

Intervals of totally nonnegative matrices

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ABSTRACT

Totally nonnegative matrices, i.e., matrices having all their minors nonnegative, and matrix intervals with respect to the checkerboard ordering are considered. It is proven that if the two bound matrices of such a matrix interval are nonsingular and totally nonnegative (and in addition all their zero minors are identical) then all matrices from this interval are also nonsingular and totally nonnegative (with identical zero minors).

Keywords:

Totally nonnegative matrix
Checkerboard ordering
Matrix interval
Cauchon diagram
Cauchon Algorithm

0. Introduction

A real matrix is called *totally nonnegative* and *totally positive* if all its minors are nonnegative and positive, respectively. Such matrices arise in a variety of ways in mathematics and its applications. For background information the reader is referred to the recently published monographs [4,13]. In this paper we solve the conjecture posed by the second author in this journal in 1982 [5], see also [4, Section 3.2] and [13, Section 3.2]. This conjecture concerns the checkerboard ordering which is obtained from the usual entry-wise ordering in the set of the square real matrices of fixed order by reversing the inequality sign for each entry in a checkerboard fashion. The conjecture states that if the two bound matrices of an interval with respect to this ordering are nonsingular and totally nonnegative then all matrices lying between the two bound matrices are nonsingular and totally nonnegative, too.

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This question has been solved for some subclasses of the totally nonnegative matrices, viz. the totally positive matrices and the nonsingular tridiagonal totally nonnegative matrices [5] and for the almost totally positive matrices [7], a class of matrices between the nonsingular totally nonnegative and the totally positive matrices. A result on the case that more than two vertex matrices of the matrix interval are involved was given in [6]. In the past, some attempts have been made to solve the general question but the conjecture remained unsettled during the last three decades. We also solve here a related problem, viz. whether an identical zero–nonzero pattern of the minors of the two bound matrices stays unchanged through such a matrix interval.

The organization of our paper is as follows. In Section 1 we introduce our notation and give some auxiliary results which we use in the subsequent sections. In Section 2 we recall from [10] the Cauchon Algorithm, specified for the case of square matrices, on which our proofs heavily rely. In Section 3 we present the proofs of our main results.

1. Notation and auxiliary results

1.1. Notation

We now introduce the notation used in our paper. For κ, n we denote by $Q_{\kappa, n}$ the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \dots, n\}$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\kappa) \in Q_{\kappa, n}$ the dispersion of α is $d(\alpha) = \alpha_\kappa - \alpha_1 - \kappa + 1$. Let A be a real $n \times n$ matrix. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\kappa)$, $\beta = (\beta_1, \beta_2, \dots, \beta_\kappa) \in Q_{\kappa, n}$, we denote by $A[\alpha | \beta]$ the $\kappa \times \kappa$ submatrix of A contained in the rows indexed by $\alpha_1, \alpha_2, \dots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \dots, \beta_\kappa$. We suppress the brackets when we enumerate the indices explicitly. We set $\det A[\alpha | \beta] = 1$ if α or β is not strictly increasing. If $d(\alpha) = d(\beta) = 0$ we call the minor $\det A[\alpha | \beta]$ *contiguous*. A matrix A is called *totally positive* (abbreviated *TP* henceforth) and *totally nonnegative* (abbreviated *TN*) if $\det A[\alpha | \beta] > 0$ and $\det A[\alpha | \beta] \geq 0$, respectively, for all $\alpha, \beta \in Q_{\kappa, n}$, $\kappa = 1, 2, \dots, n$. If a totally nonnegative matrix is also nonsingular, we write *NsTN*.

We endow $\mathbb{R}^{n, n}$, the set of the real $n \times n$ matrices, with two partial orderings: Firstly, with the usual entry-wise ordering ($A = (a_{kj}), B = (b_{kj}) \in \mathbb{R}^{n, n}$)

$$A \leq B \quad :\Leftrightarrow \quad a_{kj} \leq b_{kj}, \quad k, j = 1, \dots, n.$$

Secondly, with the *checkerboard ordering*, which is defined as follows. Let $\Delta := \text{diag}(1, -1, \dots, (-1)^{n+1})$ and $A^* := \Delta A \Delta$.

Then we define

$$A \leq^* B \quad :\Leftrightarrow \quad A^* \leq B^*.$$

1.2. Auxiliary results

In the sequel we will often make use of the following special case of Sylvester's Identity, see, e.g., [4, pp. 29–30] or [13, p. 3].

Lemma 1.1 (*Sylvester's Identity*). *Partition $A \in \mathbb{R}^{n, n}$, $n \geq 3$, as follows:*

$$A = \begin{pmatrix} c & A_{12} & d \\ A_{21} & A_{22} & A_{23} \\ e & A_{32} & f \end{pmatrix},$$

where $A_{22} \in \mathbb{R}^{n-2, n-2}$ and c, d, e, f are scalars. Define the submatrices

$$\begin{aligned} C &:= \begin{pmatrix} c & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, & D &:= \begin{pmatrix} A_{12} & d \\ A_{22} & A_{23} \end{pmatrix}, \\ E &:= \begin{pmatrix} A_{21} & A_{22} \\ e & A_{32} \end{pmatrix}, & F &:= \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & f \end{pmatrix}. \end{aligned}$$

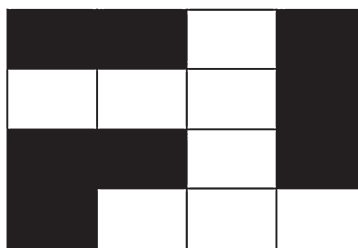


Fig. 1. Example of a 4×4 Cauchon diagram.

Then if $\det A_{22} \neq 0$

$$\det A = \frac{\det C \det F - \det D \det E}{\det A_{22}}.$$

Lemma 1.2 (Shadow property). (See [4, Corollary 7.2.11] or [13, Section 1.3].) Suppose that $A \in \mathbb{R}^{n,n}$ is NsTN and that $A[\alpha \mid \beta]$ is a $p \times p$ submatrix with $\text{rank } A[\alpha \mid \beta] < p$. If $A[\alpha \mid \beta]$ is contiguous, then one of

$$A[\alpha_1, \dots, \alpha_p, \dots, n \mid 1, \dots, \beta_1, \dots, \beta_p], \quad A[1, \dots, \alpha_1, \dots, \alpha_p \mid \beta_1, \dots, \beta_p, \dots, n]$$

has rank equal to that of $A[\alpha \mid \beta]$. In particular, it is the one to the side of the diagonal of A on which more entries of $A[\alpha \mid \beta]$ lie.

Lemma 1.3. (E.g. [13, Theorem 1.13].) All principal minors of an NsTN matrix are positive.

2. Cauchon Algorithm

In this section we first recall from [8,10] the definition of a Cauchon diagram and of the Cauchon Algorithm.²

2.1. Cauchon diagrams

Definition 2.1. An $n \times n$ Cauchon diagram C is an $n \times n$ grid consisting of n^2 squares colored black and white, where each black square has the property that either every square to its left (in the same row) or every square above it (in the same column) is black.

We denote by C_n the set of the $n \times n$ Cauchon diagrams. We fix positions in a Cauchon diagram in the following way: For $C \in C_n$ and $i, j \in \{1, \dots, n\}$, $(i, j) \in C$ if the square in row i and column j is black. Here we use the usual matrix notation for the (i, j) position in a Cauchon diagram, i.e., the square in $(1, 1)$ position of the Cauchon diagram is in its top left corner. For instance, for the Cauchon diagram C of Fig. 1, we have $(2, 3) \notin C$, whereas $(3, 2) \in C$.

Definition 2.2. Let $A \in \mathbb{R}^{n,n}$ and let $C \in C_n$. We say that A is a Cauchon matrix associated with the Cauchon diagram C if for all (i, j) , $i, j \in \{1, \dots, n\}$, we have $a_{ij} = 0$ if and only if $(i, j) \in C$. If A is a Cauchon matrix associated with an unspecified Cauchon diagram, we just say that A is a Cauchon matrix.

In passing, we note that every TN matrix is a Cauchon matrix [10, Lemma 2.3].

² This algorithm is called in [8] the deleting derivations algorithm (as the inverse of the restoration algorithm) and in [10] the Cauchon reduction algorithm.

2.2. Cauchon Algorithm

In order to formulate the Cauchon Algorithm we need the following notation.

We denote by \leq the lexicographic order on \mathbb{N}^2 , i.e., $(g, h) \leq (i, j) : \Leftrightarrow (g < i) \text{ or } (g = i \text{ and } h \leq j)$.

Set $E^\circ := \{1, \dots, n\}^2 \setminus \{(1, 1)\}$, $E := E^\circ \cup \{(n+1, 2)\}$.

Let $(s, t) \in E^\circ$. Then $(s, t)^+ := \min\{(i, j) \in E \mid (s, t) \leq (i, j)\}$, $(s, t) \neq (i, j)$.

Algorithm 2.3. Let $A \in \mathbb{R}^{n,n}$. As r runs in decreasing order over the set E , we define matrices $A^{(r)} = (a_{ij}^{(r)}) \in \mathbb{R}^{n,n}$ as follows.

1. Set $A^{(n+1,2)} := A$.
2. For $r = (s, t) \in E^\circ$ define the matrix $A^{(r)} = (a_{ij}^{(r)})$ as follows.
 - (a) If $a_{st}^{(r^+)} = 0$ then put $A^{(r)} := A^{(r^+)}$.
 - (b) If $a_{st}^{(r^+)} \neq 0$ then put

$$a_{ij}^{(r)} := \begin{cases} a_{ij}^{(r^+)} - \frac{a_{it}^{(r^+)} a_{sj}^{(r^+)}}{a_{st}^{(r^+)}} & \text{for } i < s \text{ and } j < t, \\ a_{ij}^{(r^+)} & \text{otherwise.} \end{cases}$$

3. Set $\tilde{A} := A^{(1,2)}$ ³; \tilde{A} is called *the matrix obtained from A (by the Cauchon Algorithm)*.

We conclude this subsection with some results on the application of the Cauchon Algorithm to TN matrices.

Theorem 2.4. (See [8, Theorem B4], [10, Theorem 2.6].) Let $A \in \mathbb{R}^{n,n}$. Then the following statements hold true.

- (i) If A is TN and $2 \leq s$, then for all $(s, t) \in E$, $A^{(s,t)}$ is an (entry-wise) nonnegative Cauchon matrix and $A^{(s,t)}[1, \dots, s-1 \mid 1, \dots, n]$ is TN.
- (ii) A is TN if and only if \tilde{A} is an (entry-wise) nonnegative Cauchon matrix.

In [2] we further study the Cauchon Algorithm and present new determinantal tests for total nonnegativity, new characterizations of some subclasses of the totally nonnegative matrices and shorter proofs for some classes of matrices for being (nonsingular and) totally nonnegative.

2.3. TN cells

For $\mathbb{R}^{n,n}$, fix a set Z of minors. The *TN cell corresponding to the set Z* is the set of the n -by- n TN matrices for which all their zero minors are just the ones from Z . In [10] it is proved that Algorithm 2.3 provides a bijection between the nonempty TN cells of $\mathbb{R}^{n,n}$ and C_n . The following theorem gives more details about this bijection.

Theorem 2.5. (See [10, Theorem 2.7].)

- (i) Let $A, B \in \mathbb{R}^{n,n}$ be TN. Then A, B belong to the same TN cell if and only if \tilde{A}, \tilde{B} are associated with the same Cauchon diagram. Therefore, the nonempty TN cells in $\mathbb{R}^{n,n}$ are parameterized by the Cauchon diagrams.
- (ii) Let $A \in \mathbb{R}^{n,n}$. Then A is contained in the TN cell associated with $C \in C_n$ if and only if $\tilde{a}_{ij} = 0$ if $(i, j) \in C$ and $\tilde{a}_{ij} > 0$ if $(i, j) \notin C$.

³ Note that $A^{(k,1)} = A^{(k,2)}$, $k = 1, \dots, n-1$, and $A^{(2,2)} = A^{(1,2)}$ so that the algorithm could already be terminated when $A^{(2,2)}$ is computed.

We recall from [10] the definition of a lacunary sequence.

Definition 2.6. Let $C \in C_n$. We say that a sequence $((i_k, j_k), k = 0, 1, \dots, t)$ which is strictly increasing in both arguments is a *lacunary sequence with respect to C* if the following conditions hold:

1. $(i_k, j_k) \notin C, k = 1, \dots, t$;
2. $(i, j) \in C$ for $i_t < i \leq n$ and $j_t < j \leq n$;
3. let $s \in \{1, \dots, t - 1\}$. Then $(i, j) \in C$ if
 - (a) either for all $(i, j), i_s < i < i_{s+1}$ and $j_s < j$, or for all $(i, j), i_s < i < i_{s+1}$ and $j_0 \leq j < j_{s+1}$ and
 - (b) either for all $(i, j), i_s < i$ and $j_s < j < j_{s+1}$ or for all $(i, j), i < i_{s+1}$, and $j_s < j < j_{s+1}$.

A lacunary sequence with respect to the Cauchon diagram presented in Fig. 1 is the sequence $((1, 1), (2, 3), (4, 4))$.

In [10, Section 3] an algorithm is presented which constructs for a given Cauchon diagram C and any square of C a lacunary sequence (with respect to C) starting at this square.

Theorem 2.7. (See [10, Theorem 4.4].) Let $A \in \mathbb{R}^{n,n}$ and $C \in C_n$. Then the following two statements are equivalent:

- (i) The matrix A is TN and belongs to the TN cell associated with C .
- (ii) For each $(i, j), i, j \in \{1, \dots, n\}$, fix a lacunary sequence (with respect to C) $((i_k, j_k), k = 0, \dots, t)$ starting at (i, j) , i.e., $(i_0, j_0) = (i, j)$. Then

$$\det A[i_0, i_1, \dots, i_t \mid j_0, j_1, \dots, j_t] = \begin{cases} 0 & \text{if } (i, j) \in C, \\ > 0 & \text{if } (i, j) \notin C. \end{cases}$$

Note that this test involves only n^2 minors to check whether a given matrix belongs to a specified TN cell.

In the next section we will make use of the following proposition.

Proposition 2.8. Let $A \in \mathbb{R}^{n,n}$ be a TN matrix. Then A is nonsingular if and only if $\tilde{a}_{ii} > 0, i = 1, \dots, n$.

Proof. Let C be the Cauchon diagram associated with A . Assume first that $\tilde{a}_{i_0, i_0} = 0$ and $\tilde{a}_{ii} > 0, i = i_0 + 1, \dots, n$.

Consider the lacunary sequence (with respect to C) $((i, i), i = i_0, i_0 + 1, \dots, n)$. Then by Theorem 2.7 it follows by $(i_0, i_0) \in C$ that $\det A[i_0, \dots, n] = 0$, contradicting Lemma 1.3.

Conversely, assume that $\tilde{a}_{ii} > 0, i = 1, \dots, n$. Then the sequence $((i, i), i = 1, \dots, n)$ is a lacunary sequence and by Theorem 2.7 it follows that $\det A[1, \dots, n] > 0$ since $(1, 1) \notin C$. \square

Remark 2.9. Since $a_{ss}^{(s+1,2)} = \tilde{a}_{ss}, s = 1, \dots, n$, it follows from Proposition 2.8 that the intermediate matrices $A^{(s+1,2)}$ do not contain a zero column. This property will enable us in the next subsection to apply Lemma 1.2 to the matrices $A^{(s+1,2)}$.

2.4. Representation of the intermediate matrices

If $A \in \mathbb{R}^{n,n}$ is TP, then the entries \tilde{a}_{kj} of the matrix \tilde{A} obtained from A by the Cauchon Algorithm can be represented as $(k, j = 1, \dots, n)$

$$\tilde{a}_{kj} = \frac{\det A[k, \dots, k+w \mid j, \dots, j+w]}{\det[k+1, \dots, k+w \mid j+1, \dots, j+w]}, \quad (1)$$

where $w := \min\{n - k, n - j\}$, i.e., as ratios of contiguous minors, see [10, p. 7]. If A is NsTN, then some of the minors involved in this representation may be equal to zero. In this subsection we show

that also in this case the entries of \tilde{A} can be represented as ratios of contiguous minors (with possibly different w).

Proposition 2.10. *Let $A \in \mathbb{R}^{n,n}$ be NsTN. Then the entries \tilde{a}_{kj} of the matrix \tilde{A} can be represented as ($k, j = 1, \dots, n$)*

$$\tilde{a}_{kj} = \frac{\det A[k, \dots, k+p \mid j, \dots, j+p]}{\det A[k+1, \dots, k+p \mid j+1, \dots, j+p]}, \quad (2)$$

with a suitable $0 \leq p \leq n-k$, if $j \leq k$ and $0 \leq p \leq n-j$, if $k < j$.

We call p the order of the representation (2).

Proof. By step 2 of Algorithm 2.3 we have

$$\tilde{a}_{nj} = a_{nj}, \quad j = 1, \dots, n, \quad \text{and} \quad \tilde{a}_{in} = a_{in}, \quad i = 1, \dots, n, \quad (3)$$

so that we can assume that $k, j < n$.

To simplify notation, we will write $|\alpha \mid \beta|$ to denote $\det A[\alpha \mid \beta]$.

We will show by decreasing induction on $k = n-1, \dots, 2$ that the following two statements hold for each \tilde{a}_{kj} .

- (i) The entry \tilde{a}_{kj} can be represented in the form (2) of order p .
- (ii) The entries $\tilde{a}_{k,j+1}$ for $j < k$ and $\tilde{a}_{k-1,j}$ for $1 < k < j$ can be represented in the form (2) of order p , too.

Any change in the order of the representation can be caused only by zero entries in the intermediate matrices, see step 2 in Algorithm 2.3. Consider $j < k$ and $\tilde{a}_{k+1,j} = a_{k+1,j}^{(r^+)} = 0$ with $r = (k+1, j+2)$ while $0 < \tilde{a}_{k+1,j+1} = a_{k+1,j+1}^{(r^+)}$. Then we have by Theorem 2.4(ii) $0 < \tilde{a}_{k+1,j+2} = a_{k+1,j+2}^{(r^+)}$. If we run Algorithm 2.3 then in the r th iteration the entries $a_{kj}^{(r^+)}$ and $a_{k,j+1}^{(r^+)}$ were possibly changed in step 2(b). In the next iteration, $r' = (k+1, j+1)$, the entry $\tilde{a}_{k,j+1} = a_{k,j+1}^{(r)}$ remains unchanged and $a_{kj}^{(r)}$ will possibly change. However, the calculation of $a_{kj}^{(r)}$ and $a_{k,j+1}^{(r')}$ shows that their values remain unchanged due to the fact that $\tilde{a}_{k+1,j} = 0$. So, $\tilde{a}_{kj} = a_{kj}^{(r')}$ may have a representation (2) of order p as $\tilde{a}_{k,j+1}$ as well as a representation of order $p-1$. If $\tilde{a}_{k+1,j+1} = 0$, then we have by Theorem 2.4(ii) $\tilde{a}_{k+1,j} = 0$, too, and the orders of $a_{kj}^{(\rho^+)}$ and $a_{k,j+1}^{(\rho^+)}$ were not changed in the iterations $\rho \leq (k+1, j+2)$.

We first consider the case $j \leq k$ and start with row $n-1$.

By Lemma 1.3 we have $a_{nn} \neq 0$. Set $v := 0$ if $a_{nj} \neq 0$, $j = 1, \dots, n-1$; otherwise set $v := \max\{j \in \{1, \dots, n-1\} \mid a_{nj} = 0\}$.

If $v > 1$, then by Lemma 1.2 we have $a_{nl} = 0$, $l = 1, \dots, v-1$, which yields $a_{ij}^{(n,2)} = a_{ij}$, $i = 1, \dots, n$, $j = 1, \dots, v$; in particular, $\tilde{a}_{n-1,j} = a_{n-1,j}$, $j = 1, \dots, v$.

Now we show that for $\mu = n, \dots, v+1$

$$a_{ij}^{(n,\mu)} = a_{ij} - \frac{a_{i\mu}a_{nj}}{a_{n\mu}}, \quad i = 1, \dots, n-1, \quad j = 1, \dots, \mu-1, \quad (4)$$

holds. The statement is valid for $\mu = n$ by $A^{(n+1,2)} = A$. Assume that (4) is true for a fixed but arbitrary $v+1 < \mu$. Since by (3) $a_{n,\mu-1}^{(n,\mu)} = a_{n,\mu-1} \neq 0$ we have for $i = 1, \dots, n-1$, $j = 1, \dots, \mu-2$,

$$a_{ij}^{(n,\mu-1)} = a_{ij}^{(n,\mu)} - \frac{a_{i,\mu-1}^{(n,\mu)} a_{nj}^{(n,\mu)}}{a_{n,\mu-1}^{(n,\mu)}},$$

which yields after substituting (4) and noting that $a_{nj}^{(n,\mu)} = a_{nj}$,

$$a_{ij}^{(n,\mu-1)} = a_{ij} - \frac{a_{i,\mu-1}a_{nj}}{a_{n,\mu-1}}.$$

Therefore, we have shown that (4) is valid, i.e.,

$$a_{ij}^{(n,\mu)} = \frac{|[i, n | j, \mu]|}{|[n | \mu]|}, \quad \mu = v + 1, \dots, n.$$

Specification for row $n - 1$ of \tilde{A} yields (note footnote 3 and the range of the indices in step 2(b) of Algorithm 2.3)

$$\tilde{a}_{n-1,j} = a_{n-1,j}^{(n,2)} = a_{n-1,j}^{(n,j+1)} = \frac{|[n-1, n | j, j+1]|}{|[n | j+1]|}, \quad j = v + 1, \dots, n.$$

Finally, to show (ii), we have by $a_{nv} = 0$

$$\tilde{a}_{n-1,v} = a_{n-1,v} - \frac{a_{n-1,v+1}a_{nv}}{a_{n,v+1}} = \frac{|[n-1, n | v, v+1]|}{|[n | v+1]|}.$$

We assume now that the statements (i) and (ii) are true for all rows with numbers $n - 1, \dots, k + 1$. We show (i) and consider first the case $\tilde{a}_{k+1,j+1} \neq 0$. Then by the induction hypothesis there is an integer p such that

$$\begin{aligned} \tilde{a}_{k+1,j} &= \frac{|[k+1, \dots, k+p | j, \dots, j+p-1]|}{|[k+2, \dots, k+p | j+1, \dots, j+p-1]|}, \\ \tilde{a}_{k+1,j+1} &= \frac{|[k+1, \dots, k+p | j+1, \dots, j+p]|}{|[k+2, \dots, k+p | j+2, \dots, j+p]|}. \end{aligned} \quad (5)$$

By running the steps up to $(k+2, 2)$ of Algorithm 2.3 with the matrix which is obtained from A by deleting its row $k+1$ we get the representation

$$a_{kj}^{(k+2,2)} = \frac{|[k, k+2, \dots, k+p | j, \dots, j+p-1]|}{|[k+2, \dots, k+p | j+1, \dots, j+p-1]|}, \quad (6)$$

and the similar one for $a_{k,j+1}^{(k+2,2)}$. Since by Theorem 2.4(i) $A^{(k+2,2)}[1, \dots, k+1 | 1, \dots, n]$ is TN we can proceed for row k similarly as for row $n - 1$. Application of Sylvester's Identity yields

$$\tilde{a}_{kj} = a_{kj}^{(k+2,2)} - \frac{\tilde{a}_{k+1,j}a_{k,j+1}^{(k+2,2)}}{\tilde{a}_{k+1,j+1}} = \frac{|[k, \dots, k+p | j, \dots, j+p]|}{|[k+1, \dots, k+p | j+1, \dots, j+p]|}.$$

We now assume that $\tilde{a}_{k+1,j+1} = 0$. This excludes the case $k = j$ by Proposition 2.8 and it follows by Theorem 2.4(ii) that $\tilde{a}_{k+1,j} = 0$. We obtain from (5) that

$$|[k+1, \dots, k+p | j, \dots, j+p-1]| = 0 \quad (7)$$

and

$$|[k+2, \dots, k+p | j+1, \dots, j+p-1]| > 0$$

from which it follows by Lemma 1.3 that

$$|[k+2, \dots, k+p-1 | j+1, \dots, j+p-2]| > 0.$$

Application of Sylvester's Identity to (7) yields

$$\begin{aligned} &|[k+1, \dots, k+p-1 | j, \dots, j+p-2]| |[k+2, \dots, k+p | j+1, \dots, j+p-1]| \\ &= |[k+1, \dots, k+p-1 | j+1, \dots, j+p-1]| |[k+2, \dots, k+p | j, \dots, j+p-2]| \end{aligned}$$

whence

$$\begin{aligned} & \frac{|[k+1, \dots, k+p-1 | j, \dots, j+p-2]|}{|[k+1, \dots, k+p-1 | j+1, \dots, j+p-1]|} \\ &= \frac{|[k+2, \dots, k+p | j, \dots, j+p-2]|}{|[k+2, \dots, k+p | j+1, \dots, j+p-1]|}. \end{aligned} \quad (8)$$

Note that $|[k+2, \dots, k+p | j+1, \dots, j+p-1]| \neq 0$ implies that $|[k+1, \dots, k+p-1 | j+1, \dots, j+p-1]| \neq 0$ by Lemma 1.2.

Now we apply Sylvester's Identity to (6), plug in (8), and apply again Sylvester's Identity to obtain a representation of \tilde{a}_{kj} in the form (2).

$$\begin{aligned} \tilde{a}_{kj} &= a_{kj}^{(k+2,2)} \\ &= \frac{|[k, k+2, \dots, k+p-1 | j, \dots, j+p-2]|}{|[k+2, \dots, k+p-1 | j+1, \dots, j+p-2]|} \\ &\quad - \frac{|[k, k+2, \dots, k+p-1 | j+1, \dots, j+p-1]| |[k+2, \dots, k+p | j, \dots, j+p-2]|}{|[k+2, \dots, k+p | j+1, \dots, j+p-1]| |[k+2, \dots, k+p-1 | j+1, \dots, j+p-2]|} \\ &= \frac{|[k, \dots, k+p-1 | j, \dots, j+p-1]|}{|[k+1, \dots, k+p-1 | j+1, \dots, j+p-1]|}. \end{aligned}$$

Now we prove (ii). We assume that the representations of \tilde{a}_{kj} and $\tilde{a}_{k,j+1}$ according to (2) are of different orders. By the induction hypothesis the orders can differ only by one and the representation of $\tilde{a}_{k,j+1}$ must have the greater order, see the explanation following (ii). Therefore, we have the following representations

$$\begin{aligned} \tilde{a}_{kj} &= \frac{|[k, \dots, k+p | j, \dots, j+p]|}{|[k+1, \dots, k+p | j+1, \dots, j+p]|}, \\ \tilde{a}_{k,j+1} &= \frac{|[k, \dots, k+p+1 | j+1, \dots, j+p+2]|}{|[k+1, \dots, k+p+1 | j+2, \dots, j+p+2]|}. \end{aligned} \quad (9)$$

We distinguish two cases:

Case (a): $|[k+1, \dots, k+p+1 | j+1, \dots, j+p+1]| = 0$.

Application of Sylvester's Identity to the matrix $A[k, \dots, k+p+1 | j+1, \dots, j+p+2]$ yields

$$\frac{|[k, \dots, k+p+1 | j+1, \dots, j+p+2]|}{|[k+1, \dots, k+p+1 | j+2, \dots, j+p+2]|} = \frac{|[k, \dots, k+p | j+1, \dots, j+p+1]|}{|[k+1, \dots, k+p | j+2, \dots, j+p+1]|},$$

whence the representations of \tilde{a}_{kj} and $\tilde{a}_{k,j+1}$ are both of order p .

Case (b): $|[k+1, \dots, k+p+1 | j+1, \dots, j+p+1]| > 0$.

By Lemma 1.3 it follows that

$$|[k+1, \dots, k+p | j+1, \dots, j+p]| > 0.$$

Since the order of the representations of \tilde{a}_{kj} and $\tilde{a}_{k,j+1}$ differs $\tilde{a}_{k+1,j}$ must vanish and by (5) it holds that

$$|[k+1, \dots, k+p | j, \dots, j+p-1]| = 0$$

which implies by Lemma 1.3 $|[k+1, \dots, k+p+1 | j, \dots, j+p]| = 0$. Application of Sylvester's Identity yields

$$\begin{aligned}
& \frac{|[k, \dots, k+p+1 \mid j, \dots, j+p+1]|}{|[k+1, \dots, k+p+1 \mid j+1, \dots, j+p+1]|} \\
&= \frac{|[k, \dots, k+p \mid j, \dots, j+p]|}{|[k+1, \dots, k+p \mid j+1, \dots, j+p]|} \\
&\quad - \frac{|[k, \dots, k+p \mid j+1, \dots, j+p+1]| |[k+1, \dots, k+p+1 \mid j, \dots, j+p]|}{|[k+1, \dots, k+p \mid j+1, \dots, j+p]| |[k+1, \dots, k+p+1 \mid j+1, \dots, j+p+1]|} \\
&= \frac{|[k, \dots, k+p \mid j, \dots, j+p]|}{|[k+1, \dots, k+p \mid j+1, \dots, j+p]|} = \tilde{a}_{kj}
\end{aligned}$$

by (9), whence \tilde{a}_{kj} possesses a representation of order $p+1$, too.

We now consider the case $k < j$. Since the entries \tilde{a}_{kj} with $k < j$ are identical to the entries \tilde{b}_{jk} , where $\tilde{B} = (\tilde{b}_{kj})$ is the matrix obtained from the transpose $B := A^T$ of A by the Cauchon Algorithm, cf. (1), we can reduce this case to the case $j < k$, already discussed above. This completes the proof. \square

3. Application to interval problems

We first give some auxiliary results which will be used in this section.

Lemma 3.1. (See [9, Corollary 3.5], [12, Proposition 3.6.6].) *Let $A, B, Z \in \mathbb{R}^{n,n}$, A, B be nonsingular with $0 \leq A^{-1}, B^{-1}$. If $A \leq Z \leq B$, then Z is nonsingular, and we have $B^{-1} \leq Z^{-1} \leq A^{-1}$.*

The determinantal monotonicity presented in the next lemma follows from a similar property given [11, p. 27] for matrices whose leading principal submatrices have entry-wise nonnegative inverses. We present the proof here since we will refer to it in the proofs of Proposition 3.3 and Corollary 3.7.

Lemma 3.2. *Let $A, B, Z \in \mathbb{R}^{n,n}$, A be NsTN, B be TN and $A \leq^* Z \leq^* B$. Then $\det A \leq \det Z \leq \det B$.*

Proof. We proceed by induction on n . The statement holds trivially for $n = 1$. Assume that the statement is true for fixed n and let $A, B, Z \in \mathbb{R}^{n+1, n+1}$, A be NsTN, B be TN, and $A \leq^* Z \leq^* B$. Assume first that B is nonsingular. Then by Lemma 1.3 $A[2, \dots, n+1]$, $B[2, \dots, n+1]$ are NsTN and by the induction hypothesis

$$0 < \det A[2, \dots, n+1] \leq \det Z[2, \dots, n+1] \leq \det B[2, \dots, n+1]. \quad (10)$$

Since $0 \leq (A^*)^{-1}, (B^*)^{-1}$ and $A^* \leq Z^* \leq B^*$, it follows from Lemma 3.1 that

$$(B^*)^{-1}[1] \leq (Z^*)^{-1}[1] \leq (A^*)^{-1}[1],$$

whence

$$\frac{\det B[2, \dots, n+1]}{\det B} \leq \frac{\det Z[2, \dots, n+1]}{\det Z} \leq \frac{\det A[2, \dots, n+1]}{\det A}. \quad (11)$$

From

$$\frac{\det B[2, \dots, n+1]}{\det Z[2, \dots, n+1]} \cdot \det Z \leq \det B$$

and (10) we obtain $\det Z \leq \det B$. The remaining inequality follows similarly. If B is singular set $B(\epsilon) := B + \epsilon e_{n+1} e_{n+1}^T$ for $0 < \epsilon$, where e_{n+1} denotes the last unit vector of \mathbb{R}^{n+1} . Then $B(\epsilon)$ is NsTN since by assumption

$$0 < \det A[1, \dots, n] \leq \det B[1, \dots, n]$$

and the claim follows now from the case that B is nonsingular and letting ϵ tend to zero. \square

Proposition 3.3. Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are NsTN, then $\tilde{A} \leq^* \tilde{Z} \leq^* \tilde{B}$.

Proof. We give the proof only for $j \leq k$ since as in the proof of Proposition 2.10 the case $k < j$ can be reduced to the case $j < k$ by replacing A by A^T . We show by decreasing induction on $k = n, \dots, 1$ that the representations of $\tilde{a}_{kj}, \tilde{z}_{kj}, \tilde{b}_{kj}$ are of the same order and the following inequalities hold

$$(-1)^{k+j} \tilde{a}_{kj} \leq (-1)^{k+j} \tilde{z}_{kj} \leq (-1)^{k+j} \tilde{b}_{kj}, \quad j = 1, \dots, k.$$

The statement trivially holds for $k = n$ by (3).

Assume that the statement is true for fixed $k + 1$, in particular,

$$(-1)^{k+1+j} \tilde{a}_{k+1,j} \leq (-1)^{k+1+j} \tilde{z}_{k+1,j} \leq (-1)^{k+1+j} \tilde{b}_{k+1,j}, \quad j = 1, \dots, k+1. \quad (12)$$

We want to prove the statement for k .

Set $v_A := 0$ if $\tilde{a}_{k+1,j} > 0$, $j = 1, \dots, k$; otherwise set

$$v_A := \max\{j \in \{1, \dots, k\} \mid \tilde{a}_{k+1,j} = 0\}.$$

Define v_B similarly.

If $v_A = v_B = 0$, then by (12) $\tilde{z}_{k+1,j} > 0$, $j = 1, \dots, k$, and the representations of $\tilde{a}_{kj}, \tilde{z}_{kj}, \tilde{b}_{kj}$ are all of the same order for each $j \in \{1, \dots, k\}$. Assume without loss of generality that $k + j$ is even. If $\tilde{a}_{kj} > 0$ we get similarly as in (11) (taking the reciprocal values)

$$\tilde{a}_{kj} \leq \tilde{z}_{kj} \leq \tilde{b}_{kj}. \quad (13)$$

If $\tilde{a}_{kj} = 0$ it has a representation (2), where the numerator vanishes and the denominator is positive. We replace a_{kj} by $a_{kj} + \epsilon$, where $\epsilon > 0$. Expansion of the resulting matrix along its first row or column shows that its determinant becomes positive. We replace also b_{kj} by $b_{kj} + \epsilon$ and z_{kj} by $z_{kj} + \epsilon$. Application of Lemma 3.2 and letting ϵ tend to zero yields (13).

Now we assume without loss of generality that $v := v_A \geq v_B$ and $v_A > 1$. By Theorem 2.4(ii) we conclude that $\tilde{a}_{k+1,l} = 0$, $l = 1, \dots, v$, from which it follows by (12) that $\tilde{b}_{k+1,l} = \tilde{z}_{k+1,l} = 0$, $l = 1, \dots, v - 1$. For each $l \in \{1, \dots, v - 1, v + 1, \dots, k\}$ the entries $\tilde{a}_{kl}, \tilde{z}_{kl}, \tilde{b}_{kl}$ have representations of the same order. This is also true for $l = v$ if $\tilde{b}_{k+1,v} = 0$. If $0 < \tilde{b}_{k+1,v}$ or $0 < \tilde{z}_{k+1,v}$ we proceed as in case (b) in the proof of Proposition 2.10 (with j replaced by v) to increase the degree of the representation of $\tilde{a}_{k+1,v}$ (note that $\det A[k + 1, \dots, k + p + 1 \mid v, \dots, v + p] = 0$ by $\tilde{a}_{k+1,v} = 0$) so that $\tilde{a}_{k,v}, \tilde{z}_{k,v}, \tilde{b}_{k,v}$ have representations of the same order. Proceeding as in the case $v_A = v_B = 0$ we arrive at (13). This completes the proof. \square

Theorem 3.4. Let $A, B, Z \in \mathbb{R}^{n,n}$, A, B be nonsingular and be in the same TN cell. If $A \leq^* Z \leq^* B$, then Z belongs to the same TN cell that includes A and B .

Proof. The statement follows immediately from Proposition 3.3 using Theorem 2.5(ii). \square

Remark 3.5. If A, B are TP, then both matrices are in the cell associated with the Cauchon diagram with no black squares. It follows from Theorem 3.4 that $Z, A \leq^* Z \leq^* B$, is TP, too. This result is already given in [5, Theorem 1]. Its proof is based on Lemma 3.1; for a proof see also [4, pp. 81–82], [13, pp. 84–85].

Next we settle a conjecture [5] which concerns the case that the two bound matrices A, B are not necessarily in the same cell.

Theorem 3.6. Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are NsTN, then Z is NsTN.

Proof. Since $0 \leq \tilde{Z}$ by Proposition 3.3 we have to show that \tilde{Z} is a Cauchon matrix, see Theorem 2.4(ii). By Proposition 2.8 we have $\tilde{a}_{kk} > 0$, $k = 1, \dots, n$, which implies that $\tilde{z}_{kk} > 0$, $k = 1, \dots, n$. Assume that $\tilde{z}_{kj} = 0$ and $1 < k, j$. Without loss of generality we may assume that $k + j$ is even. Then

it follows that $\tilde{a}_{kj} = 0$. Since \tilde{A} is a Cauchon matrix all entries of \tilde{A} to the left of the position (k, j) or above it are zero. Without loss of generality we may assume that $\tilde{a}_{ki} = 0$, $i = 1, \dots, j - 1$. Since $\tilde{a}_{kk} > 0$, $k = 1, \dots, n$, it follows that $j < k$. Since $A \leq^* B$ we conclude $\tilde{b}_{k,j-1} = 0$, whence all entries of \tilde{B} to the left of position $(k, j - 1)$ or above it must vanish. Again, by [Proposition 2.8](#) we can exclude the latter case. It follows that $\tilde{z}_{ki} = 0$, $i = 1, \dots, j - 1$, which concludes the proof. \square

The example in [\[5\]](#) shows that [Theorem 3.6](#) is not true if one of the bound matrices A, B is singular. Proceeding similarly as in the proof of the singular case in [Lemma 3.2](#) we obtain the following corollary as an extension of the nonsingular case.

Corollary 3.7. *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are TN and $A[2, \dots, n]$ or $A[1, \dots, n - 1]$ is nonsingular, then Z is TN.*

In the tridiagonal case the assumptions can further be weakened.

Corollary 3.8. *Let $A, B, Z \in \mathbb{R}^{n,n}$ with $A \leq^* Z \leq^* B$. If A, B are TN and tridiagonal, i.e., $a_{ij} = b_{ij} = 0$ if $1 < |i - j|$, $i, j = 1, \dots, n$, then Z is TN.*

Proof. Let $A(\epsilon) := A + \epsilon I$, where $0 < \epsilon$, and define $B(\epsilon), Z(\epsilon)$ analogously. Then by [Corollary 2.4](#) in [\[3\]](#) $A(\epsilon), B(\epsilon)$ are TN and by [Theorem 2.3](#) in [\[3\]](#)

$$0 < \det A + \epsilon^n \leq \det A(\epsilon),$$

whence $A(\epsilon)$ is nonsingular. By [Lemma 3.2](#) $B(\epsilon)$ is also nonsingular and the statement follows now from [Theorem 3.6](#) and letting ϵ tend to zero. \square

In [\[1\]](#) we give the largest amount by which the single entries of a tridiagonal TN matrix can be perturbed without losing the property of being TN.

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