; similarly as for Theorem 1.1, the necessity part can be established by localising the construction in [31, Proof of Theorem 5.2]. Indeed, if $\mathbb{A}$ is not $\mathbb{C}$-elliptic, the aforementioned construction leads to some smoothly bounded $\Omega \subset \mathbb{R}^n$ and a $\mathrm{W}^{\mathbb{A},1}$-bounded sequence $(u_j) \subset \mathrm{C}^{\infty}(\overline{\Omega}; V)$ for which $(D^{k-1}u_j)$ is unbounded in $\mathrm{L}^1(\partial\Omega; \odot^{k-1}(\mathbb{R}^n; V))$. Since $\mathrm{W}^{k,1}(\Omega; V)$ has trace space $\mathrm{B}^{k-1}_{1,1}(\partial\Omega; V)$, $\mathrm{W}^{\mathbb{A},1}(\Omega)$ cannot have the same trace space as $\mathrm{W}^{k,1}(\Omega)$. It thus suffices to give the

Proof of the sufficiency part of Theorem 1.2 Let $\mathbb{A}$ be a $\mathbb{C}$-elliptic operator and let $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$ be given. Since $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$, Lemma 5.2 (e) implies by a telescope sum argument

$$\mathrm{Tr}_{\partial\Omega}(u) - \mathrm{Tr}_{\partial\Omega}(T_{j_0}u) = \sum_{j=j_0}^{\infty} \mathrm{Tr}_{\partial\Omega}(T_{j+1}u) - \mathrm{Tr}_{\partial\Omega}(T_j u) \tag{5.9}$$

everywhere on $\partial\Omega$. We may control the right-hand side by Corollary 5.3, and so we only need to bound $\mathrm{Tr}_{\partial\Omega}(T_{j_0}u)$. Based on the decomposition Eq. 5.5, we recall that for $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$ there holds $\| \mathrm{Tr}_{\partial\Omega}(T_{j_0}^{(1)}u)\|_{\mathscr{X}(\partial\Omega)} = 0$, and so we proceed by proving that the function $T_{j_0}^{(2)}u$ belongs to $\mathrm{W}^{k,1}(\Omega; V)$ together with the requisite estimates. Since $T_{j_0}^{(2)}u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$, it possesses classical traces and so it suffices to give an estimate of the $\mathrm{W}^{k,1}$-norm of $T_{j_0}^{(2)}u$.

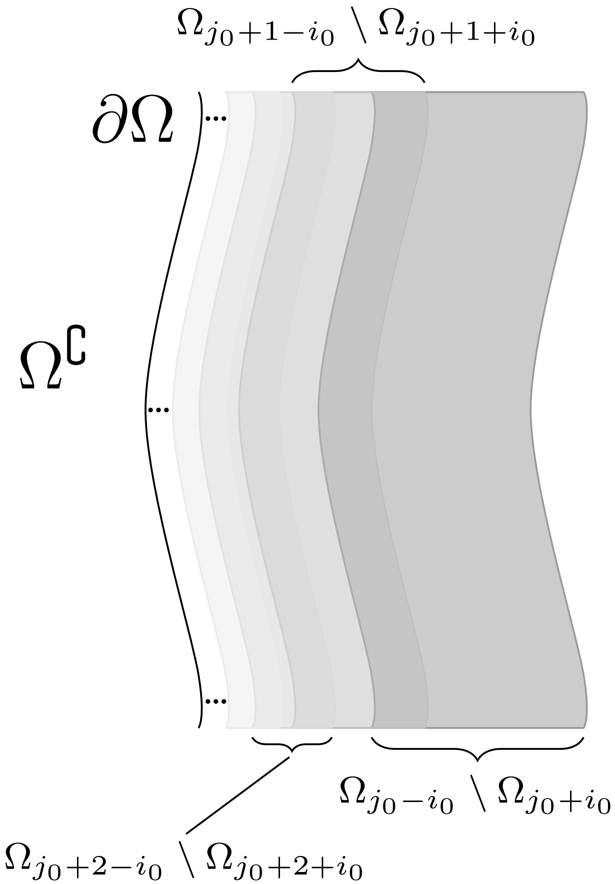

Fig. 4 Idea of the trace estimate of Theorem 1.2. In the telescope sum Eq. 5.9, the trace of $u \in \mathrm{C}^{\infty}(\overline{\Omega}; V)$ is decomposed into a contribution from $T_{j_0}u$ and a sum over $\mathrm{Tr}_{\partial\Omega}(T_{j+1}u - T_j u) = \mathrm{Tr}_{\partial\Omega}(\Pi_j[u])$. By construction, $\Pi_j[u]$ is determined by the projections of u onto $\ker(\mathbb{A})$ on reflected balls contained in the strips $\Omega_{j-i_0} \setminus \Omega_{j+i_0}$; the latter have uniformly finite mutual overlap and yield the entire Ω when united

Denoting $\mathcal{L}_{j_0}$ the set of indices $i \in \mathbb{N}$ such that $\Omega_{j_0} \cap B_{j_0,i} \neq \emptyset$,

$$
\begin{aligned}
\|T_{j_0}^{(2)}u\|_{\mathrm{W}^{k,1}(\Omega)} &\leq c\sum_{\ell=0}^{k}\sum_{|\alpha|+|\beta|=\ell}\sum_{i\in\mathcal{L}_{j_0}} \|(\partial^\alpha(\rho_{j_0}\rho_{j_0,i}))(\partial^\beta\Pi_{j_0,i}u)\|_{\mathrm{L}^1(\Omega)} \\
&\overset{Eqs.5.3,\,5.4}{\leq} c\sum_{\ell=0}^{k}\sum_{|\alpha|+|\beta|=\ell}\sum_{i\in\mathcal{L}_{j_0}} 2^{j_0|\alpha|}\times \\
&\qquad\qquad \times 2^{-j_0(n-|\beta|)}2^{-j_0|\beta|}\fint_{B_{j_0,i}}|\partial^\beta\Pi_{j_0,i}u|\,\mathrm{d}x \\
&\overset{Eq.3.18_2,\,\text{Lem. }5.1(\text{a})}{\leq} c\sum_{\ell=0}^{k}2^{j_0\ell}2^{-j_0 n}\sum_{i\in\mathcal{L}_{j_0}}\fint_{B_{j_0,i}^\sharp}|u|\,\mathrm{d}x \\
&\overset{Eq.5.1_1}{\leq} c2^{j_0k}\|u\|_{\mathrm{L}^1(\Omega_{j_0-i_0}\setminus\Omega_{j_0+i_0})},
\end{aligned}
\tag{5.10}
$$

where we additionally used the uniform finite overlap property of the $B_{j,i}$'s. Since, by assumption, $\mathrm{W}^{k,1}(\Omega;V)$ has trace space $\mathscr{X}(\partial\Omega;V)$, we consequently obtain (see Fig. 4)

$$
\|\operatorname{Tr}_{\partial\Omega}(T_{j_0}u)\|_{\mathscr{X}(\partial\Omega)} = \|\operatorname{Tr}_{\partial\Omega}(T_{j_0}^{(2)}u)\|_{\mathscr{X}(\partial\Omega)} \leq c2^{j_0k}\|u\|_{\mathrm{L}^1(\Omega)}. \tag{5.11}
$$

Combining Eqs. 5.9, 5.11 and 5.7, we then obtain

$$
\|\operatorname{Tr}_{\partial\Omega}(u)\|_{\mathscr{X}(\partial\Omega)} \leq c2^{j_0k}\|u\|_{\mathrm{L}^1(\Omega)} + c\|\mathbb{A}u\|_{\mathrm{L}^1(\Omega)} \qquad \text{for all } u\in\mathrm{C}^\infty(\overline{\Omega};V). \tag{5.12}
$$

If $u\in\mathrm{W}^{\mathbb{A},1}(\Omega)$, we invoke Lemma 5.2 (d) to find $(u_j)\subset\mathrm{C}^\infty(\overline{\Omega};V)$ such that $u_j\to u$ in $\mathrm{W}^{\mathbb{A},1}(\Omega)$. As in the proof of Lemma 4.6 and since $(\mathscr{X}(\partial\Omega),\|\cdot\|_{\mathscr{X}(\partial\Omega)})$ is assumed Banach, we may define as a limit in $\mathscr{X}(\partial\Omega;V)$

$$
\widetilde{\operatorname{Tr}}_{\partial\Omega}(u) := \lim_{j\to\infty}\operatorname{Tr}_{\partial\Omega}(u_j), \tag{5.13}
$$

being independent of the approximating sequence $(u_j)\subset\mathrm{C}^\infty(\overline{\Omega};V)$. It is then straightforward to verify that $\widetilde{\operatorname{Tr}}_{\partial\Omega}\colon\ \mathrm{W}^{\mathbb{A},1}(\Omega)\to\mathscr{X}(\partial\Omega;V)$ is bounded and linear. Clearly $\widetilde{\operatorname{Tr}}_{\partial\Omega}$ and $\operatorname{Tr}_{\partial\Omega}$ coincide on $\mathrm{W}^{k,1}(\Omega;V)$, and since $\operatorname{Tr}_{\partial\Omega}\colon\ \mathrm{W}^{k,1}(\Omega;V)\to\mathscr{X}(\partial\Omega;V)$ is surjective, $\widetilde{\operatorname{Tr}}_{\partial\Omega}\colon\ \mathrm{W}^{\mathbb{A},1}(\Omega)\to\mathscr{X}(\partial\Omega;V)$ is onto, too. The proof is complete. □

We conclude this section with the following

Remark 5.5 (Homogeneous trace inequalities) To connect with the results of Section 4, let $\Omega=\mathbb{H}$ be an open halfspace and recall that $\dot{\mathrm{W}}^{k,1}(\mathbb{H};V)$ has trace space $\dot{\mathrm{B}}^{k-1}_{1,1}(\partial\mathbb{H};V)$. Given $u\in\mathrm{C}_c^\infty(\overline{\mathbb{H}};V)$, we define T_ju as in Eq. 5.5. Note that in this particular geometric situation, the number j_0 as it appears in (C4) can be chosen to be arbitrarily negative. Proposition 5.2 directly inherits to the homogeneous situation and hence one obtains Eq. 5.7 with $j_0=-\infty$. As a substitute of Eq. 5.10 we then obtain

$$
\|T_{j_0}^{(2)}u\|_{\dot{\mathrm{W}}^{k,1}(\Omega)} \leq c2^{j_0k}\|u\|_{\mathrm{L}^1(\Omega_{j_0-i_0}\setminus\Omega_{j_0+i_0})}. \tag{5.14}
$$

Since $u\in\mathrm{C}_c^\infty(\overline{\mathbb{H}};V)$, the term on the right-hand side of Eq. 5.14 will vanish as $j_0\to-\infty$. Based on the USPENSKI estimate Eq. 1.6, an analogous argument as that for Eq. 5.12 then equally yields $\|u\|_{\dot{\mathrm{B}}^{k-1}_{1,1}(\partial\mathbb{H})}\leq c\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{H})}$ for all $u\in\mathrm{C}_c^\infty(\overline{\mathbb{H}};V)$.

6 Proof of Theorem 1.3: Korn Without Global Singular Integral Estimates

We finally turn to the Korn-type inequality of Theorem 1.3 for John domains, which form the canonical class of domains for such inequalities. Since such domains in general need not be extension domains for $\mathrm{W}^{\mathbb{A},p}$, the global Calderón-Zygmund estimates from the proof of Theorem 4.4 cannot be utilised and a local argument is required. In Section 6.1, which should be of independent interest, we provide a generalisation of a decomposition for maps on John domains by RŮŽIČKA, SCHUMACHER and the first author [23], the latter being only applicable to maps with zero mean. Building on a best approximation result and an inequality of FEFFERMAN- STEIN-type (Sections 6.2 and 6.3), we obtain even more general Korn-type inequalities by means of extrapolation (so, for instance, inequalities involving Orlicz or Lorentz norms) once Korn-type inequalities are provided for weighted Lebesgue spaces, see Sections 6.4 and 6.5. Hence, even though Theorem 1.3 is only stated for the usual L^p-spaces, we directly deal with the weighted setting throughout.

6.1 Weights and a Decomposition Theorem for John Domains

As usual, a weight $w : \mathbb{R}^n \to \mathbb{R}$ (so an $\mathscr{L}^n$-measurable function with $0 < w < \infty$ $\mathscr{L}^n$-almost everywhere) is said to be a A_q-*Muckenhoupt weight* with $1 \leq q < \infty$, in short $w \in A_q$, if

$$
\begin{aligned}
[w]_{A_q} &:= \sup_Q \fint_Q w\,\mathrm{d}x \left(\fint_Q w^{-\frac{1}{q-1}}\,\mathrm{d}x\right)^{q-1} < \infty, \qquad &&\text{if } 1 < q < \infty,\\
[w]_{A_1} &:= \sup_Q \left[\fint_Q w\ \mathrm{d}x\ \sup_{x\in Q}\frac{1}{w(x)}\right] < \infty, &&\text{if } q = 1.
\end{aligned}
\tag{6.1}
$$

The class A_∞ is defined by $A_\infty = \bigcup_{q>1} A_q$ and endowed with $[w]_{A_\infty} := \lim_{p\to\infty}[w]_{A_p}$. Now let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $w \in A_\infty$, $\mathrm{L}^q_w(\Omega; V)$ is defined in the obvious manner. For a finite dimensional real vector space E, a subspace $\mathcal{N} \subset \mathscr{P}_m(\mathbb{R}^n; E)$ and a space $X(\Omega; E) \subset \mathrm{L}^1(\Omega; E)$, we define

$$
X_{\mathcal{N}}(\Omega; E) := \left\{ f \in X(\Omega; E) \colon\ \int_\Omega f \cdot \pi\,\mathrm{d}x = 0 \quad \text{for all } \pi \in \mathcal{N} \right\}. \tag{6.2}
$$

If $w \in A_q$ with $1 \leq q < \infty$ and $f \in \mathrm{L}^q_w(\Omega; E)$, then Hölder's inequality and Eq. 6.1 imply that $f \in \mathrm{L}^1(\Omega; E)$. In particular, for such weights w, $\mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$ is well-defined. We can now state the decomposition theorem that shall later be applied to $\mathcal{N} = \nabla^k \ker(\mathbb{A})$; as usual, $\mathcal{M}$ denotes the Hardy-Littlewood maximal operator.

Theorem 6.1 (Decomposition theorem) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded John domain or, equivalently, an open and bounded domain satisfying the emanating chain condition with certain constants σ_1, σ_2 and chain-covering $\mathcal{W} = \{ W_i : i \in \mathbb{N}_0 \mid \}$. Then there exists a family of linear operators $T_i\colon \mathrm{C}^\infty_{c,\mathcal{N}}(\Omega; E) \to \mathrm{C}^\infty_{c,\mathcal{N}}(W_i; E)$, $i \in \mathbb{N}_0$, such that for all $1 < q < \infty$ and all $w \in A_q$ the following holds:*

(a) For each $i \in \mathbb{N}_0$ and all $f \in \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$ there holds

$$
|T_i f| \leq c(\sigma_2, \mathcal{N})\, \mathbb{1}_{W_i}\, \mathcal{M}(\mathbb{1}_\Omega f) \quad \mathscr{L}^n\text{-almost everywhere.} \tag{6.3}
$$

(b) *For each* $i \in \mathbb{N}_0$ *the operator* T_i *maps* $\mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$ *boundedly to* $\mathrm{L}^q_{w,\mathcal{N}}(W_i; E)$.

(c) *The family* $\{T_i f : \ i \in \mathbb{N}_0\}$ *is a decomposition of* f *in* $\mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$, *i.e.*,

$$f = \sum_{i \in \mathbb{N}_0} T_i f \qquad \textit{in } \ \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E) \tag{6.4}$$

for all $f \in \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$. *The convergence is* unconditional.

(d) *The mapping* $f \mapsto (\|T_i f\|_{\mathrm{L}^q_w(W_i)})_i$ *from* $\mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$ *to* $l^q(\mathbb{N}_0)$ *is bounded and*

$$\frac{1}{c}\, \|f\|_{\mathrm{L}^q_w(\Omega)} \leq \left(\sum_{i \in \mathbb{N}_0} \|T_i f\|^q_{\mathrm{L}^q_w(W_i)} \right)^{\frac{1}{q}} \leq c\, \|f\|_{\mathrm{L}^q_w(\Omega)} \tag{6.5}$$

with a constant $c = c(\sigma_1, \sigma_2, q, [w]_{A_q}, \mathcal{N}) > 0$.

(e) *If* $f \in \mathrm{C}^\infty_{c,\mathcal{N}}(\Omega; E)$, *then* $\{ i \in \mathbb{N}_0 \ : \ T_i f \neq 0 \,|\, \}$ *is finite.*

Proof The proof is an adaptation of the proof of [23, Thm. 4.2] by RŮŽIČKA, SCHUMACHER and the first author, and so we confine ourselves to the points that need refinement in the setting considered here.

The idea in [23], where $\mathcal{N}$ is the set of constants, is that each f is first decomposed into local functions $S_i f := \xi_i f$ by means of a partition of unity ξ_i subject to the covering $\mathcal{W} = \{W_i : \ i \in \mathbb{N}_0\}$. Then each $S_i f$ is corrected in [23] such that it has vanishing integral. To compensate this change, the corrections are transported along the chains to the central ball W_0. In the much more general setting as considered here we have to correct the $S_i f$'s by local projections to $\mathcal{N}$ and transport those corrections along the chains to the central ball W_0. Throughout, we tacitly suppose f to be extended to $\mathbb{R}^n$ by zero.

Adopting the notation of Definition 2.3, we may suppose that the balls $B_{i,l}$ as in (C2) belong to a family $\mathscr{B}$ such that $\sum_{B \in \mathscr{B}} \mathbb{1}_B \leq \sigma_2 \mathbb{1}_\Omega$; see Remark 2.4. For $B \in \mathscr{B}$ we find, by translating and scaling a fixed non-negative $\eta \in \mathrm{C}^\infty_c(B(0,1))$ with $\int_{B(0,1)} \eta \,\mathrm{d}x = 1$, a non-negative function $\eta_B \in \mathrm{C}^\infty_c(B)$ with $\int_B \eta_B \,\mathrm{d}x = 1$ and $|\eta_B| \leq c/\mathscr{L}^n(B)$, where $c = c(n) > 0$. Setting $\eta_{i,l} := \eta_{B_{i,l}}$, we may record that

$$\begin{aligned} \eta_{i,l} &\in \mathrm{C}^\infty_c(W_{i,l}) \cap \mathrm{C}^\infty_c(W_{i,l+1}), \\ \left\| \eta_{i,l} \right\|_{\mathrm{L}^\infty} &\leq c \, \min \left\{ \frac{1}{\mathscr{L}^n(W_{i,l})}, \frac{1}{\mathscr{L}^n(W_{i,l+1})} \right\} \end{aligned} \tag{6.6}$$

for all $i, l \geq 0$ with $0 \leq l \leq m_i - 1$. Equally, we pick a non-negative $\eta_0 \in \mathrm{C}^\infty_c(W_0)$ with $\int_{W_0} \eta_0 \,\mathrm{d}x = 1$ and $|\eta_0| \leq c/\mathscr{L}^n(W_0)$; since every chain ends in $W_{i,m_i} = W_0$, we put $\eta_{i,m_i} := \eta_0$ for all $i \geq 0$.

Now define an operator $\Pi_{j,l} : \mathrm{L}^1(\sigma_2 W_{j,l}; E) \to \mathcal{N}$ by

$$\int_{W_{j,l}} \eta_{j,l} (\Pi_{j,l} f)\, \pi \,\mathrm{d}x = \int_{\sigma_2 W_{j,l}} f\, \pi \,\mathrm{d}x \qquad \text{for all } \pi \in \mathcal{N}. \tag{6.7}$$

Inserting $\pi = \Pi_{j,l} f$, inverse estimates for polynomials as in Eqs. 3.16, 3.17, and using that all $\eta_{j,l}$'s are translated and scaled versions of some η, we obtain

$$\begin{aligned} \|\Pi_{j,l} f\|_{\mathrm{L}^\infty(\sigma_2 W_{j,l})} &\leq c \Big(\int_{\sigma_2 W_{j,l}} \eta_{j,l} |\Pi_{j,l} f|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} \\ &\leq c \Big(\int_{\sigma_2 W_{j,l}} f \cdot \Pi_{j,l} f \,\mathrm{d}x \Big)^{\frac{1}{2}} \leq c \|\Pi_{j,l} f\|^{\frac{1}{2}}_{\mathrm{L}^\infty(\sigma_2 W_{j,l})} \Big(\int_{\sigma_2 W_{j,l}} |f| \,\mathrm{d}x \Big)^{\frac{1}{2}}, \end{aligned}$$

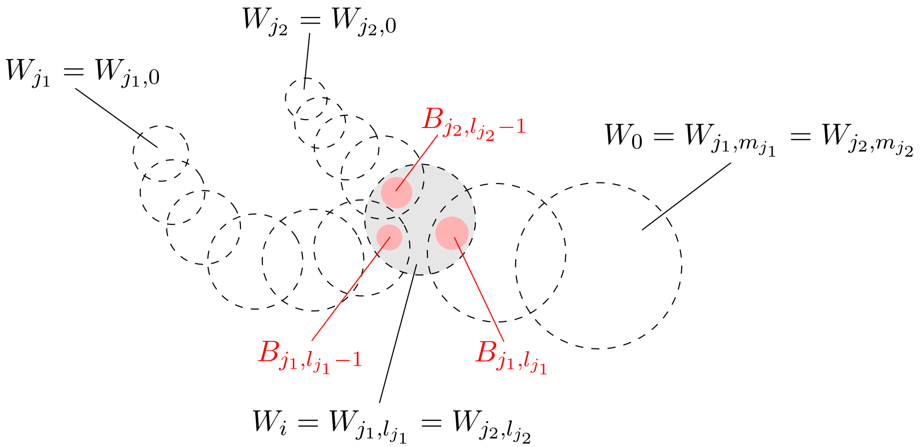

Fig. 5 Construction of the decomposition blocks T_i, cf. Eq. 6.9. In Eq. 6.9, one sums over all chains $W_{j,0} = W_j, ..., W_{j,m_j} = W_0$ that contain W_i; here, two such chains and the location of the relevant balls that matter in the definition of T_i are depicted

which implies

$$\|\Pi_{j,l} f\|_{\mathrm{L}^\infty(\sigma_2 W_{j,l})} \leq c \int_{\sigma_2 W_{j,l}} |f| \, \mathrm{d}x. \tag{6.8}$$

of [23] and, adopting the notation of Definition 2.3, set for $f \in \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$

$$\begin{aligned} T_i f &:= (S_i f - \eta_{i,0}\Pi_{i,0}(S_i f)) \\ &\quad + \sum_{\substack{j\geq 0 \\ j\neq i}} \sum_{\substack{l:0<l\leq m_j: \\ W_{j,l}=W_i}} \big(\eta_{j,l-1}\Pi_{j,l-1}(S_j f) - \eta_{j,l}\Pi_{j,l}(S_j f)\big). \end{aligned} \tag{6.9}$$

In the ultimate double sum, j effectively ranges over all numbers for which the chain $W_{j,m_j} = W_0, ..., W_{j,0} = W_j$ from (C2) connects W_j with W_0 where, in addition, $W_{j,l} = W_i$ for some $0 < l \leq m_j$ asserts that W_i must be contained in this chain (see Fig. 5). If $j \neq i$ and $0 < l \leq m_j$ are such that $W_{j,l} = W_i$, then Eq. 6.6 implies that $\mathrm{spt}(\eta_{j,l-1}) \cup \mathrm{spt}(\eta_{j,l}) \subset W_{j,l} = W_i$. We hence obtain $\mathrm{spt}(T_i f) \subset W_i$ and, by Eq. 6.6, $\eta_{j,l-1} + \eta_{j,l} \leq c\mathbb{1}_{W_i}/\mathscr{L}^n(W_i)$. Also note that the second sum over $0 < l \leq m_j$ such that $W_{j,l} = W_i$ contains only one summand.

We begin by proving that $T_i f$ is well-defined. Similarly to [23], we define the majorant

$$U_i := \sum_{\substack{j\geq 0 \\ j\neq i}} \sum_{\substack{l:0<l\leq m_j: \\ W_{j,l}=W_i}} \big(\eta_{j,l-1} \left|\Pi_{j,l-1}(S_j f)\right| + \eta_{j,l} \,|[|\,]\Pi_{j,l}(S_j f)\big). \tag{6.10}$$

Next, if $W_{j,0} = W_j, ..., W_{j,m_j} = W_0$ is a chain with $W_{j,l} = W_i$ for some $0 < l \leq m_j$, then we have by Eq. 6.6, $B_{j,l-1} \subset \sigma_2 W_{j,l-1} \subset \sigma_2^2 W_{j,l}$, $B_{j,l} \subset W_{j,l} \subset \sigma_2 W_{j,l}$ and the uniform comparability $\mathscr{L}^n(B_{j,l-1}) \simeq \mathscr{L}^n(B_{j,l}) \simeq \mathscr{L}^n(W_{j,l})$ the pointwise estimate

$$
\begin{aligned}
U_i &\le c\frac{\mathbb{1}_{W_i}}{\mathscr{L}^n(W_i)} \sum_{\substack{j\ge 0\\ j\neq i}} \sum_{\substack{l:0<l\le m_j:\\ W_{j,l}=W_i}} \big(\|\Pi_{j,l-1}(S_jf)\|_{\mathrm{L}^\infty(B_{j,l-1})} + \|\Pi_{j,l}(S_jf)\|_{\mathrm{L}^\infty(B_{j,l})}\big) \\
&\overset{Eq. 6.8}{\le} c\frac{\mathbb{1}_{W_i}}{\mathscr{L}^n(W_i)} \sum_{\substack{j\ge 0\\ j\neq i}} \sum_{\substack{l:0<l\le m_j:\\ W_{j,l}=W_i}} \|S_jf\|_{\mathrm{L}^1(\sigma_2^2 W_{j,l})} \\
&\le c\frac{\mathbb{1}_{W_i}}{\mathscr{L}^n(W_i)} \sum_{\substack{j\ge 0\\ j\neq i}} \|S_jf\|_{\mathrm{L}^1(\sigma_2 W_i)} \\
&\le c\mathbb{1}_{W_i} \fint_{\sigma_2 W_i} |f|\,\mathrm{d}x
\end{aligned}
$$

where we used in the ultimate step that the W_j's have uniformly finite overlap, cf. (C1) and, by (C2), there holds $\mathrm{spt}(S_jf) \subset W_j \subset \sigma_2 W_i$. Since $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; E)$, the sum in T_if converges absolutely $\mathscr{L}^n$ and so T_if is well-defined $\mathscr{L}^n$-a.e.. The term $S_if - \eta_{i,0}\Pi_{i,0}(S_if)$ can be estimated analogously, and so (a) follows.

Since $\mathcal{M}$ is bounded on $\mathrm{L}^q_w(\mathbb{R}^n)$, we obtain that T_i maps $\mathrm{L}^q_w(\Omega; E) \to \mathrm{L}^q_w(W_i; E)$ boundedly. In combination with (a) and recalling that $\mathrm{L}^q_w(W_i; E) \hookrightarrow \mathrm{L}^1(W_i; E)$, we obtain by Lebesgue's theorem on dominated convergence that the sum defining T_if also converges in $\mathrm{L}^1(W_i; E)$. For the full statement of (a), let $f \in \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$ and consider some $j \neq i$ such that W_i is contained in the chain connecting W_j and W_0, so $W_i = W_{j,l}$ for some $0 < l \le m_j$. By (C2), we then have $W_j \subset \sigma_2 W_{j,l-1} \cap \sigma_2 W_{j,l}$. Therefore, $\mathrm{spt}(S_jf) \subset W_j$ yields for all $\pi \in \mathcal{N}$

$$
\int_{\sigma_2 W_{j,l-1}} S_jf \cdot \pi \,\mathrm{d}x = \int_{\sigma_2 W_{j,l}} S_jf \cdot \pi \,\mathrm{d}x. \tag{6.11}
$$

The support properties of the S_jf's and the $\eta_{j,l}$'s, the definition of $\Pi_{j,l}$ and Eq. 6.11 imply that each summand in the definition of T_i has integral zero when being tested against any $\pi \in \mathcal{N}$. Since the sum defining T_if converges in $\mathrm{L}^1(\Omega; E)$, we conclude (b). We now establish that $f = \sum_{i\in\mathbb{N}_0} T_if$ holds pointwisely $\mathscr{L}^n$-a.e. for $f \in \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$; working from here, the proof evolves exactly as in [23]. By pointwise absolute convergence of $\sum_{i\in\mathbb{N}_0} T_if$ $\mathscr{L}^n$-a.e., we may change the order of summation in the following to find

$$
\begin{aligned}
\sum_{i\in\mathbb{N}_0} T_if &= \sum_{i\in\mathbb{N}_0} S_if - \sum_{i\in\mathbb{N}_0} \eta_{i,0}\Pi_{i,0}(S_if) \\
&\quad + \sum_{j\ge 0} \sum_{l:0<l\le m_j} \sum_{\substack{i\in\mathbb{N}_0\setminus\{j\}\\ W_i=W_{j,l}}} \big(\eta_{j,l-1}\Pi_{j,l-1}(S_jf) - \eta_{j,l}\Pi_{j,l}(S_jf)\big) \\
&\overset{(*)}{=} \sum_{i\in\mathbb{N}_0} S_if - \sum_{i\in\mathbb{N}_0} \eta_{i,0}\Pi_{i,0}(S_if) + \sum_{j\ge 0} \big(\eta_{j,0}\Pi_{j,0}(S_jf) - \eta_{j,m_j}\Pi_{j,m_j}(S_jf)\big) \\
&= \sum_{i\in\mathbb{N}_0} S_if - \sum_{j\ge 0} \eta_{j,m_j}\Pi_{j,m_j}(S_jf)
\end{aligned}
$$

$$\begin{aligned}&\overset{\eta_{j,m_j}=\eta_0}{=} \sum_{i\in\mathbb{N}_0} S_i f - \eta_0 \sum_{j\geq 0} \Pi_{j,m_j}(S_j f)\\ &= f - \eta_0 \sum_{j\geq 0} \Pi_{W_0}(S_j f)\\ &= f - \eta_0 \Pi_{W_0} f = f,\end{aligned}$$

since $\Pi_{W_0} := \Pi_{j,m_j}$ is independent of j and we have $\Pi_{W_0} f = 0$ as $f \in \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$. For $(*)$ note that, for fixed $j \geq 0$ and $0 < l \leq m_j$, $\sum_{i\neq j,\, W_i = W_{j,l}} 1 = 1$, cf. [23, Eq. (4.18)ff.]. Working from here, the rest of the proof is exactly as in [23]. We thus omit the details. □

Remark 6.2 (Unbounded domains) Theorem 6.1 also holds for unbounded John domains and unbounded domains satisfying the emanating chain condition. The proof requires no change. We refer to the modified definitions for unbounded domains to [23].

6.2 A Best Approximation Property

For our objectives in Section 6.3 and 6.4, we require the following best approximation property of the projections underlying Theorem 3.7:

Proposition 6.3 (Best approximation property) *Let $\mathbb{A}$ be a k-th order $\mathbb{C}$-elliptic operator of the form Eq. 1.1. Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain satisfying the emanating chain condition with constants σ_1, σ_2 and central ball B. Moreover, let $1 < p < \infty$ and $w \in A_p$. Then for all $\ell \in \{0, ..., k\}$ and all $u \in \mathrm{W}^{\ell,p}(\Omega; V)$ we have*

$$\left(\fint_{\Omega} \left|D^{\ell}(u - \Pi^B_{\mathbb{A}} u)\right|^p \, w \,\mathrm{d}x\right)^{\frac{1}{p}} \leq C \inf_{q\in\ker(\mathbb{A})} \left(\fint_{\Omega} \left|D^{\ell}(u - q)\right|^p \, w\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

where $C = C(\mathbb{A}, \sigma_2, p, [w]_{A_p}) > 0$.

Proof We start by noting that, if $B \subset \Omega$ is an open ball with $\operatorname{diam}(\Omega) \leq c_0 \operatorname{diam}(B)$, then for any $m \in \mathbb{N}_0$ there exists a constant $c_1 = c_1(c_0, m, n, V) > 0$ such that for all $\pi \in \mathscr{P}_m(\mathbb{R}^n; V)$ and all $\ell \in \mathbb{N}_0$ with $\ell \leq m + 1$ there holds

$$\begin{aligned}\left(\fint_{\Omega} |D^{\ell}\pi|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} &\leq (\sup_{\Omega} |D^{\ell}\pi|)\left(\frac{w(\Omega)}{\mathscr{L}^n(\Omega)}\right)^{\frac{1}{p}}\\ &\leq c_1 r(B)^{-\ell} \fint_B |\pi|\,\mathrm{d}x \left(\frac{w(\Omega)}{\mathscr{L}^n(\Omega)}\right)^{\frac{1}{p}}\\ &\leq c_1 [w]^{\frac{1}{p}}_{A_p} r(B)^{-\ell} \left(\fint_B |\pi|^p w\,\mathrm{d}x\right)^{\frac{1}{p}},\end{aligned} \tag{6.12}$$

where the ultimate line follows by Hölder's inequality and because w is doubling.

Now let $\ell \in \{0, ..., k\}$ and let $q \in \ker(\mathbb{A})$ be arbitrary, so that $\Pi^B_{\mathbb{A}} q = q$. Moreover, let $q_1 \in \mathscr{P}_{\ell-1} \subset \ker(\mathbb{A})$ be arbitrary. Then $\Pi^B_{\mathbb{A}} q_1 = q_1$ and therefore $D^{\ell}\Pi^B_{\mathbb{A}} q_1 = 0$. Now, $D^{\ell}(u - \Pi^B_{\mathbb{A}} u) = D^{\ell}(u - q) - D^{\ell}\Pi^B_{\mathbb{A}}(u - q - q_1)$, inverse estimates for polynomials (cf.

Eq. 3.16) in conjunction with $\operatorname{diam}(\Omega) \le c_0(\sigma_2)\operatorname{diam}(B)$ (cf. Eq. 2.1) and Eq. 6.12 imply

$$\begin{aligned}
&\left(\fint_\Omega \left|D^\ell(u - \Pi_{\mathbb{A}}^B u)\right|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad \le C\left(\fint_\Omega \left|D^\ell(u-q)\right|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} + C\left(\fint_\Omega \left|D^\ell \Pi_{\mathbb{A}}^B(u-q-q_1)\right|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\quad \overset{Eq.6.12}{\le} C\left(\fint_\Omega \left|D^\ell(u-q)\right|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} + C\operatorname{diam}(\Omega)^{-\ell}\left(\fint_B |u-q-q_1|^p w\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}$$

Now we may fix $q_1 \in \mathscr{P}_{\ell-1}(\mathbb{R}^n; V)$ such that the standard A_p-weighted Poincaré inequality for balls (see [32, Chpt. 1.4]) yields

$$\operatorname{diam}(\Omega)^{-\ell}\left(\fint_B |u-q-q_1|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} \le C\left(\fint_B \left|D^\ell(u-q)\right|^p w\,\mathrm{d}x\right)^{\frac{1}{p}}.$$

This, the previous estimate and $\operatorname{diam}(\Omega) \le c_0(\sigma_2)\operatorname{diam}(B)$ prove the claim. □

Remark 6.4 (Weighted stability estimate) For future reference, let us note that in the situation of Proposition 6.3 we have the estimate

$$\left(\fint_\Omega |\Pi_{\mathbb{A}}^B u|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} \le C\left(\fint_\Omega |u|^p w\,\mathrm{d}x\right)^{\frac{1}{p}} \tag{6.13}$$

for all $u \in \mathrm{L}^1(\Omega; V)$, where $C = C(\mathbb{A}, \sigma_2, p, [w]_{A_p}) > 0$. This is the weighted substitute of Lemma 3.5, and can be seen as follows: We replace π in Eq. 6.12 by $\Pi_{\mathbb{A}}^B u$ and repeat the first two lines of the estimation in Eq. 6.12 with $\ell = 0$. We then use Eq. 3.18 to estimate

$$\fint_B |\Pi_{\mathbb{A}}^B u|\,\mathrm{d}x \le c(\mathbb{A})\fint_B |u|\,\mathrm{d}x \le c(c_0, n, \mathbb{A})\fint_\Omega |u|\,\mathrm{d}x.$$

We then may argue as in Eq. 6.12 to deduce Eq. 6.13.

6.3 An Inequality of Fefferman-Stein-type

In this section we derive an inequality which allows to control the information of a function by its $\mathcal{N}$-sharp maximal operator. This serves as an instrumental tool in the proof of Theorem 1.3 in Section 6.4, but should be of independent interest.

We require the following restricted variant of maximal operators: Given an open subset $\Omega \subset \mathbb{R}^n$, $\sigma \ge 1$ and, for some finite dimensional real vector space E, a subspace $\mathcal{N} \subset \mathscr{P}_m(\mathbb{R}^n; E)$ for some $m \in \mathbb{N}$ and $1 \le p < \infty$, we put

$$\mathcal{M}_{\mathrm{res},\Omega,\sigma,p} f(x) := \begin{cases} \sup\limits_{Q \ni x:\, \sigma Q \subset \Omega} \left(\fint_Q |f|^p\,\mathrm{d}y\right)^{\frac{1}{p}}, & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{6.14}$$

$$\mathcal{M}^\sharp_{\mathrm{res},\Omega,\sigma,p,\mathcal{N}} f(x) := \begin{cases} \sup\limits_{Q \ni x:\, \sigma Q \subset \Omega} \inf\limits_{\pi \in \mathcal{N}} \left(\fint_Q |f-\pi|^p\,\mathrm{d}y\right)^{\frac{1}{p}}, & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases} \tag{6.15}$$

for $f \in \mathrm{L}^p_{\mathrm{loc}}(\Omega; E)$ and non-degenerate cubes Q. For brevity, we also set $\mathcal{M}_{\mathrm{res},\Omega,\sigma} := \mathcal{M}_{\mathrm{res},\Omega,\sigma,1}$ and $\mathcal{M}^\sharp_{\mathrm{res},\Omega,\sigma,\mathcal{N}} := \mathcal{M}^\sharp_{\mathrm{res},\Omega,\sigma,1,\mathcal{N}}$.

The main result of this section is a generalisation of [23, Theorem 5.23] for $\mathcal{N} = \mathbb{R}$ and its unweighted version [36, Lemma 4]. We present the proof later in this section.

Theorem 6.5 (of Fefferman-Stein-type) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain satisfying the emanating chain condition with constants $\sigma_1 > 1, \sigma_2 \geq 1$ and central ball B.*

Let $1 < q < \infty$, $w \in A_q$ and $f \in \mathrm{L}^1_{loc}(\Omega; E)$. If $\mathcal{M}^\sharp_{\mathrm{res},\Omega,\sigma_1,\mathcal{N}} f \in \mathrm{L}^q_w(\Omega)$, then $f \in \mathrm{L}^q_w(\Omega; E)$ and

$$\inf_{\pi\in\mathcal{N}} \|f - \pi\|_{\mathrm{L}^q_w(\Omega)} \leq c \left\| \mathcal{M}^\sharp_{\mathrm{res},\Omega,\sigma_1,\mathcal{N}} f \right\|_{\mathrm{L}^q_w(\Omega)}. \tag{6.16}$$

The constant c depends only on σ_1, σ_2, $\mathcal{N}$, q, and $[w]_{A_q}$.

For a cube Q (or ball) we define $\Pi^Q_{\mathcal{N}}$ as the $\mathrm{L}^2(Q; E)$-orthogonal projection to $\mathcal{N}$. It follows easily by inverse estimates as in Eq. 3.16 that $\Pi^Q_{\mathcal{N}}$ is $\mathrm{L}^1(Q; E)$-stable, i.e.,

$$⨍_Q \left|\Pi^Q_{\mathcal{N}} f\right| \mathrm{d}x \leq C_1 ⨍_Q |f| \,\mathrm{d}x, \tag{6.17}$$

and, similarly to Proposition 6.3, has the approximation property

$$⨍_Q |f - \Pi_{\mathcal{N}^Q} f| \,\mathrm{d}x \leq C_2 \inf_{q\in\mathcal{N}} ⨍_Q |f - q| \,\mathrm{d}x. \tag{6.18}$$

We need the following version of the Calderón-Zygmund decomposition by IWANIEC:

Lemma 6.6 ([36, Lemma 3]) *Let $Q_0 \subset \mathbb{R}^n$ be an open cube and let $f \in \mathrm{L}^1(Q_0; E)$. For every $\alpha \geq \langle |f| \rangle_{Q_0} = ⨍_{Q_0} |f| \,\mathrm{d}x$ there exist pairwise disjoint cubes $Q^\alpha_j \subset Q_0$ indexed by $j \in \mathbb{N}$ such that*

(a) $\alpha < ⨍_{Q^\alpha_j} |f| \,\mathrm{d}x \leq 2^n \alpha$,
(b) If $\alpha \geq \beta \geq \langle |f| \rangle_{Q_0}$, then each cube Q^α_j is contained in an element of $\{ Q^\beta_j : j \in \mathbb{N} \}$,
(c) $|f| \leq \alpha$ $\mathscr{L}^n$-almost everywhere on $Q_0 \setminus \bigcup_j Q^\alpha_j$,
(d) $\bigcup_j Q^\alpha_j \subset \{ \mathcal{M}_{\mathrm{res},Q_0,1} f > \alpha \}$ for all $\alpha \geq \langle |f| \rangle_{Q_0}$,
(e) $\{ \mathcal{M}_{\mathrm{res},Q_0,1} f > 5^n \alpha \} \subset \bigcup_j 5Q^\alpha_j$ for all $\alpha \geq \langle |f| \rangle_{Q_0}$.

We start with a version of Theorem 6.5 for the case where Ω is just a cube.

Lemma 6.7 *Let $Q_0 \subset \mathbb{R}^n$ be an open cube, $1 \leq q < \infty$, $w \in A_\infty$, and let $f \in \mathrm{L}^1(Q_0; E)$. If $\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}} f \in \mathrm{L}^q_w(Q_0)$, then $\mathcal{M}_{\mathrm{res},Q_0,1} f \in \mathrm{L}^q_w(Q_0)$ and there holds*

$$\int_{Q_0} \left|\mathcal{M}_{\mathrm{res},Q_0,1} f\right|^q w \,\mathrm{d}x \leq c \int_{Q_0} \left|\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}} f\right|^q w \,\mathrm{d}x + c\, w(Q_0) \left(⨍_{Q_0} |f| \,\mathrm{d}x \right)^q,$$

where $c = c(n, \mathcal{N}, q, [w]_{A_\infty}) > 0$.

Proof Since $w \in A_\infty$, there exists a constant $c_0 > 0$, only depending on $[w]_{A_\infty}$, such that $w(5Q) \leq c_0 w(Q)$ for any cube $Q \subset \mathbb{R}^n$, see [58, V 1.6, p. 196]. By the usual inverse estimates and Eq. 6.17, there exists $\mathtt{c} = \mathtt{c}(\mathcal{N}, n) > 0$ such that, for any open cube $Q \subset \mathbb{R}^n$,

$$\|\Pi^Q_{\mathcal{N}} f\|_{\mathrm{L}^\infty(Q)} \leq \mathtt{c} ⨍_Q |f| \,\mathrm{d}x. \tag{6.19}$$

Then we fix a number $m \in \mathbb{N}$ such that $\mathtt{C}2^{n-m} < \frac{1}{2}$. We claim that for every $\epsilon > 0$ there exists $\delta = \delta(\varepsilon, n, m, \mathcal{N}, [w]_{A_\infty}) > 0$ such that

$$w(\{\mathcal{M}_{\mathrm{res},Q_0,1}f > 5^n\alpha\,|\,)\} \le c_0 w(\{\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}}f > \delta\alpha\,|\,)\} \\ + c_0\epsilon\, w(\{\mathcal{M}_{\mathrm{res},Q_0,1}f > 2^{-m}\alpha\,|\,)\} \tag{6.20}$$

for all $f \in \mathrm{L}^1(Q_0; E)$ and all $\alpha \ge 2^m(|f|)_{Q_0}$. To prove the claim, let $\varepsilon > 0$. As $\alpha \ge 2^{-m}\alpha \ge (|f|)_{Q_0}$, we may pick $(Q_j^\alpha)_{j\in\mathbb{N}}$ as in Lemma 6.6. Toward Eq. 6.20 we note that, because of Lemma 6.6 (e) and the fact that $w(5Q) \le c_0 w(Q)$ holds for all cubes Q, it suffices to show that

$$\sum_j w(Q_j^\alpha) \le w(\{\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}}f > \delta\alpha\,|\,)\} + \epsilon\, w(\{\mathcal{M}_{\mathrm{res},Q_0,1}f > 2^{-m}\alpha\,|\,)\}. \tag{6.21}$$

By Lemma 6.6 (b), every Q_j^α is contained in some Q, $Q \in \{Q_i^{2^{-m}\alpha} : i \in \mathbb{N}\}$. Then

$$⨍_{Q_j^\alpha} |\Pi_{\mathcal{N}}^Q f|\,\mathrm{d}x \le \|\Pi_{\mathcal{N}}^Q f\|_{\mathrm{L}^\infty(Q)} \overset{Eq.6.19}{\le} \mathtt{C}⨍_Q |f|\,\mathrm{d}x \overset{\text{Lemma 6.6 (a)}}{\le} \mathtt{C}2^{n-m}\alpha \overset{Eq.6.19\text{ff.}}{<} \frac{\alpha}{2}.$$

Thus, by (a) of Lemma 6.6 we obtain for all $j \in \mathbb{N}$

$$⨍_{Q_j^\alpha} |f - \Pi_{\mathcal{N}}^Q f|\,\mathrm{d}x \ge ⨍_{Q_j^\alpha} |f|\,\mathrm{d}x - ⨍_{Q_j^\alpha} \left|\Pi_{\mathcal{N}}^Q f\right|\mathrm{d}x \ge \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

Multiplying the previous inequality by $\left|Q_j^\alpha\right|$ and summing over all $Q_j^\alpha \subset Q$, we obtain

$$\sum_{Q_j^\alpha \subset Q} \left|Q_j^\alpha\right| \le \frac{2}{\alpha}\int_Q \left|f - \Pi_{\mathcal{N}}^Q f\right|\mathrm{d}x.$$

Since $w \in A_\infty$, there exists an $\epsilon_2 > 0$ such that if $\mathcal{E} \subset Q$ with $|\mathcal{E}| \le \epsilon_2 |Q|$, then $w(\mathcal{E}) \le \epsilon w(Q)$, see [58, V 1.7, p. 196]; here, ϵ_2 only depends on ϵ and the A_∞-constant of w but is independent of Q and $\mathcal{E}$. With C_2 as in Eq. 6.18, we define $\delta := \frac{1}{2C_2}\varepsilon_2$ and then consider two cases:

- If $⨍_Q \left|f - \Pi_{\mathcal{N}}^Q f\right|\mathrm{d}x \le \frac{\alpha}{2}\epsilon_2$, then $\sum_{Q_j^\alpha \subset Q}\left|Q_j^\alpha\right| \le \epsilon_2 |Q|$. By our choice of ϵ_2, $\sum_{Q_j^\alpha\subset Q} w(Q_j^\alpha) \le \epsilon\, w(Q)$, where we have used that the Q_j^α, $j \in \mathbb{N}$, are pairwise disjoint.
- If $⨍_Q \left|f - \Pi_{\mathcal{N}}^Q f\right|\mathrm{d}x > \frac{\alpha}{2}\epsilon_2$, then we have, with C_2 as in Eq. 6.18, $\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}}f(x) > \frac{\alpha}{2C_2}\epsilon_2 = \delta\alpha$ for all points $x \in Q$. In particular, $Q \subset \{\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}}f > \delta\alpha\,|\,\}$ and

$$\sum_{Q_j^\alpha \subset Q} w(Q_j^\alpha) \le w(Q) = w(Q \cap \{\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}}f > \delta\alpha\,|\,)\}.$$

Combining both alternatives, we obtain

$$\sum_{Q_j^\alpha \subset Q} w(Q_j^\alpha) \le w(Q \cap \{\mathcal{M}^\sharp_{\mathrm{res},Q_0,1,\mathcal{N}}f > \delta\alpha\,|\,)\} + \epsilon w(Q). \tag{6.22}$$

By Lemma 6.6 (b) and since $\alpha \ge 2^m(|f|)_{Q_0}$, summing Eq. 6.22 over the pairwise disjoint cubes $Q \in \{Q_i^{2^{-m}\alpha} : i \in \mathbb{N}\,|\,\}$, all cubes from $\{Q_j^\alpha : j \in \mathbb{N}\,|\,\}$ are counted exactly once in the overall sum. We thus obtain Eq. 6.21 and hereby Eq. 6.20. Once Eq. 6.20 is established, we may exactly follow the layer cake argument of Iwaniec [36, Lemma 4] to conclude. □

Corollary 6.8 *Let $Q_0 \subset \mathbb{R}^n$ be an open cube, $1 \le q < \infty$, $w \in A_q$, and $f \in \mathrm{L}^1_{\mathcal{N}}(Q_0; E)$. If $\mathcal{M}^{\sharp}_{\mathrm{res}, Q_0, 1, \mathcal{N}} f \in \mathrm{L}^q_w(Q_0; E)$, then $f \in \mathrm{L}^q_{w,\mathcal{N}}(Q_0; E)$ and*

$$\|f\|_{\mathrm{L}^q_w(Q_0)} \le c \left\| \mathcal{M}^{\sharp}_{\mathrm{res}, Q_0, 1, \mathcal{N}} f \right\|_{\mathrm{L}^q_w(Q_0)},$$

where the constant c only depends on $\mathcal{N}$, q and $[w]_{A_q}$.

Proof Let $f \in \mathrm{L}^1_{\mathcal{N}}(Q_0; E)$ so that $\Pi^{Q_0}_{\mathcal{N}} f = 0$. In conjunction with Lemma 6.7 and Eq. 6.18, this yields

$$\begin{aligned}
\|f\|^q_{\mathrm{L}^q_w(Q_0)} &\le \left\|\mathcal{M}_{\mathrm{res}, Q_0, 1} f\right\|^q_{\mathrm{L}^q_w(Q_0)} \\
&\le c \left\| \mathcal{M}^{\sharp}_{\mathrm{res}, Q_0, 1, \mathcal{N}} f \right\|^q_{\mathrm{L}^q_w(Q_0)} + c\, w(Q_0) \left(\fint_{Q_0} \left| f - \Pi^{Q_0}_{\mathcal{N}} f \right| \mathrm{d}x \right)^q \\
&\le c \left\| \mathcal{M}^{\sharp}_{\mathrm{res}, Q_0, 1, \mathcal{N}} f \right\|^q_{\mathrm{L}^q_w(Q_0)} + c \int_{Q_0} |[|\,](\mathcal{M}^{\sharp}_{\mathrm{res}, Q_0, 1, \mathcal{N}} f)(y)^q\, w \,\mathrm{d}y \\
&\le c \left\| \mathcal{M}^{\sharp}_{\mathrm{res}, Q_0, 1, \mathcal{N}} f \right\|^q_{\mathrm{L}^q_w(Q_0)},
\end{aligned}$$

and the proof is complete. □

Before we come to the proof of Theorem 6.5, the underlying duality argument requires

Lemma 6.9 *Let $\Omega \subset \mathbb{R}^n$ be as in Theorem 6.5. Let $1 < q < \infty$ and $w \in A_q$. Then $\mathrm{C}^\infty_{c,\mathcal{N}}(\Omega; E)$ is dense in $\mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$ for the norm topology.*

Proof Let $\eta_0 \in \mathrm{C}^\infty_c(B)$ with $\eta_0 \ge 0$ and $\int_B \eta_0 \,\mathrm{d}x = 1$. Similarly as in Eq. 6.7 we define a linear operator $\Pi_0 : \mathrm{L}^1(\Omega; E) \to \mathcal{N}$ by

$$\int_B \eta_0 (\Pi_0 f) \xi \,\mathrm{d}x = \fint_\Omega f \xi \,\mathrm{d}x \qquad \text{for all } \xi \in \mathcal{N}.$$

By inverse estimates on $\mathcal{N}$ and Hölder's inequality we obtain $\|\Pi_0 f\|_{\mathrm{L}^\infty(\Omega)} \le c \fint_\Omega |f| \,\mathrm{d}x$. This and $w \in A_q$ imply $\|\eta_0 \Pi_0 f\|_{\mathrm{L}^q_w(\Omega)} \le c \|f\|_{\mathrm{L}^q_w(\Omega)}$. Now let $f \in \mathrm{L}^q_{w,\mathcal{N}}(\Omega; E)$ so that, in particular, $\Pi_0 f = 0$. Then we find $(g_j) \subset \mathrm{C}^\infty_c(\Omega; E)$ such that $g_j \to f$ in $\mathrm{L}^q_w(\Omega; E)$, cf. Miller [51, Lem. 2.1]. Put $\widetilde{\Pi}_0 := |\Omega| \Pi_0$ and define $f_j := g_j - \eta_0 \widetilde{\Pi}_0 g_j \in \mathrm{C}^\infty_{c,\mathcal{N}}(\Omega; E)$. Since $\Pi_0 f = 0$ we have $f_j = g_j - \eta_0 \widetilde{\Pi}_0 (g_j - f)$. Since $g_j \to f$ in $\mathrm{L}^q_w(\Omega; E)$ it follows with $\left\|\eta_0 \Pi_0 (g_j - f)\right\|_{\mathrm{L}^q_w(\Omega)} \le c \left\|g_j - f\right\|_{\mathrm{L}^q_w(\Omega)}$ that $f - f_j \to 0$ in $\mathrm{L}^q_w(\Omega; E)$, as desired. □

Proof of Theorem 6.5 Let $1 < q < \infty$, $w \in A_q$, and $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; E)$ with $\mathcal{M}^{\sharp}_{\mathrm{res}, \Omega, \sigma_1, \mathcal{N}} f \in \mathrm{L}^q_w(\Omega)$. Since $w \in A_q$, we have $w' := w^{\frac{1}{1-q}} \in A_{q'}$.

Let $h \in \mathrm{C}^\infty_{c,\mathcal{N}}(\Omega; E)$. By Theorem 6.1 we may then decompose h into the sum of functions $T_i h \in \mathrm{C}^\infty_{c,\mathcal{N}}(W_i; E) \subset \mathrm{L}^{q'}_{w',\mathcal{N}}(W_i; E)$ such that

$$\left(\sum_{i \ge 0} \|T_i h\|^{q'}_{\mathrm{L}^{q'}_{w'}(W_i)} \right)^{\frac{1}{q'}} \le c\, \|h\|_{\mathrm{L}^{q'}_{w'}(\Omega)}. \tag{6.23}$$

Moreover, by Theorem 6.1 (e), only finitely many summands of $h = \sum_{i\geq 0} T_i h$ are non-zero. Since $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega; E)$ and $h \in \mathrm{C}^\infty_{c,\mathcal{N}}(\Omega; E)$, $\langle f, h\rangle := \int_\Omega fh\,\mathrm{d}x$ is well-defined, and for all $i \in \mathbb{N}$, $\xi \in \mathcal{N}$ we have $\int_{W_i} \xi T_i h\,\,\mathrm{d}x = 0$. Therefore, with $\Pi^{W_i}_{\mathcal{N}}$ as in Eq. 6.17ff.,

$$\langle f, h\rangle = \sum_i \langle f, T_i h\rangle = \sum_i \langle f - \Pi^{W_i}_{\mathcal{N}} f, T_i h\rangle,$$

where we also used that $f \in \mathrm{L}^1(W_i; E)$, since $\sigma_1 > 1$ and consequently $\overline{W}_1 \Subset \sigma_1 W_i \subset \Omega$. We estimate with Corollary 6.8 in the second step

$$\begin{aligned}
|\langle f, h\rangle| &\leq \sum_i \left\| f - \Pi^{W_i}_{\mathcal{N}} f \right\|_{\mathrm{L}^q_w(W_i)} \|T_i h\|_{\mathrm{L}^{q'}_{w'}(W_i)} \\
&\leq c \sum_i \left\| \mathcal{M}^\sharp_{\mathrm{res}, W_i, 1, \mathcal{N}} f \right\|_{\mathrm{L}^q_w(W_i)} \|T_i h\|_{\mathrm{L}^{q'}_{w'}(W_i)} \\
&\leq c \Big(\sum_i \left\| \mathbb{1}_{W_i} \mathcal{M}^\sharp_{\mathrm{res}, \Omega, \sigma_1, \mathcal{N}} f \right\|^q_{\mathrm{L}^q_w(\Omega)} \Big)^{\frac{1}{q}} \Big(\sum_i \|T_i h\|^{q'}_{\mathrm{L}^{q'}_{w'}(W_i)} \Big)^{\frac{1}{q'}} \\
&\leq c \left\| \mathcal{M}^\sharp_{\mathrm{res}, \Omega, \sigma_1, \mathcal{N}} f \right\|_{\mathrm{L}^q_w(\Omega)} \|h\|_{\mathrm{L}^{q'}_{w'}(\Omega)}
\end{aligned} \tag{6.24}$$

for all $h \in \mathrm{C}^\infty_{c,\mathcal{N}}(\Omega; E)$, where we have used in the last step $\sum_{W\in\mathcal{W}} \mathbb{1}_{\sigma_1 W} \leq \sigma_2 \mathbb{1}_\Omega$ and Eq. 6.23. Therefore, invoking the smooth approximation from Lemma 6.9,

$$\sup_{\substack{h\in \mathrm{L}^{q'}_{w',\mathcal{N}}(\Omega;E) \\ \|h\|_{\mathrm{L}^{q'}_{w'}(\Omega)} \leq 1}} \langle f, h\rangle \leq c \left\| \mathcal{M}^\sharp_{\mathrm{res}, \Omega, \sigma_1, \mathcal{N}} f \right\|_{\mathrm{L}^q_w(\Omega)}. \tag{6.25}$$

By an analogous argument to that of Proposition 6.3 and because $\Pi^B_{\mathcal{N}}$ is L^q_w-stable,

$$\begin{aligned}
\inf_{\pi\in\mathcal{N}} \|f - \pi\|_{\mathrm{L}^q_w(\Omega)} &\leq c\|f - \Pi^B_{\mathcal{N}} f\|_{\mathrm{L}^q_w(\Omega)} \\
&= c \sup_{\substack{h\in \mathrm{L}^{q'}_{w'}(\Omega;E) \\ \|h\|_{\mathrm{L}^{q'}_{w'}(\Omega)} \leq 1}} \langle f - \Pi^B_{\mathcal{N}} f, h - \Pi^B_{\mathcal{N}} h\rangle + \underbrace{\langle f - \Pi^B_{\mathcal{N}} f, \Pi^B_{\mathcal{N}} h\rangle}_{=0} \\
&\leq c \sup_{\substack{\widetilde{h}\in \mathrm{L}^{q'}_{w',\mathcal{N}}(\Omega;E) \\ \|\widetilde{h}\|_{\mathrm{L}^{q'}_{w'}(\Omega)} \leq 1}} \langle f, \widetilde{h}\rangle \overset{Eq.6.25}{\leq} c \left\| \mathcal{M}^\sharp_{\mathrm{res}, \Omega, \sigma_1, \mathcal{N}} f \right\|_{\mathrm{L}^q_w(\Omega)}.
\end{aligned}$$

This completes the proof. □

6.4 Korn-type inequalities and the proof of Theorem 1.3

We now turn to Theorem 6.10 below, a special case of which is Theorem 1.3. Different from Theorem 4.4, where the focus was on the equivalence of trace and Korn-type inequalities on *smooth* domains, and previous contributions (cf. KAŁAMAJSKA [41, 42]), the previous sections now allow us to cover the natural yet vast class of John domains which, in turn, might be very irregular:

Theorem 6.10 (Weighted Korn inequality) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain that satisfies the emanating chain condition with constants $\sigma_1 > 1$ and $\sigma_2 \geq 1$ and central ball B. Moreover, let $k \in \mathbb{N}$, $1 < p < \infty$ and $w \in A_p$. Then for any k-th order $\mathbb{C}$-elliptic operator $\mathbb{A}$ of the form Eq. 1.1 there exists a constant $c = c(\sigma_1, \sigma_2, p, [w]_{A_p}, \mathbb{A}) > 0$ such that*

$$\begin{aligned} \|D^k(u - \Pi_{\mathbb{A}}^B u)\|_{\mathrm{L}^p_w(\Omega)} &\leq c\|\mathbb{A}u\|_{\mathrm{L}^p_w(\Omega)}, \\ \|D^k u\|_{\mathrm{L}^p_w(\Omega)} &\leq c(\mathrm{diam}(\Omega)^{-k}\|u\|_{\mathrm{L}^p_w(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p_w(\Omega)}) \end{aligned} \tag{6.26}$$

hold for all $u \in \mathrm{W}^{\mathbb{A}}\,\mathrm{L}^p_w(\Omega) := \{v \in \mathrm{L}^p_w(\Omega; V) : \ \mathbb{A}v \in \mathrm{L}^p_w(\Omega; W)\}$.

Proof Since $w \in A_p$, by the open-endedness property of the Muckenhoupt classes [18], there exists $\varepsilon = \varepsilon(n, p, [w]_{A_p}) > 0$ such that $w \in A_{p-\varepsilon}$, too. We then pick $1 < q < p$ such that $p - \varepsilon < \frac{p}{q} < p$ and record in advance that $w \in A_{p/q}$. With $\sigma_1 > 1$ as in the theorem, let $Q \subset \Omega$ be an open cube such that $\sigma_1 Q \subset \Omega$. Choose a cut-off function $\rho \in \mathrm{C}_c^\infty(\Omega; [0, 1])$ such that $\mathbb{1}_Q \leq \rho \leq \mathbb{1}_{\sigma_1 Q}$ and $|\nabla^\ell \rho| \leq c(\sigma_1 - 1)^{-l}\ell(Q)^{-l}$ for all $l \in \{0, ..., k\}$. Then $\varphi := \rho(u - \Pi_{\mathbb{A}}^{\sigma_1 Q} u)$ (with $\Pi_{\mathbb{A}}^{\sigma_1 Q}$ being as in Proposition 3.3) is compactly supported in $\mathbb{R}^n$. Since $\mathbb{A}$ in particular is elliptic, the Fourier multiplier argument underlying Eq. 4.29ff. yields the existence of a constant $c = c(q, \mathbb{A}) > 0$ such that

$$\|D^k \varphi\|_{\mathrm{L}^q(\mathbb{R}^n)} \leq c\|\mathbb{A}\varphi\|_{\mathrm{L}^q(\mathbb{R}^n)}. \tag{6.27}$$

Moreover, note that $\Pi_{\mathbb{A}}^Q \Pi_{\mathbb{A}}^{\sigma_1 Q} u = \Pi_{\mathbb{A}}^{\sigma_1 Q} u$ on Q and therefore, using the L^q-stability of the projections $\Pi_{\mathbb{A}}^Q$ in the second and the Poincaré-type inequalities from Theorem 3.7 in the fourth step,

$$\begin{aligned} \Big(\fint_Q |\Pi_{\mathbb{A}}^{\sigma_1 Q} u - \Pi_{\mathbb{A}}^Q u|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} &= \Big(\fint_Q |\Pi_{\mathbb{A}}^Q (u - \Pi_{\mathbb{A}}^{\sigma_1 Q} u)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \\ &\leq c\Big(\fint_Q |u - \Pi_{\mathbb{A}}^{\sigma_1 Q} u|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \\ &\leq c\Big(\fint_{\sigma_1 Q} |u - \Pi_{\mathbb{A}}^{\sigma_1 Q} u|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \overset{Eq.3.21}{\leq} c\ell(Q)^k \Big(\fint_{\sigma_1 Q} |\mathbb{A}u|^q \,\mathrm{d}x\Big)^{\frac{1}{q}}, \end{aligned} \tag{6.28}$$

where $c = c(\sigma_1, q, \mathbb{A}) > 0$. Now, using the Poincaré-type inequalities from Theorem 3.7,

$$\begin{aligned} \Big(\fint_Q |D^k(u - \Pi_{\mathbb{A}}^Q u)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} &\leq c\Big(\fint_{\sigma_1 Q} |D^k(\rho(u - \Pi_{\mathbb{A}}^{\sigma_1 Q} u))|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \\ &\quad + c\Big(\fint_Q |D^k(\Pi_{\mathbb{A}}^{\sigma_1 Q} u - \Pi_{\mathbb{A}}^Q u)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \\ \overset{Eq.6.27}{\leq} c\Big(\fint_{\sigma_1 Q} |\mathbb{A}(\rho(u - \Pi_{\mathbb{A}}^{\sigma_1 Q} u))|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} &+ \Big(\fint_Q |D^k(\Pi_{\mathbb{A}}^{\sigma_1 Q} u - \Pi_{\mathbb{A}}^Q u)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \\ \leq c\Big(\sum_{l=1}^k \ell(Q)^{-l}\Big(\fint_{\sigma_1 Q} |D^{k-l}(u - \Pi_{\mathbb{A}}^{\sigma_1 Q} u)|^q \,\mathrm{d}x\Big)^{\frac{1}{q}}\Big) &+ c\Big(\fint_{\sigma_1 Q} |\mathbb{A}u|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&+ c\ell(Q)^{-k}\sum_{l=0}^{k}\ell(Q)^{l}\Big(\fint_{Q}|D^{l}(\Pi_{\mathbb{A}}^{\sigma_1 Q}u-\Pi_{\mathbb{A}}^{Q}u)|^{q}\,\mathrm{d}x\Big)^{\frac{1}{q}}\\
&\overset{Eqs.3.21,3.18_2}{\leq} c\Big(\fint_{\sigma_1 Q}|\mathbb{A}u|^{q}\,\mathrm{d}x\Big)^{\frac{1}{q}}+c\ell(Q)^{-k}\Big(\fint_{Q}|\Pi_{\mathbb{A}}^{\sigma_1 Q}u-\Pi_{\mathbb{A}}^{Q}u|^{q}\,\mathrm{d}x\Big)^{\frac{1}{q}}\\
&\overset{Eq.6.28}{\leq} c\Big(\fint_{\sigma_1 Q}|\mathbb{A}u|^{q}\,\mathrm{d}x\Big)^{\frac{1}{q}},
\end{aligned}
$$

where again $c = c(\sigma_1, q, \mathbb{A}) > 0$. We define $\mathcal{N} := D^k \ker(\mathbb{A})$. Then, using that $D^k\Pi_{\mathbb{A}}^{Q}u \in \mathcal{N}$ for any open cube Q,

$$
\mathcal{M}^{\sharp}_{\mathrm{res},\Omega,\sigma_1,q,\mathcal{N}}(D^k u) \leq c\mathcal{M}_{\mathrm{res},\Omega,1,q}(\mathbb{A}u). \tag{6.29}
$$

Recall that B denotes the central ball of Ω, cf. Definition 2.3. We now invoke the best approximation property from Proposition 6.3 with respect to the weight w and $\ell = k$ in the first and the Fefferman-Stein-type inequality from Theorem 6.5 with $f = D^k u$, $\mathcal{N} = D^k \ker(\mathbb{A})$ in the second step to obtain

$$
\begin{aligned}
\|D^k u - D^k\Pi_{\mathbb{A}}^{B}u\|_{\mathrm{L}^p_w(\Omega)} &\overset{\text{Proposition 6.3}}{\leq} c\inf_{\pi\in\mathcal{N}}\|D^k u-\pi\|_{\mathrm{L}^p_w(\Omega)}\\
&\overset{\text{Theorem 6.5}}{\leq} c\|\mathcal{M}^{\sharp}_{\mathrm{res},\Omega,\sigma_1,q,\mathcal{N}}(D^k u)\|_{\mathrm{L}^p_w(\Omega)}\\
&\overset{Eq.6.29}{\leq} c\|\mathcal{M}_{\mathrm{res},\Omega,1,q}(\mathbb{A}u)\|_{\mathrm{L}^p_w(\Omega)}\\
&\leq c\|\mathcal{M}_{\mathrm{res},\Omega,1,1}(|\mathbb{A}u|^q)\|^{1/q}_{\mathrm{L}^{p/q}_w(\Omega)}\\
&\overset{1<q<p}{\leq} c\|\mathbb{A}u\|_{\mathrm{L}^p_w(\Omega)},
\end{aligned}
\tag{6.30}
$$

where we used that $\mathcal{M}_{\mathrm{res},\Omega,1,1}\colon \mathrm{L}^{p/q}_w(\Omega) \to \mathrm{L}^{p/q}_w(\Omega)$ is bounded provided $1 < p/q < \infty$ and $w \in A_{p/q}$. Tracking the dependencies of the constants in Eq. 6.30, we obtain $Eq.6.26_1$. To obtain $Eq.6.26_2$, the weighted inverse and stability estimates (Eq. 6.12 ff.) yield

$$
\|D^k\Pi_{\mathbb{A}}^{B}u\|_{\mathrm{L}^p_w(\Omega)} \leq c\mathrm{diam}(\Omega)^{-k}\|\Pi_{\mathbb{A}}^{B}u\|_{\mathrm{L}^p_w(\Omega)} \overset{\text{Rem. 6.4}}{\leq} c\mathrm{diam}(\Omega)^{-k}\|u\|_{\mathrm{L}^p_w(\Omega)},
$$

where $c = c(n, p, [w]_{A_p}, \sigma_1, \sigma_2, \mathbb{A}) > 0$. Combining this inequality with Eqs. 6.30, 6.26 follows and the proof is complete. □

Proof of Theorem 1.3 For a $\mathbb{C}$-elliptic k-th order operator, Theorems 6.10 and 3.7 immediately imply Theorem 1.3, direction (a)⇒(b), by taking $w \equiv 1$. For direction (b)⇒(a) of Theorem 1.3 it suffices to note that Theorem 4.4 remains valid when only requiring (d) to hold for all *connected*, open and bounded domains $\Omega \subset \mathbb{R}^n$; since every such domain is John, we thereby obtain (b)⇒(a) in Theorem 1.3. □

Remark 6.11 (Korn for unbounded John domains) Theorem 6.10 also persists for unbounded John domains which are defined in analogy with Definition 2.2. Indeed, by a result due to VÄISÄLÄ [64, Thm. 4.6], if $\Omega \subset \mathbb{R}^n$ is α-John, then there exists a sequence of relatively compact $c\alpha$-John domains $\Omega_j \subset \Omega$ with $\Omega_j \nearrow \Omega$. Since any open and bounded John domain satisfies the emanating chain condition, we obtain by Eq. 6.26

$$
\|D^k u\|_{\mathrm{L}^p_w(\Omega_j)} \leq C(\mathrm{diam}(\Omega_j)^{-k}\|u\|_{\mathrm{L}^p_w(\Omega)} + \|\mathbb{A}u\|_{\mathrm{L}^p_w(\Omega)}) \quad \text{for all } u \in \mathrm{W}^{\mathbb{A}}\mathrm{L}^p_w(\Omega)
$$

for any $1 < p < \infty$ and $w \in A_p$, with $C = C(c\alpha, p, [w]_{A_p}, \mathbb{A}) > 0$. Sending $j \to \infty$ in the preceding inequality then yields the Korn inequality

$$\|D^k u\|_{\mathrm{L}^p_w(\Omega)} \leq C\|\mathbb{A}u\|_{\mathrm{L}^p_w(\Omega)} \quad \text{for all } u \in \mathrm{W}^{\mathbb{A}}\,\mathrm{L}^p_w(\Omega).$$

6.5 Korn-type Inequalities for Other Space Scales

Theorem 6.10 easily allows for variants of Korn-type inequalities by means of extrapolation. For this, we recall a far-reaching variant of the Rubio de Francia-extrapolation theorem due to Cruz-Uribe, Martell & Pérez [20] that requires the following terminology: Given a Banach function space $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ on $\mathbb{R}^n$ with respect to $\mathscr{L}^n$, we call $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ *rearrangement invariant* provided for each $f \in \mathbb{X}$ the norm $\|f\|_{\mathbb{X}}$ exclusively depends on its distribution function $d_f(\lambda) := \mathscr{L}^n(\{x \in \mathbb{R}^n\colon\ |f(x)| > \lambda\})$. Defining for $f \in \mathbb{X}$ its *decreasing rearrangement* $f^*\colon [0,\infty) \to \mathbb{R}$ by $f^*(t) := \inf\{\lambda \geq 0\colon\ d_f(\lambda) \leq t\}$, there exists a unique rearrangement invariant Banach function space $(\overline{\mathbb{X}}, \|\cdot\|_{\overline{\mathbb{X}}})$ on $[0,\infty)$ with respect to $\mathscr{L}^1$ such that $f \in \mathbb{X}$ precisely if $f^* \in \overline{\mathbb{X}}$. In this situation, defining for $0 < t < \infty$ the dilation operator $D_t\colon \overline{\mathbb{X}} \ni g \mapsto g(\cdot/t) \in \overline{\mathbb{X}}$ and denoting $\|D_t\|_{\overline{\mathbb{X}}}$ its operator norm, the *lower* and *upper Boyd indices* of $\mathbb{X}$ are given by

$$p_{\mathbb{X}} := \lim_{t\to\infty} \frac{\log(t)}{\log(\|D_t\|_{\overline{\mathbb{X}}})} \quad \text{and} \quad q_{\mathbb{X}} := \lim_{t\searrow 0} \frac{\log(t)}{\log(\|D_t\|_{\overline{\mathbb{X}}})}, \tag{6.31}$$

respectively.

On the other hand, a differentiable function $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with right-continuous, non-decreasing derivative φ' such that $\varphi'(0) = 0$ and $\varphi'(t) > 0$ for $t > 0$ is called an *N-function*. If there exists $c_1 > 0$ such that $\varphi(2t) \leq c_1\varphi(t)$ holds for all $t > 0$, we say that φ satisfies the Δ_2-condition; in this situation, the infimum over all such possible constants c_1 is denoted $\Delta_2(\varphi)$. If the convex conjugate of φ, $\widetilde{\varphi}(t) := \sup_{s>0}(st - \varphi(s))$, satisfies the Δ_2-condition, we say that φ satisfies the ∇_2-condition and put $\nabla_2(\varphi) := \Delta_2(\widetilde{\varphi})$.

Lemma 6.12 ([20, Thm. 4.10, 4.15], [23, Prop. 6.1]) *Let $1 < p_0 < \infty$ and let $\mathcal{F} \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n) \times \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a family of pairs of non-negative functions not identically zero, such that for any $w \in A_{p_0}$ there exists $c = c(p_0, [w]_{A_{p_0}}) > 0$ with*

$$\int_{\mathbb{R}^n} f(x)^{p_0} w(x)\,\mathrm{d}x \leq c \int_{\mathbb{R}^n} g(x)^{p_0} w(x)\,\mathrm{d}x \qquad \textit{for all } (f, g) \in \mathcal{F}, \tag{6.32}$$

the inequality tacitly implying finiteness of the left-hand side. Then the following hold:

(a) If $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ is a rearrangement invariant function space whose lower and upper Boyd indices $p_{\mathbb{X}}, q_{\mathbb{X}}$ satisfy $1 < p_{\mathbb{X}} \leq q_{\mathbb{X}} < \infty$, then there holds

$$\|f\|_{\mathbb{X}} \leq c\|g\|_{\mathbb{X}} \qquad \textit{for all } (f, g) \in \mathcal{F}. \tag{6.33}$$

(b) For any N-function $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\Delta_2(\varphi), \nabla_2(\varphi) < \infty$ there exists a constant $C = C(\Delta_2(\varphi), \nabla_2(\varphi), p_0) > 0$ such that there holds

$$\int_{\mathbb{R}^n} \varphi(f)\,\mathrm{d}x \leq C \int_{\mathbb{R}^n} \varphi(cg)\,\mathrm{d}x \qquad \textit{for all } (f, g) \in \mathcal{F}. \tag{6.34}$$

By virtue of Theorem 6.10, Lemma 6.12 implies a whole family of Korn-type inequalities for $\mathbb{C}$-elliptic differential operators on John domains, out of which we point out two:

Corollary 6.13 (Korn for Lorentz) *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain that satisfies the emanating chain condition with constants $\sigma_1 > 1$ and $\sigma_2 \geq 1$ and central ball B. Let $1 < p < \infty$, $1 \leq q < \infty$. Then for any $\mathbb{C}$-elliptic operator $\mathbb{A}$ of order k there exists a constant $c = c(\sigma_1, \sigma_2, p, q, \mathbb{A}) > 0$ such that for all $u \in \mathrm{W}^{\mathbb{A}}\,\mathrm{L}^{p,q}(\Omega)$ there holds*

$$
\begin{aligned}
&\|D^k(u - \Pi^B_{\mathbb{A}} u)\|_{\mathrm{L}^{p,q}(\Omega)} \leq c\|\mathbb{A}u\|_{\mathrm{L}^{p,q}(\Omega)},\\
&\|D^k u\|_{\mathrm{L}^{p,q}(\Omega)} \leq c\mathrm{diam}(\Omega)^{-k}\|u\|_{\mathrm{L}^{p,q}(\Omega)} + c\|\mathbb{A}u\|_{\mathrm{L}^{p,q}(\Omega)}.
\end{aligned}
\tag{6.35}
$$

Proof For $p_0 := \frac{1}{2}(p+1)$, put

$$
\mathcal{F} := \{(\mathbb{1}_\Omega \mathbb{1}_{\{|D^k(u-\Pi^B_{\mathbb{A}}u)|\leq j\}}|D^k(u - \Pi^B_{\mathbb{A}}u)|, \mathbb{1}_\Omega|\mathbb{A}u|)\colon\ u \in \mathrm{W}^{\mathbb{A},p_0}(\Omega),\ j \in \mathbb{N}\}
$$

so that $\mathcal{F}$ meets the requirements of Lemma 6.12 by virtue of Theorem 6.10. The Lorentz space $\mathrm{L}^{p,q}(\mathbb{R}^n)$ is rearrangement invariant and by our assumptions on (p, q), $p_{\mathbb{X}} = q_{\mathbb{X}} = p \in (1, \infty)$ (cf. [20, p. 70]). Therefore, Eq. 6.33 and Fatou's lemma yield Eq. 6.35 for all $u \in \mathrm{W}^{\mathbb{A},p_0}(\Omega)$. Since $\mathrm{W}^{\mathbb{A}}\,\mathrm{L}^{p,q}(\Omega) \subset \mathrm{W}^{\mathbb{A},p_0}(\Omega)$ as $1 < p_0 < p$ and Ω is bounded, this implies $Eq.6.35_1$ for all $u \in \mathrm{W}^{\mathbb{A}}\,\mathrm{L}^{p,q}(\Omega)$, and $Eq.6.35_2$ follows by inverse estimates. ◻

Remark 6.14 (L^∞-estimates) If $\Omega \subset \mathbb{R}^n$ is as in Corollary 6.13 and moreover is such that the embedding $\mathrm{W}^1\,\mathrm{L}^{n,1}(\Omega; V) \hookrightarrow \mathrm{C}(\overline{\Omega}; V)$ holds (cf. [17, 43, 57]), then Corollary 6.13 immediately yields the stronger embedding $\mathrm{W}^{\mathbb{A}}\,\mathrm{L}^{n,1}(\Omega) \hookrightarrow \mathrm{C}(\overline{\Omega}; V)$ for any first order $\mathbb{C}$-elliptic differential operator $\mathbb{A}$.

In an analogous way as for Corollary 6.13, we obtain the following

Corollary 6.15 (Korn for Orlicz) *Let $\Omega \subset \mathbb{R}^n$ and $\mathbb{A}$ be as in Corollary 6.13 and let $\varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an N-function that satisfies the Δ_2- and ∇_2-conditions. Then there exists $c = c(\sigma_1, \sigma_2, \Delta_2(\varphi), \nabla_2(\varphi), \mathbb{A}) > 0$ such that for all $u \in \mathrm{W}^{\mathbb{A}}\,\mathrm{L}^{\varphi}(\Omega)$ there holds*

$$
\begin{aligned}
&\int_\Omega \varphi(|D^k(u - \Pi^B_{\mathbb{A}}u)|)\,\mathrm{d}x \leq c\int_\Omega \varphi(|\mathbb{A}u|)\,\mathrm{d}x,\\
&\int_\Omega \varphi(|D^k u|)\,\mathrm{d}x \leq c\mathrm{diam}(\Omega)^n\varphi\Big(\mathrm{diam}(\Omega)^{-k}\fint_\Omega |u|\,\mathrm{d}x\Big) + c\int_\Omega \varphi(|\mathbb{A}u|)\,\mathrm{d}x.
\end{aligned}
\tag{6.36}
$$

Remark 6.16 In Theorem 6.10 and Corollaries 6.13, 6.15, the main focus is on functions that do not vanish close to $\partial\Omega$, as otherwise the weaker ellipticity allows the use of global singular integral estimates (cf. [19, 41]). We also wish to point out that, whereas $Eqs.6.26_2$, 6.35_2 and $Eq.6.36_2$ are in fact equivalent to the $\mathbb{C}$-ellipticity of $\mathbb{A}$, this is *not* the case for inequalities $Eqs.6.26_1$, 6.35_1 and $Eq.6.36_1$, cf. Fuchs [27]. This can be seen by the operator ε^D in $n = 2$ dimensions, cf. Example 3.2, and the classical Cauchy-Pompeiu-formula.

Appendix

$\mathbb{A}$-strict Continuity of the Trace Operator on $\mathrm{BV}^{\mathbb{A}}$

The approximation argument underlying the proof of Lemma 4.6 to conclude the existence of a continuous linear trace operator $\mathrm{Tr}_{\partial\Omega}\colon\ \mathrm{W}^{\mathbb{A},p}(\Omega) \to \mathrm{B}^{k-1/p}_{p,p}(\partial\Omega; V)$ cannot be applied to $\mathrm{BV}^{\mathbb{A}}(\Omega)$. This is due to the non-density of $\mathrm{C}^\infty(\Omega; V) \cap \mathrm{BV}^{\mathbb{A}}(\Omega)$ in $\mathrm{BV}^{\mathbb{A}}(\Omega)$ for the norm topology. In the well-known case of $\mathrm{BV}(\Omega)$ (cf. Ambrosio et al. [3]), this issue is resolved by passing to the *strict metric*. The suitable substitute here is the $\mathbb{A}$-strict metric, see Section 3.1, but now we have to keep track of the intermediate derivatives as well. To this end, we require a multiplicative inequality as follows:

Lemma 7.1 (Intermediate derivatives) *Let $k \geq 1$ and let $\mathbb{A}$ be a k-th order $\mathbb{C}$-elliptic differential operator of the form Eq. 1.1. Then there exists a constant $c = c(\mathbb{A}) > 0$ such that for all $l \in \{1, ..., k-1\}$ there holds*

$$\|D^l u\|_{\mathrm{L}^1(\mathbb{R}^n)} \leq c\|u\|_{\mathrm{L}^1(\mathbb{R}^n)}^{1-\frac{l}{k}} \|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{R}^n)}^{\frac{l}{k}} \tag{7.1}$$

for all $u \in \mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n)$ with compact support.

Proof The proof is an adaptation of an argument to be found, e.g., in MAZ'YA [49, Thm. 1.4.7]. Let $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V) \setminus \{0\}$ be supported in $B(x_0, R_0)$ and $l \in \{1, ..., k-1\}$ be given. Let $x \in \mathrm{spt}(u)$, so that because of $u \in \mathrm{C}^\infty(\mathbb{R}^n; V)$, $\|u\|_{\mathrm{L}^1(B(x,R))} > 0$ for all $R > 0$. Fix $r_0 > 0$. There are two options: If $\|u\|_{\mathrm{L}^1(B(x,r_0))} < r_0^k \|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_0))}$, then we have

$$r_0^{-l} < \left(\frac{\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_0))}}{\|u\|_{\mathrm{L}^1(B(x,r_0))}}\right)^{\frac{l}{k}}, \tag{7.2}$$

and define $r_x = r_0$ in this case. Then, by virtue of the Poincaré inequality from Theorem 3.7,

$$\begin{aligned}\|D^l u\|_{\mathrm{L}^1(B(x,r_x))} &\overset{\text{Eq. 3.22}}{\leq} c r_x^{-l}\|u\|_{\mathrm{L}^1(B(x,r_x))} + c r_x^{k-l}\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_x))} \\ &\overset{\text{Eq. 7.2}}{\leq} c\|u\|_{\mathrm{L}^1(B(x,r_x))}^{1-\frac{l}{k}}\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_x))}^{\frac{l}{k}} + c r_0^{k-l}\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_x))}.\end{aligned} \tag{7.3}$$

If $\|u\|_{\mathrm{L}^1(B(x,r_0))} \geq r_0^k\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_0))}$, we then find $r_x \geq r_0$ such that $\|u\|_{\mathrm{L}^1(B(x,r_x))} = r_x^k\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_x))}$. This follows from u being compactly supported and the fact that $\|u\|_{\mathrm{L}^1(B(x,R))}$ will eventually be constant for large values of R. Fix this r_x, so that

$$\|D^l u\|_{\mathrm{L}^1(B(x,r_x))} \overset{\text{Eq. 3.22}}{\leq} c\|u\|_{\mathrm{L}^1(B(x,r_x))}^{1-\frac{l}{k}}\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_x))}^{\frac{l}{k}} \tag{7.4}$$

Adding Eqs 7.3 and 7.4, in each of the cases there holds

$$\|D^l u\|_{\mathrm{L}^1(B(x,r_x))} \leq c(r_0^{k-l}\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_x))} + \|u\|_{\mathrm{L}^1(B(x,r_x))}^{1-\frac{l}{k}}\|\mathbb{A}u\|_{\mathrm{L}^1(B(x,r_x))}^{\frac{l}{k}}). \tag{7.5}$$

Now, $\mathscr{B} := (B(x, r_x))_{x\in\mathrm{spt}(u)}$ is a covering of $\mathrm{spt}(u)$ by open balls such that $r_x \leq C < \infty$ and because $\mathrm{spt}(u)$ is compact, the sequence of radii of any sequence of mutually disjoint balls in $\mathscr{B}$ tends to zero. By the Besicovitch-type covering lemma in the form as given in MAZ'YA [49, Chpt. 1.2.1, Thm. 1], there exists a sequence of balls $B_m \in \mathscr{B}$ such that $\mathrm{spt}(u) \subset \bigcup_m B_m$, the balls $\frac{1}{3}B_m$ are mutually disjoint and $\bigcup_{B\in\mathscr{B}} B \subset \bigcup_m 4B_m$. Moreover, there exists a constant $N = N(n) \in \mathbb{N}$ such that every point belongs to at most N balls contained in $\mathscr{B}$. We then conclude by Eq. 7.5 that

$$\begin{aligned}\|D^l u\|_{\mathrm{L}^1(\mathrm{spt}(u))} &\leq \sum_m \|D^l u\|_{\mathrm{L}^1(B_m)} \\ &\leq c\sum_m \left(r_0^{k-l}\|\mathbb{A}u\|_{\mathrm{L}^1(B_m)} + \|u\|_{\mathrm{L}^1(B_m)}^{1-\frac{l}{k}}\|\mathbb{A}u\|_{\mathrm{L}^1(B_m)}^{\frac{l}{k}}\right) \\ &\leq c r_0^{k-l}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{R}^n)} + c\|u\|_{\mathrm{L}^1(\mathbb{R}^n)}^{1-\frac{l}{k}}\|\mathbb{A}u\|_{\mathrm{L}^1(\mathbb{R}^n)}^{\frac{l}{k}},\end{aligned}$$

where $c = c(n, \mathbb{A}) > 0$. Now it suffices to send $r_0 \searrow 0$ to conclude Eq. 7.1 for $u \in \mathrm{C}_c^\infty(\mathbb{R}^n; V)$; the general case of compactly supported $u \in \mathrm{W}^{\mathbb{A},1}(\mathbb{R}^n)$ follows by smooth approximation. The proof is complete. □

We now turn to the remaining part of the

Proof of Theorem 4.7 We start by noting that the existence of a norm-continuous trace operator $\mathrm{Tr}_{\partial\Omega}\colon \mathrm{W}^{\mathbb{A},1}(\Omega) \to \mathrm{B}^{k-1}_{1,1}(\partial\Omega; V)$ follows by smooth approximation along the same lines as for the case $1 < p < \infty$ (cf. Theorem 4.4, direction '(a)⇒(b)⇒(c)' and Lemma 4.6, now using Theorem 1.1), the reason being the density of $\widetilde{\mathrm{W}}^{\mathbb{A},1}(\Omega) \cap \mathrm{C}^{\infty}(\Omega; V)$ in $\mathrm{W}^{\mathbb{A},1}(\Omega)$ for the norm topology on $\mathrm{W}^{\mathbb{A},1}(\Omega)$; also note that $\mathrm{W}^{\mathbb{A},1}(\Omega) = \widetilde{\mathrm{W}}^{\mathbb{A},1}(\Omega)$ by Theorem 3.7.

By Theorem 3.7, $\mathrm{BV}^{\mathbb{A}}(\Omega) = \mathrm{W}^{k-1,1}(\Omega; V) \cap \mathrm{BV}^{\mathbb{A}}(\Omega)$. To define an $\mathbb{A}$-strictly continuous trace operator by means of smooth approximation, the localisation argument below requires $d(u, u_j) := \|u_j - u\|_{\mathrm{W}^{k-1,1}(\Omega)} + ||\mathbb{A}u_j|(\Omega) - |\mathbb{A}u|(\Omega)| \to 0$ whenever $d_{\mathbb{A}}(u, u_j) = \|u_j - u\|_{\mathrm{L}^1(\Omega)} + ||\mathbb{A}u_j|(\Omega) + |\mathbb{A}u|(\Omega)| \to 0$ as $j \to \infty$. The Poincaré inequality of Theorem 3.7 does not directly allow for this conclusion. Also note that, if we define a trace operator as the continuous extension for the metric d as above, then it is not clear that this trace operator will be continuous for $d_{\mathbb{A}}$. It is here where we require the multiplicative inequality of Lemma 7.1.

Let $u \in \mathrm{BV}^{\mathbb{A}}(\Omega) = \mathrm{W}^{k-1,1}(\Omega; V) \cap \mathrm{BV}^{\mathbb{A}}(\Omega)$ and choose $(u_j) \subset \mathrm{C}^{\infty}(\Omega; V) \cap \mathrm{BV}^{\mathbb{A}}(\Omega)$ such that $d_{\mathbb{A}}(u_j, u) \to 0$ as $j \to \infty$. We then have $L := \sup_j \|u_j\|_{\mathrm{W}^{\mathbb{A},1}(\Omega)} < \infty$. Since $\mathbb{A}$ is $\mathbb{C}$-elliptic, the Jones-type extension operator $\widetilde{E}$ from [30, Sec. 4.1] maps $\widetilde{E}\colon \widetilde{\mathrm{W}}^{\mathbb{A},1}(\Omega) \to \widetilde{\mathrm{W}}^{\mathbb{A},1}(\mathbb{R}^n)$ boundedly; clearly, we may assume that all maps $\widetilde{E}u$ are compactly supported in a fixed ball $B(0, R)$. This operator moreover satisfies $\|\widetilde{E}u\|_{\mathrm{L}^1(\mathbb{R}^n)} \leq c\|u\|_{\mathrm{L}^1(\Omega)}$. Thus, for all $\ell \in \{0, ..., k-1\}$,

$$
\begin{aligned}
\|D^{\ell}(u_j - u_m)\|_{\mathrm{L}^1(\Omega)} &\leq \|D^{\ell}(\widetilde{E}(u_j - u_m))\|_{\mathrm{L}^1(\mathbb{R}^n)} \\
&\overset{\text{Eq. } 7.1}{\leq} c\|\widetilde{E}(u_j - u_m)\|^{1-\frac{\ell}{k}}_{\mathrm{L}^1(\mathbb{R}^n)} \|\mathbb{A}(\widetilde{E}(u_j - u_m))\|^{\frac{\ell}{k}}_{\mathrm{L}^1(\mathbb{R}^n)} \\
&\leq c\|u_j - u_m\|^{1-\frac{\ell}{k}}_{\mathrm{L}^1(\Omega)} \|u_j - u_m\|^{\frac{\ell}{k}}_{\mathrm{W}^{\mathbb{A},1}(\Omega)} \\
&\leq cL^{\frac{\ell}{k}} \|u_j - u_m\|^{1-\frac{\ell}{k}}_{\mathrm{L}^1(\Omega)} \to 0
\end{aligned}
\tag{7.6}
$$

as $j, m \to \infty$, and so with d as defined above, $d(u_j, u) \to \infty$. We pick a sequence $(\rho_l) \subset \mathrm{C}^{\infty}(\overline{\Omega}; [0, 1])$ such that $\rho_l = 1$ in a $\frac{1}{l}$-neighbourhood of $\partial\Omega$ and $\rho_l = 0$ in $\Omega \setminus \Omega'_l := \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \frac{2}{l}\}$. We may then assume that $|\nabla^{\ell}\rho_l| \leq cl^{\ell}$ for some $c > 0$ and all $\ell \in \{1, ..., k\}$. For each $l \in \mathbb{N}$, we find a set Ω_l with $\operatorname{spt}(\rho_l) \subset \Omega_l \subset \Omega'_l$ and of smooth boundary such that $|\mathbb{A}u|(\partial\Omega_l) = 0$. Denoting $\mathrm{Tr}_{\partial\Omega}$ the trace operator on $\mathrm{W}^{\mathbb{A},1}(\Omega)$ that is already our disposal, we then estimate

$$
\begin{aligned}
\|\mathrm{Tr}_{\partial\Omega}(u_j) - \mathrm{Tr}_{\partial\Omega}(u_m)\|_{\mathrm{B}^{k-1}_{1,1}(\partial\Omega)} &\leq c(\|\rho_l(u_j - u_m)\|_{\mathrm{L}^1(\Omega)} + \|\mathbb{A}(\rho_j(u_l - u_m))\|_{\mathrm{L}^1(\Omega)}) \\
&\leq c\Big(\sum_{\substack{|\alpha|+|\beta|\leq k,\\ |\beta|\leq k-1}} \|\partial^{\alpha}\rho_l\partial^{\beta}(u_j - u_m)\|_{\mathrm{L}^1(\Omega)}\Big) \\
&\quad + c\|\rho_l(\mathbb{A}u_j - \mathbb{A}u_m)\|_{\mathrm{L}^1(\Omega)} =: \mathrm{I}_{l,j,m} + \mathrm{II}_{l,j,m}.
\end{aligned}
$$

For an arbitrary $l \in \mathbb{N}$, Eq. 7.6 implies that $\mathrm{I}_{l,j,m} \to 0$ as $j, m \to \infty$. Therefore, sending $j, m \to \infty$ first and then letting $l \to \infty$,

$$
\lim_{j,m\to\infty} \|\mathrm{Tr}_{\partial\Omega}(u_j) - \mathrm{Tr}_{\partial\Omega}(u_m)\|_{\mathrm{B}^{k-1}_{1,1}(\partial\Omega)} \leq c|\mathbb{A}u|(\Omega_l) \to 0
$$

as $l \to \infty$; note that $|\mathbb{A}u|$ is a finite Radon measure and $\bigcap_l \Omega_l = \emptyset$. Hence $(\mathrm{Tr}_{\partial\Omega}(u_j))$ is Cauchy in $\mathrm{B}^{k-1}_{1,1}(\partial\Omega; V)$, too. Working from here, the claimed existence of a trace operator with the requisite properties is routine, and the proof thus is complete. $\square$

Oscillation Characterisation of $\dot{\mathrm{B}}^s_{p,q}$

To the best of our knowledge, the oscillation chracterisation of Besov spaces is only available in the inhomogeneous situation, cf. TRIEBEL [62, Chpt. 3.5]. For completeness, we briefly explain how to arrive at the part of Lemma 4.1 that is required for Section 4.3.

Let $M > \lfloor s \rfloor$ and choose $2N > M$. Let $k \in \mathrm{C}^\infty_c(\mathbb{R}^n)$ be such that $\mathrm{spt}(k) \subset B(0,1)$ and consider, for $f \in \mathscr{S}(\mathbb{R}^n)$, the means

$$k(t,f)(x) := \int_{\mathbb{R}^n} k(y) f(x+ty)\,\mathrm{d}y, \qquad t>0,\ x \in \mathbb{R}^n.$$

Given some $k^0 \in \mathrm{C}^\infty_c(\mathbb{R}^n)$ with $\mathrm{spt}(k^0) \subset B(0,1)$ and $\widehat{k^0}(0) \neq 0$, we put $k := \Delta^N k^0$, where Δ^N stands for the N-fold application of the Laplacian. We now define $\varphi := k^\vee$. It is not too difficult to verify that φ satisfies nonsubstantial modifications of [61, (62),(78),(79)] (also see [62, Proof of Thm. 2.4.6]) and so, by a combination of [61, Rem. 17, Rem. 18],

$$\|f\|^{\sim}_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} := \Big(\int_0^\infty \|\mathscr{F}^{-1}[\varphi(t\cdot)\mathscr{F}f]\|^q_{\mathrm{L}^p(\mathbb{R}^n)} \frac{\mathrm{d}t}{t^{1+sq}}\Big)^{\frac{1}{q}} \tag{7.7}$$

is an equivalent norm on $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$. By definition of φ, we have

$$\|f\|^{\sim}_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} = c(n)\Big(\int_0^\infty \|k(t,f)\|^q_{\mathrm{L}^p(\mathbb{R}^n)} \frac{\mathrm{d}t}{t^{1+sq}}\Big)^{\frac{1}{q}}. \tag{7.8}$$

Since $2N > M$, we have for any $\pi \in \mathscr{P}_M(\mathbb{R}^n)$ that $\Delta^N \pi = 0$ and so, integrating by parts and Jensen's inequality imply

$$|k(t,f)(x)| \leq \int_{\mathbb{R}^n} |k(y)(f-\pi)(x+ty)|\,\mathrm{d}y \leq c(n,N,k^0)\Big(\fint_{B(x,t)} |f-\pi|^u\,\mathrm{d}y\Big)^{\frac{1}{u}}$$

and infimising the very right hand side over $\pi \in \mathscr{P}_M(\mathbb{R}^n)$ yields

$$|k(t,f)(x)| \leq c(n)\,\mathrm{osc}^M_u\, f(x,t). \tag{7.9}$$

Combining Eqs 7.9 with 7.7 and 7.8 consequently yields the existence of a constant $c = c(n,N,s,p,q,k^0) > 0$ such that

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \leq c\Big(\int_0^\infty \|\,\mathrm{osc}^M_u\, f(\cdot,t)\|^q_{\mathrm{L}^p(\mathbb{R}^n)} \frac{\mathrm{d}t}{t^{1+sq}}\Big)^{\frac{1}{q}} \qquad \text{for all } f \in \mathscr{S}(\mathbb{R}^n).$$

The statement for $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n;V)$ then follows by componentwise application.

Acknowledgements F.G. acknowledges financial support by the Hector Foundation and the University of Bielefeld. The research of L.D. was supported by the DFG through the CRC 1283. The authors are moreover thankful to the anonymous referee, whose careful reading and valuable suggestions helped to improve the manuscript.

Author Contributions The authors confirm that the present work is the work of both of the two authors and approve the file.

Funding Open Access funding enabled and organized by Projekt DEAL. Lars Diening and Franz Gmeineder wrote the main manuscript text and prepared the figures together. All authors reviewed the manuscript.

Declarations

Competing interests The authors declare no competing interests.

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