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On the forward–backward method with nonmonotone linesearch for infinite-dimensional nonsmooth nonconvex problems

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Abstract

This paper provides a comprehensive study of the nonmonotone forward–backward splitting (FBS) method for solving a class of nonsmooth composite problems in Hilbert spaces. The objective function is the sum of a Fréchet differentiable (not necessarily convex) function and a proper lower semicontinuous convex (not necessarily smooth) function. These problems appear, for example, frequently in the context of optimal control of nonlinear partial differential equations (PDEs) with nonsmooth sparsity-promoting cost functionals. We discuss the convergence and complexity of FBS equipped with the nonmonotone linesearch under different conditions. In particular, R-linear convergence will be derived under quadratic growth-type conditions. We also investigate the applicability of the algorithm to problems governed by PDEs. Numerical experiments are also given that justify our theoretical findings.

Keywords Nonsmooth nonconvex optimization · Forward–backward algorithm · Infinite-dimensional problems · Nonmonotone linesearch · Quadratic growth · PDE-constrained optimization

Mathematics Subject Classification 90C26 · 49M41 · 65K05 · 65K15 · 49M37 · 65J22

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1 Introduction

In this work, we are concerned with the following composite problem

$$\min_{u \in H} \ \Psi(u) := \mathcal{F}(u) + \mathcal{R}(u), \tag{P}$$

where H is a real Hilbert space, $\mathcal{F}$ is a continuously Fréchet differentiable function (possibly nonconvex), and $\mathcal{R}$ is a convex function whose proximal operator is assumed to be explicitly computable. The precise details will be introduced in Sect. 2. Problems of the form (**P**) appear in several fields of application such as optimal control problems [1, 2], system identification, signal, and image processing [3], machine learning, and statistics [4].

Arguably one of the most well-known algorithms for solving problem (**P**) is forward–backward splitting (FBS), also known as the proximal gradient method [3, 5], which is the generalization of the classical gradient method for problems with an additional nonsmooth term (see (7)). The iterations of FBS are defined by

$$u_{k+1} = \text{Prox}_{\frac{1}{\alpha_k}\mathcal{R}}\left(u_k - \frac{1}{\alpha_k}\nabla\mathcal{F}(u_k)\right) \quad \text{for } k \in \mathbb{N}_0, \tag{1}$$

where the step-sizes $\alpha_k > 0$ are supposed to be chosen in a way that guarantees convergence of the algorithm and accelerates it. It is known that the global convergence of FBS is to be sublinear of order $(1/k)$ for the convex case [5], where k stands for the number of iterations. This order can be improved to $(1/k^2)$ using an inertial variant of the algorithm based on Nesterov's accelerated techniques [6, 7]. Convergence of the iterates of FBS to a critical point of problem (**P**), even for the nonconvex case, has been shown for functions Ψ satisfying the Kurdyka–Łojasiewicz property e.g., [8–12] or quadratic growth error bounds e.g., [13–16]. Due to the simplicity and efficiency of FBS, its convergence, complexity, and applicability have been extensively studied, making it an ongoing area of active research. Numerous step-size strategies have been proposed and analyzed within the context of FBS under various assumptions and structures. For structured nonconvex finite-dimensional problems, works such as [17–19] have focused on the inertial method and constant step-size strategies, while [20–25] have explored linesearch strategies based on function evaluations, as well as quasi-Newton and Newton-type directions. Additionally, for problem (**P**) posed in an infinite-dimensional Hilbert space, references such as [13, 26–31] have investigated FBS assuming the convexity of $\mathcal{F}$, with [31] also addressing D.C. programming. In case where $\mathcal{F}$ is not necessarily convex, we can also mention [32, 33], which explore a class of inexact trust region methods to ensure the convergence of FBS and apply them to problems governed by PDEs.

As is well-known for smooth problems, nonmonotone linesearch strategies appear to be numerically efficient in situations where a monotone scheme is forced to propagate along the bottom of a narrow, curved valley. Additionally, because they allow for increases in function values, they can be effectively combined with spectral gradient methods, such as the Barzilai-Borwein (BB) step-sizes [34, 35], which naturally exhibit nonmonotone behavior. Building on the nonmonotone approach developed by

Grippo et al. [36], Wright et al. [37] proposed a nonmonotone FBS for nonsmooth convex composite problems posed in $\mathbb{R}^n$. The convergence of this approach, known as SpaRSA, was further studied in [38], where the authors demonstrated the order of $(1/k)$ convergence and R-linear convergence for convex and strongly convex functions, respectively. Very recently, in [39], the authors proved convergence to a critical point of this scheme for finite-dimensional nonsmooth nonconvex composite functions under milder conditions, specifically, local Lipschitz continuity of the gradient for the smooth part and uniform continuity for the objective function on its level sets.

From a computational perspective, both classical gradient and FBS methods can be accelerated if incorporated with the BB step-sizes. These step-sizes approximate the curvature of the Hessian for the smooth part of the objective function and are particularly efficient in the context of PDE-constrained optimization [34, 40]. For strongly quadratic functions $\mathcal{F}$, R-linear convergence of the BB step-sizes has been established for (**P**) provided the condition number of the quadratic operator is sufficiently small (< 2); see [34, Remark 3.4]. However, even for strongly quadratic functions $\mathcal{F}$ with condition numbers larger than 2, convergence of the BB step-sizes is not clear. Thus, to guarantee convergence, one needs to combine the BB step-sizes with a nonmonotone linesearch, which relies on function evaluations.

Weak and strong convergence can be distinguished only in the infinite-dimensional setting. In practice, numerical solutions to infinite-dimensional problems are typically obtained by implementing algorithms for finite-dimensional approximations. However, analyzing the convergence of these algorithms in the infinite-dimensional context is crucial for ensuring the numerical robustness and stability of the finite-dimensional approximations. Such properties are expressed by the so-called mesh-independent principle (MIP); see, e.g., [40–46]. Roughly speaking, MIP uses results from infinite-dimensional convergence to predict the convergence properties of finite-dimensional approximations (discretized problems). Additionally, MIP offers a theoretical foundation for developing refinement strategies; see, e.g., [47].

In light of the above discussion, we investigate the well-posedness, convergence, and complexity of the nonmonotone FBS for problems of the form (**P**) posed in infinite-dimensional Hilbert spaces, under various assumptions such as nonconvexity, convexity, and quadratic growth-type conditions. These results also apply to finite-dimensional problems, where our convergence and complexity analyses and assumptions differ from those in previous works [37–39]. Notably, the convergence of this approach under quadratic growth-type conditions, and its worst-case evaluation complexity for reaching an approximate stationary point have not been previously investigated. We will clearly outline our contributions in the next section.

In particular, we investigate the applicability of the nonmonotone FBS for problems governed by PDEs. Despite its numerical efficiency, to the best of our knowledge, this approach has not yet been studied for infinite-dimensional problems. For these problems, due to the lack of compactness in the strong topology, we need to explore convergence in the weak topology, which involves studying notions of sequential continuity for some operators with respect to the weak topology. Here, the nonmonotonicity of the approach, and working with the weak topology are two issues that we overcome with our analysis using the Lipschitz continuity of the gradient mapping (see Definition 1). We will also highlight the challenges and differences that arise when considering infinite-dimensional spaces after each proof. Additionally, we investigate and discuss the applicability of our assumptions and, consequently, our results to two examples of nonsmooth nonconvex problems with PDEs.

Contributions

More precisely, the contributions of this work can be summarized as follows:

(i) Starting with the nonconvex case, we prove well-posedness of the algorithm even without Lipschitz continuity of the gradient of $\mathcal{F}$. Under the global Lipschitz continuity of the gradient of $\mathcal{F}$, we prove the global convergence of the algorithm with complexity $(1/\sqrt{k})$. To be more precise, we show that the norm of the prox-gradient mapping of iterates vanishes with complexity $(1/\sqrt{k})$. We also establish that every weak sequential cluster point of iterates is a stationary point.

(ii) We derive the worst-case evaluation complexity of finding an ε_{tol}-stationary point. More precisely, we give estimates on the maximal number of the objective function and prox-grad operator evaluations for computing an approximate stationary point with a user-defined accuracy threshold $\varepsilon_{\text{tol}} > 0$.

(iii) In the convex setting, relying on the concept of quasi-Fejer sequences, we are able to extend previously established results to global convergence, both in terms of function values and iterates with respect to the weak sequential topology. Further, we show that the convergence is sublinear of order $(1/k)$ in function values.

(iv) Under quadratic growth-type conditions, we show global R-linear convergence, both in terms of function values and iterates. The proof of the latter is more delicate for infinite-dimensional problems since the transition from weak sequential convergence to strong convergence is not straightforward.

(v) Finally, aiming at optimization problems governed by PDEs, we discuss the validity of the convergence results of the algorithm without strong Lipschitz continuity of $\nabla\mathcal{F}$. Our theoretical framework is supported by two nonsmooth nonconvex problems governed by PDEs, including semilinear elliptic and parabolic equations. We also show that our results are applicable to these problems and report on related numerical experiments.

Outline of the paper

The rest of this paper is organized as follows: Sect. 2 presents the preliminaries, assumptions on the optimization problem (**P**), the algorithm, and the nonmonotone linesearch strategy. Section 3 investigates the convergence and complexity of the algorithm comprehensively under different conditions. Section 4 discusses the applicability of the results from the previous section to PDE-constrained optimization problems. Finally, numerical experiments are reported in Sect. 5 that justify our theoretical findings. To improve the readability of the paper, we provide proof of some results from Sect. 3 in Appendix A.

Notation

Throughout this paper, the Hilbert space H is endowed with the scalar product $(\cdot,\cdot)_H$ and the induced norm $\|\cdot\|_H$. For a radius $r > 0$ and $\bar{u} \in H$, we define $\mathbf{B}_r(\bar{u}) := \{u \in H : \|u - \bar{u}\|_H < r\}$. We also denote with $P_S : H \to H$, the orthogonal projection onto the set $S \subset H$. Further for every $\tilde{\Psi} \in \mathbb{R}$, we define $\left[\Psi \le \tilde{\Psi}\right] := \{u \in H : \Psi(u) \le \tilde{\Psi}\}$. By Argmin Ψ we denote the set of minimizers of Ψ. Sometimes we will use set-valued (in-)equalities, i.e. $A \le B$ for $A, B \subset H$, meaning that $a \le b$ for all $a \in A$ and $b \in B$.

We call a sequence $\{u_k\}_k \subset H$ quasi-Féjer monotone with respect to a non-empty set $S \subset H$, if for every $v \in S$ it holds that

$$\|u_{k+1} - v\|_H^2 \le \|u_k - v\|_H^2 + \epsilon_k,$$

where $\{\epsilon_k\}_k \subset \mathbb{R}_{\ge 0}$ is a summable sequence, i.e., $\sum_{k=0}^{\infty} \epsilon_k < \infty$. To avoid confusion in the notation, the derivative and gradient of $\mathcal{F}$ are defined by $\mathcal{F}' \colon H \to H'$ and $\nabla\mathcal{F} \colon H \to H$, respectively. In this case, for every $u \in H$ we identify $\nabla\mathcal{F}(u) \in H$ by the unique Riesz representative of $\mathcal{F}'(u) \in H'$, with H' denoting the dual space of H. We also use the notion of the convex subdifferential for convex function $\phi : H \to \mathbb{R} \cup \{+\infty\}$ at $u \in \operatorname{dom}\phi := \{u \in H : \phi(u) < +\infty\}$ defined as $\partial\phi(u) := \{w \in H : \phi(v) - \phi(u) \ge (w, v - u)_H \text{ for all } v \in H\}$.

2 Problem formulation and algorithm

In this section, the precise theoretical framework and the algorithm will be introduced. First, we state the precise assumptions on (**P**).

Assumption 1 For problem (**P**),

A1: $\mathcal{R} \colon H \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous.
A2: $\mathcal{F} \colon H \to \mathbb{R} \cup \{+\infty\}$ is continuously Fréchet differentiable on $\operatorname{int}(\operatorname{dom}\mathcal{F})$ containing $\operatorname{dom}\mathcal{R}$, that is, $\operatorname{dom}\mathcal{R} \subseteq \operatorname{int}(\operatorname{dom}\mathcal{F})$.
A3: $\nabla\mathcal{F} \colon \operatorname{int}(\operatorname{dom}\mathcal{F}) \to H$ is globally $L_{\mathcal{F}'}$-Lipschitz continuous.
A4: The gradient $\nabla\mathcal{F} \colon \operatorname{int}(\operatorname{dom}\mathcal{F}) \to H$ is weak-to-strong sequentially continuous.

Conditions **A1**–**A2** are standard in order to guarantee the well-posedness of the algorithm. **A3** will be used to derive complexity results for iterations and we will discuss the relaxation of this condition for a large class of problems governed by PDEs later in Sect. 3.4. We use **A4** to show that every weak sequential accumulation point of iterates belongs to S_*. Note that, if $\dim(H) < \infty$, **A4** follows from **A2**.

According to Assumption 1, the Fermat principle [48, Proposition 9.1.5] for (**P**) reads as follows: If $u^* \in H$ is a local minimizer of Ψ, then

$$-\nabla\mathcal{F}(u^*) \in \partial\mathcal{R}(u^*). \tag{2}$$

Further, with similar arguments as in the proof of [26, Theorem 26.2], the condition (2) can be equivalently expressed as

$$u^* = \operatorname{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u^* - \frac{1}{\alpha}\nabla\mathcal{F}(u^*)) \quad \text{for some } \alpha > 0, \tag{3}$$

where the proximal operator $\operatorname{Prox}_{\frac{1}{\alpha}\mathcal{R}} : H \to H$ is defined by

$$\operatorname{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u) := \operatorname*{argmin}_{v\in H}\left(\mathcal{R}(v) + \frac{\alpha}{2}\|v-u\|_H^2\right).$$

This operator is well-defined due to **A1**. We also define the set of critical points by

$$S_* := \{u \in H : -\nabla\mathcal{F}(u) \in \partial\mathcal{R}(u)\}.$$

Now we turn our attention towards formalizing (1) and the proposed step-size update strategy by a nonmonotone linesearch method. Therefore we introduce the prox-grad operator and the gradient mapping in the Hilbert space setting analogously to, e.g., [5, 14].

Definition 1 For every $\alpha \in \mathbb{R}_{>0}$, we define

1. the *prox-grad* operator $\mathcal{T}_\alpha\colon \operatorname{int}(\operatorname{dom}\mathcal{F}) \to \operatorname{dom}\mathcal{R}$ with $u \mapsto \operatorname{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u - \frac{1}{\alpha}\nabla\mathcal{F}(u))$.
2. the *gradient mapping* $\mathcal{G}_\alpha\colon \operatorname{int}(\operatorname{dom}\mathcal{F}) \to H$ with $u \mapsto \alpha(u - \mathcal{T}_\alpha(u))$.

Due to Definition 1 and by some simple computations, we obtain for every $u \in \operatorname{int}(\operatorname{dom}\mathcal{F})$ that

$$\mathcal{G}_\alpha(u) - \nabla\mathcal{F}(u) \in \partial\mathcal{R}(\mathcal{T}_\alpha(u)). \tag{4}$$

Further, if we define for every $u \in \operatorname{int}(\operatorname{dom}\mathcal{F})$, $w \in H$, and $\alpha \in \mathbb{R}_{>0}$

$$\mathcal{Q}_\alpha(w, u) := \mathcal{F}(u) + (\nabla\mathcal{F}(u), w-u)_H + \frac{\alpha}{2}\|w-u\|_H^2 + \mathcal{R}(w),$$

then $\mathcal{T}_\alpha(u)$ is the unique minimizer of $\mathcal{Q}_\alpha(\cdot, u)$, i.e.

$$\mathcal{T}_\alpha(u) = \operatorname*{arg\,min}_{w\in H}\mathcal{Q}_\alpha(w, u).$$

As a consequence, we can write

$$(\nabla\mathcal{F}(u), \mathcal{T}_\alpha(u) - u)_H + \frac{1}{2\alpha}\|\mathcal{G}_\alpha(u)\|_H^2 + \mathcal{R}(\mathcal{T}_\alpha(u)) \le \mathcal{R}(u). \tag{5}$$

By means of the prox-grad operator and the gradient mapping, the iterations (1) can be reformulated as follows:

$$u_{k+1} = \mathcal{T}_{\alpha_k}(u_k) \qquad \text{for } k \in \mathbb{N}_0, \tag{6}$$

$$u_{k+1} = u_k - \frac{1}{\alpha_k}\mathcal{G}_{\alpha_k}(u_k) \qquad \text{for } k \in \mathbb{N}_0. \tag{7}$$

In this case, for every $u_0 \in \operatorname{int}(\operatorname{dom}\mathcal{F})$, the iterations are well-defined, that is $\{u_k\}_k \subset \operatorname{dom}\mathcal{R} \subseteq H$. Further, due to (7), we can see the analogy between the proximal gradient method and classical gradient descent for smooth minimization. Using (3) and the notion of the gradient mapping, we also can characterize the critical points of (**P**), as follows.

Proposition 1 *For $u^* \in \operatorname{int}(\operatorname{dom}\mathcal{F})$, it holds that $\mathcal{G}_\alpha(u^*) = 0$ for some $\alpha \in \mathbb{R}_{>0}$ if and only if $u^* \in S_*$.*

Therefore $\mathcal{G}_\alpha(\bar{u}) = 0$ defines a natural termination condition analogously to the smooth case.

There are many possible choices for the step-size α_k in our iterative scheme. In this work, we will consider step-size updates by a BB-type update rule or a combination of them with a nonmonotone linesearch. To be more precise, as the initial trial step-size within the nonmonotone linesearch, we will choose one of the spectral gradient BB-types step-sizes defined by

$$\alpha_k^{\text{BB1a}} := \frac{(u_k - u_{k-1}, \nabla\mathcal{F}(u_k) - \nabla\mathcal{F}(u_{k-1}))_H}{(u_k - u_{k-1}, u_k - u_{k-1})_H},$$

$$\alpha_k^{\text{BB2a}} := \frac{(\nabla\mathcal{F}(u_k) - \nabla\mathcal{F}(u_{k-1}), \nabla\mathcal{F}(u_k) - \nabla\mathcal{F}(u_{k-1}))_H}{(u_k - u_{k-1}, \nabla\mathcal{F}(u_k) - \nabla\mathcal{F}(u_{k-1}))_H},$$

$$\alpha_k^{\text{BB1b}} := \frac{(u_k - u_{k-1}, \mathcal{G}_{\alpha_{k-1}}(u_k) - \mathcal{G}_{\alpha_{k-1}}(u_{k-1}))_H}{(u_k - u_{k-1}, u_k - u_{k-1})_H},$$

$$\alpha_k^{\text{BB2b}} := \frac{(\mathcal{G}_{\alpha_{k-1}}(u_k) - \mathcal{G}_{\alpha_{k-1}}(u_{k-1}), \mathcal{G}_{\alpha_{k-1}}(u_k) - \mathcal{G}_{\alpha_{k-1}}(u_{k-1}))_H}{(u_k - u_{k-1}, \mathcal{G}_{\alpha_{k-1}}(u_k) - \mathcal{G}_{\alpha_{k-1}}(u_{k-1}))_H}.$$

The first two strategies correspond to the BB-method presented e.g. in [40]. The last two novel BB-methods modify the classical BB-method and try to incorporate full first-order information by using the gradient mapping and not only the gradient of $\mathcal{F}$. Furthermore, we will use so-called alternating BB-type update rules given by

$$\alpha^{\text{ABBa}} := \alpha^{\text{BB1a}} \text{ for } k \text{ even and } \alpha^{\text{BB2a}} \text{ for } k \text{ odd},$$

$$\alpha^{\text{ABBb}} := \alpha^{\text{BB1b}} \text{ for } k \text{ even and } \alpha^{\text{BB2b}} \text{ for } k \text{ odd}.$$

Remark 1 In the case of $\mathcal{R} = 0$, we have $\mathcal{G}_l(u) = \nabla\mathcal{F}(u)$ for every $l > 0$ and $u \in H$. Therefore, many of the introduced step-size updates above are identical, i.e. $\alpha^{\text{BB1b}} = \alpha^{\text{BB1a}}$, $\alpha^{\text{BB2b}} = \alpha^{\text{BB2a}}$, and $\alpha^{\text{ABBb}} = \alpha^{\text{ABBa}}$.

For the nonmonotone linesearch update, we consider

$$\Psi(u_{k+1}) \leq \max_{0 \leq j \leq m(k)} \Psi(u_{k-j}) - \frac{\delta}{\alpha_k}\|\mathcal{G}_{\alpha_k}(u_k)\|_H^2, \tag{8}$$

where $0 < \delta < 1$ and memory $m\colon \mathbb{N}_0 \to \mathbb{N}_0$ satisfies

$$m(0) = 0 \quad \text{and} \quad m(k) = \min\{m(k-1)+1, m_{\max}\} \text{ for } k \in \mathbb{N}, \tag{9}$$

with a given upper bound $m_{\max} \in \mathbb{N}_0$. Similarly to [49], we also define the functions $\ell : \mathbb{N}_0 \to \mathbb{N}_0$ and $\nu : \mathbb{N}_0 \to \mathbb{N}_0$ with

$$\ell(k) := k - \arg\max_{0 \le j \le m(k)} \Psi(u_{k-j}) \quad \text{for } k \ge 0 \text{ with } k - m(k) \le \ell(k) \le k,$$
$$\nu(k) := \ell(k m_{\max} + k) \quad \text{for } k \ge 0.$$

Thus, by these notations, we have $\max_{0 \le j \le m(k)} \Psi(u_{k-j}) = \Psi(u_{\ell(k)})$ and (8) can be rewritten as

$$\Psi(u_{k+1}) \le \Psi(u_{\ell(k)}) - \frac{\delta}{\alpha_k} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2. \tag{10}$$

Further we set

$$\alpha_k = \alpha_{\text{int},k} \eta^{i_k}, \tag{11}$$

where $\eta > 1$. The initial step-size $\alpha_{\text{int},k} > 0$ is chosen by the BB-method and a lower bound $\alpha_{\inf} > 0$ is given to ensure nonnegativity. To limit the initial step-size numerically, also an upper bound $\alpha_{\sup} > \alpha_{\inf}$ is employed. In the update, we choose the smallest integer $i_k \in \mathbb{N}_0$ in (11), such that (8) is satisfied. This procedure is summarized in Algorithm 1.

Algorithm 1 Nonmonotone FBS

Input: $0 < \delta < 1$, $m_{\max} \in \mathbb{N}_0$, $\eta > 1$, $\alpha_{\sup} > \alpha_{\inf} > 0$, $u_0 \in H$, and $\alpha_0 > 0$.
Output: A stationary point $u^* \in H$ of (**P**).

Set $k = 0$;
while $\|\mathcal{G}_{\alpha_k}(u_k)\|_H > 0$ **do**
  if $k = 0$ **then**
    Set $\alpha_{\text{int},k} := \max\{\alpha_{\inf}, \min\{\alpha_{\sup}, \alpha_0\}$;
  else
    Compute α_k^{BB} according to a BB-method and set

$$\alpha_{\text{int},k} := \max\{\alpha_{\inf}, \min\{\alpha_{\sup}, \alpha_k^{\text{BB}}\}\};$$

  end if
  Set $\alpha_k = \alpha_{\text{int},k} \eta^{i_k}$, where $i_k \ge 0$ is the smallest integer for which (8) holds;
  Set $u_{k+1} = u_k - \frac{1}{\alpha_k} \mathcal{G}_{\alpha_k}(u_k)$ and $k = k + 1$;
end while

Note that by Proposition 1, the criterion $\|\mathcal{G}_{\alpha_k}(u_k)\|_H \le \varepsilon_{\text{tol}}$ with some tolerance $\varepsilon_{\text{tol}} > 0$ is a reasonable stopping criterion in the numerical realization of Algorithm 1.

3 Convergence and complexity analysis

In this section, we present a detailed convergence and complexity analysis for Algorithm 1 under different assumptions and conditions.

3.1 General case

We start by summarizing useful properties of the gradient mapping.

Lemma 2 *Suppose that* ***A1–A2*** *hold, then we have the following properties:*

P1 For every $l_1 \geq l_2 > 0$ *and* $u \in \operatorname{int}(\operatorname{dom} \mathcal{F})$ *it holds that* $\frac{1}{l_1}\|\mathcal{G}_{l_1}(u)\|_H \leq \frac{1}{l_2}\|\mathcal{G}_{l_2}(u)\|_H$.

P2 For every $l_1 \geq l_2 > 0$ *and* $u \in \operatorname{int}(\operatorname{dom} \mathcal{F})$ *it holds that* $\|\mathcal{G}_{l_1}(u)\|_H \geq \|\mathcal{G}_{l_2}(u)\|_H$.

P3 Assume in addition that ***A3*** *holds, then the gradient mapping is Lipschitz continuous, that is*

$$\|\mathcal{G}_l(u) - \mathcal{G}_l(v)\|_H \leq (2l + L_{\mathcal{F}'})\|u - v\|_H,$$

for every $l > 0$ *and* $v, u \in \operatorname{int}(\operatorname{dom} \mathcal{F})$.

Proof The proof follows using the same arguments, e.g., given in [5, Theorem 10.9, Lemma 10.10] for finite-dimensional problems. □

Furthermore, the well-known sufficient decrease condition can be formulated for the gradient mapping analogously to the finite-dimensional case presented in [5, Lemma 10.4].

Lemma 3 (Sufficient decrease lemma) *Suppose that* ***A1–A3*** *hold, then for every* $u \in \operatorname{int}(\operatorname{dom} \mathcal{F})$ *and* $l \in (\frac{L_{\mathcal{F}'}}{2}, \infty)$*, we have*

$$\Psi(\mathcal{T}_l(u)) \leq \Psi(u) - \frac{l - \frac{L_{\mathcal{F}'}}{2}}{l^2}\|\mathcal{G}_l(u)\|_H^2.$$

The previous lemmas allow us to summarize some important properties of Algorithm 1.

Lemma 4 *Suppose that* ***A1–A2*** *hold. Then for every* $k \geq 0$*, the following statements hold true:*

(i) For $\delta \in (0, \frac{1}{2})$*, the nonmonotone linesearch is well-defined. That is, there exists* $\underline{\alpha} > 0$ *such that for every* $u_k \in \operatorname{int}(\operatorname{dom} \mathcal{F})$ *and* $\alpha_k \in [\underline{\alpha}, \infty)$ *the nonmonotone rule* (8) *holds and thus, the nonmonotone linesearch terminates after finitely many iterations.*

Assume, in addition, ***A3*** *holds.*

(i) Then the statement of (i) is true for $\underline{\alpha} := \frac{L_{\mathcal{F}'}}{2(1-\delta)}$ *and every fixed* $\delta \in (0, 1)$.

(ii) The step-sizes are uniformly bounded from above. That is, for every $k \geq 1$*, we have* $\alpha_k \leq \overline{\alpha}$ *with* $\overline{\alpha} := \max\{\frac{\eta L_{\mathcal{F}'}}{2(1-\delta)}, \alpha_{\sup}\}$.

(iii) It holds $\|\mathcal{G}_{\alpha_{k+1}}(u_{k+1})\|_H \leq C_G \|\mathcal{G}_{\alpha_k}(u_k)\|_H$*, where* $C_G := \frac{3\overline{\alpha}+L_{\mathcal{F}'}}{\alpha_{\inf}}$.

Proof (i) If $u_k \in S_*$, then the claim clearly holds true for every $\underline{\alpha} > 0$. Thus, we assume that $u_k \notin S_*$. We suppose also on contrary there does not exist $\underline{\alpha} > 0$ for which the claim holds true. Then the nonmonotone linesearch generates a sequence of step-sizes $\alpha_{k,i} := \alpha_{\mathrm{int},k}\eta^i$ with $i \in \mathbb{N}_0$ satisfying $\alpha_{k,i} \to \infty$ and

$$\frac{\delta}{\alpha_{k,i}}\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H^2 > \Psi(u_{\ell(k)}) - \Psi(\mathcal{T}_{\alpha_{k,i}}(u_k)) \geq \Psi(u_k) - \Psi(\mathcal{T}_{\alpha_{k,i}}(u_k)). \tag{12}$$

First, we note that $\mathcal{T}_{\alpha_{k,i}}(u_k) \to u_k$ as $i \to \infty$. This follows from the fact that $\mathrm{Prox}_{\frac{1}{\alpha_{k,i}}\mathcal{R}}(u_k) \to u_k$ as $i \to \infty$ (see e.g., [26, Theorem 23.47]) and

$$\begin{aligned}\|\mathcal{T}_{\alpha_{k,i}}(u_k) - u_k\|_H &\leq \|\mathrm{Prox}_{\frac{1}{\alpha_{k,i}}\mathcal{R}}(u_k - \tfrac{1}{\alpha_{k,i}}\nabla\mathcal{F}(u_k)) - \mathrm{Prox}_{\frac{1}{\alpha_{k,i}}\mathcal{R}}(u_k)\|_H \\ &+ \|\mathrm{Prox}_{\frac{1}{\alpha_{k,i}}\mathcal{R}}(u_k) - u_k\|_H \leq \|\mathrm{Prox}_{\frac{1}{\alpha_{k,i}}\mathcal{R}}(u_k) - u_k\|_H + \tfrac{1}{\alpha_{k,i}}\|\nabla\mathcal{F}(u_k)\|_H,\end{aligned}$$

where we have used the firm nonexpansiveness of the proximal operator. Further, using (12) and the mean value theorem for $\mathcal{F}$, we obtain for every $i \in \mathbb{N}_0$, that

$$\begin{aligned}&\tfrac{\delta}{\alpha_{k,i}}\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H^2 \geq \Psi(u_k) - \Psi(\mathcal{T}_{\alpha_{k,i}}(u_k)) \\ &\quad\geq \mathcal{R}(u_k) - \mathcal{R}(\mathcal{T}_{\alpha_{k,i}}(u_k)) + (\nabla\mathcal{F}(u_k + t_i(\mathcal{T}_{\alpha_{k,i}}(u_k) - u_k)), u_k - \mathcal{T}_{\alpha_{k,i}}(u_k))_H,\end{aligned}$$

where $t_i \in (0, 1)$ for all $i \in \mathbb{N}_0$. Using (5) we obtain that

$$\begin{aligned}&\tfrac{\delta}{\alpha_{k,i}}\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H^2 \geq \tfrac{1}{2\alpha_{k,i}}\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H^2 \\ &\quad+ \left(\nabla\mathcal{F}\left(u_k + t_i(\mathcal{T}_{\alpha_{k,i}}(u_k) - u_k)\right) - \nabla\mathcal{F}(u_k), \mathcal{T}_{\alpha_{k,i}}(u_k) - u_k\right)_H \\ &\quad\geq \tfrac{1}{2\alpha_{k,i}}\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H^2 - \tfrac{1}{\alpha_{k,i}}\|\nabla\mathcal{F}\left(u_k + t_i(\mathcal{T}_{\alpha_{k,i}}(u_k) - u_k)\right) - \nabla\mathcal{F}(u_k)\|_H\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H.\end{aligned}$$

Together with the fact that $\delta < \frac{1}{2}$, we obtain

$$(\tfrac{1}{2} - \delta)\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H \leq \|\nabla\mathcal{F}\left(u_k + t_i(\mathcal{T}_{\alpha_{k,i}}(u_k) - u_k)\right) - \nabla\mathcal{F}(u_k)\|_H.$$

Sending $i \to \infty$ and using the continuity of $\nabla\mathcal{F}$, we obtain that

$$\lim_{i\to\infty}\|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H = 0.$$

Finally, using P2, we can infer that $\|\mathcal{G}_{\alpha_{\inf}}(u_k)\|_H \leq \|\mathcal{G}_{\alpha_{k,i}}(u_k)\|_H$ and thus $\|\mathcal{G}_{\alpha_{\inf}}(u_k)\|_H = 0$. Now Proposition 1 implies $u_k \in S_*$. This contradicts our assumption, which concludes the proof.

(ii) For every given $\alpha_k \geq \underline{\alpha} > \frac{L_{\mathcal{F}'}}{2}$ we can invoke Lemma 3 and write

$$\begin{aligned}\Psi(\mathcal{T}_{\alpha_k}(u_k)) &\leq \Psi(u_k) - \frac{\alpha_k - \frac{L_{\mathcal{F}'}}{2}}{\alpha_k^2}\|\mathcal{G}_{\alpha_k}(u_k)\|_H^2 \\ &\leq \max_{0\leq j\leq m(k)} \Psi(u_{k-j}) - \frac{\alpha_k - \frac{L_{\mathcal{F}'}}{2}}{\alpha_k^2}\|\mathcal{G}_{\alpha_k}(u_k)\|_H^2.\end{aligned}$$

Thus, (8) holds since α_k satisfies $\frac{\alpha_k - \frac{L_{\mathcal{F}'}}{2}}{\alpha_k^2} \geq \frac{\delta}{\alpha_k}$ by assumption.

(iii) To derive $\overline{\alpha}$, we consider the following cases:

- $i_k = 0$: In this case, due to (11), we have $\alpha_k = \alpha_{\text{int},k} \leq \alpha_{\text{sup}}$.
- $i_k \geq 1$: In this case, (8) holds for α_k and u_k. Then due to (11), we can write

$$\begin{aligned}\Psi(\mathcal{T}_{\frac{\alpha_k}{\eta}}(u_k)) &> \max_{0\leq j\leq m(k)} \Psi(u_{k-j}) - \frac{\delta\eta}{\alpha_k}\|\mathcal{G}_{\frac{\alpha_k}{\eta}}(u_k)\|_H^2 \\ &\geq \Psi(u_k) - \frac{\delta\eta}{\alpha_k}\|\mathcal{G}_{\frac{\alpha_k}{\eta}}(u_k)\|_H^2.\end{aligned} \tag{13}$$

Further, by assuming without loss of generality that $\frac{\alpha_k}{\eta} \geq \frac{L_{\mathcal{F}'}}{2}$, we can use Lemma 3 with $l = \frac{\alpha_k}{\eta}$ and $u = u_k$ to obtain

$$\Psi(\mathcal{T}_{\frac{\alpha_k}{\eta}}(u_k)) \leq \Psi(u_k) - \frac{\frac{\alpha_k}{\eta} - \frac{L_{\mathcal{F}'}}{2}}{(\frac{\alpha_k}{\eta})^2}\|\mathcal{G}_{\frac{\alpha_k}{\eta}}(u_k)\|_H^2. \tag{14}$$

Combining (13) and (14), we obtain $\frac{\delta\eta}{\alpha_k} > \frac{\frac{\alpha_k}{\eta} - \frac{L_{\mathcal{F}'}}{2}}{(\frac{\alpha_k}{\eta})^2}$ and as a consequence, $\alpha_k < \frac{\eta L_{\mathcal{F}'}}{2(1-\delta)}$.

Summarizing the two above cases, we can conclude the second part with $\overline{\alpha} := \max\{\frac{\eta L_{\mathcal{F}'}}{2(1-\delta)}, \alpha_{\text{sup}}\}$.

(iv) Finally for proving the last part, we use Lemma 2 and (iii) to obtain that

$$\begin{aligned}&\|\mathcal{G}_{\alpha_{k+1}}(u_{k+1})\|_H \overset{P2}{\leq} \|\mathcal{G}_{\overline{\alpha}}(u_{k+1})\|_H \leq \|\mathcal{G}_{\overline{\alpha}}(u_{k+1}) - \mathcal{G}_{\overline{\alpha}}(u_k)\|_H + \|\mathcal{G}_{\overline{\alpha}}(u_k)\|_H \\ &\overset{P3}{\leq} (2\overline{\alpha} + L_{\mathcal{F}'})\|u_{k+1} - u_k\|_H + \|\mathcal{G}_{\overline{\alpha}}(u_k)\|_H \overset{\text{Def. 1}}{\leq} \frac{(2\overline{\alpha} + L_{\mathcal{F}'})}{\alpha_k}\|\mathcal{G}_{\alpha_k}(u_k)\|_H + \|\mathcal{G}_{\overline{\alpha}}(u_k)\|_H \\ &\overset{P1}{\leq} \frac{(3\overline{\alpha} + L_{\mathcal{F}'})}{\alpha_k}\|\mathcal{G}_{\alpha_k}(u_k)\|_H \leq \frac{3\overline{\alpha} + L_{\mathcal{F}'}}{\alpha_{\text{inf}}}\|\mathcal{G}_{\alpha_k}(u_k)\|_H.\end{aligned}$$

Setting $C_G := \frac{3\overline{\alpha} + L_{\mathcal{F}'}}{\alpha_{\text{inf}}}$, the proof is complete. □

Before stating the main convergence result of this section, we present a lemma concerning the subsequence $\{u_{\nu(k)}\}_k$ and well-posedness of Algorithm 1.

Lemma 5 *Suppose that the sequences $\{u_k\}_k$ and $\{\alpha_k\}_k \subset \mathbb{R}_{>0}$ are generated by Algorithm 1. Then the following properties hold:*

L1 $\{u_{\nu(k)}\}_k$ *is a subsequence of* $\{u_k\}_k$ *with* $\nu(k) - \nu(k-1) \le 2m_{\max} + 1$ *and* $\nu(k) \le (m_{\max} + 1)k$*. Further, for every* $k \in \mathbb{N}_0$*, it holds* $\Psi(u_k) \le \Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)})$.

L2 *For every* $k \ge 1$*, we have*

$$\Psi(u_{\nu(k)}) \le \Psi(u_{\nu(k-1)}) - \tfrac{\delta}{\alpha_{\nu(k)-1}} \|\mathcal{G}_{\nu(k)-1}(u_{\nu(k)-1})\|_H^2, \tag{15}$$

and in particular the subsequence $\{\Psi(u_{\nu(k)})\}_k$ *is monotonically decreasing.*

L3 *Assume that* Ψ *is bounded from below, i.e.* $\inf_{u \in H} \Psi(u) > -\infty$*. Then we have*

$$\sum_{k=1}^{\infty} \tfrac{1}{\alpha_{\nu(k)-1}} \|\mathcal{G}_{\nu(k)-1}(u_{\nu(k)-1})\|_H^2 < \infty \quad \textit{and} \quad \liminf_{k\to\infty} \tfrac{1}{\alpha_k} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2 = 0. \tag{16}$$

In particular, if $m_{\max} = 0$*, we have*

$$\sum_{k=0}^{\infty} \tfrac{1}{\alpha_k} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2 < \infty \quad \textit{and} \quad \lim_{k\to\infty} \tfrac{1}{\alpha_k} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2 = 0. \tag{17}$$

Proof (L1) Using the fact that $\ell(k) \ge k - m_{\max}$ for every k, we obtain

$$\begin{aligned} \nu(k) &= \ell(km_{\max} + k) \ge km_{\max} + k - m_{\max} \\ &> (k-1)m_{\max} + (k-1) \ge \ell((k-1)m_{\max} + (k-1)) = \nu(k-1), \end{aligned}$$

and thus $\{u_{\nu(k)}\}_k \subset \{u_k\}_k$. Moreover, we have

$$\nu(k) = \ell(km_{\max} + k) \le km_{\max} + k,$$

and

$$\begin{aligned} \nu(k) - \nu(k-1) &= \ell(km_{\max} + k) - \ell((k-1)m_{\max} + (k-1)) \\ &\le km_{\max} + k - ((k-1)m_{\max} + (k-1) - m_{\max}) = 2m_{\max} + 1. \end{aligned}$$

Further, for a given $k \in \mathbb{N}_0$, we have $0 \le \left\lceil \frac{k}{m_{\max}+1} \right\rceil (m_{\max} + 1) - k \le m_{\max}$, and thus

$$\Psi(u_k) \le \Psi(u_{\ell(\lceil \frac{k}{m_{\max}+1} \rceil (m_{\max}+1))}) = \Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)}).$$

This completes the verification of L1.

(L2) Inserting $\nu(k) - 1$ in place of k in (10), we obtain

$$\Psi(u_{\nu(k)}) \le \Psi(u_{\ell(\nu(k)-1)}) - \tfrac{\delta}{\alpha_{\nu(k)-1}} \|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H^2. \tag{18}$$

Further, we can write

$$
\begin{aligned}
\Psi(u_{\ell(k+1)}) &= \max_{0\le j\le m(k+1)} \Psi(u_{k+1-j}) \le \max_{0\le j\le m(k)+1} \Psi(u_{k+1-j}) \\
&\le \max\left[\Psi(u_{k+1}), \max_{1\le j\le m(k)+1} \Psi(u_{k+1-j})\right] \\
&\le \max\left[\Psi(u_{\ell(k)}) - \tfrac{\delta}{\alpha_k}\|\mathcal{G}_{\alpha_k}(u_k)\|_H^2, \Psi(u_{\ell(k)})\right] \le \Psi(u_{\ell(k)}),
\end{aligned}
\tag{19}
$$

where we have used $m(k+1) \le m(k)+1$. Therefore, $\{\Psi(u_{\ell(k)})\}_k$ is decreasing and we can write

$$
\begin{aligned}
\Psi(u_{\ell(\nu(k)-1)}) &= \Psi(u_{\ell(\ell(km_{\max}+k)-1)}) \le \Psi(u_{\ell(km_{\max}+k-m_{\max}-1)}) \\
&= \Psi(u_{\ell((k-1)m_{\max}+(k-1))}) = \Psi(u_{\nu(k-1)}).
\end{aligned}
\tag{20}
$$

Together with (18) we can conclude (15) and, thus, L2 holds.

(L3) Assume that Algorithm 1 does not converge after finitely many iterations. Summing (15) up for $k = 1, \ldots, k'$, we obtain

$$
\begin{aligned}
\sum_{k=1}^{k'} \tfrac{\delta}{\alpha_{\nu(k)-1}}\|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H^2 &\le \sum_{k=1}^{k'} \Psi(u_{\nu(k-1)}) - \Psi(u_{\nu(k)}) \\
&\le \Psi(u_{\nu(0)}) - \Psi(u_{\nu(k')}).
\end{aligned}
\tag{21}
$$

Sending k' to infinity and using the fact that Ψ is bounded from below, we can conclude (16). Similarly (17) follows by using the fact that for $m_{\max} = 0$ it holds $\nu(k) = \ell(k) = k$. Thus, using (16), we can conclude the proof. □

Finally we are ready to present our main convergence result of this section.

Theorem 6 *Suppose that* ***A1–A3*** *hold and that Ψ is bounded from below. Then, for the sequence $\{u_k\}_k \subset H$ generated by Algorithm 1 with $\{\alpha_k\}_k \subset \mathbb{R}_{>0}$, the following statements hold true:*

(i) Either Algorithm 1 terminates after finitely many iterations with a stationary point of (**P**) *or the sequence $\{\|\mathcal{G}_{\alpha_k}(u_k)\|_H\}_k$ converges to zero, that is*

$$
\lim_{k\to\infty} \|\mathcal{G}_{\alpha_k}(u_k)\|_H = 0. \tag{22}
$$

(ii) The following inequality holds true

$$
\min_{0\le i\le k} \|\mathcal{G}_{\alpha_{\inf}}(u_i)\|_H \le C_G^{m_{\max}} \sqrt{\tfrac{\overline{\alpha}(m_{\max}+1)(\Psi(u_0)-\Psi(u_k))}{k\delta}}. \tag{23}
$$

(iii) If, in addition, ***A4*** *holds, every weak sequential accumulation point of $\{u_k\}_k \subset H$ is a stationary point of* (**P**).

Proof (i) Note that if Algorithm 1 converges after finitely many iterations, by definition of the stopping criterion and Proposition 1, a stationary point of (**P**) has been found. Now assume that Algorithm 1 does not converge after finitely many iterations. Then, using L3, and the fact that $\alpha_k \le \overline{\alpha}$ for all k by (iii) from Lemma 4, we arrive at

$$\lim_{k\to\infty} \|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H = 0. \tag{24}$$

It remains now to show that (22) holds. To show this, we will successively use (iv) of Lemma 4. Let $k \ge 0$ be arbitrary. Using the fact that $0 \le k - \left\lfloor \frac{k}{m_{\max}+1} \right\rfloor (m_{\max}+1) \le m_{\max}$ and

$$\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor (m_{\max}+1) - \ell\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor (m_{\max}+1)\right) \le m_{\max},$$

we can write

$$\begin{aligned}
\|\mathcal{G}_{\alpha_k}(u_k)\|_H &\le C_G^{m_{\max}} \|\mathcal{G}_{\alpha_{\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor (m_{\max}+1)}}(u_{\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor (m_{\max}+1)})\|_H \\
&\le C_G^{2m_{\max}} \|\mathcal{G}_{\alpha_{\ell\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor (m_{\max}+1)\right)}}(u_{\ell\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor (m_{\max}+1)\right)})\|_H \\
&\le C_G^{2m_{\max}} \|\mathcal{G}_{\alpha_{\nu\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor\right)}}(u_{\nu\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor\right)})\|_H \\
&\le C_G^{2m_{\max}+1} \|\mathcal{G}_{\alpha_{\nu\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor\right)-1}}(u_{\nu\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor\right)-1})\|_H.
\end{aligned} \tag{25}$$

Finally, sending k to ∞ and using (24), we obtain (22) and the proof of (i) is complete.

(ii) Due to the facts that $\Psi(u_k) \le \Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)})$ (by L1), $\Psi(u_0) = \Psi(u_{\nu(0)})$, and by successively using (8), we obtain that

$$\begin{aligned}
&\Psi(u_k) - \Psi(u_0) \le \Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)}) - \Psi(u_0) \\
&\le \Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)}) - \Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil-1)}) + \Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil-1)}) - \cdots \\
&- \Psi(u_{\nu(1)}) + \Psi(u_{\nu(1)}) - \Psi(u_0) \\
&\le \sum_{i=1}^{\left\lceil \frac{k}{m_{\max}+1} \right\rceil} -\frac{\delta}{\alpha_{\nu(i)-1}} \|\mathcal{G}_{\alpha_{\nu(i)-1}}(u_{\nu(i)-1})\|_H^2 \le \sum_{i=1}^{\left\lceil \frac{k}{m_{\max}+1} \right\rceil} -\frac{\delta}{\overline{\alpha}} \|\mathcal{G}_{\alpha_{\nu(i)-1}}(u_{\nu(i)-1})\|_H^2.
\end{aligned} \tag{26}$$

Thus, using L1, we can infer that

$$\begin{aligned}
&\frac{k\delta}{\overline{\alpha}(m_{\max}+1)} \min_{0\le i\le \left\lceil \frac{k}{m_{\max}+1}\right\rceil (m_{\max}+1)-1} \|\mathcal{G}_{\alpha_i}(u_i)\|_H^2 \\
&\le \frac{\left\lceil \frac{k}{m_{\max}+1}\right\rceil \delta}{\overline{\alpha}} \min_{1\le i\le \left\lceil \frac{k}{m_{\max}+1}\right\rceil} \|\mathcal{G}_{\alpha_{\nu(i)-1}}(u_{\nu(i)-1})\|_H^2 \\
&\le \sum_{i=1}^{\left\lceil \frac{k}{m_{\max}+1}\right\rceil} \frac{\delta}{\overline{\alpha}} \|\mathcal{G}_{\alpha_{\nu(i)-1}}(u_{\nu(i)-1})\|_H^2 \le \Psi(u_0) - \Psi(u_k),
\end{aligned}$$

and this yields

$$\min_{0\le i\le \left\lceil \frac{k}{m_{\max}+1}\right\rceil (m_{\max}+1)-1} \|\mathcal{G}_{\alpha_i}(u_i)\|_H \le \sqrt{\frac{\overline{\alpha}(m_{\max}+1)(\Psi(u_0)-\Psi(u_k))}{k\delta}}. \tag{27}$$

Further, using the second part of Lemma 4 and the fact that $\left\lceil \frac{k}{m_{\max}+1}\right\rceil (m_{\max}+1) - 1 - k < m_{\max}$, we can deduce that

$$\min_{0\le i\le k} \|\mathcal{G}_{\alpha_i}(u_i)\|_H \le C_G^{m_{\max}} \min_{0\le i\le \left\lceil \frac{k}{m_{\max}+1}\right\rceil (m_{\max}+1)-1} \|\mathcal{G}_{\alpha_i}(u_i)\|_H. \tag{28}$$

Finally, (23) follows from (27), (28), P2, and the fact that $\alpha_k \ge \alpha_{\inf}$ for all $k \ge 0$.

(iii) We show that every weak sequential accumulation point of $\{u_k\}_k \subset H$ is a stationary point of (**P**). In other words, we suppose that $u_{k_i} \rightharpoonup u^*$ for a subsequence $\{u_{k_i}\}_i \subset \{u_k\}_k$ and $u^* \in H$. Then we show that $u^* \in S_*$. For the sake of convenience, we use the same notation for the subsequence as for the sequence itself. To begin, due to (22) in (i), P2, and the fact that $\alpha_k \ge \alpha_{\inf}$, we can conclude that

$$\lim_{k\to\infty} \|\mathcal{G}_{\alpha_{\inf}}(u_k)\|_H = 0 \text{ and } \lim_{k\to\infty} \|\mathcal{T}_{\alpha_{\inf}}(u_k) - u_k\|_H = 0. \tag{29}$$

Applying (4) for $\alpha = \alpha_{\inf}$ and $u = u_k$, we obtain for every $k \in \mathbb{N}_0$ that

$$\mathcal{G}_{\alpha_{\inf}}(u_k) - \nabla\mathcal{F}(u_k) \in \partial\mathcal{R}(\mathcal{T}_{\alpha_{\inf}}(u_k)). \tag{30}$$

Due to **A4**, we can infer that $u_k \rightharpoonup u^*$ implies $\nabla\mathcal{F}(u_k) \to \nabla\mathcal{F}(u^*)$ and, thus, using (29) we have $\mathcal{G}_{\alpha_{\inf}}(u_k) - \nabla\mathcal{F}(u_k) \to -\nabla\mathcal{F}(u^*)$ as $k \to \infty$. Due to (29), we also can conclude that $\mathcal{T}_{\alpha_{\inf}}(u_k) \rightharpoonup u^*$ as $k \to \infty$. Therefore, sending $k \to \infty$ in (30) and using the fact that the graph of $\partial\mathcal{R}$ is sequentially closed [26, Proposition 16.26] under the weak topology for domain and the strong topology for codomain, we obtain $-\nabla\mathcal{F}(u^*) \in \partial\mathcal{R}(u^*)$. This completes the proof. □

Remark 2 In the case that $\dim(H) < \infty$, the statement of (iii) in Theorem 6 holds true for every (strong) accumulation point of $\{u_k\}_k$ without requiring **A4**.

In the next theorem, we derive an estimate that reflects the worst-case complexity of the required function and gradient-like evaluations of Algorithm 1 to find an ε_{tol}-stationary point. The proof is inspired by the one given in [50, Theorem 3.4] for smooth problems.

Theorem 7 (Worst-case complexity) *Suppose that **A1**–**A3** hold and that Ψ is bounded from below with $\bar{\Psi} := \inf_{u \in H} \Psi(u) > -\infty$. Then for a given tolerance $\varepsilon_{\text{tol}} > 0$, Algorithm 1 requires at most*

$$k_{\max}^{f} := \left\lfloor \frac{\gamma_{\text{comp}}^{f}(\Psi(u_0) - \bar{\Psi})}{\varepsilon_{\text{tol}}^{2}} \right\rfloor \tag{31}$$

function evaluations of Ψ and

$$k_{\max}^{g} := \left\lfloor \frac{\gamma_{\text{comp}}^{g}(\Psi(u_0) - \bar{\Psi})}{\varepsilon_{\text{tol}}^{2}} \right\rfloor \tag{32}$$

Gradient-like $\mathcal{G}_\alpha(\cdot)$ evaluations to find an iterate u_k satisfying $\|\mathcal{G}_{\alpha_k}(u_k)\|_H \le \varepsilon_{\text{tol}}$, where

$$\gamma_{\text{comp}}^{f} := \frac{(m_{\max}+1)C_G^{2m_{\max}}}{\gamma_{\text{decr}}} \quad \textit{and} \quad \gamma_{\text{comp}}^{g} := \frac{(m_{\max}+1)\overline{\alpha}C_G^{2m_{\max}}}{\delta},$$

with

$$\gamma_{\text{decr}} := \min\left\{\frac{\delta}{\alpha_{\sup}}, \frac{2(1-\delta)\delta}{n_1 \eta L_{\mathcal{F}'}}\right\} \quad \textit{and} \quad n_1 := \left\lfloor \left| \log_\eta \left(\frac{\eta L_{\mathcal{F}'}}{2\alpha_{\inf}(1-\delta)} \right) \right| \right\rfloor.$$

Proof The proof can be found in Appendix A. 1. □

This finishes our considerations of convergence and complexity in the general setting.

3.2 Convex case

In this section, higher-order convergence rates will be shown in two cases of additional structural assumptions. Firstly, we consider the case where $\mathcal{F}$ is also convex. Afterwards, the case of a quadratic growth assumption on Ψ will be investigated. We assume the following modified version of Assumption 1.

Assumption 2 Assume that **A1**–**A3** hold. Further, instead of **A4**, assume that

A'4: $\mathcal{F}\colon H \to \mathbb{R}$ is convex.

Under Assumption 2, the whole function Ψ is convex. In this case, we can conclude that the set of minimizers of Ψ coincides with S_* provided that $S_* \neq \emptyset$ and that S_* is closed and convex. Further, we have

$$S_* = \operatorname{Argmin} \Psi = (\partial \Psi)^{-1}(0).$$

The associated minimal function value is denoted by $\Psi^* \in \mathbb{R}$.

Next we prove an auxiliary lemma which will be used later.

Lemma 8 *Suppose that Assumption 2 holds, $S_* \neq \emptyset$, and $\{u_k\}_k$ is generated by Algorithm 1. Then for every $\lambda \in [0, 1]$, it holds*

$$\begin{aligned}\Psi(u_{\nu(k)}) - \Psi^* \leq (1-\lambda)\left(\Psi(u_{\nu(k-1)}) - \Psi^*\right) + \tfrac{\overline{\alpha}\lambda^2}{2}\operatorname{dist}^2(u_{\nu(k)-1}, S_*)\\ + \tfrac{\tilde{C}}{\alpha_{\nu(k)-1}}\|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H^2,\end{aligned} \tag{33}$$

where $\tilde{C} := \frac{L_{\mathcal{F}'}}{2\alpha_{\inf}}$. Further, we have the following inequality for the initial iterations

$$\Psi(u_{\nu(1)}) - \Psi^* \leq C_0 \operatorname{dist}^2(u_0, S_*), \tag{34}$$

with constant C_0 which is independent of u_0.

Proof The proof can be found in Appendix A. 2. □

Now we are ready to provide the main convergence result for the convex case. For the sake of convenience in the presentation, we set

$$\mathcal{E}_k := \Psi(u_k) - \Psi^* \quad \text{for every } k \geq 0,$$

and use this notation in the remainder of this section.

Theorem 9 (Global convergence and $O(k^{-1})$ complexity for the convex case) *Suppose that Assumption 2 holds, $S_* \neq \emptyset$, and $\{u_k\}_k$ is generated by Algorithm 1. Then the following statements hold true:*

(i) Every weak sequential accumulation point of $\{u_k\}_k$ belongs to S_*.
(ii) $\{u_k\}_k$ converges weakly to a minimizer $u^* \in S_*$ and its "shadow" sequence converges strongly to u^*, that is $P_{S_*}u_k \to u^*$.
(iii) It holds $\Psi(u_k) \to \Psi^*$ as $k \to \infty$ and for large enough $k \geq 0$, there exist constants $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\Psi(u_k) - \Psi^* \leq \frac{\rho_1}{\rho_2 + k}. \tag{35}$$

Proof (i) We suppose that a subsequence $\{u_{k_i}\}_i$ with $u_{k_i} \rightharpoonup u^*$ is given. We will show that $u^* \in S_*$. To show this, we prove that there exists a vanishing sequence of subgradients $\{w_{k_i}\}_i \subset H$, i.e. $w_{k_i} \to 0$, corresponding to $\{u_{k_i}\}_i$ with $w_{k_i} \in \partial\Psi(u_{k_i})$ for every $i \in \mathbb{N}$. Using (4), we define

$$w_{k+1} := \mathcal{G}_{\alpha_k}(u_k) + \nabla\mathcal{F}(u_{k+1}) - \nabla\mathcal{F}(u_k) \in \partial\Psi(u_{k+1}). \tag{36}$$

Further, since $S_* \neq \emptyset$, Ψ is bounded from below. Hence, we can use (i) of Theorem 6 and (22) holds. This, together with (36) and the boundedness of α_k, implies

$$\|w_{k+1}\|_H \leq \|\mathcal{G}_{\alpha_k}(u_k)\|_H + \|\nabla\mathcal{F}(u_{k+1}) - \nabla\mathcal{F}(u_k)\|_H \leq (1 + \tfrac{L_{\mathcal{F}'}}{\alpha_k})\|\mathcal{G}_{\alpha_k}(u_k)\|_H \to 0,$$

as $k \to \infty$. In particular, we can infer that $w_{k_i} \to 0$ as $i \to \infty$. Using the fact that the graph of $\partial\Psi$ is sequentially closed [26, Proposition 16.26] under the weak topology for domain and the strong topology for codomain together with $w_{k_i} \to 0$ and $u_{k_i} \rightharpoonup u^*$, we arrive at $0 \in \partial\Psi(u^*)$ and therefore $u^* \in S_*$.

(ii) We show that $u_k \rightharpoonup u^*$ with $u^* \in S_*$. In this matter, we show that $\{u_k\}_k$ is a quasi-Fejér sequence with respect to $S_* \neq \emptyset$. Using (4), we can write for every $k \in \mathbb{N}_0$ that

$$0 \in \alpha_k(u_{k+1} - u_k) + \partial\mathcal{R}(u_{k+1}) + \nabla\mathcal{F}(u_k). \tag{37}$$

Further, since $\mathcal{R}$ is convex and $\nabla\mathcal{F}$ is Lipschitz continuous, by the Haddad-Bailon Theorem [26, Corollary 18.16, p. 270], we have

$$(\nabla\mathcal{F}(v) - \nabla\mathcal{F}(w), v - w)_H \geq L_{\mathcal{F}'}{}^{-1}\|\nabla\mathcal{F}(v) - \nabla\mathcal{F}(w)\|_H^2.$$

Further, we can write for every $v, w, z \in \operatorname{int}(\operatorname{dom}\mathcal{F})$ that

$$\begin{aligned}
&(\nabla\mathcal{F}(v) - \nabla\mathcal{F}(w), z - w)_H \\
&= (\nabla\mathcal{F}(v) - \nabla\mathcal{F}(w), v - w)_H + (\nabla\mathcal{F}(v) - \nabla\mathcal{F}(w), z - v)_H \\
&\geq L_{\mathcal{F}'}{}^{-1}\|\nabla\mathcal{F}(v) - \nabla\mathcal{F}(w)\|_H^2 - \|\nabla\mathcal{F}(v) - \nabla\mathcal{F}(w)\|_H\|z - v\|_H \\
&\geq -\tfrac{L_{\mathcal{F}'}}{4}\|z - v\|_H^2,
\end{aligned} \tag{38}$$

where in the last line we have used the fact that $f(x) := L_{\mathcal{F}'}{}^{-1}x^2 - x\|z - v\|_H$ is strictly convex and attains its global minimum at $x^* = \frac{L_{\mathcal{F}'}}{2}\|z - v\|_H$.

Using (38) for u_{k+1}, u_k, and any $u^* \in S_*$ in place of v, z, and w, respectively, we obtain

$$(\nabla\mathcal{F}(u_k) - \nabla\mathcal{F}(u^*), u_{k+1} - u^*)_H \geq -\tfrac{L_{\mathcal{F}'}}{4}\|u_{k+1} - u_k\|_H^2.$$

Together with the fact that $\partial\mathcal{R}$ is monotone, cf. [26], and (2), we can write

$$(\eta + \nabla\mathcal{F}(u_k), u_{k+1} - u^*)_H \geq -\tfrac{L_{\mathcal{F}'}}{4}\|u_{k+1} - u_k\|_H^2 \quad \text{for all } \eta \in \partial\mathcal{R}(u_{k+1}). \tag{39}$$

Using (37) and (39), we can deduce that

$$(u_{k+1} - u_k, u_{k+1} - u^*)_H \leq \tfrac{L_{\mathcal{F}'}}{4\alpha_k}\|u_{k+1} - u_k\|_H^2. \tag{40}$$

Further, using (40) and the fact that

$$(w - v, w - z)_H = \tfrac{1}{2}\|w - v\|_H^2 - \tfrac{1}{2}\|v - z\|_H^2 + \tfrac{1}{2}\|w - z\|_H^2 \quad \text{for all } z, w, v \in H,$$

we can infer that

$$\begin{aligned} \tfrac{1}{2}\|u_{k+1}-u^*\|_H^2 - \tfrac{1}{2}\|u_k-u^*\|_H^2 &\leq \left(\tfrac{L_{\mathcal{F}'}}{4\alpha_k}-\tfrac{1}{2}\right)\|u_{k+1}-u_k\|_H^2 \\ &\leq \tfrac{L_{\mathcal{F}'}}{4\alpha_{\inf}^3}\|\mathcal{G}_{\alpha_k}(u_k)\|_H^2, \end{aligned} \tag{41}$$

where it can be seen from (21) and (25) that

$$\sum_{k=m_{\max}+1}^{\infty}\|\mathcal{G}_{\alpha_k}(u_k)\|_H^2 \leq (m_{\max}+1)C_G^{4m_{\max}+2}\sum_{k=1}^{\infty}\|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H < \infty.$$

Therefore, the sequence $\{u_k\}_k \subset H$ is quasi-Fejér monotone with respect to S_* and since, due to (i), every weak sequential accumulation point of $\{u_k\}_k \subset H$ belongs to $S_* \neq \emptyset$, we can conclude by [51, Proposition 1(3)] that $\{u_k\}_k$ is weakly convergent and it has a unique accumulation point.

In addition, since S_* is closed and convex, we can conclude, due to [52, Proposition 3.6 (iv)], that $\{P_{S_*}u_k\}_k$ converges strongly to a point $\hat{u} \in S_*$. Moreover, since $u^* - P_{S_*}u_k \to u^* - \hat{u}$ and $u_k - P_{S_*}u_k \rightharpoonup u^* - \hat{u}$, it follows from the definition of orthogonal projection that $\|u^* - \hat{u}\|_H^2 = \lim_{k\to\infty}(u^* - P_{S_*}u_k, u_k - P_{S_*}u_k)_H \leq 0$. Hence, we obtain that $u^* = \hat{u}$.

(iii) The proof of this part is inspired by the one in [38, Theorem 3.2.]. First, we show that $\Psi(u_k) \to \Psi^*$. Due to (ii), $u_k \rightharpoonup u^*$ for some $u^* \in S_*$. As in the proof of (i), there exists a sequence $w_k \to 0$ with $w_k \in \partial\Psi(u_k)$. Therefore, we can write

$$\Psi(u_k) \leq \Psi^* + (w_k, u_k - u^*)_H \quad \text{for every } k \in \mathbb{N}_0. \tag{42}$$

Sending $k \to \infty$ in (42) and using the facts that $u_k \rightharpoonup u^*$ and $w_k \to 0$ and the weak sequential lower semicontinuity of Ψ, we can conclude that

$$\Psi^* \leq \liminf_{k\to\infty}\Psi(u_k) \leq \limsup_{k\to\infty}\Psi(u_k) \leq \Psi^*.$$

Hence, $\Psi(u_k) \to \Psi^*$.

Next, we turn to the verification of (35) for a large enough k. Due to (33) of Lemma 8, for every $\lambda \in [0, 1]$, it holds

$$\mathcal{E}_{\nu(k)} \leq (1-\lambda)\mathcal{E}_{\nu(k-1)} + \tfrac{\overline{\alpha}\lambda^2}{2}\operatorname{dist}^2(u_{\nu(k)-1}, S_*) + \tfrac{\tilde{C}}{\alpha_{\nu(k)-1}}\|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H^2. \tag{43}$$

Since $\{u_k\}_k$ is quasi-Féjer monotone with respect to $S_* \neq \emptyset$, due to [52, Proposition 3.6 (ii)], the sequence $\operatorname{dist}^2(u_k, S_*)$ is convergent. Therefore, for a positive constant $\kappa > 0$, we have

$$\operatorname{dist}^2(u_{\nu(k)-1}, S_*) \leq \kappa, \quad \text{for all } k \geq 1. \tag{44}$$

Further, using (43), (44), and L2, we can write

$$\mathcal{E}_{\nu(k)} \leq (1-\lambda)\mathcal{E}_{\nu(k-1)} + \frac{\overline{\alpha}\lambda^2\kappa}{2} + \frac{\tilde{C}}{\delta}\left(\mathcal{E}_{\nu(k-1)} - \mathcal{E}_{\nu(k)}\right), \tag{45}$$

where the expression on the right-hand side is strictly convex in λ, since

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\left((1-\lambda)\mathcal{E}_{\nu(k-1)} + \frac{\overline{\alpha}\lambda^2\kappa}{2}\right) = \overline{\alpha}\kappa > 0.$$

Therefore, it possesses the unique minimizer $\lambda = \frac{\mathcal{E}_{\nu(k-1)}}{\overline{\alpha}\kappa}$. Since $\{\mathcal{E}_{\nu(k)}\}_k \to 0$, this implies that for large enough k, we can set $\lambda = \frac{\mathcal{E}_{\nu(k-1)}}{\overline{\alpha}\kappa} \leq 1$ in (45) and obtain that

$$\mathcal{E}_{\nu(k)} \leq \mathcal{E}_{\nu(k-1)} - \frac{\mathcal{E}^2_{\nu(k-1)}}{2\overline{\alpha}\kappa} + \frac{\tilde{C}}{\delta}\left(\mathcal{E}_{\nu(k-1)} - \mathcal{E}_{\nu(k)}\right).$$

Together with the fact that $\mathcal{E}_{\nu(k)} \leq \mathcal{E}_{\nu(k-1)}$, we can write

$$\mathcal{E}_{\nu(k)} \leq \mathcal{E}_{\nu(k-1)} - \frac{\mathcal{E}_{\nu(k-1)}\mathcal{E}_{\nu(k)}}{2\overline{\alpha}\kappa} + \frac{\tilde{C}}{\delta}\left(\mathcal{E}_{\nu(k-1)} - \mathcal{E}_{\nu(k)}\right)$$

and, thus, obtain

$$\mathcal{E}_{\nu(k)} \leq \left(1 + \frac{\delta}{2\overline{\alpha}\kappa(\tilde{C}+\delta)}\mathcal{E}_{\nu(k-1)}\right)^{-1}\mathcal{E}_{\nu(k-1)}. \tag{46}$$

Further, (46) can be expressed as

$$\frac{1}{\mathcal{E}_{\nu(k)}} \geq \frac{1}{\mathcal{E}_{\nu(k-1)}} + \frac{\delta}{2\overline{\alpha}\kappa(\tilde{C}+\delta)}.$$

Applying this inequality recursively for integers k_1 and k_2 with $k_2 \geq k_1$ and large enough k_1 satisfying $\frac{\mathcal{E}_{\nu(k_1)}}{\overline{\alpha}\kappa} \leq 1$, we obtain

$$\frac{1}{\mathcal{E}_{\nu(k_2)}} \geq \frac{1}{\mathcal{E}_{\nu(k_1)}} + \frac{\delta(k_2-k_1)}{2\overline{\alpha}\kappa(\tilde{C}+\delta)},$$

and by easy computations, also

$$\mathcal{E}_{\nu(k_2)} \leq \frac{2\overline{\alpha}\kappa(\tilde{C}+\delta)\mathcal{E}_{\nu(k_1)}}{2\overline{\alpha}\kappa(\tilde{C}+\delta)+\mathcal{E}_{\nu(k_1)}\delta(k_2-k_1)}.$$

Thus, for $k \geq 0$ large enough, we set $k_2 = \left\lceil \frac{k}{m_{\max}+1} \right\rceil$ and obtain

$$\begin{aligned}\mathcal{E}_k \leq \mathcal{E}_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)} &\leq \frac{2\overline{\alpha}\kappa(\tilde{C}+\delta)\mathcal{E}_{\nu(k_1)}}{2\overline{\alpha}\kappa(\tilde{C}+\delta)+\mathcal{E}_{\nu(k_1)}\delta(\lceil \frac{k}{m_{\max}+1} \rceil - k_1)} \\ &\leq \frac{2\overline{\alpha}\kappa(\tilde{C}+\delta)\mathcal{E}_{\nu(k_1)}}{-k_1\mathcal{E}_{\nu(k_1)}\delta + 2\overline{\alpha}\kappa(\tilde{C}+\delta)+\mathcal{E}_{\nu(k_1)}\delta(\frac{k}{m_{\max}+1})}.\end{aligned}$$

Therefore, (35) follows by setting

$$\rho_1 := (m_{\max} + 1)\delta^{-1} 2\overline{\alpha}\kappa(\tilde{C} + \delta)$$

and

$$\rho_2 := (m_{\max} + 1)(\delta \mathcal{E}_{\nu(k_1)})^{-1} \left(2\overline{\alpha}\kappa(\tilde{C} + \delta) - k_1 \mathcal{E}_{\nu(k_1)}\delta\right).$$

Thus, the proof is complete. □

Comparing Theorem 9 to Theorem 6, one obtains weak convergence of the whole sequence and convergence of the associated objective function evaluations with rate $1/k$.

Remark 3 Note that for the case where $\dim(H) < \infty$, due to the equivalence of the weak and strong topologies, it follows from (ii) of Theorem 9 that u_k converges to u^* in the strong topology. Moreover, we will later use the convergence $P_{S_*} u_k \to u^*$ of the shadow sequence to derive the strong convergence $u_k \to u^*$ under the quadratic growth conditions. This is obviously no longer necessary when $\dim(H) < \infty$.

3.3 Convergence under quadratic growth-type conditions

In this section, we turn our attention towards quadratic growth-type conditions and study convergence of Algorithm 1 under these conditions.

Definition 2 (*Quadratic growth condition*) We say that Ψ satisfies the *quadratic growth condition*, if

$$\Psi(u) - \Psi^* \geq \gamma_{\Psi,\Gamma} \operatorname{dist}^2(u, S_*) \quad \text{for all } u \in \Gamma \cap \operatorname{dom} \Psi \tag{47}$$

holds with a set $\Gamma \subset H$, a constant $\gamma_{\Psi,\Gamma} > 0$, and $S_* \neq \emptyset$. We refer to this notion as global if $\Gamma = H$, and as local, if for $u^* \in S_*$, $r \in (0, \infty]$, and $\omega > 0$, we have $\Gamma = \mathbf{B}_r(u^*) \cap [\Psi^* < \Psi + \omega]$. Additionally, Ψ is said to satisfy the *strong quadratic growth condition* at u^* if $S_* = \{u^*\}$ on Γ. That is,

$$\Psi(u) - \Psi(u^*) \geq \gamma_{\Psi,\Gamma} \|u - u^*\|_H^2 \quad \text{for all } u \in \Gamma \cap \operatorname{dom} \Psi. \tag{48}$$

The quadratic growth condition is a geometrical assumption which describes the *flatness* of the objective function around its minimizers. Roughly speaking, this condition is considered as a relaxation of the strong convexity condition and allows us to obtain faster rates of convergence (linear) and also convergence in the strong topology for the iteration sequence. It is also closely related to the notion of Tikhonov well-posedness [53]. The relationship between the quadratic growth condition and the so-called *metric subregularity of the subdifferential* has been investigated e.g. in [10, 54–57]. The strong quadratic growth condition (48) is said to be the *quadratic functional growth property* in [16] provided that Ψ is continuously differentiable over a closed convex set. In [15, 58], Ψ is also called 2-*conditional on* Γ if it satisfies the quadratic growth condition (47). This property was recently proved in [10, Theorem 5] to be equivalent with the case where Ψ satisfies the Kurdyka-Łojasiewicz inequality with order $1/2$.

Theorem 10 *Suppose that Assumption 2 and the quadratic growth condition (47) hold for* $\Gamma := [\Psi < \Psi^* + \omega]$ *with* $\omega > 0$. *Then, for the sequence of iterates* $\{u_k\}_k$ *generated by Algorithm 1, there exists* $\bar{k} \in \mathbb{N}$ *such that for every* $k \geq \bar{k}$, *we have*

$$\Psi(u_k) - \Psi^* \leq C_c \sigma^k, \tag{49}$$

and

$$\operatorname{dist}^2(u_k, S_*) \leq C_d \sigma^k, \tag{50}$$

where the constants $C_c > 0$, $C_d > 0$, *and* $0 < \sigma < 1$ *are independent of* k.

Further, there exists $u^* \in S_*$ *such that* u_k *converges in the strong topology to* u^*, *i.e.* $u_k \to u^*$, *and it holds*

$$\|u_k - u^*\|_H^2 \leq C_p \sigma^k, \tag{51}$$

with a constant $C_p > 0$ *which is independent of* u^* *and* k.

Proof First, due to L2 and (iii) from Theorem 9, the sequence $\{\Psi(u_{\nu(k)})\}_k$ is monotonically decreasing and converges to Ψ^*. Further, using (20), we can deduce for $k \geq 1$ that

$$\Psi(u_{\nu(k)-1}) \leq \Psi(u_{\ell(\nu(k)-1)}) \leq \Psi(u_{\nu(k-1)}). \tag{52}$$

Thus, for given $\omega > 0$, there exists $\bar{k}_\omega \in \mathbb{N}$ such that

$$\Psi(u_{\nu(k)-1}) \in [\Psi < \Psi^* + \omega] \quad \text{for all } k \geq \bar{k}_\omega.$$

Next, we show that

$$\mathcal{E}_{\nu(k)} \leq \theta \mathcal{E}_{\nu(k-1)} \quad \text{for all } k \geq \bar{k}_\omega, \tag{53}$$

with a constant $\theta \in (0, 1)$ independent of k.

Let an arbitrary $k \geq \bar{k}_\omega$ be given. To show (53), we choose an arbitrary ζ satisfying

$$0 < \zeta < \min\{\tfrac{1}{\delta}, \tfrac{1}{2\tilde{C}}, \tfrac{\gamma_{\Psi,\Gamma}}{2\bar{\alpha}\tilde{C}}\}.$$

Now we consider the following cases:

- The inequality $\frac{1}{\alpha_{\nu(k)-1}} \|\mathcal{G}_{\nu(k)-1}(u_{\nu(k)-1})\|_H^2 \geq \zeta \mathcal{E}_{\nu(k-1)}$ holds. In this case, using L2, we obtain

$$\mathcal{E}_{\nu(k)} \leq \mathcal{E}_{\nu(k-1)} - \frac{\delta}{\alpha_{\nu(k)-1}} \|\mathcal{G}_{\nu(k)-1}(u_{\nu(k)-1})\|_H^2 \leq (1 - \zeta\delta)\mathcal{E}_{\nu(k-1)},$$

where due to the choice of ζ we have $1 - \zeta\delta < 1$.

- The inequality

$$\frac{1}{\alpha_{\nu(k)-1}} \|\mathcal{G}_{\nu(k)-1}(u_{\nu(k)-1})\|_H^2 < \zeta \mathcal{E}_{\nu(k-1)} \tag{54}$$

holds. This case is more delicate. Using the quadratic growth condition (47) and (52), we obtain for $k \geq \bar{k}_\omega$ that

$$\operatorname{dist}^2(u_{\nu(k)-1}, S_*) \leq \frac{1}{\gamma_{\Psi,\Gamma}} \mathcal{E}_{\nu(k)-1} \leq \frac{1}{\gamma_{\Psi,\Gamma}} \mathcal{E}_{\nu(k-1)}. \tag{55}$$

Now, using Lemma 8, (33), and (55), we can write for every $\lambda \in [0, 1]$ that

$$\mathcal{E}_{\nu(k)} \leq \left(\tilde{C}\zeta + 1 - \lambda + \frac{\overline{\alpha}}{2\gamma_{\Psi,\Gamma}}\lambda^2\right) \mathcal{E}_{\nu(k-1)}, \tag{56}$$

where it can be easily seen that the minimum of $\left(\tilde{C}\zeta + 1 - \lambda + \frac{\overline{\alpha}}{2\gamma_{\Psi,\Gamma}}\lambda^2\right)$ attained at $\lambda^* := \min\{1, \frac{\gamma_{\Psi,\Gamma}}{\overline{\alpha}}\}$ is strictly smaller than 1 and therefore (53) holds for $k \geq \bar{k}_\omega$.

Let now $k \geq \bar{k} := \bar{k}_\omega(m_{\max} + 1)$ be given. By successively applying (53), we obtain

$$\begin{aligned} \mathcal{E}_k &\overset{L1}{\leq} \mathcal{E}_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)} \leq \theta^{1-\bar{k}_\omega} \theta^{\lceil \frac{k}{m_{\max}+1} \rceil} \mathcal{E}_{\nu(\bar{k}_\omega - 1)} \\ &\leq \theta^{1-\bar{k}_\omega} \theta^{\frac{k}{m_{\max}+1}} \mathcal{E}_{\nu(\bar{k}_\omega - 1)} \leq \theta^{1-\bar{k}_\omega} \left(\theta^{\frac{1}{m_{\max}+1}}\right)^k \mathcal{E}_{\nu(\bar{k}_\omega - 1)}. \end{aligned} \tag{57}$$

Together with the quadratic growth condition (47), we have

$$\operatorname{dist}^2(u_k, S_*) \leq \frac{1}{\gamma_{\Psi,\Gamma}} \mathcal{E}_k \leq \frac{\theta^{1-\bar{k}_\omega}}{\gamma_{\Psi,\Gamma}} \left(\theta^{\frac{1}{m_{\max}+1}}\right)^k \mathcal{E}_{\nu(\bar{k}_\omega - 1)}. \tag{58}$$

Thus, for every $k \geq \bar{k}$, the iterates u_k stay in Γ.

Setting $C_c := \theta^{1-\bar{k}_\omega} \mathcal{E}_{\nu(\bar{k}_\omega - 1)}$, $C_d := \gamma_{\Psi,\Gamma}^{-1} \theta^{1-\bar{k}_\omega} \mathcal{E}_{\nu(\bar{k}_\omega - 1)}$, and $\sigma := \theta^{\frac{1}{m_{\max}+1}} < 1$, we are finished with the verification of (49) and (50).

Next, we show that $u_k \to u^*$ for $u_k \rightharpoonup u^*$ with $u^* \in S_*$ given in Theorem 9 (ii). Due to (50), $\operatorname{dist}(u_k, S_*) \to 0$ and we can infer that $u_k - P_{S_*} u_k \to 0$. This together with fact that $P_{S_*} u_k \to u^*$ (see (ii) in Theorem 9) leads to $u_k \to u^*$.

Finally, we show that (51) holds true. To see this, let $p \in \mathbb{N}$ be arbitrary, using Young's inequality we have

$$\begin{aligned} \|u_k - u_{k+p}\|_H^2 &\leq 2\left(\|u_k - P_{S_*} u_k\|_H^2 + \|u_{k+p} - P_{S_*} u_k\|_H^2\right) \\ &= 2\left(\operatorname{dist}^2(u_k, S_*) + \|u_{k+p} - P_{S_*} u_k\|_H^2\right). \end{aligned} \tag{59}$$

Using (41), we obtain for an arbitrary $u^* \in S_*$ that

$$\|u_{k+p} - u^*\|_H^2 \leq \|u_k - u^*\|_H^2 + \frac{L_{\mathcal{F}'}}{2\alpha_{\inf}^2} \sum_{j=k}^{k+p-1} \|\mathcal{G}_{\alpha_j}(u_j)\|_H^2.$$

In particular, we have for $u^* = P_{S_*} u_k$ that

$$\|u_{k+p} - P_{S_*} u_k\|_H^2 \leq \operatorname{dist}^2(u_k, S_*) + \frac{L_{\mathcal{F}'}}{2\alpha_{\inf}^2} \sum_{j=k}^{k+p-1} \|\mathcal{G}_{\alpha_j}(u_j)\|_H^2. \tag{60}$$

Further, using L2 and (25), we can write for large enough k that

$$\begin{aligned}
\sum_{j=k}^{k+p-1} \|\mathcal{G}_{\alpha_j}(u_j)\|_H^2 &\leq C_G^{4m_{\max}+2} \sum_{j=k}^{k+p-1} \|\mathcal{G}_{\alpha_{\nu\left(\left\lfloor \frac{j}{m_{\max}+1} \right\rfloor\right)-1}}(u_{\nu\left(\left\lfloor \frac{j}{m_{\max}+1} \right\rfloor\right)-1})\|_H^2 \\
&\leq C_G^{4m_{\max}+2} \overline{\alpha} \delta^{-1} \sum_{j=k}^{k+p-1} \frac{\delta}{\alpha_{\nu\left(\left\lfloor \frac{j}{m_{\max}+1} \right\rfloor\right)-1}} \|\mathcal{G}_{\alpha_{\nu\left(\left\lfloor \frac{j}{m_{\max}+1} \right\rfloor\right)-1}}(u_{\nu\left(\left\lfloor \frac{j}{m_{\max}+1} \right\rfloor\right)-1})\|_H^2 \\
&\leq C_G^{4m_{\max}+2} \overline{\alpha} \delta^{-1} \left(\mathcal{E}_{\nu\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor-1\right)} - \mathcal{E}_{\nu\left(\left\lfloor \frac{k+p}{m_{\max}+1} \right\rfloor\right)} \right).
\end{aligned} \tag{61}$$

Combining (59), (60), (61), and setting $\bar{C}_p := \frac{C_G^{4m_{\max}+2} \overline{\alpha} L_{\mathcal{F}'}}{\delta \alpha_{\inf}^2}$, we arrive at

$$\|u_k - u_{k+p}\|_H^2 \leq 4 \operatorname{dist}^2(u_k, S_*) + \bar{C}_p \left(\mathcal{E}_{\nu\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor-1\right)} - \mathcal{E}_{\nu\left(\left\lfloor \frac{k+p}{m_{\max}+1} \right\rfloor\right)} \right).$$

Sending $p \to \infty$ and using Theorem 9 (iii), (50), and similar computations as in (57), we obtain for every $k \geq \bar{k}$ that

$$\begin{aligned}
\|u_k - u^*\|_H^2 &\leq 4 \operatorname{dist}^2(u_k, S_*) + \bar{C}_p \mathcal{E}_{\nu\left(\left\lfloor \frac{k}{m_{\max}+1} \right\rfloor-1\right)} \\
&\leq 4 C_d \sigma^k + \bar{C}_p \theta^{-1-\bar{k}_\omega} \sigma^k \mathcal{E}_{\nu(\bar{k}_\omega - 1)} = C_p \sigma^k.
\end{aligned}$$

Thus, (51) holds true with $C_p := 4C_d + \bar{C}_p \theta^{-1-\bar{k}_\omega} \mathcal{E}_{\nu(\bar{k}_\omega - 1)}$ and this completes the proof. □

Corollary 11 *Suppose that Assumption 2 holds and the quadratic growth condition (47) is satisfied for* $\Gamma := \mathbf{B}_r(u^*) \cap [\Psi < \Psi^* + \omega]$ *with* $r, \omega \in (0, \infty)$. *Further assume that,* $\{u_k\}_k$ *generated by Algorithm 1, converges strongly to some* $u^* \in S_*$. *Then there* *exists* $\bar{k} \in \mathbb{N}$ *such that* (49)–(51) *hold with constants* $C_c > 0$, $C_d > 0$, $C_p > 0$, *and* $0 < \sigma < 1$, *which are independent of* k *and* u^*.

Proof The proof proceeds along the lines of the proof of Theorem 10 with the difference that one also needs to be sure that $\{u_k\}_k \subset \mathbf{B}_r(u^*)$ for a large enough $\bar{k} \in \mathbb{N}$. This follows from the fact that $u_k \to u^*$. □

Remark 4 Due to the equivalence of weak and strong convergence in finite-dimensional spaces, the assumption of Corollary 11 automatically holds if $\dim(H) < \infty$. For the case that $\dim(H) = \infty$, it is not clear how to guarantee that $\{u_k\}_k \subset \mathbf{B}_r(u^*)$ for large enough $k \in \mathbb{N}$.

In many cases of PDE-constrained optimization, the assumptions of the following corollary are very likely satisfied.

Corollary 12 *Suppose that* ***A1–A3*** *in Assumption* 1 *hold and that* $\mathcal{F}$ *is convex on* $\mathbf{B}_r(u^*)$ *with* $u^* \in S_*$ *and* $r \in (0, \infty)$. *Further, we assume that the strong quadratic growth condition* (48) *holds for* $\Gamma := \mathbf{B}_r(u^*) \cap [\Psi < \Psi^* + \omega]$ *with* $\omega \in (0, \infty)$. *Then, the sequence of iterations* $\{u_k\}_k$ *generated by Algorithm* 1 *converges locally R-linear with respect to the strong topology. In other words, there exists a radius* $r_0 \leq r$ *such that for every* $u_0 \in \mathbf{B}_{r_0}(u^*) \cap [\Psi < \Psi^* + \omega]$ *and* $k \geq 1$, *we have*

$$\|u_k - u^*\|_H^2 \leq C_R \sigma^k \|u_0 - u^*\|_H^2 \tag{62}$$

and

$$\Psi(u_k) - \Psi(u^*) \leq \sigma^k \left(\Psi(u_0) - \Psi^*\right), \tag{63}$$

where the constants $C_R > 0$ *and* $0 < \sigma < 1$ *are independent of* u_0 *and* k.

Proof By similar argument as in the proof of Theorem 10, we can show that for any $u_{\nu(k)-1} \in \mathbf{B}_r(u^*) \cap [\Psi < \Psi^* + \omega]$ with $k \geq 1$, it holds

$$\mathcal{E}_{\nu(k)} \leq \theta \mathcal{E}_{\nu(k-1)}, \tag{64}$$

with $\theta \in (0, 1)$ independent of k.

Let now $k \geq 1$ be given. Assuming $u_i \in \mathbf{B}_r(u^*) \cap [\Psi < \Psi^* + \omega]$ for $i = 0, \dots, k-1$ and successively applying (64), we obtain

$$\begin{aligned} \mathcal{E}_k &\overset{L2}{\leq} \mathcal{E}_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)} \leq \theta^{-1} \theta^{\lceil \frac{k}{m_{\max}+1} \rceil} \mathcal{E}_{\nu(1)} \leq \theta^{-1} \theta^{\frac{k}{m_{\max}+1}} \mathcal{E}_{\nu(1)} \\ &\leq \theta^{-1} \left(\theta^{\frac{1}{m_{\max}+1}}\right)^k \mathcal{E}_{\nu(1)}. \end{aligned} \tag{65}$$

Together with the quadratic growth condition (48) and (34) from Lemma 8, we have

$$\begin{aligned}\|u_k - u^*\|_H^2 &\leq \tfrac{1}{\gamma_{\Psi,\Gamma}}\mathcal{E}_k \leq \tfrac{1}{\theta\gamma_{\Psi,\Gamma}}\left(\theta^{\frac{1}{m_{\max}+1}}\right)^k \mathcal{E}_{\nu(1)} \\ &\leq \tfrac{C_0}{\theta\gamma_{\Psi,\Gamma}}\left(\theta^{\frac{1}{m_{\max}+1}}\right)^k \|u_0 - u^*\|_H^2.\end{aligned} \tag{66}$$

Thus, for every $u_0 \in \mathbf{B}_{r_0}(u^*)$ with $r_0 \leq \left(\frac{\theta\gamma_{\Psi,\Gamma}}{C_0}\right)^{\frac{1}{2}} r$, iterates u_k stay in $\mathbf{B}_r(u^*)$ for $k \geq 0$ and, as a consequence, the inequalities (65) and (66) are well-defined. By setting $C_R := \frac{C_0}{\theta\gamma_{\Psi,\Gamma}}$ and $\sigma := \theta^{\frac{1}{m_{\max}+1}} < 1$, we are finished with the verification of (62).

As in (65), we can also write for every $k \geq 0$ and $u_0 \in \mathbf{B}_{r_0}(u^*) \cap [\Psi < \Psi^* + \omega]$ that

$$\mathcal{E}_k \leq \mathcal{E}_{\nu(\lceil \frac{k}{m_{\max}+1} \rceil)} \leq \theta^{\lceil \frac{k}{m_{\max}+1} \rceil}\mathcal{E}_{\nu(0)} \leq \theta^{\frac{k}{m_{\max}+1}}\mathcal{E}_{\nu(0)} \leq \sigma^k \mathcal{E}_{\nu(0)}.$$

Thus (63) holds also true and this completes the proof. □

Compared to Corollary 11, we do not need to assume the strong convergence of the sequence $\{u_k\}_k$ in Corollary 12 and the following corollary.

Corollary 13 *Suppose that* ***A1–A3*** *in Assumption 1 hold and $\mathcal{F}$ is locally strongly convex with a constant $\kappa > 0$ on $\mathbf{B}_r(u^*)$ with some $r \in (0, \infty)$ and $u^* \in S_*$. Then, the sequence of iterates $\{u_k\}_k$ generated by Algorithm 1 converges locally R-linear in the strong topology to u^*. In other words, there exists $r_0 \leq r$ such that (62) and (63) hold for every $u_0 \in \mathbf{B}_{r_0}(u^*)$.*

Proof Since Ψ is locally strongly convex, using [59, Proposition 3.23], we can write

$$\Psi(u) - \Psi(v) \geq (w, u - v)_H + \tfrac{\kappa}{2}\|u - v\|_H^2 \quad \text{for all } u, v \in \mathbf{B}_r(u^*) \text{ and } w \in \partial\Psi(v). \tag{67}$$

Setting $v = u^*$ and $w = 0$, we can easily see that the strong quadratic growth property (48) holds for $\Gamma := \mathbf{B}_r(u^*)$ and $\gamma_{\Psi,\Gamma} := \frac{\kappa}{2}$. Therefore, the proof follows by Corollary 12. □

Remark 5 Note that, if the strong convexity of $\mathcal{F}$ in Corollary 13 holds globally, then R-linear convergence is also obtained globally. More precisely, (62) and (63) hold for every $u_0 \in H$.

3.4 On relaxing the global Lipschitz continuity of $\nabla\mathcal{F}$ for problems governed by PDEs

In this section, we analyze a list of assumptions satisfied for a large class of optimization problems governed by PDEs. We will study the applicability of these assumptions in Sect. 4.

Assumption 3 For problem (**P**),

- ***H1***: $\Psi\colon H \to \mathbb{R} \cup \{+\infty\}$ is bounded from below and radially unbounded, i.e. $\lim_{\|u\|_H \to \infty} \Psi(u) = \infty$.
- ***H2***: $\mathcal{R}\colon H \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous.
- ***H3***: $\mathcal{F}\colon H \to \mathbb{R} \cup \{+\infty\}$ is continuously Fréchet differentiable on $\operatorname{int}(\operatorname{dom} \mathcal{F})$ containing $\operatorname{dom} \mathcal{R}$, that is, $\operatorname{dom} \mathcal{R} \subseteq \operatorname{int}(\operatorname{dom} \mathcal{F})$.
- ***H4***: $\nabla\mathcal{F}\colon \operatorname{int}(\operatorname{dom} \mathcal{F}) \to H$ is $L_{\mathcal{F}'}$-Lipschitz continuous on every weakly sequentially compact subset of $\operatorname{dom} \mathcal{R}$.
- ***H5***: $\mathcal{F}$ is weakly sequentially lower semicontinuous (wlsc) and $\nabla\mathcal{F}\colon \operatorname{int}(\operatorname{dom} \mathcal{F}) \to H$ is weak-to-strong sequentially continuous.

Here, compared to Assumption 1, we do not impose the global Lipschitz condition on $\nabla\mathcal{F}$. Nevertheless, Assumption 3 will be sufficient to reproduce the results of Sect. 3.1. Furthermore, **H1**, **H2**, and **H5** impose a set of conditions that ensure the existence of a global minimizer to (**P**), as will be shown in the following.

Proposition 14 *Suppose that **H1**, **H2**, and **H5** hold. Then, problem* (**P**) *possesses a global minimizer $\bar{u} \in H$, and as a consequence, $S_* \neq \emptyset$. Further, every level set of Ψ is weakly sequentially compact.*

Proof The proof uses standard arguments based on the direct methods in the calculus of variations. By **H1**, there exists

$$\bar{\Psi} := \inf_{u \in H} \Psi(u),$$

and a minimizing sequence $\{u_n\}_n \subset H$ with $\Psi(u_n) \to \bar{\Psi}$ for $n \to \infty$. Due to **H1** and [26, Proposition 11.11]), every level set of Ψ is bounded and, thus, $\{u_n\}_n$ admits a weakly convergent subsequence $\{u_{n_k}\}_k$ with $u_{n_k} \rightharpoonup \bar{u} \in H$ on $[\Psi \leq \Psi(\tilde{u})]$ for some $\tilde{u} \in \operatorname{dom} \mathcal{R}$. Properties **H2** and **H5** imply that Ψ is wlsc, and thus, $\Psi(\bar{u}) \leq \liminf_{k\to\infty} \Psi(u_{n_k}) = \bar{\Psi}$. This shows that $\Psi(\bar{u}) = \bar{\Psi}$, i.e. $\bar{u} \in H$ is a global minimizer of (**P**). Hence $S_* \neq \emptyset$. Further, since Ψ is wlsc, every level set of Ψ is weakly sequentially closed. Together, with the boundedness of the level sets, we conclude that every level set of Ψ is weakly sequentially compact. □

Comparing the properties given in Assumptions 1 and Assumption 3 in detail, we realize that besides the global Lipschitz continuity of $\nabla\mathcal{F}$, the remaining properties of Assumption 1 follow from those of Assumption 3. Further, due to Lemma 4(i) and Lemma 5, Algorithm 1 is well-defined, even without the Lipschitz continuity of $\nabla\mathcal{F}$. Further, it can be seen from (10) and Definition 1, that for every $u_0 \in \operatorname{dom} \mathcal{R}$, the whole sequence $\{u_k\}_k$ generated by Algorithm 1 stays in $U_0 := [\Psi \leq \Psi(u_0)]$. Since U_0 is weakly sequentially compact and $U_0 \subset \operatorname{dom} \mathcal{R}$, using **H4** we can deduce that $\nabla\mathcal{F}$ is $L_{\mathcal{F}'}$-Lipschitz continuous on U_0 with $\{u_k\}_k \subset U_0$. Therefore all the results and estimates in Sect. 3.1 can be easily applied in the presence of Assumption 3. Further, similar observations are also valid for the results given in Sect. 3.2, provided that we replace **H5** by **A'4** and analogously also for Sect. 3.3.

4 Application to PDE-constrained optimization

In this section, we investigate the applicability of our theoretical results to two problems governed by semilinear elliptic and parabolic PDEs. For each example, we discuss the justification of Assumption 3.

4.1 Elliptic problem

As a first example, we consider the following semilinear elliptic problem

$$\min_{(y,u)} \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\sigma}{2}\|u\|^2_{L^2(\Omega)} + \lambda\|u\|_{L^1(\Omega)} \tag{P_{E}}$$

$$\text{subject to} \begin{cases} -\kappa\Delta y + \exp(y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega, \\ u_a \le u \le u_b & \text{a.e. in } \Omega, \end{cases} \tag{68}$$

on a bounded domain $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$, which is either convex or possesses a $C^{1,1}$-boundary $\partial\Omega$. Here the parameter $\kappa > 0$ stands for the diffusion, the parameters $\sigma, \lambda > 0$ weigh the cost terms, $u_a, u_b \in \mathbb{R}$ with $u_a < 0 < u_b$ are control bounds, and $y_d \in L^2(\Omega)$ denotes the desired state. First, we show that problem (P_{E}) can be rewritten in the form (**P**). It is well-known, cf. [60, Theorems 2.7 and 2.12], that for given $u \in H := L^2(\Omega)$, the state equation (68) is uniquely solvable in the weak sense, i.e., $y(u) \in W \cap C(\overline{\Omega})$ with $W := H^1_0(\Omega)$, and the control-to-state operator $u \mapsto y(u)$ is well-defined and twice continuously Fréchet differentiable from H to $W \cap C(\overline{\Omega})$. This operator is typically constructed by applying the implicit function theorem on the following equality constraints

$$\mathcal{E}(y, u) = 0 \text{ in } Z' := H^{-1}(\Omega) \quad \text{with} \quad \mathcal{E}(y, u) := -\kappa\Delta y + \exp(y) - u.$$

Here, $\mathcal{E} : W \times H \to Z'$ is twice continuously Fréchet differentiable, for every $u \in H$, $\mathcal{E}_y(y, u) \in \mathcal{L}(W, Z')$ has a bounded inverse, and $\mathcal{E}_u(y, u) \in \mathcal{L}(H, Z')$ is continuous. Then, by defining

$$\mathcal{F}(u) := \frac{1}{2}\|y(u) - y_d\|^2_{L^2(\Omega)}, \quad \mathcal{R}(u) := \frac{\sigma}{2}\|u\|^2_{L^2(\Omega)} + \lambda\|u\|_{L^1(\Omega)} + \delta_{\mathcal{U}}(u), \tag{69}$$

with the indicator function $\delta_{\mathcal{U}}$ of $\mathcal{U} := \{u : u_a \le u \le u_b \text{ a.e. in } \Omega\} \subset H$, problem ($P_{\mathrm{E}}$) has the form (**P**). Further, by standard computations as in [61, Chapter 1.6], it can be shown that

$$\nabla\mathcal{F}(u) = -p(u) \quad \text{in } H, \tag{70}$$

where $p = p(u) \in W$ (see e.g., [60, Theorem 3.2]) is the weak solution of the adjoint equation

$$\begin{cases} -\kappa \Delta p + \exp(y)p = -(y - y_d) & \text{in } \Omega, \\ p = 0 & \text{on } \partial\Omega, \end{cases} \tag{71}$$

with $y = y(u)$. Now it remains to verify Assumption 3 for the reduced problem (**P**) with (69). Properties **H1** and **H2** are clearly satisfied. **H3** follows using the chain rule and the continuous Fréchet differentiability of the control-to-state mapping $y(u)$. Note that

$$\nabla \mathcal{F}(u) = (y'(u))^* \mathcal{J}'(y(u)) \quad \text{with} \quad \mathcal{J}(y) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)}, \tag{72}$$

where $y'(u)h = -\mathcal{E}_y^{-1}(y(u), u)\mathcal{E}_u(y(u), u)h$ for every $h \in H$ and the superscript "*" denotes the adjoint operator. It remains to verify **H4** and **H5**. Since the control-to-state operator $u \to y(u)$ and $\mathcal{J}$ are twice continuously Fréchet differentiable, using (72), and the compact embedding $W \overset{c}{\hookrightarrow} H$, one obtains that the second Fréchet derivative of $\mathcal{F}$ is bounded on bounded sets. Therefore $\nabla\mathcal{F}$ is Lipschitz continuous on any bounded set in H. Thus, **H4** holds. Finally using the weak formulation of the state (68) and adjoint (71) equations and [60, Theorem 2.11], it can be shown that $y(u_k) \rightharpoonup y(\bar{u})$ and $p(u_k) \rightharpoonup p(\bar{u})$ in W as $u_k \rightharpoonup \bar{u}$ in H. This together with $W \overset{c}{\hookrightarrow} H$, (70), and (69), implies that $\mathcal{F}(u_k) \to \mathcal{F}(\bar{u})$ and $\nabla\mathcal{F}(u_k) \to \nabla\mathcal{F}(\bar{u})$ in W as $u_k \rightharpoonup \bar{u}$ in H. Hence, **H5** holds. This finishes the verification of Assumption 3. Note that using (70) and (3), the Fermat principle can be expressed as

$$u^* = \mathrm{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u^* + \frac{1}{\alpha} p(u^*)) \quad \text{for some } \alpha > 0,$$

where $p(u^*)$ is the solution of (71) for $u = u^*$ and $\mathrm{Prox}_{\frac{1}{\alpha}\mathcal{R}}$ can be characterized in a pointwise a.e. sense. Due to [26, Proposition 24.13], we have for any $\alpha > 0$

$$\left[\mathrm{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u)\right](x) = \mathrm{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u(x)) \quad \text{for almost all } x \in \Omega,$$

where a pointwise a.e. closed form representation of the proximal operator is given by

$$\left[\mathrm{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u)\right](x) = \min\{\max\{\begin{cases} \frac{1}{C_1}(u(x) - C_2), & u(x) > C_2, \\ \frac{1}{C_1}(u(x) + C_2), & u(x) < -C_2, \\ 0, & \text{otherwise}, \end{cases}, u_a\}, u_b\},$$

with $C_1 := 1 + \sigma/\alpha$ and $C_2 := \lambda/\alpha$. The calculations are made similarly to [4, Section 6].

4.2 Parabolic problem

As a second example, we will consider the following semilinear parabolic problem

$$\min_{(y,u)} \frac{1}{2}\|y - y_d\|^2_{L^2(0,T;L^2(\Omega))} + \lambda\|u\|_{L^1(0,T;L^1(\Omega))} \tag{P_{P}}$$

$$\text{subject to} \begin{cases} \dot{y} - \kappa\Delta y + y^3 = u & \text{in } (0,T)\times\Omega, \\ y = 0 & \text{on } (0,T)\times\partial\Omega, \\ y(0) = y_0 & \text{in } \Omega, \\ u_a \le u \le u_b & \text{a.e. in } \Omega, \end{cases} \tag{73}$$

where $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$ is a bounded domain with Lipschitz boundary $\partial\Omega$ and $T > 0$. Further, $\kappa > 0$, $\lambda \ge 0$, and the control bounds $u_a, u_b \in \mathbb{R}$ are defined as in problem (P_{E}). We also consider the desired state $y_d \in L^2(0,T;L^2(\Omega))$ and initial function $y_0 \in H^1_0(\Omega)$.

Similarly to the problem (P_{E}), we define the control-to-state operator $u \mapsto y(u)$ from $H := L^2(0,T;L^2(\Omega))$ to $W := L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega))$. The well-posedness and twice continuous Fréchet differentiability of the control-to-state operator can be established by similar arguments as given in [1, 62]. Compared to the elliptic problem (P_{E}), we define

$$\mathcal{F}(u) := \frac{1}{2}\|y(u) - y_d\|^2_{L^2(0,T;L^2(\Omega))}, \quad \mathcal{R}(u) := \lambda\|u\|_{L^1(0,T;L^1(\Omega))} + \delta_{\mathcal{U}}(u), \tag{74}$$

with $\mathcal{U} := \{u : u_a \le u \le u_b \text{ a.e. in } (0,T)\times\Omega\} \subset H$ and also consider

$$\mathcal{E}(y,u) := \begin{pmatrix} \dot{y} - \kappa\Delta y + y^3 - u \\ y(0) \end{pmatrix} \quad \text{with} \quad Z' := L^2(0,T;L^2(\Omega)) \times H^1_0(\Omega).$$

In this case, (70) holds with $p = p(u) \in W$ as the strong solution of the adjoint equation

$$\begin{cases} -\dot{p} - \kappa\Delta p + 3y^2 p = -(y - y_d) & \text{in } (0,T)\times\Omega, \\ p = 0 & \text{on } \partial\Omega, \\ p(T) = 0 & \text{in } \Omega, \end{cases} \tag{75}$$

with $y = y(u)$. The verification of Assumption 3 for (P_{P}) follows by the similar arguments given for (P_{E}) and using the compact embedding $W \overset{c}{\hookrightarrow} L^2(0,T;H^1_0(\Omega))$ and consequently $W \overset{c}{\hookrightarrow} H$. Moreover, the closed form pointwise a.e. representation of the proximal operator is given by

$$\mathrm{Prox}_{\frac{1}{\alpha}\mathcal{R}}(u)(t,x) = \min\{\max\{\begin{cases} u(t,x) - C_2, & u(t,x) > C_2, \\ u(t,x) + C_2, & u(t,x) < -C_2, \\ 0, & \text{otherwise}, \end{cases}, u_a\}, u_b\},$$

where $C_2 = \lambda/\alpha$.

4.3 Discussion on the quadratic growth condition

In the remainder of the section, we present a situation in which $\mathcal{F}$ is locally strongly convex and, as a consequence, Algorithm 1 converges locally R-linear by Corollary 13. We consider the case that for $u^* \in S_*$, $\mathcal{F}$ is twice continuously Fréchet differentiable and its second derivative satisfies

$$\mathcal{F}''(u^*)(h,h) \geq C\|h\|_H^2 \quad \text{for all } h \in H, \tag{76}$$

with some $C > 0$. For the two previous problems, we express $\mathcal{F}''(u^*)$ in terms of solutions to PDEs and discuss when $\mathcal{F}$ is locally strongly convex.

For both problems (P_{E}) and (P_{P}), the control-to-state operator $u \mapsto y(u)$ from H to W is twice continuously Fréchet differentiable, and by similar computation as in [63], its second derivative can be expressed as

$$y''(u)(h,q) = -\mathcal{E}_y^{-1}(y(u),u)\mathcal{E}_{yy}(y(u),u)(y'(u)h, y'(u)q) \quad \text{for all } h, q \in H, \tag{77}$$

since the control operator is linear.

To be able to use Corollary 13 for problem (P_{E}), we first redefine $\mathcal{F}$ and $\mathcal{R}$ as follows

$$\mathcal{F}(u) := \frac{1}{2}\|y(u) - y_d\|_{L^2(\Omega)}^2 + \frac{\sigma}{2}\|u\|_{L^2(\Omega)}^2, \qquad \mathcal{R}(u) := \lambda\|u\|_{L^1(\Omega)} + \delta_{\mathcal{U}}(u).$$

In this case, with the same arguments given in Sect. 4.1, (P_{E}) can be rewritten as (**P**) for which **H1**–**H4** hold. Further, using the chain rule and (77), we can write for every $h, q \in H$ that

$$\begin{aligned}\mathcal{F}''(u)(h,q) &= \langle \mathcal{J}''(y(u))y'(u)h, y'(u)q\rangle_{W',W} \\ &\quad + \langle -\mathcal{E}_y^{-*}(y(u),u)\mathcal{J}'(y(u)), \mathcal{E}_{yy}(y(u),u)(y'(u)h, y'(u)q)\rangle_{Z,Z'} + \sigma(h,q)_H,\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ stands for the dual pairing and $\mathcal{J}$ was defined in (72). Hence, $\mathcal{F}''(u^*)$ with $u^* \in S_*$ can be expressed as

$$\mathcal{F}''(u^*)(h,h) = \|y^h\|_{L^2(\Omega)}^2 + \int_\Omega p^* \exp(y^*)(y^h)^2 \mathrm{d}x + \sigma\|h\|_H^2, \tag{78}$$

where $y^* = y(u^*)$, $p^* = p(u^*)$, and $y^h \in W$ is the weak solution of the linearized state equation

$$\begin{cases} -\kappa \Delta y^h + \exp(y^*) y^h = h & \text{in } \Omega, \\ y^h = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that, $\mathcal{F}$ is locally strongly convex on a neighborhood of $u^* \in S_*$ provided that (76) holds for some $C > 0$. Further, using (78), (76) can be equivalently rewritten as

$$\sigma \|h\|^2_{L^2(\Omega)} + \|y^h\|^2_{L^2(\Omega)} + \int_\Omega p^* \exp(y^*)(y^h)^2 \mathrm{d}x \geq C\|h\|^2_{L^2(\Omega)} \text{ for all } h \in H. \tag{79}$$

We observe that the only term in (79) that can spoil the positive definiteness of $\mathcal{F}''(u^*)$ and, hence, local strong convexity of $\mathcal{F}$, is the term involving p^*. This term originates from the nonlinearity in the state equation. Note that either a small enough adjoint p^* or a large enough parameter σ ensure that (79) and equivalently (76) hold. A small enough adjoint can occur, for instance, if $\|y^* - y_d\|^2_{L^2(\Omega)}$ is sufficiently small. Similarly, for the second example (P_P), we obtain for $\mathcal{F}$ and $\mathcal{R}$ defined in (74) that

$$\mathcal{F}''(u^*)(h, h) = \|y^h\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^T \int_\Omega 6p^* y^* (y^h)^2 \mathrm{d}x\mathrm{d}t, \tag{80}$$

where $y^h \in W$ is the weak solution to the linearized state equation

$$\begin{cases} \dot{y}^h - \kappa \Delta y^h + 3(y^*)^2 y^h = h & \text{in } (0, T) \times \Omega, \\ y^h = 0 & \text{on } (0, T) \times \partial\Omega, \\ y^h(T) = 0 & \text{in } \Omega, \end{cases}$$

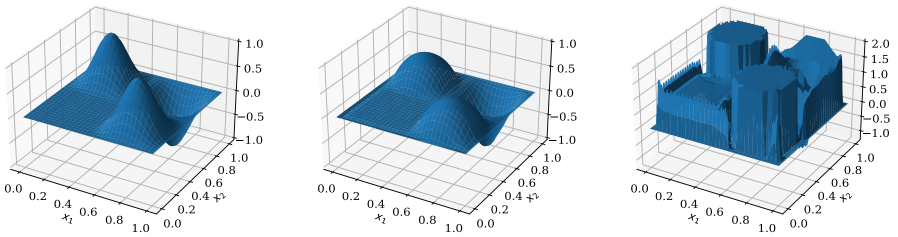
and p^* solves the weak formulation of (75) for y^*. For this problem, due to the absence of the term $\frac{\sigma}{2}\|\cdot\|^2_H$ in the objective function, the local strong convexity is not clear. For the elliptic problem (P_E), the authors in [64] have studied a weaker condition which implies that the quadratic growth condition (48) is satisfied. Furthermore from [65, Section 3], it especially follows that for $\sigma = 0$ condition (79) cannot be satisfied for the elliptic problem (P_E).

5 Numerical experiments

In this section, we report on the numerical experiments for the problems (P_E) and (P_P) in order to verify the capabilities of Algorithm 1 numerically. Throughout, we use $\|\mathcal{G}_{\alpha_k}(u_k)\|_H \leq \varepsilon_{\mathrm{tol}}$ with some tolerance $\varepsilon_{\mathrm{tol}} > 0$ as the termination condition, as it is proposed in Sect. 2. Our codes have been implemented in Python 3 and use FEniCS

Table 1 Example 1: Parameter setting

Optimization problem							Algorithm 1						
Ω	κ	σ	λ	y_d	u_a	u_b	ε_{tol}	$\alpha_{\inf}$	$\alpha_{\sup}$	α_0	η	δ	$m_{\max}$
$(0,1)^2$	10^{-2}	10^{-4}	10^{-3}	See Fig. 1a	-3	2	10^{-6}	10^{-4}	10^2	10	8	0.9	8



(a) Desired State (b) Optimal State (c) Optimal Control

Fig. 1 Example 1: The desired state y_d, the optimal state, and the optimal control (left to right)

(see [66]) for the matrix assembly. Sparse memory management and computations have been implemented with SciPy (see [67]). All computations below have been run on an Ubuntu 22.04 notebook with 32 GB main memory and an Intel Core i7-8565U CPU.

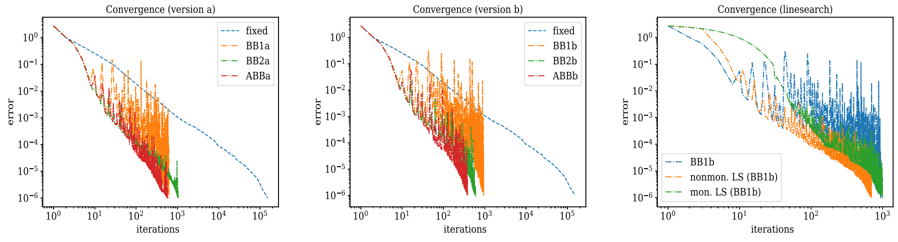
We will also compare different step-size approaches for the iterations (1) with respect to gradient-like evaluations and function evaluations as introduced in Theorem 7. We consider a fixed step-size, different combination of BB-type step-sizes presented in Sect. 2 (without linesearch method), and BB-type step-sizes incorporated with the (non-)monotone linesearch approach.

Note that for problems governed by nonlinear PDEs such as (P_{E}) and (P_{P}), any gradient-like $\mathcal{G}_\alpha$ evaluation requires solving a nonlinear state equation and a linear adjoint equation and any function Ψ evaluation is involved with solving a nonlinear state equation. Furthermore, the number of gradient-like evaluations corresponds to the number of iterations k of Algorithm 1.

Example 1 (Elliptic problem) In this example, we consider problem (P_{E}). For the spatial discretization, we follow a discretize-before-optimize approach and use P_1-type finite elements on a Friedrichs-Keller triangulation of the spatial domain Ω. To efficiently evaluate the nonlinearity, we resort to mass lumping. For the numerical tests, we choose the parameters summarized in Table 1. Note that for the fixed step-size approach, we set $\alpha_k = \alpha_0$ in (1) for all $k \in \mathbb{N}_0$.

The desired state, the optimal state, and the optimal control are illustrated in Fig. 1. We can see that the bounds are active for the control, though no strong sparsity is promoted, due to the choices of λ and σ.

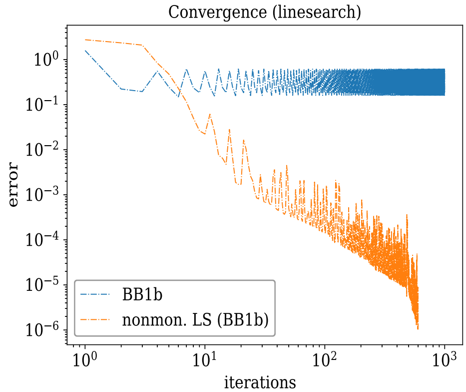
We compare the different BB-type step-sizes presented in Sect. 2 with the fixed step-size approach and with a (non-)monotone linesearch approach. The results regarding computational time, function evaluations, and gradient-like evaluations are gathered



(a) (b) (c)

Fig. 2 Example 1: Convergence of Algorithm 1. "Error" refers to $\|\mathcal{G}_{\alpha_k}(u_k)\|_H$ at the current iterate

Fig. 3 Example 1: Example of non-convergence without linesearch. "Error" refers to $\|\mathcal{G}_{\alpha_k}(u_k)\|_H$ at the current iterate



in Table 2. For the linesearch methods, the most volatile of the novel BB-type step-size updates, i.e. BB1b, is used as the initial trial step-sizes within Algorithm 1.

As can be seen from Table 2, for this example, all other approaches outperform the one with fixed step-size by a huge margin of about two orders of magnitude. The alternating BB-methods appear to be more efficient compared to the single BB-updates for both the old and novel step-sizes. Furthermore, except in the case of BB1, the novel step-sizes outperform the old ones and, overall, they appear to be competitive. All of these considerations are valid for both computational time and gradient-like evaluations. Function evaluations are only needed if a linesearch method is used. Compared to the BB1b method, the nonmonotone linesearch method needs about 250 fewer gradient-like evaluations, but this comes at the cost of 887 additional function evaluations for the linesearch and this results in an increased overall computational time. Compared to the monotone ($m_{\max} = 0$) linesearch, the nonmonotone approach performs significantly better. The convergence behavior is also visualized in Fig. 2.

Figure 3 presents an example, where a BB-type step-size update without linesearch fails to converge. In this example, we start the algorithms with $\alpha_0 = 1$ instead of $\alpha_0 = 10$. This confirms the necessity of incorporating a linesearch strategy to ensure convergence.

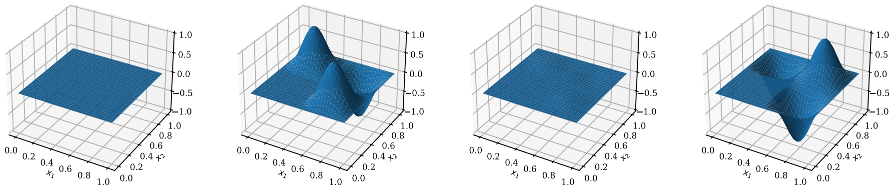
Example 2 (Parabolic problem) In this example, we consider (P_P). Here, the spatial domain is discretized in the same manner as in Example 1. For the temporal discretization, we use the Crank Nicolson/Adams Bashforth scheme [68]. In this scheme, the

Table 2 Example 1: Numerical results for fixed step-size, different spectral gradient methods as introduced in Sect. 2, and a (non-)monotone linesearch method with respect to BB1b

	Fixed	BB1a	BB2a	ABBa	BB1b	BB2b	ABBb	nonmon. LS (BB1b)	mon. LS (BB1b)
grad.-like eval.	153,662	618	1046	571	941	608	383	697	991
fun. eval.	0	0	0	0	0	0	0	887	1527
time [s]	$1.56 \cdot 10^3$	$1.01 \cdot 10^1$	$1.36 \cdot 10^1$	$1.55 \cdot 10^1$	$8.39 \cdot 10^0$	$7.92 \cdot 10^0$	$5.65 \cdot 10^0$	$2.21 \cdot 10^1$	$3.30 \cdot 10^1$

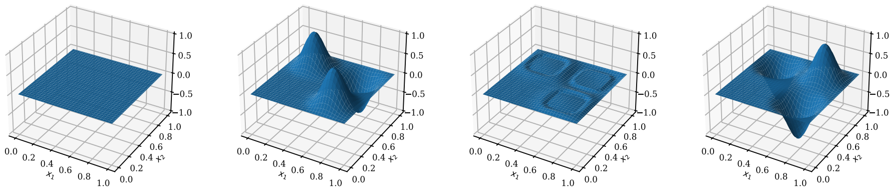
Table 3 Example 2: Parameter setting

Optimization problem							Algorithm 1						
Ω	T	κ	λ	y_d	u_a	u_b	ε_{tol}	$\alpha_{\inf}$	$\alpha_{\sup}$	α_0	η	δ	$m_{\max}$
$(0,1)^2$	1	10^{-2}	10^{-2}	See Fig. 4	-100	100	10^{-6}	10^{-4}	10^{2}	10	4	0.8	4



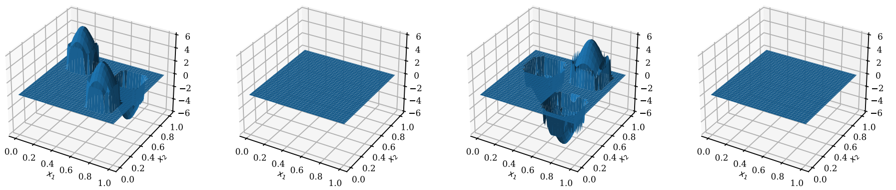
(a) $t = 0$ (b) $t = 0.25$ (c) $t = 0.5$ (d) $t = 0.75$

Fig. 4 Example 2: Snapshots of the desired state y_d at time instances 0, 0.25, 0.5 and 0.75



(e) $t = 0$ (f) $t = 0.25$ (g) $t = 0.5$ (h) $t = 0.75$

Fig. 5 Example 2: Snapshots of the optimal state at time instances 0, 0.25, 0.5 and 0.75



(i) $t = 0$ (j) $t = 0.25$ (k) $t = 0.5$ (l) $t = 0.75$

Fig. 6 Example 2: Snapshots of the optimal control at time instances 0, 0.25, 0.5 and 0.75

implicit Crank Nicolson scheme is used except for the nonlinear terms which are treated using the explicit Adams Bashforth scheme and mass lumping. For the numerical tests, we choose the parameters summarized in Table 3. For the fixed step-size approach, we choose $\alpha_k = \alpha_0$ for $k \in \mathbb{N}_0$, similarly to the previous example.

The desired state, the optimal state, and the optimal control at time instances $t = 0, 0.25, 0.5, 0.75$ are depicted in Figs. 4, 5, and 6, respectively. We can see sparsity in space for the control. Note that sparsity in time can also be observed in the sense that the control stays zero on an interval between time instances $t = 0.25$ and $t = 0.75$.

Similarly to Example 1, we compare the different BB-type step-sizes presented in Sect. 2 with the fixed step-size approach and with a (non-)monotone linesearch

Table 4 Example 2: Numerical results for fixed step-size, different spectral gradient methods as introduced in Sect. 2, and a (non-)monotone linesearch method with respect to BB1b

	Fixed	BB1a	BB2a	ABBa	BB1b	BB2b	ABBb	nonmon. LS (BB1b)	mon. LS (BB1b)
grad.-like eval.	515,044	375	758	784	470	445	221	463	471
fun. eval.	0	0	0	0	0	0	0	639	713
time [s]	$5.66 \cdot 10^4$	$4.28 \cdot 10^1$	$8.70 \cdot 10^1$	$8.89 \cdot 10^1$	$5.35 \cdot 10^1$	$5.08 \cdot 10^1$	$2.50 \cdot 10^1$	$8.65 \cdot 10^1$	$9.57 \cdot 10^1$

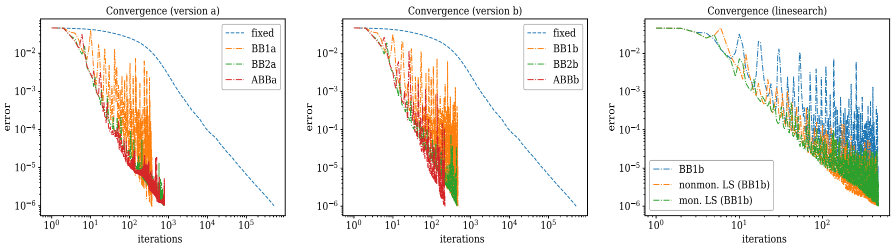


Fig. 7 Example 2: convergence of Algorithm 1. "Error" refers to $\|\mathcal{G}_{\alpha_k}(u_k)\|_H$ at the current iterate

approach, and the results regarding computational time, function evaluations, and gradient-like evaluations are presented in Table 4. To reveal the efficiency of the linesearch strategy, we incorporate the least efficient BB-type step-size, namely BB1b, with the linesearch strategy.

All considerations and observations from Example 1 hold also true for this example. That is, the iterations (1) for the choice of the BB-type step-sizes significantly outperform those with fixed step-size, as is to be expected. The novel alternating BB-method outperforms the other approaches. The nonmonotone linesearch method outperforms the monotone version. It also outperforms the BB1b version concerning gradient-like evaluations, but the cost of additional 639 function evaluations again does not pay off for overall computational time. The convergence behavior is also visualized in Fig. 7.

To summarize, all the numerical results from the above examples show the capabilities of Algorithm 1 and the necessity for the use of a linesearch method. From the above example, we can conclude that the incorporation of nonmonotone linesearch and BB step-sizes leads to an efficient algorithm for a class of nonsmooth nonconvex PDE-constrained optimization problems. Moreover, the convergence behavior is far better than what our worst-case complexity results suggest. In particular, when combined with the BB methods, this is typical behavior, see e.g., [34, 35, 40]. Whether the strong quadratic growth condition helps to accelerate convergence in the elliptic example (P_{E}) towards the end is not entirely clear, although the orange curves in Figs. 2c and 3 suggest it.

Conclusion

We have studied the nonmonotone FBS method for a class of nonsmooth infinite-dimensional composite problems in Hilbert spaces. Starting with the general nonconvex setting, we have established global convergence with complexity $(1/\sqrt{k})$ and also provided a worst-case complexity analysis. Under additional convexity assumption, convergence has been improved to the sublinear of order $(1/k)$ in function values. If additionally, a quadratic growth-type condition is satisfied, we have shown R-linear convergence, both in function values and iterates. Additional difficulties arising in the transition from weak to strong convergence in the infinite-dimensional setting have been discussed in detail. Finally, the nonmonotone FBS and the novel BB step-size rules exploiting the nonsmooth part were successfully tested for elliptic and parabolic PDE-constrained problems.

Appendix A Proofs

A. 1 Proof of Theorem 7

Here, we restrict ourselves mainly to deriving (31). Justification of (32) follows by similar arguments. The proof is carried out by determining the minimum reduction of the objective function between iteration step u_{k+1} and its maximal predecessor, i.e. $u_{\ell(k)}$, with respect to (11) as long as Algorithm 1 has not been terminated. Note that $u_{\ell(k)}$ can be calculated without further evaluations of the objective function by storing previous iterates. When using Algorithm 1, we have the following two cases:

- Assume that (8) in Step 4 of Algorithm 1 holds for $\alpha_k := \alpha_{\text{int},k}$ and therefore $i_k = 0$. Then we have a decrease

$$\Psi(u_{\ell(k)}) - \Psi(u_{k+1}) \geq \frac{\delta}{\alpha_{\text{int},k}} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2 \geq \frac{\delta}{\alpha_{\text{sup}}} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2.$$

  In this case, only one additional function evaluation is necessary to obtain this decrease.
- Assume that (8) in Step 4 of Algorithm 1 fails to hold for $\alpha_{\text{int},k}$ and therefore $i_k \geq 1$ backtracking steps are performed. Due to (iii) of Lemma 4 we have $\alpha_k = \alpha_{\text{int},k}\eta^{i_k} < \frac{\eta L_{\mathcal{F}'}}{2(1-\delta)}$, and, as a consequence, we can bound i_k as follows $i_k < \left|\log_\eta\left(\frac{\eta L_{\mathcal{F}'}}{2\alpha_{\text{int},k}(1-\delta)}\right)\right|$. Hence, Step 4 requires at most $n_1 := \left\lfloor\left|\log_\eta\left(\frac{\eta L_{\mathcal{F}'}}{2\alpha_{\inf}(1-\delta)}\right)\right|\right\rfloor$ function evaluations. At the time that the linesearch strategy terminates in this step, we have the decrease

$$\Psi(u_{\ell(k)}) - \Psi(u_{k+1}) \geq \frac{\delta}{\alpha_k} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2 > \frac{2(1-\delta)\delta}{\eta L_{\mathcal{F}'}} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2.$$

Gathering the two above cases, we can see that function value decrease per function evaluation is given, in the worst case, by

$$\Psi(u_{\ell(k)}) - \Psi(u_{k+1}) \geq \gamma_{\text{decr}} \|\mathcal{G}_{\alpha_k}(u_k)\|_H^2,$$

with $\gamma_{\mathrm{decr}} := \min\left\{\frac{\delta}{\alpha_{\mathrm{sup}}}, \frac{2(1-\delta)\delta}{n_1\eta L_{\mathcal{F}'}}\right\}$. Further, using similar arguments as in the proof of Theorem 6, we obtain

$$\begin{aligned}
\Psi(u_0)-\bar{\Psi} &\geq \Psi(u_0)-\Psi(u_k) \geq \Psi(u_{\nu(0)})-\Psi(u_{\nu(\lceil \frac{k}{m_{\max}+1}\rceil)}) \\
&= \sum_{i=1}^{\lceil \frac{k}{m_{\max}+1}\rceil} \Psi(u_{\nu(i-1)})-\Psi(u_{\nu(i)}) \geq \sum_{i=1}^{\lceil \frac{k}{m_{\max}+1}\rceil} \gamma_{\mathrm{decr}}\|\mathcal{G}_{\alpha_{\nu(i)-1}}(u_{\nu(i)-1})\|_H^2 \\
&\geq \gamma_{\mathrm{decr}}\left\lceil \frac{k}{m_{\max}+1}\right\rceil \min_{1\leq i\leq \lceil \frac{k}{m_{\max}+1}\rceil} \|\mathcal{G}_{\alpha_{\nu(i)-1}}(u_{\nu(i)-1})\|_H^2 \qquad \text{(A1)}\\
&\geq \left(\frac{\gamma_{\mathrm{decr}}k}{m_{\max}+1}\right) \min_{0\leq i\leq \lceil \frac{k}{m_{\max}+1}\rceil(m_{\max}+1)-1} \|\mathcal{G}_{\alpha_i}(u_i)\|_H^2 \\
&\geq \left(\frac{\gamma_{\mathrm{decr}}k}{C_G^{2m_{\max}}(m_{\max}+1)}\right) \min_{0\leq i\leq k} \|\mathcal{G}_{\alpha_i}(u_i)\|_H^2 .
\end{aligned}$$

To find an $\varepsilon_{\mathrm{tol}}$-stationary point, we assume that up to the current iteration k Algorithm 1 has not been terminated, i.e. $\|\mathcal{G}_{\alpha_i}(u_i)\|_H > \epsilon$ for all $i = 0,\ldots,k$. In this case, using (A1), we can write $\Psi(u_0)-\bar{\Psi} > \left(\frac{\gamma_{\mathrm{decr}}k}{C_G^{2m_{\max}}(m_{\max}+1)}\right)\varepsilon_{\mathrm{tol}}^2$, and, therefore, the total number of function evaluations of Ψ in Algorithm 1 is bounded from above by

$$k_{\max}^f \leq \left\lfloor \frac{C_G^{2m_{\max}}(m_{\max}+1)(\Psi(u_0)-\bar{\Psi})}{\gamma_{\mathrm{decr}}\varepsilon_{\mathrm{tol}}^2}\right\rfloor = \left\lfloor \frac{\gamma_{\mathrm{comp}}^f(\Psi(u_0)-\bar{\Psi})}{\varepsilon_{\mathrm{tol}}^2}\right\rfloor .$$

Thus, we are finished with the verification of (31). Similarly, using (23), it can be easily shown that the total number of $\mathcal{G}_{\alpha_k}(u_k)$ evaluations is bounded by

$$k_{\max}^g \leq \left\lfloor \frac{C_G^{2m_{\max}}\overline{\alpha}(m_{\max}+1)(\Psi(u_0)-\bar{\Psi})}{\delta\varepsilon_{\mathrm{tol}}^2}\right\rfloor = \left\lfloor \frac{\gamma_{\mathrm{comp}}^g(\Psi(u_0)-\bar{\Psi})}{\varepsilon_{\mathrm{tol}}^2}\right\rfloor ,$$

and, thus, the proof is complete.

A. 2 Proof of Lemma 8

First, we show for every $k \geq 1$ that

$$\Psi(u_k) \leq \min_{w\in H} \mathcal{Q}_{\alpha_{k-1}}(w, u_{k-1}) + \frac{L_{\mathcal{F}'}}{2\alpha_{\inf}\alpha_{k-1}}\|\mathcal{G}_{\alpha_{k-1}}(u_{k-1})\|_H^2 . \qquad \text{(A2)}$$

Due to **A3**, the descent lemma, see [59, Lemma 1.30], implies

$$\mathcal{F}(u_k) \leq \mathcal{F}(u_{k-1}) + (\nabla\mathcal{F}(u_k), u_k-u_{k-1})_H + \frac{L_{\mathcal{F}'}}{2\alpha_{k-1}^2}\|\mathcal{G}_{\alpha_{k-1}}(u_{k-1})\|_H^2 .$$

Using the definition of $\mathcal{Q}_{\alpha_{k-1}}$ and the fact that u_k is the minimizer of $\mathcal{Q}_{\alpha_{k-1}}(\cdot, u_{k-1})$, we obtain

$$\begin{aligned}\Psi(u_k) &\le \min_{w\in H} \mathcal{Q}_{\alpha_{k-1}}(w, u_{k-1}) + \left(\frac{L_{\mathcal{F}'}}{2\alpha_{k-1}^2} - \frac{1}{2\alpha_{k-1}}\right)\|\mathcal{G}_{\alpha_{k-1}}(u_{k-1})\|_H^2 \\ &\le \min_{w\in H} \mathcal{Q}_{\alpha_{k-1}}(w, u_{k-1}) + \frac{L_{\mathcal{F}'}}{2\alpha_{k-1}^2}\|\mathcal{G}_{\alpha_{k-1}}(u_{k-1})\|_H^2,\end{aligned}$$

and, thus, (A2) follows from the fact that $\alpha_k \ge \alpha_{\inf}$ for all $k \in \mathbb{N}_0$. Due to the convexity of $\mathcal{F}$, we can write

$$\min_{w\in H} \mathcal{Q}_{\alpha_{k-1}}(w, u_{k-1}) \le \min_{w\in H}\{\Psi(w) + \tfrac{\alpha_{k-1}}{2}\|w - u_{k-1}\|_H^2\}. \tag{A3}$$

Further, due to the convexity of Ψ, we obtain for every $w = (1-\lambda)u_{k-1} + \lambda u^*$ with $\lambda \in [0, 1]$ and $u^* \in S_*$ that

$$\begin{aligned}\min_{w\in H}\{\Psi(w) + \tfrac{\alpha_{k-1}}{2}\|w - u_{k-1}\|_H^2\} &\le \Psi((1-\lambda)u_{k-1} + \lambda u^*) + \tfrac{\alpha_{k-1}\lambda^2}{2}\|u_{k-1} - u^*\|_H^2 \\ &\le (1-\lambda)\Psi(u_{k-1}) + \lambda\Psi^* + \tfrac{\alpha_{k-1}\lambda^2}{2}\|u_{k-1} - u^*\|_H^2.\end{aligned}$$

Combining with (A2) and (A3), we can write

$$\begin{aligned}\Psi(u_k) &\le (1-\lambda)\Psi(u_{k-1}) + \lambda\Psi^* + \tfrac{\alpha_{k-1}\lambda^2}{2}\|u_{k-1} - u^*\|_H^2 + \tfrac{L_{\mathcal{F}'}}{2\alpha_{\inf}\alpha_{k-1}}\|\mathcal{G}_{\alpha_{k-1}}(u_{k-1})\|_H^2 \\ &\le (1-\lambda)\Psi(u_{\ell(k-1)}) + \lambda\Psi^* + \tfrac{\alpha_{k-1}\lambda^2}{2}\|u_{k-1} - u^*\|_H^2 + \tfrac{L_{\mathcal{F}'}}{2\alpha_{\inf}\alpha_{k-1}}\|\mathcal{G}_{\alpha_{k-1}}(u_{k-1})\|_H^2.\end{aligned} \tag{A4}$$

Now, inserting $\nu(k)$ in the place of k in (A4) and using (20), we have

$$\begin{aligned}\Psi(u_{\nu(k)}) \le (1-\lambda)\Psi(u_{\nu(k-1)}) + \lambda\Psi^* + \tfrac{\alpha_{\nu(k)-1}\lambda^2}{2}\|u_{\nu(k)-1} - u^*\|_H^2 \\ + \tfrac{L_{\mathcal{F}'}}{2\alpha_{\inf}\alpha_{\nu(k)-1}}\|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H^2.\end{aligned} \tag{A5}$$

Subtracting Ψ^* from both sides of (A5), setting $\tilde{C} := \frac{L_{\mathcal{F}'}}{2\alpha_{\inf}}$, and using (iii) of Lemma 4, we obtain

$$\begin{aligned}\Psi(u_{\nu(k)}) - \Psi^* \le (1-\lambda)\left(\Psi(u_{\nu(k-1)}) - \Psi^*\right) + \tfrac{\overline{\alpha}\lambda^2}{2}\|u_{\nu(k)-1} - u^*\|_H^2 \\ + \tfrac{\tilde{C}}{\alpha_{\nu(k)-1}}\|\mathcal{G}_{\alpha_{\nu(k)-1}}(u_{\nu(k)-1})\|_H^2.\end{aligned} \tag{A6}$$

Finally, (33) follows from the fact that $u^* \in S_*$ was arbitrary and S_* is non-empty, closed, and convex.

Now we deal with the verification of (34). Let an arbitrary $u^* \in S_*$ be given, setting $k = 1$ and $\lambda = 1$ in (A6), we obtain

$$\Psi(u_{\nu(1)}) - \Psi^* \le \tfrac{\overline{\alpha}}{2}\|u_{\nu(1)-1} - u^*\|_H^2 + \tfrac{\tilde{C}}{\alpha_{\nu(1)-1}}\|\mathcal{G}_{\alpha_{\nu(1)-1}}(u_{\nu(1)-1})\|_H^2. \tag{A7}$$

Using, the fact that $0 \leq \nu(1) - 1 \leq m_{\max}$ (see L1), the firm nonexpansiveness of the proximal operator, and the Lipschitz continuity of $\nabla\mathcal{F}$, we can write that

$$\begin{aligned} \|u_{\nu(1)-1} - u^*\|_H &\leq \left(1 + \frac{L_{\mathcal{F}'}}{\alpha_{\nu(1)-2}}\right) \|u_{\nu(1)-2} - u^*\|_H \\ &\leq \left(1 + \frac{L_{\mathcal{F}'}}{\alpha_{\inf}}\right) \|u_{\nu(1)-2} - u^*\|_H \leq \cdots \leq \left(1 + \frac{L_{\mathcal{F}'}}{\alpha_{\inf}}\right)^{m_{\max}} \|u_0 - u^*\|_H. \end{aligned} \tag{A8}$$

Further, using (iv) of Lemma 4 successively, we obtain that

$$\|\mathcal{G}_{\alpha_{\nu(1)-1}}(u_{\nu(1)-1})\|_H \leq C_G^{m_{\max}} \|\mathcal{G}_{\alpha_0}(u_0)\|_H. \tag{A9}$$

Combining (A7), (A8), and (A9), we can write

$$\begin{aligned} \Psi(u_{\nu(1)}) - \Psi^* &\leq \frac{\overline{\alpha}}{2}\left(1 + \frac{L_{\mathcal{F}'}}{\alpha_{\inf}}\right)^{2m_{\max}} \|u_0 - u^*\|_H^2 + \frac{\tilde{C} C_G^{2m_{\max}}}{\alpha_{\inf}} \|\mathcal{G}_{\alpha_0}(u_0)\|_H^2 \\ &\leq \frac{\overline{\alpha}}{2}\left(1 + \frac{L_{\mathcal{F}'}}{\alpha_{\inf}}\right)^{2m_{\max}} \|u_0 - u^*\|_H^2 + \frac{\overline{\alpha}^2 \tilde{C} C_G^{2m_{\max}}}{\alpha_{\inf}} \|u_1 - u_0\|_H^2. \end{aligned} \tag{A10}$$

Further, the firm nonexpansiveness of the proximal operator and the Lipschitz continuity of $\nabla\mathcal{F}$ again imply that

$$\begin{aligned} \|u_1 - u_0\|_H &\leq \|u_0 - u^*\|_H + \|u_1 - u^*\|_H \\ &\leq 2\|u_0 - u^*\|_H + \frac{L_{\mathcal{F}'}}{\alpha_0}\|u_0 - u^*\|_H \leq \left(2 + \frac{L_{\mathcal{F}'}}{\alpha_{\inf}}\right)\|u_0 - u^*\|_H. \end{aligned} \tag{A11}$$

Thus, combining (A10) and (A11) and setting

$$C_0 := \frac{\overline{\alpha}}{2}\left(1 + \frac{L_{\mathcal{F}'}}{\alpha_{\inf}}\right)^{2m_{\max}} + \frac{\overline{\alpha}^2 \tilde{C} C_G^{2m_{\max}}}{\alpha_{\inf}}\left(2 + \frac{L_{\mathcal{F}'}}{\alpha_{\inf}}\right)^2,$$

we arrive at $\Psi(u_{\nu(1)}) - \Psi^* \leq C_0\|u_0 - u^*\|_H^2$. Hence, (34) follows from the fact that $u^* \in S_*$ is arbitrary.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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