



Contents lists available at ScienceDirect

Journal of Symbolic Computation

journal homepage: www.elsevier.com/locate/jsc

Constructively describing orbit spaces of finite groups by few inequalities

Philippe Moustrou^a, Cordian Riener^b, Robin Schabert^c^a Institut de Mathématiques de Toulouse, UMR 5219, UT2J, 31058 Toulouse, France^b Department of Mathematics and Statistics, UiT - the Arctic University of Norway, 9037 Tromsø, Norway^c Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany

ARTICLE INFO

Article history:

Received 31 October 2024

Received in revised form 16 April 2025

Accepted 27 May 2025

Available online 30 May 2025

Keywords:

Semi algebraic sets

Invariant theory

Polynomial optimization

ABSTRACT

Let G be a finite group acting linearly on \mathbb{R}^n . A celebrated Theorem of Procesi and Schwarz gives an explicit description of the orbit space $\mathbb{R}^n//G$ as a basic closed semi-algebraic set. We give a new proof of this statement and another description as a basic closed semi-algebraic set using elementary tools from real algebraic geometry. Bröcker was able to show that the number of inequalities needed to describe the orbit space generically depends only on the group G . Here, we construct such inequalities explicitly for abelian groups and in the case where only one inequality is needed. Furthermore, we answer an open question raised by Bröcker concerning the genericity of his result.

© 2025 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

1. Introduction

A set $S \subset \mathbb{K}^n$ defined as the intersection of finitely many polynomial non-strict inequalities is called a *basic closed semi-algebraic set*. Sets formed by finite unions or complements of such basic closed sets are known as *semi-algebraic sets*. A key result in real algebraic geometry, due to Tarski and Seidenberg, states that the class of semi-algebraic sets is closed under polynomial maps. However, obtaining an explicit description of the image of a given semi-algebraic set under a specific polynomial map is often highly non-trivial (Andradas et al., 2012; Fernando, 2014). Moreover, while the image of a semi-algebraic set under a polynomial map remains semi-algebraic, the image of a basic closed

E-mail addresses: philippe.moustrou@math.univ-toulouse.fr (P. Moustrou), cordian.riener@uit.no (C. Riener), robin.schabert@uit.no (R. Schabert).

<https://doi.org/10.1016/j.jsc.2025.102471>

0747-7171/© 2025 The Author(s). Published by Elsevier Ltd. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

semi-algebraic set is generally not basic closed. This article examines a special class of polynomial maps that send basic closed semi-algebraic sets to other basic closed ones.

Let \mathbb{K} denote either the real numbers \mathbb{R} or the complex numbers \mathbb{C} , and let G be a finite group acting linearly on \mathbb{K}^n . A foundational result by Hilbert states that the ring of invariant polynomials is finitely generated:

$$\mathbb{K}[X_1, \dots, X_n]^G = \mathbb{K}[\pi_1, \dots, \pi_m] \subset \mathbb{K}[\underline{X}] := \mathbb{K}[X_1, \dots, X_n].$$

This inclusion defines the *Hilbert map* $\Pi : \mathbb{K}^n \rightarrow \mathbb{K}^m$, where the image lies in the variety $V_{\mathbb{K}}(I_{\Pi})$ determined by the algebraic relations (syzygies) among the generators π_1, \dots, π_m .

In the case where $\mathbb{K} = \mathbb{C}$, the algebraically closed setting, the Hilbert map is surjective and induces a homeomorphism between the orbit space \mathbb{C}^n/G and the variety $V_{\mathbb{C}}(I_{\Pi})$. More precisely, the image of the map corresponds to the *categorical quotient* $\mathbb{C}^n//G := \text{Spec}(\mathbb{C}[\underline{X}]^G)$.

For finite groups, the categorical quotient coincides with the affine variety defined by the invariant ring and reflects the orbit structure faithfully. Unlike the set-theoretic quotient \mathbb{C}^n/G , which merely identifies points lying in the same orbit, the categorical quotient is defined functorially: any G -invariant morphism from \mathbb{C}^n to an affine variety factors uniquely through the quotient map $\pi : \mathbb{C}^n \rightarrow \mathbb{C}^n//G$. Since all orbits are closed and the map π is finite, the categorical quotient separates all orbits and retains a natural algebraic structure (see, e.g., Derksen and Kemper, 2002; Procesi, 2007). The situation changes when $\mathbb{K} = \mathbb{R}$. The map $\mathbb{R}^n \rightarrow \mathbb{R}^n//G := \text{Spec}(\mathbb{R}[\underline{X}]^G)$ is typically not surjective onto the real points of the quotient variety. The image is a *semi-algebraic set*, which can be described as a finite union of basic semi-algebraic sets defined by polynomial equalities and inequalities. This follows from the Tarski–Seidenberg Theorem, which ensures that projections of semi-algebraic sets remain semi-algebraic. Remarkably, compared to general polynomial maps, this situation is very special: while the image of a general polynomial map is not necessarily basic closed, and explicitly describing it can be difficult, Procesi and Schwarz (1985) showed that the image of the Hilbert map is a basic closed semi-algebraic set. Furthermore, in the case of compact Lie groups, the defining inequalities of this image can be derived explicitly from the chosen fundamental invariants. This makes the Hilbert map a particularly tractable and structured example in the broader study of polynomial images.

For a polynomial p , we define the differential dp as

$$dp = \sum_{j=1}^n \frac{\partial p}{\partial x_j} dx_j.$$

For a finite (compact) group G , we have a G -invariant inner product $\langle \cdot, \cdot \rangle$, which, when applied to differentials, gives

$$\langle dp, dq \rangle = \sum_{j=1}^n \frac{\partial p}{\partial x_j} \cdot \frac{\partial q}{\partial x_j}.$$

Since the differentials of G -invariant polynomials are G -equivariant, the inner products $\langle d\pi_i, d\pi_j \rangle$ for $i, j \in \{1, \dots, m\}$ are G -invariant. Thus, every entry of the symmetric matrix polynomial

$$M_{\Pi} = (\langle d\pi_i, d\pi_j \rangle)_{1 \leq i, j \leq m}$$

is itself an invariant polynomial. With a slight abuse of notation, we represent it as a matrix polynomial in π_1, \dots, π_m . Using this construction, Procesi and Schwarz (1985) have established the following result.

Theorem 1 (Procesi and Schwarz). *Let $G \subseteq \text{GL}_n(\mathbb{R})$ be a finite group, and let $\Pi = (\pi_1, \dots, \pi_m)$ be the fundamental invariants of G . Then the orbit space is described by the following polynomial inequalities:*

$$\Pi(\mathbb{R}^n) = \{z \in V_{\mathbb{R}}(I_{\Pi}) \subseteq \mathbb{R}^m \mid M_{\Pi}(z) \text{ is positive semi-definite} \},$$

where $I_{\Pi} \subseteq \mathbb{R}[z_1, \dots, z_m]$ is the ideal of relations among π_1, \dots, π_m .

This theorem has numerous applications across areas such as differential geometry, dynamical systems, and mathematical physics (see, for example, Field, 2007; Dubrovin, 1998; Huisman, 1999). Combined with Artin’s solution to Hilbert’s 17th problem, the semi-algebraic description of the orbit space yields an equivariant Positivstellensatz, which has proven useful in polynomial optimization (Riener et al., 2013; Moustrou et al., 2023). This broad applicability makes the theorem a powerful tool in real algebraic geometry. However, it is worth noting that the original proof was not formulated directly in the language of real algebraic geometry. Moreover, particularly in applications, it is natural to seek a minimal or more efficient description of the orbit space, as such refinements can significantly improve the performance of algorithms based on this structure. This article contributes to both of these lines of research.

Contributions The goal and contributions of this article are twofold. In the first part, we demonstrate that the orbit space is a basic closed semi-algebraic set using only elementary results from real algebraic geometry, in the case of finite groups. Specifically, in Theorem 6, we show that basic closedness follows directly from sums-of-squares techniques combined with foundational results in invariant theory. In essence, the key insight is that the polynomial ring is a finite module over the invariant ring, which suffices to establish the result. Additionally, we provide a short proof of the Procesi–Schwarz description. While a more elementary proof for the finite group case was already given by Procesi and Schwarz (1988), we believe our version is even more direct and accessible. Furthermore, by invoking standard tools from invariant theory, such as Luna’s slice theorem, the results for finite groups can be extended to the setting of compact groups. Hence, any elementary proof in the finite case is essentially generalizable. Our motivation, however, is not only to provide a simplified proof, but also to lay the groundwork for more efficient descriptions of orbit spaces—an issue that forms the central focus of the second part of this paper.

Given any basic closed semi-algebraic set, it is natural to ask about the minimal number of inequalities required to describe it. A well-known result by Bröcker (1991); Scheiderer (1989) shows that any basic closed semi-algebraic set in \mathbb{R}^n can be described using at most $n(n-1)/2$ polynomial inequalities. In contrast, the description provided by Procesi and Schwarz is often more efficient: it involves only m inequalities, where m is the number of fundamental invariants, i.e., the dimension of the space into which the orbit space is embedded. However, as observed by Procesi and Schwarz, if the order $|G|$ of the group G is odd, the Hilbert map is surjective, and no inequalities are needed to describe the image—even though their construction still yields m inequalities. This naturally leads to the question: how is the number of inequalities related to the structure of the group G ? Bröcker (1998) provided a complete answer: the number k of inequalities needed to describe the orbit space generically (i.e., up to a lower-dimensional subset T) is precisely the maximal k such that G contains an elementary abelian 2-subgroup of order $2k$. Although this characterizes the minimal number of inequalities, Bröcker’s proof is non-constructive.

In the second part of this article, we study the first class of non-trivial examples where we explicitly construct a description using the minimal number of inequalities, as predicted by Bröcker’s theorem. For instance, in the case of abelian groups, we present in Theorem 27 an explicit description involving few inequalities of low degree, which may be particularly useful in optimization contexts. Moreover, we address a question raised by Bröcker: we construct an example where the generic description predicted by his result is not sufficient to describe the entire orbit space—that is, an example where one must explicitly include the lower-dimensional set T to obtain a complete description.

To the best of our knowledge, the literature does not offer fully constructive treatments of these questions; when addressed at all, they are typically handled only partially or in a non-constructive way. Our results represent initial steps toward a comprehensive constructive understanding of Bröcker’s theorem. It is perhaps not surprising that such constructive approaches are significantly more challenging, but we hope that our work provides a foundation for further developments in this direction.

This article is structured as follows: In the following section, we provide elementary arguments to establish that the orbit space is a basic closed semi-algebraic set and give a new proof for the description due to Procesi and Schwarz, which is obtained by going through subgroup chains. The

third section provides the construction of orbit spaces with the least number of inequalities, and we conclude with some open questions.

2. Real orbit spaces for finite groups

We now aim to provide firstly a constructive argument for the remarkable fact that $\Pi(\mathbb{R}^n)$ is a basic closed semi-algebraic set and then secondly give an elementary proof for Theorem 1. Throughout this section, G is a finite group and we use the notations of Theorem 1.

2.1. Notations

Let G act linearly on \mathbb{K}^n , then we get an action on the ring of polynomials defined by

$$h^\sigma = h(\sigma^{-1}(x)), \text{ for all } \sigma \in G.$$

Furthermore, we will use that the polynomial ring $\mathbb{K}[X]$ is a $\mathbb{K}[X]^G$ module. Notice that $\mathbb{K}[X]$ is integral over $\mathbb{K}[X]^G$. Indeed, for any $f \in \mathbb{K}[X]$, we have a monic characteristic polynomial

$$\chi_f(T) := \prod_{\sigma \in G} (T - f^\sigma) \in \mathbb{K}[X]^G[T].$$

Therefore, $\mathbb{K}[X]$ is a finitely generated $\mathbb{K}[X]^G$ -module. Moreover, we can define a simple projection operator, called the *Reynolds operator* which gives a projection from $\mathbb{K}[X]$ to $\mathbb{K}[X]^G$. It is defined by

$$\mathcal{R}_G : h \mapsto \frac{1}{|G|} \sum_{\sigma \in G} h^\sigma.$$

To work in the algebraic setting of ring extensions, it is practical to identify the points in \mathbb{K}^n with ring homomorphisms from $\mathbb{K}[X]$ to \mathbb{K} . In this way we identify $V_{\mathbb{K}}(I_\Pi)$ with $\text{hom}(\mathbb{K}[X]^G, \mathbb{K})$ and every $z \in V_{\mathbb{K}}(I_\Pi)$ is identified with the ring homomorphism ϕ_z defined to be the evaluation in z . On the other hand, since every ring homomorphism is determined uniquely by the image of X_1, \dots, X_n , we can equivalently identify every $\phi \in \text{hom}(\mathbb{K}[X]^G, \mathbb{K})$ with a unique point $z_\phi \in V_{\mathbb{K}}(I_\Pi)$.

Proposition 2. *With the above identification a point $z \in V_{\mathbb{K}}(I_\Pi)$ is in the image of Π if and only if ϕ_z can be extended to a ring homomorphism from $\mathbb{K}[X]$ to \mathbb{K} .*

One of the basic notions in real algebraic geometry is the notion of sums of squares and our proofs will rely on the set of invariant sums of squares.

Definition 3. A polynomial $f \in \mathbb{R}[X]$ is called a sum of squares if it can be decomposed into the form $f = f_1^2 + \dots + f_\ell^2$ for some polynomials $f_1, \dots, f_\ell \in \mathbb{R}[X]$. We will write $\Sigma\mathbb{R}[X]^2$ for the set of all these polynomials. Furthermore, we set $(\Sigma\mathbb{R}[X]^2)^G = \Sigma\mathbb{R}[X]^2 \cap \mathbb{R}[X]^G$. More generally, we call a symmetric matrix polynomial $A \in \mathbb{R}[X]^{k \times k}$ a sum of squares matrix polynomial, if $A = L^T L$ for some $L \in \mathbb{R}[X]^{k \times \ell}$ and we say that a sum of squares matrix polynomial is G -invariant if all of its entries are G -invariant polynomials.

Notice that the set of invariant sums of squares $(\Sigma\mathbb{R}[X]^2)^G$ defines a quadratic module in the ring $\mathbb{R}[X]^G$, which in general is not finitely generated (see Cimpric et al., 2009, Example 5.3). However, using the fact that $\mathbb{R}[X]$ is a finitely generated module it can be conveniently represented using sums of squares matrices. Indeed, let $b_1, \dots, b_l \in \mathbb{R}[X]$ be generators of $\mathbb{R}[X]$ over $\mathbb{R}[X]^G$ and define

$$B \in (\mathbb{R}[X]^G)^{l \times l} \text{ by } B_{ij} := \mathcal{R}_G(b_i b_j),$$

then we have the following characterization (see Gatermann and Parrilo, 2004; Blekherman and Riener, 2021; Moustrou et al., 2023 for details):

Proposition 4. Let $f \in \mathbb{R}[\underline{X}]^G$. Then $f \in \Sigma(\mathbb{R}[\underline{X}]^G)^2$ if and only if there exists a G -invariant sum of invariant squares matrix polynomial $A \in (\mathbb{R}[\underline{X}]^G)^{t \times t}$ with a factorization $A = L^t L$ from some $L \in (\mathbb{R}[\underline{X}]^G)^{k \times \ell}$ such that

$$f = \text{Tr}(A \cdot B)$$

Proof. We sketch the proof for the convenience of the reader. We only consider the case when f is the sum of the orbit of one square - and the general case follows directly in the same way. Let $g \in \mathbb{R}[\underline{X}]$ with $f = \mathcal{R}_G(g^2)$ and write $g = \sum_{i=1}^l a_i b_i$ for some $a_1, \dots, a_l \in \mathbb{R}[\underline{X}]^G$. Then

$$f = \mathcal{R}_G(g^2) = \mathcal{R}_G\left(\sum_{i,j=1}^l a_i a_j b_i b_j\right) = \sum_{i,j=1}^l a_i a_j \mathcal{R}_G(b_i b_j) = a^T B a = \text{Tr}(a a^T B)$$

where $a := (a_1, \dots, a_l)^T$. \square

2.2. Orbit spaces as basic closed semi-algebraic sets

Based on the previous discussions it is almost directly clear that the semi-algebraic set $\Pi(\mathbb{R}^n)$ is basic closed. Indeed, it will be a consequence of the following simple observation.

Proposition 5. Let $z \in V_{\mathbb{R}}(I_{\Pi})$ such that $\Pi^{-1}(z) \not\subseteq \mathbb{R}^n$, then there exists $f \in (\Sigma\mathbb{R}[\underline{X}]^2)^G$ such that $\phi_z(f) < 0$.

Proof. Set $\xi = \Pi^{-1}(z) \in \mathbb{C}^n \setminus \mathbb{R}^n$, let $1 \leq i \leq n$ be such that $\text{Im}(\xi_i) \neq 0$, and consider the polynomial $h := (X_i - \text{Re}(\xi_i))^2$. Clearly, $h(\xi) < 0$. Let $\mathcal{O}_G(h)$ denote the orbit of h under G . We now construct the univariate polynomial $p \in \mathbb{R}[\underline{X}]^G[T]$ by

$$p(T) = \prod_{h' \in \mathcal{O}_G(h)} (T + h').$$

Each of the coefficients of p is in $(\Sigma\mathbb{R}[\underline{X}]^2)^G$. We evaluate these coefficients in z and observe that the resulting univariate polynomial will have at least one negative coefficient by Descartes' rule of signs. Thus, for some $1 \leq k \leq l = |\mathcal{O}_G(h)|$ we have an elementary symmetric polynomial $f := e_k(h^{\sigma_1}, \dots, h^{\sigma_l}) \in (\Sigma\mathbb{R}[\underline{X}]^2)^G$ such that $\phi_z(f) = f(\xi) < 0$. \square

Combining this observation with Proposition 4, we immediately get a description of $\Pi(\mathbb{R}^n)$ as a basic closed semi-algebraic set.

Theorem 6. The semi-algebraic set $\Pi(\mathbb{R}^n)$ is basic closed, it can be represented as

$$\Pi(\mathbb{R}^n) = \{z \in V_{\mathbb{R}}(I_{\Pi}) \subseteq \mathbb{R}^m \mid \phi_z(B) \text{ is positive semi-definite}\}.$$

Proof. Clearly, the condition that $B(z)$ is positive semi-definite is necessary. Indeed, for $z \in \Pi(\mathbb{R}^n)$ there exists by definition $x \in \mathbb{R}^n$ such that $\Pi(x) = z$, and therefore $\phi_z(B) = B(x)$. Let $v \in \mathbb{R}^m$. Then $v^T B v \in (\Sigma\mathbb{R}[\underline{X}]^2)^G$, so $v^T \phi_z(B) v = (v^T B v)(x) \geq 0$. On the other hand, assume that $\phi_z(B)$ is positive semi-definite but that ϕ_z cannot be extended. By Proposition 5 we know that there exists an invariant sums of squares polynomial f with $\phi_z(f) < 0$. Furthermore, by Proposition 4 we have

$$f = \text{Tr}(L L^t B).$$

However, since $\phi_z(L)\phi_z(L^t)$ and $\phi_z(B)$ are positive semi-definite matrices, we have a contradiction. \square

Remark 7. This result can be made constructive: It suffices to find generators of $\mathbb{R}[\underline{X}]$ over $\mathbb{R}[\underline{X}]^G$. If G is of order l , $\mathbb{R}[\underline{X}]$ is generated by

$$X_1^{\alpha_1} \cdots X_n^{\alpha_n} \quad \alpha \in \{0, \dots, l-1\}^n$$

over $\mathbb{R}[\underline{X}]^G$ (use Atiyah, 2018, Proposition 2.16, Proposition 5.1 and Corollary 5.2). Furthermore, in the case of finite reflection groups these generators can be found very directly and with combinatorial methods (Hubert and Bazan, 2022; Debus and Riener, 2023). Moreover, one can also use the G -harmonic polynomials, which can be obtained as the partial derivatives of the determinant of the Jacobian of the Hilbert map (see Helgason, 1984, Chapter III 1.1, 3.6 and 3.7).

2.3. An elementary proof for Theorem 1

We now want to show an elementary proof for the concrete description given in Theorem 1. To begin, we reformulate the statement in terms of homomorphisms.

Proposition 8. *The following are equivalent:*

- (i) $\Pi(\mathbb{R}^n) = \{z \in V_{\mathbb{R}}(I_{\Pi}) \mid \phi_z(M_{\Pi}) \text{ is positive semi-definite}\}$.
- (ii) Let $\phi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ be a ring homomorphism. Then ϕ can be extended to a ring homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$ if and only if

$$\phi(\langle df, df \rangle) \geq 0 \text{ for all } f \in \mathbb{R}[\underline{X}]^G.$$

Proof. Let $z \in V_{\mathbb{R}}(I_{\Pi})$. Note that $z \in \Pi(\mathbb{R}^n)$ if and only if ϕ_z can be extended to a ring homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$. Furthermore, since π_1, \dots, π_m generate the ring of invariants we have $\phi_z(M_{\Pi})$ is positive semi-definite if and only if $\phi_z(\langle df, df \rangle) \geq 0$ for all $f \in \mathbb{R}[\underline{X}]^G$. \square

Note that $\phi_z(M_{\Pi})$ being positive semi-definite in (i) is clearly necessary for z being in the image of the real Hilbert map. Equivalently, $\phi(\langle df, df \rangle) \geq 0$ for all $f \in \mathbb{R}[\underline{X}]^G$ is necessary in order to extend a homomorphism $\phi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ to a homomorphism $\mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$. So we will show that this is also sufficient, and our strategy consists in extending homomorphisms in steps, first to the ring of invariant polynomials for some subgroup H of G . In order to do this we have to make sure that our positivity condition in (ii) extends also to $\mathbb{R}[\underline{X}]^H$ in this case. The following is in fact the core of the argument.

Proposition 9. *Let H be a subgroup of G and let $\phi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ be a homomorphism. Suppose*

$$\phi(\langle df, df \rangle) \geq 0 \text{ for all } f \in \mathbb{R}[\underline{X}]^G,$$

and ϕ can be extended to a homomorphism $\phi_H : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$. Then

$$\phi_H(\langle df, df \rangle) \geq 0 \text{ for all } f \in \mathbb{R}[\underline{X}]^H.$$

Proof. The ring of invariants $\mathbb{R}[\underline{X}]^H$ is a finitely generated \mathbb{R} -algebra, say $\mathbb{R}[\underline{X}]^H = \mathbb{R}[p_1, \dots, p_k]$ and write $\pi_i = q_i(p_1, \dots, p_k)$ for some $q_i \in \mathbb{R}[Y_1, \dots, Y_k]$. Consider

$$\Pi : \mathbb{C}^n \rightarrow V_{\mathbb{C}}(I_{\Pi}), \quad x \mapsto (\pi_1(x), \dots, \pi_m(x)),$$

$$P : \mathbb{C}^n \rightarrow V_{\mathbb{C}}(I_P), \quad x \mapsto (p_1(x), \dots, p_k(x)) \text{ and}$$

$$Q : V_{\mathbb{C}}(I_P) \rightarrow V_{\mathbb{C}}(I_{\Pi}), \quad x \mapsto (q_1(x), \dots, q_k(x))$$

and furthermore, define the points

$$z := z_{\phi} := (\phi(\pi_1), \dots, \phi(\pi_m)) \in V_{\mathbb{R}}(I_{\Pi}) \text{ and } y := y_{\phi_H} := (\phi_H(p_1), \dots, \phi_H(p_k)) \in V_{\mathbb{R}}(I_P)$$

corresponding to ϕ and ϕ_H . Since P is surjective, there is $x \in \mathbb{C}^n$ with $P(x) = y$, so $\Pi(x) = z$. Denote the corresponding total derivatives of Π , P and Q in x , respectively in y , by

$$D_Q^y : \mathbb{C}^m \rightarrow \mathbb{C}^k, D_P^x : \mathbb{C}^k \rightarrow \mathbb{C}^n \text{ and } D_\Pi^x : \mathbb{C}^m \rightarrow \mathbb{C}^n.$$

We can assume furthermore that the stabilizer G_x of x in G is trivial and therefore included in H (otherwise we can perturbate y and therefore x and z a little bit by continuity). So $\dim(\text{im}(D_P^x)) = \dim(\text{im}(D_\Pi^x))$ (see Abud and Sartori, 1983) and therefore $\text{im}(D_P^x) = \text{im}(D_\Pi^x)$, since $\text{im}(D_P^x) \supseteq \text{im}(D_\Pi^x)$ is trivial. So for every $v \in \mathbb{C}^k$ there is some $u \in \mathbb{C}^m$ with $D_\Pi^x(u) = D_P^x(v)$ and so the identity

$$v = D_Q^y(u) + (v - D_Q^y(u)) \in \text{im } D_Q^y + \ker D_P^x$$

shows that $\text{im } D_Q^y + \ker D_P^x = \mathbb{C}^k$.

Let now $f \in \mathbb{R}[\underline{X}]^H$, say $f = g(p_1, \dots, p_k)$. Then there is $u \in \ker D_P^x$ and $v \in \text{im } D_Q^y$ with $a := (dg)(P(x)) = u + v$, i.e. $D_P^x(u) = 0$ and there is $w \in \mathbb{C}^m$ with $v = D_Q^y(w)$. Because a and y are real, we can assume that $w \in \mathbb{R}^m$. Then

$$\begin{aligned} \phi_H(\langle df, df \rangle) &= \langle (df)(x), (df)(x) \rangle \\ &= \langle D_P^x(a), D_P^x(a) \rangle \\ &= \langle D_P^x(u + v), D_P^x(u + v) \rangle \\ &= \langle D_P^x(u) + D_P^x(v), D_P^x(u) + D_P^x(v) \rangle \\ &= \langle D_P^x(D_Q^y(w)), D_P^x(D_Q^y(w)) \rangle \\ &= \langle D_\Pi^x(w), D_\Pi^x(w) \rangle \\ &= \phi(\langle dh, dh \rangle) \geq 0, \end{aligned}$$

where $h := w_1\pi_1 + \dots + w_m\pi_m \in \mathbb{R}[\underline{X}]^G$. \square

Now we will collect some facts about invariants of groups of order two because we will extend our homomorphism first to the ring of invariant polynomials for some order two subgroup H of G . This order two subgroup will correspond to a complex point and its complex conjugate point.

Lemma 10. *Let H be a subgroup of G .*

- (a) $\mathbb{R}[\underline{X}]^H = \mathbb{R}[\underline{X}]^G \oplus \ker(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H})$.
- (b) If $|G/H| = 2$, then $\ker(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H})^2 \subseteq \mathbb{R}[\underline{X}]^G$.

Proof. (a) Let $g \in \ker \mathcal{R}_G \cap \mathbb{R}[\underline{X}]^G$. Then $g = \mathcal{R}_G(g) = 0$, so $\ker \mathcal{R}_G \cap \mathbb{R}[\underline{X}]^G = \{0\}$. Now let $f \in \mathbb{R}[\underline{X}]^H$. Then

$$f = \mathcal{R}_G(f) + f - \mathcal{R}_G(f) \in \mathbb{R}[\underline{X}]^G \oplus \ker(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H})$$

and

$$\mathbb{R}[\underline{X}]^G \oplus \ker(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H}) \subseteq \mathbb{R}[\underline{X}]^H$$

is clear.

(b) Denote $G/H = \{H, \sigma H\}$ and let $r \in \ker(\mathcal{R}_G|_{\mathbb{R}[\underline{X}]^H})^2$. Then

$$0 = \mathcal{R}_G(r) = \frac{1}{2}(r + r^\sigma),$$

so $r^\sigma = -r$ and therefore $(r^2)^\sigma = r^2$, i.e. r^2 is G -invariant. \square

From these elementary considerations we now can give a proof of Theorem 1.

Proof of Theorem 1. We will show property (ii) in Proposition 8: “ \Rightarrow ” is clear.

“ \Leftarrow ” by contradiction: Let $\phi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ be a ring homomorphism such that

$$\phi(\langle df, df \rangle) \geq 0 \text{ for all } f \in \mathbb{R}[\underline{X}]^G$$

and ϕ cannot be extended to $\mathbb{R}[\underline{X}]$, i.e. $z := (\phi(\pi_1), \dots, \phi(\pi_m)) \notin \Pi(\mathbb{R}^n)$. There is $x \in \mathbb{C}^n$ such that $\Pi(x) = z$. Since $\Pi(\bar{x}) = z$, there is $\sigma \in G$ such that $\sigma^{-1} \cdot x = \bar{x}$. Then σ has even order. Now consider the subgroups $C_\sigma = \langle \sigma \rangle$ and $C_{\sigma^2} = \langle \sigma^2 \rangle$ of G . Since

$$f(x) = f^\sigma(x) = f(\sigma^{-1}x) = f(\bar{x}) = \overline{f(x)}$$

for all $f \in \mathbb{R}[\underline{X}]^{C_\sigma}$, we can extend ϕ_z to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^{C_\sigma} \rightarrow \mathbb{R}$.

By Proposition 9, $\tilde{\phi}(\langle df, df \rangle) \geq 0$ for all $f \in \mathbb{R}[\underline{X}]^{C_\sigma}$.

Since $x \notin \mathbb{R}^n$, there is a linear $l \in \mathbb{R}[\underline{X}]$ with $l(x) = i$. Consider now

$$f = \mathcal{R}_{C_{\sigma^2}}(l) - \mathcal{R}_{C_\sigma}(l) \in \ker \mathcal{R}_{C_\sigma}.$$

Note that $\langle df, df \rangle \in \mathbb{R}_{>0}$, because f is linear. Since C_{σ^2} acts trivially on x and $\sigma^{-1}x = \bar{x}$ we get that $f(x) = i$ and therefore $\phi(f^2) = f^2(x) = -1 < 0$ with $f^2 \in \mathbb{R}[\underline{X}]^{C_\sigma}$ by Lemma 10.

Now

$$\tilde{\phi}(\langle df^2, df^2 \rangle) = \tilde{\phi}(\langle 2fdf, 2fdf \rangle) = \tilde{\phi}(4f^2)\langle df, df \rangle < 0,$$

which is a contradiction to $\tilde{\phi}(\langle dg, dg \rangle) \geq 0$ for all $g \in \mathbb{R}[\underline{X}]^{C_\sigma}$. \square

3. Describing orbit spaces with few inequalities

In the previous section, we obtained inequalities describing the orbit space as a basic closed semi-algebraic set. The aim of this section is to find descriptions of the orbit space involving fewer inequalities. To this end, we fix again a finite group G and introduce some notation. Recall that the dimension of a semi-algebraic set is the smallest dimension of a variety containing it.

Definition 11. We say that the orbit space $\Pi(\mathbb{R}^n)$ (or \mathbb{R}^n/G) is described by $f_1, \dots, f_k \in \mathbb{R}[\underline{X}]^G$ if

$$\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid \phi_z(f_1) \geq 0, \dots, \phi_z(f_k) \geq 0\}.$$

Furthermore, we will say that $\Pi(\mathbb{R}^n)$ is generically described by f_1, \dots, f_k if

$$\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid \phi_z(f_1) > 0, \dots, \phi_z(f_k) > 0\} \cup T,$$

where T is a semi-algebraic set of lower dimension than the orbit space.

It was already observed that if G has odd order, then one needs no inequalities to describe the orbit space \mathbb{R}^n/G , i.e. the Hilbert map is surjective. More generally, Bröcker (1998, Proposition 5.6.) proved the following about the number of inequalities needed to generically describe the orbit space. Recall that an elementary abelian 2-subgroup is a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$.

Theorem 12 (Bröcker). Let k be the maximal number for which G contains an elementary abelian subgroup of order 2^k . Then the orbit space $\Pi(\mathbb{R}^n)$ is generically described by some $f_1, \dots, f_k \in \mathbb{R}[\underline{X}]^G$.

Note that the proof of Bröcker’s Theorem is non-constructive. The goal of this section is to construct these few inequalities for the case where $k = 1$ and for abelian groups. We answer first an open question raised by Bröcker (1998) and point out an imprecision in his Theorem: Bröcker writes that he does not know any example where T is required. He also states Theorem 12 with non-strict inequalities. The following example shows at the same time that sometimes some lower-dimension T is needed and that the strict inequalities in Theorem 12 cannot be replaced by non-strict inequalities.

Example 13. Consider $C_4 = \langle (1, 2, 3, 4) \rangle$ acting by permutation on \mathbb{R}^4 . Denote by π_1, \dots, π_k the fundamental invariants and by

$$\Pi : \mathbb{C}^4 \rightarrow \mathbb{C}^k$$

the Hilbert map as usual. Suppose $\Pi(\mathbb{R}^4)$ is described by one inequality $f \geq 0$, where $f \in \mathbb{R}[X_1, \dots, X_4]^{\mathbb{C}^4}$. Since

$$g(a, b, \bar{a}, \bar{b}) = g^{(13)(24)}(a, b, \bar{a}, \bar{b}) = g(\bar{a}, \bar{b}, a, b) = \overline{g(a, b, \bar{a}, \bar{b})}$$

and

$$g(a, \bar{a}, a, \bar{a}) = g^{(1234)}(a, \bar{a}, a, \bar{a}) = g(\bar{a}, a, \bar{a}, a) = \overline{g(a, \bar{a}, a, \bar{a})}$$

for all $a, b \in \mathbb{C}$ and for all $g \in \mathbb{R}[X_1, \dots, X_4]^{\mathbb{C}^4}$, we get that

$$\Pi(a, b, \bar{a}, \bar{b}) \in \mathbb{R}^k \quad \text{and} \quad \Pi(a, \bar{a}, a, \bar{a}) \in \mathbb{R}^k,$$

for all $a, b \in \mathbb{C}$. Therefore

$$f(a, b, \bar{a}, \bar{b}) \leq 0$$

for all $a, b \in \mathbb{C}$, where the inequality is strict for a or b are not real. So we get that

$$f(a, b, a, b) = 0$$

for all $a, b \in \mathbb{R}$. This means that the 2-variate polynomial

$$g := f(X_1, X_2, X_1, X_2)$$

vanishes on \mathbb{R}^2 , so $g = 0$ and thus $f(i, -i, i, -i) = g(i, -i) = 0$, contradicting $(i, -i, i, -i) \notin \Pi(\mathbb{R}^4)$.

In a similar way one can show that the strict inequalities in Theorem 12 cannot be replaced by non-strict inequalities. Assume

$$\Pi(\mathbb{R}^n) = \{z \in V(I_\Pi) \mid \phi_z(f) \geq 0\} \cup T,$$

where $\dim(T) < \dim(\Pi(\mathbb{R}^n))$ for some G -invariant f . We get again

$$f(a, b, \bar{a}, \bar{b}) \leq 0$$

for all $a, b \in \mathbb{C}$. If

$$f(a, b, a, b) = 0$$

for all $a, b \in \mathbb{R}$, then we get the same contradiction as before. So

$$f(a, b, a, b) < 0$$

for some $a, b \in \mathbb{R}$. So

$$f < 0 \text{ on some neighbourhood of } (a, b, a, b) \text{ in } \mathbb{R}^4,$$

which is a contradiction to $\dim(T) < \dim(\Pi(\mathbb{R}^n))$.

3.1. Strategy for constructing few inequalities

In this subsection, we introduce a strategy and some tools to prove a constructive version of Bröcker’s Theorem. This is easy if G is already an elementary abelian 2-subgroup. Therefore, one might be tempted to try the following approach:

- (1) First one tries to find an elementary abelian 2-subgroup H that corresponds to all complex conjugations that might appear. If such a subgroup exists, then we can extend every homomorphism $\phi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$ (see Lemma 15).
- (2) Now one constructs H -invariant polynomials h_1, \dots, h_k describing the orbit space \mathbb{R}^n/H .
- (3) Then one tries to symmetrize these polynomials. If one finds G -invariant polynomials g_1, \dots, g_k describing the orbit space \mathbb{R}^n/H , then these will also generically describe the orbit space \mathbb{R}^n/G .

In general, the first step fails, as presented in Example 19. It is therefore interesting to have classes of groups for which this step is fulfilled. This is in particular true if G contains a so-called broad subgroup. Recall that an involution is an element $\sigma \in G$ such that $\sigma^2 = \text{id}$.

Definition 14. An elementary abelian 2-subgroup H of G is called *broad*, if every involution of G is conjugate to an element of H .

For the following Lemma, note that if G acts effectively, then almost all points have a full orbit or equivalently a trivial stabilizer. Such orbits are also called of *principal orbit type*.

Lemma 15. Let H be a broad subgroup of G . Then every homomorphism $\varphi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ of principal orbit type can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$.

Proof. Let $\varphi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ be a homomorphism of principal orbit type. There is an extension $\psi : \mathbb{R}[\underline{X}] \rightarrow \mathbb{C}$ of φ . Then $\overline{\psi}$ is also an extension of φ , i.e. there is $\sigma \in G$ such that

$$\sigma\psi = \overline{\psi}.$$

Now $\sigma^2\psi = \psi$ and therefore σ is an involution, since ψ has principal orbit type. Since H is broad, there is $h \in H$ and $g \in G$ such that $hg = g\sigma$. Consider now the extension $\tilde{\psi} := g\psi : \mathbb{R}[\underline{X}] \rightarrow \mathbb{C}$ of φ . Then

$$h\tilde{\psi} = hg\psi = g\sigma\psi = g\overline{\psi} = \overline{\tilde{\psi}}$$

and therefore

$$\tilde{\psi}(f) = \tilde{\psi}(f^h) = h\tilde{\psi}(f) = \overline{\tilde{\psi}(f)}$$

for all $f \in \mathbb{R}[\underline{X}]^H$. So $\tilde{\varphi} := \tilde{\psi}|_{\mathbb{R}[\underline{X}]^H} : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$ is a real extension of φ to $\mathbb{R}[\underline{X}]^H$. \square

There are at least three types of groups that contain broad subgroups. The first result, given by Guralnick and Robinson (2022), concerns quasi-simple groups, namely groups G that are equal to their commutator subgroup and such that the quotient of G by its center is a simple group.

Theorem 16 (Guralnick and Robinson). Let G be quasi-simple. Then G contains a broad subgroup.

Remark 17. As mentioned in Guralnick and Robinson (2022), S_n also contains a broad subgroup. Indeed

$$H := \left\langle (1, 2), (3, 4), \dots, \left(2 \left\lfloor \frac{n}{2} \right\rfloor - 1, 2 \left\lfloor \frac{n}{2} \right\rfloor \right) \right\rangle$$

is a broad subgroup of S_n .

Another particular situation in which we can find broad subgroups is the following:

Lemma 18. Assume all maximal elementary abelian 2-subgroups of G are of order 2. Then every involution of G generates a broad subgroup of G .

Proof. This follows from the fact that two involutions are either conjugate or commute with a third involution: Let $\sigma, \tau \in G$ be involutions. Then

$$(\sigma \tau)^k = 1$$

for some $k \in \mathbb{N}$. If k is even, then σ and τ commute with the involution $(\sigma \tau)^{\frac{k}{2}}$. If k is odd then

$$(\sigma \tau)^{\frac{k-1}{2}} \sigma \left((\sigma \tau)^{\frac{k-1}{2}} \right)^{-1} = \tau. \quad \square$$

In general G does not always contain a broad subgroup.

Example 19. Consider the dihedral group

$$D_4 := \left\langle \left(\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right), \left(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right) \right\rangle$$

with corresponding invariant ring

$$\mathbb{R}[X_1, X_2]^{D_4} = \mathbb{R}[\underbrace{X_1^2 + X_2^2}_{\pi_1}, \underbrace{X_1^4 + X_2^4}_{\pi_2}].$$

It is easy to check that D_4 has no broad subgroup. Furthermore, there is no elementary abelian 2-subgroup H such that every homomorphism of principal orbit type $\phi : \mathbb{R}[\underline{X}]^{D_4} \rightarrow \mathbb{R}$ can be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$. To show this it suffices to show this for all maximal elementary abelian 2-subgroups:

case 1:

$$H := \left\langle \left(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right), \left(\begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \right) \right\rangle$$

Consider the homomorphism

$$\phi : \mathbb{R}[\underline{X}]^{D_4} \rightarrow \mathbb{R}, g(\pi_1, \pi_2) \mapsto g(0, 2).$$

Now $f := X_1 X_2 \in \mathbb{R}[X_1, X_2]^H$ with $f^2 \in \mathbb{R}[X_1, X_2]^G$ and

$$\phi(f^2) = \phi\left(\frac{\pi_1^2 - \pi_2}{2}\right) = -1 < 0,$$

so ϕ cannot be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$. Furthermore, it is easy to check that ϕ has principal orbit type, since it corresponds to the complex preimage $(1, i)$.

case 2:

$$H := \left\langle \left(\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right), \left(\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix} \right) \right\rangle$$

Consider the homomorphism

$$\phi : \mathbb{R}[\underline{X}]^{D_4} \rightarrow \mathbb{R}, g(\pi_1, \pi_2) \mapsto g(0, -8).$$

Now $f := X_1^2 - X_2^2 \in \mathbb{R}[X_1, X_2]^H$ with $f^2 \in \mathbb{R}[X_1, X_2]^G$ and

$$\phi(f^2) = \phi\left(2\pi_2 - \pi_1^2\right) = -16 < 0,$$

so ϕ can not be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$. Furthermore, it is easy to check, that ϕ has principal orbit type, since it corresponds to the complex preimage $(1 + i, 1 - i)$.

3.2. Bröcker's theorem for $k = 1$

In this subsection, we will present how to generically describe the orbit space by one inequality if $k = 1$ in Bröcker's Theorem 12. To this end, we need to understand how to do step (2) in this case, i.e. we have to construct a polynomial describing the orbit space for groups of order two.

Lemma 20. *Let G be a group of order two. The orbit space $\Pi(\mathbb{R}^n)$ is described by*

$$f = \sum_{i=1}^n f_i^2$$

where $f_i := X_i - \mathcal{R}_G(X_i)$.

Proof. It suffices to show that a homomorphism $\varphi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$ if and only if $\varphi(f) \geq 0$. First note that f is G -invariant by Lemma 10, so $\varphi(f) \in \mathbb{R}$ and one implication is clear because f is a sum of squares. For the other implication, let $\varphi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ be a homomorphism such that $\varphi(f) \geq 0$. The homomorphism φ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{C}$. Note that

$$\mathbb{R}[\underline{X}] = \mathbb{R}[\underline{X}]^G[f_1, \dots, f_n]$$

Case 1: $\varphi(f_i^2) = 0$ for all $i \in \{1, \dots, n\}$. Then $\tilde{\varphi}(f_i) = 0$ for all $i \in \{1, \dots, n\}$ and therefore $\text{im } \tilde{\varphi} \subseteq \mathbb{R}$.

Case 2: $\varphi(f_i^2) > 0$ for some $i \in \{1, \dots, n\}$. Then $\tilde{\varphi}(f_i) \in \mathbb{R} \setminus \{0\}$. Now $\tilde{\varphi}(f_j) \in \mathbb{R}$ for all $j \in \{1, \dots, n\}$ since $\tilde{\varphi}(f_i)\tilde{\varphi}(f_j) = \varphi(f_i f_j) \in \mathbb{R}$ by Lemma 10. So again $\text{im } \tilde{\varphi} \subseteq \mathbb{R}$. \square

This Lemma is the first step in order to get an explicit description when $k = 1$:

Theorem 21. *Let G be a group such that all maximal elementary abelian 2-subgroups of G are of order 2. Then $\Pi(\mathbb{R}^n)$ is generically described by some $g \in \mathbb{R}[\underline{X}]^G$.*

More precisely, if $|G| = q2^l$ with q odd and

$$H_1 \subseteq H_2 \subseteq \dots \subseteq H_l \subseteq G$$

are nested subgroups with $|H_i| = 2^i$, then one can choose

$$g = \prod_{\sigma H_i \in G/H_l} \left(\mathcal{R}_{H_i} \left(\sum_{i=1}^n (X_i - \mathcal{R}_{H_1}(X_i))^2 \right) \right)^\sigma.$$

Proof. It suffices to show that there is some $g \in \mathbb{R}[\underline{X}]^G$, such that for every homomorphism $\varphi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ of principal orbit type, that $\varphi(g) \geq 0$ if and only if φ can be extended to a homomorphism $\mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$.

By Lemma 18, H_1 is a broad subgroup of G . So by Lemma 15 every homomorphism of principal orbit type $\varphi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ can be extended to a homomorphism $\varphi_H : \mathbb{R}[\underline{X}]^{H_1} \rightarrow \mathbb{R}$.

Since H_1 is of order 2, there is a H_1 -invariant sum of squares

$$f_1 = \sum_{i=1}^n (X_i - \mathcal{R}_{H_1}(X_i))^2$$

such that φ_H can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$ if and only if $\varphi_H(f_1) \geq 0$ by Lemma 20. The argument now follows from the following two lemmas.

Lemma 22. *For every $1 \leq i \leq l$ there exists a sum of squares $f_i \in \mathbb{R}[\underline{X}]^{H_i}$ such that φ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$ if and only if $\varphi_H(f_i) \geq 0$.*

Proof. We go by induction on i . The base case $i = 1$ is done above. For the induction step from i to $i + 1$ we suppose the claim holds for some $1 \leq i \leq l - 1$, i.e. there is $f_i \in \mathbb{R}[\underline{X}]^{H_i}$ such that φ can be extended if and only if $\varphi_H(f_i) \geq 0$. Either f_i is already H_{i+1} -invariant and we are done or

$$\mathbb{R}(\underline{X})^{H_i} = \mathbb{R}(\underline{X})^{H_{i+1}}(f_i) = \mathbb{R}(\underline{X})^{H_{i+1}}(f_i^\sigma).$$

Consider the sum of squares $f_{i+1} := f_i + f_i^\sigma \in \mathbb{R}[\underline{X}]_{i+1}^{H_i}$, where H_{i+1}/H_i is generated by $\sigma \in G$. If φ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$, then $\varphi_H(f_{i+1}) \geq 0$, since $\tilde{\varphi}$ is real and f_{i+1} is a sum of squares.

If $\varphi_H(f_{i+1}) \geq 0$, then $\varphi_H(f_i) \geq 0$ or $\varphi_H(f_i^\sigma) \geq 0$, because f_i^σ is H_i -invariant. In the first case we are done by the induction hypothesis and in the second case we consider the homomorphism $\sigma^{-1}\varphi_H$ instead of φ_H : The homomorphism $\sigma^{-1}\varphi_H$ is a real extension of φ and $\sigma^{-1}\varphi_H(f_i) \geq 0$, so we can use the induction hypothesis. \square

By Lemma 22 there is a H_l -invariant sum of squares f_l such that φ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$ if and only if $\varphi_H(f_l) \geq 0$.

Lemma 23. *φ can be extended to a homomorphism $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$ if and only if*

$$\varphi\left(\prod_{\sigma H_l \in G/H_l} f_l^\sigma\right) \geq 0.$$

Proof. The if case is clear, since $f := \prod_{\sigma H_l \in G/H_l} f_l^\sigma$ is a sum of squares. For the other direction assume that $\varphi(f) \geq 0$ and consider a complex extension $\tilde{\varphi} : \mathbb{R}[\underline{X}] \rightarrow \mathbb{C}$ of φ . Now

$$0 \leq \varphi(f) = \prod_{\sigma H_l \in G/H_l} \tilde{\varphi}(f_l^\sigma) = \prod_{\sigma H_l \in G/H_l} \sigma \tilde{\varphi}|_{\mathbb{R}[\underline{X}]^H}(f_l).$$

If the image of $\sigma \tilde{\varphi}|_{\mathbb{R}[\underline{X}]^H}$ is not real for some σ , then also its complex conjugate appears in the product, so their product is positive. Since $|G/H_l|$ is odd, there has to be $\sigma H_l \in G/H_l$, such that $\sigma \tilde{\varphi}(f_l) \geq 0$ and $\sigma \tilde{\varphi}|_{\mathbb{R}[\underline{X}]^H}$ is a real homomorphism. Now we apply Claim 1 to $\sigma \varphi|_H$ instead of $\varphi|_H$. \square

The proof now follows directly from Lemmas 22 and 23. \square

Note that Theorem 21 is constructive. We give some examples.

Example 24.

(1) Consider S_3 with the standard action on $\mathbb{R}[X_1, X_2, X_3]$. Following the proof of Theorem 21 we consider first some order two subgroup of S_3 , say $S_2 = \langle (12) \rangle$. We obtain using Lemma 20 the S_2 -invariant polynomial

$$f = (X_1 - \mathcal{R}_{S_2}(X_1))^2 + (X_2 - \mathcal{R}_{S_2}(X_2))^2 + (X_3 - \mathcal{R}_{S_2}(X_3))^2 = (X_1 - X_2)^2$$

in Claim 1. This polynomial is symmetrized in Claim 2 and we get, that the orbit space \mathbb{R}^3/S_3 is generically described by the discriminant

$$g = \prod_{i=0}^2 f^{(123)^i} = \prod_{1 \leq i < j \leq 3} (X_i - X_j)^2.$$

(2) Consider the Quaternion group

$$Q_8 = \left\langle \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

with the canonical action on $\mathbb{R}[X_1, X_2, X_3, X_4]$. Consider again first the unique order two subgroup $H = \langle -I_4 \rangle$. In Theorem 21 Claim 1 we obtain the H -invariant polynomial

$$f = X_1^2 + X_2^2 + X_3^2 + X_4^2$$

using Lemma 20. Since f is already Q_8 -invariant, we don't need to apply the second step. So the orbit space \mathbb{R}^4/Q_8 is already generically described by f .

(3) Consider the Dihedral group

$$D_5 = \left\langle \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \right\rangle$$

with the canonical action on $\mathbb{R}[X_1, X_2, X_3, X_4, X_5]$. Theorem 21 yields that the orbit space \mathbb{R}^5/D_5 is generically described by

$$\begin{aligned} &((X_1 - X_5)^2 + (X_2 - X_4)^2)((X_2 - X_1)^2 + (X_3 - X_5)^2)((X_3 - X_2)^2 + (X_4 - X_1)^2) \\ &((X_4 - X_3)^2 + (X_5 - X_2)^2)((X_5 - X_4)^2 + (X_1 - X_3)^2). \end{aligned}$$

3.3. Bröcker's theorem for abelian groups

In this subsection, we want to give a constructive version of Bröcker's Theorem 12 for abelian groups. To this end, we need to understand first what happens if we have a normal subgroup of G .

Lemma 25. *Let H be a normal subgroup of G . Assume that:*

- (1) *There are $g_1, \dots, g_k \in \mathbb{R}[\underline{X}]^G$ such that every homomorphism $\varphi_G : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ can be extended to a homomorphism $\varphi_H : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$, if and only if $\varphi_G(g_1) \geq 0, \dots, \varphi_G(g_k) \geq 0$.*
- (2) *There is $h \in \mathbb{R}[\underline{X}]^H$ such that every homomorphism $\varphi_H : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$ can be extended to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$, if and only if $\varphi_H(h) \geq 0$.*

Then a homomorphism $\varphi_G : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ can be extended to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$, if and only if $\varphi_G(\mathcal{R}_G(h)) \geq 0, \varphi_G(g_1) \geq 0, \dots, \varphi_G(g_k) \geq 0$.

Proof. Let $\varphi_G : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ be a homomorphism.

" \Rightarrow :" If φ_G can be extended to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$, then

$$\varphi_G(g_1) \geq 0, \dots, \varphi_G(g_k) \geq 0$$

by (1). For every $\sigma \in G$ we have that $\sigma^{-1}\varphi|_{\mathbb{R}[\underline{X}]^H}$ is an extension of φ_G , so

$$\varphi(h^\sigma) = \sigma^{-1}\varphi|_{\mathbb{R}[\underline{X}]^H}(h) \geq 0$$

by (2) and therefore $\varphi_G(\mathcal{R}_G(h)) \geq 0$.

" \Leftarrow :" Suppose $\varphi_G(\mathcal{R}_G(h)) \geq 0, \varphi_G(g_1) \geq 0, \dots, \varphi_G(g_k) \geq 0$. Then φ_G can be extended to a homomorphism $\varphi_H : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$ by (1). Now h^σ is H -invariant for all $\sigma \in G$, since H is a normal subgroup of G . Now

$$\sum_{\sigma \in G} \varphi_H(h^\sigma) = |G| \varphi_H(\mathcal{R}_G(h)) = |G| \varphi_G(\mathcal{R}_G(h)) \geq 0$$

and since all the terms are real, $\varphi_H(h^\sigma) \geq 0$ for some $\sigma \in G$. Then $\sigma^{-1} \varphi_H(h) = \varphi_H(h^\sigma) \geq 0$ and $\sigma^{-1} \varphi_H \in \text{hom}(\mathbb{R}[\underline{X}]^H, \mathbb{R})$, so we can extend $\sigma^{-1} \varphi_H$ to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$ by (2). \square

Note that Theorem 21 includes in particular cyclic groups of even order. The following example shows that Lemma 20 together with Lemma 25 give a polynomial of lower degree describing generically the orbit space in this case. Moreover, this example generalizes Example 13 above.

Example 26. Let $G = C_m$ be a cyclic group. If m is odd, the Hilbert map is surjective, so we need no inequality. If m is even, write $m = 2^k q$, where q is odd. Furthermore, consider the cyclic subgroups

$$\{\text{id}\} = C_1 \subset C_2 \subset C_4 \subset \dots \subset C_{2^k} \subset C_m$$

and the C_m -invariant polynomials

$$f_i := \mathcal{R}_{C_m} \left(\sum_{j=1}^n \left(\mathcal{R}_{C_{2^{i-1}}}(X_j) - \mathcal{R}_{C_{2^i}}(X_j) \right)^2 \right)$$

Then from Lemma 20 and Lemma 25 we obtain that the orbit space \mathbb{R}^n/C_m is generically described by f_1 . Furthermore, \mathbb{R}^n/C_m is described by f_1, \dots, f_k : For every non-real preimage $x \in \mathbb{C}^n \setminus \mathbb{R}^n$, there is σ of order 2^i with $\sigma x = \bar{x}$ (replace σ by σ^r for some odd r). Now it is easy to check that $f_i(x) < 0$:

$$\begin{aligned} f_i(x) &= \mathcal{R}_{C_m} \left(\sum_{j=1}^n \left(\mathcal{R}_{C_{2^{i-1}}}(x_j) - \mathcal{R}_{C_{2^i}}(x_j) \right)^2 \right) \\ &= \mathcal{R}_{C_m} \left(\sum_{j=1}^n \left(\frac{1}{|C_{2^{i-1}}|} \sum_{l=1}^{2^{i-1}} \sigma^{2l}(x_j) - \frac{1}{|C_{2^i}|} \sum_{l=1}^{2^i} \sigma^l(x_j) \right)^2 \right) \\ &= \frac{1}{|C_{2^i}|} \mathcal{R}_{C_m} \left(\sum_{j=1}^n \left(2 \sum_{l=1}^{2^{i-1}} \sigma^{2l}(x_j) - \sum_{l=1}^{2^{i-1}} \sigma^{2l}(x_j) + \sigma^{2l+1}(x_j) \right)^2 \right) \\ &= \frac{1}{2^i} \mathcal{R}_{C_m} \left(\sum_{j=1}^n \left(\sum_{l=1}^{2^{i-1}} \sigma^{2l}(x_j) - \sigma^{2l+1}(x_j) \right)^2 \right) \\ &= \frac{1}{2^i m} \sum_{\tau \in C_m} \tau \left(\sum_{j=1}^n \left(\sum_{l=1}^{2^{i-1}} \sigma^{2l}(x_j - \bar{x}_j) \right)^2 \right) \\ &= \frac{1}{2m} \sum_{\tau \in C_m} \left(\sum_{j=1}^n (\tau x_j - \overline{\tau x_j})^2 \right) < 0. \end{aligned}$$

In general, \mathbb{R}^n/C_m cannot be described by less than k polynomials, which can be observed in a similar way as in Example 13.

We can now prove Bröcker’s Theorem for abelian groups.

Theorem 27. Let G be abelian and choose $k \in \mathbb{N}_0$ such that the maximal abelian 2-subgroups of G are of order 2^k . Then \mathbb{R}^n/G is generically described by $f_1, \dots, f_k \in \mathbb{R}[\underline{X}]^G$.

Proof. Since G is abelian, it is isomorphic to the direct product of cyclic groups, say $G \cong H_1 \times \dots \times H_k \times H_{k+1} \times \dots \times H_m$, where H_1, \dots, H_k are of even order and H_{k+1}, \dots, H_m are of odd order.

By applying iteratively Example 26 and Lemma 25 we find $g_1, \dots, g_i \in \mathbb{R}[\underline{X}]^G$ ($1 \leq i \leq k$) such that every homomorphism $\varphi_G : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ of principal orbit type can be extended to a homomorphism $\varphi_i : \mathbb{R}[\underline{X}]^{H_{i+1} \times \dots \times H_m} \rightarrow \mathbb{R}$, if and only if $\varphi_G(g_1) \geq 0, \dots, \varphi_G(g_i) \geq 0$. Since $H_{k+1} \times \dots \times H_m$ is odd, we can extend every homomorphism $\varphi_k : \mathbb{R}[\underline{X}]^{H_{k+1} \times \dots \times H_m} \rightarrow \mathbb{R}$ to a homomorphism $\varphi : \mathbb{R}[\underline{X}] \rightarrow \mathbb{R}$. \square

The following example illustrates how to use the proof of Theorem 27 to construct inequalities describing generically the orbit space of abelian groups.

Example 28. Consider the abelian group

$$G := \left\langle \left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right) \right\rangle \cong C_4 \times C_2$$

and the subgroup $H := \left\langle \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right\rangle \cong C_2$ of H . The orbit space \mathbb{R}^4/H is generically described by the polynomial

$$h = X_1^2 + X_2^2 + X_3^2 + X_4^2$$

by Example 26. Furthermore, we can also apply Example 26 to the ring extension

$$\mathbb{R}[\underline{X}]^G \subseteq \mathbb{R}[\underline{X}]^H$$

and get that a homomorphism $\phi : \mathbb{R}[\underline{X}]^G \rightarrow \mathbb{R}$ of principal orbit type can be extended to a homomorphism $\tilde{\phi} : \mathbb{R}[\underline{X}]^H \rightarrow \mathbb{R}$ if and only if $\phi(g) \geq 0$, where

$$g = (X_1^2 - X_3^2)^2 + (X_2^2 - X_4^2)^2 + (X_1X_2 - X_3X_4)^2 + (X_1X_4 - X_2X_3)^2.$$

Now by Lemma 25 the orbit space \mathbb{R}^4/G is generically described by the polynomials g and $\mathcal{R}_G(h) = h$.

By considering first

$$H_2 = \left\langle \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle$$

instead of H , one obtains that \mathbb{R}^4/G is also generically described by

$$h_2 = (X_1 - X_3)^2 + (X_2 - X_4)^2 \text{ and}$$

$$g_2 = \left(\sum_{i=1}^4 X_i \right)^2 + \left(\sum_{i=1}^4 X_i^3 \right)^2 + (X_1X_2^2 + X_2X_3^2 + X_3X_4^2 + X_4X_1^2)^2.$$

4. Conclusion and open questions

We conclude with several open questions and directions for further investigation. These naturally arise from our study of the Hilbert map, sums of squares representations, and the structure of orbit spaces under finite and compact group actions and point towards both conceptual generalizations and algorithmic refinements.

4.1. Theorem of Procesi and Schwarz for compact Lie groups

In Section 2, we presented a new proof of the theorem of Procesi and Schwarz for finite groups. This result extends to compact Lie groups via Luna’s slice theorem, as originally shown in Procesi and Schwarz (1985). Here, we sketch an alternative approach based purely on algebraic considerations, which could lead to a direct reduction from the compact to the finite case.

When the group G is not finite, the field of invariants $\mathbb{R}(X)^G$ is no longer algebraic over $\mathbb{R}(X)$. That is, there exist transcendental elements $T_1, \dots, T_k \in \mathbb{R}(X)$ such that the extension $\mathbb{R}(X) \mid \mathbb{R}(X)^G[T_1, \dots, T_k]$ is algebraic, and $\mathbb{R}(X)^G[T_1, \dots, T_k] \mid \mathbb{R}(X)^G$ is purely transcendental. Without loss of generality, we may assume $T_i = X_i$ for $i = 1, \dots, k$. Then,

$$\mathbb{R}(X)^G[X_1, \dots, X_k] = \mathbb{R}(X)^H,$$

where $H = \text{Stab}(X_1, \dots, X_k)$, and H is finite since the extension $\mathbb{R}(X) \mid \mathbb{R}(X)^H$ is algebraic.

To extend a given homomorphism $\phi : \mathbb{R}[X]^G \rightarrow \mathbb{R}$, one may define $\phi_H : \mathbb{R}[X]^H \rightarrow \mathbb{R}$ by choosing values $x_1, \dots, x_k \in \mathbb{R}$ freely and setting

$$\phi_H(X_i) = x_i \quad \text{and} \quad \phi_H(f) = \phi(f)$$

for all $f \in \mathbb{R}[X]^G$. If one could always choose the x_i such that the corresponding *Procesi–Schwarz matrix* $M_{X_1, \dots, X_k, \Pi}$, defined with respect to H , is positive semi-definite, then this would reduce the compact case to the finite case.

Although we have not yet succeeded in proving this rigorously, such a reduction would provide an elegant algebraic approach toward a real algebraic proof of the Procesi and Schwarz Theorem beyond the finite setting.

4.2. A constructive variant of Bröcker’s theorem for the symmetric group

We have been able to verify Bröcker’s result constructively for certain classes of groups. However, the techniques developed here do not appear to directly apply to some other interesting cases. For example, note that the symmetric group S_n contains the broad subgroup

$$H_n := \left\langle (1, 2), (3, 4), \dots, \left(2 \lfloor \frac{n}{2} \rfloor - 1, 2 \lfloor \frac{n}{2} \rfloor \right) \right\rangle$$

for all n , as noted in Remark 17. Still, our current methods do not yield a generic description of the orbit space \mathbb{R}^n/S_n using $\lfloor \frac{n}{2} \rfloor$ inequalities for $n \geq 4$.

By Theorem 27, we know that \mathbb{R}^n/H_n is described by the $\lfloor \frac{n}{2} \rfloor$ polynomials

$$f_1 := (X_1 - X_2)^2, \quad f_2 := (X_3 - X_4)^2, \quad \dots, \quad f_{\lfloor \frac{n}{2} \rfloor} := (X_{2\lfloor \frac{n}{2} \rfloor - 1} - X_{2\lfloor \frac{n}{2} \rfloor})^2.$$

Since H_n is a broad subgroup, Lemma 15 implies that every homomorphism in $\text{hom}(\mathbb{R}[X]^{S_n}, \mathbb{R})$ can be extended to a homomorphism in $\text{hom}(\mathbb{R}[X]^{H_n}, \mathbb{R})$.

A natural approach might be to symmetrize the polynomials $f_1, \dots, f_{\lfloor \frac{n}{2} \rfloor}$ in such a way that their vanishing set remains unchanged. This could potentially yield a description of \mathbb{R}^n/H_n using S_n -invariant polynomials, and hence also describe \mathbb{R}^n/S_n . However, we have not yet succeeded in producing such a symmetrization.

Scheiderer has mentioned to us a construction that generically describes \mathbb{R}^4/S_4 using only two inequalities, under the action of S_4 by permutation. We include this example with his permission.

Example 29. The invariant ring $\mathbb{R}[X]^{S_4}$ is generated by the elementary symmetric polynomials e_1, e_2, e_3, e_4 . Denote the corresponding Hilbert map by E . Then $z \in E(\mathbb{R}^4)$ if and only if the univariate polynomial

$$f = T^4 - z_1 T^3 + z_2 T^2 - z_3 T + z_4$$

has only real roots. It is well known that the nature of the roots of f is governed by the eigenvalues of its Hermite matrix H_f . The number of real roots of f is equal to the signature of H_f , which is generically determined by the number of sign changes in the sequence p_1, p_2, p_3, p_4 , where p_i is the i -th leading principal minor of H_f . Since $p_1 = 4$ is constant, the matrix H_f is positive definite if and only if $p_2 p_4 > 0$ and $p_3 > 0$. Therefore, the orbit space \mathbb{R}^4/S_4 is generically described by the two inequalities $p_2 p_4 > 0$ and $p_3 > 0$.

This example can be generalized to obtain $n - 2$ inequalities that generically describe the orbit space \mathbb{R}^n/S_n . Nevertheless, it seems promising that a refined approach may lead to a description using only $\lfloor \frac{n}{2} \rfloor$ inequalities, which would match the bound given by Bröcker’s theorem. This setup is particularly interesting, as the orbit space \mathbb{R}^n/S_n can be interpreted as the space of hyperbolic univariate polynomials.

4.3. Sums of squares approaches

Another promising direction concerns the relationship between the sums of squares approach developed in Section 2 and general descriptions of orbit spaces. In Procesi and Schwarz (1985, Theorem 4.9), the orbit space is described using covariants. Our Theorem 6 encompasses this case naturally, as covariants form a natural subset of the ring extension employed to express sums of squares.

However, our current description often includes redundant inequalities. It would be interesting to investigate whether these redundancies can be systematically eliminated to yield more compact or even minimal representations.

In particular, the matrix polynomial M_Π induces a quadratic module in the invariant ring:

$$\{\text{Tr}(A \cdot M_\Pi) \mid A \text{ is a matrix polynomial that is a sum of invariant squares}\} \subseteq \mathbb{R}[X]^G.$$

This quadratic module lies within the cone of invariant sums of squares, but may in fact be significantly smaller. Quantifying this difference is a natural and potentially fruitful question, particularly in view of applications where sparse or minimal representations are critical.

CRediT authorship contribution statement

Philippe Moustrou: Writing – review & editing, Writing – original draft, Investigation. **Cordian Riener:** Writing – review & editing, Writing – original draft, Investigation. **Robin Schabert:** Writing – review & editing, Writing – original draft, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

This work has been supported by the Tromsø Research Foundation (grant agreement 17matteCR). The authors would like to express their gratitude to Igor Klep, Claus Scheiderer, and Markus Schweighofer for fruitful discussions. We also sincerely thank the four referees for their valuable feedback on an earlier version of the manuscript, which significantly improved the presentation.

Data availability

No data was used for the research described in the article.

References

- Abud, M., Sartori, G., 1983. The geometry of spontaneous symmetry breaking. *Ann. Phys.* 150 (2), 307–372.
- Andradas, Carlos, Bröcker, Ludwig, Ruiz, Jesús M., 2012. *Constructible Sets in Real Geometry*, vol. 33. Springer Science & Business Media.
- Atiyah, Michael, 2018. *Introduction to Commutative Algebra*. CRC Press.
- Blekherman, Grigoriy, Riener, Cordian, 2021. Symmetric non-negative forms and sums of squares. *Discrete Comput. Geom.* 65, 764–799.
- Bröcker, Ludwig, 1991. On basic semialgebraic sets. *Expo. Math. (ISSN 0723-0869)* 9 (4), 289–334.
- Bröcker, Ludwig, 1998. On symmetric semialgebraic sets and orbit spaces. *Banach Cent. Publ.* 44 (1), 37–50.
- Cimprič, Jaka, Kuhlmann, Salma, Scheiderer, Claus, 2009. Sums of squares and moment problems in equivariant situations. *Trans. Am. Math. Soc.* 361 (2), 735–765.
- Debus, Sebastian, Riener, Cordian, 2023. Reflection groups and cones of sums of squares. *J. Symb. Comput.* 119, 112–144.
- Derksen, Harm, Kemper, Gregor, 2002. *Computational Invariant Theory*. *Encyclopaedia of Mathematical Sciences*, vol. 130. Springer.
- Dubrovin, Boris, 1998. Differential geometry of the space of orbits of a Coxeter group, 181–211. In: *Surveys in Differential Geometry IV: Integrable Systems*. International Press, Boston.
- Fernando, José F., 2014. On the one dimensional polynomial and regular images of R^n . *J. Pure Appl. Algebra (ISSN 0022-4049)* 218 (9), 1745–1753. <https://doi.org/10.1016/j.jpaa.2014.01.011>. <https://www.sciencedirect.com/science/article/pii/S0022404914000127>.
- Field, Michael, 2007. *Dynamics and Symmetry*, vol. 3. World Scientific.
- Gattermann, Karin, Parrilo, Pablo A., 2004. Symmetry groups, semidefinite programs, and sums of squares. *J. Pure Appl. Algebra* 192 (1–3), 95–128.
- Guralnick, Robert M., Robinson, Geoffrey R., 2022. Commuting involutions and elementary abelian subgroups of simple groups. *J. Algebra* 607, 300–314.
- Helgason, Sigurdur, 1984. *Groups & Geometric Analysis: Radon Transforms, Invariant Differential Operators and Spherical Functions*, vol. 1. Academic Press.
- Hubert, Evelyne, Bazan, Erick Rodriguez, 2022. Algorithms for fundamental invariants and equivariants of finite groups. *Math. Comput.* 91 (337), 2459–2488.
- Huisman, Johannes, 1999. Real quotient singularities and nonsingular real algebraic curves in the boundary of the moduli space. *Compos. Math.* 118 (1), 43–60.
- Moustrou, Philippe, Riener, Cordian, Verdure, Hugues, 2023. Symmetries in polynomial optimization. In: Kočvara, Michal, Mourrain, Bernard, Riener, Cordian (Eds.), *Polynomial Optimization, Moments, and Applications*. Springer Nature Switzerland, Cham. ISBN 978-3-031-38659-6, pp. 53–111.
- Procesi, Claudio, 2007. *Lie Groups: An Approach Through Invariants and Representations*. Springer.
- Procesi, Claudio, Schwarz, Gerald, 1985. Inequalities defining orbit spaces. *Invent. Math.* 81 (3), 539–554.
- Procesi, Claudio, Schwarz, Gerald, 1988. Defining orbit spaces by inequalities. *Banach Cent. Publ.* 20 (1), 365–372.
- Riener, Cordian, et al., 2013. Exploiting symmetries in SDP-relaxations for polynomial optimization. *Math. Oper. Res.* 38 (1), 122–141.
- Scheiderer, Claus, 1989. Stability index of real varieties. *Invent. Math.* 97 (3), 467–483.