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Maximal L^p-regularity of an abstract evolution equation: Application to closed-loop feedback problems, with boundary controls and boundary sensors

Irena Lasiecka [a,b], Buddhika Priyasad [c,*], Roberto Triggiani [a]

[a] Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, USA
[b] IBS, Polish Academy of Sciences, Warsaw, Poland
[c] Department of Mathematics and Statistics, University of Konstanz, Konstanz, Germany

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ABSTRACT

We present an abstract maximal L^p-regularity result up to $T = \infty$ on a Banach space, that is tuned to capture (linear) PDEs of parabolic type, defined on a bounded domain and subject to finite dimensional, boundary controls and boundary sensors, in feedback form. It improves Lasiecka et al. (2021), which covered boundary controls and interior sensors. The present proof must necessarily be completely different from the one in Lasiecka et al. (2021). In applications (Section 3), the case $T < \infty$ requires no further assumptions on the boundary control/sensor vectors. Instead, the case $T = \infty$ requires, of course, the property of uniform stabilization for suitable boundary control/sensor vectors, which is geometrically sensitive.

1. Introduction, abstract setting, main results, literature

1.1. Introduction

The recent paper [1] presented an abstract maximal L^p-regularity result (see Appendix B, point #1) in a Banach space (which moreover is a UMD space, (see Appendix B, point #2) up to $T = \infty$. It was actually motivated by, and ultimately directed to, (linear) Partial Differential Equations of parabolic type, defined on a bounded domain and subject to finite dimensional, stabilizing, static, feedback *controls* acting on (a portion of) the *boundary*, with however *interior* sensors. Illustrations included, beside more classical boundary control parabolic examples, two recent settings: (i) the 3d Navier–Stokes equations on L^q, or on specific Besov spaces(see Appendix B, point #3), with finite dimensional, localized, static, *boundary*, tangential feedback stabilizing *controls* and *interior* sensors [2], as well as (ii) Boussinesq systems with finite dimensional, localized, static, feedback stabilizing Dirichlet *boundary* control for the thermal equation, and *interior* sensors [3], for which Ref. [4] provides the starting point. It turns out that for $d = 3$, the Besov space setting (“close” to $L^3(\Omega)$) was critical to solve in the positive [2] a long-standing open problem introduced in the early 2000 by Fursikov [5,6]: weather the 3d-Navier–Stokes equations can be uniformly stabilized in the vicinity of an unstable equilibrium solution, by means of localized, tangential, *boundary*-based, static feedback controls (and *interior* sensors), that moreover are *finite dimensional*. The traditional Sobolev, Hilbert setting required the additional assumption for $d = 3$ that the initial condition be compactly supported [7,8].

* Corresponding author.
E-mail address: priyasad@uni-konstanz.de (B. Priyasad).

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As stated, the breakthrough that unlocked the described 3d boundary stabilization of unstable Navier–Stokes flows was the new setting within a tight *Besov spaces* framework [2]. A critical ingredient was *maximal regularity* established for the control model under consideration with *boundary* action and with *internal* observation [1].

In the present work we go one step further: In Section 1.2, we present an abstract result that this time includes not only *boundary* controls, but also *boundary* sensors; see illustrations in Section 3. Here, the case $T < \infty$ requires no further assumptions on the boundary control/sensor vectors. Instead, the case $T = \infty$ requires, of course, the property of uniform stabilization for suitable boundary control/sensor vectors, which is geometrically sensitive [9]. For instance, it is generally false for $d = 1$, if the original free system has at least three unstable eigenvalues [9]. The above extension beside being a mathematical challenge, has a plethora of practical applications. Recovering information [sensing] on the boundary of a physical model is what is desired in modern technology. Accessing the interior may not always be feasible. But this requires a new functional analytic setup where *maximal regularity* is discovered for *boundary* action-*boundary* sensing of dynamics set in the Besov spaces framework. And, again, this will be a game changer in establishing 3d boundary stabilization with observed quantities localized on the boundary only. An original reference on maximal regularity by interpolation and extrapolation is [10].

1.2. Abstract setting

The focus of the present work is the operator

$$\begin{cases} A_F = -A(I-GF) : Y \supset \mathcal{D}(A_F) \longrightarrow Y \text{ (a)}, \\ \mathcal{D}(A_F) = \{x \in Y : \ (I-GF)x \in \mathcal{D}(A)\} \quad \text{(b)}. \end{cases} \tag{1.1}$$

and corresponding abstract equation

$$y_t = A_F y = -A(I-GF)y \tag{1.2}$$

under the following standing *assumptions*:

(**H.1**) Y is a reflexive Banach space.

(**H.2**) $-A : Y \supset \mathcal{D}(A) \longrightarrow Y$ is the maximal dissipative generator of a C_0-contraction semigroup e^{-At} on Y, $t \geq 0$, which possesses the maximal $L^p(0,T;Y)$-regularity property up to T, either $0 < T < \infty$; or else $T = \infty$, $1 < p < \infty$; in symbols [11]

$$-A \in MReg\,(L^p(0,T;Y)), \quad \text{either } 0 < T < \infty; \text{ or else } T = \infty, 1 < p < \infty;$$

so that, a fortiori, the strongly continuous (s.c.) semigroup e^{At} is analytic (holomorphic) on Y. At the price (harmless for the present note) of replacing A with a suitable translation to the right ($A_k = A + k^2 I$), the fractional powers $A^\theta, 0 < \theta < 1$, of A are well-defined [12].

(**H.3**) U is another Banach space and G is the ("Green") linear operator satisfying

$$G : \text{continuous } U \longrightarrow \mathcal{D}(A^{\alpha_0}) \subset Y, \text{ or } A^{\alpha_0} G \in \mathcal{L}(U;Y) \tag{1.3}$$

for some $0 < \alpha_0 < 1$.

(**H.4**) F is a linear ("feedback") operator of the form

$$Fz = \langle \gamma z, w \rangle_U \, g, \quad w, g \in U \tag{1.4}$$

where γ is a linear (trace) operator

$$\gamma : \text{continuous } \mathcal{D}(A^\sigma) \subset Y \longrightarrow U, \quad 0 < \sigma < \alpha_0 < 1 \tag{1.5}$$

so that

$$F : \text{continuous } \mathcal{D}(A^\sigma) \subset Y \longrightarrow U. \tag{1.6}$$

[In the applications we shall take $Fz = \sum_{k=0}^{K} \langle \gamma z, w_k \rangle_U \, g_k, \quad w_k, g_k \in U$ and we seek to identify the 'minimal' positive integer K in the case $T = \infty$.]

Remark 1.1. F is thus unbounded as an operator on Y. For the similar problem considered in [1], F was a bounded operator on Y. The purpose of this note is to extend to the operator (1.1) the result on maximal $L^p(0,T;Y)$-regularity of [1], $T \leq \infty$. The proof of [1] requires $F \in \mathcal{L}(Y;U)$. Thus, the proof of the present note is quite different from that in [1]. It is inspired by the proof in [13] showing that a PDE-specialization of the operator A_F in (1.1) generates a s.c. analytic semigroup on Y. Such specialization refers to a classic 2nd order parabolic equation on a multi-dimensional bounded domain Ω of $\mathbb{R}^N$ with boundary $\partial\Omega = \Gamma$, studied on the Hilbert space $Y = L^2(\Omega)$, where $U \equiv L^2(\Gamma)$, and where the feedback operator F is applied to the Neumann or Robin B. C. In [13], γ is (canonically) the Dirichlet trace: $\gamma z = z|_\Gamma$. Thus, the resulting problem is characterized by *boundary* feedback *control* ($g \in L^2(\Gamma)$ in (1.4)), with *boundary* actuation/sensor ($w \in L^2(\Gamma)$). In this case, $\sigma_0 = {}^3/_4 - \rho,\ \rho > 0$ arbitrarily small, [14].

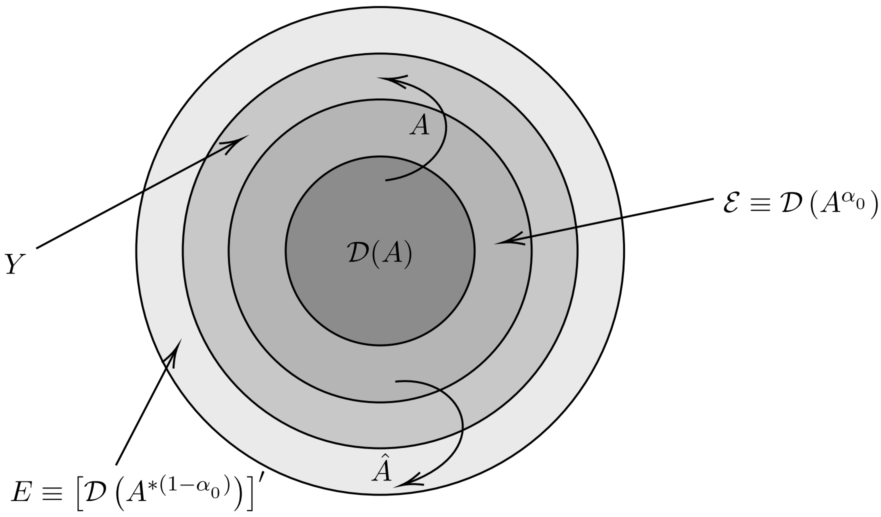

Fig. 1. Symbolic illustration of the spaces and operators involved, with the operator $\hat{A}$ being the extension of the operator A from $\mathcal{E}$ onto E, see (2.1).

Consequences of the assumptions

(i) The maximal dissipative operator $(-A)$, $0 \in \rho(A)$ = resolvent set of A, defines an isomorphism from $\mathcal{D}(A) \subset Y$ onto Y [15, p101]. By standard isomorphism techniques [16,17], then A can be extended to define an isomorphism from Y onto $[\mathcal{D}(A^*)]'$ duality with respect to Y.

(ii) The domains $\mathcal{D}(A^\theta)$ of fractional powers, $0 \le \theta \le 1$, can be expressed as complex interpolation spaces

$$\mathcal{D}(A^\theta) = [\mathcal{D}(A), Y]_{1-\theta} \tag{1.7}$$

where $[\cdot,\cdot]_\theta$ denotes the holomorphic interpolation space between $\mathcal{D}(A)$ and Y. One may quote [18, Prop 6.1, p113] (reported in [14, p5]). Alternatively the operator A^{it}, with imaginary power it, is bounded on some interval $|t| \le c,\ c > 0$, [19, Thm 5, p247], so that [15, Thm 1.15.3, p103] applies and yields (1.7).

(iii) Next, we apply interpolation theory [16, Thm 5.1, p27] as well as duality theory [16, Thm 5.2, p29] to obtain that the map

$$\begin{aligned} A : Y \subset \mathcal{D}(A^\theta) \longrightarrow \left[Y, [\mathcal{D}(A^*)]'\right]_{1-\theta} &= \left[\mathcal{D}(A^*), Y\right]'_\theta, \\ &= \left[\mathcal{D}(A^{*(1-\theta)})\right]', \quad 0 < \theta < 1 \end{aligned} \tag{1.8}$$

is also an isomorphism. Since $0 \in \rho(A)$, we have for $0 < \theta < 1$:

$$\|x\|_{\mathcal{D}(A^\theta)} \equiv \left\|A^\theta x\right\|_Y, \quad x \in \mathcal{D}(A^\theta) \subset Y, \tag{1.9}$$

$$\|z\|_{[\mathcal{D}(A^{*\theta})]'} \equiv \left\|A^{-\theta} x\right\|_Y, \quad z \in \left[\mathcal{D}(A^{*\theta})\right]' \supset Y. \tag{1.10}$$

With reference to assumption (**H.3**) centered on the constant $0 < \alpha_0 < 1$, we introduce two Banach spaces,

$$\mathcal{E} \equiv \mathcal{D}(A^{\alpha_0}), \text{ with norm } \|x\|_{\mathcal{E}} \equiv \|x\|_{\mathcal{D}(A^{\alpha_0})} \equiv \|A^{\alpha_0} x\|_Y, \tag{1.11}$$

$$E \equiv \left[\mathcal{D}(A^{*(1-\alpha_0)})\right]' \text{ with norm } \|z\|_E \equiv \|z\|_{\left[\mathcal{D}(A^{*(1-\alpha_0)})\right]'} = \left\|A^{-(1-\alpha_0)} z\right\|_Y. \tag{1.12}$$

Accordingly we introduce the following holomorphic interpolation spaces

$$[\mathcal{E}, E]_\theta \equiv \left[\mathcal{D}\left(A^{\alpha_0}\right), \left[\mathcal{D}\left(A^{*(1-\alpha_0)}\right)\right]'\right]_\theta = \begin{cases} \mathcal{D}\left(A^{\alpha_0-\theta}\right), \ 0 \le \theta \le \alpha_0, & \text{(a)} \\ \left[\mathcal{D}\left(A^{*(\theta-\alpha_0)}\right)\right]', \ \alpha_0 \le \theta \le 1. & \text{(b)} \end{cases} \tag{1.13}$$

since $\alpha_0(1-\theta) - (1-\alpha_0)\theta = \alpha_0 - \theta$, with corresponding norm according to (1.9), (1.10)

$$\|x\|_{[\mathcal{E}, E]_\theta} = \|x\|_{\mathcal{D}(A^{\alpha_0-\theta})} = \left\|A^{\alpha_0-\theta} x\right\|_Y, \ 0 \le \theta \le \alpha_0, \tag{1.14}$$

$$\|z\|_{[\mathcal{E}, E]_\theta} = \|z\|_{\left[\mathcal{D}(A^{*(\theta-\alpha_0)})\right]'} = \left\|A^{-(\theta-\alpha_0)} z\right\|_Y, \ \alpha_0 \le \theta \le 1. \tag{1.15}$$

1.3. Main result

Theorem 1.1.

(a) Let $0 < T < \infty$. The operator A_F in (1.1) defined on Y generates a s.c. semigroup $T_F(t)$, which is analytic on Y and, moreover, possesses the maximal $L^p(0,T;Y)$-regularity on Y, $1 < p < \infty$, $T < \infty$; the map

$$f \to (Lf)(t) = \int_0^t e^{A_F(t-s)} f(s)ds \text{ continuous}$$
$$L^p(0,T;Y) \to L^p(0,T;\mathcal{D}(A_F)) \cap W^{1,p}(0,T;Y), \tag{1.16a}$$

so that there exists a constant $C = C_{p,T} > 0$ such that

$$\|y_t\|_{L^p(0,T;Y)} + \|A_F y\|_{L^p(0,T;Y)} \le C \|f\|_{L^p(0,T;Y)} \tag{1.16b}$$

in symbols, [11]

$$A_F \in MReg\,(L^p(0,T;Y)), \quad 1 < p < \infty,\ T < \infty. \tag{1.16c}$$

(b) Let $T = \infty$. Assume further that the s.c. analytic semigroup $T_F(t)$ is uniformly stable on Y: there exist constants $M \ge 1$, $\delta > 0$, such that

$$\|T_F(t)\|_{\mathcal{L}(Y)} \le M e^{-\delta t},\ t \ge 0. \tag{1.17a}$$

Then, $T_F(t)$ possesses the maximal $L^p(0,\infty;Y)$-regularity on Y, $1 < p < \infty$, $T = \infty$; in symbols [11]

$$A_F \in MReg\,(L^p(0,\infty;Y)), \quad 1 < p < \infty,\ T = \infty. \tag{1.17b}$$

Actually, in the each case (a) and (b), $T_F(t)$ extends/restricts with the same properties - as s.c. analytic, uniformly stable (case (b)) semigroup, with maximal L^p-regularity ($0 < T < \infty$ in case (a), $T = \infty$ in case (b)) - on the space E in (1.12), on the space $\mathcal{E}$ in (1.11), as well as on all holomorphic interpolation spaces (1.13)–(1.15).

1.4. Orientation and comparison with the literature

As mentioned, the present paper extends the theoretical scope, as well as the range of PDE-applications of paper [1]. More precisely, in mathematical terms, in [1, Eq (3.4)], the feedback operator F was assumed to be bounded $F \in \mathcal{L}(Y;U)$. In the present paper instead, we assume $F \in \mathcal{L}(\mathcal{D}(A^\sigma);U)$, $0 < \sigma < \sigma_0 < 1$. As a result, the proof of our main abstract theorem on L^p-maximal regularity is completely different from that in [1], since the proof of [1] requires $F \in \mathcal{L}(Y;U)$. The proof of [1] was based on considering A_F^* rather than A_F. With $F \in \mathcal{L}(Y;U)$ and G satisfying $A^\gamma G \in \mathcal{L}(U;Y)$ for some γ, $0 < \gamma < 1$, the expression of A_F makes such form not directly suitable for deducing its maximal regularity on Y, as it would leave the power $A^{1-\gamma}$ on the LHS of the expression of A_F in (1.1a) unaccounted for on Y. The form of A_F^* in [1] is more amenable to show $A_F^* \in MReg(L^p(0,T;Y^*))$ by perturbation [20, Theorem 6.2, p311], [21, Remark 1, p426, for $\beta = 1$]. Next, to return from A_F^* to A_F, i.e. show that the original A_F satisfies $A_F \in MReg(L^p(0,T;Y))$ as desired, paper [1] employs the result that on the UMD space Y [Appendix B, point #2] the property that $A_F \in MReg(L^p(0,T;Y))$ is equivalent to the property that the family, $\tau \in \mathcal{L}(Y)$, see definition in Appendix A, Eq. (A.2),

$$\tau \equiv \left\{ tR(it, A_F),\ t \in \mathbb{R}\backslash\{0\} \right\} \text{ be } R-\text{bounded},$$

where $R(\cdot, A_F)$ denotes the resolvent of A_F. And in the UMD-setting for Y, the R-boundedness property for the family τ is equivalent to the property that the corresponding dual family τ' in $\mathcal{L}(Y^*)$

$$\tau' \equiv \left\{ tR(it, A_F^*),\ t \in \mathbb{R}\backslash\{0\} \right\} \text{ be } R-\text{bounded},$$

[22, Proposition 8.4.1, p211].

The proof of the present paper with F unbounded as in (1.6), $F \in \mathcal{L}(\mathcal{D}(A^\sigma);U)$, is completely different. It is inspired by a proof in [13] about analyticity of a specific parabolic semigroup in an Hilbert setting. It consists of three steps, (i) first, showing L^p-maximal regularity in the larger space E in (1.12); next, (ii) showing L^p-maximal regularity in the smaller space $\mathcal{E}$ in (1.11); and finally, (iii) showing L^p-maximal regularity on Y by interpolation.

In closing the introduction, we recall that uniform stabilization, still in Besov setting, of the $d = 2, 3$ Navier–Stokes equations and $d = 2, 3$ Boussinesq systems by means of this time interior, localized, static, finite dimensional, feedback controls are given in [23,24] respectively.

2. Proof of Theorem 1.1

Proof. *Part (a), $T<\infty$; Step 1:* desired conclusion on the space $E=\left[\mathcal{D}(A^{*(1-\alpha_0)})\right]'$ in (1.12).

1(i): Let $\hat{A}$ be the operator (depending on α_0) which is the extension by isomorphism of the operator A in (**H.2**), as (Fig. 1), see (1.11), (1.12):

$$\hat{A}\,:\,\text{continuous } E\supset\mathcal{D}(\hat{A})\equiv\mathcal{D}(A^{\alpha_0})\equiv\mathcal{E}\text{ onto } E=\left[\mathcal{D}(A^{*(1-\alpha_0)})\right]' \tag{2.1}$$

so that, for $z\in\mathcal{D}(A^{\alpha_0})$

$$\hat{A}z\equiv\hat{A}^{(1-\alpha_0)}AA^{-(1-\alpha_0)}z\in E,\;\; AA^{-(1-\alpha_0)}z\in Y,\quad z\in\mathcal{D}(A^{\alpha_0}), \tag{2.2}$$

whose domain is topologized as follows, via (1.12), respectively (2.1), and (1.11)

$$\|x\|_{\mathcal{D}(\hat{A})}=\left\|\hat{A}x\right\|_E=\|A^{\alpha_0}x\|_Y=\|x\|_{\mathcal{D}(A^{\alpha_0})}=\|x\|_{\mathcal{E}},\quad x\in\mathcal{D}(A^{\alpha_0})\equiv\mathcal{E}, \tag{2.3}$$

$$\|z\|_E=\left\|\hat{A}^{-1}z\right\|_{\mathcal{D}(\hat{A})}=\left\|\hat{A}^{-1}z\right\|_{\mathcal{D}(A^{\alpha_0})}=\left\|\hat{A}^{-1}z\right\|_{\mathcal{E}},\quad z\in E. \tag{2.4}$$

1(ii): We rewrite the operator A_F in (1.1) [the equation in (1.2)] by invoking assumption (**H.3**)

$$y_t=-\hat{A}y+\hat{A}^{(1-\alpha_0)}(A^{\alpha_0}G)Fy\in E. \tag{2.5}$$

Since by assumption (**H.2**)

$$-A\in MReg\,(L^p(0,T;Y)),\;\;1<p<\infty, T<\infty, \tag{2.6}$$

it follows by the very definition of maximal regularity via the topological relation $\|y\|_Y=\left\|\hat{A}^{(1-\alpha_0)}y\right\|_E$ which follows from (1.12) with $y=A^{-(1-\alpha_0)}z\in Y$, or $z=A^{(1-\alpha_0)}y\in E$, that

$$\begin{gathered}\hat{A}\text{ possesses the maximal } L^p(0,T;E)-\text{regularity on } E,\\ -\hat{A}\in MReg\,(L^p(0,T;E)),\;\;1<p<\infty, T<\infty.\end{gathered} \tag{2.7}$$

1(iii): With reference to Eq. (2.5), we consider the perturbation operator $\hat{P}$ on E:

$$\begin{gathered}\hat{P}z\equiv\hat{A}^{(1-\alpha_0)}(A^{\alpha_0}G)Fz\in E,\;\;z\in\mathcal{D}(A^{\sigma})\equiv\mathcal{D}(\hat{P})\subset Y,\\ Fz\in U,\;\;\left(A^{\alpha_0}G\right)Fz\in Y,\;\;0<\sigma<\alpha_0<1\end{gathered} \tag{2.8}$$

recalling (1.6), under assumptions (**H.3**), (**H.4**). We accordingly estimate, initially via (1.12):

$$\left\|\hat{P}z\right\|_E=\left\|\hat{A}^{-(1-\alpha_0)}\hat{A}^{(1-\alpha_0)}(A^{\alpha_0}G)Fz\right\|_Y=\|(A^{\alpha_0}G)Fz\|_Y \tag{2.9}$$

$$\leq\left\|A^{\alpha_0}G\right\|_{\mathcal{L}(U;Y)}\|\langle\gamma z,w\rangle_U\,g\|_U \tag{2.10}$$

$$\leq\left\|A^{\alpha_0}G\right\|_{\mathcal{L}(U;Y)}\|\gamma\|_{\mathcal{L}(\mathcal{D}(A^{\sigma});U)}\|w\|_U\,\|g\|_U\,\|z\|_{\mathcal{D}(A^{\sigma})}. \tag{2.11}$$

From (2.9) to (2.11), we have invoked assumptions (1.3), (1.4), (1.5), (1.6). Next, since $\hat{A}\in\mathcal{L}(\mathcal{D}(A^{\alpha_0})\equiv\mathcal{E};E)$ via (2.3) (in fact, (2.1) holds), and $\sigma-\alpha_0<0$ by (1.5), then for $\varepsilon>0$ small

$$\|z\|_{\mathcal{D}(A^{\sigma})}=\|A^{\sigma}z\|_Y=\left\|A^{\sigma-\alpha_0+\varepsilon}A^{\alpha_0-\varepsilon}z\right\|_Y \tag{2.12}$$

$$\leq\left\|A^{-(\alpha_0-\sigma-\varepsilon)}\right\|_{\mathcal{L}(Y)}\left\|A^{\alpha_0-\varepsilon}z\right\|_Y \tag{2.13}$$

$$\|z\|_{\mathcal{D}(A^{\sigma})}\leq\left\|A^{-(\alpha_0-\sigma-\varepsilon)}\right\|_{\mathcal{L}(Y)}\left\|\hat{A}^{1-\varepsilon}z\right\|_E \tag{2.14}$$

by (2.3), $\left\|A^{(\alpha_0-\varepsilon)}z\right\|_Y=\left\|\hat{A}^{1-\varepsilon}z\right\|_E$, by (1.12) where $A^{-(\alpha_0-\sigma-\varepsilon)}\in\mathcal{L}(Y)$, by choosing $\varepsilon>0$ small enough that $\alpha_0-\sigma-\varepsilon>0$, via (1.5), so that $\alpha_0-\varepsilon>\sigma$ and $\mathcal{D}(A^{(\alpha_0-\varepsilon)})\subset\mathcal{D}(A^{\sigma})$. Finally, substituting (2.14) in (2.11) we obtain

$$\left\|\hat{P}z\right\|_E\leq C\left\|\hat{A}^{1-\varepsilon}z\right\|_E,\quad z\in\mathcal{D}(\hat{P})\equiv\mathcal{D}(A^{\sigma})\subset E. \tag{2.15}$$

So that the perturbation $\hat{P}\,:\,E\supset\mathcal{D}(\hat{P})=\mathcal{D}(A^{\alpha})\equiv\mathcal{E}\to E$ is $\hat{A}^{1-\varepsilon}$-bounded, $\varepsilon>0$. Moreover, $-\hat{A}$ has maximal regularity as in (2.7). We can then invoke the fundamental perturbation result of [20, Theorem 6.2, p311] or [21, Remark 1i, p426 for $\beta=1$] and conclude that

$$\hat{A}_F=-\hat{A}+\hat{P}\in MReg\,(L^p(0,T;E)), \tag{2.16}$$

$$E\supset\mathcal{D}(\hat{A}_F)=\mathcal{D}(\hat{A}),\;\;\text{since } \mathcal{D}(\hat{A})\equiv\mathcal{D}(A^{\alpha_0})\equiv\mathcal{E}\subset\mathcal{D}(A^{\sigma})\equiv\mathcal{D}(\hat{P}),\;\;\sigma<\alpha_0. \tag{2.17}$$

This completes Step 1.

Remark 2.1. Estimate (2.14) and hence estimate (2.15) holds also for $\varepsilon = 0$, so that the perturbation

$$\hat{P} : E \supset \mathcal{D}(\hat{P}) = \mathcal{D}(A^{\alpha}) \to E \tag{2.18}$$

is $\hat{A}$-bounded on E, with $\hat{A}$-bound equal to zero, being of finite dimensional range [19, Problem 1.14, p196].

Step 2: desired conclusion on the space $\mathcal{E} \equiv \mathcal{D}(A^{\alpha_0})$ in (1.11).

2(i): Consider an invertible translation of the generator

$$\hat{A}_{F,0} \equiv \hat{A}_F - \lambda_0 I : E \supset \mathcal{D}(\hat{A}_F) = \mathcal{D}(\hat{A}) = \mathcal{D}(A^{\alpha_0}) \equiv \mathcal{E} \to E \tag{2.19}$$

recalling (2.17). The set $\mathcal{D}(\hat{A}_{F,0})$ coincides (set theoretically) with $\mathcal{D}(\hat{A}_F)$ hence with $\mathcal{D}(A^{\alpha_0}) \equiv \mathcal{E}$.

Claim: The norms

$$\left\|\hat{A}_{F,0}x\right\|_E \text{ and } \|A^{\alpha_0}x\|_Y \equiv \|x\|_{\mathcal{E}}, \; x \in \mathcal{E} \tag{2.20}$$

are equivalent: on the one side, by (2.19), (2.3)

$$\begin{aligned} \left\|\hat{A}_{F,0}x\right\|_E &\le \left\|\hat{A}_{F,0}\hat{A}^{-1}\right\|_{\mathcal{L}(E)} \left\|\hat{A}x\right\|_E = \left\|\hat{A}_{F,0}\hat{A}^{-1}\right\|_{\mathcal{L}(E)} \|A^{\alpha_0}x\|_Y, \\ &= \left\|\hat{A}_{F,0}\hat{A}^{-1}\right\|_{\mathcal{L}(E)} \|x\|_{\mathcal{E}}, \quad x \in \mathcal{E}. \end{aligned} \tag{2.21}$$

On the other side, still by (2.3) and (2.19)

$$\|x\|_{\mathcal{E}} \equiv \|A^{\alpha_0}x\|_Y = \left\|\hat{A}x\right\|_E \le \left\|\hat{A}\hat{A}_{F,0}^{-1}\right\|_{\mathcal{L}(E)} \left\|\hat{A}_{F,0}x\right\|_E, \quad x \in \mathcal{E}. \tag{2.22}$$

and the claim is verified.

2(ii): Instead of $\hat{A}_F$ in (2.16), we shall equivalently work with $\hat{A}_{F,0}$ in (2.19) over a finite time $T < \infty$. Consider

$$y_t = \hat{A}_{F,0}y + f, \quad y(0) = 0. \tag{2.23}$$

Apply $\hat{A}_{F,0}$ across and consider

$$\left(\hat{A}_{F,0}y\right)_t = \hat{A}_{F,0}\left(\hat{A}_{F,0}y\right) + \hat{A}_{F,0}f \tag{2.24}$$

The maximal regularity result for $\hat{A}_F$ in (2.16) of Step 1 (equivalently for $\hat{A}_{F,0}$ over $T < \infty$) says: there exists a constant $C < \infty$ (independent of the data) such that

$$\int_0^T \left\|\left(\hat{A}_{F,0}y\right)_t(t)\right\|_E^p dt + \int_0^T \left\|\hat{A}_{F,0}\left(\hat{A}_{F,0}y\right)(t)\right\|_E^p dt \le C\int_0^T \left\|\hat{A}_{F,0}f(t)\right\|_E^p dt. \tag{2.25}$$

We now invoke the norm equivalence in (2.20) and rewrite accordingly (2.25) as

$$\int_0^T \|y_t(t)\|_{\mathcal{E}}^p dt + \int_0^T \left\|\hat{A}_{F,0}y(t)\right\|_{\mathcal{E}}^p dt \le C\int_0^T \|f(t)\|_{\mathcal{E}}^p dt. \tag{2.26}$$

Then (2.26) means

$$\hat{A}_{F,0}, \text{ hence } \hat{A}_F \in MReg\left(L^p(0,T;\mathcal{E})\right), \tag{2.27}$$

and Step 2 is established.

Step 3: Desired conclusion on the spaces $[\mathcal{E}, E]_\theta = [\mathcal{D}(A^{\alpha_0}), [\mathcal{D}(A^{*(1-\alpha_0)})]']_\theta$ in (1.13), in particular $[\mathcal{E}, E]_\theta \equiv Y$ for $\theta = \alpha_0$, see (1.13).

We apply interpolation theory between estimate (2.26) of Step 2, and (2.16) of Step 1:

$$f \in L^p(0,T;\mathcal{E}) : \Longrightarrow \left\{\hat{A}_{F,0}y, y_t\right\} \in L^p(0,T;\mathcal{E} \times \mathcal{E}) \text{ continuously,} \tag{2.28}$$

$$f \in L^p(0,T;E) : \Longrightarrow \left\{\hat{A}_{F,0}y, y_t\right\} \in L^p(0,T;E \times E) \text{ continuously,} \tag{2.29}$$

imply

$$f \in L^p(0,T;[\mathcal{E},E]_\theta) : \Longrightarrow \left\{\hat{A}_{F,0}y, y_t\right\} \in L^p\left(0,T;[\mathcal{E},E]_\theta \times [\mathcal{E},E]_\theta\right) \text{ continuously,} \tag{2.30}$$

where recalling (1.13)

$$[\mathcal{E}, E]_\theta \equiv Y \text{ for } \theta = \alpha_0. \tag{2.31}$$

Part (b), $T = \infty$: The proof is exactly the same except that now, because of the uniform stability in (1.17a), we can take throughout $T = \infty$, in particular in (2.26).

The proof of Theorem 1.1 is complete. □

3. Illustrations

For simplicity and brevity of exposition, Example # 1 (for $T < \infty$ and $T = \infty$) will be restricted to a canonical case. More general results can be given by referring to [9,25–28].

3.1. Case $0 < T < \infty$

Example #1:

The PDE model. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with boundary $\partial\Omega \equiv \Gamma$, assumed to be $(d-1)$-dimensional variety with Ω locally on one side of Γ, and sufficiently smooth. We consider the following canonical full boundary closed loop parabolic system on Ω, with boundary control in the Neumann BC and boundary sensing (observations):

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = (\Delta - I)y(t,x) & \text{in } (0,T]\times\Omega, \text{(a)} \\ y(0,x) = y_0(x) & \text{in } \Omega, \text{(b)} \\ \dfrac{\partial y(t,\xi)}{\partial \nu} = f(t,\xi) \equiv \displaystyle\sum_{k=0}^{K} (\gamma y(t,\cdot), w_k(\cdot))_\Gamma g_k(\xi), & \text{(c)} \\ \qquad \equiv F y(t,\cdot) & \text{on } (0,T]\times\Gamma. \text{(d)} \end{cases} \tag{3.1}$$

(a) Let

$$Y \equiv L^q(\Omega),\ 1 < q < \infty,\ A = -\Delta + I;\ Y \supset \mathcal{D}(A) \to Y, \tag{3.2a}$$

$$\mathcal{D}(A) = \left\{ \varphi \in W^{2,q}(\Omega) :\ \left.\frac{\partial\varphi}{\partial\nu}\right|_\Gamma = 0 \right\}. \tag{3.2b}$$

Then $-A$ generates a s.c. contraction, analytic semigroup e^{-At}, $t \geq 0$ on $Y \equiv L^q(\Omega)$. The fractional powers A^θ, $0 < \theta < 1$, are well-defined.
(b) γ denotes any continuous operator [15,29]

$$\gamma : \mathcal{D}(A^\sigma) \equiv W^{2\sigma,q}(\Omega) \to U \equiv L^q(\Omega),\ 2\sigma = \frac{1}{q} + \varepsilon \tag{3.3}$$

in particular the trace operator

$$\gamma\psi \equiv \psi|_\Gamma \in L^q(\Gamma),\ \psi \in W^{2\sigma,q}(\Omega). \tag{3.4}$$

Thus, the (feedback) operator F defined in (3.1d) satisfies

$$F : \mathcal{D}(A^\sigma) \equiv W^{2\sigma,q}(\Omega) \to U \equiv L^q(\Omega),\ 2\sigma = \frac{1}{q} + \varepsilon \tag{3.5}$$

as well, for all vectors $w_k \in L^{q'}(\Gamma), g \in L^q(\Gamma), \dfrac{1}{q} + \dfrac{1}{q'} = 1$, where $(\ ,\ )_\Gamma$ denotes the duality pairing between $L^q(\Gamma)$ and $L^{q'}(\Gamma)$.
(c) We introduce the Neumann (Green) map [14]

$$Gg \equiv \varphi \iff \left\{ (\Delta - I)\varphi \equiv 0 \text{ in } \Omega,\ \frac{\partial\varphi}{\partial\nu} = g \text{ on } \Gamma \right\}, \tag{3.6a}$$

$$G : U \equiv L^q(\Omega) \to W^{1+1/q,q}(\Omega) \subset \mathcal{D}(A^{\alpha_0}),\ \alpha_0 = \frac{1}{2} + \frac{1}{2q} - \varepsilon. \tag{3.6b}$$

(d) We observe from (3.3) and (3.6b) that

$$\sigma = \frac{1}{2q} + \frac{\varepsilon}{2} < \alpha_0 = \frac{1}{2} + \frac{1}{2q} - \varepsilon. \tag{3.7}$$

The abstract model. As is well known, we can rewrite (3.1a) as

$$y_t = (\Delta - I)y = (\Delta - 1)(y - Gf),\ \text{since } (\Delta - 1)(Gf) \equiv 0 \text{ in } \Omega \tag{3.8}$$

by (3.6a), recalling f in (3.1c). Moreover

$$\frac{\partial(y - Gf)}{\partial\nu} = \frac{\partial y}{\partial\nu} - \frac{\partial(Gf)}{\partial\nu} = f - f \equiv 0 \text{ on } \Gamma \tag{3.9}$$

and so $(y - Gf)$ satisfies the boundary conditions of the operator A in (3.2b). In conclusion, recalling $f = Fy$ from (3.1d) we can rewrite (3.8) as

$$y_t = -A(I - GF)y = A_F y \tag{3.10}$$

which is the abstract model on $Y \equiv L^q(\Omega)$ of the original PDE feedback model (3.1a–d). We now verify that the abstract model (3.10) for (3.1a–d) satisfies all abstract assumptions of Section 1.

(H.1) is satisfied since $Y \equiv L^q(\Omega)$, $1 < q < \infty$ is reflexive Banach space. **(H.2)** is satisfied since the operator $-A$ in (3.2a) is the maximal dissipative generator of a C_0-contraction semigroup e^{-At} on Y, $t \geq 0$, which possesses the maximal $L^p(0,T;Y)$-regularity property, $0 < T < \infty$, $1 < p < \infty$ [30]. (**H.3**) is satisfied since $U = L^q(\Gamma)$ is a Banach space $A^{\alpha_0}G \in \mathcal{L}(U; Y)$ from (3.6b), $\alpha_0 < 1$. (**H.4**) is satisfied by (3.5).

Conclusion. Problem (3.1a–d) satisfies all assumptions of Theorem 1.1, for $0 < T < \infty$, and hence $A_F \in MReg(L^p(0,T; L^q(\Omega)))$, $1 < p < \infty$, $1 < p < \infty$, $T < \infty$ with $A_F = -A(I - GF)$ in (3.10). This conclusion is true for all $w_k \in L^{q'}(\Gamma)$, $g_k \in L^q(\Gamma)$. Below we shall consider the case $T = \infty$.

Example #2:

PDE and abstract models. We return to [2] and consider the linearized Navier–Stokes [30–35] problem over a bounded domain Ω in $\mathbb{R}^d$, $d = 2, 3$, with boundary $\partial\Omega \equiv \Gamma$ (after translation by the equilibrium solution, see [2, Eq (1.28)])

$$\begin{cases} w_t - \nu_o \Delta w + L_e(w) + \nabla \chi = 0 & \text{in } Q, \quad (a) \\ \text{div } w = 0 & \text{in } Q, \quad (b) \\ w \equiv v \equiv \sum_{k=0}^{K} \langle \gamma w, p_k \rangle_\Gamma \, g_k \equiv Fw & \text{on } \Sigma, \quad (c) \\ w(0,x) = w_0(x) & \text{on } \Omega \quad (d) \end{cases} \tag{3.11}$$

where abstract version is given by

$$\frac{dw}{dt} = \mathcal{A}_q w - \mathcal{A}_q D \left(\sum_{k=1}^{K} \langle \gamma w, p_k \rangle_\Gamma \, g_k \right) \tag{3.12a}$$

$$= \mathcal{A}_q w - \mathcal{A}_q D F w = \mathcal{A}_q (I - DF) \equiv \mathbb{A}_{F,q} w. \tag{3.12b}$$

see [2, Eq (4.3)] with $m \equiv 0$ and a feedback boundary control v in (3.11c). We have

$$Y \equiv L^q_\sigma(\Omega),\ q \geq 2,\ \mathcal{A}_q = -(\nu_o A_q + A_{o,q}),\ \mathcal{D}(\mathcal{A}_q) = \mathcal{D}(A_q) \subset L^q_\sigma(\Omega) \quad [2, \text{Eq } (2.16)], \tag{3.13}$$

$$A_q z = -P_q \Delta z,\ \mathcal{D}(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \quad [2, \text{Eq } (2.14)], \tag{3.14}$$

$$L_e(z) = (y_e \,.\, \nabla) z + (z \,.\, \nabla) y_e \quad [2, \text{Eq } (1.9)], \tag{3.15}$$

$$A_{o,q} z = P_q L_e(z) = P_q[(y_e \,.\, \nabla) z + (z \,.\, \nabla) y_e], \tag{3.16a}$$

$$\mathcal{D}(A_{o,q}) = \mathcal{D}(A_q^{1/2}) = W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega) \subset L^q_\sigma(\Omega) \quad [2, \text{Eq } (2.15)]. \tag{3.16b}$$

Helmholtz direct sum decomposition.

$$L^q(\Omega) = L^q_\sigma(\Omega) \oplus G^q(\Omega), \tag{3.17}$$

$$\begin{aligned} L^q_\sigma(\Omega) &= \overline{\{y \in C^\infty_c(\Omega) : \text{div } y = 0 \text{ in } \Omega\}}^{\|\cdot\|_q} \\ &= \{g \in L^q(\Omega) : \text{div } g = 0;\ g \cdot \nu = 0 \text{ on } \partial\Omega\}, \\ &\qquad \text{for any locally Lipschitz domain } \Omega \subset \mathbb{R}^d,\ d \geq 2 \ [31, \text{p119}], \end{aligned} \tag{3.18}$$

$$G^q(\Omega) = \{y \in L^q(\Omega) : y = \nabla p,\ p \in W^{1,q}_{loc}(\Omega) \text{ where } 1 \leq q < \infty\}. \tag{3.19}$$

See [2, Eq (1.4), (1.5)]. The unique linear bounded and idempotent (i.e. $P_q^2 = P_q$) projection operator $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ having $L^q_\sigma(\Omega)$ as its range and $G^q(\Omega)$ and its null space is the Helmholtz projection.

We now return to the abstract model (3.12b) and verify that for $0 < T < \infty$ it satisfies all the assumptions of Theorem 1.1.

(**H.1**) is verified since $Y \equiv L^q_\sigma(\Omega)$, $1 < q < \infty$, is a reflexive Banach space. (**H.2**) is verified since the operator $\mathcal{A}_q$ in (3.13) has maximal L^p-regularity on $Y \equiv L^q_\sigma(\Omega)$, for $0 < T < \infty$

$$\mathcal{A}_q \in MReg(L^p(0,T; L^q_\sigma(\Omega))),\ 1 < p < \infty,\ 1 < T < \infty \tag{3.20}$$

[2, Eq (A.32b)] and hence $\mathcal{A}_q$ generates a s.c. analytic semigroup $e^{\mathcal{A}_q t}$ on $Y \equiv L^q_\sigma(\Omega)$, $t \geq 0$ [20,28,35–37]. Moreover, $\mathcal{A}_q$ is also maximal dissipative after a suitable translation. (**H.3**) is satisfied with

$$U \equiv U_q \equiv \{g \in L^q(\Gamma) :\ g \cdot \nu = 0 \text{ on } \Gamma\}, \tag{3.21}$$

$$D : U_q \to W^{1/q,q}(\Omega) \cap L^q_\sigma(\Omega) \subset \mathcal{D}(A_q^{1/2q - \varepsilon}), \tag{3.22a}$$

$$\text{or } A_q^{1/2q - \varepsilon} D \in \mathcal{L}(U_q, L^q_\sigma(\Omega)), \quad \sigma_0 = \frac{1}{2q} - \varepsilon \tag{3.22b}$$

(**H.4**) is satisfied by taking

$$\gamma :\ \text{continuous } \mathcal{D}(A_q^\sigma) \subset Y \equiv L^q_\sigma(\Omega) \to U \text{ with } 0 < \sigma < \sigma_0 = \frac{1}{2q} - \varepsilon \tag{3.23}$$

so that then

$$F : \text{ continuous } \mathcal{D}(A_q^{\sigma}) \subset Y \to U \tag{3.24}$$

as well. Then we take $p_k \in L^{q'}(\Gamma)$ and $g \in L^q(\Gamma)$, $\frac{1}{q} + \frac{1}{q'} = 1$. Then all the assumptions of Theorem 1.1 are satisfied for the feedback operator $\mathbb{A}_{F,q} = \mathcal{A}_q(I - DF)$ in (3.12b).

3.2. Case $T = \infty$

We return to Example #1, except that, to make the problem more significant, we replace (3.1a) by the canonical equation

$$\frac{\partial y(t,x)}{\partial t} = (\Delta + k^2) y(t,x) \tag{3.25}$$

k^2 large, while keeping Eqts (3.1b–c). Thus, for $f \equiv 0$, the corresponding free dynamics operator

$$A\varphi = (\Delta + k^2)\varphi, \quad Y \equiv L^q(\Omega) \supset \mathcal{D}(A) = \left\{ \varphi \in W^{2,q}(\Omega), \left. \frac{\partial \varphi}{\partial \nu} \right|_{\Gamma} = 0 \right\} \tag{3.26}$$

is the generator of a s.c. analytic semigroup on Y which is unstable and possesses maximal $L^p(0,T;Y)$-regularity, $T < \infty$. We take the boundary vectors $g_k \in L^q(\Gamma)$ to be linearly independent. According to Theorem 1.1(b), or the basis of the analysis of Example #1 (k^2 rather than -1 is irrelevant), we only need to verify the additional assumption that, for suitable vectors $\boldsymbol{w}_k \in L^{q'}(\Gamma), g_k \in L^q(\Gamma)$, the semigroup $T_F(t) = e^{A_F t}$, $A_F = -A(I - GF)$ in (3.10), is exponentially stable

$$\left\| e^{A_F t} \right\|_{\mathcal{L}(Y)} \equiv \| T_F(t) \|_{\mathcal{L}(Y)} \le M e^{-\delta t}, \quad t \ge 0,\ \delta > 0,\ Y \equiv L^q(\Omega). \tag{3.27}$$

This statement amounts to saying that the original boundary homogeneous problem (3.25), (3.1a–d) which with $f \equiv 0$ is unstable (i.e. it has finitely many unstable eigenvalues on $\mathscr{C}^+ = \{\lambda \in \mathscr{C} : \operatorname{Re} \lambda \ge 0\}$) can be uniformly stabilized by a finite dimensional feedback control $f(t,\xi) =$ RHS of (3.1c), with suitable boundary vectors $g_k \in L^q(\Gamma)$ and boundary sensors $w_k \in L^{q'}(\Gamma)$. This problem was originally studied in early 1980s, see [9,13,38–41] and references therein. The vectors w_k have to be chosen to satisfy the algebraic condition

$$\text{rank } W_k = \ell_k = \text{geometric multiplicity of the unstable eigenvalue } \lambda_k \text{ of the self-adjoint operator } A \text{ in (3.26)} \tag{3.28}$$

where

$$W_k = \begin{bmatrix} \langle \boldsymbol{w}_1, \boldsymbol{\Phi}_{k1} \rangle_\Gamma, \langle \boldsymbol{w}_1, \boldsymbol{\Phi}_{k2} \rangle_\Gamma & \cdots & \left\langle \boldsymbol{w}_1, \boldsymbol{\Phi}_{k\ell_k} \right\rangle_\Gamma \\ \langle \boldsymbol{w}_2, \boldsymbol{\Phi}_{k1} \rangle_\Gamma, \langle \boldsymbol{w}_2, \boldsymbol{\Phi}_{k2} \rangle_\Gamma & \cdots & \left\langle \boldsymbol{w}_2, \boldsymbol{\Phi}_{k\ell_k} \right\rangle_\Gamma \\ \vdots & & \vdots \\ \langle \boldsymbol{w}_K, \boldsymbol{\Phi}_{k1} \rangle_\Gamma, \langle \boldsymbol{w}_K, \boldsymbol{\Phi}_{k2} \rangle_\Gamma & \cdots & \left\langle \boldsymbol{w}_K, \boldsymbol{\Phi}_{k\ell_k} \right\rangle_\Gamma \end{bmatrix} \tag{3.29}$$

$\langle \cdot, \cdot \rangle_\Gamma$ duality pair, where $\{\boldsymbol{\Phi}_{k1}, \dots, \boldsymbol{\Phi}_{k\ell_k}\}$ are the normalized eigenvectors in Y of the unstable eigenvalues λ_k of the (self-adjoint) operator A in (3.26). Condition (3.29) can always be satisfied by infinite choices of the vectors $\boldsymbol{w}_1, \dots, \boldsymbol{w}_K$, since for every λ_k, the Dirichlet traces $\left\{ \boldsymbol{\Phi}_{k1}|_\Gamma, \boldsymbol{\Phi}_{k2}|_\Gamma, \dots, \boldsymbol{\Phi}_{k\ell_k}|_\Gamma \right\}$ are linearly independent [42–44].

It is known [9] that if Ω is either a d-sphere or a d-parallelepiped, is it always possible to select boundary vectors g_k, $k = 1, \dots, k$ such that the exponential decay (3.27) holds true [9] and hence Theorem 1.1 holds true for $T = \infty$ for these special geometries. For other geometries, $d \ge 2$, technical conditions are available which cannot be recalled here for brevity of exposition [45]. We refer to the Ref [7]. Moreover, if $d = 1$, the uniform stability (3.27) is impossible if A has at least 3 unstable eigenvalues, with only to boundary vectors g_1, g_2 at $x = 0$, or $x = 1$ with $\Omega = (0,1)$ [9].

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Appendix A. R-boundedness on E is equivalent to R-boundedness on $\mathcal{E}$

The conclusion of Step 1, (2.16), that $\hat{A}_{F,0} \in MReg(L^p(0,T;E))$ is equivalent to the property that the family τ of bounded operators on $E, \tau \in \mathcal{L}(E)$

$$\tau \equiv \left\{ tR(it, \hat{A}_{F,0}),\ t \in \mathbb{R}\backslash\{0\} \right\} \text{ be } R-\text{bounded.} \tag{A.1}$$

This means [21,37] that there exists a constant $C < \infty$ such that: for all members $t_1 R(it_1, \hat{A}_{F,0}), \dots, t_n R(it_n, \hat{A}_{F,0})$; and all elements $e_1, \dots, e_n \in E, n \in \mathbb{N}$, we have

$$\int \left\| \sum_{j=0}^{n} r_j(u) t_j R(it_j, \hat{A}_{F,0}) e_j \right\|_E du \le C \int \left\| \sum_{j=0}^{n} r_j(u) e_j \right\|_E du \tag{A.2}$$

in the usual notation [37, p198]. We want to show that the R-boundedness condition (A.2) on E implies (in fact, is equivalent to) the R-boundedness condition on $\mathcal{E}$: for all $v_1, \dots, v_n \in \mathcal{E}, n \in \mathbb{N}$

$$\int \left\| \sum_{j=0}^{n} r_j(u) t_j R(it_j, \hat{A}_{F,0}) v_j \right\|_{\mathcal{E}} du \le \hat{C} \int \left\| \sum_{j=0}^{n} r_j(u) v_j \right\|_{\mathcal{E}} du. \tag{A.3}$$

To this end, we invoke the norm-equivalence in (2.20) along with the (critical) property that $\hat{A}_{F,0}$ commutes with its resolvent $R(it_j, \hat{A}_{F,0})$.

(A.2) implies (A.3). Write $v_j = \hat{A}_{F,0}^{-1} e_j \in \mathcal{E}$, for $e_j \in E$ or $e_j = \hat{A}_{F,0} v_j$. Rewrite accordingly (A.2) by (2.20) as

$$\int \left\| \sum_{j=0}^{n} r_j(u) \left[t_j R(it_j, \hat{A}_{F,0}) \right] v_j \right\|_{\mathcal{E}} du = \int \left\| \sum_{j=0}^{n} r_j(u) \hat{A}_{F,0} \left[t_j R(it_j, \hat{A}_{F,0}) \right] v_j \right\|_E du \tag{A.4}$$

$$= \int \left\| \sum_{j=0}^{n} r_j(u) \left[t_j R(it_j, \hat{A}_{F,0}) \right] \hat{A}_{F,0} v_j \right\|_E du \tag{A.5}$$

$$(\text{by (A.2)}) \qquad \le C \int \left\| \sum_{j=0}^{n} r_j(u) \hat{A}_{F,0} v_j \right\|_E du \tag{A.6}$$

$$(\text{by (2.20)}) \qquad = C \int \left\| \sum_{j=0}^{n} r_j(u) v_j \right\|_{\mathcal{E}} du. \tag{A.7}$$

Moreover, since $\hat{A}_{F,0}$ is an isomorphism $\mathcal{E} \to E$ (onto), with $\hat{A}_{F,0}^{-1} : E \to \mathcal{E}$ (onto), any collection $\{e_j\}_{j=0}^n \subset E$ can be obtained by a collection $\{v_j\}_{j=0}^n \in \mathcal{E}$ via $e_j = \hat{A}_{F,0} v_j$; and conversely, any collection $\{v_j\}_{j=0}^n \in \mathcal{E}$ can be obtained by a collection $\{e_j\}_{j=0}^n \in E$ by $v_j = \hat{A}_{F,0}^{-1} e_j \in \mathcal{E}$. This way we exhaust all finite collections in $\mathcal{E}$ and E. Thus, (A.7) shows (A.3).

(A.3) $\Longrightarrow$ (A.2): Similarly, with $e_j = \hat{A}_F v_j$

$$\int \left\| \sum_{j=0}^{n} r_j(u) \left[t_j R(it_j, \hat{A}_{F,0}) \right] e_j \right\|_E du = \int \left\| \sum_{j=0}^{n} r_j(u) \hat{A}_{F,0} \left[t_j R(it_j, \hat{A}_{F,0}) \right] v_j \right\|_E du \tag{A.8}$$

$$(\text{by (2.20)}) \qquad = \int \left\| \sum_{j=0}^{n} r_j(u) \left[t_j R(it_j, \hat{A}_{F,0}) \right] v_j \right\|_{\mathcal{E}} du \tag{A.9}$$

$$\le \hat{C} \int \left\| \sum_{j=0}^{n} r_j(u) \hat{v}_j \right\|_{\mathcal{E}} du \tag{A.10}$$

$$= \hat{C} \int \left\| \sum_{j=0}^{n} r_j(u) \hat{A}_{F,0} v_j \right\|_E du \tag{A.11}$$

$$= \hat{C} \int \left\| \sum_{j=0}^{n} r_j(u) \hat{e}_j \right\|_E du \tag{A.12}$$

and (A.2) is proved.

Appendix B. Review of some basic notions

At the suggestion of one referee, we give here some basic definitions/notions studied in the paper.

1. **Maximal regularity.** Consider the following abstract Cauchy problem

$$y_t(t) = Ay(t) + f(t), \quad y(0) = 0 \tag{B.1}$$

on the Banach space Y, where A is the infinitesimal generator of a s.c. (C_0) semigroup e^{At}, $t \geq 0$, on Y. We say that A has "maximal L^p-regularity" over $[0,T]$, $0 < T \leq \infty$, in case the map

$$\begin{cases} f \mapsto (Lf)(t) \equiv \int_0^t e^{A(t-s)} f(s)ds \text{(a)} \\ \text{is continuous } L^p(0,T;Y) \to L^p(0,T;\mathcal{D}(A)) \text{(b)} \end{cases} \tag{B.2}$$

i.e. in case there is a constant $K > 0$ such that

$$\int_0^T \|Ay(t)\|_Y^p \, dt \leq K \int_0^T \|f(t)\|_Y^p \, dt. \tag{B.3}$$

In which case then, via (B.1), we also have

$$\int_0^T \|y_t(t)\|_Y^p \, dt \leq C \int_0^T \|f(t)\|_Y^p \, dt \tag{B.4}$$

for a constant $C > 0$. Ultimately

$$\|y_t\|_{L^p(0,T;Y)} + \|Ay\|_{L^p(0,T;Y)} \leq \hat{C} \|f\|_{L^p(0,T;Y)} \quad \hat{C} > 0. \tag{B.5}$$

Ref. [36], 1964, showed by use of singular integrals that: on a Hilbert space Y, A generates a s.c. analytic semigroup if and only if A has maximal L^p-regularity on $[0,T]$, $T < \infty$; and on $T = \infty$ case e^{At} is uniformly stable. On a Banach space Y, A has maximal L^p-regularity implies that A is the generator of a s.c. analytic semigroup; but not conversely. The list of references includes only a limited selection of works on maximal L^p-regularity, such as [46] and the references therein.

2. **UMD Space.** From [47, p75]: A Banach space X is called a UMD space (*Unconditional Martingale Differences* property), if the Hilbert transform

$$Hf(t) = \text{pv} - \int_0^t \frac{1}{t-s} f(s)ds, \;\; f \in S(\mathbb{R}, X) \tag{B.6}$$

extends to a bounded operator on $L^p(\mathbb{R}, X)$ for some (or equivalently, for each) $p \in (1, \infty)$. All subspaces and quotient spaces of $L^p(\Omega, \mu)$ for $1 < p < \infty$ have the UMD-property, but $L^1(\Omega, \mu)$ or spaces of continuous functions $C(K)$ do not. As a general principle, Sobolev spaces, Hardy spaces, and other commonly used function spaces in analysis are UMD if they are reflexive.

3. **Definition of Besov spaces $B^s_{q,p}$ on domains of class C^1 as real interpolation of Sobolev spaces.** Let m be a positive integer, $m \in \mathbb{N}, 0 < s < m, 1 \leq q < \infty, 1 \leq p \leq \infty$, then we define [48, p1398] *the interpolation space*

$$\boldsymbol{B}^s_{q,p}(\Omega) = (\boldsymbol{L}^q(\Omega), \boldsymbol{W}^{m,q}(\Omega))_{\frac{s}{m},p}. \tag{B.7a}$$

This definition does not depend on $m \in \mathbb{N}$ [29, p xx]. This clearly gives

$$\boldsymbol{W}^{m,q}(\Omega) \subset \boldsymbol{B}^s_{q,p}(\Omega) \subset \boldsymbol{L}^q(\Omega) \quad \text{and} \quad \|y\|_{\boldsymbol{L}^q(\Omega)} \leq C \|y\|_{\boldsymbol{B}^s_{q,p}(\Omega)}. \tag{B.7b}$$

We shall be particularly interested in the following special real interpolation space [15, The K-method, p15] of the $\boldsymbol{L}^q$ and $\boldsymbol{W}^{2,q}$ spaces $\left(m = 2, s = 2 - \frac{2}{p}\right)$:

$$\boldsymbol{B}^{2-\frac{2}{p}}_{q,p}(\Omega) = \left(\boldsymbol{L}^q(\Omega), \boldsymbol{W}^{2,q}(\Omega)\right)_{1-\frac{1}{p},p}. \tag{B.8}$$

The definition of real interpolation space is as follows [15, p15], [37, p35]. Let X, Y be Banach spaces with $Y \subset X$ and continuous injection. We construct a family of intermediate spaces between X and Y, called *real interpolation spaces*, and denoted by $(X,Y)_{\theta,p}$, $0 < \theta \leq 1$, $1 \leq p \leq \infty$, using the K-method. First for every $x \in X$ and $t > 0$, set

$$K(t,x,X,Y) = \inf_{x=a+b,\, a \in X,\, b \in Y} \left(\|a\|_X + t\|b\|_Y\right). \tag{B.9}$$

Then define

$$(X,Y)_{\theta,p} = \left\{ x \in X \, : \, t \mapsto t^{-\theta-1/p} K(t,x,X,Y) \in L^p(0,+\infty) \right\}, \tag{B.10}$$

$$\|x\|_{(X,Y)_{\theta,p}} \equiv \left\| t^{-\theta-1/p} K(t,x,X,Y) \right\|_{L^p(0,+\infty)}. \tag{B.11}$$

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