Universität Konstanz

A density property for PRC-fields

Joachim Schmid

Konstanzer Schriften in Mathematik und Informatik
Nr. 124, August 2000

ISSN 1430–3558

A density property for PRC-fields

Joachim Schmid

Abstract: For an absolutely irreducible variety V defined over a pseudo real closed field K we consider the diagonal embedding of the set of regular rational points of V into the product of the set of regular points in the real closures of K. We prove that the image of this map is dense w.r.t. the product topology induced by the different orderings of K.

Introduction

In 1984 Heinemann and Prestel [4] introduced a class of fields which satisfy a local global principle for rational points on absolutely irreducible varieties referring to a finite number of local closures (i.e. real closures or henselizations) of the field. They proved ([4], Theorem 1.9) that a field K belonging to this class satisfies a certain approximation condition, namely for an absolutely irreducible variety defined over K there exists a K-rational point simultaneously close to a given sequence of regular points of V in the local closures.

Their result includes the special case of a pseudo real closed field with only finitely many orderings.

Later similar approximation and density theorems for various classes of fields and rings which satisfy certain local global principles were proven by different authors (cf. [1], [5], [6]).

It is the aim of this paper to prove that the approximation condition of Heinemann and Prestel holds for arbitrary pseudo real closed fields.

1. Preliminaries

We assume that the reader is familiar with the theory of ordered fields and we recall that a field K is called pseudo real closed if every absolutely irreducible variety defined over K which has a a regular point in each real closure of K also has a K-rational point. We fix some notations.

If K is a field we denote its algebraic closure by $\widetilde{K}$ and the real closure of K w.r.t. an ordering P is denoted by $\overline{(K,P)}$. Furthermore for $x = (x_1, \ldots, x_n) \in \overline{(K,P)}^n$ we let

$$||x||_P := \sqrt{\sum_{i=1}^{n} x_i^2}$$

By X_K we denote the set of all orderings of the field K. The set X_K is endowed with the so-called Harrison topology. A subbasis for the open sets of this topology consists of the sets $H(a) := \{P \in X_K \mid a \in P\}$ $(a \in K \setminus \{0\})$. As $H(-a) = X_K \setminus H(a)$ the sets $H(a)$ are clopen (= closed and open). It is well-known that X_K is a compact Hausdorff space w.r.t. the Harrison topology (cf. [9], Theorem 6.5). The set of all sums of squares of the field K is denoted by $\sum K^2$.

We start with an elementary lemma.

Lemma 1.1: *Let K be a formally real field and $a_1, \ldots, a_n \in \sum K^2 \setminus \{0\}$. Then there is $a \in \sum K^2 \setminus \{0\}$ such that $a_i - a \in \sum K^2 \setminus \{0\}$ $(1 \leq i \leq n)$.*

Proof: Let $a \in K$ such that $a^{-1} = \sum_{i=1}^{n} a_i^{-1}$. Now some easy arguments show the assertion. □

The following proposition states a well-known fact.

Proposition 1.2: *Let K be a formally real field and suppose that φ is a first order formula in the language of ordered fields with parameters from the field K. Then the set $\{P \in X_K \mid \overline{(K,P)} \models \varphi\}$ is clopen.*

Proof: Since the theory RCF of real closed fields admits elimination of quantifiers there is a quantifier free formula ψ with parameters from K such that

$$\mathrm{RCF} \models (\varphi \longleftrightarrow \psi)$$

W.l.o.g. we may assume that ψ is in disjunctive normal form, i.e. $\psi = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m_i} \psi_{ij}$ and each formula ψ_{ij} is either atomic or the negation of an atomic formula. Hence

$$\{P \in X_K \mid \overline{(K,P)} \models \varphi\} = \bigcup_{i=1}^{n} \bigcap_{j=1}^{m_i} Y_{ij}$$

with $Y_{ij} := \{P \in X_K \mid \overline{(K,P)} \models \psi_{ij}\}$. It suffices to show that each set Y_{ij} is clopen. If ψ_{ij} is of the form $t_1 = t_2$ or $t_1 \neq t_2$ then Y_{ij} is either $\emptyset$ or X_K depending on the case whether ψ_{ij} holds in K or not. If ψ_{ij} is of the form $P(t)$ or $\neg P(t)$ then $Y_{ij} = H(t)$ or $Y_{ij} = X_K \setminus H(t) = H(-t)$. This proves the assertion. □

2. Some approximation and density results

In 1982 Ershov [2] introduced a certain approximation property in order to give a first order axiomatization for the class of r-closed fields (=maximal pseudo real closed fields). For this kind of approximation Prestel (cf. [8]) introduced the name *block approximation* and he proved that fields satisfying a certain curve condition also have the block approximation property. With this result Prestel was able to give a characterization of PRC-fields in terms of affine plane curves. Also in Darnière's approach (cf. [1]) to the decidability of the ring of p-adic algebraic integers a certain block approximation property plays an important role.

The next theorem is a generalization of Prestel's result. It is the crucial point in the proof of our approximation theorem. The original result of Prestel is the special case $V = \mathbb{A}^1$.

In the proof of the next theorem we make essentially use of the fact that a pseudo real closed field K is a so-called SAP-field, i.e., for all $a, b \in K^\times$ there is some $c \in K^\times$ such that $H(a) \cap H(b) = H(c)$ (cf. [7], Prop. 1.3). An immediate consequence of the compactness of the space of orderings is that for an SAP-field K the clopen subsets of X_K are exactly the sets $H(a)$ $(a \in K \setminus \{0\})$ (cf. [9], Prop. 6.6).

Theorem 2.1 (Block approximation)**:** *Let K be a pseudo real closed field and suppose that $V \subset \widetilde{K}^n$ is an absolutely irreducible variety defined over K. Let further $a_1, \ldots, a_m \in K^n$, $\varepsilon_1, \ldots, \varepsilon_m \in \sum K^2 \backslash \{0\}$ and suppose that $X_K = \biguplus_{i=0}^m U_i$ for some pairwise disjoint clopen subsets $U_0, \ldots, U_m \subset X_K$. Assume that $V_{\text{reg}} \cap \overline{(K,P)}^n \neq \emptyset$ for $P \in U_0$ and suppose that for each $i \in \{1, \ldots, m\}$ and each $P \in U_i$ there is a point $x \in V_{\text{reg}} \cap \overline{(K,P)}^n$ such that $||x - a_i||_P <_P \varepsilon_i$. Then there is a point $a \in V_{\text{reg}} \cap K^n$ such that $||a - a_i||_P <_P \varepsilon_i$ for every $i \in \{1, \ldots, m\}$ and $P \in U_i$.*

Proof: As mentioned above the (possibly empty) clopen sets $U_0, \ldots, U_m$ are of the form $U_i = H(c_i)$ for certain non-zero elements $c_0, \ldots, c_m \in K^\times$. Let $f_1, \ldots, f_r \in K[X_1, \ldots, X_n]$ be a system of generators for the vanishing ideal $\mathcal{I}(V)$ of V and define

$$h(X,T) := c_0 \cdot \prod_{i=1}^{m} (T_i^2 \cdot \sum_{j=1}^{n} (X_j - a_{ij})^2 - c_i \varepsilon_i^2)$$

where a_{ij} are the coordinates of a_i, i.e., $a_i = (a_{i1}, \ldots, a_{in})$. Further we let

$$\begin{aligned} f(X,T,S_1,S_2) &:= S_1^2 + S_2^2 - h(X,T) \\ g_i(Y_i, Z_i, T_i) &:= Y_i^2 + Z_i^2 - T_i^2 + c_i \qquad (1 \le i \le m) \end{aligned}$$

Then $W := \mathrm{V}(f_1, \ldots, f_r, f, g_1, \ldots, g_m) \subset \widetilde{K}^{n+3m+2}$ is an absolutely irreducible variety defined over K. We show that W has a regular point in each real closure of K and $h \notin \mathcal{I}(W)$.

First we consider the case $P \in U_0$. By the assumption there is a point $x \in V_{\text{reg}} \cap \overline{(K,P)}^n$. We define $t_i := c_i + \frac{1}{2}$ and $t := (t_1, \ldots, t_m)$. Then

$$h(x,t) = c_0 \cdot \prod_{i=1}^{m} (t_i^2 \cdot ||x - a_i||_P^2 - c_i \varepsilon_i^2)$$

As $c_i <_P 0$ $(1 \le i \le m)$ but $0 <_P c_0$ we obtain $0 <_P h(x,t)$. Thus there are non-zero $s_1, s_2 \in \overline{(K,P)}$ such that $h(x,t) = s_1^2 + s_2^2$. and therefore $f(x,t,s_1,s_2) = 0$. If we further let $y_i := \frac{1}{2}$ and $z_i := c_i$ $(1 \le i \le m)$ then it is easy to see that (x, y, z, t, s_1, s_2) with $y := (y_1, \ldots, y_m)$ and $z := (z_1, \ldots, z_m)$ is a regular point of W.

Now let $P \in U_j = H(c_j), j \neq 0$. Then there is a point $x \in V_{\text{reg}} \cap \overline{(K,P)}^n$ such that

$$||x - a_j||_P <_P \varepsilon_j$$

Thus we may choose $t_j \in \overline{(K,P)}$ such that $\sqrt{c_j} <_P t_j$ and

$$t_j^2 \cdot ||x - a_j||_P^2 <_P c_j \varepsilon_j^2$$

Further we let $t_i := 1$ for $i \neq j$ and define $t := (t_1, \ldots, t_m)$. Since $c_i <_P 0$ for $i \neq j$, all the factors under the product sign in

$$h(x,t) = c_0 \cdot (t_j^2 \cdot ||x - a_j||_P^2 - c_j \varepsilon_j^2) \cdot \prod_{i \neq j} (||x - a_i||_P^2 - c_i \varepsilon_i^2)$$

are positive while the first and the second factors are negative. Thus $0 <_P h(x,t)$ and therefore there are non-zero $s_1, s_2 \in \overline{(K,P)}$ such that $h(x,t) = s_1^2 + s_2^2$ and thus $f(x,t,s_1,s_2) = 0$. As $c_j <_P t_j^2$ there are non-zero $y_j, z_j \in \overline{(K,P)}$ such that $t_j^2 - c_j = y_j^2 + z_j^2$. For $i \neq j$ we let $z_i = 1$ and $y_i := \sqrt{-c_i}$. Note that $c_i <_P 0$. Then it is easy to see that (x,y,z,t,s_1,s_2) with $y := (y_1, \ldots, y_m)$ and $z := (z_1, \ldots, z_m)$ is a regular point of W.

Thus we have shown that W has a regular point in each real closure of K and $h \notin \mathcal{I}(W)$. As K is pseudo real closed W has a point $(a,y,z,t,s_1,s_2) \in W_{\text{reg}} \cap K^{n+3m+2}$ such that $h(a,t) \neq 0$. Obviously $a \in V \cap K^n$ and one can easily see that even $a \in V_{\text{reg}}$.

Finally we show that a satisfies the stated approximation conditions. Fix $j \in \{1, \ldots, m\}$ and let $P \in U_j = H(c_j)$. As

$$0 = f(a,t,s_1,s_2) = s_1^2 + s_2^2 - h(a,t)$$

and $h(a,t) \neq 0$ we obtain

$$0 <_P h(a,t) = c_0 \cdot (t_j^2 \cdot ||a - a_j||_P^2 - c_j \varepsilon_j^2) \cdot \prod_{i \neq j} (t_i^2 \cdot ||a - a_i||_P^2 - c_i \varepsilon_i^2)$$

Since $c_i <_P 0$ for $i \neq j$, all the factors under the product sign are positive while c_0 is negative and thus necessarily

$$t_j^2 \cdot ||a - a_j||_P^2 <_P c_j \varepsilon_j^2$$

From $0 = g_j(y,z,t) = y_j^2 + z_j^2 - t_j^2 + c_j$ we get $c_j \leq_P t_j^2$. As $0 <_P c_j$ this finally yields

$$||a - a_j||_P^2 <_P \varepsilon_j^2$$

and thus $||a - a_j||_P <_P \varepsilon_j$ and the theorem is proved. □

Remark: A similar result was shown by Fried, Haran and Völklein (cf. [3], Prop. 1.2). Actually Prop. 1.2 of [3] and our Theorem 2.1 are equivalent as was pointed out by M. Jarden.

Theorem 2.2 (Approximation Theorem)**:** *Let K be a pseudo real closed field and suppose that $V \subset \widetilde{K}^n$ is an absolutely irreducible variety defined over K. Assume that $P_1, \ldots, P_m \in X_K$ are pairwise different orderings of K. Suppose that $V_{\text{reg}} \cap \overline{(K,P)}^n \neq \emptyset$ for each ordering P of K and let $x_i \in V_{\text{reg}} \cap \overline{(K,P_i)}^n$ $(1 \leq i \leq m)$. Then for all $\varepsilon \in \sum K^2 \setminus \{0\}$ there is a point $a \in V_{\text{reg}} \cap K^n$ such that $||a - x_i||_{P_i} <_{P_i} \varepsilon$ for every $i \in \{1, \ldots, m\}$.*

Proof: Since K is pseudo real closed it is dense in each real closure (cf. [7], Prop. 1.4). Thus there are $a_i \in K^n$ $(1 \leq i \leq m)$ such that $||x_i - a_i||_{P_i} <_{P_i} \frac{1}{2}\varepsilon$. For $1 \leq i \leq m$ we define

$$Y_i := \{P \in X_K \mid \overline{(K,P)} \models \exists x (x \in V_{\text{reg}} \wedge ||x - a_i||_P < \tfrac{1}{2}\varepsilon)\}$$

Obviously $P_i \in Y_i$ and by Proposition 1.2 the sets Y_i are clopen.

As $P_1, \ldots, P_m$ are pairwise different, there are pairwise disjoint clopen subsets $X_1, \ldots, X_m$ of X_K such that $P_i \in X_i$ $(1 \leq i \leq m)$. Thus we obtain

$$X_K = \dot{\bigcup_{i=1}^{m}} (X_i \cap Y_i) \cup Z$$

for some clopen subset $Z \subset X_K$. Now Theorem 2.1 yields a point $a \in V_{\text{reg}} \cap K^n$ such that $||a - a_i||_P <_P \frac{1}{2}\varepsilon$ for $P \in X_i \cap Y_i$. In particular $||a - a_i||_{P_i} <_{P_i} \frac{1}{2}\varepsilon$ $(1 \leq i \leq m)$ and therefore

$$||a - x_i||_{P_i} \leq_{P_i} ||a - a_i||_{P_i} + ||a_i - x_i||_{P_i} <_{P_i} \varepsilon \qquad (1 \leq i \leq m)$$

□

Finally we restate Theorem 2.2 in terms of a density property.

Theorem 2.3 (Density Theorem): *Let K be a pseudo real closed field and assume that K is formally real. Then, for every absolutely irreducible variety $V \subset \widetilde{K}^n$ defined over K the image of the canonical diagonal map*

$$V_{\text{reg}} \cap K^n \longrightarrow \prod_{P \in X_K} V_{\text{reg}} \cap \overline{(K,P)}^n$$

is dense w.r.t. the product topology.

Proof: Let $U \subset \prod_{P \in X_K} V_{\text{reg}} \cap \overline{(K,P)}^n$ be a non-empty open subset, say $(x_P)_{P \in X_K} \in U$. Then there is a family of open subsets $U_P \subset V_{\text{reg}} \cap \overline{(K,P)}^n$ $(P \in X_K)$ such that $x_P \in U_P$, $\prod_{P \in X_K} U_P \subset U$ and $U_P \neq V_{\text{reg}} \cap \overline{(K,P)}^n$ for only finitely many orderings, say $P_1, \ldots, P_m$. As K is cofinal in its real closures $\overline{(K,P_i)}$ $(1 \leq i \leq m)$ (cf. [9], Lemma 7.20) Lemma 1.1 yields $\varepsilon \in \sum K^2 \backslash \{0\}$ such that $\{x \in \overline{(K,P)} \mid ||x - x_{P_i}||_{P_i} < \varepsilon\} \subset U_{P_i} (1 \leq i \leq m)$. Now Theorem 2.2 yields the result. □

References

[1] DARNIÈRE, L.: *Nonsingular Hasse principle for rings , to appear in: Journal reine angew. Math.*

[2] ERSHOV, Y.L.: *Totally real field extensions, Soviet Math. Dokl. 25 (1982), 477-480*

[3] FRIED, M.D., HARAN, D., VÖLKLEIN, H.: *Real Hilbertianity and the Field of Totally Real Numbers, in: Contemporary Mathematics, Vol. 174, 1994*

[4] HEINEMANN, B., PRESTEL, A.: *Fields regularly closed with respect to finitely many valuations and orderings, Canadian Math. Soc., Conf. Proc. Vol. 4 (1984), 297-336*

[5] JARDEN, M., RAZON, A.: *Pseudo algebraically closed fields over rings, Israel J. of Math. 86 (1994), 25-59*

[6] JARDEN, M., RAZON, A.: *Rumely's local global principle for algebraic $\mathcal{PSC}$ fields over rings, Transactions AMS 350 (1998), 55-85*

[7] PRESTEL, A.: *Pseudo real closed fields, in: Set Theory and Model Theory, Lecture Notes in Math. 872, Springer 1982, 127-156*

[8] PRESTEL, A.: *On the axiomatization of prc-fields, in: Methods in Mathematical Logic, Lecture Notes in Math. 1130, Springer 1983, 351-359*

[9] PRESTEL, A.: *Lectures on formally real fields, Lecture Notes in Math. 1093, Springer 1984*

Universität Konstanz
Fachbereich Mathematik und Statistik
78457 Konstanz
GERMANY

schmidj @ fmi.uni-konstanz.de