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On the pitchfork bifurcation for the Chafee–Infante equation with additive noise

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Abstract

We investigate pitchfork bifurcations for a stochastic reaction diffusion equation perturbed by an infinite-dimensional Wiener process. It is well-known that the random attractor is a singleton, independently of the value of the bifurcation parameter; this phenomenon is often referred to as the "destruction" of the bifurcation by the noise. Analogous to the results of Callaway et al. (AIHP Prob Stat 53:1548–1574, 2017) for a 1D stochastic ODE, we show that some remnant of the bifurcation persists for this SPDE model in the form of a positive finite-time Lyapunov exponent. Additionally, we prove finite-time expansion of volume with increasing dimension as the bifurcation parameter crosses further eigenvalues of the Laplacian.

Keywords Stochastic partial differential equations · Singleton attractors · Lyapunov exponents · Stochastic bifurcations

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1 Introduction

We study bifurcations for the reaction diffusion equation known as the *Chafee–Infante equation*, under perturbations by infinite-dimensional additive noise. The Chafee–Infante equation without noise, in close relation to other reaction diffusion systems with cubic nonlinearity such as the Allen-Cahn or the Nagumo equation, is a well-studied parabolic partial differential equation (PDE) with global attractor whose bifurcation behaviour is fully understood as a cascade of pitchfork bifurcations. We employ the viewpoint of random dynamical systems theory (see e.g. [3]) to detect a similar bifurcation pattern for the noisy case.

In more detail, we consider the following stochastic partial differential equation (SPDE) with Dirichlet boundary conditions on a bounded domain $\mathcal{O} \subset \mathbb{R}$, say $\mathcal{O} = [0, L]$,

$$\begin{cases} \mathrm{d}u = (\Delta u + \alpha u - u^3)\,\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t, \\ u(0) = u_0 \in H, \quad u|_{\partial\mathcal{O}} = 0, \end{cases} \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is the deterministic bifurcation parameter, $H := L^2(\mathcal{O})$ is the state space and $(W_t)_{t\in\mathbb{R}}$ denotes a two-sided H-cylindrical Wiener process, with covariance operator Q as specified in Sect. 2.

In the case without noise (see e.g. [29]), the Chafee–Infante equation is well-posed, yielding a semigroup $S(t)$ on H for which one can find a global attractor A, i.e., A is a compact invariant subset of H ($S(t)A = A$ for all $t \geq 0$) which attracts the orbits of all bounded subsets of H. Starting with the homogeneous zero solution for $\alpha < \lambda_1$, where λ_i denote the eigenvalues of the Laplacian $-\Delta$, an unstable direction and two new stationary solutions are added whenever α passes λ_n; the attractor then consists of these stationary points together with their respective unstable manifolds.

In the case with noise, the attractor becomes a random object, a so-called *random attractor* $A(\omega)$, whose position in the state space depends on the noise realization. It has been shown in [9, Sect. 6] that, if Q is bounded and invertible with bounded inverse (e.g. space-time white-noise), for any value of α, the random attractor $A(\omega)$ of (1.1) consists of a single point almost surely, i.e., there exists a random variable $a = a_\alpha : \Omega \to H$ with

$$\varphi(t, \omega, a(\omega)) = a(\theta_t\omega) \quad \text{for every } t \geq 0 \quad \mathbb{P} - a.s.,$$

such that $A(\omega) = \{a(\omega)\}$. This phenomenon is often called *synchronization by noise* and has been studied thoroughly in recent years for finite-dimensional SODEs [15, 20, 27] and SPDEs [7, 9, 21]. The proof of synchronization for (1.1) in [9, Sect. 6], adapting ideas from Crauel and Flandoli [15], uses the correspondence between stationary measures and attractors and a monotonicity argument. These ideas on synchronization carry over in our setting, where the covariance operator Q of the noise will be given by a negative fractional power of the Laplacian, in order to ensure a suitable regularity of the solution according to Sect. 2. In a similar spirit, results on synchronization for (1.1) with Neumann-boundary conditions (and $\alpha = 0$) have been derived in [24] provided that the noise is $H^{2s}(\mathcal{O})$-valued for $s > \frac{1}{4}$. Finally, for (1.1) driven by space-time white-noise in two and three dimensions, a similar synchronzation phenomenon has been observed in [22].

Synchronization can be interpreted in the sense that the noise "destroys" the pitchfork bifurcation, since the attractor $A(\omega) = \{a(\omega)\}$ remains a single point for all $\alpha \in \mathbb{R}$. This interpretation was embraced by Crauel and Flandoli [15] for the stochastically-forced pitchfork normal form on $\mathbb{R}$ given by

$$\mathrm{d}x = (\alpha x - x^3)\mathrm{d}t + \mathrm{d}W_t \,. \tag{1.2}$$

There, they showed that the random attractor $A(\omega) \subset \mathbb{R}$ consists of a point for all values of α, hence all trajectories synchronize to a random equilibrium. It follows that at the level of the asymptotic dynamics, the bifurcation at $\alpha = 0$ is "destroyed" and no switch to local instability occurs.

Callaway et al. [8] challenged this point of view by measuring local stability for trajectories of the one-dimensional SODE model (1.2) on finite time scales, using *finite-time Lyapunov exponents* (FTLEs). They proved that, whereas all FTLEs are negative for $\alpha < 0$, there is always a positive probability to observe positive FTLEs for $\alpha > 0$; *this change of in FTLEs corresponds with a transition from uniform to non-uniform attractivity of the attractor and a loss of (uniform) hyperbolicity.*

The authors also showed that there is no uniformly continuous topological conjugacy between the dynamics for negative and positive α. A similar result was proved for stochastic Hopf bifurcations with additive noise in [18].

An alternative but strongly related problem is to fix the parameter $\alpha > 0$ in (1.2) (or $\alpha > \lambda_1$ in (1.1) respectively), introduce a coefficient $\epsilon > 0$ in front of the noise term $\mathrm{d}W_t$ in (1.2) [or (1.1) respectively], and determine for $\epsilon \ll 1$ the extent to which the random motion resembles that of the deterministic ($\epsilon = 0$) system. For the latter, the bulk of initial conditions relax to one of two stable solutions, while when $\epsilon > 0$ these solutions are *metastable*, with typical random trajectories transitioning from the vicinity of one to the other, reflecting the fact that the system admits a unique stationary measure. Large deviations estimates provide a way of estimating the typical timescale of these transitions, thereby giving another perspective on the "persistence" of the bifurcation in the presence of noise. For the system considered in this paper, this metastability picture was studied in [5, 6].

Statement of results

The purpose of this manuscript is to extend the mentioned ideas from Callaway et al. [8] and Doan et al. [18] to the infinite-dimensional setting, demonstrating similar bifurcation behavior for the random dynamical system induced by the SPDE (1.1). Let again $0 < \lambda_1 < \lambda_2 < \ldots$ denote the eigenvalues of $-\Delta$ on $\mathcal{O} = [0, L]$ with Dirichlet boundary conditions.

Given an initial condition $u_0 \in H$ and a sample $\omega \in \Omega$, the FTLE at time t along the trajectory $\varphi^t_\omega(u)$ is given by

$$\Lambda_1(t;\omega,u_0) = \frac{1}{t}\ln \|D_{u_0}\varphi^t_\omega\|_H \tag{1.3}$$

where $D_{u_0}\varphi^t_\omega$ is the linear operator on H obtained by Fréchet differentiating the cocycle φ^t_ω at u_0, and $\|\cdot\|_H$ denotes the norm on H.

In line with Callaway et al. [8] and in concert with our goal of assessing changes in the attractivity of the singleton attractor $a(\omega)$ of (1.1), we are primarily[1] interested in the FTLE

$$\Lambda_1(t;\omega) := \Lambda_1(t;\omega,a(\omega))$$

taken *along the attractor* $a(\omega)$.

Theorem A *The random dynamical system induced by the SPDE* (1.1) *exhibits a bifurcation at* $\alpha = 0$ *in the following sense:*

(a) *Unconditionally, we have that for all* $\alpha \in \mathbb{R}$*,*

$$\Lambda_1(t;\omega) \le \alpha - \lambda_1 \quad \text{with probability 1 for all } t \ge 0.$$

In particular, $\Lambda_1(t;\omega) < 0$ *with probability 1 for all* $\alpha < \lambda_1$.

(b) *For any* $\alpha > \lambda_1$, $0 < \delta \ll \alpha - \lambda_1$ *and* $T > 0$*, there is a positive-probability event* $\mathcal{A} \subset \Omega$ *such that*

$$\Lambda_1(t;\omega) \ge \alpha - \lambda_1 - \delta > 0 \quad \text{for all } \omega \in \mathcal{A}, t \in [0,T].$$

Statements (a) and (b) together imply loss of uniform hyperbolicity of the system at $\alpha = \lambda_1$ corresponding with the deterministic pitchfork bifurcation of the PDE.

Our second result concerns the cascade of deterministic bifurcations when α crosses more and more eigenvalues λ_k. We are able to capture this by finite-time expansion of k-dimensional volumes, as measured by the quantities

$$V_k(t;\omega,u_0) := \frac{1}{t}\log \| \wedge^k D_{u_0}\varphi^t_\omega\|_{\wedge^k H}, \quad V_k(t;\omega) := V_k(t;\omega,a(\omega)),$$

$k \ge 1$. Here, $\wedge^k$ denotes the k-fold wedge product of a linear operator; see Sect. 3.2.1 for details. Equivalently, V_k can be characterized by volume growth: for a bounded linear operator A on H, we have the identity

$$\| \wedge^k A\|_{\wedge^k H} = \max\{|\det(A|_E)| : E \subset H, \dim E = k\}$$

relating $\|\wedge^k A\|_{\wedge^k H}$ with the maximal volume growth A exhibits along a k-dimensional subspace of H. Here, $\det(A|_E)$ is the determinant of $A|_E : E \to A(E)$ regarded as a linear operator between E and $A(E)$, and we follow the convention $\det(A|_E) = 0$ if $\dim A(E) < \dim E$.

[1] See Remark 1.3 below for a discussion of FTLEs at more general initial data.

Theorem B *Let $k \geq 1$ be arbitrary.*

(a) *Unconditionally, we have that for all $\alpha \in \mathbb{R}$,*

$$V_k(t;\omega) \leq \sum_{i=1}^{k}(\alpha - \lambda_i) \quad \text{with probability 1 for all } t \geq 0.$$

(b) *For any $\alpha > \frac{1}{k}\sum_{i=1}^{k}\lambda_i$, $0 < \delta \ll \sum_{i=1}^{k}(\alpha - \lambda_i)$ and $T > 0$, there is a positive probability event $\mathcal{A} \subset \Omega$ such that*

$$V_k(t;\omega) \geq \sum_{i=1}^{k}(\alpha - \lambda_i) - \delta \quad \text{for all } \omega \in \mathcal{A}, t \in [0, T].$$

Remarks and comments on the proof

Estimation of FTLE for the stochastic ODE (1.2) is straightforward and proceeds roughly as follows. Analogously to our setting, FTLE are unconditionally $\leq \alpha$ for any value of $\alpha \in \mathbb{R}$. When $\alpha > 0$, the origin 0 is linearly unstable, yet one can find a positive probability event that the point attractor remains close to 0 on arbitrarily long timescales, accumulating a positive FTLE $\approx \alpha$.

Our proof in the SPDE setting (1.1) follows roughly the same lines: the events $\mathcal{A}$ we construct in Theorems A and B steer the random point attractor $a_\alpha(\omega)$ towards 0, where the linearization is "close" to the shifted heat semigroup $e^{(\alpha+\Delta)t}$ with singular values $e^{t(\alpha-\lambda_i)}$, $i \geq 1$. However, making this rigorous entails several challenges not present in the finite-dimensional case, e.g.:

(a) Due to the nonlinear term in (1.1), the linearization is only close to the semigroup $e^{(\alpha+\Delta)t}$ when the point attractor $a_\alpha(\omega)$ is small in $\mathcal{C}(\mathcal{O})$, not just $H = L^2(\mathcal{O})$ (c.f. Proposition 3.1). For details, see Proposition 2.7 and its proof in the Appendix.
(b) Even when the linearization is steered to be close to the semigroup $e^{(\alpha+\Delta)t}$, lower bounds on FTLE do not follow from naive L^2 energy estimates. Instead, we derive a lower bound using a careful invariant cones argument delineated by an appropriate quadratic form Q_δ on $H \times H$. This approach, inspired by techniques for ODEs [26], avoids verifying the abstract criteria developed in [1] in order to establish cone invariance for parabolic evolution operators. The lower bounds on k-dimensional volume growth are obtained by extending this technique to the wedge spaces $\Lambda^k H$. See Sect. 3.2.1 for details.

Remark 1.1 The collection of Lyapunov exponents usually refers to the set of asymptotic exponential growth rates realized by different tangent directions. By the Multiplicative Ergodic Theorem (see, e.g., [28, 30] or the survey [34]), the rates achieved are precisely asymptotic exponential growth rates of singular values, and so a natural way to consider 'lower' *finite-time Lyapunov exponents* is to use finite-time singular values:

$$\Lambda_k(t;\omega,u_0) := \frac{1}{t}\log\sigma_k(D_{u_0}\varphi_\omega^t)$$

where for $k \geq 1$ we write $\sigma_k(A)$ for the k-th singular value of a bounded linear operator A on H. It is straightforward to show that for such a bounded operator A, we have the identity $\| \wedge^k A\|_{\wedge^k H} = \prod_{i=1}^k \sigma_i(A)$, hence

$$V_k(t;\omega) = \sum_{i=1}^{k} \Lambda_i(t;\omega)\,.$$

For $T > 0$, $0 < \delta \ll 1$, Theorem B implies there is a positive probability event $\mathcal{A} \subset \Omega$ so that

$$\alpha - \lambda_k - \delta \leq \Lambda_k(t;\omega) \leq \alpha - \lambda_k + \delta \quad \text{for all } \omega \in \mathcal{A}, t \in [0,T]\,,$$

suggesting as before a bifurcation occurring as α moves past λ_k.

Remark 1.2 Our main results concern estimates of *finite-time* Lyapunov exponents on positive probability events, i.e., fixing a time $T > 0$ and estimating $\Lambda_k(t,\omega)$ for $t \in [0,T]$ and for ω drawn from a positive-probability set in Ω. As $t \to \infty$, it follows from the subadditive ergodic theorem and uniqueness of the stationary measure for the Markov process defined by (1.1) that the *asymptotic Lyapunov exponents*

$$\lim_{t\to\infty} \Lambda_k(t;\omega) \tag{1.4}$$

exist for each $k \geq 1$ and are deterministic (independent of ω) with probability 1 (see, e.g., [30]). Since the random attractor of (1.1) consists of a single point with full probability for all values of the bifurcation parameter, it is likely that the asymptotic exponents as in (1.4) are all negative (or at least nonpositive). Indeed, in [15] the asymptotic Lyapunov exponent of the one-dimensional system (1.2) was shown to be negative, using an explicit calculation, and in [18] a quantitative negative upper bound was derived for the two-dimensional extension to Hopf bifurcations in certain parameter regimes. Perhaps unsurprisingly, though, the arguments from these papers do not carry over to the infinite dimensional setting of our work. Providing such an estimate remains an open problem for future work.

Remark 1.3 Our estimates of FTLEs along the random attractor, carried out in Sect. 3, are relatively versatile and make no direct use of assumptions about the nondegeneracy of the covariance (as in [9]), otherwise used to ensure uniqueness of stationary measures or synchronization by noise– see Remark 3.3 for further discussion. Moreover, it is not hard to check that our estimates carry over to trajectories initiated at any sufficiently small initial $u_0 \in H$. However, our proof as-is does not extend to arbitrary initial data: see Sect. 4 at the end of the paper for further discussion along these lines.

Structure of the paper

Section 2 recalls regularity properties of the solution to equation (1.1) and the generation of an associated random dynamical system with random attractor and sample measures corresponding with the unique stationary measure of (1.1). Section 3 is dedicated to proving Theorems A and B, firstly taking care of estimating the top FTLE with the cone technique described above, and secondly bounding the k-volume growth rates via the wedge spaces $\wedge^k H$. In the Appendix we provide the proof of the crucial Proposition 2.7, yielding control of the random equilibrium in $V_\gamma = H_0^{2\gamma}(\mathcal{O})$ for $\gamma \in (\frac{1}{4}, \frac{1}{2})$. For an outlook on future problems along the lines of the results in this paper, see Sect. 4.

2 Preliminaries

We study bifurcations for the following reaction diffusion equation with additive noise. Let $\alpha > 0$, set $H := L^2(\mathcal{O})$ and $V := H_0^1(\mathcal{O})$ for a bounded domain $\mathcal{O} \in \mathbb{R}$, e.g. $\mathcal{O} = [0, L]$, and consider again the SPDE (1.1)

$$\begin{cases} \mathrm{d}u = (\Delta u + \alpha u - u^3)\,\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t, \\ u(0) = u_0 \in H, \quad u|_{\partial\mathcal{O}} = 0. \end{cases}$$

2.1 Background and basic properties

Notations, assumptions, and basic results Throughout, we regard the Laplacian Δ as a closed linear operator on $H = L^2(\mathcal{O})$ with Dirichlet boundary conditions. It is well-known that this generates a compact, analytic C_0-semigroup $(S(t))_{t\geq 0}$ on H and the domain of its fractional powers can be identified with fractional Sobolev spaces [1]. More precisely we introduce the spaces $V_\gamma := (D(-\mathcal{L})^\gamma, \langle\cdot,\cdot\rangle_{V_\gamma})$, where $\langle x, y\rangle_{V_\gamma} = \langle(-\mathcal{L})^\gamma x, (-\mathcal{L})^\gamma y\rangle$ for $x, y \in V_\gamma$. Then we can identify the fractional power spaces with Sobolev spaces [33, Theorem 16.15]

$$V_\gamma = \begin{cases} H^{2\gamma}(\mathcal{O}), & \gamma \in \left[0, \frac{1}{4}\right), \\ H_0^{2\gamma}(\mathcal{O}), & \gamma \in \left(\frac{1}{4}, 1\right] \setminus \left\{\frac{3}{4}\right\}. \end{cases}$$

Note that in particular, $V_{\frac{1}{2}} = H_0^1(\mathcal{O}) = D((-\mathcal{L})^{1/2})$ and that V_γ is continuously embedded in $\mathcal{C}(\mathcal{O})$ for all $\gamma > \frac{1}{4}$ by the Sobolev embedding theorem [33, Theorem 1.36].

In this paper, we will utilize a two-sided H-cylindrical Wiener process $(W_t)_{t\in\mathbb{R}}$ with covariance operator Q (to be specified shortly) on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is well-known that there exists a Hilbert space $\tilde{H}$ such that $H \subset \tilde{H}$ where $(W_t)_{t\in\mathbb{R}}$ is trace-class. Since we are working in a random dynamical systems framework, it is expedient to use the canonical space $\Omega := C_0(\mathbb{R}, \tilde{H})$ with the compact-open toplogy, $\mathcal{F} = \mathrm{Bor}(\Omega)$, and Wiener measure $\mathbb{P}$, so that $\mathbb{P}$-typical $\omega \in \Omega$ correspond to two-sided Brownian paths $\omega : \mathbb{R} \to \tilde{H}$ with $\omega(0) = 0$. We abuse notation somewhat and will write $W_t = W_t(\omega) = \omega(t)$ in the following. The space $(\Omega, \mathcal{F}, \mathbb{P})$ will be equipped with the two-sided filtration $(\mathcal{F}_s^t)_{s<t}, \mathcal{F}_s^t := \omega(W_t - W_s)$, as well as the time-shift

$\theta_t : \Omega \circlearrowleft$ given by $\theta^t(\omega)(s) := \omega(t+s)$, so that $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic measure-preserving transformation (see, e.g., [13]).

We will assume in what follows that Q is given by

$$Q := (-\Delta)^{-2\beta}, \tag{2.1}$$

where $\beta \in (0, \frac{1}{4})$. While other choices are possible to make the forthcoming arguments work, this choice is made for ease of exposition. See Remark 3.3 below for further discussion on precisely what is needed regarding the covariance Q.

The following is used to ensure propagation of V_γ-regularity, a crucial ingredient of the proofs in Sect. 3.

Lemma 2.1 *Let $\gamma \in (\frac{1}{4}, \frac{1}{4}+\beta)$. With Q as above, there exists an almost-sure, $\mathcal{F}_0^t$-adapted modification $(z_t)_{t\geq 0}$ of the stochastic convolution*

$$\int_0^t S(t-s)\sqrt{Q}dW_s \tag{2.2}$$

for which $t \mapsto z_t$ is continuous with values in V_γ for $t \geq 0$.

Proof sketch This follows from the same argument as that of Chueshov and Scheutzow [13, Proposition 3.1], using only the assumption that

$$\mathrm{tr}_H(Q(-\mathcal{L})^{2\gamma-1+\varepsilon}) < \infty, \tag{2.3}$$

for some $\varepsilon > 0$, which is equivalent to the upper bound $\gamma < \beta + \frac{1}{4}$. Here, tr_H refers to the standard trace of an operator on H. □

Well-posedness, C^1 semiflow and RDS formulation

Let $\gamma \in (\frac{1}{4}, \frac{1}{4}+\beta)$ be fixed for now. By standard techniques, it now holds (see, e.g., [16]) that

(1) there exists $\mathbb{P}$-a.s. a unique mild solution of (1.1)

$$u \in L^2(\Omega \times (0,T); V_\gamma) \cap L^2(\Omega; C([0,T]; H))$$

for all $T > 0$; and

(2) the first variation equation along the trajectory (u_t), given by

$$\mathrm{d}v = (\Delta v + \alpha v - 3u^2 v)\mathrm{d}t \tag{2.4}$$

is well-posed $\mathbb{P}$-almost surely and for arbitrary initial data $v \in H$.

Below, we collect various properties that allow to realize mild solutions of (1.1) as the trajectories of a random C^1 semiflow $\varphi_\omega^t : H \to H$. The first three statements are well-known [10] and the Fréchet differentiablity follows from Debussche [17, Lemma 4.4].

Proposition 2.2 *There is a θ^t-invariant subset $\Omega' \subset \Omega$ of full probability*[2] *such that for all $\omega \in \Omega'$ and any $t \geq 0$, there is a (Fréchet differentiable) C^1 semiflow*

$$u_0 \mapsto \varphi(t, \omega, u_0) =: \varphi^t_\omega(u_0)$$

on $H = L^2(\mathcal{O})$ with the following properties for each $\omega \in \Omega'$:

(a) *For all $T > 0$ and fixed initial $u_0 \in H$, the mapping $\Omega \times [0, T] \mapsto H$ given by $(\omega, t) \mapsto \varphi^t_\omega(u_0)$ is the (unique) pathwise mild solution to (1.1).*
(b) *The semiflow φ satisfies the* cocycle property*: for $s, t > 0$ we have*

$$\varphi^{t+s}_\omega = \varphi^t_{\theta^s\omega} \circ \varphi^s_\omega$$

(c) *For any $u \in H$ and $s, t \in \mathbb{R}$, $s < t$, we have that $\varphi^{t-s}_{\theta^s\omega}(u)$ is $\mathcal{F}^t_s$-measurable as an H-valued random variable.*
(d) *For all $T > 0$ and fixed initial conditions $u_0, v_0 \in H$, the mapping $\Omega \times [0, T] \mapsto H$ given by $(\omega, t) \mapsto D_{u_0}\varphi^t_\omega(v_0)$ is the unique solution to the first variation Eq. (2.4).*

Here, given a C^1 Fréchet-differentiable mapping $\psi : H \to H$, we write $D_u\psi \in L(H)$ for the derivative of ψ evaluated at $u \in H$.

Markov process formulation

For fixed initial $u_0 \in H$ and for $\omega \in \Omega$, we will write $(u_t)_{t\geq 0}$ for the random process in H defined by $u_t := \varphi^t_\omega(u_0)$. We note two important properties of this process.

Lemma 2.3 *The process (u_t) is an $\mathcal{F}^t_0$-adapted, Feller Markov process.*

Proof That (u_t) is $\mathcal{F}^t_0$-adapted follows from its definition and Proposition 2.2(c). The fact that it is a Feller Markov process follows from continuity of $u_0 \mapsto \varphi^t_\omega(u_0)$ for almost all $\omega \in \Omega'$. □

Furthermore, we obtain the following statement regarding the existence, uniqueness and regularity of the invariant measure associated to (1.1).

Proposition 2.4 *The process (u_t) admits a unique, locally positive stationary measure ρ on H for which $\rho(V_\gamma) = 1$.*

Proof The existence of an invariant measure is an application of Es-Sarhir and Stannat [19, Theorem 4.4], whereas its uniqueness can be inferred from Cerrai [11, Sect. 5]. The hypothesis in [11] can be verified in our setting, due to the structure of the covariance operator of the noise $Q = (-\Delta)^{-2\beta}$ for $\beta \in (0, \frac{1}{4})$. More precisely letting $(\lambda_k)_{k\in\mathbb{N}}$ denote the eigenvalues of the Dirichlet–Laplacian and $(q_k)_{k\in\mathbb{N}}$ stand for the eigenvalues of Q, the condition

$$\sum_{k=1}^{\infty} \frac{q_k^2}{\lambda_k^{1-p}} < \infty$$

[2] In what follows, we will intentionally abuse notation and conflate Ω and Ω'.

holds for some $p \in (0, 1)$. This reduces to

$$\sum_{k=1}^{\infty} \frac{1}{k^{4\beta+2-2\gamma}} < \infty,$$

which is satisfied provided that $\beta > \frac{p}{2} - \frac{1}{4}$. Therefore it is always possible to find $p \in (0, 1)$ satisfying the above inequality. That the resulting invariant measure fully charges V_γ follows by construction relying on the Krylov–Bogoliubov method [19, Sect. 2] and regarding the compact embeddings $V_\gamma \hookrightarrow V_{\gamma'}$ for $\gamma' \leq \gamma$.

Lastly, this measure is locally positive on H by Cerrai [11, Proposition 8.3.6]. Since $\rho(V_\gamma) = 1$, it is straightforward to check that local positivity on V_γ also holds. □

2.2 Properties of attractor and sample measures

Given a Borel-measurable mapping $\psi : H \to H$ and a Borel probability μ on H, define $\psi_*\mu := \mu \circ \psi^{-1}$ to be the pushforward of μ by ψ. We recall a, by now, classical result from the RDS literature, sometimes also coined the *correspondence theorem* between stationary measures ρ and sample measures ρ_ω that are measurable with respect to the past, also named *Markov measures*.

Lemma 2.5 (Theorem 4.2.9 of [25]) *For $\mathbb{P}$ a.e. $\omega \in \Omega'$, the weak* limit*

$$\rho_\omega = \lim_{t\to\infty} (\varphi^t_{\theta^{-t}\omega})_*\rho$$

exists and is $\mathcal{F}_- := \mathcal{F}^0_{-\infty}$-measurable. The sample measures ρ_ω satisfy $(\varphi^t_\omega)_*\rho_\omega = \rho_{\theta^t\omega}$ with probability 1 for all $t \geq 0$ as well as $\mathbb{E}(\rho_\omega) = \rho$.

The sample measures can be associated with a unique attracting random equilibrium for the situation of (1.1), by combining Proposition 2.4, Lemma 2.5, the almost sure order-preservation $\varphi^t_\omega u \leq \varphi^t_\omega v$ for all $u \leq v$ [14, Theorem 5.8] and the characterization of random attractors by Arnold and Chueshov [4].

Proposition 2.6 *Consider the RDS induced by the SPDE (1.1) for any value of $\alpha \in \mathbb{R}$. We have that*

(a) *with probability 1, the sample measure ρ_ω is atomic, i.e., $\rho_\omega = \delta_{a(\omega)}$ where $a = a_\alpha : \Omega \to H$ is an $\mathcal{F}_-$-measurable H-valued random variable such that almost surely*

$$\varphi^t_\omega(a(\omega)) = a(\theta^t(\omega)),$$

(b) *the set valued map $\omega \mapsto \{a(\omega)\}$ is the unique random attractor of the RDS induced by the SPDE (1.1), i.e. for all bounded $D \subset H$ we have almost surely*

$$\lim_{t\to\infty} \sup_{d\in D} \|\varphi^t_{\theta_{-t}\omega}(d) - a(\omega)\|_H = 0.$$

Proof While the covariance operator Q is slightly different in our case, the main arguments from Caraballo et al. [9, Theorem 6.1] carry over. The existence of the attractor follows upon reducing the SPDE (1.1) into a PDE with random coefficients, which utilizes the stochastic convolution from Lemma 2.1. Then, one can easily derive an absorbing set using a-priori estimates and the fact that $F : V_\gamma \to V_\gamma$ is locally Lipschitz. The compactness of the absorbing set follows from the compact embeddings $V_\gamma \hookrightarrow V_{\gamma'}$ for $\gamma' \leq \gamma$. The fact that the random attractor is a singleton is implied by the order-preservation of the system together with the existence of a unique invariant measure, recalling Proposition 2.4. □

For proving the bifurcations in terms of finite-time Lyapunov exponents, we crucially require the following lemma on regularity of the random attractor $a = a_\alpha$.

Proposition 2.7 (a) *With probability 1, we have that* $a(\theta^t\omega) \in V_\gamma$ *for all* $t \in \mathbb{R}$.
(b) *For any* $T > 0$, *there exists a* $\mathcal{F}_{-\infty}^T$*-measurable set* $\mathcal{A} \subset \Omega$ *with* $\mathbb{P}(\mathcal{A}) > 0$ *such that*

$$\|a_\alpha(\theta^s\omega)\|_{V_\gamma} \in (0, \varepsilon) \quad \text{for all } s \in [0, T] \text{ and } \omega \in \mathcal{A}.$$

Roughly, Proposition 2.7 will follow from (i) the fact that $\rho(V_\gamma) = 1$, hence $a_\alpha(\omega) \in V_\gamma$ with probability 1 and (ii) that V_γ-regularity is propagated[3] in time by the evolution Eq. (1.1), and that $\|a(\theta^t\omega)\|_{V_\gamma}$ can be made small by taking $\|a(\omega)\|_{V_\gamma}$ small. Point (i) is immediate from the preceding discussion in this section, while point (ii) is ensured by the existence of the V_γ-continuous modification of the stochastic convolution in Lemma 2.1 together with regularizing properties of analytic semigroups. See Appendix A for further details.

Remark 2.8 We refer the reader to Chueshov [12] and Zhao [35] for further details regarding the regularity of random attractors for stochastic reaction diffusion equations with finite-dimensional additive noise, based on a random dynamical systems approach (without using the correspondence between attractors and invariant measures).

3 Proofs of Theorems A and B

Notation: in the following, the vectors $e_1, e_2, \ldots$ denote the orthonormal Fourier basis of the Laplacian Δ on $\mathcal{O} = [0, L]$ with Dirichlet boundary conditions,

$$e_k(x) = \sqrt{\frac{2}{L}} \sin(2\pi kx/L).$$

We write π_1 for the orthogonal projection onto the span of e_1, and $(\cdot, \cdot) = (\cdot, \cdot)_{L^2}$ for the L^2 inner product. Moreover, we write $\lambda_k = (2\pi k/L)^2$ for the corresponding eigenvalues of $(-\Delta)$, so that $\Delta e_k = -\lambda_k e_k$. Lastly, in this section we will write $\|\cdot\| = \|\cdot\|_H$ for clarity of notation.

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[3] Note that this regularity issue is inherent to the infinite-dimensional problem and does not arise in the previous SODE works [8, Proposition 4.1] and [18, Proposition 5.1].

3.1 Proof of Theorem A: estimate of top FTLE

The following summarizes our estimates of finite-time Lyapunov exponents along the attracting random equilibrium $a_\alpha(\omega)$ and immediately implies Theorem A.

Proposition 3.1

(a) *Unconditionally,*

$$\|D_{a_\alpha(\omega)}\varphi_\omega^t\| \le e^{t(\alpha-\lambda_1)}$$

for all $t > 0$ *and with probability 1.*

(b) *Assume* $\alpha - \lambda_1 > 0$. *For all* $0 < \eta \ll \alpha - \lambda_1$, $T > 0$ *there exists* $\mathcal{A} \in \mathcal{F}_{-\infty}^T$ *with* $\mathbb{P}(\mathcal{A}) > 0$ *such that*

$$\|D_{a_\alpha(\omega)}\varphi_\omega^t\| \ge (1-\eta)e^{t(\alpha-\lambda_1-\eta)}$$

for all $t \in [0, T]$, $\omega \in \mathcal{A}$.

Proof of the upper bound (a)

The bound from above in part (a) is straightforward from the energy estimate derived by taking a time derivative of $\|v_t\|^2$, where $v_t := D_{a(\omega)}\varphi_\omega^t v_0$ for fixed $v_0 \in L^2$. Recalling with (2.4) the variational random PDE

$$\dot{v}_t = (\alpha + \Delta)v_t - 3a_\alpha(\theta_t\omega)^2 v_t\,, \tag{3.1}$$

we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t\|^2 = (v_t, \dot{v}_t) = (v_t, (\alpha+\Delta)v_t) - (v_t, 3a_\alpha(\omega)^2 v_t) \le (v_t, (\alpha+\Delta)v_t)\,.$$

Since $(\alpha + \Delta)$ is self-adjoint with eigenvalues $\alpha - \lambda_i$, $i \ge 1$, we see in view of the min-max principle for closed self-adjoint operators that

$$(w, (\alpha+\Delta)w) \le (\alpha - \lambda_1)\|w\|^2$$

for all $w \in L^2$. In conclusion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\|v_t\| \le \alpha - \lambda_1\,,$$

from which the estimate in (a) follows.

Proof of the lower bound (b)

Our primary tool in this proof will be the family of quadratic forms on $L^2 \times L^2$

$$Q_\delta(v, w) = \delta(\pi_1 v, \pi_1 w)_{L^2} - (\pi_1^\perp v, \pi_1^\perp w)_{L^2}\,, \quad \delta > 0\,.$$

In what follows, we will abuse notation and write $Q_\delta(v) = Q_\delta(v, v)$. Conceptually, quadratic forms such as these specify closed cones

$$\mathcal{C}_\delta = \{v \in L^2 : Q_\delta(v) \geq 0\} = \left\{\|\pi_1^\perp v\|^2 \leq \delta \|\pi_1 v\|^2\right\}\,, \quad \delta > 0\,,$$

roughly parallel to the span of the first eigenmode e_1. It is evident that the shifted heat semigroup $e^{(\alpha+\Delta)t}$ leaves these cones $\mathcal{C}_\delta$ invariant, expanding vectors within them to order $e^{t(\alpha-\lambda_1)}$.

In summary, our lower bound on $\|D_{a(\omega)}\varphi^t_\omega\|$ will come from showing the following: (1) the operator $D_{a(\omega)}\varphi^t_\omega$ is close to the heat semigroup $e^{(\alpha+\Delta)t}$ conditioned on an event $\mathcal{A}$ along which the perturbation factor $3a_\alpha(\theta_t\omega)^2$ in (3.1) is small in an appropriate sense; and (2) we can transfer cone preservation and vector expansion of the heat semigroup to the nearby time-t cocycle $D_{a(\omega)}\varphi^t_\omega$. The following makes this more precise:

Lemma 3.2 *Assume $\alpha > \lambda_1$. Let $T > 0$ and $\varepsilon > 0$ with $\sqrt{\varepsilon} \ll \alpha - \lambda_1$ be fixed, and let $\omega \in \Omega$ be a noise path with the property that the nonlinear term $B^t_\omega := -3a_\alpha(\theta^t\omega)^2$ satisfies*

$$\|B^t_\omega\|_{\mathcal{C}(\mathcal{O})} \leq \varepsilon \tag{3.2}$$

for all $t \in [0, T]$.

Finally, assume $v_0 \in L^2$ satisfies $Q_\delta(v_0) > 0$, where

$$\delta := \sqrt{\varepsilon}.$$

Under these conditions, the time-t solution $v_t = D_{a(\omega)}\varphi^t_\omega(v_0)$ to the first variation Eq. (3.1) satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} Q_\delta(v_t) \geq (\alpha - \lambda_1 - 2\delta)\; Q_\delta(v_t)\,. \tag{3.3}$$

While in Sect. 3.2 we will prove a more general result, we have included the following proof for convenience of the reader.

Proof To start, observe that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} Q_\delta(v_t) = Q_\delta(v_t, \dot{v}_t) = \delta(\pi_1 v_t, \pi_1 \dot{v}_t) - (\pi_1^\perp v_t, \pi_1^\perp \dot{v}_t)\,,$$

hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q_\delta(v_t) \geq \delta(\alpha-\lambda_1)\|\pi_1 v_t\|^2 - (\alpha-\lambda_2)\|\pi_1^\perp v_t\|^2 - (1+\delta)\varepsilon\|v_t\|^2$$
$$\geq \left(\alpha-\lambda_1-\frac{(1+\delta)\varepsilon}{\delta}\right)\delta\|\pi_1 v_t\|^2 - (\alpha-\lambda_2+(1+\delta)\varepsilon)\,\|\pi_1^\perp v_t\|^2\,,$$

having used the estimates

$$(\pi_1 v, (\alpha+\Delta)\pi_1 v) \geq (\alpha-\lambda_1)\|\pi_1 v\|^2\,,$$
$$(\pi_1^\perp v, (\alpha+\Delta)\pi_1^\perp v) \leq (\alpha-\lambda_2)\|\pi_1^\perp v\|^2$$

and decomposing $\|v\|^2 = \|\pi_1 v\|^2 + \|\pi_1^\perp v\|^2$. On assuming that

$$(1+\delta)\varepsilon \leq \alpha-\lambda_1-\frac{(1+\delta)\varepsilon}{\delta},$$

which can be arranged with $\delta = \sqrt{\varepsilon}$ and ε taken sufficiently small, it follows that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q_\delta(v_t) \geq \left(\alpha-\lambda_1-\frac{(1+\delta)\varepsilon}{\delta}\right)Q_\delta(v_t)\,,$$

which implies the desired bound. □

Completing the proof of Proposition 3.1(b) Assume for the moment that Eq. (3.3) holds for all $t \in [0, T]$. Then,

$$\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}t}Q_\delta(v_t)}{Q_\delta(v_t)} \geq \alpha-\lambda_1-2\delta\,,$$

hence,

$$Q_\delta(v_t) \geq Q_\delta(v_0)\exp\{2t\,(\alpha-\lambda_1-2\delta)\}. \tag{3.4}$$

To translate this to a lower bound on norms: Take $M > 1$ and assume $Q_{\delta/M}(v_0) \geq 0$. Then,

$$Q_\delta(v_0) = \delta\|\pi_1 v_0\|^2 - \|\pi_1^\perp v_0\|^2 = \delta\left(1-\frac{1}{M}\right)\|\pi_1 v_0\|^2 + Q_{\delta/M}(v_0)$$
$$\geq \frac{\delta(M-1)}{M}\|\pi_1 v_0\|^2.$$

Using that $v \in \mathcal{C}_\eta$ implies $\|v\|^2 \leq (1+\eta)\|\pi_1 v\|^2$ for $\eta > 0$, we have

$$Q_\delta(v_0) \geq \frac{\delta(M-1)/M}{1+\delta/M}\|v_0\|^2 = \frac{\delta(M-1)}{M+\delta}\|v_0\|^2\,. \tag{3.5}$$

Since $Q_\delta(w) \le \delta \|w\|^2$ unconditionally for all $w \in L^2$, we combine (3.4) and (3.5) to obtain

$$\|v_t\|^2 \ge \frac{M-1}{M+\delta} \exp\{2t\,(\alpha - \lambda_1 - 2\delta)\} \|v_0\|^2 \quad \text{for all} \quad v_0 \in \mathcal{C}_{\delta/M} .$$

We can clearly make the prefactor as close to 1 as desired on taking M sufficiently large.

To complete the proof, it suffices to arrange for (3.3) to hold for all $t \in [0, T]$ on a positive-probability event $\mathcal{A} \in \mathcal{F}^T_{-\infty}$. This follows by Lemma 3.2 as long as $\|B^t_\omega\|_{L^\infty}$ can be made small as in (3.2) for all $t \in [0, T]$. This, in turn, is implied by Proposition 2.7, which allows to make $a_\alpha(\theta^t\omega)$ small in V_γ, hence also in $\mathcal{C}(\mathcal{O})$, for all $t \in [0, T]$. □

Remark 3.3 Note that the only information we used in the preceding argument was that $a_\alpha(\theta^t\omega)$ could be made sufficiently small in V_γ (hence also in $\mathcal{C}(\mathcal{O})$) for all $t \in [0, T]$. Indeed, the argument goes through with $a_\alpha(\theta^t\omega)$ replaced by any trajectory $(u_t)_{t\ge 0}$, as long as $\|u_t\|_{V_\gamma}$ remains sufficiently small for all $t \in [0, T]$. By Lemma A.1 in the Appendix, this can be arranged with positive probability for all fixed initial u_0 with $\|u_0\|_{V_\gamma}$ sufficiently small. Indeed, this alternative version of Proposition 3.1 makes no use at all of the stationary measure or the point attractor, and relies only on the RDS framework of Proposition 2.2, hence only on the assumption (2.3) on the covariance operator Q. These observations apply equally well to the forthcoming arguments of Sect. 3.2 controlling finite-time k-dimensional volumes.

3.2 Proof of Theorem B: bounding volume growth

We now wish to apply similar ideas to estimate volume growth rates. We begin with some background on wedge spaces and norms in Sect. 3.2.1, allowing us to state Proposition 3.4 which summarizes the volume growth bounds required in Theorem B. The proof of Proposition 3.4 via a cones argument will occupy the remainder of the Section.

3.2.1 Background on wedge spaces

Let H be a separable Hilbert space. Given $v_1, \dots, v_k \in H$ we write

$$v_1 \wedge \cdots \wedge v_k$$

for the *wedge product* of $\{v_i\}$ (sometimes referred to as a k-blade), and write $\wedge^k H$ for the closure of the set of finite linear combinations of k-blades under the norm induced by the inner product

$$(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \det\left[(v_i, w_j)_{ij}\right] ,$$

where $(v_i, w_j)_{ij}$ denotes the $k \times k$ matrix with i, j-th entry (v_i, w_j). Let $\|\cdot\|_{\wedge^k H}$ denote the norm induced by this inner product.

We recall the following elementary properties of $\wedge^k H$:

(1) If $e_1, e_2, \ldots \in H$ is a complete orthonormal system, then an orthonormal basis for $\wedge^k H$ is given by the set of k-blades

$$e_{\mathbf{i}} = e_{i_1} \wedge \cdots \wedge e_{i_k},$$

as $\mathbf{i} = (i_1, \ldots, i_k)$ ranges over the set of all distinct indices $i_1 < i_2 < \cdots < i_k$.

(2) A bounded linear operator A on H gives rise to an operator $\wedge^k A$ in $\mathcal{B}(\wedge^k H)$, i.e. the space of bounded linear operators from $\wedge^k H$ into itself, via

$$\wedge^k A(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k.$$

This operator has the property that

$$\begin{aligned} \| \wedge^k A\|_{\wedge^k H} &= \sup\{|\det(A|_E)| : E \subset H, \dim E = k\} \\ &= \sup\left\{\frac{\|Av_1 \wedge \cdots \wedge Av_k\|_{\wedge^k H}}{\|v_1 \wedge \cdots \wedge v_k\|_{\wedge^k H}} : v_1, \ldots, v_k \in H \text{ linearly independent}\right\} \end{aligned} \tag{3.6}$$

where for the purposes of defining det we view $A|_E : E \to A(E)$ as a linear operator of finite-dimensional inner product spaces, and set $\det(A|_E) = 0$ if $\dim A(E) < \dim E$.

Below, we will abuse notation somewhat and write $\| \cdot \| = \| \cdot \|_{\wedge^k H}$.

We are now in position to formulate the estimates needed in the proof of Theorem B.

Proposition 3.4

(a) *Unconditionally, for all* $k \geq 1$ *we have that*

$$\| \wedge^k D_{a(\omega)}\varphi^t_\omega\| \leq e^{t \sum_{i=1}^k (\alpha - \lambda_i)}$$

for all $t > 0$ *with probability 1.*

(b) *Assume* $\alpha - \lambda_r > 0 > \alpha - \lambda_{r+1}$ *for some* $r \geq 1$. *Let* $\eta > 0$ *be sufficiently small,* $T > 0$ *arbitrary. Then, there exists* $\mathcal{A} \in \mathcal{F}^T_{-\infty}$ *with* $\mathbb{P}(\mathcal{A}) > 0$ *such that for all* $\omega \in \mathcal{A}$, $t \in [0, T]$, *we have*

$$\| \wedge^k D_{a(\omega)}\varphi^t_\omega\| \geq (1 - \eta)e^{t \sum_{i=1}^k (\alpha - \lambda_i - \eta)}$$

for each $k \in \{1, \ldots, r\}$.

3.2.2 Proof of Proposition 3.4(a): upper bound on k-dimensional volume growth

In using wedge products to derive upper bounds we follow a long tradition of authors, e.g., Temam [32], Debussche [17]).

To this end, let $k \geq 1$ and fix $\mathbf{v}_0 = v_0^1 \wedge \cdots \wedge v_0^k \in \wedge^k H$, writing

$$\mathbf{v}_t = v_t^1 \wedge \cdots \wedge v_t^k = \wedge^k D_{a(\omega)} \varphi_\omega^t (\mathbf{v}_0).$$

Recalling that $\partial_t v_t^i = (\alpha + \Delta + B_\omega^t) v_t^i$ for all $i = 1, \ldots, k$, we compute

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{v}_t\|^2 = \sum_{j=1}^{k} (\mathbf{v}_t, v_t^1 \wedge \cdots \wedge (\alpha + \Delta + B_\omega^t) v_t^j \wedge \cdots \wedge v_t^k). \tag{3.7}$$

We will use the following linear algebra lemma.

Lemma 3.5 *Let B be a bounded, negative semi-definite operator on a separable Hilbert space H and let $k \geq 1$. Then, the operator $\hat{B}^{(k)}$ on $\wedge^k H$ defined by*

$$\hat{B}^{(k)}(v_1 \wedge \cdots \wedge v_k) = \sum_{j=1}^{k} v_1 \wedge \cdots \wedge B v_j \wedge \cdots \wedge v_k$$

is negative semi-definite as an operator on $\wedge^k H$.

Proof To start, observe that

$$\hat{B}^{(k)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \wedge^k e^{Bt}$$

where on the RHS is the Frechet derivative at $t = 0$ of $t \mapsto \wedge^k e^{Bt}$. Immediately it follows that $\hat{B}^{(k)}$ is self-adjoint. To check it is negative semidefinite, we will show that $\wedge^k e^{Bt}$ is a contraction semigroup, i.e.,

$$\| \wedge^k e^{Bt} \| \leq 1$$

for all $t > 0$. If this is the case, then

$$(\hat{B}^{(k)}(v_1 \wedge \cdots \wedge v_k), v_1 \wedge \cdots \wedge v_k) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \| \wedge^k e^{Bt}(v_1 \wedge \cdots \wedge v_k)\|^2 \leq 0$$

for all $v_1 \wedge \cdots \wedge v_k \in \Lambda^k H$, hence $\hat{B}^{(k)}$ is negative semidefinite.

To check that $\wedge^k e^{Bt}$ is a contraction semigroup, it suffices by the characterization in (3.6) to estimate $\| \wedge^k e^{Bt}(v_1 \wedge \cdots \wedge v_k)\|$ for some linearly independent set $\{v_1, \ldots, v_k\} \subset H$. Applying the Gram Schmidt process to this set of vectors, let $\{w_1, \ldots, w_k\} \subset H$ be an orthogonal set for which $w_i \in \mathrm{Span}\{v_1, \ldots, v_i\}$ for each $1 \leq i \leq k$. On cancelling repeated wedge terms of the form $v_i \wedge v_i$, it follows that

$$v_1 \wedge \cdots \wedge v_k = w_1 \wedge \cdots \wedge w_k \,.$$

Then,

$$\| \wedge^k e^{Bt}(v_1 \wedge \cdots \wedge v_k)\| = \| \wedge^k e^{Bt}(w_1 \wedge \cdots \wedge w_k)\| \\ \leq \prod_{i=1}^{k} \|e^{Bt} w_i\| \leq \prod_{i=1}^{k} \|w_i\| ,$$

having used that B negative semi-definite implies that e^{Bt} is a contraction semigroup. Note now that since the $\{w_i\}$ are orthogonal, $(w_i, w_j) = \|w_i\|^2 \delta_{ij}$ (here $\delta_{ij} = 1$ if $i = j$ and $= 0$ if $i \neq j$), hence $\det(w_i, w_j) = \prod_{i=1}^k \|w_i\|^2$. In view of the definition of $\| \cdot \| = \| \cdot \|_{\wedge^k H}$, we conclude that

$$\prod_i \|w_i\| = \|w_1 \wedge \cdots \wedge w_k\| = \|v_1 \wedge \cdots \wedge v_k\| ,$$

completing the proof. □

To complete the upper bound as in part (a) of Proposition 3.4, observe from (3.7) and the previous Lemma that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{v}_t\|^2 \leq \sum_{j=1}^{k}(\mathbf{v}_t, v_t^1 \wedge \cdots \wedge (\alpha + \Delta)v_t^j \wedge \cdots \wedge v_t^k) \leq \left(\sum_{j=1}^{k}(\alpha - \lambda_j)\right)\|\mathbf{v}_t\|^2$$

by the min-max principle. The estimate in (a) now follows.

3.2.3 Proof of Proposition 3.4(b): Quadratic forms on $\wedge^k H$

For $\delta > 0$ we define the quadratic form $Q_\delta^{(k)}$ on $\wedge^k H$

$$Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \delta\langle \wedge^k \Pi_k(v_1 \wedge \cdots \wedge v_k), w_1 \wedge \cdots \wedge w_k\rangle \\ - \langle \wedge^k \Pi_k^\perp(v_1 \wedge \cdots \wedge v_k), w_1 \wedge \cdots \wedge w_k\rangle$$

where Π_k denotes orthogonal projection onto the span of the first k eigenmodes $\{e_1, \ldots, e_k\}$ of Δ, and $\Pi_k^\perp = I - \Pi_k$. Equivalently, $\wedge^k \Pi_k$ is the orthogonal projection onto the span of $e_{\mathbf{i}_0} = e_1 \wedge \cdots \wedge e_k$, $\mathbf{i}_0 := (1, \ldots, k)$. In what follows, we will again abuse notation and write

$$Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k) = Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k, v_1 \wedge \cdots \wedge v_k).$$

The following elaboration on Lemma 3.2 above extends that result to the operator $\wedge^k D_{a(\omega)}\varphi_\omega^t$ as a perturbation of $\wedge^k e^{(\alpha+\Delta)t}$ in view of Proposition 2.7. Below, given $\mathbf{i} = (i_1, \ldots, i_k)$ we write $\Lambda_{\mathbf{i}} := \sum_{j=1}^k (\alpha - \lambda_{\mathbf{i}_j})$.

Lemma 3.6 *Assume $\alpha - \lambda_k > 0$. Let $T > 0$ and $\varepsilon > 0$, $\varepsilon \ll \alpha - \lambda_k$ be fixed, and let $\omega \in \Omega$ be a noise path with the property that the nonlinear term $B^t_\omega := -3a_\alpha(\theta^t\omega)^2$ satisfies*

$$\|B^t_\omega\|_{\mathcal{C}(\mathcal{O})} \le \varepsilon$$

for all $t \in [0, T]$. Finally, assume $\mathbf{v}_0 = v^1_0 \wedge \cdots \wedge v^k_0 \in \wedge^k L^2$ satisfies $Q^{(k)}_\delta(\mathbf{v}_0) > 0$ where $\delta > 0$ satisfies

$$\varepsilon(1+\delta)k \le \Lambda_{\mathbf{i}_0} - \Lambda_{\mathbf{i}} - \frac{\varepsilon(1+\delta)k}{\delta},$$

for all $\mathbf{i} \ne \mathbf{i}_0$.

Under these conditions, the k-blade $\mathbf{v}_t := \wedge^k D_{a(\omega)}\varphi^t_\omega(\mathbf{v}_0)$, corresponding with the time-t solutions $v^j_t = D_{a(\omega)}\varphi^t_\omega(v^j_0)$ to the first variation Eq. (3.1), satisfies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} Q^{(k)}_\delta(\mathbf{v}_t) \ge \left(\Lambda_{\mathbf{i}_0} - \frac{\varepsilon(1+\delta)k}{\delta}\right) Q^{(k)}_\delta(\mathbf{v}_t).$$

Lemma 3.2 above is a special case of Lemma 3.6 with $k = 1$ and $\delta = \sqrt{\varepsilon}$. Fixing this value of δ, we see that parallel to the argument presented around Eq. (3.4), $Q^{(k)}_\delta(\mathbf{v}_0) > 0$ implies $\mathbf{v}_t = \wedge^k D_{a(\omega)}\varphi^t_\omega(\mathbf{v}_0)$ satisfies

$$Q^{(k)}_\delta(\mathbf{v}_t) \ge Q^{(k)}_\delta(\mathbf{v}_0)\exp\{2t(\Lambda_{\mathbf{i}_0} - 2k\delta)\}.$$

In particular, for $M > 1$ we have that if $Q^{(k)}_{\delta/M}(\mathbf{v}_0) > 0$, then

$$\|\mathbf{v}_t\|^2 \ge \frac{M-1}{M+\delta}\exp\{2t(\Lambda_{\mathbf{i}_0} - 2k\delta)\}\|\mathbf{v}_0\|^2.$$

The proof of Proposition 3.4(b) is now complete on taking M sufficiently large and δ sufficiently small, and appealing to Proposition 2.7 to ensure $\|B^t_\omega\|_{\mathcal{C}(\mathcal{O})}$ is sufficiently small along the time window $[0, T]$ (c.f. the end of the proof of Proposition 3.1(b)).

Proof of Lemma 3.6

To start, note the unconditional estimate

$$|Q^{(k)}_\delta(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k)| \le (1+\delta)\|v_1 \wedge \cdots \wedge v_k\|\|w_1 \wedge \cdots \wedge w_k\|.$$

Let $v_i = v_i(t) = D_{a(\omega)}\varphi_\omega^t(v_i(0))$. Assuming $B = B_\omega^t$ is such that $\|B\|_V \leq \varepsilon$, it follows that $\|Bv\| \leq \varepsilon\|v\|$ for any $v \in H$. Therefore,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k) &= \sum_{j=1}^{k} Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k, v_1 \wedge \cdots \wedge \dot{v}_j \wedge \cdots \wedge v_k)\\ &= \sum_{j=1}^{k} Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k, v_1 \wedge \cdots \wedge (\alpha + \Delta + B)v_j \wedge \cdots \wedge v_k)\\ &\geq \sum_{j=1}^{k} Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k, v_1 \wedge \cdots \wedge (\alpha + \Delta)v_j \wedge \cdots \wedge v_k)\\ &\quad - \varepsilon(1+\delta)k\|v_1 \wedge \cdots \wedge v_k\|^2 .\end{aligned}$$

We can write

$$v_1 \wedge \cdots \wedge v_k = \sum_{\mathbf{i}} v_{\mathbf{i}} e_{\mathbf{i}},$$

such that

$$Q_\delta^{(k)}(v_1 \wedge \cdots \wedge (\alpha + \Delta)v_j \wedge \cdots \wedge v_k) = \sum_{\mathbf{i},\mathbf{i}'} v_{\mathbf{i}} v_{\mathbf{i}'}(\alpha - \lambda_{i'_j}) Q_\delta^{(k)}(e_{\mathbf{i}}, e_{\mathbf{i}'}).$$

For the summands we have

$$Q_\delta^{(k)}(e_{\mathbf{i}}, e_{\mathbf{i}'}) = \begin{cases} \delta & \mathbf{i} = \mathbf{i}' = \mathbf{i}_0,\\ -1 & \mathbf{i} = \mathbf{i}' \neq \mathbf{i}_0,\\ 0 & \text{else,}\end{cases}$$

which implies

$$Q_\delta^{(k)}(v_1 \wedge \cdots \wedge (\alpha + \Delta)v_j \wedge \cdots \wedge v_k) = \delta(\alpha - \lambda_j)v_{\mathbf{i}_0}^2 - \sum_{\mathbf{i} \neq \mathbf{i}_0} v_{\mathbf{i}}^2(\alpha - \lambda_{i_j})$$

such that we obtain

$$\sum_{j=1}^{k} Q_\delta^{(k)}(v_1 \wedge \cdots \wedge (\alpha + \Delta)v_j \wedge \cdots \wedge v_k) = \delta\Lambda_{\mathbf{i}_0} v_{\mathbf{i}_0}^2 - \sum_{\mathbf{i} \neq \mathbf{i}_0} \Lambda_{\mathbf{i}} v_{\mathbf{i}}^2 .$$

Altogether, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k) &\geq \delta\Lambda_{\mathbf{i}_0} v_{\mathbf{i}_0}^2 - \sum_{\mathbf{i} \neq \mathbf{i}_0} \Lambda_{\mathbf{i}} v_{\mathbf{i}}^2 - \varepsilon(1+\delta)k\sum_{\mathbf{i}} v_{\mathbf{i}}^2\\ &= \delta\left(\Lambda_{\mathbf{i}_0} - \frac{\varepsilon(1+\delta)k}{\delta}\right)v_{\mathbf{i}_0}^2 - \sum_{\mathbf{i} \neq \mathbf{i}_0}(\Lambda_{\mathbf{i}} + \varepsilon(1+\delta)k)v_{\mathbf{i}}^2 .\end{aligned}$$

Therefore, we may conclude

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k) \geq \left(\Lambda_{\mathbf{i}_0} - \frac{\varepsilon(1+\delta)k}{\delta}\right) v_{\mathbf{i}_0}^2 Q_\delta^{(k)}(v_1 \wedge \cdots \wedge v_k),$$

as long as

$$\varepsilon(1+\delta)k \leq \Lambda_{\mathbf{i}_0} - \Lambda_{\mathbf{i}} - \frac{\varepsilon(1+\delta)k}{\delta}$$

for all $\mathbf{i} \neq \mathbf{i}_0$. This finishes the proof of Lemma 3.6.

4 Outlook

In this paper we considered a scenario where stochastic driving destroyed a bifurcation and showed that some "signature" of the bifurcation persisted in the form of a positive FTLE on finite timescales and with positive probability. The results in this paper suggest several possible areas of future work, some of which we list below:

Broader class of models It should be possible to extend the techniques of this paper to a broader class of SPDEs exhibiting synchronization phenomena "destroying" bifurcations, e.g., the class of higher-order models exhibited in [7], or systems undergoing Hopf bifurcations.

Quantitative estimates on FTLE Unaddressed by our work is the concrete value of the probability of 'seeing' a positive FTLE on a given timescale for a statistically stationary initial condition. In view of the convergence of the FTLE to the asymptotic Lyapunov exponent with probability 1, this kind of quantitative information amounts to a large deviations estimate. This is naturally tied to the ergodic properties of the Markov process $(u_t, \hat{v}_t)$, where u_t is a solution to the SPDE, v_t is a solution to the first variation equation, and $\hat{v}_t = v_t/\|v_t\|$, see e.g. [2] in the context of SODE. However, the ergodic properties of $(u_t, \hat{v}_t)$ are difficult to study due to the normalization by $\|v_t\|$ and the difficult-to-rule-out possibility that the asymptotic Lyapunov exponent is $-\infty$. A rigorous proof of the fact that the asymptotic Lyapunov exponent is negative was recently obtained in [23]. However, a better understanding of the ergodic properties of this process remains an interesting open problem for a large class of systems.

Arbitrary initial data. It is an interesting question, outside the scope of this work, to provide lower bounds on the FTLE Λ_1 for a trajectory initiated away from 0 (large initial data). Suppose, for instance, that one could arrange so that the stationary measure ρ is fully supported in V_γ, and that the corresponding Markov semigroup is strong/ultra Feller [31]. According to Cerrai [11, Sect. 8.3.1] we know that these properties hold in H under our assumptions. It would then hold that *any* initial data $u_0 \in V_\gamma$ enters a V_γ-small neighborhood of 0 given enough time (with probability 1). This, in conjunction with the arguments in Sect. 3, would imply the following: for all $u_0 \in V_\gamma$ and $T > 0$, there exists $t > 0$ and a set $\mathcal{A} \in \mathcal{F}_0^{t+T}$ such that for $\omega \in \mathcal{A}$,

$$\|D_{u_t}\varphi^s_{\theta^t\omega}\| \approx e^{s(\alpha-\lambda_1)} \quad \text{for all} \quad s \in [0, T],$$

along the lines of Theorem A. A significantly harder question, however, is to provide an estimate of $\| D_{u_0} \varphi^s_\omega \|$, initiated at time zero, for potentially large initial data u_0 and for times $s > T(u_0)$. While it should be possible to steer the trajectory of (u_t) close to 0 and apply the arguments of Sect. 3, what is missing is an argument to bound $D\varphi^t_\omega$ from below during the 'transient' time period before u_t has been 'steered' toward 0.

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Data Availability No new data were created or analyzed in this study. Data sharing is not applicable to this article.

A Appendix

A.1 Proof of Proposition 2.7

Proposition 2.7 will be deduced from the following.

Lemma A.1 *For any $T > 0$ and $\varepsilon > 0$ there is an event $\mathcal{A}_T \in \mathcal{F}_0^T$ of positive probability and a real number $\eta = \eta(\varepsilon, T) > 0$ such that for any $u_0 \in V_\gamma$ with $\|u_0\|_{V_\gamma} < \eta$, we have that $\|u_t\|_{V_\gamma} < \varepsilon$ for all $t \in [0, T]$ and all $\omega \in \mathcal{A}_T$.*

Proof of Proposition 2.7 assuming Lemma A.1 By Proposition 2.4, the process (u_t) admits a unique stationary measure ρ on H with $\rho(V_\gamma) = 1$, hence $a_\alpha(\omega) \in V_\gamma$ with probability 1. Let $\varepsilon, T > 0$ be fixed let and $\eta = \eta(\varepsilon, T) > 0$, $\mathcal{A}_T \in \mathcal{F}_0^T$ be as in Lemma A.1. Local positivity of ρ ensures $\mathbb{P}(\mathcal{A}_-) > 0$, where

$$\mathcal{A}_- = \{\|a_\alpha\|_{V_\gamma} < \eta\} .$$

Let now $\mathcal{A} = \mathcal{A}_- \cap \mathcal{A}_T$, and note that since $\mathcal{A}_- \in \mathcal{F}^0_{-\infty}$, we have that $\mathcal{A}_-, \mathcal{A}_T$ are independent, hence

$$\mathbb{P}(\mathcal{A}) = \mathbb{P}(\mathcal{A}_-) \cdot \mathbb{P}(\mathcal{A}_T) > 0,$$

which completes the proof. □

Proof of Lemma A.1 The main idea is to exploit regularizing properties of analytic semigroups. Recalling the SPDE (1.1)

$$\begin{aligned} \mathrm{d}u &= [\Delta u + \alpha u \underbrace{-u^3}_{:= f(u)}] \,\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t \\ &= [\Delta u + \alpha u] \,\mathrm{d}t + f(u) \,\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t, \end{aligned} \tag{A.1}$$

we subtract the Ornstein-Uhlenbeck process, i.e. the solution of the linear SPDE

$$\mathrm{d}z = [\Delta z + \alpha z]\,\mathrm{d}t + \sqrt{Q}\mathrm{d}W_t.$$

This solution is given by the convolution

$$z(t) = \int_0^t T(t-r)\,\sqrt{Q}\mathrm{d}W_t,$$

where $(T(t))_{t\geq 0}$ is the shifted heat semigroup with α, i.e. $T(t) := e^{(\Delta+\alpha)t}$. According to Lemma 2.1 this belongs to $C([0,T]; V_\gamma\ \mathbb{P})$-a.s. Therefore the set

$$\mathcal{A}_T := \left\{\omega \in \Omega : \sup_{t\in[0,T]} \|z(t)\|_{V_\gamma} \leq \eta\right\} \in \mathcal{F}_0^T$$

has positive probability. This is sufficient for our aims since we are only interested in a finite-time statement. We further fix $\omega \in \mathcal{A} = \mathcal{A}_- \cap \mathcal{A}_T$ as in Proposition 2.7 and study on V_γ the PDE with random non-autonomous coefficients

$$\mathrm{d}\tilde{u} = [\Delta\tilde{u} + \alpha\tilde{u}]\,\mathrm{d}t + f(\tilde{u}+z)\,\mathrm{d}t. \tag{A.2}$$

(Since $(T(t))_{t\geq 0}$ is an analytic semigroup, $\tilde{u}$ is differentiable on V_γ for $t > 0$). We assume that the initial data $\tilde{u}_0 := \tilde{u}(0) = a_\alpha(\omega)$. We now prove that for any given $\varepsilon > 0$ and $\omega \in \mathcal{A}$ we have

$$\|a_\alpha(\theta^s\omega)\|_{V_\gamma} \in (0,\varepsilon),\ \text{ for } s \in [0,T]. \tag{A.3}$$

To this aim we use the fact the semigroup $(T(t))_{t\geq 0}$ acts on all function spaces V_γ together with the classical estimate

$$\|T(t)\|_{\mathcal{B}(V_\gamma)} \leq e^{(-\lambda_1+\alpha)t}, \quad t > 0. \tag{A.4}$$

The cubic nonlinearity $f : V_\gamma \to V_\gamma$ is locally Lipschitz for $\gamma \in (\frac{1}{4}, \frac{1}{2})$, i.e. there exists a constant $\widetilde{l} := \widetilde{l}(\|\tilde{u}_1\|_{V_\gamma}, \|\tilde{u}_2\|_{V_\gamma})$ such that

$$\|f(\tilde{u}_1) - f(\tilde{u}_2)\|_{V_\gamma} \leq \tilde{l}\|\tilde{u}_1 - \tilde{u}_2\|_{V_\gamma}, \quad \text{for } \tilde{u}_1, \tilde{u}_2 \in B \subset V_\gamma,$$

where B is a bounded subset of V_γ. Moreover there exists an increasing function $a : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\langle F(y+w), y\rangle_{V_\gamma} \leq a(\|w\|)(1+\|y\|), \quad \text{for all } y, w \in V_\gamma, \tag{A.5}$$

see [19, (H3), p. 130] and [16, Sect. 7.2.1]. Regarding this together with $\|\cdot\|_H \le \|\cdot\|_{V_\gamma}$ further results in

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{u}(t)\|_{V_\gamma}^2 &\le \langle \Delta\tilde{u}(t) + \alpha\tilde{u}(t), \tilde{u}(t)\rangle_{V_\gamma} + \langle F(\tilde{u}(t) + z(t)), \tilde{u}(t)\rangle_{V_\gamma} \\ &\le (-\lambda_1 + \alpha)\|\tilde{u}(t)\|_{V_\gamma}^2 + a(\|z(t)\|_{V_\gamma})(1 + \|\tilde{u}(t)\|_{V_\gamma}) \\ &\le (-\lambda_1 + \alpha + a(\|z(t)\|_{V_\gamma}))\|\tilde{u}(t)\|_{V_\gamma}^2 + a(\|z(t)\|_{V_\gamma}).\end{aligned}$$

Here we estimated $\|\tilde{u}(t)\|_{V_\gamma} \le \|\tilde{u}(t)\|_{V_\gamma}^2$, since if $\|\tilde{u}(t)\|_{V_\gamma} \le 1$, then the statement automatically holds. Gronwall's inequality now entails

$$\begin{aligned}\|\tilde{u}(t)\|_{V_\gamma}^2 &\le e^{(-\lambda_1+\alpha)t+\int_0^t a(\|z(s)\|_{V_\gamma})\,\mathrm{d}s}\|\tilde{u}_0\|_{V_\gamma} + \int_0^t e^{(-\lambda_1+\alpha)(t-s)+\int_s^t a(\|z(r)\|_{V_\gamma})\,\mathrm{d}r} a(\|z(s)\|_{V_\gamma})\,\mathrm{d}s \\ &\le e^{(-\lambda_1+\alpha+a(\eta))t}\|\tilde{u}_0\|_{V_\gamma} + a(\eta)\int_0^t e^{(-\lambda_1+\alpha+a(\eta))(t-s)}\,\mathrm{d}s,\end{aligned}$$

where we used the fact that a is increasing and that $\|z(s)\|_{V_\gamma} \le \sup_{s\in[0,T]} \|z(s)\|_{V_\gamma} \le \eta$ for $\omega \in \mathcal{A}$. Regarding that $\|u(t)\|_{V_\gamma} \le \|\tilde{u}(t)\|_{V_\gamma} + \|z(t)\|_{V_\gamma}$, $u_0 = a_\alpha(\omega)$ and consequently $u(t) = a_\alpha(\theta^t\omega)$, the statement follows. □

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