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Introduction

In a previous article (see [AC]), we obtained tight lower bounds for the so–called *generalized h–vector* of a rational polytope with a certain type of symmetry generalizing a previous result of R. Adin for rational simplicial polytopes (see [Ad]). For the special case of a centrally symmetric polytope these bounds are due to R. Stanley (see [St1]).

In this article, we consider centrally symmetric polytopes that are not required to be rational. The generalized h–vector of a convex polytope is a combinatorial invariant defined by recursion over the faces. The invariant has been introduced by R. Stanley (see [St2]), generalizing the usual definition of the h–vector of a simplicial polytope.

If the polytope K is rational then there is an associated projective toric variety X_K, and the coefficients of the generalized h–vector of K have a topological interpretation as Betti numbers of the intersection cohomology of X_K. If K in addition is simplicial then X_K is rationally smooth, and then the intersection cohomology coincides with ordinary singular cohomology and the generalized h–vector coincides with the usual h–vector.

It was discovered by G. Barthel, J.-P. Brasselet, K. Fieseler and L. Kaup, that one can completely characterize the equivariant intersection cohomology of a toric variety by combinatorial and algebraic data associated to the corresponding fan, namely in terms of a *minimal extension sheaf* on the fan considered as a topological space where the subfans are the open subsets (see [BBFK1]). Associating an analogous object to a non–rational fan, one can define a "virtual" intersection cohomology satisfying similar formal properties as the usual intersection cohomology. If Δ is the fan through the faces of a polytope K then under certain natural vanishing conditions for the minimal extension sheaf on Δ the coefficients of the generalized h–vector of K coincide with the even "Betti–numbers" of this "virtual" or "combinatorial" intersection cohomology (see [BBFK2]).

We apply these results to a centrally symmetric polytope K, and denoting the polynomial defined by the generalized h–vector by h_K, we prove the following:

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If certain natural vanishing conditions are satisfied for the minimal extension sheaf on Δ_K*, then the polynomial*

$$h_K(x) - (1+x)^n$$

is palindromic, its coefficients are even and they increase up to the middle one(s) and then decrease again. In particular, if $\dim K = n$ *this implies the following bounds for the coefficients* h_j *of* h_K*:*

$$h_j \geq \binom{n}{j} \quad \text{for } j = 0, \ldots, n.$$

1 Preliminaries

Let K denote a convex polytope of dimension n in $V := \mathbb{R}^n$. Assume that zero lies in the interior of K. Then the polytope K defines a complete fan in V consisting of the cones through its proper faces:

$$\Delta_K := \{\text{cone}(F); F \text{ proper face of } K\} \cup \{0\},$$

Moreover, this fan is equipped with a *strictly concave support function*, that means a concave function whose restriction to any cone in Δ_K is linear and such that for any two different maximal cones the linear functions obtained by restriction are different. To define this function, consider the dual polytope $K^* := \{u \in V^*; \langle u, v\rangle \leq -1 \text{ for all } v \in K\}$ of K. There is an order–reversing one–to–one–correspondence between the proper faces of K and the proper faces of K^* defined by

$$F \mapsto s_F := \{u \in K^*; \langle u, v\rangle = -1 \text{ for all } v \in F\}.$$

So the vertices of K^* are of the form s_F, where F is a one–codimensional face of K, and s_F defines a linear function on cone(F). These linear functions glue together to a well–defined concave function $s_K: V \to \mathbb{R}$.

The *generalized h–vector* of the polytope K is a combinatorial invariant defined by recursion over the faces of K (see [St2]). In fact, this invariant only depends on the fan Δ_K, and it makes sense to define a generalized h–vector for arbitrary complete fans using the same recursion formulae. So let us state the definition here in terms of fans. A *fan* in a real vector space V is a nonempty set Δ of strictly convex polyhedral cones intersecting pairwise in common faces and such that if a cone belongs to the set Δ then all its faces also belong to Δ. The fan Δ is called *complete* if its support $|\Delta| := \bigcup_{\sigma\in\Delta} \sigma$ equals V, and Δ is *rational* with respect to a lattice N in $\mathbb{R}$, if all the cones are generated by vectors in N. For a given cone σ, let V_σ denote the linear span of σ in V. Let Λ_σ denote the fan that we obtain by projecting the boundary of σ to V_σ/L, where L is a one–dimensional subspace generated by a vector in the relative interior of σ.

We introduce two polynomials, namely h_Δ for each complete fan Δ and g_σ for each strictly convex polyhedral cone σ, satisfying the following recursion:

i) $g_0 \equiv 1$

ii) $h_\Delta(x) = \sum_{\sigma\in\Delta}(x-1)^{\dim\Delta - \dim\sigma} g_\sigma(x)$

iii) $g_\sigma(x) = \tau_{<[(\dim\sigma)/2]}((1-x)h_{\Lambda_\sigma}(x))$,

where $\tau_{\leq r}$ denotes the truncation operator $\tau_{\leq r}(\sum_{i=0}^{n} a_i x^i) := \sum_{i=0}^{r} a_i x^i$. The vector formed by the coefficients of the polynomial h_Δ is called the *generalized h–vector* of the fan Δ.

2 Combinatorial Intersection Cohomology

For later use, in this section we very briefly summarize the main results on the combinatorial intersection cohomology for fans that are presented in [BBFK2], and at the same time we fix the notation. Let Δ be a (not necessarily rational) fan in a real vectorspace $V = \mathbb{R}^n$.

If a subset of cones $\Lambda \subset \Delta$ is again a fan, then we speak of a *subfan* and write $\Lambda \prec \Delta$. In the rational case, where Δ defines a toric variety, the subfans of Δ correspond to the open invariant subsets of the toric variety, so they define a "T–stable topology". That is the motivation for considering the set of all subfans of an arbitrary fan Δ together with the empty set as the open sets of a topology, namely the *fan topology* on Δ. A basis of this topology is formed by the *affine* subfans, i.e. the subfans that are fans of faces of single cones. For a cone $\sigma \in \Delta$, we denote the fan of faces of σ by $\langle\sigma\rangle$ and its boundary fan by $\partial\sigma$.

Let $A^\bullet := S^\bullet(V^*)$ denote the algebra of real–valued polynomial functions on V, together with the grading defined by associating to each linear function the degree 2. The algebra $A^\bullet$ defines a *sheaf of graded algebras* $\mathcal{A}^\bullet$ on Δ (with the fan topology), where for $\sigma \in \Delta$ the algebra $\mathcal{A}^\bullet(\langle\sigma\rangle) =: A^\bullet_\sigma$ consists of the elements of $S^\bullet(V^*_\sigma)$ viewed as polynomial functions on σ. The restriction homomorphisms of $\mathcal{A}^\bullet$ are given by restriction of polynomial functions. For $\Lambda \prec \Delta$, the sections in $\mathcal{A}^\bullet(\Lambda)$ correspond to those functions on Λ that are conewise polynomial. Instead of $\mathcal{A}^\bullet(\Lambda)$ we also write $A^\bullet_\Lambda$.

Now consider a sheaf $\mathcal{E}^\bullet$ of graded $\mathcal{A}^\bullet$–modules on Δ. To denote the sections $\mathcal{E}^\bullet(\Lambda)$ of $\mathcal{E}^\bullet$ on $\Lambda \prec \Delta$ we also write E_Λ, and we abbreviate $E_{\langle\sigma\rangle}$ to E_σ. Let $\mathfrak{m}$ denote the unique homogeneous maximal ideal of $\mathcal{A}^\bullet$. Then forming residue classes modulo $\mathfrak{m}$ we obtain a sheaf of graded real vector spaces $\overline{\mathcal{E}}^\bullet$ on Δ, where $\overline{\mathcal{E}}^\bullet(\Lambda) := \overline{E}_\Lambda$.

The sheaf $\mathcal{E}^\bullet$ is called a *minimal extension sheaf* if the following properties hold:

i) $E^\bullet_0 \simeq \mathbb{R}^\bullet$, where $\mathbb{R}^\bullet$ denotes $\mathbb{R}$ viewed as a graded algebra with trivial zero grading.

ii) For every $\sigma \in \Delta$, the module $E^\bullet_\sigma$ is free over $A^\bullet_\sigma$.

iii) For each cone $\sigma \in \Delta\backslash\{0\}$, the restriction map $\rho_\sigma\colon E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$ induces an isomorphism

$$\overline{\rho}_\sigma\colon \overline{E}^\bullet_\sigma \to \overline{E}^\bullet_{\partial\sigma}$$

of graded real vector spaces.

In [BBFK2], the authors prove that for any given fan Δ a minimal extension sheaf exists and up to isomorphism is unique. Moreover, for any cone $\sigma \neq 0$ in Δ they consider the *vanishing condition* $V(\sigma)$, defined by

$$\overline{E}^q_\sigma = 0 \quad \text{for all } q \geq \dim \sigma,$$

and they prove the following:

2.1 Theorem. *(*Barthel, Brasselet, Fieseler, Kaup*) Let Δ be a complete fan.*

i) *$E^\bullet_\Delta$ is a free $A^\bullet$–module, and therefore $E^\bullet_\Delta = A^\bullet \otimes_{\mathbb{R}} \overline{E}^\bullet_\Delta$.*

ii) *If for all $\sigma \in \Delta \setminus \{0\}$ the vanishing condition $V(\sigma)$ is satisfied, then*

$$h_\Delta(t^2) = \sum_{q=0}^{2n} (\dim \overline{E}^q_\Delta)\, t^q\,.$$

If the fan Δ is rational, then we have an associated toric variety X_Δ. The equivariant intersection cohomology of open subsets of X_Δ defines a minimal extension sheaf on Δ by the assignment $\Lambda \mapsto IH^*_T(X_\Lambda;\mathbb{R})$ for $\Lambda \prec \Delta$, so in particular $E^\bullet_\Delta \simeq IH^*_T(X_\Delta;\mathbb{R})$. If Δ is complete then moreover, $\overline{E}^\bullet_\Delta$ is isomorphic to the intersection cohomology $IH^*(X_\Delta;\mathbb{R})$ of X_Δ (see [BBFK1]).

For an arbitrary fan Δ, one can therefore view $\overline{E}^\bullet_\Delta$ as the *virtual* intersection cohomology of Δ. If Δ is complete and the vanishing conditions $V(\sigma)$ are satisfied, then the entries of the generalized h–vector correspond to the even "Betti numbers" of the virtual intersection cohomology.

For $\mathcal{E}^\bullet$ and $\overline{\mathcal{E}}^\bullet$ we can define a Poincaré–series Q_Δ and a Poincaré–polynomial P_Δ respectively as follows:

$$Q_\Delta(t) := \sum_{q\geq 0} (\dim E^q_\Delta)\, t^q \quad \text{and} \quad P_\Delta(t) := \sum_{q\geq 0} (\dim \overline{E}^q_\Delta)\, t^q\,.$$

Note that here we do not follow the convention used in [BBFK2] in order to be consistent with the notation used in [AC]. One obtains the Poincaré–series used in [BBFK2] from ours by viewing it as a function in t^2.

If Δ is complete, then as a consequence of the first part of the above theorem we obtain:

(1) $$Q_\Delta(t) = \frac{1}{(1-t^2)^n} \cdot P_\Delta(t)\,.$$

The authors of [BBFK2] also show that for a complete fan, the minimal extension sheaf satisfies a “combinatorial Poincaré–duality”. So in particular, the Poincaré–polynomial P_Δ in that case is palindromic. Moreover, under an additional vanishing condition, for a polytopal fan they prove a “combinatorial hard Lefschetz” theorem. Assume that $\Delta = \Delta_K$ is the fan associated to a polytope, and let s_K denote the corresponding strictly concave support function on Δ. Let $\gamma^+(s_K)$ denote the convex hull of the graph in $V \times \mathbb{R}$ of s_K. Then the following holds:

2.2 Theorem. *(*Barthel, Brasselet, Fieseler, Kaup*) If the vanishing condition $V(\gamma^+(s_K))$ is satisfied then the map $\overline{\mu}^{2q}\colon \overline{E}_\Delta^{2q} \to \overline{E}_\Delta^{2q+2}$ induced by the multiplication with $s_K \in \mathcal{A}^2(\Delta)$ is injective for $2q \le n-1$ and surjective for $2q \ge n-1$.*

3 Refined Poincaré–Series

From now on let Δ denote a complete fan, and assume that for every $\sigma \in \Delta$ also $-\sigma \in \Delta$, in other words assume that Δ is *centrally symmetric*. Let $\varphi = -\operatorname{id}_V$ denote the central symmetry. Being an invertible linear transformation, φ induces an $\mathbb{R}$–linear automorphism of the graded algebra $A^\bullet = S^\bullet(V^*)$. Note that for every cone $\sigma \in \Delta$, we have $V_\sigma = V_{-\sigma}$. Since $A_\sigma^\bullet$ is the algebra of polynomial functions on V_σ restricted to σ, the algebras $A_\sigma^\bullet$ and $A_{-\sigma}^\bullet$ are not identical, but canonically isomorphic. The action of φ on V_σ induces an isomorphism of graded algebras from $A_\sigma^\bullet$ to $A_{-\sigma}^\bullet$ that is compatible with this canonical isomorphism. Moreover, the induced isomorphisms are compatible with the restriction homomorphisms $\rho_\tau^\sigma\colon A_\sigma^\bullet \to A_\tau^\bullet$ for every $\tau \prec \sigma$. That means that in fact φ defines a natural automorphism of Δ as a ringed space equipped with the sheaf of graded algebras $\mathcal{A}^\bullet$. We can also define an action of φ on the minimal extension sheaf $\mathcal{E}^\bullet$ on Δ.

3.1 Lemma. *There are isomorphisms of graded vector spaces*

$$\varphi\colon E_\sigma^\bullet \to E_{-\sigma}^\bullet$$

for every $\sigma \in \Delta$ that are equivariant with respect to the module structure over $A_\sigma^\bullet$ and $A_{-\sigma}^\bullet$ respectively and compatible with the restriction homomorphisms of the sheaf $\mathcal{E}^\bullet$.

Proof. To define the required isomorphisms, we proceed by recursion over the k–skeleton $\Delta^{\le k}$ of Δ following the recursive construction of $\mathcal{E}^\bullet$ as in Section 1 of [BBFK2]. On $E_0^\bullet = \mathbb{R}^\bullet$, where 0 denotes the zero cone, φ acts as the identity. Now assume that the isomorphisms have been defined for $\Delta^{<k}$, and let $\sigma \in \Delta^k$. We can assume that $E_\sigma^\bullet = A_\sigma^\bullet \otimes_{\mathbb{R}} \overline{E}_{\partial\sigma}^\bullet$. By induction, we already have an isomorphism $\varphi\colon E_{\partial\sigma}^\bullet \to E_{-\partial\sigma}^\bullet$, and since the maximal ideal $\mathfrak{m}$ of $\mathcal{A}^\bullet$ is φ–stable, φ induces an isomorphism $\overline{\varphi}\colon \overline{E}_{\partial\sigma}^\bullet \to \overline{E}_{-\partial\sigma}^\bullet$. Together with the map from $A_\sigma^\bullet$ to $A_{-\sigma}^\bullet$ determined by φ, that provides us with an isomorphism of graded vector spaces $\varphi\colon E_\sigma^\bullet \to E_{-\sigma}^\bullet$. By construction, this map is equivariant as a map from an $A_\sigma^\bullet$–module to an $A_{-\sigma}^\bullet$–module.

In the construction of $\mathcal{E}^\bullet$, the restriction homomorphism from $E^\bullet_\sigma = A^\bullet_\sigma \otimes_{\mathbb{R}} \overline{E}^\bullet_{\partial\sigma}$ to $E^\bullet_{\partial\sigma}$ is defined using the restriction homomorphism $\rho^\sigma_{\partial\sigma}\colon A^\bullet_\sigma \to A^\bullet_{\partial\sigma}$ and an $\mathbb{R}$–linear section $s_\sigma\colon \overline{E}^\bullet_{\partial\sigma} \to E^\bullet_{\partial\sigma}$ of the residue class map $E^\bullet_{\partial\sigma} \to \overline{E}^\bullet_{\partial\sigma}$. The section s_σ can be chosen freely. So we can assume that for any pair $\sigma, -\sigma$ of antipodal cones in Δ the corresponding sections have been chosen such that the following diagram is commutative:

$$\begin{array}{ccc} \overline{E}^\bullet_{\partial\sigma} & \xrightarrow{s_\sigma} & E^\bullet_{\partial\sigma} \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ \overline{E}^\bullet_{-\partial\sigma} & \xrightarrow{s_{-\sigma}} & E^\bullet_{-\partial\sigma} \end{array} .$$

That implies compatibility of φ with the restriction homomorphisms of $\mathcal{E}^\bullet$. □

In particular, we obtain an induced automorphism φ on the module $E^\bullet_\Delta$ of global sections of $\mathcal{E}^\bullet$, and though the automorphism is not canonical, the dimensions of the eigenspaces in each graded piece are uniquely determined and therefore the so–called *refined Poincaré–series* for the action of φ on $\mathcal{E}^\bullet$ depends only on Δ. This series is defined as a polynomial over the group ring $\mathbb{Z}[G]$ of the character group $G := \{\pm 1\}$ of the group generated by φ in $\mathrm{GL}(V)$, namely:

$$Q^\varphi_\Delta(t) := \sum_{q\geq 0} (\dim(E^q_\Delta)^+ + \dim(E^q_\Delta)^-\chi)\ t^q ,$$

where χ denotes the element corresponding to -1 in $\mathbb{Z}[G]$, and the superscripts $+$, $-$ indicate the eigenspaces for the eigenvalues $+1$ and -1 respectively. The refined Poincaré–polynomial P^φ_Δ for the action of φ on $\overline{\mathcal{E}}^\bullet$ is defined analogously.

We have assumed that Δ is complete, and therefore $E^\bullet_\Delta$ is a free $A^\bullet$–module (see Theorem 2.1). The action of φ on $\overline{E}^\bullet_\Delta$ is induced by taking residue classes. So if we choose a homogeneous basis for $\overline{E}^\bullet_\Delta$ and preimages under the residue class map in $E^\bullet_\Delta$ to define an isomorphism

$$E^\bullet_\Delta \to \overline{E}^\bullet_\Delta \otimes_{\mathbb{R}} A^\bullet ,$$

then this isomorphism is automatically compatible with the action of φ. That implies

$$Q^\varphi_\Delta(t) = \frac{1}{(1-\chi t^2)^n} \cdot P^\varphi_\Delta(t) . \tag{2}$$

To obtain a relation between the Poincaré–series Q_Δ and its refined version Q^φ_Δ, we can use the fact that the minimal extension sheaf $\mathcal{E}^\bullet$ as a sheaf of real vector spaces can be written as a direct sum of simpler subsheafs. Note that $\mathcal{E}^\bullet$ is a *flabby* sheaf on Δ. Here that means that the restriction homomorphism $\rho^\sigma_{\partial\sigma}\colon E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$ is surjective for all $\sigma \in \Delta$.

For $\sigma \in \Delta$, let $\mathcal{J}_\sigma$ denote the *characteristic sheaf* of σ defined on $\Lambda \prec \Delta$ by

$$\mathcal{J}_\sigma(\Lambda) := \begin{cases} \mathbb{R} & \text{if } \sigma \in \Lambda \\ 0 & \text{otherwise} \end{cases} .$$

Then there is an isomorphism of sheafs of graded real vector spaces

$$\mathcal{E}^\bullet \simeq \bigoplus_{\sigma\in\Delta} \mathcal{J}_\sigma \otimes_{\mathbb{R}} K_\sigma \,, \tag{3}$$

where K_σ denotes the kernel of the restriction homomorphism $\rho^\sigma_{\partial\sigma}\colon E^\bullet_\sigma \to E^\bullet_{\partial\sigma}$ (see Section 3, [BBFK2]).

For every $\sigma \in \Delta$, φ induces a map from $\mathcal{J}_\sigma$ to $\mathcal{J}_{-\sigma}$ that is compatible with the action of φ on Δ. And using these maps together with the action of φ on $\mathcal{E}^\bullet$, we obtain an induced φ–action on the direct sum on the righthandside of (3), where φ maps $\mathcal{J}_\sigma$ to $\mathcal{J}_{-\sigma}$ and K_σ to $K_{-\sigma}$. Modifying the proof of the decomposition theorem from [BBFK2], we can show the following:

3.2 Lemma. *The isomorphism of sheafs of graded real vector spaces on Δ in (3) can be chosen as φ–equivariant.*

Proof. We prove our claim by induction on the number of antipodal pairs of cones in Δ. Suppose that there is a φ–equivariant decomposition of $\mathcal{E}^\bullet$ into φ–stable flabby sheafs

$$\mathcal{E}^\bullet \simeq \mathcal{F} \oplus \left(\bigoplus_{\sigma\in\Lambda} \mathcal{J}_\sigma \otimes_{\mathbb{R}} K_\sigma \right) \,,$$

where the sum is taken over a φ–stable subset Λ of Δ (that is not necessarily a subfan), such that $\mathcal{F}(\sigma) = 0$ for all $\sigma \in \Lambda$.

Then choose a pair of antipodal cones $\sigma, -\sigma \in \Delta \setminus \Lambda$ of minimal dimension k with $\mathcal{F}(\sigma) \neq 0 \neq \mathcal{F}(-\sigma)$. We have to show that we can write $\mathcal{F}$ as a direct sum of φ–stable flabby subsheafs $\mathcal{F} = \mathcal{G} \oplus \mathcal{H}$ such that $\mathcal{H}(\sigma) = 0$ and if $\sigma = 0$ then $\mathcal{G} \simeq \mathcal{J}_0 \otimes_{\mathbb{R}} K_0$ and if $\sigma \neq 0$ then $\mathcal{G} \simeq (\mathcal{J}_\sigma \otimes_{\mathbb{R}} K_\sigma) \oplus (\mathcal{J}_{-\sigma} \otimes_{\mathbb{R}} K_{-\sigma})$. We define $\mathcal{G}$ and $\mathcal{H}$ on the k–skeleton $\Delta^{\le k}$ by

$$\mathcal{G}(\tau) := \begin{cases} K_\tau & \text{if } \tau = \pm\sigma \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mathcal{H}(\tau) := \begin{cases} 0 & \text{if } \tau = \pm\sigma \\ \mathcal{F}(\tau) & \text{otherwise} \end{cases} .$$

Now suppose, that $\mathcal{F} = \mathcal{G} \oplus \mathcal{H}$ is already defined on $\Delta^{\le m}$, and consider a pair of antipodal cones $\pm\tau$ of dimension $m+1$. If σ is neither a face of τ nor of $-\tau$, then set $\mathcal{G}(\pm\tau) = 0$ and $\mathcal{H}(\pm\tau) := \mathcal{F}(\pm\tau)$. Otherwise say $\sigma \prec \tau$ and $-\sigma \prec -\tau$. Note that τ cannot contain both σ and $-\sigma$ as a face.

By assumption, we have a decomposition $\mathcal{F}(\partial\tau) = \mathcal{G}(\partial\tau) \oplus \mathcal{H}(\partial\tau)$, and $\mathcal{G}(-\partial\tau) = \varphi(\mathcal{G}(\partial\tau))$ and $\mathcal{H}(-\partial\tau) = \varphi(\mathcal{H}(\partial\tau))$. Since $\mathcal{F}$ is flabby, we can choose a decomposition $\mathcal{F}(\tau) = U \oplus W$ such that the restriction homomorphism $\rho^\tau_{\partial\tau}\colon \mathcal{F}(\tau) \to \mathcal{F}(\partial\tau)$ induces an isomorphism $U \to \mathcal{G}(\partial\tau)$ and a surjective homomorphism from W to $\mathcal{H}(\partial\tau)$. Since the action of φ is compatible with the restriction homorphisms, for the decomposition $\mathcal{F}(-\tau) = \varphi(U) \oplus \varphi(W)$ the following holds: The restriction homomorphism

$\rho^{-\tau}_{-\partial\tau}\colon \mathcal{F}(-\tau) \to \mathcal{F}(-\partial\tau)$ induces an isomorphism $\varphi(U) \to \mathcal{G}(-\partial\tau)$ and a surjective homomorphism from $\varphi(W)$ to $\mathcal{H}(-\partial\tau)$. Now set $\mathcal{G}(\tau) = U$ and $\mathcal{G}(-\tau) = \varphi(U)$, $\mathcal{H}(\tau) = W$ and $\mathcal{H}(-\tau) = \varphi(W)$. Then $\mathcal{G}$ and $\mathcal{H}$ have the required properties. □

Now consider the action of φ on the direct sum on the righthandside of (3). Apparently, for every cone $\sigma \neq 0$, the action of φ interchanges the summands $\mathcal{J}_\sigma \otimes_{\mathbb{R}} K_\sigma$ and $\mathcal{J}_{-\sigma} \otimes_{\mathbb{R}} K_{-\sigma}$. So for every q, in $\bigoplus_{\sigma\neq 0} \mathcal{J}_\sigma \otimes_{\mathbb{R}} K_\sigma$ the eigenvalue $+1$ and -1 occur with the same multiplicity. We obtain the relation

$$Q^{\varphi}_{\Delta}(t) - 1 = \frac{1}{2}((Q_\Delta(t) - 1) + (Q_\Delta(t) - 1)\cdot\chi) = \frac{1}{2}(1+\chi)(Q_\Delta(t) - 1)\,. \tag{4}$$

4 Lower Bounds for the Generalized h–Vector

Summarizing the considerations in the previous section, we obtain the following description of the refined Poincaré–polynomial:

4.1 Proposition. *Let Δ be a centrally symmetric complete fan. Then*

$$P^{\varphi}_{\Delta}(t) = \frac{1}{2}(P_\Delta(t) + (1+t^2)^n) + \frac{1}{2}\chi(P_\Delta(t) - (1+t^2)^n)\,.$$

Proof. Inserting (1) in (4), we obtain

$$Q^{\varphi}_{\Delta}(t) = \frac{1}{2}(1+\chi)Q_\Delta(t) + \frac{1-\chi}{2}\,.$$

Using (2), that implies

$$P^{\varphi}_{\Delta}(t) = \frac{1}{2}\frac{(1+\chi)(1-\chi t^2)^n}{(1-t^2)^n}P_\Delta(t) + \frac{1-\chi}{2}(1-\chi t^2)^n\,.$$

Note that since $\chi^2 = 1$, we have $(1-\chi t^2)(1+\chi) = (1-t^2)(1+\chi)$ and $(1-\chi t^2)(1-\chi) = (1+t^2)(1-\chi)$. This implies

$$P^{\varphi}_{\Delta}(t) = \frac{1-\chi}{2}(1+t^2)^n + \frac{1+\chi}{2}P_\Delta(t)\,. \quad \square$$

We now apply this proposition to polytopal centrally symmetric fans. Let K be a centrally–symmetric polytope, and let $\Delta := \Delta_K$ denote the fan through the faces of K. Let s_K denote the Δ–strictly convex support function defined by K and set $h_K = h_\Delta$. A polynomial is called *symmetric* if the coefficients are palindromic, and it is called *unimodal* if its coefficients increase up to a certain index and then decrease again.

4.2 Theorem. *Assume that $V(\sigma)$ holds for all $\sigma \in \Delta\setminus\{0\}$ and moreover that $V(\gamma^+(s_K))$ holds. Then the polynomial $P_\Delta(t) - (1+t^2)^n$, or equivalently the polynomial $h_K(x) - (1+x)^n$, is symmetric and unimodal, and all its coefficients are even. In particular, we obtain the following lower bounds for the coefficients h_j of h_K:*

$$h_j \geq \binom{n}{j} \quad \text{for } j = 0, \dots, n.$$

Proof. The assumptions imply that $P_\Delta(t) = h_K(t^2)$. The symmetry follows immediately from the combinatorial Poincaré–duality. Moreover, by the above proposition, we have

$$\frac{1}{2}(P_\Delta(t) - (1+t^2)^n) = \sum_{q\geq 0} (\dim(\overline{E}^q_\Delta)^-)\, t^q\,.$$

This implies in particular, that all the coefficients of $p(t) := P_\Delta(t) - (1+t^2)^n$ are even.

Since the support function $s_k \in \mathcal{A}^2(\Delta)$ is invariant under φ, we have

$$s_k \cdot (\overline{E}^q_\Delta)^- \subset \overline{E}^{q+2}_\Delta)^-\,.$$

Now it follows from the combinatorial hard Lefschetz theorem (see Theorem 2.2) that $\dim(\overline{E}^{2q}_\Delta)^- \leq \dim(\overline{E}^{2q+2}_\Delta)^-$ for $2q \leq n-1$, and that means that the polynomial p is unimodal. Obviously, the coefficients p_0 and p_{2n} are zero. So the unimodality implies that all other coefficients are nonnegative, and we obtain the lower bounds for the coefficients of h_K as asserted. □

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