




Maxitive functions with respect to general orders [☆]

Michael Kupper ^a, José M. Zapata ^{b,  *}

^a Department of Mathematics and Statistics, University of Konstanz, Germany

^b Universidad de Murcia, Dpto. de Estadística e Investigación Operativa, 30100 Espinardo, Murcia, Spain

ARTICLE INFO

MSC:
62C86
47H05
28A25
91B05

Keywords:

Maxitive functions
Representation theorem
Stochastic orderings

ABSTRACT

In decision-making, maxitive functions are used for worst-case and best-case evaluations. Maxitivity gives rise to a rich structure that is well-studied in the context of the pointwise order. In this article, we investigate maxitivity with respect to general preorders and provide a representation theorem for such functions. The results are illustrated for different stochastic orders in the literature, including the usual stochastic order, the increasing convex/concave order, and the dispersive order.

1. Introduction

In decision-making under risk and uncertainty, one typically considers a set of gambles X and a function $\psi : X \rightarrow \mathbb{R}$, which assigns to each gamble $f \in X$ a value $\psi(f)$, representing its price, risk, or utility. In the context of imprecise probabilities, such functions are referred to as lower/upper previsions; see [31]. In financial mathematics, they are commonly called risk measures [12] or nonlinear expectations [23].

In many cases, the function ψ takes the form of an expectation, where the value $\psi(f)$ represents the weighted average of the values associated with the gamble f . However, decisions are often based on assessments of best-case or worst-case scenarios, rather than relying on averages. These evaluations are captured by *maxitive functions*. A function ψ is called maxitive if it satisfies the condition $\psi(f \vee g) = \max\{\psi(f), \psi(g)\}$ for any two gambles f and g , where $f \vee g$ denotes the pointwise supremum of f and g .

Maxitive functions play a crucial role in *possibility theory* [10,11,34] and serve as aggregation functions in various fields such as economics, information fusion, and control theory [13,29]. Under reasonable assumptions, these functions can be represented in terms of maxitive integrals, such as the Shilkret integral discussed in [3,16,24,28]. Beyond decision theory, maxitive functions also arise in other areas, including idempotent or tropical analysis [17] and large deviations theory [15,16,25].

In probability theory, statistics, and particularly decision theory, a wide range of criteria exists to define order relations between gambles beyond the pointwise order [22,27]. While most of the literature has focused on maxitivity with respect to pointwise order, less attention has been paid to maxitivity with respect to more general orders. The aim of this work is to fill this gap by studying

[☆] We wish to express our gratitude to two anonymous referees for their careful reading of the manuscript and for their comments, which significantly improved the presentation. The second author was partially supported by grants 21955/PI/22 (funded by Fundación Séneca - ACyT Región de Murcia), PID2021-122126NB-C32 (funded by MICIU/AEI/10.13039/501100011033/ and ERDF "A way of making Europe"), and PID2022-137396NB-I00 (funded by MICIU/AEI/10.13039/501100011033/ and ERDF "A way of making Europe").

* Corresponding author.

E-mail addresses: kupper@uni-konstanz.de (M. Kupper), jmzg1@um.es (J.M. Zapata).

<https://doi.org/10.1016/j.fss.2025.109430>

Received 15 March 2024; Received in revised form 3 March 2025; Accepted 21 April 2025

Available online 30 April 2025

0165-0114/© 2025 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>).

maxitivity in the context of preorders, which enables the extension of the maxitive structure and its beneficial properties to a wider class of functions. By passing from the pointwise order to weaker *stochastic orders*, new maxitive functions are obtained. For example, in [32], it is shown that the value-at-risk is maxitive with respect to the usual stochastic order.

The main result of this paper provides a characterization and *representation of countably maxitive functions* on a preordered space. The proof relies on recent representation results from [7,18] for convex monotone functions on spaces of bounded continuous functions, and incorporates dual representations of sublevel sets as discussed in [9]. A key assumption is that the preorder \leq admits a numerical representation given by $f \leq g$ if and only if $\phi(f) \leq \phi(g)$ for all ϕ in some Polish space¹ Φ , where the mapping $\phi \mapsto \phi(f)$ is lower semicontinuous² for all $f \in X$. This assumption is fulfilled for typical stochastic orders. As an application of the main result, we obtain characterizations of maxitive functions with respect to the usual stochastic order, the increasing convex order, the convex order, the increasing concave order, and the dispersive order; see [22,27].

The paper is structured as follows: In Section 2, we provide some motivation for maxitivity in decision-making and possibility theory. Section 3 introduces the basic concepts and definitions. In Section 4, we present and prove the main result. Section 5 explores variations of the main result in the context of different stochastic orders. Finally, the appendix provides representation results for maxitive risk measures on C_b , along with auxiliary results related to quantile functions.

2. Maxitive functions in decision-making

Let X be a set of gambles of the form $f : S \rightarrow [-\infty, \infty]$, where each gamble f assigns a loss $f(s)$ to each possible state $s \in S$.³ We consider a decision maker who evaluates the gambles using a decision function $\psi : X \rightarrow (-\infty, \infty]$, which in risk analysis is commonly referred to as a *risk measure*; see [12]. We adopt the convention that $\psi(f) \leq \psi(g)$ indicates that the gamble g is considered riskier than the gamble f . The set of acceptable gambles for the decision maker is defined by the *acceptance set*, given by $A := \{f \in X : \psi(f) \leq 0\}$.

We assume that the decision maker believes in an ordering \leq , thus evaluating gambles $f, g \in X$ satisfying $f \leq g$ with $\psi(f) \leq \psi(g)$. This could be the *pointwise order*, in which case $f \leq_{pw} g$ if and only if $f(s) \leq g(s)$ for all $s \in S$. In decision-making, however, there are numerous methods for comparing gambles beyond pointwise ordering. These include comparisons based on quantiles, the spread of quantiles, and other distributional properties. Such considerations lead to various stochastic orders, including the *usual stochastic order*, the *increasing convex stochastic order*, or the *dispersive order*. In the first case, we assign probabilities to the states in S and order gambles $f, g \in X$ using their quantile functions, i.e., $f \leq_{st} g$ if and only if $q_f(u) \leq q_g(u)$ for all $u \in (0, 1)$ (or equivalently, if the cumulative distribution function of f is pointwise greater than the cumulative distribution function of g). For further examples of stochastic orders and their definitions, we refer to the monographs [22,27].

In this article, we focus on decision functions whose acceptance sets remain invariant under *worst-case aggregation*. To be more precise, consider a decision maker with an acceptance set A . For two acceptable positions $f, g \in A$, we require that the supremum $f \vee g$ with respect to the decision maker's order \leq exists and remains acceptable, i.e., $f \vee g \in A$. Equivalently, if $f \vee g$ is not acceptable, then either f or g is not acceptable. Intuitively, this means that losses in certain states cannot be compensated by gains in other states. Suppose, in addition, that the decision function ψ is monotone, i.e., $f \leq g$ implies $\psi(f) \leq \psi(g)$, and satisfies the translation property, i.e., $\psi(f + c) = \psi(f) + c$ for all $f \in X$ and $c \in \mathbb{R}$, where $f + c$ denotes the function f shifted by a constant $c \in \mathbb{R}$. We then obtain that ψ is *maxitive*, i.e.,⁴

$$\psi(f \vee g) = \max\{\psi(f), \psi(g)\} \quad \text{for all } f, g \in X.$$

We also want to mention that in the literature, there are different versions of maxitivity, all of which require $\psi(\bigvee_{i \in I} f_i) = \sup_{i \in I} \psi(f_i)$, but for different families $(f_i)_{i \in I}$, for which the supremum $\bigvee_{i \in I} f_i$ is assumed to exist; see e.g. [3,28]. For example, ψ is called *completely maxitive* if this holds for all families $(f_i)_{i \in I}$, and ψ is called *countably maxitive* if this holds for countable families $(f_n)_{n \in \mathbb{N}}$. Furthermore, a topological perspective on maxitivity, which is closely linked to the large deviation principle, was recently investigated in [16]; see also [15,25].

2.1. Maxitivity for the pointwise order in the context of possibility theory; see [10,11,34]

As a simple example, consider a loss f that depends on three possible scenarios $S = \{\text{good, average, bad}\}$, each with a certain plausibility given by $\pi : S \rightarrow [0, 1]$. The weighted worst-case criterion then leads to the completely maxitive function

$$\psi(f) = \max_{s \in S} f(s)\pi(s)$$

on the space of positive gambles $f : S \rightarrow [0, \infty]$. We note that this example generalizes to arbitrary state spaces S . Moreover, the mapping $\sup_{s \in S} f(s)\pi(s)$ coincides with the *Shilkret integral* $\int^S f \, d\Pi$ with respect to the possibility measure Π , which assigns each subset $B \subset S$ the value $\Pi(B) := \sup_{s \in B} \pi(s)$, where π is a possibility distribution; see e.g. [3,5,19,28]. Another closely related

¹ I.e., a separable completely metrizable topological space.

² I.e., $\{\phi \in \Phi : \phi(f) \leq s\}$ is closed for all $s \in \mathbb{R}$.

³ Here, we view negative losses as gains. We note that decision functions are usually defined on spaces of positions. However, for our purposes, describing them in terms of losses seems to be more transparent from a mathematical perspective. Since positions and losses only differ by their sign, all results can be easily transformed.

⁴ Indeed, since both f and g are smaller than $f \vee g$, we obtain the inequality \geq by monotonicity. As for the other inequality \leq , let $s := \max\{\psi(f), \psi(g)\}$ so that $f - s, g - s \in A$ due to the translation property, which implies $\psi(f \vee g) \leq s$ as the supremum of $f - s$ and $g - s$ remains acceptable.

example is the penalized worst-case evaluation $\sup_{s \in S} \{f(s) - \alpha(s)\}$ for some penalty $\alpha : S \rightarrow [-\infty, \infty]$. Again this is a completely maxitive function on the space X of all gambles $f : S \rightarrow [-\infty, \infty]$. Conversely, it is known that every completely maxitive function $\psi : X \rightarrow [-\infty, \infty]$, which has the translation property and satisfies $\psi(0) \in \mathbb{R}$, admits the representation

$$\psi(f) = \sup_{s \in S} \{f(s) - \alpha(s)\} \quad \text{for all } f \in X, \tag{2.1}$$

for some penalty $\alpha : S \rightarrow [-\infty, \infty]$; see [3, Corollary 7]. The right-hand side in the previous equation corresponds to the so-called convex integral, which is obtained as a transformation of the Shilkret integral; see [3]. In the translation-invariant case (particularly for monetary risk measures), these considerations show that maxitive functions with respect to the pointwise order take the form (2.1) of a penalized worst-case evaluation over the state space.

2.2. Maxitivity with respect to other orders

As an illustration, we start with the usual stochastic order. In this case, we order the distributions of the gambles according to their quantile functions. We recall that for two gambles f and g , there exists a gamble $f \vee_{\text{st}} g$ whose quantile function is the smallest one that dominates the quantiles q_f and q_g ; see [22]. A decision function is then maxitive if $\psi(f \vee_{\text{st}} g) = \max\{\psi(f), \psi(g)\}$ for all gambles f, g . Intuitively, maxitivity still ensures that losses in certain states cannot be compensated by gains in other states. However, in this context, the relevant ‘states’ refer to the levels of the quantile functions. More formally, suppose we have two gambles f and g , where f is acceptable but $f \vee_{\text{st}} g$ is not acceptable, for a decision function ψ with acceptance set A . If ψ is maxitive, then any gamble h that satisfies $q_s(h) = q_s(g)$ for all s in the ‘bad states’ $U = \{s : q_g(s) \geq q_f(s)\}$ is not acceptable, regardless of its values $q_s(h)$ in the ‘good states’ $s \in U^c$. Indeed, since $f \vee_{\text{st}} g \leq f \vee_{\text{st}} h$, it follows that

$$0 < \psi(f \vee_{\text{st}} g) \leq \psi(f \vee_{\text{st}} h) = \max\{\psi(f), \psi(h)\}.$$

Given that $\psi(f) \leq 0$, this implies $\psi(h) > 0$, meaning that h is not acceptable. In other words, maxitivity with respect to the usual stochastic order implies that risky quantiles at certain levels (‘bad states’) cannot be compensated by less risky quantiles at other levels (‘good states’). When dealing with probability distributions, it seems natural to consider quantities (that are closely linked to the distribution) as the relevant ‘states’. A typical example of such a maxitive function is the *Value at Risk*, which is widely used in the financial industry. Formally, the Value at Risk at level $s \in (0, 1)$, defined by $\text{VaR}_s(f) := q_f(s)$, is maxitive with respect to the usual stochastic order but is not maxitive with respect to the pointwise order; see [32].

In the context of model aggregation in risk analysis, an interesting motivation for maxitivity with respect to the usual stochastic order was recently presented in [32], where the topic is discussed in great detail. Simplifying their example, consider a risk analyst who evaluates random losses using different methods, resulting in a set of possible risk models $\{f_1, \dots, f_n\}$, which are aggregated into a worst-case model $f^* := f_1 \vee_{\text{st}} \dots \vee_{\text{st}} f_n$. Under maxitivity, we obtain the desirable property that the risk of the worst-case model $\psi(f^*)$ equals the worst-case risk $\max\{\psi(f_1), \dots, \psi(f_n)\}$. Moreover, it is shown in [32] that every real-valued function ψ , which has the translation property, is lower semicontinuous with respect to the convergence in distribution⁵ and is completely maxitive (with respect to the usual stochastic order) admits a representation

$$\psi(f) = \sup_{s \in (0,1)} \{q_f(s) - \alpha(s)\} \quad \text{for all } f, \tag{2.2}$$

for some penalty $\alpha : (0, 1) \rightarrow [-\infty, \infty]$. The representations in (2.2) and (2.1) are closely related, as (2.2) can be interpreted as an analogue of (2.1), where quantile evaluation replaces scenario-wise evaluation.

Another widely used preorder in decision-making is the *dispersive order*, which is employed to compare the dispersion or variability of two gambles. Specifically, for two gambles f, g , we write $f \leq_{\text{disp}} g$ if and only if

$$q_f(v) - q_f(u) \leq q_g(v) - q_g(u) \quad \text{whenever } 0 < u < v < 1.$$

For any two gambles f and g , there exists a gamble $f \vee_{\text{disp}} g$, which is the least upper bound of f and g with respect to \leq_{disp} . Intuitively, $f \vee_{\text{disp}} g$ is the least dispersed gamble that is more dispersed than both f and g ; see [22]. We say that a function $\psi : X \rightarrow \mathbb{R}$ is maxitive with respect to \leq_{disp} if $\psi(f \vee_{\text{disp}} g) = \max\{\psi(f), \psi(g)\}$ for any two gambles f, g . In this context, maxitivity once again ensures that losses in certain states cannot be offset by gains in others. Here, the relevant ‘states’ correspond to all pairs of levels (u, v) with $0 < u < v < 1$, at which the quantile spread $q_g(v) - q_g(u)$ is evaluated. Thereby, maxitivity with respect to the dispersive order implies that risky quantile spreads at certain pairs of levels cannot be compensated by less risky quantile spreads at other levels.

These ideas naturally extend to other partial orders. Consider a decision-maker who compares gambles based on the evaluations of certain relevant features represented by a set Φ of functions, i.e., $f \leq g$ if and only if $\phi(f) \leq \phi(g)$ for all $\phi \in \Phi$.⁶ In this situation, assuming that any two gambles have a least upper bound, a decision function $\psi : X \rightarrow \mathbb{R}$ is called *maxitive* if it satisfies $\psi(f \vee_{\Phi} g) =$

⁵ I.e., $\liminf_{n \rightarrow \infty} \psi(f_n) \geq \psi(f)$ whenever f_n converges to f in distribution.

⁶ For the pointwise order, we consider the evaluation $\phi_s(f) = f(s)$ for each state $s \in S$. For the usual stochastic order, the evaluation is given by $\phi_s(f) = q_f(s)$ at each quantile level $s \in (0, 1)$. For the dispersive order, we use the evaluation $\phi_{(u,v)}(f) = q_f(v) - q_f(u)$ for each pair $(u, v) \in \Delta$ of quantile levels.

$\max\{\psi(f), \psi(g)\}$ for all gambles f, g . We prove in this paper that, in the translation-invariant case and under reasonable assumptions, maxitive functions can be represented in the form

$$\psi(f) = \sup_{\phi \in \Phi} \{\phi(f) - \alpha(\phi)\}, \tag{2.3}$$

for some penalty function $\alpha : \Phi \rightarrow [-\infty, \infty]$. However, in general, a function that admits a representation (2.3) is not necessarily maxitive. To ensure this, the penalty has to satisfy an additional stability condition. Furthermore, we want to emphasize that the ϕ 's generally do not satisfy the translation property (e.g. for the dispersive order), in which case the representation in (2.3) is more involved.

3. Basic concepts and definitions

In this section, we provide the formal definitions of *maxitivity with respect to general orders* and introduce the relevant related concepts. Let X be a non-empty set,⁷ which is endowed with a preorder

$$x \preceq y \text{ if and only if } \phi(x) \leq \phi(y) \text{ for all } \phi \in \Phi,$$

where Φ is a non-empty set of functions $\phi : X \rightarrow \mathbb{R}$. Throughout this article, the notation $Y \subset X$ denotes non-strict inclusion, meaning that every element of Y is also an element of X , allowing for the possibility that $Y = X$.

For the moment, we assume that (X, \preceq) is an upper semilattice, i.e., for any two elements $x, y \in X$, the set $\{x, y\}$ has a unique supremum $x \vee y \in X$. In that case, the preorder \preceq is a partial order.⁸ Our focus is on functions $\psi : X \rightarrow (-\infty, \infty]$ that are maxitive with respect to the order \preceq , i.e.,

$$\psi(x \vee y) = \max\{\psi(x), \psi(y)\} \text{ for all } x, y \in X.$$

Note that the supremum, and thus maxitivity, depends on the order \preceq . Such maxitive functions are characterized as follows:

Lemma 3.1. *Let (X, \preceq) be an upper semilattice. For a function $\psi : X \rightarrow (-\infty, \infty]$, the following conditions are equivalent:*

- (i) ψ is maxitive.
- (ii) ψ is monotone⁹ and has upwards directed sublevel sets.¹⁰
- (iii) ψ is monotone and for every partition $\{\Phi_1, \Phi_2\}$ of Φ and $x_1, x_2 \in X$ with $\phi(x_1) \geq \phi(x_2)$ for all $\phi \in \Phi_1$ and $\phi(x_2) \geq \phi(x_1)$ for all $\phi \in \Phi_2$, there exists $i \in \{1, 2\}$ such that

$$\phi(y) \geq \phi(x_i) \text{ for all } \phi \in \Phi_i \text{ implies } \psi(y) \geq \psi(x_1 \vee x_2).$$

Proof. (i) \Rightarrow (ii): Given $x, y \in X$ with $x \preceq y$, we have that $y = x \vee y$. Since ψ is maxitive, we obtain that $\psi(y) = \psi(x \vee y) = \max\{\psi(x), \psi(y)\} \geq \psi(x)$, which shows that ψ is monotone. Moreover, given $x, y \in A_s$ for some $s \in \mathbb{R}$, we fix $z^* = x \vee y$. By maxitivity, it follows that $\psi(z^*) = \psi(x) \vee \psi(y) \leq s$, so that $z^* \in A_s$. This shows that ψ has upwards directed sublevel sets.

(ii) \Rightarrow (i): Given $x, y \in X$, by monotonicity, we have that $\psi(x \vee y) \geq \max\{\psi(x), \psi(y)\}$. Suppose by contradiction that $\psi(x \vee y) > s := \max\{\psi(x), \psi(y)\}$. Since $x, y \in A_s$ and ψ has upwards directed sublevel sets, there exists $z^* \in A_s$ such that $x, y \preceq z^*$. Hence, $x \vee y \preceq z^*$, which by monotonicity implies that $\psi(x \vee y) \leq \psi(z^*) \leq s$. This contradicts that $\psi(x \vee y) > s$.

(i) \Rightarrow (iii): By contradiction, assume that there exist a partition $\{\Phi_1, \Phi_2\}$ of Φ and $x_1, x_2 \in X$ with $\phi(x_1) \geq \phi(x_2)$ for all $\phi \in \Phi_1$ and $\phi(x_2) \geq \phi(x_1)$ for all $\phi \in \Phi_2$, for which there exist $y_1, y_2 \in X$ with

$$\phi(y_i) \geq \phi(x_i) \text{ for all } \phi \in \Phi_i \text{ and } \psi(y_i) < \psi(x_1 \vee x_2),$$

for $i \in \{1, 2\}$. Then, for any $\phi \in \Phi$, either $\phi \in \Phi_1$ so that $\phi(y_1 \vee y_2) \geq \phi(x_1) \geq \phi(x_2)$, or $\phi \in \Phi_2$ so that $\phi(y_1 \vee y_2) \geq \phi(x_2) \geq \phi(x_1)$, which shows that $x_1 \vee x_2 \preceq y_1 \vee y_2$. Since ψ is maxitive, it is also monotone, which leads to the following contradiction:

$$\psi(x_1 \vee x_2) \leq \psi(y_1 \vee y_2) = \max\{\psi(y_1), \psi(y_2)\} < \psi(x_1 \vee x_2).$$

(iii) \Rightarrow (i): For $x_1, x_2 \in X$, we consider the partition $\Phi_1 := \{\phi : \phi(x_1) \geq \phi(x_2)\}$ and $\Phi_2 := \{\phi : \phi(x_1) < \phi(x_2)\}$. Then, there exists $i \in \{1, 2\}$ such that

$$\phi(y) \geq \phi(x_i) \text{ for all } \phi \in \Phi_i \text{ implies } \psi(y) \geq \psi(x_1 \vee x_2).$$

⁷ In the previous motivation, X was a set of gambles.

⁸ Indeed, for $x, y \in X$ with $x \preceq y$ and $y \preceq x$, there exists a unique supremum $x \vee y$ satisfying $x \preceq x \vee y$ and $y \preceq x \vee y$. Since x, y are upper bounds for $\{x, y\}$, we conclude that $x = y = x \vee y$.

⁹ I.e., $x \preceq y$ implies $\psi(x) \leq \psi(y)$.

¹⁰ I.e., if $x, y \in A_s = \{z \in X : \psi(z) \leq s\}$ for some $s \in \mathbb{R}$, then there exists $z^* \in A_s$ with $x \preceq z^*$ and $y \preceq z^*$.

By choosing $y := x_i$ in the previous relation, we obtain that $\psi(x_1 \vee x_2) \leq \psi(x_i) \leq \max\{\psi(x_1), \psi(x_2)\}$. The other inequality $\psi(x_1 \vee x_2) \geq \max\{\psi(x_1), \psi(x_2)\}$ follows directly from the monotonicity of ψ . \square

Discussion 3.2. We note that our concept includes the classical notion of maxitivity. To see this, let X be the space of all functions $f : S \rightarrow [-\infty, \infty]$ on some state space S , and consider the family $\Phi := \{\phi_s : s \in S\}$, where $\phi_s : X \rightarrow \mathbb{R}$ is given by $\phi_s(f) = \arctan(f(s))$.¹¹ In this case, $(f \vee g)(s) = \max\{f(s), g(s)\}$, and the concept of maxitivity corresponds to the usual notion of maxitivity studied in the literature; see e.g. [3,5,17,28]. Moreover, the state space S can be identified with the space Φ , which defines the order \leq . In line with the motivation presented in the previous section, Lemma 3.1(iii) guarantees that losses in certain states cannot be compensated by gains in others. For a general order, this result remains valid when interpreting the elements of Φ as the relevant states.

Representation results in terms of a weighted or a penalized worst-case evaluation for maxitive functions generally require stronger forms of maxitivity; see e.g. [3,5]. For instance, ψ is called completely maxitive if

$$\psi\left(\bigvee_{i \in I} x_i\right) = \sup_{i \in I} \psi(x_i)$$

for every family $(x_i)_{i \in I} \subset X$. However, for such a notion to make sense, the supremum $\bigvee_{i \in I} x_i$ must exist for arbitrary families. Under additional regularity assumptions, complete maxitivity is already ensured by (finite) maxitivity [1,15,16,20]. For our purposes, and to achieve the most elegant mathematical representation results, we will equip the ‘state space’ Φ with a Polish topology and assume that the mapping $\Phi \rightarrow \mathbb{R}, \phi \mapsto \phi(x)$, is lower semicontinuous for all $x \in X$; see Section 4. In this context, we work with the following version of maxitivity:

Definition 3.3. Suppose that (X, \leq) is countably complete.^{12,13} Then, a function $\psi : X \rightarrow (-\infty, \infty]$ is *bounded countably maxitive* if for every countable subset $Y \subset X$, we have

$$\sup_{y \in Y} \psi(y) = \begin{cases} \psi(\bigvee Y), & \text{if } Y \text{ is bounded from above,} \\ \infty, & \text{else.} \end{cases} \tag{3.1}$$

As in the finite case described in Lemma 3.1, we obtain the following characterization of bounded countable maxitivity:

Lemma 3.4. Suppose that (X, \leq) is countably complete. Then, a function $\psi : X \rightarrow (-\infty, \infty]$ is bounded countably maxitive if and only if it is monotone and has countably upwards directed sublevel sets.¹⁴

Proof. First, suppose that ψ is bounded countably maxitive. Then, for $x, y \in X$ with $x \leq y$, it follows from bounded countably maxitivity with $Y := \{x, y\}$ that $\psi(x) \leq \psi(x \vee y) = \psi(y)$. Moreover, for any countable subset $Y \subset A_s$, it holds $\sup_{y \in Y} \psi(y) \leq s$ and bounded countable maxitivity implies that Y is bounded from above with $\bigvee Y \in A_s$.

Second, if ψ is monotone, then the right hand side of (3.1) is greater than the left hand side. If ψ has countably upwards directed sublevel sets, then either $\sup_{y \in Y} \psi(y) = \infty$ or $\sup_{y \in Y} \psi(y) = s < \infty$ in which case $\bigvee Y \in A_s$ so that $\psi(\bigvee Y) \leq s$, i.e., the right hand side of (3.1) is smaller than the left hand side. \square

Our representation results are based on duality arguments. Hereby, maxitivity translates dually into the following stability property.

Definition 3.5. A function $\alpha : \Phi \rightarrow [-\infty, \infty]$ is *countably stable* if for every countable subset $Y \subset X$ with $\sup_{y \in Y} \phi(y) \leq \alpha(\phi)$ for all $\phi \in \Phi$, there exists $y^* \in X$ with $y \leq y^*$ for all $y \in Y$ and $\phi(y^*) \leq \alpha(\phi)$ for all $\phi \in \Phi$.

At the level of sublevel sets, we have the following characterization.

Proposition 3.6. Suppose that

$$A = \{x \in X : \phi(x) \leq \alpha(\phi) \text{ for all } \phi \in \Phi\},$$

for some $\alpha : \Phi \rightarrow [-\infty, \infty]$. Then, we have

¹¹ We set $\arctan(\infty) := \pi/2$ and $\arctan(-\infty) := -\pi/2$.
¹² We say that (X, \leq) is countably complete if every countable subset¹³ $Y \subset X$ that is bounded from above has a unique supremum $\bigvee Y$ in X . Recall that Y is bounded from above, if there exists $y^* \in X$ such that $y \leq y^*$ for all $y \in Y$.
¹³ Recall that we use the convention that $A \subset B$ means that every element of A is an element of B .
¹⁴ A function ψ has countably upwards directed sublevel sets if for every $s \in \mathbb{R}$, the level set $A_s := \{x \in X : \psi(x) \leq s\}$ is countably upwards directed, i.e., for every countable subset $Y \subset A_s$, there exists $y^* \in A_s$ with $y \leq y^*$ for all $y \in Y$.

$$A = \{x \in X : \phi(x) \leq \alpha_{\min}(\phi) \text{ for all } \phi \in \Phi\},$$

where $\alpha_{\min}(\phi) := \sup_{x \in A} \phi(x)$ for all $\phi \in \Phi$.

In this case, A is countably upwards directed if and only if α is countably stable if and only if α_{\min} is countably stable.

Proof. First, since $\phi(x) \leq \alpha(\phi)$ for all $x \in A$ and $\phi \in \Phi$, it holds that $\alpha_{\min}(\phi) \leq \alpha(\phi)$ for all $\phi \in \Phi$, which shows that $\{x \in X : \phi(x) \leq \alpha_{\min}(\phi) \text{ for all } \phi \in \Phi\} \subset A$. Conversely, if $x \in A$, then $\phi(x) \leq \alpha_{\min}(\phi)$ for all $\phi \in \Phi$, which shows that $A \subset \{x \in X : \phi(x) \leq \alpha_{\min}(\phi) \text{ for all } \phi \in \Phi\}$.

Second, if A is countably upwards directed and the subset $Y \subset X$ is countable with $\sup_{y \in Y} \phi(y) \leq \alpha(\phi)$ for all $\phi \in \Phi$, then $Y \subset A$, so there exists $y^* \in A$ with $y \leq y^*$ for all $y \in Y$, and therefore $\phi(y^*) \leq \alpha(\phi)$ for all $\phi \in \Phi$, i.e., α is countably stable. Conversely, if α is countably stable and $Y \subset X$ is countable such that $Y \subset A$, then $\sup_{y \in Y} \phi(y) \leq \alpha(\phi)$ for all $\phi \in \Phi$, and countable stability implies that Y is bounded from above by some $y^* \in A$, i.e., A is countably upwards directed. The argument for α_{\min} is analogous. \square

Remark 3.7. In general, for an arbitrary countably upwards directed subset $A \subset X$, it does not follow that $\alpha_{\min}(\phi) := \sup_{x \in A} \phi(x)$ is countably stable, even if X is countably complete.

To give a counterexample, let X be the set of all functions $f : \mathbb{R} \rightarrow [0, 1]$ such that there exists a bounded interval $I \subset \mathbb{R}$ for which $\{x \in \mathbb{R} \setminus I : f(x) > 0\}$ is at most countable. We consider the family $\Phi := \{\phi_x : x \in \mathbb{R}\}$, where $\phi_x(f) = f(x)$. The ordered set (X, \leq) is countably complete. Now, let A be the set of all functions $f \in X$ such that $\{x \in \mathbb{R} : f(x) > 0\}$ is at most countable. Then, the set A is countably upwards directed, and

$$\alpha_{\min}(\phi_x) = \sup_{f \in A} f(x) = 1 \quad \text{for all } x \in \mathbb{R}.$$

To show that α_{\min} is not countably stable, consider for every $n \in \mathbb{N}$ the indicator function $1_{[-n, n]}$ which assumes the value 1 on $[-n, n]$ and 0 on $[-n, n]^c$. Then, it holds

$$\sup_{n \in \mathbb{N}} 1_{[-n, n]}(x) \leq \alpha_{\min}(\phi_x) \quad \text{for all } x \in \mathbb{R},$$

but there is no function $f \in X$ with $1_{[-n, n]}(x) \leq f(x)$ for all $x \in \mathbb{R}$.

4. Main result

In this section, we present our representation results for maxitive functions with respect to general orders. Since we do not assume any form of completeness on the order, we consider monotone functions with countably upwards directed sublevel sets. In contrast to bounded countable maxitivity, these concepts do not require that the order is countably complete, allowing us to derive more general statements without relying on the assumption of countable completeness.

Throughout this section, we assume that X is a non-empty set, and Φ is a non-empty set of functions $\phi : X \rightarrow \mathbb{R}$ which induces the preorder

$$x \leq y \quad \text{if and only if} \quad \phi(x) \leq \phi(y) \text{ for all } \phi \in \Phi.$$

In addition, we assume that the set Φ is endowed with a topology such that Φ is a Polish space, and the functions $f_x : \Phi \rightarrow \mathbb{R}$ defined by $f_x(\phi) := \phi(x)$ are lower semicontinuous¹⁵ for all $x \in X$.

The following is the main result of this article.

Theorem 4.1. Let $\psi : X \rightarrow (-\infty, \infty]$ be a function with sublevel sets $(A_s)_{s \in \mathbb{R}}$ given by

$$A_s := \{x \in X : \psi(x) \leq s\}.$$

Then, the following statements are equivalent:

- (i) ψ is monotone and has countably upwards directed sublevel sets.
- (ii) There exists a function $\alpha : \mathbb{R} \times \Phi \rightarrow [-\infty, \infty]$ which is non-decreasing in the first argument and countably stable in the second argument such that

$$\psi(x) = \inf \left\{ s \in \mathbb{R} : \sup_{\phi \in \Phi} \{\phi(x) - \alpha(s, \phi)\} \leq 0 \right\} \quad \text{for all } x \in X. \tag{4.1}$$

The infimum is attained for all $x \in X$ with $\psi(x) < \infty$.

¹⁵ I.e., $\{\phi \in \Phi : f_x(\phi) \leq s\}$ is closed for all $s \in \mathbb{R}$.

(iii) The function $\alpha_{\min} : \mathbb{R} \times \Phi \rightarrow [-\infty, \infty]$, $\alpha_{\min}(s, \phi) := \sup_{x \in A_s} \phi(x)$ is countably stable in the second argument and

$$\psi(x) = \inf \left\{ s \in \mathbb{R} : \sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}(s, \phi) \} \leq 0 \right\} \quad \text{for all } x \in X. \tag{4.2}$$

The infimum is attained for all $x \in X$ with $\psi(x) < \infty$.

(iv) The function $\alpha_{\min}^+ : \mathbb{R} \times \Phi \rightarrow [-\infty, \infty]$, $\alpha_{\min}^+(s, \phi) := \inf_{s < t} \alpha_{\min}(t, \phi)$ is countably stable in the second argument and

$$\psi(x) = \inf \left\{ s \in \mathbb{R} : \sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}^+(s, \phi) \} \leq 0 \right\} \quad \text{for all } x \in X. \tag{4.3}$$

The infimum is attained for all $x \in X$ with $\psi(x) < \infty$.

(v) The function α_{\min}^+ is countably stable in the second argument and

$$\psi(x) = \sup_{\phi \in \Phi} G(\phi(x), \phi) \quad \text{for all } x \in X, \tag{4.4}$$

where $G : \mathbb{R} \times \Phi \rightarrow [-\infty, \infty]$, $G(t, \phi) := \inf \{ s \in \mathbb{R} : t \leq \alpha_{\min}^+(s, \phi) \}$.

Proof. Let $L(\Phi)$ denote the set of all lower semicontinuous functions $f : \Phi \rightarrow \mathbb{R}$ endowed with the pointwise order. Recall that $f_x : \Phi \rightarrow \mathbb{R}$ is defined by $f_x(\phi) := \phi(x)$ for all $x \in X$. Moreover, let $A_s := \{x \in X : \psi(x) \leq s\}$ for all $s \in \mathbb{R}$.

(i) \Rightarrow (ii): We can assume w.l.o.g. that $\inf_{\phi \in \Phi} \phi(x) > -\infty$ for all $x \in X$.¹⁶ For $I := \{s \in \mathbb{R} : A_s \neq \emptyset\}$, we define

$$F(s, f) := \inf_{x \in A_s} \sup_{\phi \in \Phi} \{ f(\phi) - \phi(x) \} \quad \text{for all } s \in I \text{ and } f \in L(\Phi). \tag{4.5}$$

Then, the mapping $F : I \times L(\Phi) \rightarrow [-\infty, \infty]$ is well-defined and satisfies the following properties. For every $f, g \in L(\Phi)$ and $s, t \in I$, it holds

- (1) $F(s, f) \leq F(t, f)$ for all $t \leq s$,
- (2) $F(s, f) \leq F(s, g)$ whenever $f \leq g$,
- (3) $F(s, f + c) = F(s, f) + c$ for all $c \in \mathbb{R}$,
- (4) $F(s, \sup_{n \in \mathbb{N}} f_n) = \sup_{n \in \mathbb{N}} F(s, f_n)$ for every sequence $(f_n)_{n \in \mathbb{N}} \subset L(\Phi)$ with $\sup_{n \in \mathbb{N}} f_n \in L(\Phi)$,
- (5) $F(s, f) \in (-\infty, \infty]$ and $F(s, 0) \in \mathbb{R}$,
- (6) $A_s = \{x \in X : F(s, f_x) \leq 0\}$,
- (7) $\psi(x) = \inf \{s \in I : F(s, f_x) \leq 0\}$ for all $x \in X$, and the infimum is attained whenever $\psi(x) < \infty$.

The properties (1)–(3) follow directly from the definition of F .

To show (4), let $(f_n)_{n \in \mathbb{N}} \subset L(\Phi)$ with $f := \sup_{n \in \mathbb{N}} f_n \in L(\Phi)$. Then, it follows from condition (2) that $\sup_{n \in \mathbb{N}} F(s, f_n) \leq F(s, f)$. By contradiction, we assume there exists $c \in \mathbb{R}$ with

$$\sup_{n \in \mathbb{N}} F(s, f_n) < c < F(s, f).$$

Then for every $n \in \mathbb{N}$ there exists $x_n \in A_s$ with $f_n(\phi) - \phi(x_n) < c$ for all $\phi \in \Phi$. Since A_s is countably upwards directed, there exists $x^* \in A_s$ with $x_1, x_2, \dots \leq x^*$. We obtain that $f_n(\phi) - \phi(x^*) < c$ for all $\phi \in \Phi$ and $n \in \mathbb{N}$. Taking the supremum first over all $n \in \mathbb{N}$, and second over all $\phi \in \Phi$, we obtain that

$$\sup_{\phi \in \Phi} \{ f(\phi) - \phi(x^*) \} \leq c < F(s, f) = \inf_{x \in A_s} \sup_{\phi \in \Phi} \{ f(\phi) - \phi(x) \},$$

which is a contradiction to $x^* \in A_s$.

To verify (5), suppose by contradiction that $F(s, f) = -\infty$. Then, for $\phi \in \Phi$ there exists $(x_n)_{n \in \mathbb{N}} \subset A_s$ with $f(\phi) - \phi(x_n) \leq -n$ for all $n \in \mathbb{N}$. Since A_s is countably upwards directed, there is $x^* \in A_s$ with $x_1, x_2, \dots \leq x^*$, and therefore

$$n + f(\phi) \leq \phi(x^*) \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \rightarrow \infty$ yields $\phi(x^*) = \infty$ in contrast to the assumption that ϕ is real-valued. This shows that $F(s, f) \in (-\infty, \infty]$. In particular, we obtain that for every $x \in A_s$,

$$-\infty < F(s, 0) = \inf_{y \in A_s} \sup_{\phi \in \Phi} \{ -\phi(y) \} \leq - \inf_{\phi \in \Phi} \phi(x) < \infty,$$

where the last inequality follows from the boundedness assumption on the family Φ , i.e., $F(s, 0) \in \mathbb{R}$.

¹⁶ Otherwise, we prove (ii) for the set $\Phi' := \{e^\phi : \phi \in \Phi\}$, which also induces the preorder \leq . In fact, if (ii) holds for the set Φ' and some function $\alpha' : \mathbb{R} \times \Phi' \rightarrow [-\infty, \infty]$, then it follows by inspection that (ii) is also satisfied by the original family Φ and the function $\alpha : \mathbb{R} \times \Phi \rightarrow [-\infty, \infty]$ defined by $\alpha(s, \phi) := \log(0 \vee \alpha'(s, e^\phi))$.

Next, we show (6). If $x \in A_s$, then

$$F(s, f_x) = \inf_{y \in A_s, \phi \in \Phi} \sup \{ \phi(x) - \phi(y) \} \leq \sup_{\phi \in \Phi} \{ \phi(x) - \phi(x) \} = 0.$$

If $x \in X$ satisfies $F(s, f_x) \leq 0$, then there exists $(x_n)_{n \in \mathbb{N}} \subset A_s$ with $\phi(x) - \phi(x_n) < 1/n$ for all $\phi \in \Phi$. Since A_s is countably upwards directed, there exists $x^* \in A_s$ with $x_1, x_2, \dots \leq x^*$, and therefore $\phi(x) - \phi(x^*) \leq 0$ for all $\phi \in \Phi$, which shows $x \leq x^*$. Since ψ is monotone, we conclude that $\psi(x) \leq \psi(x^*) \leq s$, and therefore $x \in A_s$.

To show (7), we notice that condition (6) implies that for every $x \in X$,

$$\psi(x) = \inf \{ s \in I : x \in A_s \} = \inf \{ s \in I : F(s, f_x) \leq 0 \},$$

and the infimum is attained whenever $\psi(x) < \infty$. Hence, we have verified the conditions (1)–(7).

Due to conditions (3)–(5), for each $s \in I$, the function $F(s, \cdot) : L(\Phi) \rightarrow (-\infty, \infty]$ satisfies the assumptions of Corollary A.3, and we obtain the representation

$$F(s, f) = \sup_{\phi \in \Phi} \{ f(\phi) - \alpha(s, \phi) \} \quad \text{for all } s \in I, f \in L(\Phi), \tag{4.6}$$

where $\alpha(s, \phi) := \sup_{f \in C_b(\Phi)} \{ f(\phi) - F(s, f) \}$ for all $s \in I$ and $\phi \in \Phi$. As a result of condition (7), we obtain that

$$\psi(x) = \inf \left\{ s \in \mathbb{R} : \sup_{\phi \in \Phi} \{ \phi(x) - \alpha(s, \phi) \} \leq 0 \right\} \quad \text{for all } x \in X, \tag{4.7}$$

where we set $\alpha(s, \phi) := -\infty$ whenever $s \notin I$. Moreover, the infimum is attained whenever $\psi(x) < \infty$. Due to condition (1), the mapping α is non-decreasing in the first argument. For $s \in \mathbb{R}$, the representation (4.7) implies that $x \in A_s$ if and only if $\phi(x) \leq \alpha(s, \phi)$ for all $\phi \in \Phi$. Since A_s is countably upwards directed, it follows from Proposition 3.6 that α is countably stable in the second argument.

(ii) \Rightarrow (iii): Fix $s \in \mathbb{R}$. The representation (4.1) implies that $x \in A_s$ if and only if $\sup_{\phi \in \Phi} \{ \phi(x) - \alpha(s, \phi) \} \leq 0$. Since α is countably stable in the second argument, it follows from Proposition 3.6 that A_s is countably upwards directed. Moreover, Proposition 3.6 implies that $x \in A_s$ if and only if $\sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}(s, \phi) \} \leq 0$, where $\alpha_{\min}(s, \phi) := \sup_{x \in A_s} \phi(x)$, and α_{\min} is countably stable in the second argument.

(iii) \Rightarrow (iv): Fix $s \in \mathbb{R}$. It follows from (iii) together with Proposition 3.6 that A_s is countably directed and $x \in A_s$ if and only if $\sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}(s, \phi) \} \leq 0$. Hence, $x \in X$ satisfies $\sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}(s, \phi) \} \leq 0$, if and only if $x \in A_t$ for all $t > s$, if and only if $\sup_{t > s} \sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}(t, \phi) \} \leq 0$, if and only if $\sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}^+(t, \phi) \} \leq 0$. In particular, it follows from Proposition 3.6 that α^+ is countably stable in the second argument.

(iv) \Rightarrow (v): Since α_{\min}^+ is right-continuous and non-decreasing in the first argument, it holds

$$\alpha_{\phi}^{-1}([t, \infty)) = [G(t, \phi), \infty) \quad \text{for all } t \in \mathbb{R} \text{ and } \phi \in \Phi,$$

where $\alpha_{\phi}^{-1}([t, \infty)) := \{ s \in \mathbb{R} : \alpha_{\min}^+(s, \phi) \in [t, \infty) \}$ is the preimage of $[t, \infty)$ under α_{\min}^+ . Fix $x \in X$. Suppose first that $\psi(x) < \infty$. It follows from the representation (4.3) that $s \in [\psi(x), \infty)$ if and only if $\sup_{\phi \in \Phi} \{ \phi(x) - \alpha_{\min}^+(s, \phi) \} \leq 0$, if and only if $\phi(x) \leq \alpha_{\min}^+(s, \phi)$ for all $\phi \in \Phi$, if and only if $s \in \alpha_{\phi}^{-1}([\phi(x), \infty))$ for all $\phi \in \Phi$. Hence,

$$[\psi(x), \infty) = \bigcap_{\phi \in \Phi} \alpha_{\phi}^{-1}([\phi(x), \infty)) = \bigcap_{\phi \in \Phi} [G(\phi(x), \phi), \infty).$$

This shows $\psi(x) = \sup_{\phi \in \Phi} G(\phi(x), \phi)$ for all $x \in X$. If $\psi(x) = \infty$, then the representation (4.3) implies that for every $n \in \mathbb{N}$, there is $\phi_n \in \Phi$ such that $n \notin \alpha_{\phi_n}^{-1}([\phi_n(x), \infty))$. Hence, for every $n \in \mathbb{N}$,

$$n \notin \bigcap_{\phi \in \Phi} \alpha_{\phi}^{-1}([\phi(x), \infty)) = \bigcap_{\phi \in \Phi} [G(\phi(x), \phi), \infty).$$

This implies that $\psi(x) = \sup_{\phi \in \Phi} G(\phi(x), \phi) = \infty$.

(v) \Rightarrow (i) Since α_{\min}^+ is countably stable in the second argument, we argue as in the implication from (iii) to (iv) that the level sets of ψ are countably upwards directed. Since G is non-decreasing in the first argument, the function ψ is monotone. \square

Remark 4.2. A modification of the previous result also leads to the representation of bounded completely maxitive functions.¹⁷ Indeed, Theorem 4.1 remains valid if we replace countably upwards directed sublevel sets with completely upwards directed sublevel sets¹⁸ and countable stability with complete stability.¹⁹ The proof follows with the same arguments, subject to the following modifications. In the implication (i) \Rightarrow (ii), we do not rely on Corollary A.3, but rather extend the definition (4.5) of $F(s, f)$ for all $s \in I$ and $f \in [-\infty, \infty)^\Phi$, where $[-\infty, \infty)^\Phi$ denotes the set of all functions from Φ to $[-\infty, \infty)$. Direct verification shows that F satisfies

¹⁷ Defined similarly to Definition 3.3, but for arbitrary subsets rather than countable subsets.

¹⁸ I.e., for every $s \in \mathbb{R}$, there exists $y^s \in A_s$ with $y \leq y^s$ for all $y \in A_s$.

¹⁹ A function $\alpha : \Phi \rightarrow [-\infty, \infty]$ is *completely stable* if for every subset $Y \subset X$ with $\sup_{y \in Y} \phi(y) \leq \alpha(\phi)$ for all $\phi \in \Phi$, there exists $y^* \in X$ with $y \leq y^*$ for all $y \in Y$ and $\phi(y^*) \leq \alpha(\phi)$ for all $\phi \in \Phi$.

the conditions (1)–(3) on $I \times [-\infty, \infty)^\Phi$ and $F(s, \sup_{j \in J} f_j) = \sup_{j \in J} F(s, f_j)$ for every family $(f_j)_{j \in J}$ in $[-\infty, \infty)^\Phi$. From the latter property, one obtains the representation (4.6) for all $s \in I$ and $f \in [-\infty, \infty)^\Phi$ for a suitable function $\alpha : I \times \Phi \rightarrow (-\infty, \infty]$. From there, the argumentation is similar as in the proof of (i) \Rightarrow (ii) in Theorem 4.1.

Remark 4.3. In line with (2.1), (2.2), and (2.3), the function α in (4.1) can be interpreted as a penalty function. In the present setting, there may exist different penalties α for which the representation (4.1) holds.

We note that α_{\min} is the smallest penalty α for which the representation (4.1) is valid. Indeed, if (4.1) holds for a penalty α , then for any fixed $\phi \in \Phi$ and $s \in \mathbb{R}$, it follows that $\phi(x) \leq \alpha(s, \phi)$ for all $x \in A_s$. Taking the supremum over all $x \in A_s$, we obtain that $\alpha_{\min}(s, \phi) = \sup_{x \in A_s} \phi(x) \leq \alpha(s, \phi)$. Furthermore, by construction, α_{\min}^+ is the smallest penalty for which the representation (4.1) holds and is also right-continuous in the first argument. Note that the right continuity of α_{\min}^+ is necessary in the implication (iv) \Rightarrow (v) in Theorem 4.1.

For the remainder of this section, X is endowed with an operation $+$: $X \times \mathbb{R} \rightarrow X$. A function $\psi : X \rightarrow [-\infty, \infty]$ satisfies the *translation property* if $\psi(x + c) = \psi(x) + c$ for all $x \in X$ and $c \in \mathbb{R}$. A function $\psi : X \rightarrow [-\infty, \infty]$ satisfies the *deviation property* if $\psi(x + c) = \psi(x)$ for all $x \in X$ and $c \in \mathbb{R}$.

Corollary 4.4. Suppose that every $\phi \in \Phi$ has the translation property. Let $\psi : X \rightarrow (-\infty, \infty]$ be a function with $A_s := \{x \in X : \psi(x) \leq s\}$ for all $s \in \mathbb{R}$. Then, the following statements are equivalent:

- (i) ψ has the translation property, is monotone, and has countably upwards directed sublevel sets.
- (ii) There exists a countably stable function $\gamma : \Phi \rightarrow [-\infty, \infty]$ with

$$\psi(x) = \sup_{\phi \in \Phi} \{\phi(x) - \gamma(\phi)\} \quad \text{for all } x \in X.$$

- (iii) The mapping $\phi \mapsto \alpha_{\min}(0, \phi) := \sup_{x \in A_0} \phi(x)$ is countably stable and

$$\psi(x) = \sup_{\phi \in \Phi} \{\phi(x) - \alpha_{\min}(0, \phi)\} \quad \text{for all } x \in X.$$

Proof. (i) \Rightarrow (iii): Since ψ has the translation property, it holds $\psi(x) \leq s$ if and only if $\psi(x - s) \leq 0$, so that $A_s = A_0 - s := \{x - s : x \in A_0\}$ for all $s \in \mathbb{R}$. Hence, since every $\phi \in \Phi$ has the translation property, we obtain for every $s \in \mathbb{R}$ and $\phi \in \Phi$,

$$\alpha_{\min}(s, \phi) = \sup_{x \in A_s} \phi(x) = \sup_{x \in A_0} \phi(x + s) = \sup_{x \in A_0} \phi(x) + s = \alpha_{\min}(0, \phi) + s.$$

It follows from Theorem 4.1 that for every $x \in X$,

$$\begin{aligned} \psi(x) &= \inf \left\{ s \in \mathbb{R} : \sup_{\phi \in \Phi} \{\phi(x) - \alpha_{\min}(s, \phi)\} \leq 0 \right\} \\ &= \inf \left\{ s \in \mathbb{R} : \sup_{\phi \in \Phi} \{\phi(x) - \alpha_{\min}(0, \phi)\} \leq s \right\} \\ &= \sup_{\phi \in \Phi} \{\phi(x) - \alpha_{\min}(0, \phi)\}. \end{aligned}$$

(iii) \Rightarrow (i) is obvious.

(ii) \Rightarrow (i): That ψ is monotone and has countably upwards directed sublevel sets follows from Theorem 4.1. Moreover, since every $\phi \in \Phi$ has the translation property, for every $x \in X$ and $c \in \mathbb{R}$,

$$\psi(x + c) = \sup_{\phi \in \Phi} \{\phi(x + c) - \gamma(\phi)\} = \sup_{\phi \in \Phi} \{\phi(x) + c - \gamma(\phi)\} = \psi(x) + c,$$

which shows that ψ has the translation property. \square

Corollary 4.5. Suppose that every $\phi \in \Phi$ satisfies the deviation property, and let $\psi : X \rightarrow (-\infty, \infty]$ be a monotone function with countably upwards directed sublevel sets. If ψ additionally satisfies the translation property, then $\psi(x) = \infty$ for all $x \in X$.

Proof. It follows from Theorem 4.1 that for every $x \in X$,

$$\psi(x) = \inf \{s \in \mathbb{R} : \phi(x) \leq \alpha_{\min}(s, \phi) \text{ for all } \phi \in \Phi\}.$$

By similar arguments as in Corollary 4.4, it holds $\alpha_{\min}(s, \phi) = \alpha_{\min}(0, \phi)$ for all $s \in \mathbb{R}$. Hence, it necessary holds $\psi(x) = \infty$. \square

5. Maxitivity with respect to stochastic orders

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space.²⁰ We denote by \mathcal{M} the set of all random variables $\xi : \Omega \rightarrow \mathbb{R}$, where two of them are identified if they have the same law. Further, let \mathcal{M}_∞ and \mathcal{M}_1 be the sets of all random variables in \mathcal{M} which are essentially bounded, and have finite first moments, respectively. In the following, we analyze the main results of the previous section for several stochastic orders. For definitions and further details on stochastic orders, we refer to [21,27].

5.1. Usual stochastic order

For $\xi \in \mathcal{M}$, the *quantile function* $q_\xi : (0, 1) \rightarrow \mathbb{R}$ is defined as the left-inverse of the cumulative distribution function, i.e.,

$$q_\xi(u) := \inf \{ v \in \mathbb{R} : u \leq \mathbb{P}(\xi \leq v) \}.$$

On the one hand, the quantile function q_ξ is non-decreasing and left-continuous, see Lemma B.1. On the other hand, every function $q : (0, 1) \rightarrow \mathbb{R}$ which is non-decreasing and left-continuous satisfies $q = q_\xi$ for some $\xi \in \mathcal{M}$, see Lemma B.3. Hence, there is a one-to-one relation between the set \mathcal{M} and quantile functions.

The *usual stochastic order* on \mathcal{M} is defined by

$$\xi \preceq_{st} \eta \quad \text{if and only if} \quad q_\xi(u) \leq q_\eta(u) \text{ for all } u \in (0, 1).$$

The ordered space $(\mathcal{M}, \preceq_{st})$ is a complete lattice, see [14,22]. In particular, every sequence $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}$ which is bounded from above has a supremum $\xi^* = \vee_{n \in \mathbb{N}} \xi_n$ in \mathcal{M} which is determined by $q_{\xi^*}(u) = \sup_{n \in \mathbb{N}} q_{\xi_n}(u)$ for all $u \in (0, 1)$.

Let $\Phi_{st} := \{ \phi_u : u \in (0, 1) \}$ denote the family of functions $\phi_u : \mathcal{M} \rightarrow \mathbb{R}$ given by $\phi_u(\xi) := q_\xi(u)$. We identify Φ_{st} with $(0, 1)$, and endow $(0, 1)$ with the Euclidean topology.²¹ Then, it holds $\xi \preceq_{st} \eta$ if and only if $\phi_u(\xi) \leq \phi_u(\eta)$ for all $u \in (0, 1)$. Moreover, for every $\xi \in \mathcal{M}$, the mapping $u \mapsto \phi_u(\xi)$ is lower semicontinuous. Therefore, the partially ordered space $(\mathcal{M}, \preceq_{st})$ satisfies the assumptions of Section 4.

Lemma 5.1. *Every function $\gamma : (0, 1) \rightarrow [-\infty, \infty)$ is countably stable.*

Proof. Let $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}$ with $q^*(u) := \sup_{n \in \mathbb{N}} q_{\xi_n}(u) \leq \gamma(u)$ for all $u \in (0, 1)$. Then, the mapping $u \mapsto q^*(u)$ is real-valued, non-decreasing and left-continuous. Hence, there exists $\xi^* \in \mathcal{M}$ with $q^*(u) = q_{\xi^*}(u)$ for all $u \in (0, 1)$. In particular, $\xi^* = \vee_{n \in \mathbb{N}} \xi_n$ and $\phi_u(\xi^*) = q^*(u) \leq \gamma(u)$ for all $u \in (0, 1)$. \square

For a function $\psi : \mathcal{M} \rightarrow (-\infty, \infty]$ with sublevel sets $A_s := \{ \xi \in \mathcal{M} : \psi(\xi) \leq s \}$, we define $\alpha_{\min}(s, u) := \sup_{\xi \in A_s} q_\xi(u)$ and $\alpha_{\min}^+(s, u) := \inf_{s < t} \alpha_{\min}(t, u)$ for all $s \in \mathbb{R}$ and $u \in (0, 1)$. If ψ is countably maxitive, then the sublevel sets are countably upwards directed, so that $u \mapsto \alpha_{\min}(s, u)$ is real-valued for all $s \in \mathbb{R}$ with $A_s \neq \emptyset$. As a direct application of Theorem 4.1 and Lemma 3.4, we obtain the following representation result for maxitive functions w.r.t. the usual stochastic order. A related representation result for maxitive functionals with respect to the usual stochastic order was investigated in parallel to our work in [4, Theorem 1].

Corollary 5.2. *Let $\psi : \mathcal{M} \rightarrow (-\infty, \infty]$ be a function. Then, the following conditions are equivalent:*

- (i) ψ is bounded countably maxitive w.r.t. \preceq_{st} .
- (ii) ψ is monotone and has countably upwards directed sublevel sets w.r.t. \preceq_{st} .
- (iii) There exists a function $\alpha : \mathbb{R} \times (0, 1) \rightarrow [-\infty, \infty)$ which is non-decreasing in the first argument such that

$$\psi(\xi) = \inf \left\{ s \in \mathbb{R} : \sup_{u \in (0, 1)} \{ q_\xi(u) - \alpha(s, u) \} \leq 0 \right\} \quad \text{for all } \xi \in \mathcal{M}.$$

The infimum is attained for all $\xi \in \mathcal{M}$ with $\psi(\xi) < \infty$.

- (iv) The mapping $u \mapsto \alpha_{\min}(s, u)$ is real-valued for all $s \in \mathbb{R}$ with $A_s \neq \emptyset$. Moreover, it holds

$$\psi(\xi) = \sup_{u \in (0, 1)} G(q_\xi(u), u) \quad \text{for all } \xi \in \mathcal{M},$$

where $G(t, u) := \inf \{ s \in \mathbb{R} : t \leq \alpha_{\min}^+(s, u) \}$ for all $(t, u) \in \mathbb{R} \times (0, 1)$.

²⁰ Recall that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless if for all $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$ there exists $B \subset A$ and $0 < \mathbb{P}(B) < \mathbb{P}(A)$; see e.g. [6].

²¹ $(0, 1)$ is not complete with the Euclidean distance. However, it is completely metrizable, that is, there exists an equivalent metric on $(0, 1)$ such that $(0, 1)$ with the relative Euclidean topology is a complete metric space. In fact, every countable intersection of open subsets of a complete metric space is completely metrizable, see [33, Theorem 24.12].

If ψ also has the translation property, then we get the following version of the previous result. We notice that $q_{\xi+c} = q_\xi + c$ for all $\xi \in \mathcal{M}$ and $c \in \mathbb{R}$, i.e., the mapping $\phi_u : \mathcal{M} \rightarrow \mathbb{R}$ has the translation property for all $u \in (0, 1)$. Hence, we can apply Corollary 4.4 and obtain the following result.

Corollary 5.3. *Let $\psi : \mathcal{M} \rightarrow (-\infty, \infty]$ be a function with the translation property. Then, the following conditions are equivalent:*

- (i) ψ is bounded countably maxitive w.r.t. \leq_{st} .
- (ii) There exists a function $\alpha : \mathbb{R} \times (0, 1) \rightarrow [-\infty, \infty)$ such that

$$\psi(\xi) = \sup_{u \in (0,1)} \{q_\xi(u) - \alpha(u)\} \quad \text{for all } \xi \in \mathcal{M}.$$

- (iii) The mapping $u \mapsto \alpha_{\min}(0, u) = \sup_{\xi \in A_0} q_\xi(u)$ is real-valued, and

$$\psi(\xi) = \sup_{u \in (0,1)} \{q_\xi(u) - \alpha_{\min}(0, u)\} \quad \text{for all } \xi \in \mathcal{M}.$$

Recall that the value at risk $\text{VaR}_u : \mathcal{M} \rightarrow \mathbb{R}$ at level $u \in (0, 1)$, which is defined by $\text{VaR}_u(\xi) := q_\xi(u)$, is a monetary risk measure; see [12]. The previous Corollary 5.3 is closely related to [32, Theorem 4] which provides a representation result for lower semicontinuous completely maxitive functions $\mathcal{M}_1 \rightarrow \mathbb{R}$ having the translation property. A related representation result was also shown in [2, Proposition 4] for the class of benchmark-loss VaR, which were introduced by Bignozzi et al. [2].

Remark 5.4. For every sequence $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}$ such that $\bigvee_{n \in \mathbb{N}} \xi_n$ exists, it holds $\text{VaR}_u(\bigvee_{n \in \mathbb{N}} \xi_n) = \sup_{n \in \mathbb{N}} \text{VaR}_u(\xi_n)$. Nevertheless, the function $\text{VaR}_u : \mathcal{M} \rightarrow \mathbb{R}$ is not countably bounded maxitive because it may happen that $\sup_{n \in \mathbb{N}} \text{VaR}_u(\xi_n) < \infty$ for an unbounded sequence $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}$. However, VaR_u can also be defined on the set \mathcal{M}^* of all extended random variables $\xi : \Omega \rightarrow (-\infty, \infty]$. Then, the supremum $\bigvee_{i \in I} \xi_i$ exists for every family $(\xi_i)_{i \in I} \subset \mathcal{M}^*$ and $\text{VaR}_u : \mathcal{M}^* \rightarrow (-\infty, \infty]$ is completely maxitive.

5.2. Increasing convex stochastic order

For $\xi \in \mathcal{M}_1$, we define the integrated quantile function $Q_\xi : (0, 1) \rightarrow \mathbb{R}$ by

$$Q_\xi(u) := \int_u^1 q_\xi(v) dv.$$

The function $u \mapsto Q_\xi(u)$ is concave, with $Q_\xi(0^+) = \mathbb{E}[\xi]$ and $Q_\xi(1^-) = 0$. Conversely, every concave function $Q : (0, 1) \rightarrow \mathbb{R}$ with $Q(0^+) \in \mathbb{R}$ and $Q(1^-) = 0$ is of the form $Q = Q_\xi$ for some random variable $\xi \in \mathcal{M}_1$, see Lemma B.4. The increasing convex order on \mathcal{M}_1 is defined by

$$\xi \leq_{icx} \eta \quad \text{if and only if} \quad Q_\xi(u) \leq Q_\eta(u) \quad \text{for all } u \in (0, 1).$$

The ordered space $(\mathcal{M}_1, \leq_{icx})$ is a complete lattice, see [14,22]. In particular, every sequence $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}_1$ which is bounded from above has a supremum $\xi^* = \bigvee_{n \in \mathbb{N}} \xi_n$ which is determined by

$$Q_{\xi^*}(u) = \text{conc}(Q_{\xi_1}, Q_{\xi_2}, \dots)(u) \quad \text{for all } u \in (0, 1).$$

Here, $\text{conc}(Q_{\xi_1}, Q_{\xi_2}, \dots)$ denotes the concave hull of the sequence $(Q_{\xi_n})_{n \in \mathbb{N}}$, i.e., the smallest concave function $Q^* : (0, 1) \rightarrow [-\infty, \infty]$ with $\sup_{n \in \mathbb{N}} Q_{\xi_n}(u) \leq Q^*(u)$ for all $u \in (0, 1)$.²²

Let $\Phi_{icx} := \{\phi_u : u \in (0, 1)\}$ be the family of functions $\phi_u : \mathcal{M}_1 \rightarrow \mathbb{R}$ given by $\phi_u(\xi) := Q_\xi(u)$. We identify Φ_{icx} with $(0, 1)$, and endow $(0, 1)$ with the Euclidean topology. For every $\xi \in \mathcal{M}_1$, the function $u \mapsto \phi_u(\xi)$ is continuous. Hence, the partially ordered space $(\mathcal{M}_1, \leq_{icx})$ satisfies the assumptions of Section 4.

Lemma 5.5. *Every concave function $\gamma : (0, 1) \rightarrow [-\infty, \infty)$ is countably stable.*

Proof. Let $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}_1$ with $\bigvee_{n \in \mathbb{N}} Q_{\xi_n}(u) \leq \gamma(u)$ for all $u \in (0, 1)$. Since γ is concave, it holds $Q^*(u) := \text{conc}(Q_{\xi_1}, Q_{\xi_2}, \dots)(u) \leq \gamma(u)$ for all $u \in (0, 1)$. Since $u \mapsto Q^*(u)$ is real-valued, we obtain $Q_{\xi^*} = Q^*$, where $\xi^* = \bigvee_{n \in \mathbb{N}} \xi_n \in \mathcal{M}_1$, and $Q_{\xi^*}(u) = Q^*(u) \leq \gamma(u)$ for all $u \in (0, 1)$. \square

For a function $\psi : \mathcal{M}_1 \rightarrow (-\infty, \infty]$ with sublevel sets $A_s := \{\xi \in \mathcal{M}_1 : \psi(\xi) \leq s\}$, we define $\alpha_{\min}(s, u) := \sup_{\xi \in A_s} Q_\xi(u)$ and $\alpha_{\min}^+(s, u) := \inf_{s < t} \alpha_{\min}(t, u)$ for all $s \in \mathbb{R}$ and $u \in (0, 1)$. If ψ is countably maxitive, then the sublevel sets are countably upwards

²² Notice that $\text{conc}(Q_{\xi_1}, Q_{\xi_2}, \dots)$ is the pointwise infimum over all concave functions $Q : (0, 1) \rightarrow (-\infty, \infty]$ with $Q_{\xi_n}(u) \leq Q(u)$ for all $u \in (0, 1)$ and $n \in \mathbb{N}$.

directed, from which it is straightforward to verify that the mapping $u \mapsto \alpha_{\min}(s, u)$ is real-valued and concave for all $s \in \mathbb{R}$ with $A_s \neq \emptyset$.²³ As a direct application of Theorem 4.1 and Lemma 3.4, we obtain the following representation result for maxitive functions w.r.t. the increasing convex stochastic order.

Corollary 5.6. *Let $\psi : \mathcal{M}_1 \rightarrow (-\infty, \infty]$ be a function. Then, the following conditions are equivalent:*

- (i) ψ is bounded countably maxitive w.r.t. \leq_{icx} .
- (ii) ψ is monotone and has countably upwards directed sublevel sets w.r.t. \leq_{icx} .
- (iii) There exists a function $\alpha : \mathbb{R} \times (0, 1) \rightarrow [-\infty, \infty)$ which is non-decreasing in the first argument and concave in the second argument such that

$$\psi(\xi) = \inf \left\{ s \in \mathbb{R} : \sup_{u \in (0,1)} \{ Q_\xi(u) - \alpha(s, u) \} \leq 0 \right\} \quad \text{for all } \xi \in \mathcal{M}_1.$$

The infimum is attained for all $\xi \in \mathcal{M}_1$ with $\psi(\xi) < \infty$.

- (iv) The mapping $u \mapsto \alpha_{\min}(s, u)$ is real-valued and concave for all $s \in \mathbb{R}$ with $A_s \neq \emptyset$. Moreover, it holds

$$\psi(\xi) = \sup_{u \in (0,1)} G(Q_\xi(u), u) \quad \text{for all } \xi \in \mathcal{M}_1,$$

where $G(t, u) := \inf \{ s \in \mathbb{R} : t \leq \alpha_{\min}^+(s, u) \}$ for all $(t, u) \in \mathbb{R} \times (0, 1)$.

If ψ in addition has the translation property, then we can argue as follows. Since the function $\xi \mapsto Q_\xi$ does not have the translation property, we consider the modified family $\Phi'_{\text{icx}} := \{ \text{ES}_u : u \in (0, 1) \}$. Here, $\text{ES}_u : \mathcal{M}_1 \rightarrow \mathbb{R}$ is the expected shortfall given by

$$\text{ES}_u(\xi) := \frac{1}{1-u} Q_\xi(u) \quad \text{for all } u \in (0, 1).$$

The expected shortfall is a convex risk measure; see [12]. In particular, the functions ES_u have the translation property and also induce the partial order \leq_{icx} . However, by changing to the family Φ'_{icx} , we also change the set of countably stable functions. More precisely, a function $\gamma : (0, 1) \rightarrow [-\infty, \infty]$ is countably stable w.r.t. Φ'_{icx} if and only if the function $u \mapsto \frac{1}{1-u} \gamma(u)$ is countably stable w.r.t. Φ'_{icx} . Moreover, the minimal penalty function α'_{\min} w.r.t. Φ'_{icx} is given by

$$\alpha'_{\min}(s, u) = \frac{1}{1-u} \alpha_{\min}(s, u).$$

As a consequence of Corollary 4.4, we obtain the following result.

Corollary 5.7. *Let $\psi : \mathcal{M}_1 \rightarrow (-\infty, \infty]$ be a function with the translation property. Then, the following conditions are equivalent:*

- (i) ψ is bounded countably maxitive w.r.t. \leq_{icx} .
- (ii) There exists a concave function $\alpha : (0, 1) \rightarrow [-\infty, \infty)$ such that

$$\psi(\xi) = \sup_{u \in (0,1)} \left\{ \text{ES}_u(\xi) - \frac{1}{1-u} \alpha(u) \right\} \quad \text{for all } \xi \in \mathcal{M}_1.$$

- (iii) The mapping $u \mapsto \alpha_{\min}(0, u)$ is real-valued and concave. Moreover, it holds

$$\psi(\xi) = \sup_{u \in (0,1)} \left\{ \text{ES}_u(\xi) - \frac{1}{1-u} \alpha_{\min}(0, u) \right\} \quad \text{for all } \xi \in \mathcal{M}_1.$$

The previous Corollary 5.7 is closely related to [32, Theorem 5]. Moreover, it is shown in [32] that the function ES_u does not always preserve the maximum of two random variables w.r.t. \leq_{icx} , i.e., the function ES_u is not maxitive w.r.t. \leq_{icx} .

5.3. Convex stochastic order

The convex order on \mathcal{M}_1 is defined by

$$\xi \leq_{\text{cx}} \eta \quad \text{if and only if} \quad \mathbb{E}[\xi] = \mathbb{E}[\eta] \quad \text{and} \quad \xi \leq_{\text{icx}} \eta.$$

The partially ordered space $(\mathcal{M}_1, \leq_{\text{cx}})$ is not a lattice on \mathcal{M}_1 because it is not possible to define the maximum of two random variables $\xi, \nu \in \mathcal{M}_1$ for which $\mathbb{E}[\xi] \neq \mathbb{E}[\nu]$. However, for $r \in \mathbb{R}$, the convex order agrees with the increasing convex order on the set

²³ As for the concavity, we notice that for $\xi_1, \xi_2 \in A_s$, there exists $\xi^* \in A_s$ with $Q_{\xi^*} = \text{conc}(Q_{\xi_1}, Q_{\xi_2})$ such that $\lambda Q_{\xi_1}(u_1) + (1-\lambda)Q_{\xi_2}(u_2) \leq Q_{\xi^*}(\lambda u_1 + (1-\lambda)u_2)$ for all $u_1, u_2, \lambda \in (0, 1)$.

$$\mathcal{M}'_1 := \{ \xi \in \mathcal{M}_1 : \mathbb{E}[\xi] = r \}.$$

In particular, $(\mathcal{M}'_1, \preceq_{\text{cx}})$ is a complete lattice. For $\psi : \mathcal{M}'_1 \rightarrow (-\infty, \infty]$ with sublevel sets $A_s := \{ \xi \in \mathcal{M}'_1 : \psi(\xi) \leq s \}$, we define $\alpha_{\min} : \mathbb{R} \times (0, 1) \rightarrow [-\infty, \infty]$ by $\alpha_{\min}(s, u) := \sup_{\xi \in A_s} Q_\xi(u)$ and $\alpha_{\min}^+(s, u) := \inf_{s < t} \alpha_{\min}(t, u)$. As a direct application of Theorem 4.1 and Lemma 3.4, we conclude that a function $\psi : \mathcal{M}'_1 \rightarrow \mathbb{R}$ is countably maxitive with respect to \preceq_{cx} , if it admits a representation similar to that in Corollary 5.6, with \mathcal{M}_1 replaced by \mathcal{M}'_1 .

5.4. Increasing concave stochastic order

For $\xi \in \mathcal{M}_1$, we define the function $\bar{Q}_\xi : (0, 1) \rightarrow \mathbb{R}$ by

$$\bar{Q}_\xi(u) := -Q_{-\xi}(u) = - \int_u^1 q_{-\xi}(v) dv.$$

The mapping $t \mapsto \bar{Q}_\xi(t)$ is convex and satisfies $\bar{Q}_\xi(0^+) = \mathbb{E}[\xi]$ and $\bar{Q}_\xi(1^-) = 0$. Conversely, every convex function $\bar{Q} : (0, 1) \rightarrow \mathbb{R}$ with $\bar{Q}(0^+) \in \mathbb{R}$ and $\bar{Q}(1^-) = 0$ is of the form $\bar{Q} = \bar{Q}_\xi$ for some random variable $\xi \in \mathcal{M}_1$. The increasing concave stochastic order is defined by

$$\xi \preceq_{\text{icv}} \eta \quad \text{if and only if} \quad -\eta \preceq_{\text{icx}} -\xi.$$

That is, $\xi \preceq_{\text{icv}} \eta$ if and only if $\bar{Q}_\xi(u) \leq \bar{Q}_\eta(u)$ for all $u \in (0, 1)$. The partially ordered space $(\mathcal{M}_1, \preceq_{\text{icv}})$ is a complete lattice, see [14,22]. In particular, every sequence $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}_1$ which is bounded from above w.r.t. \preceq_{icv} has a supremum $\xi^* = \bigvee_{n \in \mathbb{N}} \xi_n$ which is determined by $\bar{Q}_{\xi^*}(u) = \bigvee_{n \in \mathbb{N}} \bar{Q}_{\xi_n}(u)$ for all $u \in (0, 1)$. We consider the family $\Phi_{\text{icv}} := \{ \phi_u : u \in (0, 1) \}$ of functions $\phi_u : \mathcal{M}_1 \rightarrow \mathbb{R}$ given by $\phi_u(\xi) := \bar{Q}_\xi(u)$. Hence, the partially ordered space $(\mathcal{M}_1, \preceq_{\text{icv}})$ satisfies the conditions of Section 4. With arguments similar to those for the usual stochastic order, we get the following result.

Lemma 5.8. Every function $\gamma : (0, 1) \rightarrow [-\infty, \infty)$ is countably stable.

For a function $\psi : \mathcal{M} \rightarrow (-\infty, \infty]$ with sublevel sets $A_s := \{ \xi \in \mathcal{M}_1 : \psi(\xi) \leq s \}$, we define $\alpha_{\min}(s, u) := \sup_{\xi \in A_s} \bar{Q}_\xi(u)$ and $\alpha_{\min}^+(s, u) := \inf_{s < t} \alpha_{\min}(t, u)$ for all $s \in \mathbb{R}$ and $u \in (0, 1)$. If ψ is countably maxitive, then the sublevel sets are countably upwards directed, so that $u \mapsto \alpha_{\min}(s, u)$ is real-valued for all $s \in \mathbb{R}$ with $A_s \neq \emptyset$. As a direct application of Theorem 4.1 and Lemma 3.4, we obtain the following representation result for maxitive functions w.r.t. the increasing concave stochastic order.

Corollary 5.9. Let $\psi : \mathcal{M}_1 \rightarrow (-\infty, \infty]$ be a function. Then, the following conditions are equivalent.

- (i) ψ is bounded countably maxitive w.r.t. \preceq_{icv} .
- (ii) ψ is monotone and has countably upwards directed sublevel sets w.r.t. \preceq_{icv} .
- (iii) There exists a function $\alpha : \mathbb{R} \times (0, 1) \rightarrow [-\infty, \infty)$ which is non-decreasing in the first argument such that

$$\psi(\xi) = \inf \left\{ s \in \mathbb{R} : \sup_{u \in (0, 1)} \{ \bar{Q}_\xi(u) - \alpha(s, u) \} \leq 0 \right\} \quad \text{for all } \xi \in \mathcal{M}_1.$$

The infimum is attained for all $\xi \in \mathcal{M}_1$ with $\psi(\xi) < \infty$.

- (iv) The mapping $u \mapsto \alpha_{\min}(s, u)$ is real-valued for all $s \in \mathbb{R}$ with $A_s \neq \emptyset$. Moreover, it holds

$$\psi(\xi) = \sup_{u \in (0, 1)} G(\bar{Q}_\xi(u), u) \quad \text{for all } \xi \in \mathcal{M},$$

where $G(t, u) := \inf \{ s \in \mathbb{R} : t \leq \alpha_{\min}^+(s, u) \}$ for all $(t, u) \in \mathbb{R} \times (0, 1)$.

In case that ψ has the translation property, we can proceed as follows. Since the function $\xi \mapsto \bar{Q}_\xi$ does not have the translation property, we consider the modified family $\Phi'_{\text{icv}} := \{ \bar{\text{ES}}_u : u \in (0, 1) \}$, where $\bar{\text{ES}}_u : \mathcal{M}_1 \rightarrow \mathbb{R}$ is defined by $\bar{\text{ES}}_u(\xi) := -\text{ES}_u(-\xi)$. The functions $\bar{\text{ES}}_u$ are translation invariant and induce the order \preceq_{icv} . The families Φ'_{icv} and Φ_{icv} result in the same countably stable functions. In addition, we obtain

$$\alpha'_{\min}(s, u) = \frac{1}{1-u} \alpha_{\min}(s, u),$$

where α'_{\min} is the minimal penalty function w.r.t. Φ'_{icv} . As a consequence of Corollary 4.4, we conclude that a function $\psi : \mathcal{M}_1 \rightarrow (-\infty, \infty]$ with the translation property is countably maxitive with respect to \preceq_{icv} , if it admits a representation similar to that in Corollary 5.7, with ES_u replaced by $\bar{\text{ES}}_u$.

5.5. Dispersive stochastic order

The *dispersive stochastic order* on \mathcal{M} is defined by

$$\xi \preceq_{\text{disp}} \eta \quad \text{if and only if} \quad q_{\xi}(v) - q_{\xi}(u) \leq q_{\eta}(v) - q_{\eta}(u) \text{ for all } 0 < u \leq v < 1.$$

The dispersive order is not antisymmetric. However, by considering the quotient space \mathcal{M}/\sim w.r.t. the equivalence relation $\xi \sim \eta$ if and only if $\xi = \eta + c$ for some $c \in \mathbb{R}$, we obtain that the partially ordered space $(\mathcal{M}/\sim, \preceq_{\text{disp}})$ is a lattice, see [22].

For a non-empty set Q of quantile functions and $[u, v] \subset [0, 1]$, we define its *total variation* on $[u, v]$ as

$$\text{TV}_{[u,v]}(Q) := \sup_{\pi \in \Pi[u,v]} S_{\pi}(Q),$$

where $\Pi[u, v]$ consists of all partitions $\pi = \{u = t_0 < t_1 < \dots < t_n = v\}$ for some $n \in \mathbb{N}$. Here, we set $S_{\pi}(Q) := \sum_{i=1}^n \sup_{q \in Q} (q(t_i) - q(t_{i-1}))$ with $q(0) := q(0^+)$ and $q(1) := q(1^-)$. We say that the set Q is of *bounded variation* if $\text{TV}_{[0,1]}(Q) < \infty$. It is straightforward to verify that $\text{TV}_{[0,v]} = \text{TV}_{[0,u]} + \text{TV}_{[u,v]}$ for all $0 < u \leq v < 1$.

Proposition 5.10. *Suppose that $Q := \{q_{\xi} : \xi \in A\}$ is of bounded variation for some $A \subset \mathcal{M}$. Then, there exists $\xi^* \in \mathcal{M}$ such that $q_{\xi^*}(t) = \text{TV}_{[0,t]}(Q)$ for all $t \in (0, 1]$. Moreover, ξ^* is a supremum of A w.r.t. \preceq_{disp} .*

Proof. By Proposition C.1 and the fact that Q is of bounded variation, the function $t \mapsto \text{TV}_{[0,t]}(Q)$ is real-valued, non-decreasing and left-continuous. Hence, there exists $\xi^* \in \mathcal{M}$ such that $q_{\xi^*}(t) = \text{TV}_{[0,t]}(Q)$ for all $t \in (0, 1]$. Given $u, v \in \mathbb{R}$ with $0 < u \leq v < 1$, for every $\xi \in A$, we have that

$$q_{\xi}(v) - q_{\xi}(u) \leq \text{TV}_{[u,v]}(Q) = \text{TV}_{[0,v]}(Q) - \text{TV}_{[0,u]}(Q) = q_{\xi^*}(v) - q_{\xi^*}(u).$$

Thus, ξ^* is an upper bound of A w.r.t. \preceq_{disp} . On the other hand, if η is another upper bound of A w.r.t. \preceq_{disp} , we necessarily have that

$$q_{\xi^*}(v) - q_{\xi^*}(u) = \text{TV}_{[u,v]}(Q) \leq \text{TV}_{[u,v]}(q_{\eta}) = q_{\eta}(v) - q_{\eta}(u),$$

where we have used in the last equality that q_{η} is non-decreasing. This implies that $\xi^* \preceq_{\text{disp}} \eta$. \square

Corollary 5.11. *The partially ordered space $(\mathcal{M}_{\infty}/\sim, \preceq_{\text{disp}})$ is a complete lattice.*

Proof. Let $A \subset \mathcal{M}_{\infty}/\sim$ be a set which is bounded from above w.r.t. \preceq_{disp} , and denote by A_0 the set of respective representatives ξ with $q_{\xi}(0^+) = 0$. Suppose that ξ^* is an upper bound of A_0 . Then

$$\text{TV}_{[0,1]}(A_0) \leq \text{TV}_{[0,1]}(q_{\xi^*}) = q_{\xi^*}(1^-) - q_{\xi^*}(0^+) < \infty.$$

Hence, A_0 is of bounded variation and has a supremum in \mathcal{M}_{∞} w.r.t. \preceq_{disp} by Proposition 5.10. Then passing to the quotient, A has a unique supremum in $\mathcal{M}_{\infty}/\sim$. \square

It follows from Proposition 5.10 that every sequence $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}_{\infty}$ which is bounded from above w.r.t. \preceq_{disp} has a supremum $\xi^* \in \mathcal{M}_{\infty}$ which is determined by

$$q_{\xi^*}(u) := \text{TV}_{[0,u]}(\{q_{\xi_n} : n \in \mathbb{N}\}) \quad \text{for all } u \in (0, 1].$$

Here, for $0 \leq u < v \leq 1$, we denote by $\text{TV}_{[u,v]}(Q) := \sup_{\pi \in \Pi[u,v]} S_{\pi}(Q)$ the *total variation* along $[u, v]$ of a non-empty set Q of quantile functions, where $\Pi[u, v]$ is the set of all partitions of the form $\pi = \{u = t_0 < t_1 < \dots < t_n = v\}$ for some $n \in \mathbb{N}$, for which we set $S_{\pi}(Q) := \sum_{i=1}^n \sup_{q \in Q} (q(t_i) - q(t_{i-1}))$ with $q(0) := q(0^+)$ and $q(1) := q(1^-)$. It follows from Proposition B.2 that

$$\xi \preceq_{\text{disp}} \eta \quad \text{if and only if} \quad q_{\xi}(v) - q_{\xi}^+(u) \leq q_{\eta}(v) - q_{\eta}^+(u) \text{ for all } 0 < u < v < 1, \tag{5.1}$$

where $q_{\xi}^+(u) := \inf \{x \in \mathbb{R} : \mathbb{P}(\xi \leq x) > u\}$ is the right quantile function of ξ .

In the following, we consider the family $\Phi_{\text{disp}} := \{\phi_{u,v} : 0 < u < v < 1\}$ of functions $\phi_{u,v} : \mathcal{M}_{\infty} \rightarrow \mathbb{R}$ defined by $\phi_{u,v}(\xi) := q_{\xi}(v) - q_{\xi}^+(u)$. We identify Φ_{disp} with the set $\Delta := \{(u, v) \in \mathbb{R}^2 : 0 < u < v < 1\}$ which we endow with the Euclidean topology.²⁴ It follows from (5.1) that the family Φ_{disp} induces the dispersive order on \mathcal{M}_{∞} . By Proposition B.1, the mapping $(u, v) \mapsto \phi_{u,v}(\xi)$ is lower semicontinuous on Δ for all $\xi \in \mathcal{M}_{\infty}$. Hence, the ordered space $(\mathcal{M}_{\infty}, \preceq_{\text{disp}})$ satisfies the assumptions of Section 4.

²⁴ Although Δ is not a complete metric space, it is completely metrizable as it is an open subset, see [33, Theorem 24.12]. In particular, Δ is a Polish space with the Euclidean topology.

Lemma 5.12. A function $\alpha : \Delta \rightarrow [0, \infty)$ is countably stable if there exists a non-decreasing function $\beta : [0, 1] \rightarrow \mathbb{R}$ such that $\alpha(u, v) = \beta(v) - \beta(u)$ for all $(u, v) \in \Delta$.

Proof. Suppose that $(\xi_n)_{n \in \mathbb{N}} \subset \mathcal{M}_\infty$ satisfies $\sup_{n \in \mathbb{N}} \{q_{\xi_n}(v) - q_{\xi_n}^+(u)\} \leq \alpha(u, v)$ for all $(u, v) \in \Delta$. As an application of Proposition C.2 and Proposition C.1, we obtain that

$$\text{TV}_{[0,1]}(\{q_{\xi_n} : n \in \mathbb{N}\}) = \lim_{t \rightarrow 1^-} \text{TV}_{[0,t]}(\{q_{\xi_n} : n \in \mathbb{N}\}) \leq \beta(1) - \beta(0) < \infty.$$

By Proposition 5.10, the sequence $(\xi_n)_{n \in \mathbb{N}}$ has a supremum $\xi^* \in \mathcal{M}_\infty$ which is determined by

$$q_{\xi^*}(u) = \text{TV}_{[0,u]}(\{q_{\xi_n} : n \in \mathbb{N}\}) \quad \text{for all } u \in (0, 1].$$

Then for $(u, v) \in \Delta$, we obtain from $\text{TV}_{[0,v]} = \text{TV}_{[0,u]} + \text{TV}_{[u,v]}$ that

$$\begin{aligned} q_{\xi^*}(v) - q_{\xi^*}^+(u) &\leq \text{TV}_{[0,v]}(\{q_{\xi_n} : n \in \mathbb{N}\}) - \text{TV}_{[0,u]}(\{q_{\xi_n} : n \in \mathbb{N}\}) \\ &= \text{TV}_{[u,v]}(\{q_{\xi_n} : n \in \mathbb{N}\}) \leq \beta(v) - \beta(u) = \alpha(u, v), \end{aligned}$$

which shows that α is countable stable. \square

A set Q of quantile functions is of bounded variation if $\text{TV}_{[0,1]}(Q) < \infty$.

Lemma 5.13. Let A be a non-empty subset of \mathcal{M}_∞ . If A is countably directed w.r.t. \preceq_{disp} , then the set $Q := \{q_\xi : \xi \in A\}$ is of bounded variation and satisfies

$$\sup_{\xi \in A} (q_\xi(v) - q_\xi^+(u)) = \text{TV}_{[u,v]}(Q) \quad \text{for all } (u, v) \in \Delta.$$

Proof. By contradiction, suppose that $\text{TV}_{[0,1]}(Q) = \infty$. Then there exists a sequence $(\xi_n)_{n \in \mathbb{N}} \in A$ and partitions $\pi_1 \subset \pi_2 \subset \dots$ of $[0, 1]$ with $\lim_{n \rightarrow \infty} S_{\pi_n}(q_{\xi_n}) = \infty$. Since A is countably directed, the sequence $(\xi_n)_{n \in \mathbb{N}}$ has a supremum $\xi^* \in \mathcal{M}_\infty$. Then

$$\infty = \lim_{n \rightarrow \infty} S_{\pi_n}(q_{\xi_n}) \leq \sup_{n \rightarrow \infty} S_{\pi_n}(q_{\xi^*}) \leq \text{TV}_{[0,1]}(q_{\xi^*}),$$

in contradiction to $\text{TV}_{[0,1]}(q_{\xi^*}) < \infty$ since ξ^* is essentially bounded. This shows that Q is of bounded variation.

Define $\alpha(u, v) := \sup_{\xi \in A} (q_\xi(v) - q_\xi^+(u))$. For $(u, v) \in \Delta$, it follows from Proposition C.2 that

$$\alpha(u, v) \leq \text{TV}_{[u,v]}(Q).$$

To prove the equality, suppose by contradiction that $\alpha(u, v) < \text{TV}_{[u,v]}(Q)$. Then by Proposition C.2, there exists a partition $\pi := \{u = t_0 < t_1 < \dots < t_n = v\}$ with

$$\alpha(u, v) < \sum_{i=1}^n \sup_{\xi \in A} (q_\xi(t_i) - q_\xi^+(t_{i-1})) < \sum_{i=1}^n (q_{\xi_i}(t_i) - q_{\xi_i}^+(t_{i-1}))$$

for some $\xi_1, \xi_2, \dots, \xi_n \in A$. Since A is countably directed, there exists $\xi^* \in A$ such that $\xi_1, \dots, \xi_n \preceq_{\text{disp}} \xi^*$. Then

$$\alpha(u, v) < \sum_{i=1}^n (q_{\xi^*}(t_i) - q_{\xi^*}^+(t_{i-1})) \leq q_{\xi^*}(v) - q_{\xi^*}^+(u) \leq \alpha(u, v),$$

which is a contradiction. \square

For a function $\psi : \mathcal{M}_\infty \rightarrow (-\infty, \infty]$ with sublevel sets $A_s := \{\xi \in \mathcal{M}_\infty : \psi(\xi) \leq s\}$, we define $\alpha_{\min}(s, u, v) := \sup_{\xi \in A_s} (q_\xi(v) - q_\xi^+(u))$ and $\alpha_{\min}^+(s, u, v) := \inf_{s < t} \alpha_{\min}(t, u, v)$ for all $s \in \mathbb{R}$ and $(u, v) \in \Delta$.

Let $I := \{s \in \mathbb{R} : A_s \neq \emptyset\}$. If ψ has countably upwards directed sublevel sets, it follows from Lemma 5.13 that for every $s \in I$, the set $Q_s := \{q_\xi : \xi \in A_s\}$ is of bounded variation, and $\alpha_{\min}(s, u, v) = \beta(s, v) - \beta(s, u)$ for all $(u, v) \in \Delta$ where $\beta(s, u) := \text{TV}_{[0,u]}(Q_s)$. Moreover, the mapping $\beta : \mathbb{R} \times (0, 1) \rightarrow \mathbb{R}$ is non-decreasing in the second argument and satisfies $\beta(s, v) - \beta(s, u) \leq \beta(s', v) - \beta(s', u)$ for all $s \leq s'$. As a direct application of Theorem 4.1, we obtain the following representation result for maxitive functions w.r.t. the dispersive stochastic order.

Corollary 5.14. Let $\psi : \mathcal{M}_\infty \rightarrow (-\infty, \infty]$ be a function with sublevel sets $(A_s)_{s \in \mathbb{R}}$. Then, the following conditions are equivalent:

- (1) ψ is monotone and $(A_s)_{s \in \mathbb{R}}$ are countably upwards directed w.r.t. \preceq_{disp} .
- (2) There exists a function $\beta : I \times (0, 1) \rightarrow \mathbb{R}$ which is non-decreasing in the second argument, and satisfies $\beta(s, v) - \beta(s, u) \leq \beta(s', v) - \beta(s', u)$ for all $s \leq s'$ and $(u, v) \in \Delta$. Moreover, it holds

$$\psi(\xi) = \inf \left\{ s \in \mathbb{R} : \sup_{(u,v) \in \Delta} \{q_\xi(v) - q_\xi(u^+) - \alpha(s, u, v)\} \leq 0 \right\} \quad \text{for all } \xi \in \mathcal{M}_\infty,$$

where $\alpha(s, u, v) := \beta(s, v) - \beta(s, u)$ for all $s \in I$, and $\alpha(s, u, v) := \infty$ otherwise. The infimum is attained for all $\xi \in \mathcal{M}_\infty$ with $\psi(\xi) < \infty$.
 (3) For every $s \in I$, the set $Q_s := \{q_\xi : \xi \in A_s\}$ is of bounded variation and $\alpha_{\min}(s, u, v) = \text{TV}_{[u,v]}(Q_s)$ for all $(u, v) \in \Delta$. Moreover, it holds

$$\psi(\xi) = \sup_{(u,v) \in \Delta} G(q_\xi(v) - q_\xi^+(u), u, v) \quad \text{for all } \xi \in \mathcal{M}_\infty,$$

where $G(t, u, v) := \inf \{s \in \mathbb{R} : t \leq \alpha_{\min}^+(s, u, v)\}$ for all $(t, u, v) \in \mathbb{R} \times \Delta$.

We conclude with an example of a maxitive function for the dispersive order.

Example 5.15. Consider the function $\beta : [0, \infty) \times (0, 1) \rightarrow \mathbb{R}$, defined by $\beta(s, u) := su$. The function

$$\psi(\xi) := \sup_{(u,v) \in \Delta} \frac{q_\xi(v) - q_\xi(u^+)}{v - u} \in [0, \infty]$$

satisfies the equivalent conditions of Corollary 5.14 and is therefore bounded countably maxitive on \mathcal{M}_∞/\sim . The value $\psi(\xi)$ corresponds to the Lipschitz constant of q_ξ . In particular, if q_ξ is differentiable, it holds that

$$\psi(\xi) = \sup_{s \in (0,1)} q'_\xi(s).$$

In reliability analysis, the derivative $q'_\xi(s)$ is referred to as the *quantile density function* and measures the rate of change of the quantile function; see [30]. When ξ models the uncertain time to failure of a system, $q_\xi(u)$ corresponds to the time at which the survival probability of the system is $1 - u$. A high value of $q'_\xi(u)$ implies that small reductions in survival probability correspond to larger increases in time, indicating a slower rate of reliability decline over time. Consequently, $\psi(\xi)$ acts as a measure of the speed at which the reliability of the system decreases, capturing the steepest rate of decline.

6. Conclusion

Maxitive functions are widely used in decision-making under risk and uncertainty, particularly for worst-case or best-case evaluations. In this paper, we extend the concept of maxitive functions to a broad class of preorders, providing a unified framework that goes beyond the classical pointwise order. Specifically, our results address the main stochastic orders in the literature, including the usual stochastic order, the increasing convex order, and the dispersive order.

By transitioning from the pointwise order to weaker stochastic orders, we develop a framework that covers a broader class of maxitive functions. Our main result is a representation theorem for bounded countably maxitive functions. In the translation-invariant case, we obtain representation results in terms of penalized worst-case evaluations. Notably, this generalization also includes widely used decision-making tools such as the value at risk.

CRedit authorship contribution statement

Michael Kupper: Investigation. **José M. Zapata:** Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Auxiliary results for maxitive risk measures on $C_b(S)$

Let (S, d) be a Polish space, and denote by \mathcal{X} a convex cone of functions $f : S \rightarrow \mathbb{R}$ with $C_b(S) \subset \mathcal{X}$. Here, $C_b(S)$ denotes the space of all bounded continuous functions. The set \mathcal{X} is endowed with the pointwise order. For a function $\psi : \mathcal{X} \rightarrow (-\infty, \infty]$, we frequently deal with the following properties:

- (T) $\psi(f + c) = \psi(f) + c$ for all $c \in \mathbb{R}$.
- (M) $\psi(f) \leq \psi(g)$ whenever $f \leq g$.
- (C) $\psi(\lambda f + (1 - \lambda)g) \leq \lambda\psi(f) + (1 - \lambda)\psi(g)$ for all $\lambda \in [0, 1]$.

The function ψ is *continuous from below* if $\psi(f) = \lim_{n \rightarrow \infty} \psi(f_n)$ for all sequences $(f_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ with $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $f = \sup_{n \in \mathbb{N}} f_n \in \mathcal{X}$.

For $\psi : \mathcal{X} \rightarrow (-\infty, \infty]$, we consider the *acceptance set* $\mathcal{A} := \{f \in \mathcal{X} : \psi(f) \leq 0\}$, and define the *convex conjugate* $\psi_{C_b(S)}^* : \text{ca}_1^+(S) \rightarrow [0, \infty]$ by

$$\psi_{C_b(S)}^*(\mu) := \sup_{f \in C_b(S)} \left\{ \int f \, d\mu - \psi(f) \right\},$$

where $\text{ca}_1^+(S)$ denotes the set of all Borel probability measures on S . In the particular, for the Dirac measure δ_x , we get

$$I_{\min}(x) := \psi_{C_b(S)}^*(\delta_x) = \sup_{f \in C_b(S)} \{f(x) - \psi(f)\} \quad \text{for all } x \in S.$$

The following result was recently shown by Delbaen [7,8]; see also [18] for an alternative proof.

Theorem A.1. *Let $\psi : C_b(S) \rightarrow \mathbb{R}$ be a function satisfying (T), (M) and (C). Then, the following statements are equivalent:*

- (i) ψ is continuous from below.
- (ii) ψ admits the representation

$$\psi(f) = \sup_{\mu \in \text{ca}_1^+(S)} \left\{ \int f \, d\mu - \psi_{C_b(S)}^*(\mu) \right\}.$$

In combination with [15, Theorem 2.1], we obtain the following result. Whereas the representation result in [15] relies on continuity from above, here we rely on the weaker condition of continuity from below, which is guaranteed by countable maxitivity.

Theorem A.2. *Let $\psi : C_b(S) \rightarrow \mathbb{R}$ be a function satisfying (T) and (M). Then, the following statements are equivalent:*

- (i) ψ is countably maxitive, i.e., $\psi(f) = \sup_{n \in \mathbb{N}} \psi(f_n)$ for every $(f_n)_{n \in \mathbb{N}} \subset C_b(S)$ with $f = \sup_{n \in \mathbb{N}} f_n \in C_b(S)$.
- (ii) $\sup_{n \in \mathbb{N}} f_n \in \mathcal{A}$ for every $(f_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ with $\sup_{n \in \mathbb{N}} f_n \in C_b(S)$.
- (iii) ψ admits the representation

$$\psi(f) = \sup_{x \in S} \{f(x) - I_{\min}(x)\} \quad \text{for all } f \in C_b(S).$$

(iv) ψ satisfies (C) and is continuous from below. Moreover,

$$\psi_{C_b(S)}^*(\mu) = \int I_{\min}(x) \mu(dx) \quad \text{for all } \mu \in \text{ca}_1^+(S).$$

(v) There exists a function $\gamma : S \rightarrow [0, \infty]$ such that

$$\mathcal{A} = \{f \in C_b(S) : f \leq \gamma\}.$$

Proof. The argumentation is similar to that in [15, Theorem 2.1]. Similar to the proof of [15, Theorem 2.1], it holds

$$\psi^*(\delta) = \text{ess. sup}_{\mu} \mathcal{A} \quad \mu\text{-almost surely} \tag{A.1}$$

for all $\mu \in \text{ca}_1^+(S)$, where $\text{ess. sup}_{\mu} \mathcal{A}$ denotes the essential supremum of \mathcal{A} with respect to the probability measure μ .

(i) \Rightarrow (ii): For $(f_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ with $\sup_{n \in \mathbb{N}} f_n \in C_b(S)$, we have

$$\psi\left(\sup_{n \in \mathbb{N}} f_n\right) = \sup_{n \in \mathbb{N}} \psi(f_n) \leq 0,$$

and therefore $\sup_{n \in \mathbb{N}} f_n \in \mathcal{A}$.

(ii) \Rightarrow (i): Let $(f_n)_{n \in \mathbb{N}} \subset C_b(S)$ with $f := \sup_{n \in \mathbb{N}} f_n \in C_b(S)$. Since by (T), $f_n - \sup_{m \in \mathbb{N}} \psi(f_m) \in \mathcal{A}$ for all $n \in \mathbb{N}$, we obtain that $f - \sup_{m \in \mathbb{N}} \psi(f_m) \in \mathcal{A}$. By (T) and (M), it holds

$$\psi(f) \leq \sup_{n \in \mathbb{N}} \psi(f_n) \leq \psi(f).$$

(i) \Rightarrow (iv): Condition (i) ensures that ψ is (finitely) maxitive and continuous from below. In particular, since ψ is maxitive, it satisfies (C) due to [15, Proposition 2.1]. Fix $\mu \in \text{ca}_1^+(S)$. Since \mathcal{A} is directed upwards, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ with $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ such that $f_n \nearrow \text{ess. sup}_{\mu} \mathcal{A}$ μ -almost surely. By (A.1), we have that $\text{ess. sup}_{\mu} \mathcal{A} = I_{\min}$ μ -almost surely. Hence, by monotone convergence,

$$\psi_{C_b(S)}^*(\mu) = \sup_{f \in \mathcal{A}} \int f \, d\mu \geq \sup_{n \in \mathbb{N}} \int f_n \, d\mu = \int I_{\min}(x) \mu(dx).$$

Moreover,

$$\psi_{C_b(S)}^*(\mu) = \sup_{f \in \mathcal{A}} \int f \, d\mu \leq \int \text{ess. sup}_{\mu} \mathcal{A} \, d\mu = \int I_{\min}(x) \mu(dx).$$

(iv) \Rightarrow (iii): For every $f \in C_b(S)$, it follows from Theorem A.1 that

$$\begin{aligned} \psi(f) &= \sup_{\mu \in \text{ca}_1^+(S)} \left\{ \int f \, d\mu - \psi_{C_b(S)}^*(\mu) \right\} = \sup_{\mu \in \text{ca}_1^+(S)} \left\{ \int f \, d\mu - \int I_{\min}(x) \mu(dx) \right\} \\ &= \sup_{\mu \in \text{ca}_1^+(S)} \left\{ \int f(x) - I_{\min}(x) \mu(dx) \right\} = \sup_{x \in S} \{f(x) - I_{\min}(x)\}. \end{aligned}$$

(iii) \Rightarrow (v): From the representation, $f \in \mathcal{A}$ if and only if $f(x) \leq I_{\min}(x)$ for all $x \in S$. Hence, $\mathcal{A} = \{f \in C_b(S) : f \leq \gamma\}$ with $\gamma = I_{\min}$.

(v) \Rightarrow (ii): Let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{A}$ with $\sup_{n \in \mathbb{N}} f_n \in C_b(S)$. Since $f_n \leq \gamma$ for all $n \in \mathbb{N}$, it follows that $\sup_{n \in \mathbb{N}} f_n \leq \gamma$. This shows that $f \in \mathcal{A}$. \square

We denote by $L(S)$ the set of all lower semicontinuous functions $f : S \rightarrow \mathbb{R}$.

Corollary A.3. Suppose that $\psi : L(S) \rightarrow (-\infty, \infty]$ satisfies $\psi(0) \in \mathbb{R}$, properties (T) and (M), and that $\psi(f) = \sup_{n \in \mathbb{N}} \psi(f_n)$ for every sequence $(f_n)_{n \in \mathbb{N}} \subset L(S)$ such that $f = \sup_{n \in \mathbb{N}} f_n \in L(S)$. Then,

$$\psi(f) = \sup_{x \in S} \{f(x) - I_{\min}(x)\} \quad \text{for all } f \in L(S).$$

Proof. Since $\psi(0) \in \mathbb{R}$, property (T) implies that the restriction of ψ to $C_b(S)$ is real-valued. Thus, by Theorem A.2, we have

$$\psi(f) = \sup_{x \in S} \{f(x) - I_{\min}(x)\} \quad \text{for all } f \in C_b(S).$$

Now, let $f \in L(S)$ be bounded from below. In this case, there exists a sequence $(f_n)_{n \in \mathbb{N}} \subset C_b(S)$ such that $f_n \nearrow f$. For instance, the sequence defined by

$$f_n(x) := \inf \{f(y) + nd(x, y) : y \in S\} \wedge n$$

satisfies this property. Since ψ is continuous from below, we have

$$\begin{aligned} \psi(f) &= \sup_{n \in \mathbb{N}} \psi(f_n) = \sup_{n \in \mathbb{N}} \sup_{x \in S} \{f_n(x) - I_{\min}(x)\} \\ &= \sup_{x \in S} \sup_{n \in \mathbb{N}} \{f_n(x) - I_{\min}(x)\} = \sup_{x \in S} \{f(x) - I_{\min}(x)\}. \end{aligned}$$

Finally, consider a general $f \in L(S)$. Since ψ is maxitive, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} \psi(f) \vee \psi(-n) &= \psi(f \vee (-n)) \\ &= \sup_{x \in S} \{(f(x) \vee (-n)) - I_{\min}(x)\} \\ &= \sup_{x \in S} \{(f(x) - I_{\min}(x)) \vee (-n - I_{\min}(x))\}. \end{aligned}$$

Letting $n \rightarrow \infty$, we obtain that $\psi(f) = \sup_{x \in S} \{f(x) - I_{\min}(x)\}$. \square

Appendix B. Auxiliary results for quantile functions

Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is an atomless probability space. Recall that \mathcal{M} denotes the set of all random variables $\xi : \Omega \rightarrow \mathbb{R}$, where two of them are identified if they have the same law. Given $\xi \in \mathcal{M}$, we define the cumulative distribution function by $F_\xi(x) := \mathbb{P}(\xi \leq x)$ and set $F_\xi(x^-) = \sup_{y < x} F_\xi(y)$ for all $x \in \mathbb{R}$. Moreover, the quantile function $q_\xi : (0, 1) \rightarrow \mathbb{R}$ and upper quantile function $q_\xi^+ : (0, 1) \rightarrow \mathbb{R}$ are defined by

$$q_\xi(t) := \inf \{x \in \mathbb{R} : t \leq F_\xi(x)\} \quad \text{and} \quad q_\xi^+(t) := \inf \{x \in \mathbb{R} : t < F_\xi(x)\}.$$

Lemma B.1. For all $\xi \in \mathcal{M}$ and $t \in (0, 1)$, the following statements hold:

- (i) $[q_\xi(t), \infty) = \{x \in \mathbb{R} : t \leq F_\xi(x)\}$.
- (ii) $(-\infty, q_\xi^+(t)] = \{x \in \mathbb{R} : t \geq F_\xi(x^-)\}$.
- (iii) $\sup_{s < t} q_\xi(s) = \sup_{s < t} q_\xi^+(s) = q_\xi(t)$.
- (iv) $\inf_{s > t} q_\xi(s) = \inf_{s > t} q_\xi^+(s) = q_\xi^+(t)$.
- (v) The function $q_\xi : (0, 1) \rightarrow \mathbb{R}$ is left-continuous.
- (vi) The function $q_\xi^+ : (0, 1) \rightarrow \mathbb{R}$ is right-continuous.

(vii) The function $(s, r) \mapsto q_\xi(r) - q_\xi^+(s)$ is lower semicontinuous on $(0, 1)^2$.

Proof. (i) and (ii) follow from the definitions of q and q^+ , using that F_ξ^- is non-decreasing and right-continuous.

For $a := \sup_{s < t} q_\xi(s)$, it follows from (i) that

$$[a, \infty) = \bigcap_{s < t} [q_\xi(s), \infty) = \bigcap_{s < t} F_\xi^{-1}([s, \infty)) = F_\xi^{-1}([t, \infty)) = [q_\xi(t), \infty),$$

where $F_\xi^{-1}([s, \infty)) = \{x \in \mathbb{R} : s \leq F_\xi(x)\}$. This shows $\sup_{s < t} q_\xi(s) = q_\xi(t)$. For $s < t$, it holds $q_\xi(s) \leq q_\xi^+(s) \leq q_\xi(t)$, so that $q_\xi(t) = \sup_{s < t} q_\xi(s) \leq \sup_{s < t} q_\xi^+(s) \leq q_\xi(t)$, which shows (iii).

Define $G(s) := F_\xi(s^-)$ for all $s \in \mathbb{R}$. For $b := \inf_{s > t} q_\xi^+(s)$, it follows from (ii) that

$$(-\infty, b] = \bigcap_{s > t} (-\infty, q_\xi^+(s)] = \bigcap_{s > t} G^{-1}((-\infty, s]) = G^{-1}((-\infty, t]) = (-\infty, q_\xi^+(t)],$$

where $G^{-1}([s, \infty)) = \{x \in \mathbb{R} : s \leq G(x)\}$. This shows $\inf_{s > t} q_\xi^+(s) = q_\xi^+(t)$. For $s > t$, it holds $q_\xi^+(s) \geq q_\xi(s) \geq q_\xi^+(t)$, so that $q_\xi^+(t) = \inf_{s > t} q_\xi^+(s) \geq \inf_{s > t} q_\xi(s) \geq q_\xi^+(t)$, which shows (iv).

Finally, (v), (vi) and (vii) are consequences of (iii) and (iv). \square

Lemma B.2. For $\xi, \eta \in \mathcal{M}$, the following statements are equivalent:

- (i) $q_\xi(u) - q_\xi(t) \leq q_\eta(u) - q_\eta(t)$ for all $0 < t \leq u < 1$.
- (ii) $q_\xi(u) - q_\xi^+(t) \leq q_\eta(u) - q_\eta^+(t)$ for all $0 < t < u < 1$.

Proof. (ii) \Rightarrow (i): Let $0 < t \leq u < 1$. For $s \in (0, t)$, it holds $q_\xi(u) - q_\xi^+(s) \leq q_\eta(u) - q_\eta^+(s)$. Letting $s \rightarrow t^-$, it follows from (iii) in Lemma B.1 that

$$q_\xi(u) - q_\xi(t) \leq q_\eta(u) - q_\eta(t).$$

(i) \Rightarrow (ii): Let $0 < t < u < 1$. For $s \in (t, u)$, it holds $q_\xi(u) - q_\xi(s) \leq q_\eta(u) - q_\eta(s)$. Letting $s \rightarrow t^+$, it follows from (iv) in Lemma B.1 that $q_\xi(u) - q_\xi^+(t) \leq q_\eta(u) - q_\eta^+(t)$. \square

Lemma B.3. If $q : (0, 1) \rightarrow \mathbb{R}$ is non-decreasing and left-continuous, then there exists $\xi \in \mathcal{M}$ such that $q(u) = q_\xi(u)$ for all $u \in (0, 1)$.

Proof. If $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then there exists a uniformly distributed random variable U with values in $(0, 1)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, see e.g. [6, Theorem 1] and [12, Proposition A.27]. Then $\xi = q(U)$ is as desired. To that end, define $F : \mathbb{R} \rightarrow [0, 1]$ by

$$F(x) := 0 \vee \sup\{s \in (0, 1) : x \geq q(s)\},$$

which satisfies

- (1) $q(F(x)) \leq x$ for all $x \in \mathbb{R}$ (where we set $q(0) := q(0^+)$ and $q(1) := q(1^-)$),
- (2) $F(q(s)) \geq s$ for all $s \in (0, 1)$.

For $x \in \mathbb{R}$, it follows from (2) that $\{q(U) \leq x\} \subset \{F(q(U)) \leq F(x)\} \subset \{U \leq F(x)\}$. Hence, for every $u \in (0, 1)$,

$$\begin{aligned} q_\xi(u) &= \inf\{x \in \mathbb{R} : \mathbb{P}(q(U) \leq x) \geq u\} \\ &\geq \inf\{x \in \mathbb{R} : \mathbb{P}(U \leq F(x)) \geq u\} \\ &= \inf\{x \in \mathbb{R} : F(x) \geq u\}. \end{aligned}$$

Let $a := \inf\{x \in \mathbb{R} : F(x) \geq u\}$, and $x_n \downarrow a$ with $F(x_n) \geq u$. It follows from (1) that $x_n \geq q(u)$, and therefore $q_\xi(u) \geq a \geq q(u)$ by letting $n \rightarrow \infty$.

As for the other inequality, we have that $\mathbb{P}(q(U) \leq q(u)) \geq \mathbb{P}(U \leq u) = u$. Hence, it follows from the definition of $q_\xi(u)$ that $q_\xi(u) \leq q(u)$. \square

Recall that for $\xi \in \mathcal{M}_1$, the integrated quantile function $Q_\xi : (0, 1) \rightarrow \mathbb{R}$ is defined by

$$Q_\xi(u) := \int_u^1 q_\xi(v) dv.$$

Lemma B.4. *If $Q : (0, 1) \rightarrow \mathbb{R}$ is concave and satisfies $Q(0^+) \in \mathbb{R}$ and $Q(1^-) = 0$, then there exists $\xi \in \mathcal{M}_1$ such that $Q(u) = Q_\xi(u)$ for all $u \in (0, 1)$.*

Proof. Since Q is concave, there exist a non-decreasing function $q : (0, 1) \rightarrow \mathbb{R}$ and $c \in (0, 1)$ such that

$$Q(u) = Q(c) - \int_c^u q(v)dv,$$

see e.g. [26, Theorem 12A]. By replacing q by its left-continuous modification, we can assume that q is left-continuous. Using that $Q(1^-) = 0$, we get that

$$Q(u) = \int_u^1 q(v)dv.$$

Since q is left-continuous, it follows from Lemma B.3 that $q = q_\xi$ for some $\xi \in \mathcal{M}$. It remains to show that $\xi \in \mathcal{M}_1$. We have

$$\begin{aligned} \mathbb{E}_\mathbb{P}[|\xi|] &= \int_0^1 |q_\xi(v)|dv \\ &= - \int_0^{F_\xi(0)} q_\xi(v)dv + \int_{F_\xi(0)}^1 q_\xi(v)dv \\ &= -Q(0^+) + 2Q(F_\xi(0)) < \infty. \end{aligned}$$

Hence, ξ has finite first moment, and consequently $\xi \in \mathcal{M}_1$. \square

Appendix C. Auxiliary results for the total variation

We consider the setting of Subsection 5.5.

Proposition C.1. *The mapping $t \mapsto \text{TV}_{[0,t]}(Q)$ is non-decreasing and left-continuous on $(0, 1]$.*

Proof. By definition, the mapping $t \mapsto \text{TV}_{[0,t]}(Q)$ is non-decreasing. Given $t \in (0, 1]$, we fix a partition $\pi = \{t_0 < t_1 < \dots < t_n\}$ of $[0, t]$. For every $u \in (0, t)$, we consider the partition $\pi_u := (\pi \cap [0, u]) \cup \{u\} \in \Pi[0, u]$. Then since every $q \in Q$ is non-decreasing and left-continuous, we have that

$$\begin{aligned} \sup_{u < t} S_{\pi_u}(Q) &= \sup_{u \in (t_{n-1}, t)} S_{\pi_u}(Q) \\ &= \sum_{i=1}^{n-1} \sup_{q \in Q} (q(t_i) - q(t_{i-1})) + \sup_{u \in (t_{n-1}, t)} \sup_{q \in Q} (q(u) - q(t_{n-1})) \\ &= \sum_{i=1}^{n-1} \sup_{q \in Q} (q(t_i) - q(t_{i-1})) + \sup_{q \in Q} \sup_{u \in (t_{n-1}, t)} (q(u) - q(t_{n-1})) \\ &= \sum_{i=1}^{n-1} \sup_{q \in Q} (q(t_i) - q(t_{i-1})) + \sup_{q \in Q} (q(t) - q(t_{n-1})) \\ &= S_\pi(Q). \end{aligned}$$

Since $\text{TV}_{[0,u]}(Q) = \sup_{\pi \in \Pi[0,u]} S_\pi(Q) = \sup_{\pi \in \Pi[0,t]} S_{\pi_u}(Q)$ for all $u < t$, it follows that

$$\begin{aligned} \lim_{u \uparrow t} \text{TV}_{[0,u]}(Q) &= \sup_{u < t} \sup_{\pi \in \Pi[0,t]} S_{\pi_u}(Q) = \sup_{\pi \in \Pi[0,t]} \sup_{u < t} S_{\pi_u}(Q) \\ &= \sup_{\pi \in \Pi[0,t]} S_\pi(Q) = \text{TV}_{[0,t]}(Q). \quad \square \end{aligned}$$

For $[u, v] \subset [0, 1]$ and $\pi \in \Pi[u, v]$, we define $\underline{S}_\pi(Q) := \sum_{i=1}^n \sup_{q \in Q} (q_\xi(t_i) - q_\xi^+(t_{i-1}))$.

Proposition C.2. *For every $[u, v] \subset [0, 1]$, it holds*

$$\text{TV}_{[u,v]}(Q) = \sup_{\pi \in \Pi[u,v]} \underline{S}_\pi(Q).$$

Proof. By contradiction, we assume that $\sup_{\pi \in \Pi[u,v]} S_{\pi}(Q) > \sup_{\pi \in \Pi[u,v]} \underline{S}_{\pi}(Q)$. Then, there exists $\bar{\pi} = \{u = t_0 < t_1 < \dots < t_n = v\} \in \Pi[u, v]$ with

$$S_{\bar{\pi}}(Q) > \sup_{\pi \in \Pi[0,t]} \underline{S}_{\pi}(Q).$$

Fix $\varepsilon > 0$ such that $S_{\bar{\pi}}(Q) - \varepsilon > \sup_{\pi \in \Pi[0,t]} \underline{S}_{\pi}(Q)$. For every $i \in \{1, 2, \dots, n\}$, the mapping $t \mapsto \sup_{q \in Q} (q(t) - q(t_{i-1}))$ is left-continuous. Hence, there exists $u_i \in [t_{i-1}, t_i)$ such that

$$\sup_{q \in Q} (q(u_i) - q(t_{i-1})) > \sup_{q \in Q} (q(t_i) - q(t_{i-1})) - \frac{\varepsilon}{n}.$$

Since $q^+(u_i) \leq q(t_i)$ for all i , we get

$$\sup_{q \in Q} (q(u_i) - q^+(u_{i-1})) \geq \sup_{q \in Q} (q(t_i) - q(t_{i-1})) - \frac{\varepsilon}{n} \quad \text{for all } i \in \{1, 2, \dots, n\},$$

where $u_0 := t_0 = u$. For the partition $\pi' := \{u = u_0 < \dots < u_n < t_n = v\} \in \Pi[u, v]$, we obtain

$$\underline{S}_{\pi'}(Q) \geq S_{\bar{\pi}}(Q) - \varepsilon > \sup_{\pi \in \Pi[0,t]} \underline{S}_{\pi}(Q),$$

which is a contradiction. \square

Data availability

No data was used for the research described in the article.

References

- [1] M.Z. Arslanov, E.E. Ismail, On the existence of a possibility distribution function, *Fuzzy Sets Syst.* 148 (2004) 279–290.
- [2] V. Bignozzi, M. Burzoni, C. Munari, Risk measures based on benchmark loss distributions, *J. Risk Insur.* 87 (2020) 437–475.
- [3] M. Cattaneo, On maxitive integration, *Fuzzy Sets Syst.* 304 (2016) 65–81.
- [4] C. Chambers, A. Müller, R. Wang, Q. Wu, Max- and min-stability under first-order stochastic dominance, Preprint, arXiv:2403.13138, 2024.
- [5] G. de Cooman, D. Aeyels, Supremum preserving upper probabilities, *Inf. Sci.* 118 (1999) 173–212.
- [6] F. Delbaen, *Monetary Utility Functions*, Osaka Univ. Press, Osaka, 2012.
- [7] F. Delbaen, Convex increasing functionals on $C_b(X)$ spaces, *Stud. Math.* 271 (2023) 107–120.
- [8] F. Delbaen, Monetary utility functions on $C_b(X)$ spaces, *Int. J. Theor. Appl. Finance* 27 (2024) 2350033.
- [9] S. Drapeau, M. Kupper, Risk preferences and their robust representation, *Math. Oper. Res.* 38 (2013) 28–62.
- [10] D. Dubois, H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.
- [11] D. Dubois, H. Prade, Possibility theory and its applications: where do we stand?, in: *Springer Handbook of Computational Intelligence*, Springer, Berlin, Heidelberg, 2015, pp. 31–60.
- [12] H. Föllmer, A. Schied, *Stochastic Finance. An Introduction in Discrete Time*, 2nd ed., Walter de Gruyter, Berlin, 2011.
- [13] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, *Aggregation Functions*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, Cambridge, 2009.
- [14] R.P. Kertz, U. Rösler, *Complete Lattices of Probability Measures with Applications to Martingale Theory*, IMS Lecture Notes Monograph Series, vol. 35, 2000, pp. 153–177.
- [15] M. Kupper, J.M. Zapata, Large deviations built on max-stability, *Bernoulli* 27 (2021) 1001–1027.
- [16] M. Kupper, J.M. Zapata, Weakly maxitive set functions and their possibility distributions, *Fuzzy Sets Syst.* 467 (2023) 108506.
- [17] V.N. Kolokoltsov, V.P. Maslov, *Idempotent Analysis and Its Applications*, Springer, 1997.
- [18] M. Nendel, Lower semicontinuity of monotone functionals in the mixed topology on C_b , *Finance Stoch.* (2024) 1–27.
- [19] R. Mesiar, Possibility measures, integration and fuzzy possibility measures, *Fuzzy Sets Syst.* 92 (1997) 191–196.
- [20] T. Murofushi, M. Sugeno, Continuous-from-above possibility measures and f -additive fuzzy measures: characterization and regularity, *Fuzzy Sets Syst.* 54 (1993) 351–354.
- [21] A. Müller, D. Stoyan, *Comparison Methods for Stochastic Models and Risks*, John Wiley & Sons, 2002.
- [22] A. Müller, M. Scarsini, Stochastic order relations and lattices of probability measures, *SIAM J. Optim.* 16 (2006) 1024–1043.
- [23] S. Peng, *Nonlinear Expectations and Stochastic Calculus Under Uncertainty*, Springer, Berlin, Heidelberg, 2019.
- [24] P. Poncet, Representation of maxitive measures: an overview, *Math. Slovaca* 67 (2017) 121–150.
- [25] A. Puhalskii, *Large Deviations and Idempotent Probability*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 119, CRC Press, Boca Raton, FL, 2001.
- [26] A.W. Roberts, D.E. Varberg, *Convex Functions*, Academic Press, New York, 1973.
- [27] M. Shaked, J.G. Shanthikumar, *Stochastic Orders*, Springer Series in Statistics, Springer, New York, 2007.
- [28] N. Shilkret, Maxitive measure and integration, *Indag. Math.* 33 (1971) 109–116.
- [29] Y. Torra, Y. Narukawa, *Modeling Decisions: Information Fusion and Aggregation Operators*, Springer, 2007.
- [30] N. Unnikrishnan, P.G. Sankaran, N. Balakrishnan, *Quantile-Based Reliability Analysis*, Birkhäuser, Basel, 2013.
- [31] P. Walley, *Statistical Reasoning with Imprecise Probabilities*, Monographs on Statistics and Applied Probability, vol. 42, Chapman & Hall, London, 1991.
- [32] T. Mao, R. Wang, Q. Wu, Model aggregation for risk evaluation and robust optimization, Preprint, arXiv:2201.06370, 2022.
- [33] S. Willard, *General Topology*, Addison Wesley, Reading, 1970.
- [34] L.A. Zadeh, Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.* 1 (1978) 3–28.