Erschienen in: Journal of Philosophical Logic ; 49 (2020), 2. - S. 283-313
https://dx.doi.org/10.1007/s10992-019-09518-8

An Accuracy Argument in Favor of Ranking Theory

Eric Raidl[1] · Wolfgang Spohn[2]

Abstract
Fitelson and McCarthy (2014) have proposed an accuracy measure for confidence orders which favors probability measures and Dempster-Shafer belief functions as accounts of degrees of belief and excludes ranking functions. Their accuracy measure only penalizes mistakes in confidence comparisons. We propose an alternative accuracy measure that also rewards correct confidence comparisons. Thus we conform to both of William James' maxims: "Believe truth! Shun error!" We combine the two maxims, penalties and rewards, into one criterion that we call prioritized accuracy optimization (PAO). That is, PAO punishes wrong comparisons (preferring the false to the true) and rewards right comparisons (preferring the true to the false). And it requires to prioritize being right und avoiding to be wrong in a specific way. Thus PAO is both, a scoring rule and a decision rule. It turns out that precisely confidence orders representable by two-sided ranking functions satisfy PAO. The point is not to argue that PAO is the better accuracy goal. The point is only that ranking theory can also be supported by accuracy considerations. Thus, those considerations by themselves cannot decide about rational formats for degrees of belief, but are part and parcel of an overall normative assessment of those formats.

Keywords Comparative belief · Accuracy · Accuracy-first epistemology · Degrees of belief · Probability theory · Ranking theory · Representation theorems

---

Eric Raidl's work was supported by the Deutsche Forschungsgemeinschaft (DFG) Research Unit FOR 1614.

✉ Wolfgang Spohn
wolfgang.spohn@uni-konstanz.de

Eric Raidl
eric.raidl@uni-tuebingen.de

1 Cluster of Excellence "Machine Learning: New Perspectives for Science", Eberhard Karls University Tübingen, Tübingen, Germany

2 Department of Philosophy, University Konstanz, 78457 Konstanz, Germany

1 Introduction

Joyce [7, 8] has proposed a novel justification of probability theory in terms of accuracy dominance: A possible epistemic state, i.e., an assignment of degrees of belief, is an assignment f of numbers between 0 and 1 to propositions as a possible epistemic state. For each possible world w, truth in w (truth = 1, falsity = 0) is also such an assignment i_w, the perfectly accurate one. The given assignment f deviates from each i_w to some extent $\delta(f, i_w)$. Now, one may argue that this inaccuracy measure δ has to satisfy some plausible conditions; the quality of that argument is crucial, of course. Joyce then shows that if f violates the axioms of probability theory, then there is a probability measure p that weakly dominates f in accuracy, i.e., for which $\delta(f, i_w) \geq \delta(p, i_w)$ for all w and $\delta(f, i_w) > \delta(p, i_w)$ for some w. This is an exciting kind of argument. This argument and the broader philosophical position defended by that argument is now referred to as accuracy-first epistemology. It may be summarized as claiming that we must accept probability theory for purely epistemic reasons, i.e., only in order to approximate the truth as well as possible. There are variations and extensions of that argument; the discussion is vigorous and ramified. This theoretical field is most comprehensively presented by Pettigrew [10]. It should be noted, though, that it is always about justifying probability.

Easwaran and Fitelson [2] have developed a more general kind of accuracy dominance argument applicable to belief sets. Fitelson and McCarthy [4] did the same regarding comparative judgments. For them, the basic notion is that of a confidence order $\preceq$: $B \preceq A$ iff A is at least as plausible or credible as B. They go on defining an inaccuracy measure for such orders. They argue that this measure must have certain properties. And then they can run the same kind of argument. That is, such an order is required not to be weakly dominated in accuracy by some other order, i.e., to satisfy weak accuracy dominance avoidance (WADA). And so they study which properties those orders must have in order to satisfy WADA and which properties guarantee WADA to be satisfied. One significant result is, e.g., that WADA is satisfied by qualitative probability. However, they widen the perspective and discuss also other epistemic formats.

Spohn [14] has proposed another representation of degrees of belief, which he claims also to represent belief (in contrast to probability theory, which is prima facie plagued by the lottery paradox). This is called ranking theory nowadays and widely developed and applied in Spohn [16]. Spohn [16, ch. 5 and 8] presents his own justifications. But it is an obvious question how ranking theory fares under this type of justificatory argument. This is the topic of this paper.

At first, it seems that Joyce's kind of argument is inapplicable to ranking theory, simply because ranks measure degrees of belief not by real numbers between 0 and 1, but by (non-negative) integers. However, the generalized argument is applicable. Ranking functions also induce confidence orders, and then one may check how they relate to WADA. Indeed, Fitelson [3] has done so and shown that rank-induced confidence orders violate WADA. For Fitelson this throws doubt, to say the least, on the normative foundations of ranking theory.

One may evade this criticism by pointing to all the other normative arguments in favor of ranking theory. And then one may dispute which kind of justification weighs more. This is not what we want to do here; that would just mean repeating the old justifications. Rather we want to take the bull by the horns. Again, the crucial issue is which kind of inaccuracy measure we should accept. And we would like to propose a novel kind of inaccuracy measure that confirms rank-induced confidence orders and disconfirms, e.g., qualitative probability.

Moreover, we want to argue that this novel inaccuracy measure is at least plausible. We don't claim that it is superior to Fitelson's and McCarthy's measure; this would be presumptuous. But we do claim that the discussion is symmetric. Each plausible epistemic format can be confirmed by its own plausible accuracy measure. Accuracy dominance arguments are not decisive. Thus, other normative arguments need not fear to be dominated by those accuracy arguments. That's the moral we would like to draw in this paper.

Although our paper provides an accuracy argument for ranking theory and could thus be seen as extending the accuracy-first epistemology from probability theory to ranking theory, we conceive of this rather as a criticism of the accuracy-first strategy: Each epistemic format will require its particular notion of accuracy, so that in the end, accuracy-first arguments are part of a holistic picture and non-detachable from the epistemic format they are supposed to ground.

For this purpose, Section 2 introduces Fitelson's and McCarthy's accuracy dominance argument concerning confidence orders. Section 3 briefly introduces ranking theory, shows which kind of confidence orders it entails, and explains Fitelson's criticism, the bad behavior of rank-induced confidence orders according to his criteria. Thus the problem we want to deal with is exposed in detail. Section 4 then develops a different kind of accuracy measure. Of course, it is tailored to ranking theory, but we will argue that it has an independent kind of plausibility. Section 5 proves our main result: namely that weak accuracy dominance avoidance with respect to this measure entails and is in fact equivalent to the properties of rank-induced confidence orders. Our case is summarized in the final Section 6. Longer proofs can be found in the Appendix A.

2 Confidence Orders and Their Accuracy

Fitelson and McCarthy [4] take comparative orders to be the basic epistemic level, with good reason. Beliefs seem to be an all-or-nothing affair, providing too coarse-grained a description, and quantitative degrees of belief seem to derive from epistemic orders by some measurement procedure. So, let's accept their starting point. Let W be a non-empty finite set of possibilities or possible worlds. The subsequent considerations don't work for infinitely many possibilities. Throughout the paper let $\mathscr{A}$ be the power set of W, that is, the set of all subsets of W. We interpret $\mathscr{A}$ as the set of all propositions over W and consider these to be the objects of epistemic attitudes. We could take sentences instead, which have those propositions as truth conditions. However, speaking directly about propositions simplifies our business. Then we say:

Definition 1 $\preceq$ is a *confidence order* over $\mathscr{A}$ iff

(1) $\preceq$ is a weak order (transitive and total relation) over $\mathscr{A}$,[1]
(2) $\emptyset \prec W$ (non-triviality), and
(3) if $A \subseteq B$, then $A \preceq B$ (monotonicity).

Moreover, $\preceq$ is *regular* iff

(4) $\emptyset \prec A \prec W$ for all $A \neq \emptyset, W$.

Transitivity of confidence orders has never been under dispute. Totality, however, is a strong requirement, which we do not want to defend, but assume here for the sake of theorizing. One could try to do without it, as Halpern [5, sect. 2.8] does with his plausibility measures. (2) seems indispensable. The tautology W can't be just as credible as the contradiction $\emptyset$. And (3) seems indispensable, too. If A entails B, B cannot be less credible than A. These (together with transitivity) are also the two basic properties of Halpern's plausibility measures. Let's not further argue about them. Regularity (4) is often a most reasonable additional requirement. However, Definition 1 leaves open a lot of properties of confidence orders that might be desirable.

In order to get a normative guideline to further properties, Fitelson and McCarthy observe that a confidence order can be more or less accurate in a world. Perhaps it is perfectly accurate in w if it orders all propositions true in w above all propositions false in w. Perhaps it should also order all truths equally, or even equal to the tautology W, and correspondingly for the falsehoods; but we may leave this open. Clearly, if the order is perfectly accurate in one world, it can't be so in another world. And usually it will be less than perfectly accurate everywhere.

Let us study the inaccuracies that might occur. Fitelson and McCarthy suggest that it may not suffice to count the number of inaccuracies a confidence order makes in a world; it may be necessary to weigh them. Let us first distinguish the strict order $\prec$ and the indifference $\sim$, because they may be guilty of different kinds of inaccuracy. And let $\iota(B \prec A, w)$ and $\iota(B \sim A, w)$ respectively measure the size of the inaccuracy made by the judgments $B \prec A$ and $B \sim A$ in the world w.

First, it seems that it is not a mistake in w at all to judge $B \prec A$ if A is true and B is false in w. Second, it doesn't seem to be a mistake in w to judge $B \sim A$ if A and B have the same truth value in w. Thus, we have $\iota(B \prec A, w) = 0$ if $w \in A \cap \overline{B}$ and $\iota(B \sim A, w) = 0$ if $w \in A \cap B$ or $w \in \overline{A} \cap \overline{B}$.

However, it does seem to be at least a little mistake in w to judge $B \sim A$ if A and B differ in truth value in w. So let's normalize and assume $\iota(B \sim A, w) = 1$ if $w \in A \cap \overline{B}$ or $w \in \overline{A} \cap B$. Now, is it a mistake in w to judge $B \prec A$ if A and B have the same truth value in w? Perhaps A and B should not be distinguished then? Let us assume that $\iota(B \prec A, w) = x$ if $w \in A \cap B$ or $w \in \overline{A} \cap \overline{B}$ and leave open the size of x for a moment. Finally, it is definitely a mistake in w, indeed the worst of all, to judge $B \prec A$ while A is false and B is true in w. So, say that $\iota(B \prec A, w) = y$

[1] $\prec$ denotes the corresponding strict order, defined by $B \prec A$ iff $B \preceq A$ and not $A \preceq B$. And $\sim$ denotes the corresponding indifference relation, defined by $A \sim B$ iff $A \preceq B$ and $B \preceq A$.

if $w \in \overline{A} \cap B$ and leave open the size of y for the moment. Thereby we have listed all possible cases. Later on we shall doubt, however, that one must see things in this way; it is only one of various plausible ways.

Obviously, the further course of argument crucially depends on x and y. How should we go about determining them? Here, Fitelson and McCarthy advance a clever argument. They require the inaccuracy measure ι of a confidence order $\preceq$ to be *evidentially proper*, as they call it. This is to say that there is some probability measure P on $\mathscr{A}$ with the following properties: if $P(B) < P(A)$, then the expected inaccuracy of judging $B \prec A$ is not larger than that of any alternative judgment ($B \sim A$ or $A \prec B$), and likewise, if $P(B) = P(A)$, then the expected inaccuracy of judging $B \sim A$ is not larger than that of any alternative judgment ($B \prec A$ or $A \prec B$). Here, the expectation of ι is taken with respect to P itself. They prove then that we must have $x = 0$ and $y = 2$ for any evidentially proper inaccuracy measure ι [4, slide 25].

This solves the problem above. Fitelson and McCarthy then proceed with the inaccuracy measure ι thus determined. Again, the result is at least plausible. $x = 0$ means that $B \prec A$ is not a mistake at all if A and B have the same truth value; accuracy does not require to take all truths and, respectively, all falsehoods as equally credible. Furthermore, it seems obvious that mistakes with $\prec$ have more weight than mistakes with $\sim$, although the exact size of $y = 2$ is a bit surprising. It is not clear to us how much their further course depends on the specific choice of $y = 2$. In any case, one must grant that by accepting this argument, i.e., by requiring the evidential propriety of the inaccuracy measure ι, this measure is already geared to probability theory. Thereby the further course gets a probabilistic bias, although, as said, we do not know how strong it really is.

What is Fitelson's and McCarthy's further argument? First, we may summarize the above by defining

$$\iota(B \preceq A, w) := \begin{cases} \iota(B \prec A, w) & \text{if } B \prec A, \text{ (i.e., } B \preceq A \text{ and not } A \preceq B), \\ \iota(B \sim A, w) & \text{if } B \sim A, \text{ (i.e., } B \preceq A \text{ and } A \preceq B). \end{cases}$$

They then define the overall inaccuracy of a confidence order in a world w by summing up all the inaccuracies made in comparing any two propositions, i.e., $\iota(\preceq, w) := \sum \iota(B \preceq A, w)$, where the sum is taken over all pairs $A, B \in \mathscr{A}$ of propositions.

This is not inevitable. For instance, one might think that mistakes involving the propositions A and B somehow entail mistakes involving the propositions $\overline{A}$, $\overline{B}$, $A \cap B$, or $A \cup B$. And then the latter mistakes might not count as much as the original mistakes with A and B. So, it may be doubtful that all propositions get the same weight. Be this as it may, their definition of $\iota(\preceq, w)$ is perhaps the most natural one, and it is only deviations from it that would require special argument.[2]

[2]In agreement with their general 'comparative first' philosophy, Fitelson and McCarthy have announced an axiomatic defense of their criterion of evidential propriety and thus of conceiving overall inaccuracy simply as the sum of all local inaccuracies. However, our reservation about the total inaccuracy still stands in some form, since the mentioned result makes an extensionality assumption. These remarks are based on a personal communication with David McCarthy.

Now, as mentioned above, a perfectly accurate confidence order would perfectly capture the truth. And the more inaccurate it is, the farther it is from the truth. For this reason, accuracy arguments offer the exciting perspective of a purely epistemic justification of certain epistemic principles. Therefore, we should definitely try to minimize inaccuracy. The problem is: we can't do so. A confidence order that is perfectly accurate in one world makes a lot of mistakes in another world. Still, inaccuracy minimization has some bite. Fitelson and McCarthy say that our confidence order should be such that it cannot be dominated in accuracy, i.e., that there is no other confidence order that is at least as accurate in all worlds and even more accurate in some worlds. Taking ι as the natural and so far well-argued inaccuracy measure, we obtain

Definition 2 The confidence order $\preceq$ *weakly accuracy-dominates* the confidence order $\preceq'$ iff $\iota(\preceq, w) \leq \iota(\preceq', w)$ for all worlds w and $\iota(\preceq, w) < \iota(\preceq', w)$ for some world w.

They then suggest the following, most convincing rationality requirement for confidence orders:

> **Weak Accuracy-Dominance Avoidance (WADA):** Rationally, a confidence order over $\mathscr{A}$ should not be weakly accuracy-dominated by any other confidence order over $\mathscr{A}$.

They do not specify properties of confidence orders that are equivalent to satisfying WADA. However, they nest WADA as closely as they can. And this is already illuminating. On the one hand, they give a necessary condition. That is, WADA entails:

(5) if $B \subseteq A$ and $A \cap C = \emptyset$, and $B \prec A$, then $B \cup C \prec A \cup C$.

(5) is interesting in the following way. A numerical function f from $\mathscr{A}$ into a set of numbers (for example [0, 1]) *represents* a given confidence order $\preceq$ over $\mathscr{A}$ iff for all $A, B \in \mathscr{A}$, $B \preceq A$ iff $f(B) \leq f(A)$. Let's call a confidence order $\preceq$ *representable* by a type of numerical function (given as a class of such functions or by axioms for these functions) iff there is a numerical function f of that type that represents $\preceq$.[3] Then Fitelson and McCarthy refer to a theorem stating that a confidence order satisfies (1)–(3) and (5) if and only if it is representable by a Dempster-Shafer belief function.[4] Thus, if we accept WADA, we should conform at least to that epistemic format.

On the other hand, Fitelson and McCarthy specify a sufficient condition for WADA. That is, they prove that if a confidence order is representable by a regular probability measure on $\mathscr{A}$, then it must satisfy WADA. We don't know, though, how much this result depends on the specific definition of WADA, i.e., on the particular weights given to the various inaccuracies embodied in a confidence order. However,

---

[3] Fitelson and McCarthy [4] speak of full representability here, because they also define partial representability, which is not relevant for us.

[4] For a definition see Shafer [13, ch. 2], for a proof see Wong et al. [17, thm. 4].

Fitelson's and McCarthy's use of evidential propriety does seem to involve some kind of 'prior friendliness' to probability theory.

Does this sufficient condition translate into specific properties of confidence orders? Yes. It entails that a confidence order must at least be a *qualitative probability* in the sense of satisfying (1)–(3) and:

(6) if $A \cap C = \emptyset$ and $B \cap C = \emptyset$, then $B \preceq A$ iff $B \cup C \preceq A \cup C$.

(6) obviously strengthens (5) in several ways. However, as is well known since Kraft et al. [9], it is not sufficient for representability by a probability measure. This is only guaranteed by an additional more complicated property, which we need not quote here [cf. 12]. According to Fitelson and McCarthy, (1)–(3) and (6) do not entail WADA, nor is (6) entailed by WADA. However, (6), the characteristic axiom of qualitative probability, is a crucial part of guaranteeing WADA.

This may suffice as a background for Fitelson's [3] criticism of ranking theory. So, let's turn to ranking theory and see how it fares with respect to WADA. We will see that Fitelson is right: it fares badly.

3 Rank-Induced Confidence Orders

The basic notion of ranking theory is this:

Definition 3 κ is a *negative ranking function* for $\mathscr{A}$ iff κ is a function from $\mathscr{A}$ into $\mathbb{N} \cup \{\infty\}$ (the set of natural numbers augmented by infinity) such that:

(7) $\kappa(W) = 0$ and $\kappa(\emptyset) = \infty$,
(8) $\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}$ for all $A, B \in \mathscr{A}$.

Moreover, we call κ *regular* iff

(9) $\kappa(A) < \infty$ for all $A \neq \emptyset$.

The basic point of ranking theory is that it accounts for both, belief and degrees of belief. Whereas probability theory has great problems in accommodating belief—see the extensive discussion about the lottery-paradox—negative ranking functions express it directly in terms of *disbelief* (therefore the qualification 'negative'). If $\kappa(A) = 0$, then A is not disbelieved at all. If $\kappa(A) > 0$, then A is disbelieved or taken to be false. Positive belief in A, taking A to be true, is expressed by disbelief in $\overline{A}$, i.e., by $\kappa(\overline{A}) > 0$. This leaves room for suspense of judgment about A, which is expressed by $\kappa(A) = \kappa(\overline{A}) = 0$. Still, (dis-)belief is not an all-or-nothing affair; it may be more or less firm. Thus disbelief in A is ever stronger, the larger $\kappa(A)$. Regularity then means that only the contradiction is disbelieved to the maximal degree.

This interpretation explains axioms (7) and (8). (7) says that the tautology is not disbelieved, and hence the contradiction is not believed. This entails that beliefs are *consistent* according to a negative ranking function. (7) moreover says that the contradiction is indeed maximally disbelieved. (8) is the characteristic axiom of ranking theory. It says that you cannot disbelieve a disjunction less strongly than any of its disjuncts. Why? In a moment we will see a fuller justification. Right now we may observe that it obviously entails that if you believe two conjuncts (by disbelieving their negations), you also believe their conjunction (by disbelieving the disjunction of these negations). Hence, beliefs are *deductively closed* according to a ranking function.

Ranking theory thus embodies the two basic rationality postulates from doxastic logic. There is no point in discussing these postulates now. It is clear, however, that Fitelson's and McCarthy's work is strongly motivated by a rejection of those postulates, on the basis of such obstinate paradoxes like the lottery and the preface paradox. This is our basic disagreement, which we won't try to dissolve, but which will entail all further differences. In a way, this paper is only to argue that this disagreement can be resolved neither by resorting to confidence orders and their properties nor in particular by alluding to accuracy dominance arguments.

We can also introduce *conditional negative ranks* by defining:

(10) $\kappa(B|A) = \kappa(A \cap B) - \kappa(A)$, provided $\kappa(A) < \infty$.

When $\kappa(A) < \infty$, $\kappa(.|A)$ is again a ranking function, the *conditionalization* of κ by A. Then the characteristic axiom (8) turns out to be equivalent to the assertion that even our conditional beliefs have to be consistent under all entertainable conditions A (i.e., with $\kappa(A) < \infty$). This is a most reasonable requirement and hence the most direct justification of (8).

(7), (8), and (10) display an obvious close analogy to probability theory, and this is the basic reason in presenting ranking theory in negative terms. Conditional ranks are absolutely essential for the power and beauty of ranking theory. In particular, they allow stating completely general rules of epistemic change, i.e., of changing beliefs and disbeliefs [cf. 16, sect. 5.4]. Similar claims hold just as well for probability theory. Hence, for a full assessment of ranking theory these extensions must be carefully taken into account. However, in this paper conditional (degrees of) belief will play no further role, and hence we will not deepen the issue here.

Of course, we could directly introduce the positive counterpart.

Definition 4 β is a *positive ranking function* for $\mathscr{A}$ iff there is a negative ranking function κ such that $\beta(A) = \kappa(\overline{A})$ for all $A \in \mathscr{A}$, or, equivalently, iff β is a function from $\mathscr{A}$ into $\mathbb{N} \cup \{\infty\}$ such that:

(11) $\beta(W) = \infty$ and $\beta(\emptyset) = 0$,
(12) $\beta(A \cap B) = \min\{\beta(A), \beta(B)\}$, for all $A, B \in \mathscr{A}$.

Correspondingly, β represents *degrees of belief*. A proposition A is believed or taken to be true according to β iff $\beta(A) > 0$, and the more strongly, the larger $\beta(A)$. $\beta(A) = 0$ says that A is not believed. Finally, we may represent degrees of belief and degrees of disbelief also in a single two-sided function:

Definition 5 τ is a *two-sided ranking function* for $\mathscr{A}$ iff there is a negative ranking function κ and the corresponding positive ranking function β for $\mathscr{A}$ such that $\tau(A) = \beta(A) - \kappa(A) = \kappa(\overline{A}) - \kappa(A)$ for all $A \in \mathscr{A}$.

Note that we always have $\tau(\overline{A}) = -\tau(A)$. Additionally, we have $\tau(A) > 0$, < 0, or $= 0$ according to whether A is believed, disbelieved, or neither in τ. For this reason this is perhaps the most intuitive notion. However, it is formally more awkward since it behaves differently on the positive and on the negative side. The mathematics is best done in terms of negative ranking functions. It is clear, though, that the three functions are equivalent and interdefinable. Let us study now how those functions can be represented by confidence orders. This is our topic here. This representation is indeed more or less straightforward.

Theorem 1 *Let $\mathscr{A} = \wp(W)$ be finite and $\preceq$ a binary relation over $\mathscr{A}$. Then $\preceq$ is κ-representable iff $\preceq$ is a disconfidence order over $\mathscr{A}$ (weak order, inversely non-trivial and inversely monotone) satisfying*

(13) *for all $A, B, C \in \mathscr{A}$, $A \preceq B$ implies $A \cup C \preceq B \cup C$.*

Proof $\preceq$ is κ representable iff the inverse $\preceq^-$ (defined by $A \preceq^- B$ iff $B \preceq A$) is representable by a plausibility measure ρ.[5] The latter holds iff $\preceq^-$ is a confidence order over $\mathscr{A}$ which satisfies (13) [1]. This holds iff $\preceq$ is a disconfidence order over $\mathscr{A}$ which satisfies (13). Because the corresponding axioms are respectively inverses of each other. In particular, axiom (13) is its own inverse. An independent constructive proof is given in Raidl [11]. □

Theorem 2 *Let $\mathscr{A} = \wp(W)$ be finite and $\preceq$ a binary relation over $\mathscr{A}$. Then $\preceq$ is β-representable iff $\preceq$ is a confidence order satisfying:*

(14) *for all $A, B, C \in \mathscr{A}$, $A \preceq B$ implies $A \cap C \preceq B \cap C$.*

Proof $\preceq$ is β-representable iff the complement order $\preceq^c$ (defined by $A \preceq^c B$ iff $\overline{A} \preceq \overline{B}$) is κ-representable. But the above axioms are the complements to the axioms characteristic for a negative ranking function. In particular (14) and (13) are complementary axioms. □

Note that we could have replaced (13) and (14) respectively by

(15) $\min_{\preceq}\{A, B\} \preceq A \cup B$ (weak $\cup$-minimativity)[6]

(16) $\min_{\preceq}\{A, B\} \preceq A \cap B$. (weak $\cap$-minimativity)[7]

---

[5]From κ to ρ take $\rho(A) = a^{\kappa(A)}$ for $a \in (0, 1)$. Conversely, consider $K(A) = \log_a \rho(A)$ which is a real valued negative ranking function from which an order-isomorphic natural valued negative ranking function κ can be obtained.

[6]Together with inverse monotonicity this is equivalent to the postulate that the inverse order $\preceq^-$ satisfies the following qualitative version of the max-T-conorm: $\max_{\preceq^-}\{A, B\} \sim^- A \cup B$.

[7]Together with monotonicity this is equivalent to the following qualitative version of the min-T-norm: $\min_{\preceq}\{A, B\} \sim A \cap B$.

Theorem 3 *Let $\mathscr{A} = \wp(W)$ be finite and $\preceq$ a binary relation over $\mathscr{A}$. $\preceq$ is τ-representable iff $\preceq$ is a confidence order satisfying*

(17) *If $A \preceq B$ then $\overline{B} \preceq \overline{A}$,* (contraposition)
(18) *If $A \preceq B \prec \overline{B}$ then $A \cup B \preceq B$.* (weak max-law)

Proof The proof can be found in Raidl [11] The order induced from a two-sided ranking function clearly satisfies these axioms. Conversely, the idea goes as follows: transform $\preceq$ into a relation $\preceq'$ defined by $A \preceq' B$ iff $(B \preceq A \prec \overline{A}) \vee \overline{A} \preceq A$. Show that $\preceq'$ is κ-representable. Induce τ from such a κ and show that τ represents $\preceq$. □

Note that the above (18) can be replaced by

(19) If $\overline{B} \prec B \preceq A$ then $B \preceq A \cap B$.[8]

Fitelson [3] is critical of confidence orders representing ranking functions. What are his objections? First he has a positive result:

Theorem 4 *If a confidence order $\preceq$ (over an algebra $\mathscr{A}$) is β-representable then $\preceq$ satisfies (5).*

Recall that (5) was the characteristic axiom for the representability by a DS belief function. This is no surprise, since it is well known that a positive ranking function may be conceived as a so-called consonant DS belief function.[9] In any case, β-representability is stronger than DS-representability. However, a negative observation immediately follows:

Theorem 5 *There is a β-representable confidence order $\preceq$ that violates* (6).

Proof Consider the worlds w_0, w_1, w_2, v_0 and the negative ranking function $\kappa(w_i) = i, \kappa(v_0) = 0$. The corresponding β induces an order which is β-representable. Choose $A = \{w_2\}$, $B = \{v_0, w_2\}$ and $C = \{w_0\}$. We certainly have $A \cap C = \emptyset = B \cap C$. Additionally $\beta(A) = \kappa(\overline{A}) = 0 = \kappa(\overline{B}) = \beta(B)$. Thus $B \sim A$. However $\beta(B \cup C) = \kappa(\overline{B} \cap \overline{C}) = \kappa(\{w_1\}) = 1$ and $\beta(A \cup C) = \kappa(\overline{A} \cap \overline{C}) = \kappa(\{v_0\}) = 0$. Thus $A \cup C \prec B \cup C$, contradicting (6). □

Fitelson moreover observes that the order $\emptyset \prec A \sim \overline{A} \prec W$ is perfectly acceptable; he calls it the 'fair coin ordering'. However, it is not β-representable; β-representability would require either $A \sim \emptyset$ or $\overline{A} \sim \emptyset$ or both. Indeed, the latter order shows:

Theorem 6 *There is a β-representable confidence order $\preceq$ that violates* WADA.

[8] For other equivalent formulations, see Raidl [11].

[9] However, as Spohn [15] argues, the two notions diverge in their dynamic behavior, i.e., in their conditionalization rules. Therefore the one is not reducible to the other.

Proof We just saw that the order $\emptyset \sim A \sim \overline{A} \prec W$ is β-representable. However, it violates WADA; it makes two $\sim$-mistakes in each world and is thus weakly and indeed strictly accuracy-dominated by the fair coin ordering, which makes only one $\sim$-mistake in each world. □

However, this only shows that we should not look at confidence orders induced by positive ranking functions, because they treat non-belief and disbelief in the very same way, namely as positive rank 0, and thus do not distinguish various degrees of disbelief. It's only two-sided ranking functions that express degrees of belief and of disbelief at once, and hence we should rather attend to τ-representability. The last problem then immediately disappears. The fair coin ordering is obviously τ-representable. However, things get worse on the other score. For τ-representability something weaker than (6) holds, namely:

Theorem 7 *If a confidence order $\preceq$ is τ-representable then $\preceq$ satisfies:*

(20) *if $A \cap C = \emptyset$, $B \cap C = \emptyset$, and $A \prec B$, then $A \cup C \preceq B \cup C$.*

Proof Let A, B, C be as assumed and $A \prec B$. Either (i) $\overline{A} \preceq A$ or (ii) $A \prec \overline{A}$.

(i) Suppose, for reductio not $A \cup C \preceq B \cup C$. Thus $B \cup C \prec A \cup C$ (totality). Hence $\overline{A} \preceq A \prec B \preceq B \cup C \prec A \cup C$ (i, monotonicity) and hence $\overline{B} \prec B \prec A \cup C$ (contraposition, transitivity). Therefore $B \preceq B \cap (A \cup C) = B \cap A$ (19). But then $A \prec B \preceq B \cap A \preceq A$ (monotonicity) and thus $A \prec A$, contradicting reflexivity.

(ii) Either (a) $C \prec A$ or (b) $A \preceq C$ and $C \prec \overline{C}$ or (c) $A \preceq C$ and $\overline{C} \preceq C$. (ii,a) yields $C \prec A \prec \overline{A}$ and thus $A \cup C \preceq A \prec B \preceq B \cup C$ (weak max law (18), monotonicity). (ii,b) yields $A \preceq C \prec \overline{C}$ and thus $A \cup C \preceq C \preceq B \cup C$ (weak max law, monotonicity).

(ii, c) Let τ be a two-sided ranking function representing $\preceq$ and κ the associated negative ranking function. By (ii) $A \prec \overline{A}$ we have $\kappa(A) > 0$ and $\kappa(\overline{A}) = 0$. by (c) $\overline{C} \preceq C$, we have $\kappa(C) = 0$ and $\kappa(\overline{C}) \geq 0$. But $C \subseteq \overline{B}$. Thus $\kappa(\overline{B}) = 0$. By $A \prec B$, we have $-\kappa(B) = \tau(B) > \tau(A) = -\kappa(A)$ and thus $\kappa(B) < \kappa(A)$. We have $B \subseteq \overline{C}$. Thus $\kappa(\overline{C}) \leq \kappa(B)$. Thus either $\kappa(\overline{C}) < \kappa(B)$, in which case $\kappa(\overline{C}) = \kappa(\overline{C} \cap \overline{A})$ and thus $\kappa(\overline{C} \cap \overline{A}) = \kappa(\overline{C}) = \kappa(\overline{C} \cap \overline{B})$, which implies $A \cup C \preceq B \cup C$. Or $\kappa(B) = \kappa(\overline{C})$ and since the first B-worlds are not A-worlds, we have $\kappa(\overline{A} \cap \overline{C}) = \kappa(B) = \kappa(\overline{C}) \leq \kappa(\overline{B} \cap \overline{C})$, which again implies $A \cup C \preceq B \cup C$ (since $\kappa(A \cup C) = 0 = \kappa(B \cup C)$). □

The difference to (5) is that the consequent of (20) may only amount to indifference, even under the stricter algebraic presupposition of (5) (as it becomes clear in the proof). Hence:

Theorem 8 *There is a τ-representable confidence order $\preceq$ that violates:*

(21) *if $\emptyset \prec A$ and $A \cap B = \emptyset$ then $B \prec A \cup B$,* (lower probability)

and thus $\preceq$ violates (5), *and also violates* WADA.

Proof Consider w_0, w_1, w_2 and $B = \{w_1\}$, $A = \{w_2\}$. The worlds have their indices as negative ranks. Thus $\kappa(B) = 1$, $\kappa(A) = 2$. Let τ be the corresponding two-sided ranking function and $\preceq$ the induced order. We have $\emptyset \prec A$ and $A \cap B = \emptyset$. But $\tau(A \cup B) = -1 = \tau(B)$ violating (21). It is known that confidence orders satisfying (5) also satisfy (21). Thus (5) is violated as well. And since WADA implies (5), WADA is violated. □

In other words, Fitelson's [3] criticism stands even against our preferred rank-induced confidence orders, i.e., those representing two-sided ranking functions. If we accept WADA, we can't accept ranking theory at the same time. Something must give way.

4 Prioritized Accuracy

Of course, we want to argue that WADA may give way. By critically examining WADA this section attempts to develop four criteria for an alternative notion of accuracy. This will lead to a new notion of accuracy optimization conforming to those criteria.

A preliminary point which went unnoticed so far, but proves to be important in the sequel, is this: Let's call $\leq$ a *basic order* over W iff $\leq$ is a weak order over W. Obviously, each confidence order $\preceq$ on propositions induces such a basic order $\leq_\preceq$ by defining $w \leq_\preceq w'$ iff $\{w\} \preceq \{w'\}$. For this reason we will often not distinguish $\preceq$ and the induced basic order $\leq_\preceq$. We may leave it presently open whether the reverse entailment is desirable, although we will find later on that it does hold according to our proposal. Our present concern is whether accuracy considerations should be allowed to constrain basic orders. We think: no. That's our

Criterion 1: Accuracy considerations should not discriminate among basic orders.

Hence, we should only compare confidence orders which are compatible with the same basic order, where a confidence order $\preceq$ is *compatible* with a basic order $\leq$ iff $\leq_\preceq$ is $\leq$.

We have no strict argument in favor of criterion 1, but we cannot think of any reason against it. How could any accuracy consideration imply that one world should be more, or equally, or less credible than another? It even cannot imply that there should be exactly one most plausible or exactly one most implausible world. Epistemic states must be free in their basic order. We may, of course, argue about whether initial or a priori confidence orders should satisfy suitable symmetries a priori and should hence treat some worlds alike (in analogy to symmetry principles in probability theory). However, we are here discussing confidence orders in general and not specifically a priori ones.

It should be noted, moreover, that Fitelson and McCarthy [4] also comply with criterion 1. For, $w' \prec w$ is a big mistake in w', but none in w, conversely for the reverse judgment $w \prec w'$, and $w \sim w'$ is a little mistake in both worlds. Hence, no judgment about w and w' can dominate another one in the sense of Definition 2. We will thus accept criterion 1 for later use.

Let us inspect WADA a bit more closely. A first surprise is that WADA doesn't deliver contraposition (17). This says that one cannot have more or equal trust in the truth of A than in the truth of B and at the same time put more trust in the falsity of A than in the falsity of B. This seems intuitively required, and it is required by probabilistic as well as by τ-representability. However, it is not entailed by WADA. This may be a symptom that something is missing in WADA.

The observation generalizes. We may consider it desirable that a confidence order $\preceq$ is at least not *guaranteed* to make big $\preceq$-mistakes, i.e., that it satisfies *consistency*:

(22) There is a world w such that $B \prec A$ for no A false in w and no B true in w.[10]

WADA does not entail consistency, either. We only get that no consistent confidence order can be weakly accuracy-dominated by any inconsistent confidence order. But inconsistent confidence orders may turn out undominated by any (consistent or inconsistent) confidence order. Indeed, probabilistic and τ-representability divide over consistency:

Theorem 9 *Any τ-representable confidence order $\preceq$ satisfies consistency* (22). *However, there is a probabilistically representable confidence order $\preceq$ which violates* (22).

Proof Let $\preceq$ be τ-representable, τ a two-sided ranking function representing it and κ the corresponding negative ranking function. Either (a) there is a unique world w at 0 (i.e., $\kappa(\{w\}) = 0$), or (b) more than one, say w, v. (a) entails $\tau(\{w\}) = \kappa(\overline{\{w\}}) - \kappa(\{w\}) = \kappa(\overline{\{w\}}) > 0 > -\tau(\{w\}) = \tau(\overline{\{w\}})$. (b) entails $\tau(\{w\}) = \kappa(\overline{\{w\}}) - \kappa(\{w\}) = \kappa(\overline{\{w\}}) = 0 = -\tau(\{w\}) = \tau(\overline{\{w\}})$, idem for v. In both cases there is a world w, such that $\overline{\{w\}} \preceq \{w\}$. Consider now A, B such that $w \in B \setminus A$. Then $\{w\} \subseteq B, \overline{A}$, i.e., $A \subseteq \overline{\{w\}}$. Therefore $A \preceq \overline{\{w\}} \preceq \{w\} \preceq B$ (monotonicity). Hence $A \preceq B$ (transitivity), i.e., not $B \prec A$.

Let P be a probability measure over $\mathscr{A}$ and $\preceq$ the associated order. Assume $P(\{w\}) < 0.5$ for all $w \in W$, which is perfectly possibly as long as $|W| > 2$. Then for all w, $\{w\} \prec \overline{\{w\}}$. Let w be arbitrary and consider $B = \{w\}, A = \overline{B}$. Then $w \in B \setminus A$, but $B \prec A$. In fact, the only P-representable orders satisfying consistency are *weakly atomically opinionated*, i.e. have $\overline{\{w\}} \preceq \{w\}$ for at least one w. Indeed, take such a w, then $P(w) \geq 0.5$. Let $w \in B \setminus A$. Then $P(B) \geq P(\{w\}) \geq P(\overline{\{w\}}) \geq P(A)$, since $\{w\} \subseteq B$ and $A \subseteq \overline{\{w\}}$. Thus not $B \prec A$. □

Should we accept consistency (22)? Let's not unroll the rich discussion about this issue. Of course, one may cite the preface and the lottery paradox (or the undecidability of consistency in first-order logic, which is irrelevant, though, in our finite framework) in order to argue that consistency is unattainable (on the qualitative as

[10] Equivalently: there is a world w such that if $w \in B \setminus A$ then not $B \prec A$.

well as on the comparative level, as expressed by (22)) and that we should therefore seek for rationality criteria below consistency. Or we may still be convinced of sticking to consistency despite those objections.

There is no point now in starting this discussion. The only point we want to make is that, in our view, accuracy arguments cannot decide the issue. The intuitive appeal of accuracy is strong indeed. And it may be explicated along the lines of WADA. However, one may also say that the accuracy intuition is not served at all if perfect accuracy is excluded a priori, if one is bound to be inaccurate in all worlds. It appears that approximation to the truth must at least admit of the possibility of avoiding all inaccuracies.

What we want to do in the sequel is to elaborate an explication of the accuracy intuition that does justice to consistency. We don't claim that this is the better explication. But certainly we cannot take one explication for granted and rely on what it entails. Rather, we have to assess the explications as well. This then is part and parcel of the holistic normative assessment of our epistemic constitution. We don't strive here for this overall assessment, but we want to open the accuracy discussion for it.

How should we then develop a notion of accuracy and of accuracy dominance in this spirit? To begin with, let us change the terminology in order to avoid getting mixed up with Fitelson's and McCarthy's account, and let us say that the judgment $B \prec A$ is *wrong* in the world w iff A is false and B is true in w (i.e., $w \in B \setminus A$) and that the judgment $B \prec A$ is *right* in the world w iff A is true and B is false in w (i.e., $w \in A \setminus B$). If A and B have the same truth value in w, this judgment is neither right nor wrong. And the judgment $A \sim B$ is never right or wrong.

Initially, we share Fitelson's and McCarthy's approach in accepting

Criterion 2: Try to minimize wrong comparisons!

This seems to be a natural accuracy requirement. Note that it is what inspires consistency (22). Consistency says that you reach zero wrong comparisons at least in one world, i.e., that there is at least one possibility to optimally satisfy criterion 2. Criterion 2 also justifies monotonicity (3). Conformity to (3) never produces a wrong comparison, but a violation of (3) may be wrong.

However, criterion 2 cannot be our only aim. It is perfectly reached in too easy a way, namely by being indifferent between any two contingent propositions. Thereby you can avoid being wrong in any world. Clearly, though, that's not a recommendable attitude. It would also be in conflict with criterion 1 by dictating a specific basic order.

There are two ways out, and here we part from Fitelson and McCarthy: Their way consists in claiming that indifferences can be mistaken as well. They thus penalize the flight into indifference. However, this is not the way we want to go. First, note that Fitelson and McCarthy need to measure the size of mistakes; this is a potential gateway of criticism. Therefore we try to get along with the more elementary way of simply *counting* instances of being wrong. Second, the idea that being indifferent may be an inaccuracy does not seem very convincing to us. If it is a mistake not to distinguish between propositions of distinct truth values, we should say in the same spirit that it is a mistake to distinguish propositions not distinct in truth value. However, the latter idea is certainly much too strong. In general, Fitelson's and MacCarthy's procedure appears defensive to us; it focuses exclusively on minimizing mistakes.[11]

Our idea is more positive; we think that accuracy should also be about *trying to be right*. Recall the famous slogan of James [6, section VII]: "Believe truth! Shun error!" Accordingly, our idea is that there should not only be a malus for being wrong, but also a bonus for being right.[12] This seems entirely natural, indeed compelling. At the same time, it provides a means to counteract the flight into indifference promoted by criterion 2. Moreover, there is no need then to give $\sim$-judgments a positive or negative value. We may take them to be neutral, not wrong, but not right, either. Thus, we have arrived at our

Criterion 3: Try to maximize right comparisons!

Note that this criterion inspires contraposition (17). When you violate (17), at most one of your two strict comparisons can be right. By contrast, when you satisfy (17), possibly both of your comparisons can be right.

However, it is unclear how to satisfy criterion 3. James was explicit in stating that his two maxims are independent. Till the present day it is unclear how to combine them into one coherent rule. On our level of comparative notions we have the analogous problems. The more comparisons are right in one world, the less of them are right in another world. Indeed, the more of them are false in another world, and thus we get a conflict with criterion 2. Again, there is also an incompatibility with criterion 1, since criterion 3 requires that there should be one world w that is more credible than any other world; this would maximize the number of basic comparisons that are right in w.

These problems suggest that accuracy dominance avoidance doesn't carry us very far. However, if we balance right and wrong comparisons, there will be too many undominated confidence orders. Rather we should prioritize where to maximize right comparisons; we cannot attempt to maximize across the board. How?

Well, it seems to be a good idea, first to follow criterion 2 and then criterion 3, i.e., first to minimize wrong and then to maximize right comparisons. However, this cannot mean to first minimize wrong comparisons across the board, since this would again force us into indifferences. The idea rather is first to look for the worlds with the least wrong comparisons, then to maximize right comparisons there, then again to look how we can minimize wrong comparisons in the remaining worlds, and then try to maximize right comparisons there, and so on.

This is not yet completely precise, though. It may allow, for instance, to go for a world w with 0 wrong comparisons and to maximize right comparisons there, at

[11]This claim would need further qualification in light of McCarthy's and Fitelson's axiomatic defense of evidential propriety (see footnote 2) which may provide a response to some of our criticisms. However, the issue of the probabilistic bias and the criticism about indifference, remain. This remark is based on a personal communication by McCarthy.

[12]Note that in a purely probabilistic framework there is no such distinction between a malus for being wrong and a bonus for being right. There is only a more or less bad distance from the truth values 1 and 0.

the cost of making wrong comparisons in other worlds which would otherwise also make 0 wrong comparisons. This seems again to be an inadequate way of prioritizing criterion 2 over criterion 3.

So, the right way seems this: First minimize wrong comparisons in as many worlds as possible. Then maximize right comparisons in those worlds. Then again minimize wrong comparisons in as many of the remaining worlds as possible, and then maximize right comparisons among those worlds. And so on. Hence, the idea is to bring criteria 2 and 3 into a nested lexicographic order and not to somehow arithmetically balance the number of right and wrong comparisons.

To put it very sloppily: we are trying to make our best horses (= worlds) as strong as possible. Only after that we try to optimize our second best horses, and so on. So, each step in this procedure lexicographically dominates the subsequent ones. All in all, this is, as it were, an active way of accuracy optimization, not a defensive one. It is our way of combining James' two maxims into one on a comparative level. Is it really reasonable? Not quite. However, let's preliminarily formalize the ideas laid out so far. Then the residual problems will appear more clearly.

So far, we have the following construction: We start with a weak order $\preceq$ over $\mathscr{A}$. We need not assume that $\preceq$ is a confidence order, since monotonicity and non-triviality will be seen to fall out from accuracy optimization. Moreover, $\preceq$ may or may not be regular.

For any weak order $\preceq$ over $\mathscr{A}$ and world $w \in W$ define $F(\preceq, w) = \{\langle B, A \rangle \,|\, B \prec A, w \in B \setminus A\}$; $|F(\preceq, w)|$ (the cardinality of $F(\preceq, w)$) counts how often $\preceq$ is wrong (false) in w. Similarly define $R(\preceq, w) = \{\langle B, A \rangle \,|\, B \prec A, w \in A \setminus B\}$; $|R(\preceq, w)|$ counts how often $\preceq$ is right in w. Please beware: Below we will introduce the functions F^* and R^* that slightly modify F and R. The intention, then, is to proceed with these modified functions in the very same way as we do now with F and R.

The functions F and R give us the *accuracy* of $\preceq$ in w defined as $a(\preceq, w) = \langle |F(\preceq, w)|, |R(\preceq, w)| \rangle$. Thus, accuracies are pairs of non-negative integers. Obviously they can be better or worse. Our informal considerations suggested to define that $\langle f, r \rangle \geq \langle f', r' \rangle$, i.e., $\langle f, r \rangle$ is at *least as good* as $\langle f', r' \rangle$ iff $f < f'$, or $f = f'$ and $r \geq r'$. Hence $\langle f, r \rangle > \langle f', r' \rangle$, i.e., $\langle f, r \rangle$ is *better* than $\langle f', r' \rangle$ iff $\langle f, r \rangle \geq \langle f', r' \rangle$ and $\langle f, r \rangle \neq \langle f', r' \rangle$. Thereby, it is of first importance to minimize wrong comparisons and it is of second importance to maximize right comparisons. This is one aspect of prioritization.

On this basis we can assign accuracy profiles to weak orders over $\mathscr{A}$ as such. If there are k worlds in W, then $\preceq$ has k accuracies $\langle f_i, r_i \rangle$ $(i = 1, \ldots, k)$. Let them be ordered in such a way that $\langle f_i, r_i \rangle \geq \langle f_j, r_j \rangle$ for $i \leq j$. Then we define $a(\preceq) = \langle \langle f_1, r_1 \rangle, \ldots, \langle f_k, r_k \rangle \rangle$ to be the *accuracy profile* of $\preceq$.

We need to compare prioritized accuracy profiles as well. This is a second aspect of prioritization. Recall, as we have explained above, that it is indeed of first importance to maximize the number of worlds minimizing wrong comparisons and then of second importance to maximize right comparisons. This makes our definition slightly complicated.

Definition 6 Let $\preceq$ and $\preceq'$ be two weak orders over $\mathscr{A}$, with

$$a(\preceq) = \langle \langle f_1, r_1 \rangle, \ldots, \langle f_k, r_k \rangle \rangle \text{ and } a(\preceq') = \langle \langle f'_1, r'_1 \rangle, \ldots, \langle f'_k, r'_k \rangle \rangle$$

their accuracy profiles. We say that $a(\preceq)$ is *at least as good* as $a(\preceq')$, i.e., $a(\preceq) \geq a(\preceq')$ iff

(i) either $\langle f_i, r_i \rangle = \langle f'_i, r'_i \rangle$ for all $i = 1, \ldots, k$,
(ii) or there is j such that $f_j < f'_j$ and for all $i < j$, $f_i < f_j$ implies $\langle f_i, r_i \rangle = \langle f'_i, r'_i \rangle$ and $f_i = f_j$ implies $f_i = f'_i$,
(iii) or there is j such that $f_j = f'_j$ and $r_j > r'_j$, and for all $i < j$ we have $\langle f_i, r_i \rangle = \langle f'_i, r'_i \rangle$, whereas for all $i > j$, $f'_i = f'_j$ implies $f_i = f'_i$.

This captures precisely the idea laid out above.

What, then, is the aim confidence orders should strive for? Optimizing prioritized accuracy profiles in the sense defined? Not quite. This would violate criterion 1 by requiring one world to be more credible than all other worlds in the basic order. In order to respect criterion 1 we must compare only confidence orders with the same basic order. We cannot discard one basic order because a better prioritized accuracy profile would be attainable on the basis of a different basic order. Hence our conclusion so far must be the following maxim of

> **Prioritized Accuracy Optimization (PAO):** Rationally, a weak order $\preceq$ on $\mathscr{A}$ can be chosen as confidence order only if $a(\preceq) \geq a(\preceq')$ for all weak orders $\preceq'$ *basically compatible* with $\preceq$, i.e., for all $w, w' \in W$, $w \preceq' w'$ iff $w \preceq w'$.

We will later see (Theorem 10) that every weak order satisfying PAO is indeed a confidence order. Is this a satisfactory formulation of the epistemic goal of confidence orders? Almost.[13] It seems that we have put criteria 2 and 3 into an adequate relation. However, we have slightly overdosed, we think, the incentive of making right comparisons. Even if we prioritize criterion 2 over criterion 3, we cannot allow that $\prec$-judgments are amassed in the hope to subsequently maximize right $\prec$-judgments. It also seems to be a sort of mistake to assume more $\prec$-comparisons than can possibly be right.

To take an example: We might judge $A \prec A \cup B$ and $B \prec A \cup B$, in the hope that the first is right in $\overline{A}$-worlds and the second is right in $\overline{B}$-worlds. If we would change one (or both) of these comparisons into indifference, we would be right less often. However, haven't we overdone it now? The two comparisons could not possibly be right at once; there cannot be a world in which both are right. Let's call this a *disjunctive flaw*.

There is the symmetric case of a *conjunctive flaw*. We might want to judge $A \cap B \prec A$ and $A \cap B \prec B$, where the first is right in certain $\overline{B}$-worlds and the second is right in certain $\overline{A}$-worlds. Again, the two comparisons could not possibly be right at once. Such disjunctive and conjunctive flaws—let's call them *logical flaws*—are not as bad as being inconsistent, as a guarantee of being wrong. Still, it seems that we should

[13]It may be interesting to study where PAO as presently stated leads us. We didn't study this issue and preferred to proceed with the amendment we are about to explain.

not tolerate a guarantee of not being right, since this is also a guarantee of not being perfectly accurate. This leads us to our final

Criterion 4: Try to minimize logical flaws!

We should thus also set an incentive in favor of criterion 4, i.e., against overdoing with satisfying criterion 3. Note that this incentive cannot consist in penalizing to be wrong. The comparisons in our example can't be wrong. Therefore it won't do only to negatively count instances of wrong $\prec$-judgments. We need to find another negative score.

Already here you may be suspicious about what's to come. It was our declared aim to push confidence orders towards τ-representability. And now we are about to do so by biasing the notion of accuracy with criteria of logical correctness. We cannot ward off that suspicion. But recall that Fitelson and McCarthy directly installed a probabilistic bias by their requirement of evidential propriety. We are at least on a par in this respect. And we do claim that it is intuitively plausible that criterion 4 constrains maximization of right comparisons.

However, how exactly should we conform to criterion 4? Again, we get a conflict with criterion 1. According to criterion 4, we would make a conjunctive flaw when $\emptyset \prec \{w\}, \{w'\}$. So, we could have $\emptyset \prec \{w\}$ only for one world w, thus violating criterion 1. In fact, when we look more closely, we find that such flaws are entirely unavoidable. Giving priority to criterion 4 would again entail much more indifferences than are tolerable. Should we therefore give up criterion 4? We think no. We can't see that the intuition behind it is entirely wrong. The mildest form to do justice to criterion 4 which we have found is the following one.

Consider the comparison $C \prec A$, where $C \subseteq A$, and let us assume that it is right in the world w. According to criterion 3 this should count in favor of w. However, it might be involved in a conjunctive flaw. So, suppose that there is a proposition B such that $C = A \cap B$ and also $C \prec B$. Thus at most one of the two comparisons can be right. Should we therefore count $C \prec A$ rather against w? Not automatically; this would give too much weight to criterion 4. Conversely we might say that w gets at least one comparison right. It cannot do better. So everything is fine. But then we would give no weight at all to criterion 4. Is there a middle course? Yes.

Let us look at w from the point of view of another world w'. The conjunctive flaw is present also in w', but w' may deal with it in another way. There are two ways to be distinguished. First w' may be a 'worse' world than w, i.e., $\{w'\} \prec \{w\}$. Then getting $C \prec A$ right in w should get priority over however w' deals with the logical flaw. This agrees with our general strategy to make the 'better' worlds as accurate as possible before we turn to the 'worse' worlds.

The other case is that w' is a 'better' world than w, i.e., $\{w\} \prec \{w'\}$. Then there are two cases in turn. Either, C is false in w', just as it is in w. Then w' may get one of the two comparisons right, just as w actually does. So, again, w does not something w' can object to. Or, secondly, C is true in w'. Then A and B must be true in w' as well, and not even one of the two comparisons can be right in w'. Then judging $C \prec A$ and $C \prec B$ would mean overdoing trying to make right comparisons from the point of view of w'; from that point of view we should rather judge $C \sim A$ and $C \sim B$, thus avoiding the conjunctive flaw. And since w' is 'better' than w, this point of view should take priority over that of w. Therefore, in this case, and only in this case, the judgment $C \prec A$ should rather count against w, even though it is right in w. That is, we should count this case as a conjunctive flaw in w, i.e., as a flaw for which w is accountable.

Isn't there the third case in which w and w' are 'equally good', i.e., $\{w\} \sim \{w'\}$? Yes. It turns out, however, that it does not matter how we decide this case. Therefore we choose the weaker rule that there can be a conjunctive flaw in w only if there is a better world with the features just explained.

Similarly for disjunctive flaws. Suppose again that $C \subseteq A$, that $C \prec A$, and that this comparison is right in w. This might also involve a disjunctive flaw, i.e., there might be a proposition B such that $A = B \cup C$ and $B \prec A$, so that at most one of these comparisons can be right. A world w' which is worse than w cannot be used as an objection against w, since w has priority over w'. A world w' which is better than w cannot be used as an objection against w, either, if A is true in w'. This is so because w' allows one of the two comparisons being right. However, if there is a better world w' in which A is false and in which not even one of the two comparisons can be right, then this situation should take priority over w and count against w.

Let's summarize these considerations.

Definition 7 Let $\preceq$ be a weak order over $\mathscr{A}$. Then

(a) $C \prec A$ is a *conjunctive flaw* or a $\cap$-*flaw* in w iff $C \subseteq A$, $C \prec A$ is right in w, and there exist a proposition B and a world w' such that $C = A \cap B$, $C \prec B$, C is true in w', and $\{w\} \prec \{w'\}$.
In this case let's call $\langle B, w' \rangle$ a *witness* of (the conjunctive flaw) $C \prec A$ in w.

(b) $C \prec A$ is a *disjunctive flaw* or a $\cup$-*flaw* in w iff $C \subseteq A$, $C \prec A$ is right in w, and there exist a proposition B and a world w' such that $A = B \cup C$, $B \prec A$, A is false in w', and $\{w\} \prec \{w'\}$.
Again, let's call $\langle B, w' \rangle$ a *witness* of that disjunctive flaw in w.

(c) $C \prec A$ is a *logical flaw* in w iff it is a conjunctive or disjunctive flaw in w.

$L_{\cup}(\preceq, w)$ is the set of $\cup$-flaws in w, $L_{\cap}(\preceq, w)$ is the set of $\cap$-flaws in w and $L(\preceq, w) = L_{\cap}(\preceq, w) \cup L_{\cup}(\preceq, w)$ is the set of logical flaws in w.

The idea then is that such a logically flawed comparison should count against a world together with its wrong comparisons and that they should not count in favor of a world, even though it is right in this world. That is, we must add these flaws to the mistakes of making wrong comparisons, and we must subtract them from the number of right comparisons. This amounts to redefining our functions F and R. That is, define $F^*(\preceq, w) = F(\preceq, w) \cup L(\preceq, w)$ and $R^*(\preceq, w) = R(\preceq, w) \setminus L(\preceq, w)$.[14] With $|F^*|$ and $|R^*|$ thus defined, we can proceed as above (with $|F|$ and $|R|$) and PAO thus understood is our official conception of prioritized accuracy optimization.

[14] Note that since $L(\preceq, w) \subseteq R(\preceq, w)$ and $L(\preceq, w)$ is disjoint of $F(\preceq, w)$, we have $|F^*(\preceq, w)| = |F(\preceq, w)| + |L(\preceq, w)|$ and $|R^*(\preceq, w)| = |R(\preceq, w)| - |L(\preceq, w)|$, which we will be using in the proofs.

5 How to Represent Prioritized Accuracy

The final statement of PAO certainly looks a bit complicated. However, we think we have well motivated each step leading to it. Of course, we have stated it also in order to be able to prove the following theorems about it. This was the hidden main motivation. Still, the intuitive motivation given so far is completely independent of the theorems we are about to prove. We first have:

Theorem 10 *Each weak order $\preceq$ over $\mathscr{A}$ conforming to PAO satisfies non-triviality* (2) *and monotonicity* (3)*, and is hence a confidence order over $\mathscr{A}$.*

Proof Corollary to the next Theorem 11. □

This is pleasing. PAO thus justifies the characteristic properties (2) and (3) of confidence orders. Maybe they were not in need of justification. Still this is a nice surplus of PAO. And note once more that PAO proceeds only by counting cases of being wrong, being right, and being a logical flaw. There is no need to weigh mistakes in any delicate way.

Our main theorem finally is:

Theorem 11 *A weak order $\preceq$ over finite $\mathscr{A}$ satisfies PAO if and only if it is representable by a regular two-sided ranking function τ.*

Proof In the Appendix, we establish the following result (Theorem 13): for any basic order $\leq$ over W, there is a weak order $\preceq_{\leq}$ extending $\leq$ (the *canonical extension*, see Definition 8 and the remarks thereafter) such that if $\preceq^*$ is another weak order which is basically compatible with $\preceq_{\leq}$ then $a(\preceq^*) \ngeq a(\preceq_{\leq})$. But profile comparison between basically compatible weak orders is total, i.e., for $\preceq$ and $\preceq^*$ basically compatible, if $a(\preceq^*) \ngeq a(\preceq)$ then $a(\preceq) > a(\preceq^*)$. Thus $\preceq_{\leq}$ is the best weak order extension of $\leq$, i.e., for all other transitive total extensions $\preceq^*$, $a(\preceq_{\leq}) > a(\preceq^*)$. Additionally $\preceq_{\leq}$ is representable by a regular two-sided ranking function $\tau_{\leq}$ (by construction, see Definition 8 and the remarks thereafter).

Let $\preceq$ be a weak order over $\mathscr{A}$ satisfying PAO, and $\leq$ its underlying basic order. Then for any other weak order $\preceq^*$ which is basically compatible with $\preceq$, we have $a(\preceq) \geq a(\preceq^*)$. By the above result, $\preceq$ must be $\preceq_{\leq}$ (else $a(\preceq_{\leq}) > a(\preceq)$). But $\preceq_{\leq}$ is representable by a regular two-sided ranking function, namely $\tau_{\leq}$.

Conversely, assume $\preceq$ to be representable by a regular two-sided ranking function and $\leq$ its underlying basic order. Then $\preceq$ is $\preceq_{\leq}$ (appendix, Lemma 1.3). By the above mentioned result, $\preceq$ *strictly* dominates all other weak order extensions of $\leq$, thus it satisfies PAO. □

It should be noted that we may restrict the counting in R (and L) to $\mathscr{A} \setminus \{W, \emptyset\}$. This has as a consequence that neither $A \prec W$ will be rewarded as right in non-A worlds, nor will $\emptyset \prec A$ be rewarded as being right in A-worlds. The previous $\prec$-judgments can also not lead to a disjunctive or a conjunctive flaw.[15] The overall consequence is that PAO, restricted in the above sense, will not entail regularity. Thus we obtain a slightly weaker correspondence, namely: A weak order $\preceq$ over finite $\mathscr{A}$ satisfies PAO with R (and L) restricted to $\mathscr{A} \setminus \{W, \emptyset\}$) if and only if it is representable by a two-sided ranking function τ (which now may or may not be regular).[16]

A direct consequence of this is

Theorem 12 *If a confidence order* $\preceq$ *satisfies PAO, then it satisfies consistency* (22) *and contraposition* (17).

Proof Since the order induced by a two-sided ranking function satisfies consistency and contraposition. □

Note that according to our Theorem 13, there are exactly as many weak orders over $\mathscr{A}$ satisfying PAO as there are basic orders over W, since by PAO the basic order $\leq$ over W determines the algebraic extension $\preceq$ to $\mathscr{A}$ completely.

6 Conclusion

Theorem 11 is the success we aimed to achieve. So, there is an accuracy argument which favors confidence orders conforming to ranking theory and thus conflicting with probability theory. The accuracy measure applied here is quite different from those considered elsewhere. But it looks attractive, since it does not only defensively try to avoid mistakes or being wrong, but also actively attempts to be right. It is thus a comparative version of James' two qualitative maxims: accept true beliefs and avoid false beliefs! And it is a constructive proposal for combining these two antagonistic maxims into one. Finally, the amendment about logical flaws was required and intuitively motivated, though perhaps less convincingly than the rest.

It was not our aim, anyway, to argue that our accuracy measure would be superior. We only claim that it is one plausible measure among many, with its pros and cons. Accuracy is no way to decide about the normatively most adequate epistemic format. Rather, accuracy measures are part and parcel of an overall normative assessment of those formats. It remains an open issue whether a single format can and need be distinguished as the most appropriate one.

Acknowledgements We are indebted to Branden Fitelson for making us think about the topics that we develop in this paper and to David MacCarthy for most valuable comments.

---

[15] This is in fact true in general for $A \prec W$ and $\emptyset \prec A$.

[16] See the footnote to our main Theorem 13 in the appendix and compare the last two lines in the proof of this Theorem.

Appendix A: Proofs

A.1 Canonical Order

$k : W \to \mathbb{N} \cup \{\infty\}$ is a *ranking mass* (or *point ranking function*) iff it is a total function and $k^{-1}(0) \neq \emptyset$. Given a ranking mass, the *induced negative ranking function* over $\mathscr{A}$ is $\kappa_k(A) = \min_{w \in A} k(w)$ if $A \neq \emptyset$ and else $\kappa_k(\emptyset) = \infty$. κ_k is indeed a negative ranking function and in fact *complete*, i.e., for all $S \subseteq \mathscr{A}$, $\kappa_k(\bigcup S) = \min_{A \in S} \kappa_k(A)$. Conversely, every complete negative ranking function κ over $\mathscr{A}$ has a unique ranking mass $k_\kappa(w) = \kappa(\{w\})$ inducing it.

Definition 8 Let $W \neq \emptyset$ be finite. Given a basic order $\leq$ over W (i.e., weak order), the *canonical two-sided ranking function* $\tau_\leq$ is recursively defined as follows:

(1) $W_0 = W$, $[0] =: \max_\leq W_0$,
(2) $W_{n+1} := W_n \setminus [n]$ (if non-empty, else stop), $[n+1] := \max_\leq W_{n+1}$,
(3) $k(w) := x$ iff $w \in [x]$, $\kappa = \kappa_k$,
(4) $\tau_\leq(A) = \kappa(\overline{A}) - \kappa(A)$.

The *canonical extension of* $\leq$ to $\mathscr{A}$ is the order $\preceq$ induced by $\tau_\leq$: $A \preceq B$ iff $\tau_\leq(A) \leq \tau_\leq(B)$ for all $A, B \in \mathscr{A}$.

Call $[m]$ the *m-worlds*. It is clear that k is a ranking mass over W (since $[0] \neq \emptyset$, by finiteness and totality), thus $\kappa = \kappa_k$ is a negative ranking function and $\tau_\leq$ a two-sided ranking function over $\mathscr{A}$, and all are regular. The canonical extension $\preceq$ of $\leq$ is a weak order, $\leq$-compatible and clearly representable by a regular two-sided ranking function (namely by $\tau_\leq$).

Let $A \in \mathscr{A} \setminus \{\emptyset, W\}$ and $n > 0$: A is a *0-set*, iff A partitions $[0]$ non-voidly ($A \cap [0] \neq \emptyset$ and $\overline{A} \cap [0] \neq \emptyset$). A is an *$-n$-set* iff the smallest world in A is an n-world. A is an *n-set* iff A is the complement of a $-n$-set. Note: there are 0-sets iff $|[0]| > 1$.

Lemma 1 *Let $\leq$ be a basic order over finite W, $\preceq$ its canonical extension and A_x, A_y x- and y-sets. Then*

(1) *for $A \in \mathscr{A} \setminus \{\emptyset, W\}$: A is an x-set iff $\tau_\leq(A) = x$,*
(2) *for $A_x, A_y \in \mathscr{A} \setminus \{\emptyset, W\}$: $A_x \preceq A_y$ iff $x \leq y$,*
(3) *an extension $\preceq^*$ of $\leq$ to $\mathscr{A}$ is representable by a regular two-sided ranking function iff $\preceq^*$ is $\preceq$.*

Proof (1) holds by construction, (2) follows. (3) holds since $\tau_\leq$ is, up to order isomorphism, the only regular two-sided ranking function representing $\leq$. □

Let $n \in \mathbb{N}$: A $-n$-set A_{-n} is *full* iff $[n] \subseteq A_n$, it is *non-full* iff it is not full. An n-set A_n is *void* iff it is the complement of a full $-n$-set, it is *non-void* iff it is not void. $\dot{A}_{-n}, \dot{A}_n$ will stand for full $-n$-sets and void n-sets, and be contrasted with A_{-n}, A_n, where these are non-full $-n$-sets and non-void n-sets.

Denote $PF(w) := \{\langle A, B\rangle : w \in A \setminus B\}$, resp. $PR(w) := \{\langle A, B\rangle : w \in B \setminus A\}$, the set of *potentially wrong, resp. right, comparisons in* w . In the whole section $\preceq$ is the canonical extension of the basic $\leq$.

Lemma 2 (Wrongness) *Let $\preceq$ be the canonical extension of the basic $\leq$. Then*

(1) $F(\preceq, w) = PF(w) \cap \{\langle A_x, A_y\rangle : -n \leq x < y \leq n\}$ *for* $w \in [n]$,
(2) $|F(\preceq, w)| = 0$ *for* $w \in [0]$,
(3) $|F(\preceq, w)| = |F(\preceq, v)|$ *for all* $w, v \in [n]$, $n \geq 0$.

Proof Assume $\leq, \preceq$ as given. (1) Let $\langle A_x, A_y\rangle \in F(\preceq, w)$ for $w \in [n]$. Then (i) $w \in A_x \setminus A_y$, i.e., $\langle A_x, A_y\rangle \in PF(w)$ and (ii) $A_x \prec A_y$. By (i) $-n \leq x$ and $y \leq n$. By (ii) $x < y$ (Lemma 1.2). Thus $-n \leq x < y \leq n$. Conversely, let $\langle A_x, A_y\rangle \in PF(w) \cap \{\langle A_x, A_y\rangle : -n \leq x < y \leq n\}$. Thus $w \in A_x \setminus A_y$ and $-n \leq x < y \leq n$. Hence $A_x \prec A_y$ (Lemma 1.2). (2) $-0 \leq x < y \leq 0$ is impossible. (3) follows by permutation invariance within classes $[n]$. □

Lemma 3 (Logical flaws) *Let $\preceq$ be the canonical extension of the basic $\leq$. Then*

(1) $|L(\preceq, w)| = 0$ *for* $w \in [0]$,
(2) *if* $\langle A_y, A_x\rangle \in L_\cup(\preceq, w)$ *for* $w \in [n]$, *then the witnesses* A_z, $v \in [k]$ *are such that* $y, z < x \leq k < n$ *and* $0 < x$,
(3) *if* $\langle A_x, A_y\rangle \in L_\cap(\preceq, w)$ *for* $w \in [n]$, *then the witnesses* A_z, $v \in [k]$ *are such that* $-n < -k \leq x < y, z$ *and* $x < 0$,
(4) $|L(\preceq, w)| = 0$ *for* $w \in [1]$,
(5) $|L(\preceq, w)| = |L(\preceq, v)|$ *for all* $w, v \in [n]$.

Proof (1) There are no better worlds than 0-worlds (in the sense of $\leq$ or F), thus there can be no world-witness for flaws. (2) Let $\langle A_y, A_x\rangle \in L_\cup(\preceq, w)$ for $w \in [n]$. Thus, according to Definition 7, the witnesses A_z, $v \in [k]$ for $k < n$ satisfy (i) $A_y \prec A_x$, (ii) $A_z \prec A_x$, (iii) $w \in A_x \setminus A_y$, (iv) $A_x = A_y \cup A_z$, (v) $v \notin A_x$. $k < x$ contradicts (v). Hence $x \leq k < n$. We know $x > 0$: For suppose (for reductio) $x \leq 0$. Since $y, z < x$, we have $-x < -y, -z$. But then A_x contains a $-x$-world and A_y, A_z none, contradicting (iv). (3) follows by a similar reasoning as for (2). (4) is a corollary to (2) and (3). (5) is due to permutation invariance within classes $[n]$. □

Lemma 4 (Rightness) *Let $\preceq$ be the canonical extension of the basic $\leq$. Then for* $w \in [n]$, $n \geq 0$*:*

(1) $R(\preceq, w) = PR(w) \setminus \{\langle A_y, A_x\rangle : -n \leq x \leq y \leq n\}$,
(2) $R(\preceq, w) = PR(w) \setminus \{\langle A_x, A_y\rangle : x = y = 0\}$, *if* $n = 0$,
(3) $|R(\preceq, w)| = |R(\preceq, v)|$ *for all* $w, v \in [n]$, $n \geq 0$.

Proof (1) Let $\langle A_y, A_x\rangle \in R(\preceq, w)$. Then (i) $A_y \prec A_x$ and (ii) $w \in A_x \setminus A_y$, i.e., $\langle A_y, A_x\rangle \in PR(w)$. We show that, given (ii), (i) is equivalent to $\langle A_y, A_x\rangle \notin$

$\{\langle A_y, A_x\rangle : -n \leq x \leq y \leq n\}$. Since $w \in [n]$, (ii) implies (a) $x \geq -n$ and (b) $y \leq n$. Therefore:

$$\begin{aligned} &\langle A_y, A_x\rangle \notin \{\langle A_y, A_x\rangle : -n \leq x \leq y \leq n\} \\ \text{iff } & x < -n \vee y < x \vee n < y && \text{(logic)} \\ \text{iff } & y < x && \text{(ii a, b)} \\ \text{iff } & A_y \prec A_x && \text{(canonical } \preceq) \end{aligned}$$

(2) is a corollary to (1). (3) is due to permutation invariance within classes $[n]$. □

Definition 9 Let $\preceq^*$ be a relation over $\mathscr{A}$ and $\preceq$ the canonical extension of the basic $\leq$.

– $\preceq^*$ *agrees weakly with* $\preceq$ *up to n* iff $\preceq^*$ agrees with the transitive closure of the relation determined by:

$$A_y \prec A_z, \tag{A.1}$$

$$A_{-t} \prec A_{-n} \quad , \quad A_n \prec A_t, \tag{A.2}$$

$$A_x \sim A'_x, \tag{A.3}$$

where $-n \leq y < z \leq n$, $n < t$, $-n < x < n$.
– $\preceq^*$ *agrees strongly with* $\preceq$ *up to n* iff $\preceq^*$ agrees with the transitive closure of the relation determined by Eqs. A.1, A.2, A.3 and

$$A_{-n} \sim A'_{-n} \quad , \quad A_n \sim A'_n. \tag{A.4}$$

– $\preceq^*$ *agrees with* $\preceq$ *up to n* iff agreement is strong in case $|[n]| > 1$ and agreement is weak in case $|[n]| = 1$.

Lemma 5 *Let $\leq$ be a basic order over W, $\preceq$ its canonical extension and $\preceq^*$another extension which agrees weakly with $\preceq$ up to n. Then for $w \in [n+1]$ and $v \in [t]$ where $t > n + 1$, we have $|F(\preceq^*, v)| > |F(\preceq, w)|$.*

Proof Let $\preceq^*, \preceq, \leq, w, v, t$ as assumed. Define $\psi : F(\preceq, w) \to F(\preceq^*, v)$ by

$$\psi(\langle A_x, A_y\rangle) = \begin{cases} \langle A_x, A_y\rangle & \text{if } v \in A_x, v \notin A_y, \\ \langle A_x, (A_y \cup \{w\}) \setminus \{v\}\rangle & \text{if } v \in A_x, v \in A_y, \\ \langle (A_x \cup \{v\}) \setminus \{w\}, A_y\rangle & \text{if } v \notin A_x, v \notin A_y, \\ \langle (A_x \cup \{v\}) \setminus \{w\}, (A_y \cup \{w\}) \setminus \{v\}\rangle & \text{if } v \notin A_x, v \in A_y. \end{cases}$$

We show: ψ is a total, injective and non-surjective function. Denote $\langle B_{x'}, B_{y'}\rangle = \psi(\langle A_x, A_y\rangle)$.

Total function: Let $\langle A_x, A_y\rangle \in F(\preceq, w)$. Thus $w \in A_x \setminus A_y$ and $-(n+1) \leq x < y \leq n+1$ (Lemma 2.1). By definition of ψ, (i) $v \in B_{x'} \setminus B_{y'}$. To establish (ii) $B_{x'} \prec^* B_{y'}$, we show that the *type shift* $\langle x, y\rangle \mapsto \langle x', y'\rangle$ satisfies $x' \in \{x, -(n+2)\}$, $y' \in \{y, n+2\}$, where $x' = -(n+2)$ iff $x = -(n+1)$ and A_x contains only w as $n+1$-world, and $y' = n+2$ iff $y = n+1$ and A_y misses only the $n+1$-world w among the $n+1$-worlds.

Case 1: the identity involves no shift.

Case 2: Note $-(n+1) < y \leq n+1$. Removing the t-world v from A_y $(t > n+1)$, implies no shift. Adding the $n+1$-world w changes the type, iff A_y is an $n+1$-set which misses only the $n+1$-world w, and in this case $y = n+1$ goes to $y' = n+2$.
Case 3: Note $-(n+1) \leq x < n+1$. Adding the t-world v to A_x $(t > n+1)$ implies no shift. Removing the $n+1$-world w changes the type iff A_x is a $-(n+1)$-set which contains only w as $n+1$-world, and in this case $x = -(n+1)$ goes to $x' = -(n+2)$.
Case 4: $-n \leq x < y \leq n$: Combine cases 2 and 3.

This, together with weak agreement up to n, establishes (ii) and thus the totality of ψ.

Injective: For reductio, assume (a) $\langle B_{x'}, B_{y'}\rangle = \psi(\langle A_x, A_y\rangle)$ for some $\langle A_x, A_y\rangle \in F(\preceq, w)$ and (b) there is $(A^*_{x^*}, A^*_{y^*}) \in F(\preceq, w)$, such that (c) $(A^*_{x^*}, A^*_{y^*}) \neq \langle A_x, A_y\rangle$, but (d) $\langle B_{x'}, B_{y'}\rangle = \psi((A^*_{x^*}, A^*_{y^*}))$. Note: either $x' \geq -(n+1)$, in which case there is no type-shift (cf. cases 1-4), hence $x = x' = x^*$; or $x' = -(n+2)$, in which case (cf. cases 2-4) $x = -(n+1) = x^*$. Similarly for y, y^*. Thus $x = x^*$ and $y = y^*$. By (c) either (c1) $A^*_x \neq A_x$ or (c2) $A^*_y \neq A_y$.

(c1). Either (i) $B_{x'} = A_x$ (iff $v \in A_x$) or (ii) $B_{x'} = (A_x \cup \{v\}) \setminus \{w\}$ (iff $v \notin A_x$). Similarly, either (i*) $B_{x'} = A^*_x$ (iff $v \in A^*_x$) or (ii*) $B_{x'} = (A^*_x \cup \{v\}) \setminus \{w\}$ (iff $v \notin A^*_x$) (equivalently $A^*_x = (B_{x'} \cup \{w\}) \setminus \{v\}$). (i,i*) contradicts (c1). (i, ii*) implies $A_x = (A^*_x \cup \{v\}) \setminus \{w\}$ and thus $w \notin A_x$, contradicting (a). (ii, i*) implies $A^*_x = (A_x \cup \{v\}) \setminus \{w\}$ and thus $w \notin A^*_x$, contradicting (b). (ii, ii*) implies $A^*_x = (((A_x \cup \{v\}) \setminus \{w\}) \cup \{w\}) \setminus \{v\} = A_x$, contradicting (c1). Hence $A^*_x = A_x$.

(c2) follows similarly, with the simultaneous replacements: y/x, w/v, v/w. Hence $A^*_y = A_y$, proving injectivitiy.

Non-Surjective: $\{v\} \prec^* \{w\}$ (because $\preceq^*$ extends $\leq$) and $\langle\{v\}, \{w\}\rangle \in F(\preceq^*, v)$. However, that comparison is not reached by ψ. Else there would be $\langle A_x, A_y\rangle$ such that $A_x = \{w\}$ and $A_y = \{v\}$, i.e. $x = -(n+1)$ and $y = -t$ for $t > n+1$. Yet $\langle A_x, A_y\rangle \notin F(\preceq, w)$, since we do not have $\{w\} \prec \{v\}$ (because $\preceq$ also extends $\leq$). □

Lemma 6 *Given the same assumptions as in Lemma 5:* $|L(\preceq^*, v)| \geq |L(\preceq, w)|$.[17]

Proof Let $\preceq^*, \preceq, \leq, w, u, t$ as described. Define $\chi : L(\preceq, w) \to L(\preceq^*, u)$ by

$$\chi(\langle A_y, A_x\rangle) = \begin{cases} \langle A_y, A_x\rangle & \text{if } u \in A_x, u \notin A_y, \\ \langle (A_y \cup \{w\}) \setminus \{u\}, A_x\rangle & \text{if } u \in A_x, u \in A_y, \\ \langle A_y, (A_x \cup \{u\}) \setminus \{w\}\rangle & \text{if } u \notin A_x, u \notin A_y. \end{cases}$$

If $\langle A_y, A_x\rangle \in L(\preceq, w)$ we have $A_y \subseteq A_x$. Thus $u \notin A_x, u \in A_y$ is impossible. We show that χ is a total, injective function. Denote $\langle B_{y'}, B_{x'}\rangle = \chi(\langle A_y, A_x\rangle)$. We establish, by a similar reasoning as in Lemma 5: $y' = y$ and $x' = x$. From Lemma

[17] One could in fact establish >, showing non-surjectivity.

3.2–3: either (a) $z, y < x \leq k < n+1$ and $0 < x$ or (b) $-(n+1) < -k \leq y < x, z$ and $y < 0$.

Case 1: Trivially $y' = y, x' = x$.

Case 2: A_y *gains* a $n+1$-world and *looses* a t-world ($t > n+1$). The gain has an effect iff (i) $y \leq -(n+2)$ or (ii) $y = n+1$. (a) excludes (ii), and (i) and $A_x = A_z \cup A_y$ would imply that $A_z \cap [\leq x] = A_x \cap [\leq x]$, i.e., $z \geq x$, contrary to the assumption $z < x$ in (a). (b) excludes (i) and (ii). The loss has an effect iff (i) $y = -t$ or (ii) $y = t+1$. (a) excludes (ii), and (i) would imply the same contradiction as above, since $-t \leq -(n+2)$. (b) excludes (i) and (ii). Thus there is no type shift.

Case 3: A_x gains a t-world and looses a $n+1$-world. The gain has an effect iff (i) $x \leq -(t+1)$ or (ii) $x = t$. (a) excludes (i) and (ii). (b) excludes (i); and (ii) and $y < z$ imply that there is $v \in A_z$ with $v \in [m]$ for $m < -y$ and $v \in A_x$ (since $x = t > n+1 > m$), contradicting $A_y = A_x \cap A_z$. The loss has an effect only if (i) $x = -(n+1)$ or (ii) $x = n+2$. (a) excludes (i) and (ii). (b) excludes (i) and (ii) would imply a similar contradiction as above. Thus there is no type shift.

We can therefore write $\langle B_y, B_x\rangle = \psi(\langle A_y, A_x\rangle)$. For $\cup$-*flaws*, define $B_{z'}$ as: *Case 1*: $B_{z'} = A_z$. *Case 2*: $B_{z'} = A_z \cup \{u\}$. *Case 3*: $B_{z'} = (A_z \cup \{u\}) \setminus \{w\}$. One shows $z' = z$ (cf. case 2 above, exchanging t and $n+1$). For $\cap$-*flaws* define $B_{z'}$ as : *Case 1*: $B_{z'} = A_z$. *Case 2*: $B_{z'} = (A_z \cup \{w\}) \setminus \{u\}$. *Case 3*: $B_{z'} = A_z \setminus \{u\}$. One shows $z' = z$ (cf. case 2 above). Thus in general $z' = z$.

Total function: If $\langle A_y, A_x\rangle \in L_\cup(\preceq, w)$ has witnesses A_z, v, then $\langle B_y, B_x\rangle \in L_\cup(\preceq^*, u)$ has witnesses B_z, v. I.e., we show: (0) $B_y \prec^* B_x$, (1) $B_z \prec^* B_x$, (2) $u \in B_x \setminus B_y$, (3) $B_x = B_y \cup B_z$ and (4) $v \notin B_x$. (0) and (1) hold since $z, y < x$ for $0 < x < n+1$ (weak agreement up to n). (2) holds by definition of χ. (3) Case 1 is clear. Case 2: $(A_z \cup \{u\}) \cup ((A_y \cup \{w\}) \setminus \{u\}) = A_x$ since $w \in A_x$ (by assumption) and the removal of u from A_y was compensated with the addition of u to A_z. Case 3: $A_y \cup ((A_z \cup \{u\}) \setminus \{w\}) = (A_x \cup \{u\}) \setminus \{w\}$, since the addition of u and removal of w from A_x was reproduced for A_z. (4) $u \in [t]$ for $t > n+1$ is distinct from $v \in [k]$, since $k < n+1 < t$ and thus $v \notin B_x$.

By a dual reasoning: If $\langle A_y, A_x\rangle \in L_\cap(\preceq, w)$ has witnesses A_z, v, then $\langle B_y, B_x\rangle \in L_\cap(\preceq^*, u)$ has witnesses B_z, v.

Injective: For reductio, assume (a) $\langle B_y, B_x\rangle = \psi(\langle A_y, A_x\rangle)$ for $\langle A_y, A_x\rangle \in L(\preceq, w)$ (i.e. $w \in A_x \setminus A_y$) and (b) there is $\langle A^*_{y^*}, A^*_{x^*}\rangle \in L(\preceq, w)$ (i.e. $w \in A^*_{x^*} \setminus A^*_{y^*}$), such that (c) $\langle A^*_{y^*}, A^*_{x^*}\rangle \neq \langle A_y, A_x\rangle$ but (d) $\langle B_y, B_x\rangle = \psi(\langle A^*_{y^*}, A^*_{x^*}\rangle)$. We have $x^* = x$ and $y^* = y$ (c.f. above). By (c) either (c1) $A^*_x \neq A_x$ or (c2) $A^*_y \neq A_y$.

(c1): Either (i) $B_x = A_x$ (iff $u \in A_x$) or (ii) $B_x = (A_x \cup \{u\}) \setminus \{w\}$ (iff $u \notin A_x$). Similarly, either (i*) $B_x = A^*_x$ (iff $u \in A^*_x$) or (ii*) $B_x = (A^*_x \cup \{u\}) \setminus \{w\}$ (iff $u \notin A^*_x$) (equivalently $A^*_x = (B_{x'} \cup \{w\}) \setminus \{u\}$). (i,i*) contradicts (c1). (i,ii*), implies $A_x = (A^*_x \cup \{u\}) \setminus \{w\}$ and thus $w \notin A_x$, contradicting (a). (ii, i*) implies $A^*_x = (A_x \cup \{u\}) \setminus \{w\}$ and thus $w \notin A^*_x$, contradicting (b). (ii,ii*) implies $A^*_x = (((A_x \cup \{u\}) \setminus \{w\}) \cup \{w\}) \setminus \{u\} = A_x$, contradicting (c1). Thus we have a contradiction in each case. Hence $A^*_x = A_x$.

(c2) similarly leads to a contradiction, applying the simultaneous replacements: $y/x, w/u, u/w$. □

A.2 Main Theorem

Definition 10 Let $\preceq^*$ be a relation over $\mathscr{A}$ and $\preceq$ the canonical extension of $\leq$ (see Definition 8). $a(\preceq^*)$ *agrees totally with* $a(\preceq)$ *up to n* iff for all $w \in [\leq n]$:

$$F(\preceq^*, w) = F(\preceq, w) \quad , \quad L(\preceq^*, w) = L(\preceq, w) \quad , \quad R(\preceq^*, w) = R(\preceq, w)$$

$a(\preceq^*)$ *agrees partially with* $a(\preceq)$ *up to n* iff $a(\preceq^*)$ agrees with $a(\preceq)$ up to $n-1$ and for all $w \in [n]$:

$$F(\preceq^*, w) = F(\preceq, w) \quad , \quad L(\preceq^*, w) = L(\preceq, w) \quad , \quad |R(\preceq^*, w)| \geq |R(\preceq, w)|$$

Clearly, total agreement implies partial agreement.

Definition 11 Let $\preceq$ be a weak order over $\mathscr{A}$ and $a(\preceq) = \langle\langle f_1, r_1\rangle, \ldots, \langle f_k, r_k\rangle\rangle$ an accuracy profile for $\preceq$. For $f \in \{f_1, \ldots f_k\}$, the f*-block* of $a(\preceq)$ is the restriction of $a(\preceq)$ to $\langle f_i, r_i\rangle$, where $f_i = f$.

Note that Definition 6 can now be understood as saying that $a(\preceq^*) \geq a(\preceq)$ iff (1) for all f the f-block in $a(\preceq^*)$ and in $a(\preceq)$ are equal (if it exists), or (2) there is a first f such that the f-block in $a(\preceq^*)$ is unequal to that of $a(\preceq)$, being either (a) longer or (b) better, i.e., equally long, but for a first i in the block, $r_i^* > r_i$.

Theorem 13 *Let $\leq$ be a basic order over W (finite) and $\preceq$ its canonical extension. Then for $\preceq^*$ a transitive total extension of $\leq$, $a(\preceq^*) \geq a(\preceq)$ implies that $\preceq^*$ is $\preceq$.*[18]

Proof Assume (I) $\preceq^*$ is a transitive, total extension of $\leq$ to $\mathscr{A}$ and (II) $a(\preceq^*) \geq a(\preceq)$. Thus $a(\preceq^*) = \langle\langle f_1^*, r_1^*\rangle, \ldots, \langle f_k^*, r_k^*\rangle\rangle$ and $a(\preceq) = \langle\langle f_1, r_1\rangle, \ldots, \langle f_k, r_k\rangle\rangle$ satisfy Definition 6. We recursively show profile- and order-agreement up to n, for all n, and thus $\preceq^*$ is $\preceq$ (cf. Definitions 10 and 9).

Base case: Note that for all $v \notin [0]$: $|F(\preceq^*, v)| > 0$ ($\leq$-compatibility). Thus we must have $|F(\preceq^*, w)| = 0$ and $|L(\preceq^*, w)| = 0$ for all $w \in [0]$, else (II) would be violated, i.e., the 0-block would be longer for $\preceq$ (Lemma 2, 3) which contradicts clauses (i)–(iii) of Definition 6. Since the 0-block for $\preceq^*$ cannot be longer than [0], (II) also implies $|R(\preceq^*, w)| \geq |R(\preceq, w)|$ (by (i) and (iii) of Definition 6). Thus $a(\preceq^*)$ *agrees partially* with $a(\preceq)$ up to 0.

– We show that $|F(\preceq^*, w)| = 0$ for $w \in [0]$ implies that for all $n, m > 0$ all n-sets A_n, all $-m$-sets A_{-m} and all 0-sets A_0, A_0' (which exist only if $|[0]| > 1$):

$$A_{-m} \preceq^* A_0 \sim^* A_0' \preceq^* A_n$$

(1) $A_0 \sim^* A_0'$: Let A be a 0-set, (i.e., $|[0]| > 1$). Then there is a witness $v_A \in [0] \setminus A$ and $w \in A \cap [0]$, so that minimising F yields $\{v_A\} \preceq^* A$ in w and

[18] Restricting R (and L) to $\mathscr{A} \setminus \{\emptyset, W\}$, we get the weaker claim: any optimal total transitive extension of $\leq$ is either the canonical $\preceq$ or its irregular alternative. Compare the last two lines in the proof below.

$A \preceq^* \{v_A\}$ in v_A, i.e., $\{v_A\} \sim^* A$. Therefore $A_0 \sim^* \{v_{A_0}\} \sim^* \{v_{A'_0}\} \sim^* A'_0$ ($\preceq$-compatibility).

(2) $A_i \preceq^* A_n$, $i \leq 0$: $[0] \subseteq A_n$, but $[0] \not\subseteq A_i$. Hence there is a 0-world $v \in A_n \setminus A_i$ and minimising F yields $A_i \preceq^* A_n$.

(3) $A_{-m} \preceq^* A_j$, $j \geq 0$: $A_{-m} \cap [0] = \emptyset$ but $A_j \cap [0] \neq \emptyset$. Minimising F yields $A_{-m} \preceq^* A_j$.

- $|L(\preceq^*, w)| = 0$ for $w \in [0]$ implies no new constraints.
- $|R(\preceq^*, w)| \geq |R(\preceq, w)|$ implies $R(\preceq^*, w) = R(\preceq, w)$. $R(\preceq^*, w) \subseteq PR(w)$ (by definition). Additionally, if $(A_x, A_y) \in R(\preceq^*, w)$, then $A_x \prec^* A_y$ and hence not $x = y = 0$ (due to (1) above and because if $|[0]| = 1$ there are no 0-sets). Therefore $R(\preceq^*, w) \subseteq R(\preceq, w)$ (Lemma 4). Thus $|R(\preceq^*, w)| = |R(\preceq, w)|$, and in fact $R(\preceq^*, w) = R(\preceq, w)$. This shows *total profile agreement* up to 0. Therefore all F-undecided $\preceq^*$ (not turned into $\sim^*$) are turned into $\prec^*$ (Lemma 4), i.e.:

$$A_{-m} \prec^* A_0 \sim^* A'_0 \prec^* A_n$$

By transitivity, $\preceq^*$ *agrees strongly* with $\preceq$ up to 0 *if* $|[0]| > 1$ and *weakly if* $|[0]| = 1$.

Induction step: Suppose $a(\preceq^*)$ agrees with $a(\preceq)$ totally up to n (IH1) and $\preceq^*$ agrees with $\preceq$ up to n (IH2) (i.e., strongly if $|[n]| > 1$, and weakly if $|[n]| = 1$). We lift this to $n + 1$, by first using in addition to (IH1) only weak order agreement, denoted (IH2*).

By (IH1), $X(\preceq, w) = X(\preceq^*, w)$ for $w \in [\leq n]$ and $X = F, L, R$. We first show that for $w \in [n + 1]$ and $v \in [> n + 1]$:

(1) $|F(\preceq, w)| \leq |F(\preceq^*, w)|$,
(2) $|L(\preceq, w)| \leq |L(\preceq^*, w)|$,
(3) $|F(\preceq, w)| < |F(\preceq^*, v)|$,
(4) $|L(\preceq, w)| \leq |L(\preceq^*, v)|$.

(1): It suffices to show $F(\preceq, w) \subseteq F(\preceq^*, w)$: Let $(A_x, A_y) \in F(\preceq, w)$ for $w \in [n + 1]$. Then $(A_x, A_y) \in PF(w)$ and $-(n + 1) \leq x < y \leq n + 1$ (Lemma 2). Thus $A_x \prec^* A_y$ (IH2*), i.e., $(A_x, A_y) \in F(\preceq^*, w)$. (2): It suffices to show $L_\cup(\preceq, w) \subseteq L_\cup(\preceq^*, w)$, $L_\cap(\preceq, w) \subseteq L_\cap(\preceq^*, w)$: Let $(A_y, A_x) \in L_\cup(\preceq, w)$ for $w \in [n + 1]$. Then $(A_y, A_x) \in R(\preceq, [m])$, where $0 \leq m < x \leq k < n$ and $y < x$ (Lemma 3). IH2* implies $A_y \prec^* A_x$. Thus $(A_y, A_x) \in L_\cup(\preceq^*, w)$ (using the same witnesses). An analogous reasoning applies to for $L_\cap$. (3) follows from Lemma 5, IH2* and $\preceq$-compatibility. (4) follows from Lemma 6 and IH2*.

Thus for $w \in [n + 1]$ and $v \in [> n + 1]$:

- $|F(\preceq^*, w)| + |L(\preceq^*, w)| \geq |F(\preceq, w)| + |L(\preceq, w)|$,
- $|F(\preceq^*, v)| + |L(\preceq^*, v)| > |F(\preceq, w)| + |L(\preceq, w)|$.

Hence: we must have $|F(\preceq^*, w)| + |L(\preceq^*, w)| = |F(\preceq, w)| + |L(\preceq, w)|$, else the $(|F(\preceq, w)| + |L(\preceq, w)|)$-block would be longer for $\preceq$ than for $\preceq^*$, contradicting Assumption (II) by Definition 6. (1) and (2) above then imply $|F(\preceq^*, w)| = |F(\preceq, w)|$, $|L(\preceq^*, w)| = |L(\preceq, w)|$ and in fact $F(\preceq^*, w) = F(\preceq, w)$, $L(\preceq^*, w) = L(\preceq, w)$ and thus $|R(\preceq^*, w)| \geq |R(\preceq, w)|$. This shows *partial profile agreement* up to $n + 1$ (using IH1).

By IH2* and transitivity, Eq. A.1 extends from n to $n+1$ (set $t = n+1$ in Eq. A.1). It thus suffices to establish Eqs. A.2, A.3 for $n+1$ and, if $|[n+1]| > 1$ (14) for $n+1$.

– We show that $F(\preceq^*, w) = F(\preceq, w)$ implies:

$$A_{-k} \preceq^* A'_{-(n+1)} \sim^* A_{-(n+1)} \preceq^* \dot{A}_{-(n+1)} \quad , \quad \dot{A}_{n+1} \preceq^* A_{n+1} \sim^* A'_{n+1} \preceq^* A_k, \tag{A.5}$$

where $k > n+1$, $A'_{-(n+1)}, A_{-(n+1)}$ are non-full, $\dot{A}_{-(n+1)}$ is full, $\dot{A}_{n+1}$ is void, and in the case $|[n+1]| > 1$ the sets $A'_{-(n+1)}, A_{-(n+1)}$ are non-full and A'_{n+1}, A_{n+1} are non-void, whereas in the case $|[n+1]| = 1$, A_x and A'_x for $x \in \{-(n+1), n+1\}$ with the previously required specifications don't exist and can be discarded. The reasoning for the claimed non-existence is as follows: Assume $|[n+1]| = 1$ and consider the case $x = -(n+1)$. An $-(n+1)$-set, contains as smallest world an $n+1$ world (by definition). But since there is only one such world (by assumption), such a set must be full. Thus it can only be of the type $\dot{A}_{-(n+1)}$. Hence only $A_{-k} \preceq \dot{A}_{-(n+1)}$ needs to be established (for $k > n+1$). The reasoning is similar for $x = n+1$.

Additionally, we also show:

$$A_{-n} \preceq^* A^*_{-n} \sim^* A'_{-n} \preceq^* \dot{A}_{-n} \quad , \quad A_n \preceq^* A^*_n \sim^* A'_n \preceq^* \dot{A}_n, \tag{A.6}$$

where now A_{-n}, A_n are $(n+1)$-empty, $A^*_{-n}, A^*_n, A'_{-n}, A'_n$ are not $(n+1)$-empty and not $(n+1)$-full and $\dot{A}_{-n}, \dot{A}_n$ are $n+1$-full. Here a set A is $(n+1)$-*empty* iff $A \cap [n+1] = \emptyset$ and $(n+1)$-*full* iff $[n+1] \subseteq A$.

The following establishes Eq. A.5: $A_{n+1} \sim^* A'_{n+1}$ is shown as (1) in the base case (replace $n+1/0$). $A_{n+1} \preceq^* A_k$, $A_{-k} \preceq^* A_{-(n+1)}$ is shown as (2) and (3) in the base case (replace $n+1/i, k/n, k/m$). $\dot{A}_{n+1} \preceq^* A_{n+1}$ is proved as (3) in the base case. $A_{-(n+1)} \preceq^* \dot{A}_{-(n+1)}$ is proved as (2) in the base case. If $|[n+1]| = 1$: $A_{-k} \preceq^* \dot{A}_{-(n+1)}$, $\dot{A}_{n+1} \preceq^* A_k$ are also obtained as in the base case (2) and (3).

The following establishes Eq. A.6: Let $|[n]| = 1$ (else there is nothing to prove by strong agreement IH2). $A^*_{-n} \sim^* A'_{-n}$, $A^*_n \sim^* A'_n$ follows as (1) in the base case. $A'_{-n} \preceq^* \dot{A}_{-n}$, $A'_n \preceq^* \dot{A}_n$ follow as (2) in the base case. $A_{-n} \preceq^* A^*_{-n}$, $A_n \preceq^* A^*_n$ follow as (3) in the base case.

– We show that $L(\preceq^*, w) = L(\preceq, w)$ for all $w \in [n+1]$ implies: for all $-n$ sets (full or not) and all n-sets (void or not):

$$A_{-n} \sim^* A'_{-n} \quad , \quad A'_n \sim^* A_n. \tag{A.7}$$

And if $|[n+1]| > 1$ then for all $-(n+1)$ sets (full or not) and all $n+1$-sets (void or not):

$$A_{-(n+1)} \sim^* A'_{-(n+1)} \quad , \quad A'_{n+1} \sim^* A_{n+1}. \tag{A.8}$$

Let us establish Eq. A.7: Consider $|[n]| = 1$ (else Eq. A.7 holds by strong agreement IH2). Assume $n > 0$ (else there is nothing to prove).

(1) $A_{-n} \sim^* A'_{-n}$: We first show that $A_{-n} \sim^* \dot{A}_{-n}$ in case A_{-n} is $n+1$-empty and $\dot{A}_{-n}$ is $n+1$-full.

Let $\dot{A}$ be a $-n$-set which is $n+1$-full. Define $A := \dot{A} \setminus [n+1]$ and $A' := [n+1]$. Thus A is a $-n$-set which is $n+1$-empty and A' is a $-(n+1)$-set. Then $A \prec^* \dot{A}$ would be a $\cup$-flaw in any $w \in [n+1]$. Indeed: (1)

$A' \prec^* \dot{A}$ by IH2*, (2) $w \in \dot{A} \setminus A$ and (3) $\dot{A} = A \cup A'$ and for $v \in [n-1]$ (non-empty, since $n > 0$), we have (4) $v \notin \dot{A}$. But $(A, \dot{A}) \notin L(\preceq, w)$, hence $(A, \dot{A}) \notin L(\preceq^*, w)$. Thus $\dot{A} \preceq^* A$. Since $A \preceq^* \dot{A}$, we obtain $A \sim^* \dot{A}$.

Let now $A', \dot{A}$ be two $-n$ sets, such that $\dot{A}$ is $n+1$-full and A' is not $n+1$-empty and not $n+1$ full. Then we can find an $n+1$-empty witness A (cf. above) such that $A \sim^* \dot{A}$. But $A \preceq^* A' \preceq^* \dot{A}$ (by F-minimisation). Thus $A \preceq^* A' \preceq^* \dot{A} \sim^* A$. Squeezing yields $A' \sim^* \dot{A}$.

Similarly, let A, A' be two $-n$-sets, such that A is $n+1$-empty and A' is not $n+1$-empty and not $n+1$-full. Again, we can find a witness $\dot{A}$ $-n$-set that is $n+1$-full, ($\dot{A} = A \cup [n+1]$), such that $\cup$-flaw-avoidance yields $A \sim^* \dot{A}$ (choose witness $A' = [n+1]$). Hence $A \sim^* \dot{A} \sim^* A'$, i.e., $A \sim^* A'$ (transitivity). Thus equivalence extends to all $-n$-sets.

(2) $A_n \sim^* A'_n$ follows for all n-sets by a similar reasoning, using $\cap$-flaw avoidance.

We have thus extended Eq. A.3 from n to $n+1$ (whether $|[n]| = 1$ or > 1).

Let us now establish Eq. A.8, provided $|[n+1]| > 1$:

(1) $A_{n+1} \sim A'_{n+1}$: Let $\dot{A}$ be a void $n+1$-set and let C_1, C_2 partition $[n+1]$ (assuming $|[n+1]| > 1$). Then $A = \dot{A} \cup C_1$ as $A' = \dot{A} \cup C_2$ are non-void $n+1$-sets. Let $w \in C_1$. Then $\dot{A} \prec^* A$ would be a $\cap$-flaw in w, for witnesses A' and $v \in [n]$ (using $A \sim^* A'$ by Eq. A.5). But $(\dot{A}, A) \notin L(\preceq, w)$. Thus $A \preceq^* \dot{A}$. Since $\dot{A} \preceq^* A$, we obtain $A \sim^* \dot{A}$. By the same reasoning $B \sim^* \dot{B}$ where $\dot{B}$ is another void $n+1$-set. But $B \sim^* A$, and hence equivalence extends to all $n+1$-sets.

(2) $A_{-(n+1)} \sim^* A_{-(n+1)}$ follows by the same reasoning, using $\cup$-flaws.

Thus we have also obtained Eq. A.4 for $n+1$ (provided $|[n+1]| > 1$).

– We show $R(\preceq^*, w) = R(\preceq, w)$. We already have $|R(\preceq^*, w)| \geq |R(\preceq, w)|$. We show $R(\preceq^*, w) \subseteq R(\preceq, w)$. Let $(A_y, A_x) \in R(\preceq^*, w)$, then $(A_y, A_x) \in PR(w)$ and $A_y \prec^* A_x$. Hence not $x = y$ for $-n \leq x, y \leq n$ (by Eq. A.3). We also obtain not $x = y$ for $x \in \{-(n+1), n+1\}$, by (14) established above if $|[n+1]| > 1$; and if $|[n+1]| = 1$ the sets A_y, A_x are both void or both full and $A_y \prec^* A_x$ cannot be a right comparison. Thus: not $x = y$ for $-(n+1) \leq x, y \leq n+1$. Similarly not $-(n+1) \leq x < y \leq (n+1)$ (by IH2, i.e., Eqs. A.1 and A.2 for n). But this implies $(A_y, A_x) \in R(\preceq, w)$ (Lemma 4). Hence $|R(\preceq^*, w)| = |R(\preceq, w)|$ and thus $R(\preceq^*, w) = R(\preceq, w)$. This proves *total profile agreement* up to $n+1$.

$R(\preceq^*, w) = R(\preceq, w)$ then implies that all $\preceq^*$ obtained by F-minimisation in $[n+1]$ and left undecided (not turned into $\sim^*$) are now turned into $\prec^*$. In particular

$$A_{-t} \prec^* A_{-(n+1)} \quad A_{n+1} \prec^* A_t \quad \text{for } t > n+1$$

This is Eq. A.2 for $n+1$. Therefore $\preceq^*$ *agrees with* $\preceq$ up to $n+1$, where *agreement is strong if* $|[n+1]| > 1$ and *weak if* $|[n+1]| = 1$.

The recursively constructed $\preceq^*$ over $\mathscr{A} \setminus \{\emptyset, W\}$ satisfies

$$A_x \prec^* A_y \text{ iff } x < y$$

In the last step n (for n maximal), we also obtain by R-maximisation $\emptyset \prec^* A_{-n}$ and $A_n \prec^* W$, since these cannot be flaws. The transitive closure of $\preceq^*$ over $\mathscr{A}$ is thus the canonical $\preceq$. $\square$

References

1. Dubois, D. (1984). Steps to a theory of qualitative possibility. In *Proceedings of the 6th international congress on cybernetics and systems* (pp. 147–152). Paris: AFCET Publication.
2. Easwaran, K., & Fitelson, B. (2015). Accuracy, coherence and evidence. In T. Szabo Gendler, & J. Hawthorne (Eds.) *Oxford studies in epistemology* (Vol. 5). Oxford: Oxford University Press.
3. Fitelson, B. (2017). Two approaches to doxastic representation, *presentation* at the Eastern APA, Baltimore (January 2017), available at: http://fitelson.org/spohn_apa_handout.pdf.
4. Fitelson, B., & McCarthy, D. (2014). Toward an epistemic foundation for comparative confidence, presentation, University of Wisconsin-Madison (April 2014), available at: http://fitelson.org/cc_handout.pdf.
5. Halpern, J. (2003). *Reasoning about uncertainty*. Cambridge: MIT Press.
6. James, W. (1896). The will to believe. In *The will to believe and other essays in popular philosophy* (p. 1956). New York: Dover Publications.
7. Joyce, J. (1998). A nonpragmatic vindication of probabilism. *Philosophy of Science*, *65*, 575–603.
8. Joyce, J. (2009). Accuracy and coherence: prospects for an alethic epistemology of partial belief. In F. Huber, & C. Schmidt-Petri (Eds.) *Degrees of Belief*, Synthese library (Vol. 342, pp. 263–297). Berlin: Springer.
9. Kraft, C., Pratt, J., Seidenberg, A. (1959). Intuitive probability on finite sets. *Annals of Mathematical Statistics*, *30*(2), 408–419.
10. Pettigrew, R. (2016). *Accuracy and the laws of credence*. Oxford: Oxford University Press.
11. Raidl, E. (ms). Comparative belief and T-norms: a model based on ordinal valued ranking functions, Manuscript (March 2019). Earlier version available at https://www.researchgate.net/publication/313842085_Representation_Theory_for_Ranking_Functions.
12. Scott, D. (1964). Measurement structures and linear inequalities. *Journal of Mathematical Psychology*, *1*, 233–247.
13. Shafer, G. (1976). *A mathematical theory of evidence*. Princeton: Princeton University Press.
14. Spohn, W. (1988). Ordinal conditional functions. A dynamic theory of epistemic states. In W.L. Harper, & B. Skyrms (Eds.) *Causation in decision, belief change, and statistics* (Vol. 2, pp. 105–134). Dordrecht: Kluwer.
15. Spohn, W. (1990). A general non-probabilistic theory of inductive reasoning. In R.D. Shachter, T.S. Levitt, J. Lemmer, L.N. Kanal (Eds.) *Uncertainty in artificial intelligence 4* (pp. 149–158). Amsterdam: Elsevier.
16. Spohn, W. (2012). *The laws of belief: ranking theory and its philosophical applications*. Oxford: Oxford University Press.
17. Wong, S.K.M., Yao, Y.Y., Bollmann, P., Bürger, H.C. (1991). Axiomatization of qualitative belief structure. *IEEE Transactions on System, Man, and Cybernetics*, *21*, 726–734.