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Long-time behavior of stochastic evolution equations with multiplicative rough noise

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Contents

Chapter 1

Introduction

The present thesis is primarily dealing with stochastic partial differential equations in a Banach space $\mathcal{E}$ with nonlinear multiplicative noise of the form

$$\begin{cases} \mathrm{d}y_t &= \big(A(t)y_t + F(t,y_t)\big)\ \mathrm{d}t + G(t,y_t)\ \mathrm{d}\mathbf{X}_t, \\ y_0 &\in \mathcal{E}. \end{cases} \tag{1-1}$$

Here, $\big(A(t)\big)_{t\in\mathbb{R}}$ is a family of linear operators, F, G are time-dependent nonlinearities, y_0 the initial value, and $(\mathbf{X}_t)_{t\in\mathbb{R}}$ is a stochastic noise which is specified later. Stochastic equations of parabolic type, such as the one given in equation (1-1), are of significant mathematical importance and are also employed in various scientific disciplines. The incorporation of randomness enables the modeling of uncertainty, a feature that is advantageous in numerous applications. Before we discuss the detailed content of this thesis, it is beneficial to provide a general overview of the motivation behind stochastic modeling.

For example, stochastic is widely used to model asset pricing in mathematical finance. The first appearance of such stochastic models in finance was introduced by Bachelier in 1900 in his doctoral thesis [Bac00]. He modeled the stock prices at the Paris Bourse, assuming that the increments should be independent, normally distributed, and without memory. Here, the increments are just the difference in the asset price between two fixed points in time. Independence, therefore, means that the past at the stock market does not influence whether the price falls or rises in the next period. These assumptions on the process of a stock price lead to the Brownian motion as the model of choice. Today, Bachelier is considered one of the pioneers of modern mathematical finance, as his early considerations led to the creation of numerous models to investigate the dynamics of financial markets. Over the years, the importance of stochastic modeling has risen. Today, there are also many stochastic (partial) differential equations in use, for example, the Heath–Jarrow–Morton model for interest rates [HJM92] or the Heston model, which describes the evolution of volatility [Hes93].

More recently, stochastic models are also increasingly being used to describe natural phenomena. They have proven to be particularly useful in climate science. Hasselmann primarily influenced the idea of incorporating randomness into climate models [Has76, FH77], suggesting that climate fluctuations should not be attributed exclusively to external deterministic forcings. Rather, the climate is influenced by noisy forces due to the random development of weather patterns. This phenomenon can be illustrated by considering the motion of a heavy ball, representing the slowly responding climate, being forced by randomly moving smaller balls, representing the rapidly changing weather. These observations coincide with the original considerations regarding Brownian motion, in which the seemingly random movement of pollen in water was caused by constant collisions with microscopic water particles. Accepting this framework of Brownian motion in climate dynamics leads, besides being a more suitable model, to a much more complicated, high-dimensional nonlinear system. For his contribution to climate modeling, Hasselmann received the 2021 Nobel Prize in physics. In light of these findings, it has been established that a significant number of climate models use stochastic forcings, despite their fundamental deterministic structure [FO17, Pal19, ABCR21, FBOL22]. Following this trend, an increasing number of mathematical publications are also dealing with (stochastic) climate models [IVS01, MTVE01, HIP10, DSF24].

Another recent development is the consideration of stochastic processes with a memory effect, which are mathematically modeled by so-called non-Markovian processes, like the fractional Brownian motion. In the case of a Brownian motion, which is a Markov process, future events are independent of what has happened in the past, so we say it has no memory. In numerous instances, this assumption of memorylessness of a system represents a substantial reduction in the complexity. Indeed, many recent studies have demonstrated the involvement of memory effects in various applications. One of the earliest examples of such a phenomenon comes from the hydrologist Hurst [Hur51]. He demonstrated that the probability of a Nile flood increases if high water levels and floods have occurred more frequently in previous years. This is referred to as a positive correlation of increments, meaning that the previous trend is more likely to continue in the same direction. The Hurst parameter H measures this correlation and was determined to be $H = 0.74$ in the case of Nile floods. Values above $1/2$ indicate a positive, values below $1/2$ a negative correlation, and $H = 1/2$ means there is no correlation. The stochastic process with a general Hurst parameter $H \in (0, 1)$ is known as fractional Brownian motion, where we recover the Brownian motion for $H = 1/2$. However, non-Markovian processes are more complex to handle than a Brownian motion. This complexity makes them, in particular, interesting from a purely mathematical perspective. Indeed, there has been an increasing body of research into considering stochastic partial differential equations with more general, non-Markovian processes as noise, see for example [HL20, LS22, CMO22, BN22, LNZ24] and the references cited therein. Furthermore, the fractional Brownian motion has established itself beyond mathematical interest as the model of choice in some applications, such as mathematical finance. One reason for this is that volatility has been found to behave like a fractional Brownian motion with a small Hurst parameter, resulting in so-called rough volatility models [BFG16, GJR18, FSW25].

This brief overview aims to provide a general insight into the motivation for incorporating stochastic influences into models. So, from an application point of view, we are convinced that stochastic partial differential equations are worth considering in many cases. However, the resulting questions, such as considering more general noise, are also very interesting and challenging from a purely mathematical perspective. Recent studies are therefore increasingly moving away from Brownian motion as the preferred model towards more general noise terms. The objective of this thesis is to investigate, in particular, the long-term behavior of stochastic partial differential equations of the form (1-1) with such general noise, modeled by a rough path that allows a pathwise analysis of the solution. In addition, special attention is given to equations in a bounded domain $\mathcal{O}$. This is relevant insofar as many results in the stochastic literature focus on the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n \setminus \mathbb{Z}^n$, which simplifies many calculations as no boundary conditions need to be taken into account. In this case, it is particularly interesting if the noise affects only the boundary of the area $\partial\mathcal{O}$ and not the interior.

In the remainder of this introduction, these three primary aspects of this thesis are examined in greater detail: boundary noise, dynamical behavior, and pathwise integration. The significance of these aspects is highlighted below, as are the latest developments in the individual subject areas and the contribution made by this dissertation.

Equations with boundary noise

First of all, it is worth mentioning that from an application point of view, it makes sense to consider equations on a bounded domain as well as on the torus or the whole space. For bounded domains $\mathcal{O}$, it is necessary to include boundary conditions on $\partial\mathcal{O}$, which makes the analysis more involved. The torus is analogous to periodic boundary conditions and is applicable in many situations where the system extends to infinity, such as the ocean or the atmosphere, at least in one direction. On the other hand, both the atmosphere and the ocean have boundaries, such as the seabed or the surface, for which periodic boundary conditions are not suitable. More useful here are, for example, Dirichlet conditions, where a fixed value is prescribed on the boundary $y_t|_{\partial\mathcal{O}} = u_t$, for example, if the system is bounded by a solid wall, or Neumann conditions, where flows can be modeled as they can occur on a water surface. In addition to Dirichlet and Neumann conditions, many other conceivable boundary conditions correspond to a wide variety of physical meanings, such as dynamical, Robin, or Wentzell conditions, to name a few.

Adding noise on the boundary introduces another layer of complexity and a potential realism to models that traditional internal noise models may overlook. The general idea behind this concept is that in many real-world applications, the interaction of a system with its environment primarily occurs at its boundaries, as this is the only way external influences can be introduced. For instance, the interface between the ocean and atmosphere can be viewed as the boundary of the ocean, which is subject to perturbations caused by wind or other weather effects [TM20]. This influence can significantly affect weather patterns, ocean and atmospheric circulations, and climate dynamics. Other examples, such as random boundary excitations of a Stokes flow, an elastic half-plane, and reaction-diffusion equations, are examined in a series of papers [SS08, Sab08, SS09a, SS09b, Sab24] by Shalimova and Sabelfeld using the Karhunen–Loève expansion.

Besides these potential applications, the impact of boundary noise on the corresponding system is a challenging question. From a mathematical point of view, the consideration of boundary noise originated from the field of control theory, which deals with boundary controls [Las80, LT81, Bal81]. The motivation here is the same as above: The easiest way to influence a system with external control is to add it via the boundary. The semigroup approach used here also builds the basis for the solution concept for equations with boundary noise, which is explained in more detail in Section 3.3.

Since optimal control problems and stochastic evolution equations are related, as seen in [DPZ14, (9.60)], the first results transferring the theory of boundary controls to boundary noise were obtained only a few years later [Ich85a, Ich85b]. A first milestone was the paper by Da Prato and Zabczyk [DPZ93], who proved that in the case of Neumann conditions, a function-valued solution exists. However, for the Dirichlet condition, the solution only exists in the space of distributions.

Over the following years, articles on equations with boundary noise were published repeatedly [FW92, MM93, Sow94, Mas95, DMPD98]. However, these dealt exclusively with Neumann or Robin boundary conditions. The first breakthrough on Dirichlet conditions comes from Alós and Bonaccorsi [AB02a, AB02b]. They used techniques from Malliavin calculus in a weighted Hilbert space setting to treat the semilinear stochastic heat equation with Dirichlet boundary noise. Based on this idea, Brzeźniak, Fabbri, Goldys, Russo, and Peszat in particular have advanced the theory of Dirichlet boundary noise [FG09, BGPR15].

A different way of preventing the problems concerning Dirichlet conditions is to treat the fractional Brownian motion instead of the Brownian motion on the boundary. If the Hurst parameter H is bigger than $1/2$, the resulting process $\left(B_t^H\right)_{t\geq 0}$ is more regular then the original Brownian motion. To be precise, the sample paths $t \mapsto B_t^H(\omega)$ are Hölder continuous with parameter $H - \varepsilon$ for every small $\varepsilon > 0$. In particular, if $H > 3/4$ then also Dirichlet conditions are possible to treat [DPDM02] and Neumann conditions can be considered for every $H > 1/4$ [DPDM06].

Other notable achievements concerning boundary noise include the consideration of dynamical boundary conditions [CS07], the numerical solution of the Burgers equation [GHB17], optimal control with boundary noise [DFT07, Mun17], and the comprehensive analysis of non-autonomous stochastic partial differential equations with values in Banach spaces and Neumann boundary noise [SV11].

In recent years, interest in systems with boundary noise has increased [CMO22, GP23, AL24, ABL25]. Indeed, a growing body of research has begun to examine equations in fluid dynamics, adding noise on the boundary. In consideration of the motivation previously outlined, this approach is also logical, as the mentioned wind-driven boundary conditions are suitable for stochastic modeling. This particular situation was the subject of a detailed investigation focused on the primitive equation in [BHHS24].

In terms of long-term behavior, equations with boundary noise have only been studied sparsely so far. The problem here is that no flow transformations are possible when Dirichlet or Neumann conditions are considered. What is known are some results on the (exponential) stability of such systems [Ich85b, Mas95, AB02b, GP23], which do not utilize the random dynamical system approach. In the case of dynamical boundary conditions, the situation is different, as the special structure of the boundary conditions allows for the use of a random dynamical system, see [CS07, FSTT19]. At this point in the research, we contribute by enabling the formulation of more extensive statements regarding the dynamics of such equations. Specifically, we demonstrate the existence of random dynamical systems and attractors, as well as Lyapunov exponents, for a substantial class of equations subject to Neumann boundary noise. In the context of regular noise, such as the fractional Brownian motion with $H > 3/4$, the consideration of Dirichlet noise is also included. All these results are based on the articles [NS23, BS24, BGVS25].

Dynamical behavior

A particularly important aspect in investigating (partial) differential equations is the long-term behavior of solutions. By this, we mean the analysis of solutions that exist over an infinitely long period, as well as the sensitivity of the initial data. The relevance of such investigations is omnipresent. For example, we want to know when a so-called tipping point of a system occurs, i.e., a point in time at which the system moves out of its current stable state into a new one. A prominent tipping point is the cessation of the Atlantic meridional overturning circulation (AMOC) [vWKD24, CAW25, Noo25]. In its current state, this current in the North Atlantic transports heat and salt water northwards and colder water southwards. This mechanism plays a key role in regulating climatic conditions in the Northern Hemisphere. Reaching the tipping point would mean that this current would weaken or even cease, which could have far-reaching consequences for the European climate. Mathematically, these tipping points are so-called bifurcations, and warning signals are used to investigate the extent to which it is possible to predict when these bifurcations will occur [KLN22, BDK24, BK25].

To study bifurcations and other dynamical aspects, such as attractors, Lyapunov exponents, and invariant manifolds, we rely on the theory of (random) dynamical systems, which is used to model the evolution over time of an underlying system. Given an initial datum and a time t, the dynamical system then provides the system's status at time t, allowing us to analyze its behavior with respect to its initial value. The theory of dynamical systems goes back to Newtonian mechanics and has been further developed over time by both mathematicians and physicists due to its relevance in applications. Pioneers of the first hour were Poincaré [Poi92], who investigated the dynamics of the three-body problem in particular, and Lyapunov [Lya92], who decisively shaped the stability analysis and established the use of Lyapunov functions and exponents.

In essence, Lyapunov exponents quantify the divergence or convergence of nearby trajectories. In the case of simple differential equations like

$$\dot{y}_t = Ay_t,$$

the Lyapunov exponents are just the eigenvalues of A, which are known to characterize the dynamical behavior of the solution. In the non-autonomous case, it turns out that the eigenvalues of the family $\big(A(t)\big)_{t\in[0,T]}$ do not describe the asymptotic behavior sufficiently. This gap is filled by the Lyapunov exponents of a system.

The existence of Lyapunov exponents for random dynamical systems is ensured by so-called ergodic theorems. The most important of these are Kingman's subadditive [Kin68] and Oseledec's celebrated multiplicative ergodic theorem [Ose68]. Furthermore, Ruelle generalized the ergodic theorem to Hilbert spaces [Rue82], and Lian and Lu to Banach spaces [LL10]. More recently, Riedel and Varzaneh proved an ergodic theorem for fields of Banach spaces [GVR23], which can be applied to delay equations. In this thesis, we mostly use the theorem by Riedel and Varzaneh and apply it to a single Banach space.

The area of Lyapunov exponents is an active field of research, investigating the existence and the resulting conclusions for the corresponding equations [ELR19, BPS23, GT24]. It is particularly interesting to establish bounds for Lyapunov exponents and thus obtain sign information [BY22, BBPS22, BBPS23]. This information on the sign is used to make a concrete statement about stability, such as the existence of a stable manifold [GVR25, BGVS25] if the top Lyapunov exponent is negative. In fact, in many cases, it is difficult to check which sign the exponents have, which is why there are also some weaker concepts, as the finite-time Lyapunov exponents. They, as the name suggests, are only defined on a finite time interval and therefore represent the local behavior of the solution [BN23, BB25].

The main object of interest in this thesis is the global attractor. In simple terms, an attractor is a compact subset of the state space that "attracts" specific solutions of the system. This property of attracting solutions can be defined in many ways, which will be discussed in more detail in Section 4.1. For an autonomous system, the differences between those approaches are not as substantial and lead to analogous results. However, in the context of time-dependent systems, the situation differs, and it plays a crucial role which type of attraction is chosen. In this thesis, we address the so-called pullback approach, which involves fixing the observation time of the system and sending the initial time against $-\infty$. In contrast to the forward attraction, where the initial time is fixed and the observation time converges to ∞, the resulting attractor depends on time. Given that the noise is modeled as a stochastic process, its behavior is particularly dependent on time. Consequently, the pullback approach can also be utilized in the stochastic case. This approach has proven successful in many cases, see [CSG11].

There are numerous results on the existence of attractors for stochastic partial differential equations, starting with the famous works by Crauel, Debussche, Flandoli and Schmalfuss [Sch92, CF94, FS96, CDF97] for equations driven by a Brownian motion followed by more recent articles [GLR11, Wan12, Wan14, Ges14, GLS20] dealing also with more general noise terms. It should be noted that their focus is either on additive, i.e., $G(t, y_t) \equiv \sigma$ is a constant, or linear multiplicative noise, i.e., $G(t, y_t) = \sigma y_t$. In these situations, there exists a flow transformation from the stochastic partial differential equation to a random partial differential equation, sometimes referred to as Doss–Sussmann transformation. This is necessary, as it is often unclear whether a stochastic partial differential equation generates a random dynamical system. If such a transformation exists, the random partial differential equation admits a random dynamical system since it can be solved pathwise. Then, after transforming back, we can conclude that the original equation also admits a random dynamical system, see [IS01] and Subsection 4.2.3. There are only a few papers that circumvent this procedure and deal with nonlinear multiplicative noise directly, using Wong–Zakay approximations [ZL24], mean-square random dynamical systems [Wan19], or pathwise integration [GALS16].

As mentioned in the general motivation, we also consider non-Markovian noise to account for memory effects. The problem here is that non-Markovian processes are also not semi-martingales, so the analysis of the associated stochastic partial differential equations is generally more complex [DPDM02, DPDM06, DMPD12]. Recent results on the long-term behavior of equations where the noise is modeled by a fractional Brownian motion are the existence of a random dynamical system [MS04, GALS10], ergodicity [Hai05, GAS11], exponential stability [BGAHS17, DGANS18] and the existence of attractors [GAMS10, GGAS14, CG24].

One of the main contributions of this thesis is the existence of a random attractor. We consider a pathwise approach, outlined in the next subsection, which allows the investigation of general nonlinear multiplicative noise terms, including the Brownian and fractional Brownian motion.

Pathwise integration

Throughout this thesis, we investigate mild solutions for equations of type (1-1). This solution concept involves a convolution of the form

$$\int_s^t S_{t,r} G(r, y_r) \, \mathrm{d}\mathbf{X}_r, \tag{1-2}$$

where the parabolic evolution family, denoted by $(S_{s,t})_{s\leq t}$, is equivalent to a semigroup in the autonomous case. In order to define solutions, it is necessary to first define this integral. This issue is especially complex in the context of $\mathbf{X}$ being a stochastic process, such as Brownian motion and fractional Brownian motion. In particular, we aim to define the integral in a pathwise sense, as this offers several advantages in the study of long-term behavior, as mentioned earlier. For a more comprehensive overview of the various integration concepts, we refer to Section 2.1.

One of the first approaches to define integrals pathwise, where $\mathbf{X}$ is a stochastic process, is due to Föllmer [Fö81]. He developed a pathwise alternative to the Itô calculus by using the Itô formula to define the integral against a path with quadratic variation along a sequence of partitions, which is not to be confused with the bounded 2-variation. Several authors later extended this concept [AC17, CP19, CJ24]. However, the breakthrough in the pathwise integration theory came with the seminal paper by Lyons [Lyo98]. In this paper, the concept of a rough path is introduced, which is a pair $\mathbf{X} = (X, \mathbb{X})$ of a path X and a second-order process $\mathbb{X}$. The idea is that $\mathbb{X}$ provides additional information about the path which is necessary to define integrals of the type $\int X \, \mathrm{d}X$.

Over the following years, the theory was developed further and applied to differential equations [LCL07, Dav08]. Gubinelli and Tindel were able to define rough integrals of the form (1-2) with the help of controlled rough paths, which also made it possible to investigate rough partial differential equations [Gub04, GT10]. This was followed by further work on semilinear [HN19, GHN21] and quasilinear problems [OW19, HN24]. In 2013, Hairer was able to solve the Kardar–Parisi–Zhang equation with his regularity structures, which contain the theory of rough paths as a special case [Hai13, Hai14] and have proven to be a breakthrough in the study of singular stochastic partial differential equations.

All of the articles mentioned so far only deal with local well-posedness; global solutions were only investigated in [HN20, HN22]. Due to the rough path techniques, quadratic terms appear in the estimates, which generally pose a problem for proving global solutions. Under what conditions does a global solution exist is still a prominent question [LY25]. Furthermore, many existing works consider the equations on the torus and do not treat time-dependent diffusion coefficients. In this thesis, we address all of these aspects. Among others, we consider equations on a domain $\mathcal{O}$, where the noise due to the rough path only acts on the boundary [NS23]. Moreover, as in [BGVS25], we deal extensively with non-autonomous rough partial differential equations, allowing for time-dependent diffusion coefficients.

In addition to the abstract results for equations, the use of rough paths has also increased in many areas of application. Since the theory of rough paths does not rely on semi-martingales as noise, rough paths are also used where memory effects are investigated, such as in mathematical finance [FGP21, ACLP23, ALP24, BBFP25] or stochastic control theory [DFG17, FLZ24]. Due to the pathwise integration, the rough paths also provide an ideal framework for investigating the long-term behavior of dynamical systems. Where previously only additive, or linear multiplicative, noise could be investigated, the use of rough paths also makes it possible to consider nonlinear multiplicative noise using the random dynamical system approach. The foundation for investigating the long-time behavior of rough partial differential equations was laid by Bailleul, Riedel, and Scheutzow [BRS17], who successfully unified random dynamical systems and the theory of rough paths. Subsequently, further results on the long time behavior of rough (partial) differential equations have been established, such as the existence of random dynamical systems [HN22, CHNR22, GGH25], stability properties [DHC19, Hes22, Duc22a, DHC25], the existence of attractors [Duc22b, DK23, YLZ23, CG24] and invariant manifolds [NK21, KN23, GVR25].

In this thesis, we contribute to the area of global attractors and the existence of Lyapunov exponents. The existing works on attractors are either dealing with linear multiplicative noise [CG24], rough differential equations [Duc22b], or numerical attractors [DK23, CDH25]. As in [YLZ23], we can show the existence of attractors for nonlinear multiplicative noise using different techniques under natural assumptions on the coefficients, which also allows us to obtain optimal regularity results. This framework can be applied to partial differential equations with nonlinear multiplicative rough boundary noise [BS24]. To the best of our knowledge, this is the first result concerning attractors for equations with Neumann or Dirichlet boundary noise, which was not possible so far, since for such equations, no flow transformations are possible, and thus no random dynamical system could be proven before [NS23]. For equations with dynamical boundary conditions, the work of Chuesov and Schmalfuss [CS07] treats the equation in the product space $\mathrm{L}^2(\mathcal{O}) \times \mathrm{L}^2(\partial\mathcal{O})$. This framework enables the use of a flow transformation, allowing for the proof of the existence of a global pullback attractor.

In order to prove the existence of Lyapunov exponents, we establish a mild Gronwall for equations of the form (1-1). This inequality is necessary to prove the integrability assumption of the multiplicative ergodic theorem established by [GVR25] and is the first Gronwall inequality for mild solutions of rough partial differential equations, which makes it interesting in its own right. So far, the only other Gronwall-type estimate is due to Deya, Gubinelli, Hofmanová, and Tindel, who used weak solutions of rough partial differential equations [DGHT19] in the framework of unbounded rough drivers. However, this result can only be applied to rough partial differential eqautions with transport type noise instead of nonlinear multiplicative noise as in our situation. The second ingredient, which is also used for the global attractor, is an integrable solution bound in the controlled rough path norm for non-autonomous rough partial differential equations [BGVS25].

Outline of the thesis

This thesis builds upon the following four articles. In addition to the mentioned publications, certain results have been generalized.

"Stochastic evolution equations with rough boundary noise"
Alexandra Neamţu and Tim Seitz
Partial Differential Equations and Applications, Vol. 4, No. 6, 2023, 1–27

"Existence and regularity of random attractors for stochastic evolution equations driven by rough noise"
Alexandra Blessing (Neamţu) and Tim Seitz
Journal of Dynamics and Differential Equations, 2024, 1–24

"A mild rough Gronwall lemma with applications to non-autonomous evolution equations"
Alexandra Blessing (Neamţu), Mazyar Ghani Varzaneh and Tim Seitz
Preprint: arXiv:2503.03628, to appear in "Stochastic Partial Differential Equations: Analysis and Computation".

"Pathwise mild solutions for superlinear stochastic evolution equations and their attractors"
Alexandra Blessing (Neamţu), Tim Seitz, Stefanie Sonner and Bao Quoc Tang
Preprint: arXiv:2502.01209, under review in "Journal of Differential Equations".

The thesis is structured as follows:

Part I In the first part, the primary emphasis is laid on rough partial differential equations, establishing a foundational framework for the subsequent investigation of their long-term behavior.

We begin by examining the fundamental motivations behind rough paths in Section 2.1 and the resulting rough integrals, highlighting the distinctions between these concepts and other integration theories.
The basic definitions and results are then introduced and explained in Section 2.2 using standard textbooks on the subject, such as [FV10, FH20]. Specifically, we also mention the construction of rough paths using Gaussian processes, such as the Brownian motion, the fractional Brownian motion, and Volterra processes in Subsection 2.2.1. Subsection 2.2.2 features a brief exploration of the Cameron–Martin space of a rough path, as studied in [BGVS25], which plays a pivotal role in the integrability of solutions to rough differential equations. Building upon the theory of rough paths, we introduce the rough integral in Section 2.3. Utilizing the Sewing lemma, we focus on the concept developed by Gerasimovičs, Hocquet, and Nilssen [GHN21], which leads to controlled rough paths and is particularly well-suited for parabolic equations. This section augments the established results by incorporating non-autonomous diffusion coefficients, as detailed in [BGVS25]. Additionally, we investigate in Subsection 2.3.4 the interplay between controlled rough paths and extrapolated operators, which is a necessary step for the subsequent study of evolution equations with rough boundary noise.

Chapter 3 treats rough evolution equations with non-autonomous coefficients. First, the rough partial differential equations in the Sections 3.1 and 3.2 are analyzed for local and global solutions. The results presented here are based on [GHN21, HN22] and have been extended to the non-autonomous case in [BGVS25].
In particular, a rough evolution equation with boundary noise is considered in Section 3.3, which is the main topic of the article [NS23].
At the end of this chapter, the important properties of the solutions are examined, which play an essential role in the investigation of their long-term behavior. Section 3.4 deals with integrable bounds for solutions of rough partial differential equations and combines the results from [BS24, BGVS25], and in Section 3.5 a mild Gronwall inequality is proven, which is also part of the article [BGVS25].

Part II The second part of this thesis is devoted to the long-time behavior of rough evolution equations.
Chapter 4 introduces the theory of random dynamical systems. Specifically, Section 4.1

explains the motivation behind the concept of random dynamical systems, in which the autonomous and non-autonomous deterministic case is examined first to highlight the differences. Afterwards, Section 4.2 addresses this theory's most significant definitions and theorems. In particular, Subsection 4.2.3 examines under which conditions rough differential equations, or random partial differential equations, generate a random dynamical system.
The subsection 4.2.2 deals with non-autonomous random dynamical systems, which are relevant for the study of time-dependent coefficients. Here, we also introduce the concept of an extended probability space, as described in [BGVS25], which combines random and non-autonomous influences.

Chapter 5 now contains the main results of this work. Section 5.1 is largely based on [BS24] and deals with the existence of attractors for rough evolution equations. The results from [BS24] are presented for the more general non-autonomous case. In particular, the results are again applicable for equations with boundary noise, which is explained in Subsection 5.1.4.
Furthermore, in section 5.2, the existence of Lyapunov exponents is proved and applied to equations with boundary noise, which is, to the best of our knowledge, the first result in this direction.

Appendices Before the appendices, Chapter 7 provides a summary and an outlook indicating possible research directions and open questions.

The appendices primarily serve to introduce the foundational concepts and tools utilized throughout the thesis. These appendices are designed to present the results in a manner that aligns with the thesis's framework.
Appendix A introduces the Sobolev spaces as well as some results regarding interpolation theory. Here, we gather, for example, the Sobolev inequalities or important properties about interpolating Sobolev spaces. Appendix B gives an overview of sectorial operators and analytic semigroups. These two concepts are particularly important for stochastic partial differential equations, as we frequently utilize the smoothing property of analytic semigroups. Further, the fractional power spaces and the concept of bounded imaginary powers are introduced. In order to treat equations of boundary noise, it is useful to work in Banach scales, which is the topic of Appendix C. The last Appendix D contains an overview of deterministic parabolic evolution equations. In particular, the conditions of Acquistapace–Terreni are discussed here, in order to treat non-autonomous equations as well as some assumptions to solve parabolic boundary value problems.

Part I

Non-autonomous rough partial differential equations

Chapter 2

The theory of rough paths

2.1 Motivation

To briefly introduce the theory of rough paths, we take a step back and start with an old and widely studied problem: integration along non-smooth paths. The purpose of this is to illuminate the development of the theory over the years, highlighting both the positive and negative aspects of existing approaches, and to clarify the contribution of rough path theory to it.

As also mentioned in the introduction, the fundamental motivation behind this thesis is to analyze systems that evolve over time. In the simplest case, such an evolution can be modeled by an ordinary differential equation of the form

$$\dot{y}_t = G(y_t). \tag{ODE}$$

The evolution is then encoded in the solution $y\colon [0,T] \to \mathbb{R}$ of the problem (ODE). Under suitable conditions on $G\colon \mathbb{R} \to \mathbb{R}$, for example, Lipschitz continuity, and a fixed initial value $y_0 \in \mathbb{R}$, it is known that there exists a unique solution of the form

$$y_t = y_0 + \int_0^t G(y_r)\,\mathrm{d}r,$$

for $t \in \mathbb{R}$. It is, therefore, essential that we first have a suitable definition of the integral on the right-hand side before studying the evolution itself. In this particular situation, this definition is given by the Lebesgue integral or the Bochner integral if G has values in a general Banach space. Nonetheless, not all problems of interest are as straightforward as this one. In particular, we want to study controlled differential equations

$$\dot{y}_t = G(y_t)\dot{X}_t, \tag{CODE}$$

where $\dot{X}$ is the derivative of a known function $X\colon [0,T] \to \mathbb{R}$, called the control of y. The name originates from the fact that, in this case, the small-scale fluctuations of y look like a rescaled version of X, so X has control over the behavior of y. For $X\colon \mathbb{R} \to \mathbb{R}, t \mapsto X_t := t$, the ordinary differential equation is a special case of a controlled differential equation.

To solve (CODE), we examine the equivalent integral equation

$$y_t = y_0 + \int_0^t G(y_r)\,\mathrm{d}X_r, \tag{2-1}$$

as similarly done for (ODE). A key step is to establish the integral with respect to the control X. In the case where the control is smooth enough, which means at least $X \in \mathcal{C}^1$, we can formally write

$$y_t = y_0 + \int_0^t G(y_r)\dot{X}_r\,\mathrm{d}r,$$

using the chain rule. The question now arises whether less smooth control terms, such as Hölder continuous paths $X \in \mathcal{C}^\gamma$ for $\gamma < 1$, can be allowed? In particular, can we allow random controls, such as Brownian motion? In the following subsections, we investigate different approaches to define the integral for less smooth integrators.

Remark 2.1. If X is not differentiable, one usually considers the integral equation (2-1) instead of (CODE). In this context

$$\mathrm{d}y_t = G(y_t)\,\mathrm{d}X_t,$$

is commonly used as a shortened form of (2-1).

Riemann–Stieltjes integration

As part of his habilitation, Riemann worked on the construction of the integral [Rie67]. He approximated the area under the curve by summing up rectangles with a fixed width, which is then chosen to be arbitrarily small. The height of each rectangle is determined by the function values within that rectangle's width. To be more precise, let

$$\pi \coloneqq \big\{[t_i, t_{i+1}] : 0 \coloneqq t_0 \le t_1 \le \ldots \le t_N \coloneqq t\big\}$$

be a partition of the interval $[0, t]$ with mesh size $|\pi| \coloneqq \max_{i=0,\ldots,N} |t_{i+1} - t_i|$. Then a path $f\colon [0, T] \to \mathbb{R}$ is called Riemann integrable if the following limit exists

$$\int_0^t f_r\,\mathrm{d}r \coloneqq \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} f_{w_{u,v}}(v-u), \tag{2-2}$$

where $w_{u,v}$ is an intermediate point of $[u, v]$. The integral is independent of the choice of $w_{u,v}$, but usually, one chooses them in a way such that $f_{w_{u,v}}$ is the supremum or the infimum of f on the interval $[u, v]$. These choices of $w_{u,v}$ are going back to Darboux [Dar75], and the resulting sums are called upper (Darboux) or lower (Darboux) sums.

The extension of this concept to a more general integrator $X\colon \mathbb{R} \to \mathbb{R}$, firstly appeared in [Sti94] and is known as the Riemann–Stieltjes integral. Here, the author defines the integral for a control X as

$$\int_0^t f_r\,\mathrm{d}X_r \coloneqq \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} f_{w_{u,v}}(X_v - X_u), \tag{2-3}$$

replacing the difference $v - u$ by $X_v - X_u$, which takes up the aforementioned interpretation that the solution of (CODE) can be seen as a rescaled version of f. Stieltjes proved that (2-3) is well-defined if X has bounded variation and f is bounded or the other way around. However, the approximation used in (2-3) can be extended to even rougher paths.

Young integration

It took around half a century to determine the precise conditions under which the Riemann–Stieltjes integral exists and when it does not. 1936 Young showed in his article [You36] that the right-hand side in (2-3) is finite even for paths with finite p-variation. Recall that a Banach valued path $f\colon J \to \mathcal{V}$ has finite p-variation with $p \in [1, \infty)$ if

$$[f]_{p\text{-var},J,\mathcal{V}} \coloneqq \left(\sup_{\pi} \sum_{[u,v]\in\pi} \|f_v - f_u\|_{\mathcal{V}}^p\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all partitions of the interval $J \subset \mathbb{R}$. The space of finite p-variation, denoted by $\mathcal{C}^{p\text{-var}}(J;\mathcal{V})$, is a Banach space using the norm

$$\|f\|_{p\text{-var},J,\mathcal{V}} \coloneqq \sup_{s\in J}\|f_s\|_{\mathcal{V}} + [f]_{p\text{-var},J,\mathcal{V}}.$$

If $\mathcal{V} = \mathbb{R}^d$ we shorten the notation by $\|f\|_{p\text{-var},J}$ and $[f]_{p\text{-var},J}$.

Theorem 2.2. ([You36]) *Let $f \in \mathcal{C}^{p\text{-var}}([0,T];\mathbb{R}), X \in \mathcal{C}^{q\text{-var}}([0,T];\mathbb{R})$ such that $1/p + 1/q > 1$. Then (2-3) is finite, and $\int_0^t f_s \,\mathrm{d}X_s$ is well-defined as a Young integral. Further, we obtain*

$$\left|\int_s^t f_r \,\mathrm{d}X_r - f_s(X_t - X_s)\right| \lesssim [f]_{p\text{-var},[s,t]}\,[X]_{q\text{-var},[s,t]}\,,$$

for $0 \le s \le t \le T$.

Remark 2.3. It is also possible to state Theorem 2.2 in the setting of Hölder continuous functions. Therefore recall that a Banach valued continuous path $f\colon J \to \mathcal{V}$ on an interval $J \subset \mathbb{R}$, is called Hölder continuous of parameter $\gamma \in (0,1]$ if the Hölder semi-norm

$$[f]_{\gamma,J,\mathcal{V}} \coloneqq \sup_{s,t\in J, s\neq t}\frac{\|f_t - f_s\|_{\mathcal{V}}}{|t-s|^{\gamma}}$$

is finite. The space of Hölder continuous functions, denoted by $\mathcal{C}^{\gamma}(J;\mathcal{V})$, is a Banach space using the norm

$$\|f\|_{\gamma,J,\mathcal{V}} \coloneqq \sup_{s\in J}\|f_s\|_{\mathcal{V}} + [f]_{\gamma,J,\mathcal{V}}.$$

If $\mathcal{V} = \mathbb{R}^d$ we shorten the notation by $\|f\|_{\gamma,J}$ and $[f]_{\gamma,J}$. It is known that $\mathcal{C}^{\gamma} \subset \mathcal{C}^{1/\gamma\text{-var}}$ and therefore, Theorem 2.2 can be expressed as follows:

Let $f \in \mathcal{C}^{\gamma'}([0,T];\mathbb{R}), X \in \mathcal{C}^{\gamma}([0,T];\mathbb{R})$ be such that $\gamma' + \gamma > 1$. Then (2-3) is finite, and $\int_0^t f_s \,\mathrm{d}X_s$ is well-defined as a Young integral. Further, we obtain

$$\left|\int_s^t f_r \,\mathrm{d}X_r - f_s(X_t - X_s)\right| \lesssim \|f\|_{\gamma',[s,t]}\,\|X\|_{\gamma,[s,t]}\,(t-s)^{\gamma'+\gamma},$$

for $0 \le s \le t \le T$.

This result by Young is sharp. Indeed, there are paths f and X where $1/p + 1/q = 1$, but the Riemann–Stieltjes approximation in (2-3) does not converge, see [You36, Section 7].

Stochastic integration

As mentioned in the introduction, we also want to examine the situation where the control X is random. Therefore, consider a scalar-valued Brownian motion $(B_t)_{t\ge 0}$ on the filtered probability space $\big(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\ge 0}\big)$ which is a stochastic process with $B_0 = 0$, $\mathbb{P}$-almost surely continuous paths, and independent Gaussian increments, that means $B_t - B_s$ is independent of $\mathcal{F}_s$ and normally distributed with mean 0 and variance $t - s$ for $0 \le s \le t$.

It is known that the sample paths $t \mapsto B_t(\omega)$ for $\omega \in \Omega$ are almost surely Hölder continuous with parameter $\gamma < 1/2$, which is a direct consequence of Kolmogorov's continuity theorem [Kun90, Theorem 1.4.1]. Therefore, the integral

$$\int_0^T B_r(\omega)\,\mathrm{d}B_r(\omega) \tag{2-4}$$

is, in general, not well-defined as a Young integral, where $T > 0$.

One way to resolve this problem is to use the Itô calculus introduced in [Itô44]. In particular, Itô introduced a stochastic concept to define integrals with respect to the Brownian motion, for example, (2-4). We won't go into further detail here, but the basic concept is to use the same Riemann approximation as in (2-3) and replace the type of convergence with a stochastic convergence. For example, consider a left-continuous stochastic process $(Y_t)_{t\in[0,T]}$ such that Y_s is $\mathcal{F}_s$-measurable for every $s \in [0,T]$ and the following $\mathrm{L}^2(\Omega)$-norm is finite

$$\mathbb{E}\left[\int_0^T |Y_r|^2 \, \mathrm{d}r\right] < \infty.$$

Then, the Itô integral is defined as

$$\int_0^T Y_r \, \mathrm{d}B_r \coloneqq \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} Y_u(B_v - B_u),$$

where the limit is taken in $\mathrm{L}^2(\Omega)$. Unlike in the deterministic case, the value of the integral depends now on the choice of the intermediate point, where the left endpoint Y_u leads to the Itô integral. Another prominent choice is to replace Y_u by $1/2(Y_v - Y_u)$ which leads to the Stratonovich integral [Fis66, Str66].

More generally, the Itô and Stratonovich integral can be defined for semi-martingales. We will state the existence of these integrals in the continuous case; for more information, see [RY99, Chapter IV] and [Kal21, Chapter 18 & 20].

Theorem 2.4. *Let $T > 0$, $\big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]}\big)$ be a filtered probability space, $(X_t)_{t\in[0,T]}$ a continuous semi-martingale and $(Y_t)_{t\in[0,T]}$ an adapted, left continuous, and locally bounded stochastic process. Then, the Itô integral is given by*

$$\int_0^t Y_r \, \mathrm{d}X_r \coloneqq \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} Y_u(X_v - X_u).$$

Similarly, the Stratonovich integral is defined as

$$\int_0^t Y_r \circ \mathrm{d}X_r \coloneqq \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \frac{Y_u + Y_v}{2}(X_v - X_u),$$

where both limits are taken in probability.

This stochastic integration theory was a breakthrough in the analysis of stochastic (partial) differential equations; see, for example, the monographs [Øks03, IW81] for stochastic ordinary differential equations and [DPZ14, LR15] for stochastic partial differential equations, along with the references cited therein. However, these integrals are stochastic by nature and, in general, not pathwise!

Also, the Itô calculus is only applicable for semi-martingales and fails in the non-Markovian setting, so a fractional Brownian motion $\big(B^H_t\big)_{t\geq 0}$ is not in the scope of the Itô calculus. It is also possible to use stochastic methods to extend the integral concept to a fractional Brownian motion; however, these concepts become more complex, as seen in [Zä98, AMN00, PT00, CCM03, AN03, Nua03] and the references therein. The integral for more general fractional stochastic processes has been investigated more recently in [CMO22], which also includes the analysis of non-Gaussian processes.

In addition to this restriction to semi-martingales, the Itô calculus leads to another issue, which involves the well-posedness of stochastic differential equations. To be more precise, the following negative result exists.

Proposition 2.5 ([Lyo91]). *There exists no separable Banach space $\mathcal{V} \subset \mathcal{C}([0,T];\mathbb{R})$ such that $\mathcal{V}$ contains the paths of a Brownian motion and the map*

$$\mathcal{C}^\infty([0,T];\mathbb{R}) \times \mathcal{C}^\infty([0,T];\mathbb{R}) \to \mathcal{C}^\infty([0,T];\mathbb{R}), (f,g) \mapsto \int_0^T f_r g_r' \,\mathrm{d}r,$$

extends to a continuous map on $\mathcal{V} \times \mathcal{V}$ into the continuous functions.

The consequence of this proposition is that even simple differential equations driven by a Brownian motion are possibly ill-posed in the Itô sense. For example, consider the equations

$$\begin{aligned} \mathrm{d}y_t^{(1)} &= \mathrm{d}B_t^{(1)}, \\ \mathrm{d}y_t^{(2)} &= y_t^{(1)} \,\mathrm{d}B_t^{(2)}, \end{aligned}$$

where $\big(B_t^{(1)}\big)_{t\geq 0}$ and $\big(B_t^{(2)}\big)_{t\geq 0}$ are two independent Brownian motions. Formally, the second component $y^{(2)}$ has the form $\int_0^t B_r^{(1)} \dot{B}_r^{(2)} \,\mathrm{d}r$, so due to Proposition 2.5 the solution does not depend continuously on the noise, which makes the problem ill-posed.

Even if the addition of stochastic properties to the definition of the integral has proved to be very helpful, we still want a pathwise integral concept.

Pathwise integration of rough controls

There is one component that combines all the integral concepts we have seen so far: They all use Riemann sums as an approximation for the integral. It turns out that this approximation is no longer good enough if the integrand and integrator are too rough. In a sense, the Riemann sum does not have enough information to cover rapidly changing paths. Itô's theory addresses this lack of information by employing the martingale property. In this subsection, we establish an alternative approach to compensate for the missing information, which motivates the theory of rough paths.

To better understand what we mean by this missing information, consider the smooth path $X \in \mathcal{C}^\gamma([0,T];\mathbb{R})$ for $\gamma > 1/2$ and a smooth function $G \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$. For every $r \in [u,v]$ the Taylor expansion of $G(X)$ yields

$$G(X_r) \approx G(X_u) + \mathrm{D}G(X_u)X_{u,r}, \tag{2-5}$$

where $X\colon [0,T]^2 \to \mathbb{R}, (s,t) \mapsto X_{s,t} := X_t - X_s$ is the two-parameter function associated to the path X. Integrating (2-5) with respect to X, which is well-defined as a Young integral, leads to

$$\int_u^v G(X_r) \,\mathrm{d}X_r \approx G(X_u)X_{u,v} + \mathrm{D}G(X_u) \int_u^v X_{u,r} \,\mathrm{d}X_r.$$

Summing over an arbitrary partition π of $[s,t]$, the left-hand side equals the integral over $[s,t]$. Taking the limit $|\pi| \to 0$ of this expression yields

$$\int_s^t G(X_r) \,\mathrm{d}X_r = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \Big(G(X_u)X_{u,v} + \mathrm{D}G(X_u) \int_u^v X_{u,r} \,\mathrm{d}X_r \Big), \tag{2-6}$$

which is known as the compensated Riemann sum. In comparison to the approximation used in (2-2) with $w_{u,v} := u$, it turns out that

$$\lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \mathrm{D}G(X_u) \int_u^v X_{u,r} \,\mathrm{d}X_r = 0. \tag{2-7}$$

This can be interpreted as follows: The second-order terms do not contribute any information to the definition of the integral. However, this changes when X is only Hölder continuous with parameter $\gamma < 1/2$. Then, the Riemann and compensated Riemann sums are not necessarily identical, and (2-7) is further expected to be nonzero.

The idea is to replace the Riemann sum when it stops being convergent with the compensated Riemann sum. But, the key issue is that (2-6) is not usable for rougher inputs since it involves the iterated integral

$$\int_u^v X_{u,r}\, \mathrm{d}X_r, \tag{2-8}$$

which is not well-defined for $\gamma < 1/2$. To overcome this major problem, we have to replace this integral with an algebraically similar object.

This is the fundamental idea of a rough path and the resulting rough integrals. Based on this, it is then possible to use an approximation inspired by the compensated Riemann sum (2-6) to establish a pathwise integration theory and, therefore, a theory to investigate equations like (CODE) for rough paths.

2.2 Hölder rough paths

This section provides a more detailed consideration of the theory of rough paths. The notation and presentation of the results are based on [FH20]. For further reading, we also refer to [LCL07, Dav08, FV10] and the original article by Lyons [Lyo98]. Fix from now on $T > 0$.

The initial step is to specify what we mean by an algebraically similar object to (2-8). For this, note that every concept presented in Section 2.1 fulfills basic additivity properties. For example, let $X\colon [0,T] \to \mathbb{R}$ be a real-valued smooth path, then the additivity of the Young integral leads to

$$\int_s^t X_{s,r}\, \mathrm{d}X_r - \int_s^u X_{s,r}\, \mathrm{d}X_r - \int_u^t X_{u,r}\, \mathrm{d}X_r = X_{s,u}X_{u,t}, \tag{2-9}$$

where $(s,u,t) \in \Delta^{(3)}_{[0,T]}$ with

$$\Delta^{(k)}_J := \big\{(s_1,\dots,s_k) : s_1,\dots,s_k \in J, s_1 \le s_2 \le \dots \le s_k\big\}, \tag{2-10}$$

for an arbitrary interval $J \subset \mathbb{R}$ and $k \in \mathbb{N}$, for $k = 2$ the superscript is omitted. It is shown by Chen that (2-9) encodes abstractly the additivity of iterated integrals [Che54, Che57]. With this in mind, the basic idea behind the theory of rough paths can be formulated: The objective is not to find an integral that is directly analogous to (2-8), but rather a two-parameter function $\mathbb{X}\colon \Delta_{[0,T]} \to \mathbb{R}$, which is Hölder continuous, such that Chen's relation

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u}X_{u,t}, \tag{2-11}$$

holds for $(s,u,t) \in \Delta^{(3)}_{[0,T]}$. Of course, a smooth path fulfills (2-11) by choosing $\mathbb{X}$ to be the Young integral, as in (2-9). But in rougher cases, the possible abstract, two-parameter function $\mathbb{X}$ then defines the iterated integral

$$\mathbb{X}_{s,t} =: \int_s^t X_{s,r}\, \mathrm{d}X_r,$$

and not the other way around!

Remark 2.6. Note that it is possible to use stochastic properties of the path X to define $\mathbb{X}$. For example, in the case of a Brownian motion, $\mathbb{X}$ is the iterated Itô or Stratonovich integral. The main difference to the original Itô theory is that it is now possible to use the compensated Riemann sum (2-6) to construct integrals as in (2-1) pathwise, which means without a convergence in probability.

This can be applied to general paths with values in Banach spaces, but to formalize it, we need to introduce tensor product spaces. Let $\mathcal{U}$ and $\mathcal{V}$ be two separable Banach spaces over a scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In short, the tensor product space $\mathcal{U} \otimes \mathcal{V}$ characterizes all bilinear maps on $\mathcal{U} \times \mathcal{V}$ linearly. To be more precise, the elements in $\mathcal{U} \otimes \mathcal{V}$ represent every bilinear functional on the Cartesian product $\mathcal{U} \times \mathcal{V}$ as a linear functional on the tensor product space.

Define $u \otimes v$ for $(u, v) \in \mathcal{U} \times \mathcal{V}$ on the space of bilinear functionals $\mathcal{L}^{(2)}(\mathcal{U} \times \mathcal{V}; \mathbb{K})$ by the evaluation at the point (u, v), this means

$$(u \otimes v)h := h(u, v),$$

for every $h \in \mathcal{L}^{(2)}(\mathcal{U} \times \mathcal{V}; \mathbb{K})$. This function is a bilinear map itself.

Definition 2.7. The tensor product space of $\mathcal{U}$ and $\mathcal{V}$ is defined by

$$\mathcal{U} \otimes \mathcal{V} := \big\{u \otimes v : u \in \mathcal{U}, v \in \mathcal{V}\big\}.$$

In particular, if $(e_i)_{i \in I}$ and $(f_j)_{j \in J}$ are bases for $\mathcal{U}$ respectively $\mathcal{V}$, the family of tensor products $\big(e_i \otimes f_j\big)_{i \in I, j \in J}$ is a basis of $\mathcal{U} \otimes \mathcal{V}$, so every element $x \in \mathcal{U} \otimes \mathcal{V}$ can be represented as

$$x = \sum_{i \in I, j \in J} a_{ij} \big(e_i \otimes f_j\big),$$

for coefficients $(a_{ij})_{i \in I, j \in J} \subset \mathbb{K}$ which are equal to 0 except for finitely many indices.

Example 2.8. *Suppose that $\mathcal{U} := \mathbb{R}^n$ and $\mathcal{V} := \mathbb{R}^m$. In this situation, the tensor product space can be identified with the space of all $n \times m$-matrices. The tensor product $u \otimes v$ of two vectors $u \in \mathbb{R}^n, v \in \mathbb{R}^m$ is then a matrix with entries $(u \otimes v)_{ij} = u_i v_i$ for $i = 1, \dots, n$ and $j = 1, \dots, m$, which means $\mathcal{U} \otimes \mathcal{V} = \mathbb{R}^{n \times m}$. It is easy to see that $\dim(\mathbb{R}^n \otimes \mathbb{R}^m) = nm$.*

In particular, it can be shown that the dimension of the tensor product space is generally given by

$$\dim(\mathcal{U} \otimes \mathcal{V}) = \dim(\mathcal{U}) \dim(\mathcal{V}).$$

We want to highlight the fact that the tensor product space linearizes all bilinear mappings on $\mathcal{U} \times \mathcal{V}$. In the finite-dimensional setting, this is straightforward. For $\mathcal{U} = \mathbb{R}^n$ and $\mathcal{V} = \mathbb{R}^m$ we have $\mathcal{U} \otimes \mathcal{V} = \mathbb{R}^{n \times m}$ and obtain

$$\mathcal{L}^{(2)}\big(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^m\big) \simeq \mathcal{L}\big(\mathbb{R}^n; \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)\big) \simeq \mathcal{L}\big(\mathbb{R}^{n \times n}; \mathbb{R}^m\big) \simeq \mathcal{L}\big(\mathbb{R}^n \otimes \mathbb{R}^n; \mathbb{R}^m\big). \tag{2-12}$$

The first identification is, in fact, also true for general Banach spaces. The other identification requires more information about the structure of the Banach spaces. For this note, that $\mathcal{U} \otimes \mathcal{V}$ can be constructed using different bases, which leads to different spaces, but which are all isomorphic, [Rya02, Proposition 1.15]. To ensure (2-12), equip $\mathcal{U} \otimes \mathcal{V}$ with the projective norm

$$\|x\|_{\text{proj}} := \inf \left\{ \sum_{i \in I, j \in J} \|a_i\|_{\mathcal{U}} \|b_i\|_{\mathcal{V}} \ : \ x = \sum_{i \in I, j \in J} a_i b_j (e_i \otimes f_j) \right\}, \tag{2-13}$$

where the infimum is taken over all representations of x. This is the largest norm such that $\|u \otimes v\|_{\mathcal{U}\otimes\mathcal{V}} = \|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}$ holds. This means the norm generates the smallest of all possible tensor product spaces; see [Rya02, Proposition 2.1].

Theorem 2.9. ([Rya02, Theorem 2.9]) *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be three Banach spaces. If the tensor product space $\mathcal{U} \otimes \mathcal{V}$ is endowed with the projective norm defined in (2-13), then we obtain*

$$\mathcal{L}^{(2)}(\mathcal{U} \times \mathcal{V}; \mathcal{W}) \simeq \mathcal{L}\big(\mathcal{U}; \mathcal{L}(\mathcal{V}; \mathcal{W})\big) \simeq \mathcal{L}(\mathcal{U} \otimes \mathcal{V}; \mathcal{W}).$$

To close this interlude, define the symmetric and antisymmetric parts of a tensor. This is particularly relevant for the definition of a geometric rough path.

Definition 2.10. Let $x \in \mathcal{U} \otimes \mathcal{U}$ be a tensor. Then the symmetric and antisymmetric parts of x are given by

$$\mathrm{Sym}(x) \coloneqq \frac{x + x^T}{2}, \qquad \mathrm{Anti}(x) \coloneqq \frac{x - x^T}{2},$$

where $x^T \coloneqq \sum_{i\in I, j\in J} a_i b_j (e_j \otimes e_i)$ is the transposed tensor of $x = \sum_{i\in I, j \in J} a_i b_j (e_i \otimes e_j)$.

It is now possible to proceed with the definition of a rough path with values in an arbitrary, separable Banach space.

Definition 2.11. A pair $\mathbf{X} = (X, \mathbb{X})$ is called a γ-Hölder rough path for $\gamma \in (1/3, 1/2]$ if $X \in \mathcal{C}^\gamma([0,T]; \mathcal{V}), \mathbb{X} \in \mathcal{C}_2^{2\gamma}([0,T]; \mathcal{V} \otimes \mathcal{V})$, and they satisfy

$$\mathbb{X}_{s,t} - \mathbb{X}_{s,u} - \mathbb{X}_{u,t} = X_{s,u} \otimes X_{u,t}, \tag{2-14}$$

for $(s,u,t) \in \Delta^{(3)}_{[0,T]}$. Denote by $\mathscr{C}^\gamma([0,T]; \mathcal{V}) =: \mathscr{C}^\gamma$ the space of all γ-Hölder rough paths.

Remark 2.12. i) For a smooth path $X \in \mathcal{C}^\infty$, with the canonical choice for $\mathbb{X}$ as

$$\mathbb{X}_{s,t} \coloneqq \int_s^t X_{s,r} \, \mathrm{d}X_r,$$

Chen's relation (2-14) follows immediately as in (2-11). Then $(X, \mathbb{X})$ is called the canonical rough path lift of X, and $\mathscr{L}(\mathcal{C}^\infty)$ denotes the space of all rough paths obtained that way.

iii) The two-parameter function $\mathbb{X}$ is not uniquely determined. Let $(X, \mathbb{X}) \in \mathscr{C}^\gamma$ and define $\widehat{\mathbb{X}}_{s,t} \coloneqq \mathbb{X}_{s,t} + F_t - F_s$ for some $F \in \mathcal{C}^{2\gamma}([0,T]; \mathcal{V} \otimes \mathcal{V})$, then $\big(X, \widehat{\mathbb{X}}\big)$ also satisfies (2-14).

The space of rough paths $\mathscr{C}^\gamma$ is not separable and, due to the nonlinear nature of Chen's relation, not even a linear space. However, it is a metric space, where the metric is defined as follows.

Definition 2.13. Let $J \subset \mathbb{R}$ be a compact interval and $\mathbf{X} = (X, \mathbb{X})$, $\widehat{\mathbf{X}} = \big(\widehat{X}, \widehat{\mathbb{X}}\big)$ be two γ-Hölder rough paths. Then the γ-Hölder rough path metric is given by

$$\varrho_{\gamma,J}\big(\mathbf{X}, \widehat{\mathbf{X}}\big) \coloneqq \sup_{(s,t)\in\Delta_J} \frac{\Big\|X_{s,t} - \widehat{X}_{s,t}\Big\|_{\mathcal{V}}}{|t-s|^\gamma} + \sup_{(s,t)\in\Delta_J} \frac{\Big\|\mathbb{X}_{s,t} - \widehat{\mathbb{X}}_{s,t}\Big\|_{\mathcal{V}\otimes\mathcal{V}}}{|t-s|^{2\gamma}},$$

and set $\varrho_{\gamma,J}(\mathbf{X}) \coloneqq \varrho_{\gamma,J}(\mathbf{X}, 0)$.

Apart from the additivity of the integral, one could think of other properties for the definition of $\mathbb{X}$. For instance, the integration by parts rule, which for a real-valued smooth path X implies that

$$\int_s^t X_{s,r} \, \mathrm{d}X_r = \int_s^t X_{s,r} \dot{X}_r \, \mathrm{d}r = \frac{1}{2} X_{s,t} X_{s,t},$$

for $(s,t) \in \Delta_{[0,T]}$. In comparison to the additivity, this "first order calculus" rule holds not for every integral concept presented in the motivation. For example, for a real-valued Brownian motion $(B_t)_{t\in[0,T]}$ the Itô and Stratonovich integrals are respectively given by

$$\int_0^T B_r \,\mathrm{d}B_r = \frac{1}{2}B_T^2 - \frac{T}{2} \qquad \text{and} \qquad \int_0^T B_r \circ \mathrm{d}B_r = \frac{1}{2}B_T^2.$$

So, the Stratonovich integral fulfills the first-order calculus rule, but the Itô integral does not. This means that this property is not suitable to replace Chen's relation in the definition, but it can work as an addition to it since integrals satisfying the integration by parts rule are also needed in some situations.

Definition 2.14. i) A pair $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^\gamma([0,T];\mathcal{V})$ with $\gamma \in (1/3, 1/2]$ such that

$$\mathrm{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2}\big(X_{s,t} \otimes X_{s,t}\big) \tag{2-15}$$

holds for all $(s,t) \in \Delta_{[0,T]}$, is called a weakly geometric γ-Hölder rough path. Then $\mathscr{C}_g^\gamma([0,T];\mathcal{V}) =: \mathscr{C}_g^\gamma$ denotes the space of all weakly geometric γ-Hölder rough paths.

ii) For $\gamma \in (1/3, 1/2]$, the space of geometric rough paths is defined as the closure of $\mathscr{L}\big(\mathcal{C}^\infty([0,T];\mathcal{V})\big)$ in $\mathscr{C}^\gamma([0,T];\mathcal{V})$ and denoted by $\mathscr{C}_g^{0,\gamma}([0,T];\mathcal{V}) =: \mathscr{C}_g^{0,\gamma}$.

It is important to emphasize that $\mathscr{C}_g^{0,\gamma}$, unlike $\mathscr{C}^\gamma$, is a separable Banach space, see [FH20, Exercise 2.8].

The relationship between the different rough path spaces is relatively straightforward since it works similarly to the corresponding Hölder spaces. For $\gamma \in (1/3, 1/2]$ and $\varepsilon > 0$ we obtain for instance the (strict) embeddings

$$\mathscr{C}^{\gamma+\varepsilon} \hookrightarrow \mathscr{C}_g^{0,\gamma} \hookrightarrow \mathscr{C}_g^\gamma \hookrightarrow \mathscr{C}^\gamma. \tag{2-16}$$

The last two are trivial, and the first one is proven in [FH20, Proposition 2.8, Exercise 2.12]. Another feature that the rough path space inherits from the Hölder spaces is that the Hölder norms fulfill a certain continuity. To see this, define $\mathcal{C}^{0,\gamma}([0,T];\mathcal{V})$ as the closure of smooth functions in $\mathcal{C}^\gamma([0,T];\mathcal{V})$. Then the Wiener characterization of $\mathcal{C}^{0,\gamma}([0,T];\mathcal{V})$ yields

$$\lim_{\delta\to 0} \sup_{|t-s|<\delta} [X]_{\gamma,[s,t],\mathcal{V}} = 0, \tag{2-17}$$

for every $X \in \mathcal{C}^{0,\gamma}([0,T];\mathcal{V})$, see [FV10, Theorem 5.33]. The limit (2-17) holds also for any $X \in \mathcal{C}^{\gamma'}([0,T];\mathcal{V})$ with $\gamma' > \gamma$, due to the embedding $\mathcal{C}^{\gamma'} \hookrightarrow \mathcal{C}^{0,\gamma}$, but not for $X \in \mathcal{C}^\gamma([0,T];\mathcal{V})$! A counterexample for the latter statement is the path $t \mapsto X_t := t^\gamma$.

Lemma 2.15. ([BS24, Remark 2.3]) *For any $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}_g^{0,\gamma}([0,T];\mathcal{V})$ the maps*

$$\Delta_{[0,T]} \to \mathbb{R}, (s,t) \mapsto [X]_{\gamma,[s,t],\mathcal{V}} \qquad \text{and} \qquad \Delta_{[0,T]} \to \mathbb{R}, (s,t) \mapsto [\mathbb{X}]_{\gamma,[s,t],\mathcal{V}\otimes\mathcal{V}}$$

are continuous. Both maps are also continuous for $\mathbf{X} = (X, \mathbb{X}) \in \mathscr{C}^{\gamma'}([0,T];\mathcal{V})$ with $\gamma' > \gamma$.

Proof. Let $\mathbf{X} \in \mathscr{C}_g^{0,\gamma}([0,T];\mathcal{V})$ be a geometric rough path. By definition, there exists a sequence $\big(\mathbf{X}^{(k)}\big)_{k\in\mathbb{N}} \subset \mathscr{L}(\mathcal{C}^\infty([0,T];\mathcal{V}))$ such that $\varrho_{\gamma,[0,T]}\big(\mathbf{X}^{(k)}, \mathbf{X}\big) \to 0$ for $k \to \infty$. In particular, we have

$$\big[X^{(k)} - X\big]_{\gamma,[0,T],\mathcal{V}} \to 0 \qquad \text{and} \qquad \big[\mathbb{X}^{(k)} - \mathbb{X}\big]_{2\gamma,[0,T],\mathcal{V}\otimes\mathcal{V}} \to 0$$

for $k \to \infty$. This means $X \in \mathcal{C}^{0,\gamma}([0,T];\mathcal{V})$, so (2-17) holds. It is easy to verify that $\Delta_{[0,T]} \to \mathbb{R}, (s,t) \mapsto [X]_{\gamma,[s,t]}$ is continuous, indeed for $0 < s_2 - s_1 < \delta_1$ and $0 < t_2 - t_1 < \delta_2$

we obtain

$$
\begin{aligned}
&|[X]_{\gamma,[s_1,t_1],\mathcal{V}} - [X]_{\gamma,[s_2,t_2],\mathcal{V}}| \\
&\qquad \le \left|[X]_{\gamma,[s_1,t_1],\mathcal{V}} - [X]_{\gamma,[s_1,t_2],\mathcal{V}}\right| + \left|[X]_{\gamma,[s_1,t_2],\mathcal{V}} - [X]_{\gamma,[s_2,t_2],\mathcal{V}}\right| \\
&\qquad \le \sup_{\substack{s_1 \le u < v \le t_2 \\ |v-u|<\delta_2}} [X]_{\gamma,[u,v],\mathcal{V}} + \sup_{\substack{s_1 \le u < v \le t_2 \\ |v-u|<\delta_1}} [X]_{\gamma,[u,v],\mathcal{V}} \to 0,
\end{aligned}
$$

for $\delta_1, \delta_2 \to 0$ using (2-17). A similar proof can be used for the two-parameter function $\mathbb{X}$. The second statement holds due to the embeddings in (2-16). □

2.2.1 Construction of rough paths

The objective of this section is to construct a second-order process $\mathbb{X}$ for a given path X such that Chen's relation (2-14) is satisfied. It can be proven that every path $X \in \mathcal{C}^\gamma([0,T];\mathcal{V})$ can be lifted to a rough path, even to a geometric rough path. This Lyons–Victoir extension theorem is initially stated for p-variation rough paths in [LV07] and was later proven for Hölder continuous paths in [FH20].

Theorem 2.16. ([LV07, Corollary 19], [FH20, Exercise 2.14]) *Let $X \in \mathcal{C}^\gamma([0,T];\mathcal{V})$ be a Hölder continuous path with parameter $\gamma \in (1/3, 1/2)$. Then there exists a two parameter function $\mathbb{X} \in \mathcal{C}_2^{2\gamma}([0,T];\mathcal{V}\otimes\mathcal{V})$ such that $(X,\mathbb{X}) \in \mathscr{C}_g^{0,\gamma}([0,T];\mathcal{V})$.*

Even though this theorem yields the existence of a two-parameter function $\mathbb{X}$ such that (2-14) is fulfilled, it does not provide a general canonical construction. For a smooth path $X \in \mathcal{C}^\gamma$ with $\gamma > 1/2$, the construction is straightforward, by choosing the Young integral

$$
(s,t) \mapsto \mathbb{X}_{s,t} := \int_s^t X_{s,r} \, \mathrm{d}X_r.
$$

For any one-dimensional continuous path $X \in \mathcal{C}^\gamma([0,T];\mathbb{R})$ with $\gamma \in (1/3, 1/2]$ we can define

$$
\mathbb{X}_{s,t} := \frac{1}{2} (X_{s,t})^2 ,
$$

which fulfills (2-14).

For multidimensional, non-smooth paths, the choice of $\mathbb{X}$ is not so clear, especially in the case of stochastic processes. Fortunately, it is possible to achieve a meaningful candidate for $\mathbb{X}$ in the case where X is a Brownian motion or a Gaussian process satisfying a certain covariance condition.

Brownian rough paths

As a first non-trivial example, consider the d-dimensional Brownian motion $(B_t)_{t\in[0,T]}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that we have already observed two possibilities to define the iterated integral with probabilistic methods: the Itô and Stratonovich integral. Therefore, define

$$
\mathbb{B}^{\text{Itô}}_{s,t} := \int_s^t B_{s,r} \, \mathrm{d}B_r, \qquad \mathbb{B}^{\text{Strat}}_{s,t} := \int_s^t B_{s,r} \circ \mathrm{d}B_r,
$$

for $(s,t) \in \Delta_{[0,T]}$. Due to the additivity, it is easy to show that both $\left(B, \mathbb{B}^{\text{Itô}}\right)$ and $\left(B, \mathbb{B}^{\text{Strat}}\right)$ fulfill (2-14) almost everywhere, where the null set does not depend on time. It remains to show that $\mathbb{B}^{\text{Itô}}$, respectively $\mathbb{B}^{\text{Strat}}$ is 2γ-Hölder continuous.

Theorem 2.17 ([FH20, Theorem 3.1]). *Let $q \geq 2, \beta > 1/q$. Further, suppose $X\colon [0,T]\times\Omega \to \mathcal{V}$ and $\mathbb{X}\colon \Delta_{[0,T]}\times\Omega\to\mathcal{V}$ fulfill (2-14) as well as*

$$\mathbb{E}\left[\|X_{s,t}\|_{\mathcal{V}}^{q}\right] \lesssim (t-s)^{q\beta} \qquad \text{and} \qquad \mathbb{E}\left[\|\mathbb{X}_{s,t}\|_{\mathcal{V}\otimes\mathcal{V}}^{\frac{q}{2}}\right] \lesssim (t-s)^{q\beta},$$

for all $(s,t)\in\Delta_{[0,T]}$. Then there exists for every $\gamma\in(1/3,\beta-1/q)$ a modification $\mathbf{X} := (X,\mathbb{X})$ such that $\mathbf{X}(\omega)\in\mathscr{C}^{\gamma}$ almost surely.

Using elementary properties of the Brownian motion, the conditions of Theorem 2.17 can be easily checked.

Proposition 2.18. ([FH20, Proposition 3.4 - 3.5])

i) For any $\gamma\in(1/3,1/2)$, $\big(B,\mathbb{B}^{\text{Itô}}\big)\in\mathscr{C}^{\gamma}\big([0,T];\mathbb{R}^d\big)$ holds almost surely.
ii) For any $\gamma\in(1/3,1/2)$, $\big(B,\mathbb{B}^{\text{Strat}}\big)\in\mathscr{C}_g^{\gamma}\big([0,T];\mathbb{R}^d\big)$ holds almost surely.

Proof. i) Let $q\geq 2$ and $(s,t)\in\Delta_{[0,T]}$. Due to the scaling properties of the Brownian motion, we have

$$\mathbb{E}\big[|B_{s,t}|_{\mathbb{R}^d}^{q}\big] = (t-s)^{\frac{q}{2}}\mathbb{E}\big[|B_{0,1}|_{\mathbb{R}^d}^{q}\big].$$

To bound $\mathbb{B}^{\text{Itô}}$, note that the quadratic variation process of $\mathbb{B}^{\text{Itô}}_{s,t}$ is given by

$$\int_s^t |B_{s,r}|_{\mathbb{R}^d}^2 \,\mathrm{d}r.$$

Using the Burkholder–Davis–Gundy inequality, see [Kal21, Theorem 18.7], we obtain

$$\mathbb{E}\Big[|\mathbb{B}^{\text{Itô}}_{s,t}|_{\mathbb{R}^d}^{\frac{q}{2}}\Big] \lesssim \mathbb{E}\left[\left|\int_s^t |B_{s,r}|_{\mathbb{R}^d}^2\,\mathrm{d}r\right|^{\frac{q}{4}}\right] \lesssim (t-s)^{\frac{q}{4}}\mathbb{E}\left[\sup_{r\in[s,t]}|B_{s,r}|_{\mathbb{R}^d}^{\frac{q}{2}}\right] \lesssim (t-s)^{\frac{q}{2}}.$$

Since this holds for any $q\geq 2$, Kolmogorov's Theorem 2.17 yields that $\big(B,\mathbb{B}^{\text{Itô}}\big)$ is a γ-Hölder rough path for every $\gamma\in(1/3,1/2)$.

ii) The well-known relation between the Itô and Stratonovich integral is given by

$$\mathbb{B}^{\text{Strat}}_{s,t} = \int_s^t B_{s,r}\circ\mathrm{d}B_r = \int_s^t B_{s,r}\,\mathrm{d}B_r + \frac{1}{2}\mathrm{Id}_{\mathbb{R}^d}(t-s) = \mathbb{B}^{\text{Itô}}_{s,t} + \frac{1}{2}\mathrm{Id}_{\mathbb{R}^d}(t-s),$$

for $(s,t)\in\Delta_{[0,T]}$. Due to this fact and $i)$ we obtain $\mathbb{B}^{\text{Strat}}\in\mathcal{C}_2^{2\gamma}\big([0,T];\mathbb{R}^d\otimes\mathbb{R}^d\big)$ and therefore $\big(B,\mathbb{B}^{\text{Strat}}\big)\in\mathscr{C}^{\gamma}\big([0,T];\mathbb{R}^d\big)$.
That $\big(B,\mathbb{B}^{\text{Strat}}\big)$ fulfills (2-15) can be proven using the integration by parts rule, which is known to be true for the Stratonovich integral. Therefore, $\big(B,\mathbb{B}^{\text{Strat}}\big)$ is even a weakly geometric γ-Hölder rough path.

□

Gaussian rough paths

With the possibility to enhance the Brownian motion to a rough path, it is now possible to treat one of the most important stochastic processes in this setting. However, a key motivation for exploring rough paths, as mentioned in the motivation, is also the extension beyond the Brownian and, consequently, the martingale framework as a potential integrator. In this subsection, a covariance criterion is presented for enhancing a Gaussian process to a rough path.

Suppose that $(X_t)_{t\in[0,T]}$ is a d-dimensional, centered Gaussian process with independent components $\big(X^{(i)}\big)_{i=1}^{d}$. This means that $(X_{t_1},\dots,X_{t_k})$ is a Gaussian random variable for all $t_1,\dots,t_k \in [0,T]$ and $\mathbb{E}[X]=0$. The covariance function of X is given by

$$R_X\colon [0,T]^2 \to \mathbb{R}^{d\times d},\, (s,t)\mapsto \mathbb{E}[X_s\otimes X_t].$$

We further define the q-variance of this covariance function by

$$[R_X]^q_{q\text{-var},[0,T]^2} \coloneqq \sup_{\pi,\pi'\subset[0,T]} \sum_{\substack{[u,v]\in\pi\\ [\tilde u,\tilde v]\in\tilde\pi}} \mathbb{E}[X_{u,v}\otimes X_{\tilde u,\tilde v}]^q,$$

where the supremum is taken over all partitions $\pi,\tilde\pi\subset[0,T]$.

Similar to the Brownian case, we need a candidate for $\mathbb{X}$ fulfilling Chen's relation (2-14). Such a process can be constructed as a suitable iterated integral using stochastic tools. Under certain conditions on the covariance, which are specified in the next theorem, it is possible to define

$$\mathbb{X}^{i,j}_{s,t} \coloneqq \int_s^t X^{(i)}_{s,r}\,\mathrm{d}X^{(j)}_r \coloneqq \lim_{|\pi|\to 0}\sum_{[u,v]\in\pi} X^{(i)}_{s,u}X^{(j)}_{u,v}, \tag{2-18}$$

for $(s,t)\in\Delta_{[0,T]}$ as an L^2-limit for $1\le i<j\le d$, see [FH20, Proposition 10.3]. The independence of the components is crucial at this point. But how do we define the other components of $\mathbb{X}$? To this aim, we investigate the properties of a (geometric) rough path. In particular, to fulfill (2-15) and (2-14), the remaining components are necessarily given by

$$\mathbb{X}^{i,i}_{s,t} \coloneqq \frac{1}{2}\Big(X^{(i)}_{s,t}\Big)^2 \qquad\text{and}\qquad \mathbb{X}^{j,i}_{s,t} \coloneqq -\mathbb{X}^{i,j}_{s,t} + X^{(i)}_{s,t}X^{(j)}_{s,t}, \tag{2-19}$$

for $(s,t)\in\Delta_{[0,T]}$ and $1\le i<j\le d$.

Theorem 2.19. ([FH20, Theorem 10.4 c)]) *Let $(X_t)_{t\in[0,T]}$ be a d-dimensional process which is centered, Gaussian, and has independent components. Further, suppose that there exists a constant $q\in[1,{}^3\!/_2)$ such that every component $X^{(i)}$ satisfies*

$$\big[R_{X^{(i)}}\big]_{q\text{-var},[s,t]^2} \lesssim |t-s|^{\frac{1}{q}}, \tag{2-20}$$

for $i\in\{1,\dots,d\}$. Then, there exists a continuous modification of $\mathbb{X}$ defined in (2-18)-(2-19), which is denoted again by $\mathbb{X}$. Furthermore, $\mathbf{X}\coloneqq(X,\mathbb{X})\in\mathscr{C}^\gamma_g\big([0,T];\mathbb{R}^d\big)$ almost surely for every $\gamma\in({}^1\!/_3,{}^1\!/_{2q})$.

We know that a Gaussian process $(X_t)_{t\in[0,T]}$ has finite moments, i.e. $\mathbb{E}[|X|^p]<\infty$ for every $p\ge 1$. This property carries over to $\mathbb{X}$ and therefore also to the rough path $\mathbf{X}$.

Corollary 2.20. ([FH20, Theorem 10.4 b)]) *Assume the same conditions as in Theorem 2.19. Then, all moments of X and $\mathbb{X}$ are finite. In particular, we have*

$$\varrho_{\gamma,[0,T]}(\mathbf{X}) \in \bigcap_{p\ge 1}\mathrm{L}^p(\Omega)$$

for every $\gamma\in({}^1\!/_3,{}^1\!/_{2q})$.

Example 2.21. *Using Theorem 2.19, the fractional Brownian motion can be lifted to a rough path, similar to the case of a Brownian motion. The covariance of a scalar-valued fractional Brownian motion $\big(B^H_t\big)_{t\in[0,T]}$ with Hurst parameter $H\in({}^1\!/_3,{}^1\!/_2)$ is given by*

$$\mathbb{E}\Big[B^H_t\otimes B^H_s\Big] = \frac{1}{2}\left(|t|^{2H}+|s|^{2H}-|t-s|^{2H}\right). \tag{2-21}$$

Due to the stationarity of the increments of $\left(B_t^H\right)_{t\in[0,T]}$*, it is enough to investigate*

$$\varsigma^2(h) := \mathbb{E}\left[\left(B^H_{t,t+h}\right)^2\right].$$

In particular, if ς^2 *is concave, non-decreasing and fulfills*

$$\left|\varsigma^2(h)\right| \lesssim |h|^{\frac{1}{q}},$$

for all $h \in [0, \tilde{h}]$ *with* $\tilde{h} > 0$ *and some* $q \in [1, 2)$*, it is possible to verify the assumptions of Theorem 2.19, see [FH20, Corollary 10.10], which leads to the existence of a rough path lift. Using (2-21), we obtain* $\varsigma^2(h) = h^{2H}$*, which is clearly concave and non-decreasing for* $H \le 1/2$*, which means that* $\left(B^H, \mathbb{B}^H\right)$ *is a geometric rough path. The same can be said of a d-dimensional fractional Brownian motion with independent components, where the aforementioned considerations can be applied componentwise.*

In particular, a Gaussian Volterra process can be enhanced to a rough path.

Definition 2.22. Let $K\colon [0,T]\times[0,T] \to \mathbb{R}$ be a kernel such that $K(0,s) = 0$ for all $s \in [0,T]$, $K(t,s) = 0$ for $0 \le t < s \le T$ and there exists a parameter $\iota > 0$ such that

$$\int_0^T \left(K(t,r) - K(s,r)\right)^2 \,\mathrm{d}r \lesssim |t-s|^{\iota}, \tag{2-22}$$

holds for all $s,t \in [0,T]$. Then a centered, Gaussian process $(V_t)_{t\in[0,T]}$ is called Volterra process if V_t has the form

$$V_t := \int_0^t K(t,r)\,\mathrm{d}B_r,$$

where the integral is defined as an Itô integral with respect to a scalar-valued Brownian motion $(B_t)_{t\in[0,T]}$. Then, the covariance function is given by

$$\mathbb{E}\big[V_s V_t\big] = \int_0^{\min\{s,t\}} K(t,r)K(s,r)\,\mathrm{d}r.$$

Due to (2-22), there exists a Hilbert–Schmidt operator $\mathcal{K}\colon \mathrm{L}^2([0,T];\mathbb{R}) \to \mathrm{L}^2([0,T];\mathbb{R})$, associated to a Volterra process with kernel K, defined by

$$\mathcal{K}f := \int_0^T K(\cdot,r)f(r)\,\mathrm{d}r,$$

for $f \in \mathrm{L}^2([0,T];\mathbb{R})$.

Lemma 2.23. *Let* $(V_t)_{t\ge0}$ *be a Volterra process with kernel* K*. Then there exists a* β*-Hölder continuous modification for every* $\beta \in (0, \iota/2)$*. If additionally* $\iota \in (2/3, 1]$*, then there exists a two-parameter function* $\mathbb{V}$ *such that* $(V,\mathbb{V}) \in \mathscr{C}_g^{\beta}([0,T];\mathbb{R})$ *for every* $\beta \in (1/3, \iota/2)$*.*

Proof. The existence of a Hölder-continuous modification follows directly from (2-22) and Kolmogorov's continuity theorem [Kun90, 1.4.1].

To prove the existence of a rough path lift, we verify (2-20). Let $(u,v),(\tilde{u},\tilde{v}) \in \Delta_{[0,T]}$, then we have $\min\{u,\tilde{u}\}, \min\{u,\tilde{v}\}, \min\{\tilde{u},v\} \le \min\{v,\tilde{v}\}$, which leads to

$$\mathbb{E}\big[V_{u,v}V_{\tilde{u},\tilde{v}}\big] = \int_0^{\min\{v,\tilde{v}\}} K(v,r)K(\tilde{v},r)\,\mathrm{d}r - \int_0^{\min\{v,\tilde{u}\}} K(v,r)K(\tilde{u},r)\,\mathrm{d}r$$

$$
\begin{aligned}
&- \int_0^{\min\{u,\tilde v\}} K(u,r)K(\tilde v,r)\,\mathrm{d}r + \int_0^{\min\{u,\tilde u\}} K(u,r)K(\tilde u,r)\,\mathrm{d}r \\
&= \int_0^{\min\{v,\tilde v\}} \big(K(v,r)-K(u,r)\big)\big(K(\tilde v,r)-K(\tilde u,r)\big)\,\mathrm{d}r,
\end{aligned}
$$

using $K(s,t) = 0$ for $s < t$. With this equality, (2-22) as well as the Hölder and Young inequalities, we obtain

$$
\begin{aligned}
\mathbb{E}\big[V_{u,v}V_{\tilde u,\tilde v}\big] &\lesssim \left(\int_0^{\min\{v,\tilde v\}} \big(K(v,r)-K(u,r)\big)^2\,\mathrm{d}r\right)^{\frac12} \left(\int_0^{\min\{v,\tilde v\}} \big(K(\tilde v,r)-K(\tilde u,r)\big)^2\,\mathrm{d}r\right)^{\frac12} \\
&\lesssim (v-u)^{\frac{\iota}{2}}(\tilde v-\tilde u)^{\frac{\iota}{2}} \lesssim (v-u)^{\iota} + (\tilde v-\tilde u)^{\iota}.
\end{aligned}
$$

In particular, this implies that

$$
[R_V]^{\frac{1}{\iota}}_{1/\iota\text{-var},[s,t]} \lesssim \sup_{\pi\subset[s,t]} \sum_{[u,v]\in\pi} |v-u|^{\frac{1}{\iota}\iota} = |t-s|\,.
$$

Due to $\iota \in (2/3, 1]$, the assumptions of Theorem 2.19 are fulfilled, which means that V can be enhanced to a weakly geometric rough path. □

Example 2.24. *i)* (Fractional Brownian motion). *The fractional Brownian motion with Hurst parameter $H \in (0,1)$ can be represented as a Volterra process using the kernel*

$$
K(t,s) := \frac{(t-s)^{H-\frac12}}{\Gamma\big(H+\frac12\big)} f_h\left(\frac12 - H, H-\frac12, H+\frac12, 1-\frac{t}{s}\right)\mathbb{1}_{[0,t)}(s),
$$

for $s,t \in [0,T]$ where Γ is the Gamma- and f_h the hypergeometric function. For $a,b,c \in \mathbb{R}$ and $|x| \le 1$, f_h is defined by the power series

$$
f_h(a,b,c,d) := \sum_{k=0}^{\infty} \frac{\prod_{l=0}^{k-1}(a-l)\prod_{l=0}^{k-1}(b-l)}{\prod_{l=0}^{k-1}(c-l)}\frac{x^k}{k!},
$$

for complex values $|x| \ge 1$ it can be analytically continued. Then, K satisfies (2-22) for $\iota = 2H$, see for example [DU99] and [Hul03, Example 8]. This implies, using Lemma 2.23, that the fractional Brownian motion can be lifted to a rough path for $H \in (1/3, 1/2)$ which is consistent with Example 2.21.

ii) (Liouville fractional Brownian motion). *Given the complexity of the Volterra kernel from the fractional Brownian motion, often a modification is used, where the hypergeometric function f_h is omitted. In the literature, this modification is sometimes called fractional Brownian motion of type II or Liouville fractional Brownian motion, see for example [FdLP99, BvNS12]. The kernel is given by*

$$
K(t,s) = \frac{1}{\Gamma\big(H+\frac12\big)}(t-s)^{H-\frac12}\mathbb{1}_{[0,t)}(s),
$$

for $s,t \in [0,T]$ and $H \in (0,1)$. For $H \le 1/2$ and $s \le t$ we obtain

$$
\begin{aligned}
&\int_0^T \Big((t-r)^{H-\frac12}\mathbb{1}_{[0,t)}(r) - (s-r)^{H-\frac12}\mathbb{1}_{[0,s)}(r)\Big)^2\,\mathrm{d}r \\
&\quad= \int_0^t (t-r)^{2H-1}\,\mathrm{d}r + \int_0^s (s-r)^{2H-1}\,\mathrm{d}r - 2\int_0^s (t-r)^{H-\frac12}(s-r)^{H-\frac12}\,\mathrm{d}r \\
&\quad\le \int_0^t (t-r)^{2H-1}\,\mathrm{d}r + \int_0^s (s-r)^{2H-1}\,\mathrm{d}r - 2\int_0^s (t-r)^{H-\frac12}(t-r)^{H-\frac12}\,\mathrm{d}r
\end{aligned}
$$

$$= \int_s^t (t-r)^{2H-1}\,\mathrm{d}r + \int_0^s (s-r)^{2H-1} - (t-r)^{2H-1}\,\mathrm{d}r$$
$$\leq \frac{(t-s)^{2H}}{2H} + \frac{(t-s)^{2H} - \big(t^{2H} - s^{2H}\big)}{2H} \lesssim (t-s)^{2H},$$

and analogously for $s \geq t$. This means that (2-22) is fulfilled by choosing $\iota = 2H$, which leads to the fact that there is a rough path lift of the Liouville fractional Brownian motion, provided that $H \in (1/3, 1/2]$.

iii) (Ornstein–Uhlenbeck process). *The Ornstein–Uhlenbeck process has the kernel*

$$K(t,s) := e^{a(t-s)}\mathbb{1}_{[0,t)}(s),$$

for $s,t \in [0,T]$ and some $a < 0$. Then (2-22) holds for $\iota = 1$. Indeed, for $s \leq t$ we obtain

$$\int_0^T \Big(e^{a(t-r)}\mathbb{1}_{[0,t)}(r) - e^{a(s-r)}\mathbb{1}_{[0,s)}(r)\Big)^2\,\mathrm{d}r = \int_0^s \Big(e^{a(t-r)} - e^{a(s-r)}\Big)^2\,\mathrm{d}r + \int_s^t e^{2a(t-r)}\,\mathrm{d}r$$
$$= \big(e^{at} - e^{as}\big)^2 \int_0^s e^{-2ar}\,\mathrm{d}r + \frac{1 - e^{2a(t-s)}}{-2a} = \big(e^{at} - e^{as}\big)^2 \frac{e^{-2as} - 1}{-2a} + \frac{1 - e^{2a(t-s)}}{-2a}$$
$$= \big(e^{a(t-s)} - 1\big)^2 \frac{1 - e^{2as}}{-2a} + \frac{1 - e^{2a(t-s)}}{-2a}.$$

The second term is bounded by $t-s$, using the inequality $1 + x \leq e^x$, which holds for all $x \in \mathbb{R}$. Further, we have $1 - e^{2as} \leq 1$ and $e^{a(t-s)} \leq 1 - a(t-s)$, since $a < 0$. This leads to

$$\int_0^T \Big(e^{a(t-r)}\mathbb{1}_{[0,t)}(r) - e^{a(s-r)}\mathbb{1}_{[0,s)}(r)\Big)^2\,\mathrm{d}r \lesssim (t-s)^2 + (t-s) \lesssim t-s,$$

for $t - s \leq 1$. For $t - s > 1$, we use that

$$\big(e^{a(t-s)} - 1\big)^2 \frac{1 - e^{2as}}{-2a} \leq \frac{1}{-2a},$$

to obtain

$$\int_0^T \Big(e^{a(t-r)}\mathbb{1}_{[0,t)}(r) - e^{a(s-r)}\mathbb{1}_{[0,s)}(r)\Big)^2\,\mathrm{d}r \lesssim 1 + (t-s) \lesssim t-s.$$

2.2.2 The Cameron–Martin space associated to a Volterra process

This subsection is part of the article [BGVS25] which is based on a collaboration with Alexandra Blessing and Mazyar Ghani Varzaneh.

Since we deal mostly with Gaussian rough paths, we know from Theorem 2.19 that all moments are finite. However, it turns out that this assumption is not enough to bound the moments of the solution of a rough evolution equation. However, these bounds are of significant value when analyzing the long-term behavior. In order to establish them in Section 3.4, we need some properties of the Cameron–Martin space associated with the noise.

For a Volterra process $(V_t)_{t\in[0,T]}$ with kernel K on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, as in Definition 2.22, the Cameron–Martin space is denoted by $\mathcal{H}_V$. To be precise, the Cameron–Martin space is associated with the law of V and identifies every direction such that the translation in this direction is absolutely continuous with respect to the original law. For a Volterra process, it is possible to show that the Cameron–Martin space is given by the image of $\mathrm{L}^2([0,T];\mathbb{R})$ under the operator $\mathcal{K}$, i.e. $\mathcal{H}_V = \mathcal{K}\big(\mathrm{L}^2[0,T];\mathbb{R}\big)$. In particular, for every

$h \in \mathcal{H}_V$ there exists a $f \in \mathrm{L}^2([0,T];\mathbb{R})$ such that

$$h(t) = \int_0^t K(t,r)f(r)\,\mathrm{d}r,$$

with the norm $\|h\|_{\mathcal{H}_V} = \|f\|_{\mathrm{L}^2([0,T];\mathbb{R})}$.

Before analyzing the Cameron–Martin space in more detail, we introduce a control function, which is essential for establishing the existence of an integrable solution bound.

Definition 2.25. Let $\mathbf{X} \in \mathscr{C}^\gamma([0,T];\mathcal{V})$ be a γ-Hölder rough path for $\gamma \in (1/3, 1/2]$. Then define for $\eta < \gamma$ the function

$$\begin{aligned} W_{\mathbf{X},\gamma,\eta}\colon \Delta_{[0,T]} &\to \mathbb{R} \\ (s,t) &\mapsto \sup_{\pi \subset [s,t]} \left\{ \sum_{[u,v]\in\pi} (v-u)^{\frac{-\eta}{\gamma-\eta}} \left(\|X_{u,v}\|_{\mathcal{V}}^{\frac{1}{\gamma-\eta}} + \|\mathbb{X}_{u,v}\|_{\mathcal{V}\otimes\mathcal{V}}^{\frac{1}{2(\gamma-\eta)}} \right) \right\}, \end{aligned} \tag{2-23}$$

where the supremum is taken over all partitions π of $[s,t]$. If it is clear from the context, we omit the dependence on γ in the notation $W_{\mathbf{X},\eta}(s,t) = W_{\mathbf{X},\gamma,\eta}(s,t)$.

Lemma 2.26. ([BGVS25, Lemma 3.15]) *Let $\mathbf{X} \in \mathscr{C}^\gamma([0,T];\mathcal{V})$ be a γ-Hölder rough path for $\gamma \in (1/3, 1/2]$ and $\eta < \gamma$.*

i) Then $W_{\mathbf{X},\eta}$ is continuous and super-additive, this means

$$W_{\mathbf{X},\eta}(s,u) + W_{\mathbf{X},\eta}(u,t) \le W_{\mathbf{X},\eta}(s,t),$$

holds for $s \le u \le t$. Therefore, $W_{\mathbf{X},\eta}$ is a control function.

ii) For $s \le t$ it holds that

$$W_{\mathbf{X},\eta}(s,t) \le (t-s)\left([X]_{\gamma,[s,t],\mathcal{V}}^{\frac{1}{\gamma-\eta}} + [\mathbb{X}]_{2\gamma,[s,t],\mathcal{V}\otimes\mathcal{V}}^{\frac{1}{2(\gamma-\eta)}} \right).$$

Proof. i) The proof is similar to [FV10, Proposition 1.15]. Let $(s,u,t) \in \Delta_{[0,T]}^{(3)}$ and define the abbreviation

$$d(s,t) := (t-s)^{-\frac{\eta}{\gamma-\eta}} \left(\|X_{s,t}\|_{\mathcal{V}}^{\frac{1}{\gamma-\eta}} + \|\mathbb{X}_{s,t}\|_{\mathcal{V}\otimes\mathcal{V}}^{\frac{1}{2(\gamma-\eta)}} \right).$$

For two partitions $\pi_1 \subset [s,u]$ and $\pi_2 \subset [u,t]$ we obtain

$$\sum_{[t_i,t_{i+1}]\in\pi_1} d(t_i,t_{i+1}) + \sum_{[s_i,s_{i+1}]\in\pi_2} d(s_i,s_{i+1}) \le \sum_{[r_i,r_{i+1}]\in\pi_1\cup\pi_2} d(r_i,r_{i+1}).$$

Since $\pi_1 \cup \pi_2$ is a partition of $[s,t]$, this proves the super-additivity of $W_{\mathbf{X},\eta}$. To prove the continuity, we show that $W_{\mathbf{X},\eta}$ is left and right continuous. Therefore define

$$W_{\mathbf{X},\eta}(s^{\pm}, t^{\mp}) := \lim_{h_1,h_2 \searrow 0} W_{\mathbf{X},\eta}(s \pm h_1, t \mp h_2),$$

where the limit exists due to the monotonicity of $W_{\mathbf{X},\eta}$. Further, we have

$$W_{\mathbf{X},\eta}(s^+, t^-) \le W_{\mathbf{X},\eta}(s,t) \le W_{\mathbf{X},\eta}(s^-, t^+).$$

The super-additivity of $W_{\mathbf{X},\eta}$ leads to

$$W_{\mathbf{X},\eta}(s+h_1, u-h_2) + W_{\mathbf{X},\eta}(u+h_3, t-h_4)$$

$$\leq W_{\mathbf{X},\eta}(s+h_1,u-h_2)+W_{\mathbf{X},\eta}(u-h_2,t-h_4)$$
$$\leq W_{\mathbf{X},\eta}(s+h_1,t-h_4),$$

for small constants $h_1,h_2,h_3,h_4>0$, which shows that $W_{\mathbf{X},\eta}(s^+,t^-)$ is super-additive. Further, we have $d(s,t)\leq W_{\mathbf{X},\eta}(s^+,t^-)$, and therefore

$$\sum_{[t_i,t_{i+1}]\subset\pi} d(t_i,t_{i+1})\leq \sum_{[t_i,t_{i+1}]\subset\pi} W_{\mathbf{X},\eta}(t_i^+,t_{i+1}^-)\leq W_{\mathbf{X},\eta}(s^+,t^-),$$

for an arbitrary partition $\pi\subset[s,t]$. Taking the supremum over all partitions π yields $W_{\mathbf{X},\eta}(s,t)=W_{\mathbf{X},\eta}\big(s^+,t^-\big)$. A similar computation shows that $W_{\mathbf{X},\eta}(s,t)=W_{\mathbf{X},\eta}\big(s^-,t^+\big)$, which proves the the continuity of the control.

ii) For every partition $\pi\subset[s,t]$, we obtain the bound

$$\begin{aligned}
&\sum_{[t_i,t_{i+1}]\in\pi}(t_{j+1}-t_j)^{-\frac{\eta}{\gamma-\eta}}\left(\left\|X_{t_j,t_{j+1}}\right\|_{\mathcal{V}}^{\frac{1}{\gamma-\eta}}+\left\|\mathbb{X}_{t_j,t_{j+1}}\right\|_{\mathcal{V}\otimes\mathcal{V}}^{\frac{1}{2(\gamma-\eta)}}\right)\\
&\quad=\sum_{[t_i,t_{i+1}]\in\pi}(t_{j+1}-t_j)^{1-\frac{\gamma}{\gamma-\eta}}\left(\left\|X_{t_j,t_{j+1}}\right\|_{\mathcal{V}}^{\frac{1}{\gamma-\eta}}+\left\|\mathbb{X}_{t_j,t_{j+1}}\right\|_{\mathcal{V}\otimes\mathcal{V}}^{\frac{1}{2(\gamma-\eta)}}\right)\\
&\quad\leq\sum_{[t_i,t_{i+1}]\in\pi}(t_{j+1}-t_j)\left(\left(\frac{\left\|X_{t_j,t_{j+1}}\right\|_{\mathcal{V}}}{(t_{j+1}-t_j)^{\gamma}}\right)^{\frac{1}{\gamma-\eta}}+\left(\frac{\left\|\mathbb{X}_{t_j,t_{j+1}}\right\|_{\mathcal{V}\otimes\mathcal{V}}}{(t_{j+1}-t_j)^{2\gamma}}\right)^{\frac{1}{2(\gamma-\eta)}}\right)\\
&\quad\leq\sum_{[t_i,t_{i+1}]\in\pi}(t_{j+1}-t_j)\left([X]_{\gamma,[t_j,t_{j+1}],\mathcal{V}}^{\frac{1}{\gamma-\eta}}+[\mathbb{X}]_{2\gamma,[t_j,t_{j+1}],\mathcal{V}\otimes\mathcal{V}}^{\frac{1}{2(\gamma-\eta)}}\right)\\
&\quad\leq(t-s)\left([X]_{\gamma,[s,t],\mathcal{V}}^{\frac{1}{\gamma-\eta}}+[\mathbb{X}]_{2\gamma,[s,t],\mathcal{V}\otimes\mathcal{V}}^{\frac{1}{2(\gamma-\eta)}}\right).
\end{aligned}$$

Since the right-hand side is independent of the choice of the partition, this yields the claim. □

Recall that $(V_t)_{t\in[0,T]}$ has a β-continuous modification for $\beta<1/2$. Then, the following lemma characterizes the elements of $\mathcal{H}_V$ via the control defined in (2-23). A similar statement for the Cameron–Martin space of the fractional Brownian motion is shown in [FV06, Theorem 3]. For simplicity, we fix $T=1$ for the next lemma.

Lemma 2.27. ([BGVS25, Lemma 3.15]) *Let $(V_t)_{t\in[0,1]}$ be a scalar-valued Volterra process with kernel K such that*

$$\sup_{s\in[0,1-t]}\int_0^1|K(t+s,r)-K(s,r)|\;\mathrm{d}r=O\big(t^{\beta+\frac{1}{2}}\big),\tag{K1}$$

$$\sup_{s\in[0,1]}\int_0^{1-t}|K(t+r,s)-K(r,s)|\;\mathrm{d}r=O\big(t^{\beta+\frac{1}{2}}\big),\tag{K2}$$

holds for all $s,t\in[0,1]$, where $f=O\big(t^{\beta+1/2}\big)$ means that $\limsup_{t\to 0}\left|\frac{f(t)}{t^{\beta+\frac{1}{2}}}\right|<\infty$. Then, for every $1/2<\alpha<\beta+1/2$, we obtain

$$[h]_{W^{\alpha,2}([0,1])}\lesssim_{\alpha,\beta}\|h\|_{\mathcal{H}_V},\tag{2-24}$$

for every $h\in\mathcal{H}_V$. In addition, for every $0\leq\eta<\alpha-1/2$

$$W_{\boldsymbol{h},\eta}(0,1)\lesssim_{\beta,\eta}\|h\|_{\mathcal{H}_V}^{\frac{1}{\alpha-\eta}},\tag{2-25}$$

holds.

Proof. Recall, that $f = O\big(t^{\beta+\frac{1}{2}}\big)$ means that $\limsup_{t\to 0}\left|\frac{f(t)}{t^{\beta+\frac{1}{2}}}\right| < \infty$, i.e. there exists a constant $C > 0$ and a constant $\varepsilon > 0$ such that $|f(t)| \leq Ct^{\beta+\frac{1}{2}}$ for every $|t| < \varepsilon$. We will assume without loss of generality that $\varepsilon > 0$, which is in fact possible if f is bounded on $(\varepsilon, 1]$. Since we have

$$\int_0^1 |K(t+s,r) - K(s,r)|\ \mathrm{d}r \lesssim \left(\int_0^1 |K(t+s,r) - K(s,r)|^2\ \mathrm{d}r\right)^{\frac{1}{2}} \lesssim t^{\frac{t}{2}},$$

due to the definition of a Volterra process, we obtain therefore

$$\sup_{s\in[0,1-t]} \int_0^1 |K(t+s,r) - K(s,r)|\ \mathrm{d}r \lesssim t^{\beta+\frac{1}{2}},$$
$$\sup_{s\in[0,1]} \int_0^{1-t} |K(t+r,s) - K(r,s)|\ \mathrm{d}r \lesssim t^{\beta+\frac{1}{2}},$$

for every $t \in [0,1]$.

Let $h \in \mathcal{H}_V = \mathcal{K}\big(\mathrm{L}^2([0,1];\mathbb{R})\big)$. Then there exists a $g \in \mathrm{L}^2([0,1];\mathbb{R})$ such that

$$h(t) = \int_0^t K(t,r)g(r)\ \mathrm{d}r.$$

The difference is given by

$$h(u) - h(v) = \begin{cases} \int_0^v \big(K(u,r) - K(v,r)\big)g(r)\ \mathrm{d}r + \int_v^u K(u,r)g(r)\ \mathrm{d}r, & 1 \geq u \geq v \geq 0, \\ \int_0^u \big(K(u,r) - K(v,r)\big)g(r)\ \mathrm{d}r + \int_u^v K(v,r)g(r)\ \mathrm{d}r, & 0 \leq u < v \leq 1, \end{cases},$$

which leads to

$$\begin{aligned} [h]^2_{\mathrm{W}^{\alpha,2}([0,1])} \leq 2\int_0^1\int_v^1 &\frac{\big(\int_0^v (K(u,r) - K(v,r))g(r)\ \mathrm{d}r\big)^2}{|u-v|^{1+2\alpha}}\ \mathrm{d}u\mathrm{d}v \\ &+ 2\int_0^1\int_v^1 \frac{\big(\int_v^u K(u,r)g(r)\ \mathrm{d}r\big)^2}{|u-v|^{1+2\alpha}}\ \mathrm{d}u\mathrm{d}v \\ &+ 2\int_0^1\int_0^v \frac{\big(\int_0^u (K(u,r) - K(v,r))g(r)\ \mathrm{d}r\big)^2}{|u-v|^{1+2\alpha}}\ \mathrm{d}u\mathrm{d}v \\ &+ 2\int_0^1\int_0^v \frac{\big(\int_u^v K(v,r)g(r)\ \mathrm{d}r\big)^2}{|u-v|^{1+2\alpha}}\ \mathrm{d}u\mathrm{d}v. \end{aligned} \tag{2-26}$$

Due to the Cauchy–Schwarz inequality and (K1) we further obtain for $v \leq u \leq 1$

$$\begin{aligned} &\Big(\int_0^v (K(u,r) - K(v,r))g(r)\ \mathrm{d}r\Big)^2 \\ &\quad\leq \int_0^v |K(u,r) - K(v,r)|\ \mathrm{d}r \int_0^v |K(u,r) - K(v,r)|\,(g(r))^2\ \mathrm{d}r \\ &\quad\leq \sup_{s\in[0,1-(u-v)]} \int_0^1 |K(u-v+s,r) - K(s,r)|\ \mathrm{d}r \int_0^v |K(u,r) - K(v,r)|\,(g(r))^2\ \mathrm{d}r \\ &\quad\lesssim |u-v|^{\beta+\frac{1}{2}} \int_0^v |K(u,r) - K(v,r)|\,(g(r))^2\ \mathrm{d}r, \end{aligned} \tag{2-27}$$

and similarly for $u < v \leq 1$

$$\left(\int_0^u (K(u,r) - K(v,r))g(r)\, \mathrm{d}r\right)^2 \lesssim |u-v|^{\beta+\frac{1}{2}} \int_0^u |K(u,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}r.$$

Using (K2), (2-27) and Tonelli's theorem the first term in (2-26) can be estimated by

$$\begin{aligned}
&\int_0^1 \int_v^1 \frac{\left(\int_0^v (K(u,r) - K(v,r))g(r)\, \mathrm{d}r\right)^2}{|u-v|^{1+2\alpha}} \, \mathrm{d}u\mathrm{d}v \\
&\qquad \lesssim \int_0^1 \int_v^1 \frac{\int_0^v |K(u,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}r}{|u-v|^{2\alpha-\beta+1/2}} \, \mathrm{d}u\mathrm{d}v \\
&\qquad = \int_0^1 \int_0^{1-v} \frac{\int_0^v |K(v+x,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}r}{x^{2\alpha-\beta+1/2}} \, \mathrm{d}x\mathrm{d}v \\
&\qquad = \int_0^1 (g(r))^2 \int_0^{1-r} \frac{\int_r^{1-x} |K(v+x,r) - K(v,r)| \, \mathrm{d}v}{x^{2\alpha-\beta+1/2}} \, \mathrm{d}x\mathrm{d}r \\
&\qquad \leq \int_0^1 (g(r))^2 \int_0^{1-r} \frac{\int_0^{1-x} |K(v+x,r) - K(v,r)| \, \mathrm{d}v}{x^{2\alpha-\beta+1/2}} \, \mathrm{d}x\mathrm{d}r \\
&\qquad \leq \|h\|^2_{\mathcal{H}_V} \int_0^1 x^{2\beta-2\alpha} \, \mathrm{d}x \lesssim \|h\|^2_{\mathcal{H}_V}.
\end{aligned}$$

A similar computation can be used to estimate the third term in (2-26) as

$$\begin{aligned}
&\int_0^1 \int_0^v \frac{\left(\int_0^u (K(u,r) - K(v,r))g(r)\mathrm{d}r\right)^2}{|u-v|^{1+2\alpha}} \, \mathrm{d}u\mathrm{d}v \\
&\qquad \lesssim \int_0^1 \int_0^v \frac{\int_0^u |K(u,r) - K(v,r)|\, (g(r))^2 \mathrm{d}r}{|u-v|^{1/2+2\alpha-\beta}} \, \mathrm{d}u\mathrm{d}v \\
&\qquad = \int_0^1 \int_u^1 \frac{\int_0^u |K(u,r) - K(v,r)|\, (g(r))^2 \mathrm{d}r}{|u-v|^{1/2+2\alpha-\beta}} \, \mathrm{d}v\mathrm{d}u \lesssim \|h\|^2_{\mathcal{H}_V}.
\end{aligned}$$

To estimate the second and fourth term in (2-26), we use the fact that $K(s,t) = 0$ for $s \geq t$. Then, similar to (2-27), we obtain

$$\begin{aligned}
&\left(\int_v^u (K(u,r) - \underbrace{K(v,r)}_{=0})g(r)\, \mathrm{d}r\right)^2 \\
&\qquad \leq \int_v^u |K(u,r) - K(v,r)| \, \mathrm{d}r \int_v^u |K(u,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}r \\
&\qquad \lesssim |u-v|^{\beta+\frac{1}{2}} \int_0^u |K(u,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}r,
\end{aligned}$$

for $u > v$. This leads to

$$\begin{aligned}
\int_0^1 \int_v^1 \frac{\left(\int_v^u K(u,r)g(r)\, \mathrm{d}r\right)^2}{|u-v|^{1+2\alpha}} \, \mathrm{d}u\mathrm{d}v &= \int_0^1 \int_v^1 \frac{\left(\int_v^u (K(u,r) - K(v,r))g(r)\, \mathrm{d}r\right)^2}{|u-v|^{1+2\alpha}} \, \mathrm{d}u\mathrm{d}v \\
&\lesssim \int_0^1 \int_v^1 \frac{\int_0^u |K(u,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}r}{|u-v|^{1/2+2\alpha-\beta}} \, \mathrm{d}u\mathrm{d}v \\
&= \int_0^1 \int_v^1 \frac{\int_0^{v+x} |K(v+x,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}r}{x^{1/2+2\alpha-\beta}} \, \mathrm{d}x\mathrm{d}v \\
&= \int_0^1 \int_0^1 \frac{\int_{\max\{r-x,0\}}^{1-x} |K(v+x,r) - K(v,r)|\, (g(r))^2 \, \mathrm{d}v}{x^{1/2+2\alpha-\beta}} \, \mathrm{d}x\mathrm{d}r
\end{aligned}$$

$$\leq \int_0^1 \int_0^1 \frac{\int_0^{1-x} |K(v+x,r) - K(v,r)| \left(g(r)\right)^2 \,\mathrm{d}v}{x^{1/2+2\alpha-\beta}} \,\mathrm{d}x\mathrm{d}r \lesssim \|h\|_{\mathcal{H}_V}^2 .$$

A similar computation yields

$$\int_0^1 \int_0^v \frac{\left(\int_u^v K(v,r)g(r)\mathrm{d}r\right)^2}{|u-v|^{1+2\alpha}} \,\mathrm{d}u\mathrm{d}v \lesssim \|h\|_{\mathcal{H}_V}^2 ,$$

which proves (2-24).

To show (2-25), note that (2-24) together with the Besov-variation embedding (A-3), yields that the $1/\alpha$-variation of every $h \in \mathcal{H}_V$ is finite. Since $\alpha > 1/2$, the Young integral

$$\Delta_{[0,T]} \to \mathbb{R}, (s,t) \mapsto \mathbb{h}_{s,t} := \int_s^t h(r) - h(s) \,\mathrm{d}h(r)$$

is well-defined. Using the Besov–Hölder embedding (A-2) we obtain $[h]_{\alpha-1/2,[s,t]} \lesssim [h]_{W^{\alpha,2}((s,t))}$ and therefore

$$|h(t) - h(s)|^2 \lesssim |t-s|^{2\alpha-1} [h]^2_{\mathrm{W}^{\alpha,2}([s,t])} ,$$

for any $s,t \in [0,1]$. This yields

$$\frac{|h(t)-h(s)|^{\frac{1}{\alpha-\eta}}}{|t-s|^{\frac{\eta}{\alpha-\eta}}} \lesssim |t-s|^{\frac{\alpha-1/2-\eta}{\alpha-\eta}} \left([h]^2_{\mathrm{W}^{\alpha,2}([s,t])}\right)^{\frac{1}{2(\alpha-\eta)}} . \tag{2-28}$$

Note that the right-hand side is a control function. Indeed, $(s,t) \mapsto t-s$ and $(s,t) \mapsto [h]^2_{\mathrm{W}^{\alpha,2}([s,t])}$ are controls and then the product is also a control function due to

$$\frac{\alpha - \frac{1}{2} - \eta}{\alpha - \eta} + \frac{1}{2(\alpha-\eta)} = 1$$

and $\alpha > 1/2 + \eta$, see [FV10, Exercise 1.10]. In particular, the right-hand side is super-additive, which leads to

$$\begin{aligned}
W_{\mathbf{h},\alpha,\eta}(0,1) &= \sup_{\pi \subset [0,1]} \left\{ \sum_{[u,v]\in\pi} (v-u)^{\frac{-\eta}{\alpha-\eta}} \left(|h(v)-h(u)|^{\frac{1}{\alpha-\eta}} + |\mathbb{h}_{u,v}|^{\frac{1}{2(\alpha-\eta)}} \right) \right\} \\
&\lesssim \sup_{\pi \subset [0,1]} \left\{ \sum_{[u,v]\in\pi} (v-u)^{\frac{-\eta}{\alpha-\eta}} |h(v)-h(u)|^{\frac{1}{\alpha-\eta}} \right\} \\
&\lesssim \sup_{\pi \subset [0,1]} \left\{ \sum_{[u,v]\in\pi} (v-u)^{\frac{\alpha-1/2-\eta}{\alpha-\eta}} \left([h]^2_{\mathrm{W}^{\alpha,2}([u,v])}\right)^{\frac{1}{2(\alpha-\eta)}} \right\} \\
&\leq \left([h]^2_{\mathrm{W}^{\alpha,2}([0,1])}\right)^{\frac{1}{2(\alpha-\eta)}} \lesssim \|h\|_{\mathcal{H}_V}^{\frac{1}{\alpha-\eta}} ,
\end{aligned}$$

where we used (2-24) and (2-28). □

It is known that the fractional Brownian motion satisfies the statement of Lemma 2.27, see [GVR25, Proposition 2.12]. In fact, the proof uses only that the kernel fulfills (K1) and (K2) and builds on the embedding stated in [FV06, Theorem 3]. For this reason, it has already been conjectured that this property of the Cameron–Martin space can be extended to Volterra processes fulfilling these assumptions, see [FV06, Appendix A]. It turns out, that we can replace (K1) by

$$\int_0^1 |K(t,r) - K(s,r)| \, \mathrm{d}r = O\big(|t-s|^{\beta+\frac{1}{2}}\big), \tag{2-29}$$

and the statement of Lemma (2.27) is still true. But (2-29) is harder to verify in applications, which is the reason why we stick to the original condition (K1).

In the following, we will revisit Example 2.24 and show that Lemma 2.27 applies to more Volterra processes than the fractional Brownian motion.

Example 2.28. (Liouville Fractional Brownian motion). *Consider the kernel given by*

$$K(t,s) = \frac{1}{\Gamma(H+\frac{1}{2})}(t-s)^{H-\frac{1}{2}} \mathbb{1}_{[0,t)}(s),$$

for $s,t \geq 0$ *and* $H < 1/2$*. Furthermore, recall that the resulting Volterra process is Hölder continuous with parameter* $\beta < H$*.*

In order to verify (K1)*, note that*

$$\begin{aligned} \int_0^1 \big|(t+s-r)^{H-\frac{1}{2}} \mathbb{1}_{[0,t+s)}(r) - (s-r)^{H-\frac{1}{2}} \mathbb{1}_{[0,s)}(r)\big| \, \mathrm{d}r \\ = \int_0^s (s-r)^{H-\frac{1}{2}} - (t+s-r)^{H-\frac{1}{2}} \, \mathrm{d}r + \int_s^{t+s} (t+s-r)^{H-\frac{1}{2}} \, \mathrm{d}r \\ = \frac{s^{H+\frac{1}{2}} + t^{H+\frac{1}{2}} - (t+s)^{H+\frac{1}{2}} + t^{H+\frac{1}{2}}}{H+\frac{1}{2}} \\ = \frac{2t^{H+\frac{1}{2}} - (t+s)^{H+\frac{1}{2}} + s^{H+\frac{1}{2}}}{H+\frac{1}{2}}, \end{aligned} \tag{2-30}$$

holds for every $s \in [0, 1-t]$*, which implies* $t+s \leq 1$*. In particular, we obtain*

$$\sup_{s\in[0,1-t]} \int_0^t \big|(t+s-r)^{H-\frac{1}{2}} - (s-r)^{H-\frac{1}{2}} \mathbb{1}_{[0,s)}(r)\big| \, \mathrm{d}r = \frac{2t^{H+\frac{1}{2}} + (1-t)^{H+\frac{1}{2}} - 1}{H+\frac{1}{2}}.$$

since

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(2t^{H+\frac{1}{2}} - (t+s)^{H+\frac{1}{2}} + s^{H+\frac{1}{2}}\Big) = \left(H+\frac{1}{2}\right)\Big(s^{H-\frac{1}{2}} - (s+t)^{H+\frac{1}{2}}\Big) > 0,$$

holds for every $s \in [0, 1-t]$*, so the right-hand side of* (2-30) *is increasing, which means the maximum is attained at* $s = 1-t$*. To analyze the behavior for* $t \to 0$*, we consider the first-order series expansion of the numerator. We obtain*

$$(1-t)^{H+\frac{1}{2}} = 1 - \left(H+\frac{1}{2}\right) t + o(t)$$

and in particular

$$(1-t)^{H+\frac{1}{2}} - 1 = -\left(H+\frac{1}{2}\right) t + o(t).$$

In this framework $f = o(t)$ *means that* $\lim_{t\to 0}\left|\frac{f(t)}{t}\right| = 0$*. Using this, we have*

$$\frac{2t^{H+\frac{1}{2}} + (1-t)^{H+\frac{1}{2}} - 1}{t^{H-\varepsilon+\frac{1}{2}}} = \frac{2t^{H+\frac{1}{2}} - \big(H+\frac{1}{2}\big)\, t + o(t)}{t^{H-\varepsilon+\frac{1}{2}}}$$

$$= 2t^{\varepsilon} - \left(H + \frac{1}{2}\right) t^{1-\left(H-\varepsilon+\frac{1}{2}\right)} + o\left(t^{1-\left(H-\varepsilon+\frac{1}{2}\right)}\right) \to 0,$$

for some arbitrarily small $\varepsilon > 0$ *and* $t \to 0$*, since* $1 - (H - \varepsilon + 1/2) > 0$.

Verifying (K2) *works similarly. We fix* $t < 1/2$ *and obtain*

$$\begin{aligned}
&\int_0^{1-t} \left|(t+r-s)^{H-\frac{1}{2}} \mathbb{1}_{[0,t+r)}(s) - (r-s)^{H-\frac{1}{2}} \mathbb{1}_{[0,r)}(s)\right| \,\mathrm{d}r \\
&= \begin{cases} \int_0^s (t+r-s)^{H-\frac{1}{2}} \,\mathrm{d}r + \int_s^{1-t} (r-s)^{H-\frac{1}{2}} - (t+r-s)^{H-\frac{1}{2}} \,\mathrm{d}r, & s < t \\ \int_{s-t}^s (t+r-s)^{H-\frac{1}{2}} \,\mathrm{d}r + \int_s^{1-t} (r-s)^{H-\frac{1}{2}} - (t+r-s)^{H-\frac{1}{2}} \,\mathrm{d}r, & t \le s \le 1-t \\ \int_{s-t}^{1-t} (t+r-s)^{H-\frac{1}{2}} \,\mathrm{d}r, & 1-t < s \end{cases} \\
&= \frac{1}{H+\frac{1}{2}} \begin{cases} 2t^{H+\frac{1}{2}} - (t-s)^{H+\frac{1}{2}} + (1-t-s)^{H+\frac{1}{2}} - (1-s)^{H+\frac{1}{2}}, & s < t \\ 2t^{H+\frac{1}{2}} + (1-t-s)^{H+\frac{1}{2}} - (1-s)^{H+\frac{1}{2}}, & t \le s \le 1-t \\ (1-s)^{H+\frac{1}{2}}, & 1-t < s \end{cases} \\
&=: g(s)
\end{aligned} \tag{2-31}$$

In order to find the supremum of (2-31)*, we analyze the piecewise-defined function* g *on every part separately. It is clear that* g *is decreasing on* $[1-t, 1]$*. On the interval* $[t, 1-t]$ *we obtain*

$$\frac{\mathrm{d}}{\mathrm{d}s} g(s) = \left(H + \frac{1}{2}\right) \left((1-s)^{H-\frac{1}{2}} - (1-(s+t))^{H-\frac{1}{2}}\right) < 0,$$

which means that g *is decreasing for* $s \ge t$*. On* $[0, t]$*, we obtain the derivative*

$$\frac{\mathrm{d}}{\mathrm{d}s} g(s) = \left(H + \frac{1}{2}\right) \left((1-s)^{H-\frac{1}{2}} + (t-s)^{H-\frac{1}{2}} - (1-(s+t))^{H-\frac{1}{2}}\right),$$

which has a-priori no clear sign. However, due to $(t-s)^{H+1/2} \to \infty$ *for* $s \to t$*, the right-hand side tends to* $+\infty$ *for* $s \to t$*, which suggests that* g *increases on* $[0, t]$*. Numerically, one can prove that the supremum of* g *is attained at* t*. This leads to*

$$\begin{aligned}
\sup_{s \in [0,1]} \int_0^{1-t} &\left|(t+r-s)^{H-\frac{1}{2}} \mathbb{1}_{[0,t+r)}(s) - (r-s)^{H-\frac{1}{2}} \mathbb{1}_{[0,r)}(s)\right| \,\mathrm{d}r \\
&= \frac{2t^{H+\frac{1}{2}} + (1-2t)^{H+\frac{1}{2}} - (1-t)^{H+\frac{1}{2}}}{H + \frac{1}{2}}.
\end{aligned}$$

In order to analyze the behavior for $t \to 0$*, we consider again the first-order series expansion of the numerator, which yields*

$$(1-2t)^{H+\frac{1}{2}} - (1-t)^{H+\frac{1}{2}} = -\left(H + \frac{1}{2}\right) t + o(t).$$

Finally, for some arbitrary $\varepsilon > 0$ *we have*

$$\begin{aligned}
\frac{2t^{H+\frac{1}{2}} + (1-2t)^{H+\frac{1}{2}} - (1-t)^{H+\frac{1}{2}}}{t^{H-\varepsilon+\frac{1}{2}}} &= \frac{2t^{H+\frac{1}{2}} - \left(H + \frac{1}{2}\right) t + o(t)}{t^{H-\varepsilon+\frac{1}{2}}} \\
&= 2t^{\varepsilon} - \left(H + \frac{1}{2}\right) t^{1-\left(H-\varepsilon+\frac{1}{2}\right)} + o\left(t^{1-\left(H-\varepsilon+\frac{1}{2}\right)}\right) \to 0,
\end{aligned}$$

for $t \to 0$*, since* $1 - (H - \varepsilon + 1/2) > 0$*. In conclusion, the Liouville fractional Brownian motion fulfills* (K2).

Example 2.29. (Ornstein–Uhlenbeck process). *The Ornstein–Uhlenbeck process has the kernel*

$$K(t,s) := e^{a(t-s)}\mathbb{1}_{[0,t)}(s),$$

for $s,t \geq 0$ and some $a < 0$. Recall that the resulting Volterra process is Hölder continuous with parameter $\beta < 1/2$. In order to verify (K1)*, note that*

$$\begin{aligned}\int_0^1 &\left|e^{a(t+s-r)}\mathbb{1}_{[0,t+s)}(r) - e^{a(s-r)}\mathbb{1}_{[0,s)}(r)\right| \,\mathrm{d}r \\ &= \int_0^s e^{a(s-r)} - e^{a(t+s-r)} \,\mathrm{d}r + \int_s^{t+s} e^{a(t+s-r)} \,\mathrm{d}r \\ &= \frac{1 - e^{at} - \left(e^{as} - e^{a(t+s)}\right) + 1 - e^{at}}{-a} = \frac{(2 - e^{as})\left(1 - e^{at}\right)}{-a},\end{aligned}$$

holds for every $s \leq 1 - t$. Taking the supremum over the interval $[0, 1-t]$ yields

$$\begin{aligned}\sup_{s\in[0,1-t]} \int_0^1 &\left|e^{a(t+s-r)}\mathbb{1}_{[0,t+s)}(r) - e^{a(s-r)}\mathbb{1}_{[0,s)}(r)\right| \,\mathrm{d}r \\ &= \frac{\left(2 - e^{a(1-t)}\right)\left(1 - e^{at}\right)}{-a}.\end{aligned}$$

As in the previous example, we examine the behavior for $t \to 0$ by considering the series representation of the numerator. Note that the exponential function

$$e^x = 1 + x + o(x),$$

holds for every $x \in \mathbb{R}$. In particular, this leads to

$$\begin{aligned}\left(2 - e^{a(1-t)}\right)\left(1 - e^{at}\right) &= \Big(2 - e^a\left(1 - at + o(t)\right)\Big)\left(-at + o(t)\right) \\ &= \Big((2 - e^a) + e^a at + o(t)\Big)\left(-at + o(t)\right) \\ &= -a\,(2 - e^a)\,t - e^a a^2 t^2 + o\big(t^2\big).\end{aligned}$$

For some arbitrarily small $\varepsilon > 0$, we obtain

$$\frac{\left(2 - e^{a(1-t)}\right)\left(1 - e^{at}\right)}{t^{1-\varepsilon}} = -a\,(2 - e^a)\,t^{\varepsilon} - e^a a^2 t^{1-\varepsilon} + o\big(t^{1-\varepsilon}\big) \to 0,$$

for $t \to 0$, which proves (K1)*.*

To verify (K2)*, we can rewrite the integral as in* (2-31)*. Therefore, fix $t < 1/2$ and obtain*

$$\begin{aligned}\int_0^{1-t} &\left|e^{a(t+r-s)}\mathbb{1}_{[0,t+r)}(s) - e^{a(r-s)}\mathbb{1}_{[0,r)}(s)\right| \,\mathrm{d}r \\ &= \begin{cases} \int_0^s e^{a(t+r-s)} \,\mathrm{d}r + \int_s^{1-t} e^{a(r-s)} - e^{a(t+r-s)} \,\mathrm{d}r, & s < t \\ \int_{s-t}^s e^{a(t+r-s)} \,\mathrm{d}r + \int_s^{1-t} e^{a(r-s)} - e^{a(t+r-s)} \,\mathrm{d}r, & t \leq s \leq 1-t \\ \int_{s-t}^{1-t} e^{a(t+r-s)} \,\mathrm{d}r, & 1-t < s \end{cases} \\ &= \frac{1}{a}\begin{cases} e^{at} - e^{a(t-s)} + \left(e^{a(1-t-s)} - 1\right) - \left(e^{a(1-s)} - e^{at}\right), & s < t \\ e^{at} - 1 + \left(e^{a(1-t-s)} - 1\right) - \left(e^{a(1-s)} - e^{at}\right), & t \leq s \leq 1-t \\ e^{a(1-s)} - 1, & 1-t < s \end{cases}\end{aligned}$$

$$= \frac{1}{a}\begin{cases} 2e^{at} - e^{a(t-s)} + \left(e^{a(1-t-s)} - 1\right) - e^{a(1-s)}, & s < t \\ 2e^{at} - 2 + e^{a(1-t-s)} - e^{a(1-s)}, & t \le s \le 1-t \\ e^{a(1-s)} - 1, & 1-t < s \end{cases}.$$

The analysis of the supremum of this function is similar to the previous example, so we omit the details. The result is that the supremum is again attained at t, which leads to

$$\sup_{s\in[0,1]} \int_0^{1-t} \left| e^{a(t+r-s)}\mathbb{1}_{[0,t+r)}(s) - e^{a(r-s)}\mathbb{1}_{[0,r)}(s)\right| \, \mathrm{d}r$$
$$= \frac{2e^{at} - 2 + e^{a(1-2t)} - e^{a(1-t)}}{a} = \frac{2(1-e^{at}) + e^{a(1-t)}\left(1 - e^{-at}\right)}{-a}.$$

Finally, we use once more the series expansion of the numerator, which yields

$$\begin{aligned} 2(1-e^{at}) + e^{a(1-t)}\left(1 - e^{-at}\right) &= 2\left(-at + o(t)\right) + e^{a}\left(1 - at + o(t)\right)\left(at + o(t)\right) \\ &= at(e^a - 2) + o(t) \end{aligned}$$

and

$$\frac{2(1-e^{at}) + e^{a(1-t)}\left(1 - e^{-at}\right)}{t^{1-\varepsilon}} = at^{\varepsilon}(e^a - 2) + o(t^{\varepsilon}) \to 0,$$

for $t \to 0$, where $\varepsilon > 0$ is arbitrarily small. In conclusion, the Ornstein–Uhlenbeck process fulfills (K2).

2.2.3 Alternative concepts of rough paths

In this thesis, we focus exclusively on γ-Hölder rough paths. However, various notions of rough paths have been developed over time. In this section, we provide a brief overview of alternative definitions. We do not aim to offer an exhaustive list or detailed explanation of the various concepts; rather, the purpose of this section is to highlight the activity of this field and the diverse range of settings and applications that have emerged, thanks to the contributions of numerous researchers.

Rough paths of lower order

Everything obtained up to now is under the restriction that the regularity γ of the path X is bounded from below by $1/3$, but in principle we can keep adding more iterated integrals to allow even rougher paths, though generalizing Chen's relation is not straightforward. Therefore introduce the signature of a path $X\colon [0,T] \to \mathbb{R}$ by

$$\mathcal{S}(X)_{s,t} := \left(1, \int_s^t \mathrm{d}X_r, \int_{\Delta_{[s,t]}} \mathrm{d}X_r \mathrm{d}X_{s_2}, \dots, \int_{\Delta^{(k)}_{[s,t]}} \underbrace{\mathrm{d}X_r \mathrm{d}X_{s_2} \dots \mathrm{d}X_{s_k}}_{k\text{-times}}, \dots \right),$$

for $(s,t) \in \Delta_{[0,T]}$, provided all the iterated integrals are well-defined. Then Chen's relation has the form

$$\mathcal{S}(X)_{s,u} \otimes \mathcal{S}(X)_{u,t} = \mathcal{S}(X)_{s,t},$$

for every $(s,u,t) \in \Delta^{(3)}_{[0,T]}$. Given the complexity of handling the full signature, we only consider it up to the level necessary for the analysis. Specifically, if we have a path $X \in \mathcal{C}^{\gamma}([0,T];\mathbb{R})$ with $\gamma \in (0, 1/3]$, then only the first $N+1$ components of the signature must be considered, where $N \coloneqq \lfloor 1/\gamma \rfloor$. As in Definition 2.11, we now search for a $N+1$-tupel

$$\mathbf{X} \coloneqq \left(1, \mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \dots, \mathbb{X}^{(N)}\right)$$

that satisfies

$$\mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t} = \mathbf{X}_{s,t}, \tag{2-32}$$

and $\mathbb{X}^{(k)} \in \mathcal{C}_k^{k\gamma}([0,T];\mathbb{R})$, where $\mathcal{C}_k^{k\gamma}$ denotes the space of $k\gamma$-Hölder continuous k-parameter functions $f \colon \Delta_{0,T]}^{(k)} \to \mathbb{R}$. More precisely, (2-32) is equivalent to

$$\mathbb{X}_{s,t}^{(k)} = \sum_{j=0}^{k} \mathbb{X}_{s,u}^{(j)} \mathbb{X}_{u,t}^{(k-j)},$$

for every $k = 0, \dots, N$ and $(s,u,t) \in \Delta_{[0,T]}^{(3)}$, with the convention $\mathbb{X}^{(0)} \equiv 1$. Then, $\mathbf{X}$ is a γ-Hölder rough path.

This definition can be further extended to paths with values in a Banach space $\mathcal{V}$, though this requires further background of tensor product spaces, which is, for example, possible using Hilbert–Schmidt operators, see [HN19]. For further details, we refer to [Lyo98], [FV10, Part II] and [FH20, Section 2.4].

p-variation rough paths

The advantage of the variational approach compared to the Hölder continuous paths is that X does not necessarily have to be continuous. So, in this framework, it is possible to investigate jump processes. Historically, the variational approach is the notion Lyons used in his original article about rough paths [Lyo98]. However, since many processes of interest are continuous, like the fractional Brownian motion, numerous articles choose to treat Hölder rough paths for simplicity.

For some applications on rough paths with jumps, see, for example, [FS17, FZ18, ALP21, AP24].

Besov rough paths

Recall that the Hölder spaces can be seen as a special case of the Besov scale since $\mathrm{B}_{\infty,\infty}^{\gamma} = \mathcal{C}^{\gamma}$ for $\gamma \in (0,1)$. It is also possible to link the space of finite q-variation with Besov spaces, since $\mathcal{C}^{q\text{-var}}$ lies in between $\mathrm{B}_{q,1}^{1/q}$ and $\mathrm{B}_{q,\infty}^{1/q}$, see [BLdCS06, Theorem 5]. So it seems reasonable to bring the Hölder and the p-variation rough paths together using a Besov space approach. This is the underlying idea of a Besov rough path, which Prömel and Trabs introduce in [PT16]. In this article, the authors consider $\mathbb{R}^d$-valued paths which have Besov regularity in time, meaning that $X \in \mathrm{B}_{p,q}^{s}\big([0,T];\mathbb{R}^d\big)$. Similar approaches consider paths in Sobolev [LPT21] or Besov–Nikolskii type spaces [FP18]. A more recent treatment of this topic can be found in [FSZK22].

Unbounded rough drivers

An unbounded rough driver is an extension of the concept of rough paths to scenarios where the driving signals are not bounded, allowing for the study of more complex and realistic systems. Unlike the Hölder rough paths, the unbounded rough driver relies on a variational formulation of rough partial differential equations. The main application is given by equations with transport-type rough noise, see for example [BFHZ22, BG17, HHN25] and the references therein. However, nonlinear multiplicative noise is not yet covered in the framework of unbounded rough drivers.

2.3 Rough integration

This section is about finding a suitable class of paths y and functions f such that integrals of the form

$$\int_0^T f(r, y_r)\, \mathrm{d}\mathbf{X}_r,$$

are well-defined. In particular, we want to treat rough convolutions, i.e. $f(r, y_r) = S_{T,r}G(r, y_r)$ for some parabolic evolution family $(S_{t,s})_{s\leq t}$ and nonlinearity G, in order to treat parabolic equations in Chapter 3.

This section introduces the necessary fundamentals of the rough integration theory, but also expands on them with new findings. Parts of the expansions have already been published in [NS23] and [BGVS25] which is based on a collaboration with Alexandra Blessing and Mazyar Ghani Varzaneh.

2.3.1 Integration against one forms

Let $\mathcal{V}, \mathcal{W}$ be two Banach spaces, $T \leq 1$ and $\mathbf{X} \in \mathscr{C}^\gamma([0,T];\mathcal{V})$ a γ-Hölder rough path with values in a Banach space $\mathcal{V}$ for $\gamma \in (1/3, 1/2]$. Since $\mathbb{X}$ postulates the value of the iterated integral, the logical next step is to extend the integral to functions that are "looking like X". Consider the composition $f\circ X$ for $f\colon \mathcal{V} \to \mathcal{L}(\mathcal{V};\mathcal{W})$ which is at least $\mathcal{C}^1$ such that the Fréchet derivative $\mathrm{D}f(X_u) \in \mathcal{L}\big(\mathcal{V};\mathcal{L}(\mathcal{V};\mathcal{W})\big)$ exists. Then Taylor's approximation yields

$$f(X_r) = f(X_u) + \mathrm{D}f(X_u)X_{u,r} + R_{u,r},$$

for some $u \leq r \leq v$ and a remainder $R_{u,r}$. So on small scales, $f \circ X$ can be seen as a perturbation of X, which justifies saying that "$f \circ X$ looks like X".

Now, recall one of the primary motivations underpinning our discussion in Section 2.1 to define the integral

$$\int_0^T f(X_r)\, \mathrm{d}\mathbf{X}_r. \tag{2-33}$$

We have shown that (2-33) can be approximated by a compensated Riemann sum (2-6), provided that the iterated integral is well-defined. Given that $\mathbb{X}$ prescribes the value of this iterated integral, it can be expected to define the rough integral by

$$\int_s^t f(X_r)\, \mathrm{d}\mathbf{X}_r := \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \Big(f(X_u)X_{u,v} + \mathrm{D}f(X_u)\mathbb{X}_{u,v}\Big), \tag{2-34}$$

where we use the fact $\mathrm{D}f(X_u) \in \mathcal{L}\big(\mathcal{V};\mathcal{L}(\mathcal{V};\mathcal{W})\big) = \mathcal{L}(\mathcal{V}\otimes\mathcal{V};\mathcal{W})$ in order to make sense of $\mathrm{D}f(X_u)\mathbb{X}_{u,v}$, see Theorem 2.9. That this is an appropriate approximation follows from the sewing lemma, which provides a sufficient condition for when an approximation is "good enough" to define an integral. We refrain from giving a proof here, even though it is elementary for the theory of rough paths, and refer to Theorem 2.42, which uses a similar technique and fits better in the framework of parabolic equations, which is used later on.

Define $\mathcal{C}_2^{\alpha,\beta}([0,T];\mathcal{W})$ as the space of all two-parameter functions $\Xi \in \mathcal{C}_2^{\alpha}([0,T];\mathcal{W})$ such that

$$\sup_{(s,u,t)\in\Delta^{(3)}_{[0,T]}} \frac{\left\|(\delta\Xi)_{s,u,t}\right\|_{\mathcal{W}}}{|t-s|^{\beta}} < \infty$$

where $(\delta\Xi)_{s,u,t} := \Xi_{s,t} - \Xi_{s,u} - \Xi_{u,t}$.

Theorem 2.30. ([FdLP06, Lemma 2.1]) *Let $0 < \alpha \le 1 < \beta$. Then there exists a unique continuous map $\mathcal{I}\colon \mathcal{C}_2^{\alpha,\beta}([0,T];\mathcal{W}) \to \mathcal{C}^{\alpha}([0,T];\mathcal{W})$ such that*

$$\|(\mathcal{I}\Xi)_{s,t} - \Xi_{s,t}\|_{\mathcal{W}} \lesssim_{\beta,\Xi} (t-s)^{\beta},$$

holds for $(s,t)\in\Delta_{[0,T]}$. Note that $\mathcal{I}\Xi$ is itself a path.

A similar version of the sewing lemma is proven in [Gub04, Proposition 1]. Now, Theorem 2.30 yields

$$(\mathcal{I}\Xi)_{s,t} = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \Xi_{u,v}.$$

In order to apply this to (2-34), set $\Xi_{u,v} := f(X_u)X_{u,v} + \mathrm{D}f(X_u)\mathbb{X}_{u,v}$. It is easy to see that

$$(\delta\Xi)_{s,u,t} = -R^y_{s,u}X_{u,t} - y'_{s,u}\mathbb{X}_{u,t},$$

holds for $(s,u,t)\in\Delta^{(3)}_{[0,T]}$. The following lemma then yields $\Xi\in\mathcal{C}_2^{2\gamma,3\gamma}([0,T];\mathcal{W})$.

Lemma 2.31. *Let $f\colon\mathcal{V}\to\mathcal{L}(\mathcal{V};\mathcal{W})$ be a bounded $\mathcal{C}^2$-function, $\mathbf{X}\in\mathscr{C}^{\gamma}([0,T];\mathcal{V})$ and set*

$$y := f\circ X, y' := \mathrm{D}f\circ X, R^y_{s,t} := y_{s,t} - y'_s X_{s,t},$$

for $(s,t)\in\Delta_{[0,T]}$. Then $y\in\mathcal{C}^{\gamma}([0,T];\mathcal{W}), y'\in\mathcal{C}^{\gamma}\big([0,T];\mathcal{L}(\mathcal{V};\mathcal{W})\big)$ and $R^y\in\mathcal{C}_2^{2\gamma}([0,T];\mathcal{W})$ follows.

In particular, the limit in (2-34) is well-defined and

$$\begin{aligned}&\left\|\int_s^t f(X_r)\,\mathrm{d}\mathbf{X}_r - \big(f(X_s)X_{s,t} + \mathrm{D}f(X_s)\mathbb{X}_{s,t}\big)\right\|_{\mathcal{W}}\\ &\qquad\lesssim_{\gamma}\left([X]_{\gamma,[s,t],\mathcal{V}}\left[R^{f\circ X}\right]_{2\gamma,[s,t],\mathcal{W}} + [\mathbb{X}]_{2\gamma,[s,t],\mathcal{V}\otimes\mathcal{V}}\,[\mathrm{D}f\circ X]_{\gamma,[s,t],\mathcal{W}}\right)(t-s)^{3\gamma}\end{aligned}$$

holds for $(s,t)\in\Delta_{[0,T]}$. This rough integral against a one-form f is the one Lyons introduced in the variational setting in [Lyo98, Section 3.2].

2.3.2 Integration along controlled rough paths

It turns out that the integrand does not need to have the exact form $f\circ X$ to define the rough integral. It is about the pair (y,y') which we introduced as $y = f\circ X, y' = \mathrm{D}f\circ X$ in Lemma 2.31. It suffices for (y,y') to satisfy the regularity conditions in Lemma 2.31 to define the integral. In this context, y is said to be controlled by the path X. This concept traces back to Gubinelli [Gub04], that's why y' is often referred to as the Gubinelli derivative of y.

Definition 2.32. Let $X\in\mathcal{C}^{\gamma}([0,T];\mathcal{V})$ be a path with $\gamma\in(1/3,1/2]$ and $\overline{\mathcal{W}}$ an additional Banach space. A pair (y,y') consisting of the path $y:[0,T]\to\overline{\mathcal{W}}$ and the Gubinelli derivative $y'\colon[0,T]\to\mathcal{L}\big(\mathcal{V};\overline{\mathcal{W}}\big)$ is a controlled rough path if $y\in\mathcal{C}^{\gamma}\big([0,T];\overline{\mathcal{W}}\big)$, $y'\in\mathcal{C}^{\gamma}\big([0,T];\mathcal{L}\big(\mathcal{V};\overline{\mathcal{W}}\big)\big)$

and the remainder

$$R^y_{s,t} := y_{s,t} - y'_s X_{s,t},$$

for $(s,t) \in \Delta_{[0,T]}$ satisfies $R^y \in \mathcal{C}_2^{2\gamma}\big([0,T];\overline{\mathcal{W}}\big)$. Then $\mathcal{D}_X^\gamma\big([0,T];\overline{\mathcal{W}}\big)$ denotes the space of controlled rough paths endowed with the norm

$$\left\|y,y'\right\|_{\mathcal{D}_X^\gamma([0,T];\overline{\mathcal{W}})} := \|y_0\|_{\overline{\mathcal{W}}} + \left\|y'_0\right\|_{\mathcal{L}(\mathcal{V};\overline{\mathcal{W}})} + \left[y'\right]_{\gamma,[0,T],\mathcal{L}(\mathcal{V};\overline{\mathcal{W}})} + [R^y]_{2\gamma,[0,T],\mathcal{L}(\mathcal{V};\overline{\mathcal{W}})}\,.$$

Remark 2.33. i) The usual choice of the additional Banach space is $\overline{\mathcal{W}} = \mathcal{L}(\mathcal{V};\mathcal{W})$. In particular, this is necessary to define the rough integral, as we have seen for the integration against one-forms.

ii) As the name suggests, y' is some sort of a derivative to capture the first-order approximation appearing in Taylor's theorem. However, this should not be confused with the time derivative of y, which is sometimes in the literature also denoted by y'. To avoid confusion, the time derivative of paths is denoted in the course of this manuscript by $\dot{y}, \partial_t y$ or $\frac{\mathrm{d}}{\mathrm{d}t}y$.

iii) It is worth noting that the Gubinelli derivative is not unique. It depends on the "roughness" of the underlying path, which controls y. If $X \in \mathcal{C}^\gamma([0,T];\mathcal{V})$ is truly rough, meaning that

$$\lim_{t\searrow s} \frac{|v^*(X_{s,t})|}{|t-s|^{2\gamma}} = \infty$$

for every s in a dense subset of $[0,T]$ and $v^* \in \mathcal{V}'$, then the Gubinelli derivative is uniquely defined, see [FH20, Proposition 6.4]. For example, the d-dimensional Brownian motion is truly rough; see [FH20, Theorem 6.6].

iv) Since the remainder has to fulfill $R^y \in \mathcal{C}_2^{2\gamma}$ and the path component only $y' \in \mathcal{C}^\gamma$, there has to be some cancellation in the definition of the remainder to obtain that higher regularity. But if, on the other hand, the path is smooth $y \in \mathcal{C}^{2\gamma}$, it is possible to choose $y' \equiv 0$ since the path component is smooth enough to provide $R^y \in \mathcal{C}_2^{2\gamma}$. If, in addition, X is truly rough, $y' \equiv 0$ is the unique Gubinelli derivative.

v) Unlike the space of γ-Hölder continuous rough paths, $\mathcal{D}_X^\gamma$ is a Banach space endowed with the norm $\|y,y'\|_{\mathcal{D}_X^\gamma([0,T];\overline{\mathcal{W}})}$.

The following remarkable theorem is originally due to Gubinelli [Gub04]. The theory he established is a little more general than the version presented here from [FH20].

Theorem 2.34. ([FH20, Theorem 4.10]) *Let $\mathbf{X} \in \mathscr{C}^\gamma([0,T];\mathcal{V})$ be a γ-Hölder rough path for $\gamma \in (1/3, 1/2]$ and $(y,y') \in \mathcal{D}_X^\gamma\big([0,T];\mathcal{L}(\mathcal{V};\mathcal{W})\big)$. Then, the rough integral*

$$\int_0^T y_r \,\mathrm{d}\mathbf{X}_r := \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} \big(y_u X_{u,v} + y'_u \mathbb{X}_{u,v}\big)\,,$$

is well-defined and for every $(s,t) \in \Delta_{[0,T]}$ we have

$$\left\|\int_s^t y_r \,\mathrm{d}\mathbf{X}_r - \big(y_s X_{s,t} + y'_s \mathbb{X}_{s,t}\big)\right\|_{\mathcal{W}} \lesssim_\gamma \Big([X]_{\gamma,[s,t],\mathcal{V}}\,[R^y]_{2\gamma,[s,t],\mathcal{L}(\mathcal{V};\mathcal{W})} + [\mathbb{X}]_{2\gamma,[s,t],\mathcal{V}\otimes\mathcal{V}}\,\left[y'\right]_{\gamma,[s,t],\mathcal{L}(\mathcal{V}\otimes\mathcal{V};\mathcal{W})}\Big)\,(t-s)^{3\gamma}.$$

In particular, the map given by

$$(y,y') \mapsto \left(\int_0^\cdot y_r \,\mathrm{d}\mathbf{X}_r, y\right),$$

is continuous and linear from $\mathcal{D}_X^\gamma\big([0,T];\mathcal{L}(\mathcal{V};\mathcal{W})\big)$ *to* $\mathcal{D}_X^\gamma([0,T];\mathcal{W})$*.*

2.3.3 Controlled rough paths according to a family of Banach spaces

We have now reached the point where we can introduce the integral notion that is central to the remainder of this thesis. The reasoning of this entire consideration is the question of solving a differential equation that is controlled by an irregular path. As of now, we can consider equations of the form (CODE) with the rough integral established in the previous subsection. The objective is now to extend the analysis by considering partial differential equations, which entails incorporating a differential operator A. Under specific conditions, as detailed in Appendix B, this operator generates an analytic semigroup. Specifically, the concept of mild solutions is studied here, which involves a convolution integral

$$\int_0^T S_{T-r} y_r \,\mathrm{d}\mathbf{X}_r,$$

where $S_r \in \mathcal{L}(\mathcal{E})$ for every $r \in [0,T]$ and some Banach space $\mathcal{E}$. The problem now is that the analytic semigroup $(S_r)_{r\in[0,T]}$ is generally not Hölder continuous in 0, which makes it impossible to find a suitably controlled rough path in the sense of Definition 2.32. To get around the lack of Hölder continuity in 0, it is possible to exploit the known trade-off between time and space regularity of analytic semigroups (B-2). However, this requires a reformulation of the controlled rough paths.

In recent years, several approaches have been proposed for extending controlled rough paths to parabolic equations. In this thesis, we rely on the definition of Gerasimovičs, Hocquet, and Nilssen [GHN21]; other possibilities are briefly discussed in Subsection 2.3.5.

From now on, fix a $\mathbb{R}^d$-valued γ-Hölder rough path $\mathbf{X} = (X, \mathbb{X})$ for $\gamma \in (1/3, 1/2]$. Since the goal is to consider parabolic equations using the smoothing properties of analytic semigroups or parabolic evolution families, it is necessary to define families of function spaces. These families are often defined using the fractional powers or the abstract Banach scale defined by the linear part of the equation. These two topics are discussed in Appendix C and B.2.

Definition 2.35. A family $(E_\beta)_{\beta\in\mathbb{R}}$ of function spaces with norms $\|\cdot\|_{E_\beta} =: \|\cdot\|_\beta$ for $\beta \in \mathbb{R}$ is called a monotone scale of interpolation spaces, if the continuous embedding $E_{\alpha_2} \hookrightarrow E_{\alpha_1}$ is dense and

$$[E_{\alpha_1}, E_{\alpha_2}]_\alpha = E_{(\alpha_2-\alpha_1)\alpha+\alpha_1}, \tag{2-35}$$

holds for $\alpha_1 \le \alpha_2$ and $\alpha \in (0,1)$.

Remark 2.36. This definition is stronger than originally stated in [GHN21, Definition 2.1]. There, only the interpolation inequality

$$\|x\|_\alpha^{\alpha_2-\alpha_1} \lesssim \|x\|_{\alpha_1}^{\alpha_2-\alpha}\,\|x\|_{\alpha_2}^{\alpha-\alpha_1} \tag{2-36}$$

for $\alpha_1 \le \alpha \le \alpha_2$ and $x \in E_{\alpha_2}$ is considered. Actually, (2-35) implies (2-36), but not the other way around. However, this is not a restriction, as most relevant examples that fulfill (2-36) also fulfill (2-35).

We can now state the notion of a controlled rough path tailored to parabolic problems. It is a slightly more general version of the one used in [GHN21].

Definition 2.37. ([GHN21, Definition 4.3]) Let $\alpha \in \mathbb{R}, \gamma \in (1/3, 1/2], \gamma_1, \gamma_2 \in (0,\gamma]$ such that

$$\min\{\gamma_1 + 2\gamma, 2\gamma_2 + \gamma\} \ge 1,$$

and $X \in \mathcal{C}^\gamma\big([0,T];\mathbb{R}^d\big)$. We call a pair (y,y') a controlled rough path according to a monotone family of interpolation spaces $(E_\beta)_{\beta\in\mathbb{R}}$ if

$$\begin{aligned} &y \in \mathcal{C}([0,T];E_\alpha),\\ &y' \in \mathcal{C}\big([0,T];\mathcal{L}\big(\mathbb{R}^d;E_{\alpha-\gamma}\big)\big)\cap\mathcal{C}^{\gamma_1}\big([0,T];\mathcal{L}\big(\mathbb{R}^d;E_{\alpha-2\gamma}\big)\big)\end{aligned}$$

and the remainder

$$\Delta_{[0,T]} \to E_\alpha, (s,t)\mapsto R^y_{s,t} := y_{s,t} - y'_s X_{s,t}$$

belongs to $\mathcal{C}_2^{\gamma_2}([0,T];E_{\alpha-\gamma})\cap\mathcal{C}_2^{2\gamma_2}([0,T];E_{\alpha-2\gamma})$. The space of controlled rough paths is denoted by $\mathcal{D}^{\gamma,\gamma_1,\gamma_2}_{X,\alpha}([0,T])$ and endowed with the norm $\|\cdot,\cdot\|_{\mathcal{D}^{\gamma,\gamma_1,\gamma_2}_{X,\alpha}([0,T])}$ given by

$$\begin{aligned}\big\|y,y'\big\|_{\mathcal{D}^{\gamma,\gamma_1,\gamma_2}_{X,\alpha}([0,T])} := \|y\|_{\infty,[0,T],E_\alpha} &+ \big\|y'\big\|_{\infty,[0,T],\mathcal{L}(\mathbb{R}^d;E_{\alpha-\gamma})}\\ &+ \big[y'\big]_{\gamma_1,[0,T],\mathcal{L}(\mathbb{R}^d;E_{\alpha-2\gamma})} + [R^y]_{\gamma_2,[0,T],E_{\alpha-\gamma}}\\ &+ [R^y]_{2\gamma_2,[0,T],E_{\alpha-2\gamma}},\end{aligned}$$

where $\|y\|_{\infty,J,\mathcal{E}} := \sup_{s\in J}\|y_s\|_{\mathcal{E}}$ is the supremum norm for an arbitrary interval $J\subset\mathbb{R}$ and Banach space $\mathcal{E}$. For $\gamma=\gamma_1=\gamma_2$, define $\mathcal{D}^\gamma_{X,\alpha}([0,T]) := \mathcal{D}^{\gamma,\gamma_1,\gamma_2}_{X,\alpha}([0,T])$

Remark 2.38. The space $\mathcal{D}^\gamma_{X,\alpha}([0,T])$ is the one defined originally in [GHN21]. Incorporating the two new regularity parameters γ_1,γ_2 adds more flexibility in the sewing lemma for the definition of the integral. This approach is useful to investigate the continuous dependence on the noise. Most of the time, we work in the space $\mathcal{D}^\gamma_{X,\alpha}([0,T])$, and mention the refined one only if needed, to not overload the notation even more.

Note that it is possible to identify $y'\in\mathcal{L}\big(\mathbb{R}^d;E_\alpha\big)$ with an element in E_α^d, writing $y' = (y'^{,i})_{1\le i\le d}$. Therefore, the norm is given by

$$\big\|y'_r\big\|_{\mathcal{L}(\mathbb{R}^d;E_\alpha)} = \sup_{1\le i\le d}\big\|y'^{,i}_r\big\|_{E_\alpha}.$$

Due to better readability, we simplify the notation by $\|y\|_{\infty,[0,T],\alpha}$, $\|y'\|_{\infty,[0,T],\alpha-\gamma}$ and $[y']_{\gamma,[0,T],\alpha-2\gamma}$ any time where there is no risk of confusion, and analogously for the remainder. Also, the time interval is often omitted in the index of all (semi-)norms. Then, the first index of the (semi-)norms always indicates the time regularity, and the second one stands for the space regularity. Furthermore, we omit the time interval when specifying the function space; for example, we use $\mathcal{C}^\gamma(E_\alpha)$ instead of $\mathcal{C}^\gamma([0,T];E_\alpha)$ to highlight the trade-off between space and time regularity.

Remark 2.39. If the path component itself is d-dimensional, the resulting Gubinelli derivative is matrix-valued, where the entries are in a Banach space. Then write $(y,y')\in\big(\mathcal{D}^\gamma_{X,\alpha}\big)^d$, meaning that every component of y together with the corresponding column is a controlled rough path. In particular, it holds that

$$\begin{aligned} y &\in \mathcal{C}\big(\mathcal{L}\big(\mathbb{R}^d;E_\alpha\big)\big),\\ y' &\in \mathcal{C}\big(\mathcal{L}\big(R^{d\times d};E_{\alpha-\gamma}\big)\big)\cap\mathcal{C}^\gamma\big(\mathcal{L}\big(\mathbb{R}^{d\times d};E_{\alpha-2\gamma}\big)\big),\\ R^y &\in \mathcal{C}_2^\gamma\big(\mathcal{L}\big(\mathbb{R}^d;E_{\alpha-\gamma}\big)\big)\cap\mathcal{C}_2^{2\gamma}\big(\mathcal{L}\big(\mathbb{R}^d;E_{\alpha-2\gamma}\big)\big),\end{aligned}$$

where the identification of $\mathcal{L}\big(\mathbb{R}^d;E_\alpha\big)$ with E^d_α is used again, as well as (2-12). These properties justify simply writing $y_sX_{s,t}$ or $y'_s\mathbb{X}_{s,t}$ even in the d-dimensional setting. However, keep in mind that for $d>1$, these two terms are given by the scalar product for vectors, respectively, for matrices.

A direct consequence of the definition of $\mathcal{D}^{\gamma}_{X,\alpha}$ is that the path component is, in particular, Hölder continuous. Let $(y,y') \in \mathcal{D}^{\gamma}_{X,\alpha}$ and obtain for $i = 1,2$

$$[y]_{\gamma,\alpha-i\gamma} \leq \left\|y'\right\|_{\infty,\alpha-i\gamma}[X]_{\gamma,\mathbb{R}^d} + [R^y]_{\gamma,\alpha-i\gamma} \leq \left(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\right) \left\|y,y'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}}, \tag{2-37}$$

using $y_{s,t} = R^y_{s,t} + y'_s X_{s,t}$.

Remark 2.40. For every $\varsigma \in [0,\gamma)$, we obtain from the interpolation inequality (2-36) in particular

$$\begin{aligned}
\frac{\|y_{s,t}\|_{\alpha+\varsigma-\gamma}}{(t-s)^{\gamma-\varsigma}} &\lesssim \left(\frac{\|y_{s,t}\|_{\alpha+\varsigma-\gamma}}{(t-s)^{\gamma}}\right)^{\frac{\gamma-\varsigma}{\gamma}} \|y_{s,t}\|_{\alpha}^{\frac{\varsigma}{\gamma}}, \\
\frac{\left\|y'_{s,t}\right\|_{\alpha+\varsigma-2\gamma}}{(t-s)^{\gamma-\varsigma}} &\lesssim \left(\frac{\left\|y'_{s,t}\right\|_{\alpha-2\gamma}}{(t-s)^{\gamma}}\right)^{\frac{\gamma-\varsigma}{\gamma}} \|y'_{s,t}\|_{\alpha-\gamma}^{\frac{\varsigma}{\gamma}}, \\
\frac{\left\|R^y_{s,t}\right\|_{\alpha+\varsigma-2\gamma}}{(t-s)^{\gamma}} &\lesssim \left(\frac{\left\|R^y_{s,t}\right\|_{\alpha-2\gamma}}{(t-s)^{2\gamma}}\right)^{\frac{\gamma-\varsigma}{\gamma}} \left(\frac{\left\|R^y_{s,t}\right\|_{\alpha-\gamma}}{(t-s)^{\gamma}}\right)^{\frac{\varsigma}{\gamma}} (t-s)^{\gamma-\varsigma},
\end{aligned}$$

for $(s,t) \in \Delta_{[0,T]}$, which leads to

$$\begin{aligned}
[y]_{\gamma-\varsigma,\alpha+\varsigma-\gamma} &\lesssim \left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right) \left\|y,y'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}}, \\
\left[y'\right]_{2\gamma-\varsigma,\alpha+\varsigma-2\gamma} &\lesssim \left\|y,y'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}}, \\
[R^y]_{\gamma,\alpha+\varsigma-2\gamma} &\lesssim \left\|y,y'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}} T^{\gamma-\varsigma}.
\end{aligned} \tag{2-38}$$

Before stating the existence of the rough convolution, the following auxiliary lemma is important to estimate the approximation.

Lemma 2.41. *Let $a,b \in \mathbb{R}$ and $c \geq 0$ such that $a+b > 0$, then the sum*

$$\sum_{k\geq 0} \left(\sum_{0\leq m<2^k} \left(1 - \frac{2m}{2^{k+1}}\right)^{a/c} \left(2^{-(k+1)}\right)^{b/c} \right)^c < \infty,$$

is finite.

Proof. Choose a small $\delta > 0$ such that $a+b > \delta$ and obtain

$$\begin{aligned}
&\sum_{k\geq 0} \left(\sum_{0\leq m<2^k} \left(1 - \frac{2m}{2^{k+1}}\right)^{a/c} \left(2^{-(k+1)}\right)^{b/c} \right)^c \\
&\qquad \leq \sum_{k\geq 0} \left(2^{-(k+1)}\right)^{\delta} \left(\sum_{0\leq m<2^k} \left(1 - \frac{2m}{2^{k+1}}\right)^{\frac{a+b-\delta}{c}-1} 2^{-(k+1)} \right)^c,
\end{aligned}$$

since $2^{-(k+1)} \leq 1 - \frac{2m}{2^{k+1}}$ holds for every $0 \leq m < 2^k$. The sum over m is a Riemann approximation, which leads to

$$\begin{aligned}
\sum_{0\leq m<2^k} \left(1 - \frac{2m}{2^{k+1}}\right)^{\frac{a+b-\delta}{c}-1} 2^{-(k+1)} &\leq \lim_{k\to\infty} \sum_{0\leq m<2^k} \left(1 - \frac{2m}{2^{k+1}}\right)^{\frac{a+b-\delta}{c}-1} 2^{-(k+1)} \\
&= \int_0^1 (1-x)^{\frac{a+b-\delta}{c}-1} \,\mathrm{d}x = \frac{c}{a+b-\delta} < \infty,
\end{aligned}$$

since $a+b-\delta>0$ and the Riemann and Lebesgue integrals coincide if the Riemann sum converges. Using this, the claim follows

$$\sum_{k\geq 0}\left(\sum_{0\leq m<2^k}\left(1-\frac{2m}{2^{k+1}}\right)^{a/c}\left(2^{-(k+1)}\right)^{b/c}\right)^c \\ \leq \left(\frac{c}{a+b-\delta}\right)^c\sum_{k\geq 0}\left(2^{-(k+1)}\right)^{\delta}=\left(\frac{c}{a+b-\delta}\right)^c 2^{1-\delta}.$$

□

Now, the existence of the rough convolution can be proven with the help of all the ideas collected in the previous sections.

Theorem 2.42. ([GHN21, Theorem 4.5]) *Let $\alpha\in\mathbb{R},\gamma\in(1/3,1/2],\gamma_1,\gamma_2\in(0,\gamma]$ be such that*

$$\min\{\gamma_1+2\gamma,2\gamma_2+\gamma\}\geq 1,$$

$(E_\beta)_{\beta\in\mathbb{R}}$ a monotone scale of interpolation spaces, $\mathbf{X}=(X,\mathbb{X})\in\mathscr{C}^\gamma([0,T];\mathbb{R})$ a γ-Hölder rough path, and $(y,y')\in\left(\mathcal{D}_{X,\alpha}^{\gamma,\gamma_1,\gamma_2}\right)^d$ a controlled rough path. Further, let for every $(s,t)\in\Delta_{[0,T]}$ and $\beta\in\mathbb{R}$

$$S_{t,s}\colon E_\beta\to E_\beta$$

be a linear operator. Assume that $S_{t,s}$ fulfills the following conditions for every $(s,u,t)\in\Delta^{(3)}_{[0,T]}$.

(S1) *The operators are continuous in time $S\in\mathcal{C}_2([0,T];\mathcal{L}(E_\beta))$ and fulfill*

$$\lim_{t\searrow s}\|S_{t,s}x-x\|_{E_\beta}=0,$$

for every $x\in E_\beta$.

(S2) *The semigroup property $S_{t,s}=S_{t,u}S_{u,s}$ is satisfied and $S_{t,t}=\mathrm{Id}_{E_\beta}$ holds.*

(S3) *The operator $S_{t,s}$ has the following smoothing properties*

$$\left\|S_{t,s}-\mathrm{Id}_{E_{\beta+1}}\right\|_{\mathcal{L}(E_{\beta+1};E_\beta)}\lesssim_{\beta,T}(t-s), \\ \|S_{t,s}\|_{\mathcal{L}(E_\beta;E_{\beta+1})}\lesssim_{\beta,T}(t-s)^{-1}.$$

Then the integral

$$\int_0^T S_{T,r}y_r\,\mathrm{d}\mathbf{X}_r:=\lim_{|\pi|\to 0}\sum_{[u,v]\in\pi}S_{T,u}\left(y_uX_{u,v}+y'_u\mathbb{X}_{u,v}\right), \tag{2-39}$$

exists as an element of $E_{\alpha-2\gamma+\beta}$ for $\beta\in[0,\min\{\gamma_1+2\gamma,2\gamma_2+\gamma\})$. Moreover

$$\left\|\int_s^t S_{t,r}y_r\,\mathrm{d}\mathbf{X}_r-S_{t,s}\left(y_sX_{s,t}+y'_s\mathbb{X}_{s,t}\right)\right\|_{\alpha-2\gamma+\beta} \\ \lesssim \varrho_{\gamma,[s,t]}(\mathbf{X})\left\|y,y'\right\|_{(\mathcal{D}^\gamma_{X,\alpha}([s,t]))^d}(t-s)^{\min\{\gamma_1+2\gamma,2\gamma_2+\gamma\}-\beta}. \tag{2-40}$$

holds true for every $\beta\in[0,\min\{\gamma_1+2\gamma,2\gamma_2+\gamma\})$ and $(s,t)\in\Delta_{[0,T]}$.

Remark 2.43. Note that condition **(S3)** together with (2-36) implies

$$\left\| S_{t,s} - \text{Id}_{E_{\beta_2}} \right\|_{\mathcal{L}(E_{\beta_2};E_{\beta_1})} \lesssim_{\beta_1,\beta_2} (t-s)^{\beta_2-\beta_1}, \tag{2-41}$$

$$\|S_{t,s}\|_{\mathcal{L}(E_{\beta_1};E_{\beta_2})} \lesssim_{\beta_1,\beta_2} (t-s)^{-(\beta_2-\beta_1)}, \tag{2-42}$$

for $0 \leq \beta_2 - \beta_1 \leq 1$. It is known that analytic semigroups and parabolic evolution families satisfy **(S1)-(S3)**, see Appendix B and Theorem D.1 respectively.

Proof. For the sake of completeness, the proof of this statement is sketched based on the classical sewing lemma, see also [GHN21, Theorem 4.1] and [GH19, Theorem 2.4], to emphasize the differences. Let

$$\pi^k := \left\{ s_k^m := s + \frac{m}{2^k}(t-s) : m = 0, \dots, 2^k \right\}$$

be the k-th dyadic partition of the interval $[s,t]$, and

$$\mathcal{I}_{s,t}^{\pi^k} := \sum_{[u,v]\in\pi^k} S_{t,u}\left(y_u X_{u,v} + y'_u \mathbb{X}_{u,v}\right) := \sum_{[u,v]\in\pi^k} S_{t,u}\xi_{u,v}$$

the approximation associated with the partition π^k. Set $m_{u,v} := \frac{v+u}{2}$ as the midpoint of $[u,v]$, then the difference $\mathcal{I}_{s,t}^{\pi^k} - \mathcal{I}_{s,t}^{\pi^{k+1}}$ can be rewritten as

$$\begin{aligned} \mathcal{I}_{s,t}^{\pi^k} - \mathcal{I}_{s,t}^{\pi^{k+1}} &= \sum_{[u,v]\in\pi^k} S_{t,u}\xi_{u,v} - \sum_{[u,v]\in\pi^k} S_{t,u}\xi_{u,m_{u,v}} + S_{t,m_{u,v}}\xi_{m_{u,v},v} \\ &= \sum_{[u,v]\in\pi^k} S_{t,u}(\xi_{u,v} - \xi_{u,m_{u,v}} - \xi_{m_{u,v},v}) + S_{t,m_{u,v}}(S_{m_{u,v},u} - \text{Id})\xi_{m_{u,v},v}. \end{aligned} \tag{2-43}$$

With the help of Chen's relation, it can be shown that

$$\xi_{u,v} - \xi_{u,m_{u,v}} - \xi_{m_{u,v},v} = -R^y_{u,m_{u,v}} X_{m_{u,v},v} - y'_{u,m_{u,v}} \mathbb{X}_{v,m_{u,v}}.$$

Choose now $k \in \mathbb{N}$ big enough, such that the intervals of the partition π^k are smaller than 1. Using the regularity property (2-42), $(y,y') \in \left(\mathcal{D}_{X,\alpha}^{\gamma,\gamma_1,\gamma_2}\right)^d$ and $v - m_{u,v} = m_{u,v} - u$ we obtain

$$\begin{aligned} &\left\| \sum_{[u,v]\in\pi^k} S_{t,u}(\xi_{u,v} - \xi_{u,m_{u,v}} - \xi_{m_{u,v},v}) \right\|_{\alpha-2\gamma+\beta} \\ &\leq \sum_{[u,v]\in\pi^k} \left\| S_{t,u} R^y_{u,m_{u,v}} X_{m_{u,v},v} \right\|_{\alpha-2\gamma+\beta} + \left\| S_{t,u} y'_{u,m_{u,v}} \mathbb{X}_{m_{u,v},v} \right\|_{\alpha-2\gamma+\beta} \\ &\lesssim \varrho_{\gamma,[0,T]}(\mathbf{X}) \left\| y, y' \right\|_{(\mathcal{D}_{X,\alpha}^{\gamma,\gamma_1,\gamma_2})^d} \\ &\quad \times \sum_{[u,v]\in\pi^k} (t-u)^{-\beta} \left((v-m_{u,v})^{\gamma} (m_{u,v}-u)^{2\gamma_2} + (v-m_{u,v})^{2\gamma} (m_{u,v}-u)^{\gamma_1} \right) \\ &\lesssim \varrho_{\gamma,[0,T]}(\mathbf{X}) \left\| y, y' \right\|_{(\mathcal{D}_{X,\alpha}^{\gamma,\gamma_1,\gamma_2})^d} \sum_{[u,v]\in\pi^k} (t-m_{u,v})^{-\beta} (v-m_{u,v})^{\tilde{\gamma}-1} (m_{u,v}-u), \end{aligned}$$

for $\tilde{\gamma} = \min\{\gamma_1 + 2\gamma, 2\gamma_2 + \gamma\}$ and similarly by using (2-41)

$$\left\| \sum_{[u,v]\in\pi^k} S_{t,m_{u,v}}(S_{m_{u,v},u} - \text{Id})\xi_{m_{u,v},v} \right\|_{\alpha-2\gamma+\beta}$$

$$\leq \sum_{[u,v]\in\pi^k} \Big(\left\| S_{t,m_{u,v}}(S_{m_{u,v},u} - \mathrm{Id}) y_{m_{u,v}} X_{m_{u,v},v} \right\|_{\alpha-2\gamma+\beta} + \left\| S_{t,m_{u,v}}(S_{m_{u,v},u} - \mathrm{Id}) y'_{m_{u,v}} \mathbb{X}_{m_{u,v},v} \right\|_{\alpha-2\gamma+\beta} \Big)$$
$$\lesssim \varrho_{\gamma,[0,T]}(\mathbf{X}) \left\| y, y' \right\|_{(\mathcal{D}^{\gamma,\gamma_1,\gamma_2}_{X,\alpha})^d} \sum_{[u,v]\in\pi^k} (t - m_{u,v})^{-\beta} (v - m_{u,v})^{\tilde{\gamma}-1} (m_{u,v} - u),$$

Recall, that

$$\frac{t-s}{2^{k+1}} = v - m_{u,v} = m_{u,v} - u \leq t - m_{u,v},$$

and choose some δ with $\beta - 1 < \delta < \tilde{\gamma} - 1$. Then we obtain using Lemma 2.41

$$\sum_{[u,v]\in\pi^k} (t - m_{u,v})^{-\beta} (v - m_{u,v})^{\tilde{\gamma}-1} (m_{u,v} - u)$$
$$\leq \sum_{[u,v]\in\pi^k} (t - m_{u,v})^{\delta-\beta} (v - m_{u,v})^{\tilde{\gamma}-1-\delta} (m_{u,v} - u)$$
$$\lesssim 2^{-k(\tilde{\gamma}-1-\delta)} (t-s)^{\tilde{\gamma}-1-\delta} \sum_{[u,v]\in\pi^k} (t - m_{u,v})^{\delta-\beta} (m_{u,v} - u)$$
$$\leq 2^{-k(\tilde{\gamma}-1-\delta)} (t-s)^{\tilde{\gamma}-1-\delta} \int_s^t (t-r)^{\delta-\beta} \, \mathrm{d}r \lesssim 2^{-k(\tilde{\gamma}-1-\delta)} (t-s)^{\tilde{\gamma}-\beta},$$

which leads to

$$\left\| \mathcal{I}^{\pi^k}_{s,t} - \mathcal{I}^{\pi^{k+1}}_{s,t} \right\|_{\alpha-2\gamma+\beta} \lesssim 2^{-k(\tilde{\gamma}-1-\delta)} (t-s)^{\tilde{\gamma}-\beta} \varrho_{\gamma,[0,T]}(\mathbf{X}) \left\| y, y' \right\|_{(\mathcal{D}^{\gamma,\gamma_1,\gamma_2}_{X,\alpha})^d}. \tag{2-44}$$

Since the right-hand side of (2-44) is summable over k, the sequence $\left(\mathcal{I}^{\pi^k}_{s,t}\right)_{k\in\mathbb{N}}$ is Cauchy in $E_{\alpha-2\gamma+\beta}$ and therefore has a limit $\mathcal{I}_{s,t} \in E_{\alpha-2\gamma+\beta}$. The additivity of $\mathcal{I}_{s,t}$ can be proven as in [GHN21, Theorem 4.1] and [FH20, Theorem 4.2], which shows the existence of (2-39) and (2-40). □

It is an inherent property of convolution integrals that they possess a smoothing property. This is also true for the rough convolution, as we can see in the next corollary.

Corollary 2.44. ([GHN21, Corollary 4.6]) *Assume that the same conditions as in Theorem 2.42 are satisfied. Then we have*

$$\left(\mathcal{D}^{\gamma'}_{X,\alpha}\right)^d \to \mathcal{D}^{\gamma'}_{X,\alpha+\varsigma}, (y, y') \mapsto \left(\int_0^{\cdot} S_{\cdot,r} y_r \, \mathrm{d}\mathbf{X}_r, y \right),$$

for every $\varsigma \in [0, \gamma')$ and $0 < \gamma' \leq \gamma$. Further, we obtain the bound

$$\left\| \int_0^{\cdot} S_{\cdot,r} y_r \, \mathrm{d}\mathbf{X}_r, y \right\|_{\mathcal{D}^{\gamma'}_{X,\alpha+\varsigma}} \leq C_I (1 + \varrho_{\gamma,[0,T]}(\mathbf{X})) \left(\|y_0\|_\alpha + \|y'_0\|_{\alpha-\gamma'} + T^\kappa \left\| y, y' \right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d} \right), \tag{2-45}$$

where $\kappa := \min\{\gamma - \gamma', \gamma' - \varsigma\}$ and $C_I := C_I(\gamma, \varsigma) > 0$.

Proof. For the sake of completeness, the proof of this fact is presented in rather detail, even if it is a longer technical calculation. However, the techniques and estimates used here are important throughout this thesis.

Let $(s,t) \in \Delta_{[0,T]}$. First, define

$$\mathscr{R}_{s,t} := \int_s^t S_{t,r} y_r \, \mathrm{d}\mathbf{X}_r - S_{t,s}\left(y_s X_{s,t} + y'_s \mathbb{X}_{s,t}\right), \tag{2-46}$$

and note that due to $T \le 1$ we obtain $\varrho_{\gamma',[0,T]}(\mathbf{X}) \le \varrho_{\gamma,[0,T]}(\mathbf{X})$. Rewrite the remainder of z as

$$\begin{aligned} R^z_{s,t} &= \mathscr{R}_{s,t} + (S_{t,s} - \mathrm{Id}\,) y_s X_{s,t} + (S_{t,s} - \mathrm{Id}\,) \int_0^s S_{s,r} y_r \, \mathrm{d}\mathbf{X}_r + S_{t,s} y'_s \mathbb{X}_{s,t} \\ &= \mathscr{R}_{s,t} + \mathrm{I}_{s,t} + \mathrm{II}_{s,t} + \mathrm{III}_{s,t}, \end{aligned} \tag{2-47}$$

where the smoothing of the semigroup can be used to bound every term separately. In fact, using (2-40) for $\beta = \varsigma + (2-i)\gamma'$ entails

$$\|\mathscr{R}_{s,t}\|_{\alpha+\varsigma-i\gamma'} \lesssim \varrho_{\gamma,[0,T]}(\mathbf{X}) \left\|y, y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d} (t-s)^{i\gamma'} T^{\gamma'-\varsigma}. \tag{2-48}$$

The smoothing property (2-41) and (2-42) yields

$$\begin{aligned} \|\mathrm{I}_{s,t}\|_{\alpha+\varsigma-i\gamma'} &\lesssim (t-s)^{i\gamma'} T^{\gamma-\varsigma} \varrho_{\gamma,[0,T]}(\mathbf{X}) \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d}, \\ \|\mathrm{II}_{s,t}\|_{\alpha+\varsigma-i\gamma'} &\lesssim (t-s)^{i\gamma'} \left(\left\|S_{s,0} y_0 X_{0,s} + S_{s,0} y'_0 \mathbb{X}_{0,s} + \mathscr{R}_{0,s}\right\|_{\alpha-\varsigma}\right) \\ &\lesssim (t-s)^{i\gamma'} \varrho_{\gamma,[0,T]}(\mathbf{X}) \left(T^{\gamma-\varsigma} \|y_0\|_\alpha + T^{2\gamma-\gamma'-\varsigma} \left\|y'_0\right\|_{\alpha-\gamma'} + T^{\gamma'-\varsigma} \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d}\right) \\ &\lesssim (t-s)^{i\gamma'} \varrho_{\gamma,[0,T]}(\mathbf{X}) T^{\gamma'-\varsigma} \left(\|y_0\|_\alpha + \left\|y'_0\right\|_{\alpha-\gamma'} + \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d}\right), \end{aligned}$$

for $i = 1, 2$. For the third term, we need to distinguish between the cases $i = 1$ and $i = 2$

$$\begin{aligned} \|\mathrm{III}_{s,t}\|_{\alpha+\varsigma-\gamma'} &\le \varrho_{\gamma,[0,T]}(\mathbf{X})(t-s)^{2\gamma} \left\|S_{t,s} y'_s\right\|_{\alpha+\varsigma-\gamma'} \\ &\lesssim \varrho_{\gamma,[0,T]}(\mathbf{X})(t-s)^{\gamma'} T^{2\gamma-\gamma'-\varsigma} \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d}, \\ \|\mathrm{III}_{s,t}\|_{\alpha+\varsigma-2\gamma'} &\lesssim \varrho_{\gamma,[0,T]}(\mathbf{X})(t-s)^{2\gamma} \left\|S_{t,s} y'_s\right\|_{\alpha-\gamma'} \\ &\lesssim \varrho_{\gamma,[0,T]}(\mathbf{X})(t-s)^{2\gamma'} T^{2\gamma-2\gamma'} \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d}. \end{aligned}$$

The next step is to estimate the Gubinelli derivative $z' = y$. Using (2-38) and

$$y_{s,t} = y'_s X_{s,t} + R^y_{s,t}$$

leads to

$$\begin{aligned} \left\|z'_{s,t}\right\|_{\alpha+\varsigma-2\gamma'} &\lesssim \left(\varrho_{\gamma,[0,T]}(\mathbf{X}) \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d} T^{\gamma-\gamma'} + \left\|y,y'\right\|_{\mathcal{D}^{\gamma'}_{X,\alpha}} T^{\gamma'-\varsigma}\right)(t-s)^{\gamma'} \\ &\lesssim \left(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\right) \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d} T^\kappa (t-s)^{\gamma'}, \\ \left\|z'_t\right\|_{\alpha+\varsigma-\gamma'} &\le \|y_0\|_{\alpha+\varsigma-\gamma'} + \|y_{0,t}\|_{\alpha+\varsigma-\gamma'} \lesssim \|y_0\|_\alpha + T^{\gamma'-\varsigma} [y]_{\gamma'-\varsigma,\alpha+\varsigma-\gamma'} \\ &\lesssim \|y_0\|_\alpha + \left(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\right) \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d} T^\kappa. \end{aligned}$$

Finally, the path component needs to be estimated, which can be done using (2-47) and (2-48) for $s = 0$

$$\|z_t\|_{\alpha+\varsigma} \lesssim \varrho_{\gamma,[0,T]}(\mathbf{X}) \left\|y,y'\right\|_{(\mathcal{D}^{\gamma'}_{X,\alpha})^d} T^\kappa.$$

Altogether, this proves the inequality (2-45), and that (z, z') is a controlled rough path. □

A useful fact about the rough integral is that it depends continuously on the noise, as mentioned in Remark 2.38. With this property, it is later easy to prove that the solution of a rough evolution equation depends continuously on the data, which is known to be mostly false for evolution equations driven by Itô or Stratonovich noise, recall Proposition 2.5. However, due to the pathwise construction of the integral, this follows immediately for rough noise. To this aim, define for $0 \leq \gamma' < \gamma$, two γ-Hölder rough paths $\mathbf{X} = (X, \mathbb{X})$, $\widehat{\mathbf{X}} = (\widehat{X}, \widehat{\mathbb{X}})$ and two controlled rough paths $(y, y') \in \mathcal{D}^{\gamma}_{X,\alpha}(J)$, $(\hat{y}, \hat{y}') \in \mathcal{D}^{\gamma}_{\widehat{X},\alpha}(J)$ the (semi-)metric

$$
\begin{aligned}
d^{\gamma',\gamma}_{\alpha,J}(y, \hat{y}) = \|y - \hat{y}\|_{\infty,J,\alpha} + \|y' - \hat{y}'\|_{\infty,J,\alpha-\gamma} + [y' - \hat{y}']_{\gamma',J,\alpha-2\gamma} \\
+ \left[R^y - R^{\hat{y}}\right]_{\gamma',J,\alpha-\gamma} + \left[R^y - R^{\hat{y}}\right]_{2\gamma',J,\alpha-2\gamma},
\end{aligned} \tag{2-49}
$$

for an arbitrary interval $J \subset \mathbb{R}$. The dependence of the (semi-)metric on $\mathbf{X}$ $\widehat{\mathbf{X}}$, y' and $\hat{y}'$ is not displayed here for notational simplicity.

Lemma 2.45. *Assume that the same conditions as in Theorem 2.42 are satisfied with $\gamma' := \gamma_1 = \gamma_2 \leq \gamma$. Further assume there exists $\widehat{\mathbf{X}} \in \mathscr{C}^{\gamma}([0,T]; \mathbb{R}^d)$ and $(\hat{y}, \hat{y}') \in \mathcal{D}^{\gamma,\gamma',\gamma'}_{\widehat{X},\alpha}$ such that there exists a constant $M \geq 0$ with*

$$
\varrho_{\gamma,[0,T]}(\mathbf{X}), \varrho_{\gamma,[0,T]}(\widehat{\mathbf{X}}), \|y, y'\|_{\mathcal{D}^{\gamma,\gamma',\gamma'}_{X,\alpha}}, \|\hat{y}, \hat{y}'\|_{\mathcal{D}^{\gamma,\gamma',\gamma'}_{\widehat{X},\alpha}} \leq M.
$$

Define $z := \int_0^{\cdot} S_{\cdot,r} y_r \, \mathrm{d}\mathbf{X}_r$, $z' := y$ and $\hat{z} := \int_0^{\cdot} S_{\cdot,r} \hat{y}_r \, \mathrm{d}\widehat{\mathbf{X}}_r$, $\hat{z}' := \hat{y}$, then

$$
d^{\gamma',\gamma}_{[0,T],\alpha+\varsigma}(z, \hat{z}) \lesssim_M \|y_0 - \hat{y}_0\|_{\alpha} + \varrho_{\gamma,[0,T]}(\mathbf{X}, \widehat{\mathbf{X}}) + d^{\gamma',\gamma}_{[0,T],\alpha}(y, \hat{y}) T^{\kappa}, \tag{2-50}
$$

holds for every $\varsigma \in [0, \gamma')$ where $\kappa := \min\{\gamma - \gamma', \gamma' - \varsigma\}$.

Proof. The idea of the proof is to analyze the difference of the stochastic convolutions $z - \hat{z}$. Let $(s,t) \in \Delta_{[0,T]}$.

As in the proof of Theorem 2.42 one needs an approximation $\Xi := \xi - \hat{\xi}$, where $\xi_{s,t} := y_s X_{s,t} + y'_s \mathbb{X}_{s,t}$ and $\hat{\xi}_{s,t} := \hat{y}_s \widehat{X}_{s,t} + \hat{y}'_s \widehat{\mathbb{X}}_{s,t}$ are the approximations of the individual integrals z and $\hat{z}$. Then, Ξ can be written as

$$
\Xi_{s,t} = (y_s - \hat{y}_s) X_{s,t} + \hat{y}_s (X_{s,t} - \widehat{X}_{s,t}) + (y'_s - \hat{y}'_s) \mathbb{X}_{s,t} + \hat{y}'_s (\mathbb{X}_{s,t} - \widehat{\mathbb{X}}_{s,t}). \tag{2-51}
$$

Combining now (2-51) with the proof of Theorem 2.42 we derive

$$
\begin{aligned}
&\|z_{s,t} - \hat{z}_{s,t} - S_{t,s}\Xi_{s,t}\|_{\alpha-2\gamma+\beta} \\
&\lesssim \left(\|\hat{y}, \hat{y}'\|_{\mathcal{D}^{\gamma,\gamma',\gamma'}_{\widehat{X},\alpha}} \varrho_{\gamma,[0,T]}(\mathbf{X}, \widehat{\mathbf{X}}) + d^{\gamma',\gamma}_{[0,T],\alpha}(y, \hat{y}) \varrho_{\gamma,[0,T]}(\mathbf{X}) \right) (t-s)^{\min\{\gamma'+2\gamma, 2\gamma'+\gamma\}-\beta},
\end{aligned}
$$

for every $(s,t) \in \Delta_{[0,T]}$ and $\beta \in [0, \min\{\gamma' + 2\gamma, 2\gamma' + \gamma\})$.

To receive now (2-50), every term of the distance (2-49) has to be estimated separately. For simplicity, denote $\mathscr{R}_{s,t} := z_{s,t} - \hat{z}_{s,t} - S_{t,s}\Xi_{s,t}$, then the path component is bounded by

$$
\|z_t - \hat{z}_t\|_{\alpha+\varsigma} \leq \|\mathscr{R}_{0,t}\| + \|S_{t,0}\Xi_{0,t}\|_{\alpha} \lesssim d^{\gamma',\gamma}_{[0,T],\alpha}(y, \hat{y}) T^{\gamma'-\varsigma} + \varrho_{\gamma,[0,T]}(\mathbf{X}, \widehat{\mathbf{X}}).
$$

Using the same representation as in (2-47), the remainder difference has the bound

$$
\max_{i=1,2} \left[R^y - R^{\hat{y}}\right]_{i\gamma,\alpha-i\gamma+\varsigma} \lesssim \varrho_{\gamma,[0,T]}(\mathbf{X}, \widehat{\mathbf{X}}) + d^{\gamma',\gamma}_{[0,T],\alpha}(y, \hat{y}) T^{\kappa}.
$$

For the Gubinelli derivative, use once again (2-38) and $z'_{s,t} = y_{s,t} = y'_s X_{s,t} + R^y_{s,t}$ to obtain

$$\begin{aligned}\big[z' - \hat{z}'\big]_{\gamma',\alpha-2\gamma+\varsigma} &\lesssim \varrho_{\gamma,[0,T]}\big(\mathbf{X}, \widehat{\mathbf{X}}\big) + d^{\gamma',\gamma}_{[0,T],\alpha}(y,\hat{y})T^{\kappa},\\ \big\|z' - \hat{z}'\big\|_{\infty,\alpha-\gamma+\varsigma} &\lesssim \|y_0 - \hat{y}_0\|_\alpha + \varrho_{\gamma,[0,T]}\big(\mathbf{X}, \widehat{\mathbf{X}}\big) + d^{\gamma',\gamma}_{[0,T],\alpha}(y,\hat{y})T^{\kappa},\end{aligned}$$

which proves the claim. □

Another significant property of the rough integral is its consistency with classical stochastic integration theories, specifically the Itô and Stratonovich integrals, for a suitable class of integrands. More precisely, when both the rough and the stochastic integral are well-defined, they end up being identical. This is proven in [GH19], using a different concept of controlled rough paths, where the space of controlled rough paths depends on the analytic semigroup itself. However, the proof fundamentally relies on the inherent properties of $\mathbb{B}^{\text{Itô}}$, respectively $\mathbb{B}^{\text{Strat}}$, so it is directly transferable to Definition 2.37.

Lemma 2.46. ([GH19, Proposition 4.8]) *Let $(y, y') \in \big(\mathcal{D}^{\gamma}_{B,\alpha}\big)^d$ be a controlled rough path where $(B_t)_{t\geq 0}$ is the d-dimensional Brownian motion on the filtered probability space $\big(\Omega, \Sigma, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\big)$ and consider the Itô lift $\mathbf{B} := \big(B, \mathbb{B}^{\text{Itô}}\big)$ such that $\mathbf{B} \in \mathscr{C}^{\gamma}$ almost surely. Further, assume that there exists for every $M > 0$ a time $T_M > 0$ such that $\|y_t\|_\alpha + \|y'_t\|_{\alpha-\gamma} \leq M$ holds for $t \leq T_M$ and (y, y') is adapted to $(\mathcal{F}_t)_{t\geq 0}$. Then*

$$\int_0^t S_{t,r} y_r \,\mathrm{d}\mathbf{B}_r = \int_0^t S_{t,r} y_r \,\mathrm{d}B_r,$$

holds almost surely. The same holds for the Stratonovich lift of the Brownian motion.

2.3.4 Operations on the space of controlled rough paths

As mentioned before, the Gubinelli derivative is not necessarily unique. Fortunately, controlled rough paths behave very nicely under many operations, such as compositions. This means there is then a canonical way to define the Gubinelli derivative.

First, investigate the composition with smooth functions. In contrast to [Gub04, GH19, GHN21], the functions under consideration do not necessarily map within a Banach scale; rather, they change between them. This is of particular interest in the context of equations with boundary noise, where one scale corresponds to the functions defined on the boundary $\big(\widetilde{E}_\beta\big)_{\beta\in\mathbb{R}}$ and the other represents the solution scale $(E_\beta)_{\beta\in\mathbb{R}}$. The respective spaces of controlled rough paths are denoted by $\mathcal{D}^{\gamma}_{X,\alpha}$ and $\widetilde{\mathcal{D}}^{\gamma}_{X,\alpha}$. Typically, the situation is that both scales coincide $\widetilde{E}_\beta = E_\beta$. Therefore, the case of using only one scale for the presentation of the results is presented, and it is emphasized when two different scales are required.

Furthermore, the nonlinearities included in this work are time-dependent. For autonomous functions G, it is straightforward to choose a suitable Gubinelli derivative, namely $\mathrm{D}G(y)y'$. In the non-autonomous case, the canonical choice is

$$t \mapsto \mathrm{D}_1 G(t, y_t) + \mathrm{D}_2 G(t, y_t) y'_t,$$

where differentiability in time is needed. It turns out that it is enough to consider only the second term, so the time derivative is neglected in the following computations.

Many results in this thesis can be achieved for quite general mappings, but others require additional properties. The conditions are made as transparent as possible, such that it become clear which results require which conditions. The following assumption serves as the basis for all subsequent conditions and is the minimum requirement for G.

(G1) There exists a, potentially negative, constant $\sigma_G < \gamma$ and $C_G > 0$ such that

$$G\colon [0,T] \times E_{\alpha-i\gamma} \to \widetilde{E}^d_{\alpha-i\gamma-\sigma_G}$$

satisfies for every $i = 0,1,2$ the following conditions:

i) For every $t \in [0,T]$, $G(t,\cdot)$ is three times continuously Fréchet differentiable where the derivatives are bounded uniformly in time, this means

$$\sup_{t\in[0,T]} \left\| \mathrm{D}_2^k G(t,\cdot) \right\|_{\mathcal{L}(E^{\otimes k}_{\alpha-i\gamma}; E^d_{\alpha-i\gamma-\sigma_G})} \leq C_G < \infty,$$

is fulfilled for every $i = 0,1,2$ and $k = 1,2,3$.

ii) For every $x \in E_{\alpha-i\gamma}$, $G(\cdot,x)$, as well as all existing Fréchet derivatives, are 2γ-Hölder continuous and the Hölder constants are uniformly bounded in $E_{\alpha-i\gamma}$, such that

$$\left\| \mathrm{D}_2^k G(t,x) - \mathrm{D}_2^k G(s,x) \right\|_{\alpha-i\gamma-\sigma_G} \leq C_G (t-s)^{2\gamma},$$

for every $k = 0,1,2,3$ and $i = 0,1,2$.

Remark 2.47. A positive constant σ_G means that G loses spatial regularity, which is the reason why it is needed to bound it from above. If σ_G is negative, G gains spatial regularity, which is needed for boundary value problems later on.

Recall, that E^d_α can be identified with $\mathcal{L}(\mathbb{R}^d; E_\alpha)$.

Lemma 2.48. ([NS23, Lemma 3.10]) *Let $\alpha \in \mathbb{R}$ and G be a nonlinearity satisfying **(G1)**. For two controlled rough paths $(y,y'), (\hat{y},\hat{y}') \in \mathcal{D}^\gamma_{X,\alpha}$ we define*

$$(z_t, z'_t) := \big(G(t,y_t), \mathrm{D}_2 G(t,y_t) y'_t\big), \quad (\hat{z}_t, \hat{z}'_t) := \big(G(t,\hat{y}_t), \mathrm{D}_2 G(t,\hat{y}_t)\hat{y}'_t\big),$$

for $t \in [0,T]$.

i) The composition fulfills $(z,z') \in \left(\widetilde{\mathcal{D}}^\gamma_{X,\alpha-\sigma_G}\right)^d$ and the estimate

$$\|z,z'\|_{(\widetilde{\mathcal{D}}^\gamma_{X,\alpha-\sigma_G})^d} \leq C_G + C_G\big(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\big)^2 \|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}} \left(1 + \|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\right), \tag{2-52}$$

holds. This statement is also true if G is only two times Fréchet differentiable with bounded derivatives.

ii) The difference between the two compositions can be bound by

$$\begin{aligned} \|z-\hat{z}, z'-\hat{z}'\|_{(\widetilde{\mathcal{D}}^\gamma_{X,\alpha-\sigma_G})^d} \leq{}& C_G\big(1+\varrho_\gamma(\mathbf{X})\big)^2 \|y-\hat{y}, y'-\hat{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}} \\ &\times \left(1 + \|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}} + \|\hat{y},\hat{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\right)^2. \end{aligned} \tag{2-53}$$

Proof. As in Corollary 2.44, the proof is provided in greater detail, even if it is a longer technical calculation, since the estimates are used throughout this thesis.

In the following it is useful to view the derivative $\mathrm{D}^k G(y_t)$ as an element of $\mathcal{L}\big(E^{\otimes k}_{\alpha-i\gamma}; \widetilde{E}^d_{\alpha-i\gamma-\sigma_G}\big)$ for $k = 1,2,3$ and $i = 1,2$.

i) Let $(s,t) \in \Delta_{[0,T]}$. Due to $E_\alpha \hookrightarrow E_{\alpha-\gamma}$ and **(G1)** we have

$$(z,z') \in \mathcal{C}\big(\widetilde{E}^d_{\alpha-\sigma_G}\big) \times \mathcal{C}\big(\widetilde{E}^d_{\alpha-\gamma-\sigma_G}\big).$$

Using the boundedness of D_2G and the uniform continuity of $t \mapsto G(t, y_0)$ entails

$$\begin{aligned}\|z\|_{\infty,\widetilde{E}^d_{\alpha-\sigma_G}} &\leq \sup_{t\in[0,T]} \|G(t,0)\|_{\widetilde{E}^d_{\alpha-\sigma_G}} + \|y\|_{\infty,\alpha} \lesssim 1 + \|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\,, \\ \|z'\|_{\infty,\widetilde{E}^d_{\alpha-\gamma-\sigma_G}} &= \|\mathrm{D}_2G(\cdot,y)y'\|_{\infty,\widetilde{E}^d_{\alpha-\gamma-\sigma_G}} \lesssim \|y'\|_{\infty,\alpha-\gamma} \lesssim \|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\,.\end{aligned}$$

To establish the Hölder continuity of the Gubinelli derivative, **(G1)** ii) leads to

$$\begin{aligned}&\|\mathrm{D}_2\ G(t,y_t)y'_t - \mathrm{D}_2G(s,y_s)y'_s\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} \\ &\quad\leq \big\|\big(\mathrm{D}_2G(t,y_t) - \mathrm{D}_2G(t,y_s)\big)y'_t\big\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} \\ &\qquad + \big\|\big(\mathrm{D}_2G(t,y_s) - \mathrm{D}_2G(s,y_s)\big)y'_t\big\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} + \big\|\mathrm{D}_2G(s,y_s)\big(y'_t - y'_s\big)\big\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} \\ &\quad\lesssim \|y_{s,t}\|_{\alpha-2\gamma}\,\|y'_t\|_{\alpha-2\gamma} + (t-s)^{2\gamma}\,\|y'_t\|_{\alpha-2\gamma} + \|y'_{s,t}\|_{\alpha-2\gamma} \\ &\quad\lesssim (t-s)^\gamma\big(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\big)\,\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big) + (t-s)^{2\gamma}\,\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\,,\end{aligned}$$

and therefore

$$\|\mathrm{D}_2G(\cdot,y)y'\|_{\gamma,\widetilde{E}^d_{\alpha-2\gamma-\sigma}} \lesssim \big(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\big)\,\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big) + \|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\,.$$

For the remainder, a similar calculation as in [GHN21, Lemma 4.7] leads to the representation

$$\begin{aligned}R^z_{s,t} &= G(t,y_t) - G(s,y_s) - \mathrm{D}_2G(s,y_s)\big(y'_sX_{s,t}\big) \\ &= G(t,y_t) - G(s,y_t) + G(s,y_t) - G(s,y_s) - \mathrm{D}_2G(s,y_s)\big(y'_sX_{s,t}\big) \\ &= G(t,y_t) - G(s,y_t) + G(s,y_t) - G(s,y_s) - \mathrm{D}_2G(s,y_s)\big(y_{s,t} - R^y_{s,t}\big) \\ &= G(t,y_t) - G(s,y_t) + \mathrm{D}_2G(s,y_s)R^y_{s,t} \\ &\quad + \int_0^1\int_0^1 \tilde{r}\mathrm{D}_2^2G(s,y_s + r\tilde{r}y_{s,t})[y_{s,t},y_{s,t}]\ \mathrm{d}r\mathrm{d}\tilde{r}.\end{aligned} \tag{2-54}$$

The first term can then be estimated using the 2γ-Hölder continuity of $G(\cdot, y_t)$

$$\|G(t,y_t) - G(s,y_t)\|_{\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} \lesssim (t-s)^{2\gamma},$$

for $i = 1, 2$. Further we obtain for $i = 1, 2$

$$\|\mathrm{D}_2G(s,y_s)R^y_{s,t}\|_{\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} \lesssim (t-s)^{i\gamma}\,\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\,,$$

since the Fréchet derivative of G is bounded, and R^y is 2γ-Hölder continuous. Finally, for the integral, we obtain

$$\begin{aligned}&\left\|\int_0^1\int_0^1 \tilde{r}\mathrm{D}_2^2G(s,y_s + r\tilde{r}y_{s,t})[y_{s,t},y_{s,t}]\ \mathrm{d}r\mathrm{d}\tilde{r}\right\|_{\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} \\ &\qquad\lesssim \|y_{s,t}\|^2_{\alpha-i\gamma} \lesssim (t-s)^{2\gamma}\,[y]^2_{\gamma,\alpha-i\gamma} \lesssim (t-s)^{2\gamma}\,\big(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\big)^2\,\|y,y'\|^2_{\mathcal{D}^\gamma_{X,\alpha}}\,,\end{aligned}$$

for $i = 1, 2$ using (2-37). Putting this together leads to

$$\|R^z\|_{i\gamma,\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} \lesssim 1 + \big(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\big)^2\,\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big),$$

proving the claim.

ii) The difference can be estimated similarly to (2-52). Let $(s,t) \in \Delta_{[0,T]}$. The bounds for the supremum norms are straightforward

$$\begin{aligned}
\|z - \hat{z}\|_{\infty, \widetilde{E}^d_{\alpha-\sigma_G}} &= \|G(\cdot, y) - G(\cdot, \hat{y})\|_{\infty, \widetilde{E}^d_{\alpha-\sigma_G}} \lesssim \|y - \hat{y}\|_{\infty,\alpha} \lesssim \left\|y - \hat{y}, y' - \hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}}, \\
\left\|z' - \hat{z}'\right\|_{\infty, \widetilde{E}^d_{\alpha-\gamma-\sigma_G}} &\leq \left\|\left(\mathrm{D}_2 G(\cdot, y) - G(\cdot, \hat{y})\right) y'\right\|_{\infty, \widetilde{E}^d_{\alpha-\gamma-\sigma_G}} + \left\|G(\cdot, \hat{y})\left(y' - \hat{y}'\right)\right\|_{\infty, \widetilde{E}^d_{\alpha-\gamma-\sigma_G}} \\
&\lesssim \|y - \hat{y}\|_{\infty,\alpha-\gamma} \left\|y'\right\|_{\infty,\alpha-\gamma} + \left\|y' - \hat{y}'\right\|_{\infty,\alpha-\gamma} \\
&\lesssim \left\|y - \hat{y}, y' - \hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left(1 + \left\|\hat{y}, \hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}}\right).
\end{aligned}$$

The Hölder seminorms for the Gubinelli derivative and the remainder are more involved. For the remainder, consider the representation

$$\begin{aligned}
R^z_{s,t} - R^{\hat{z}}_{s,t} &= G(t, y_t) - G(s, y_t) - \big(G(t, \hat{y}_t) - G(s, \hat{y}_t)\big) \\
&\quad + \big(\mathrm{D}_2 G(s, y_s) - \mathrm{D}_2 G(s, \hat{y}_s)\big) R^y_{s,t} + \mathrm{D}_2 G(s, \hat{y}_s)\big(R^y_{s,t} - R^{\hat{y}}_{s,t}\big) \\
&\quad + \int_0^1 \int_0^1 \tilde{r}\mathrm{D}_2^2 G(s, y_s + r\tilde{r} y_{s,t})[y_{s,t}, y_{s,t}] - \tilde{r}\mathrm{D}_2^2 G(s, \hat{y}_s + r\tilde{r}\hat{y}_{s,t})[\hat{y}_{s,t}, \hat{y}_{s,t}] \,\mathrm{d}r\mathrm{d}\tilde{r} \\
&= \mathrm{I}_{s,t} + \mathrm{II}_{s,t} + \mathrm{III}_{s,t},
\end{aligned} \tag{2-55}$$

which is the difference of the respective representations in (2-54). The second line is bounded by

$$\begin{aligned}
\|\mathrm{II}_{s,t}\|_{\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} &\lesssim \left(\|y - \hat{y}\|_{\infty,\alpha-i\gamma} [R^y]_{i\gamma,\alpha-i\gamma} + \left[R^y - R^{\hat{y}}\right]_{i\gamma,\alpha-i\gamma}\right)(t-s)^{i\gamma} \\
&\lesssim \left\|y - \hat{y}, y' - \hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left(1 + \left\|y, y'\right\|_{\mathcal{D}^\gamma_{X,\alpha}}\right)(t-s)^{i\gamma}.
\end{aligned}$$

The difference of the integrals can be rewritten as

$$\begin{aligned}
\mathrm{III}_{s,t} &= \int_0^1 \int_0^1 \tilde{r}\left(\mathrm{D}_2^2 G(s, y_s + r\tilde{r} y_{s,t}) - \mathrm{D}_2^2 G(s, \hat{y}_s + r\tilde{r}\hat{y}_{s,t})\right)\left[y_{s,t}, y_{s,t}\right] \,\mathrm{d}r\mathrm{d}\tilde{r} \\
&\quad + \int_0^1 \int_0^1 \tilde{r}\mathrm{D}_2^2 G(s, \hat{y}_s + r\tilde{r}\hat{y}_{s,t})\left[y_{s,t} - \hat{y}_{s,t}, y_{s,t}\right] \,\mathrm{d}r\mathrm{d}\tilde{r} \\
&\quad + \int_0^1 \int_0^1 \tilde{r}\mathrm{D}_2^2 G(s, \hat{y}_s + r\tilde{r}\hat{y}_{s,t})\left[\hat{y}_{s,t}, y_{s,t} - \hat{y}_{s,t}\right] \,\mathrm{d}r\mathrm{d}\tilde{r},
\end{aligned}$$

which leads then to the estimate

$$\begin{aligned}
\|\mathrm{III}_{s,t}\|_{\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} &\lesssim \|y - \hat{y}\|_{\infty,\alpha-i\gamma} \|y_{s,t}\|^2_{\alpha-i\gamma} + \|y_{s,t} - \hat{y}_{s,t}\|_{\alpha-i\gamma} \|y_{s,t}\|_{\alpha-i\gamma} \\
&\quad + \|y_{s,t} - \hat{y}_{s,t}\|_{\alpha-i\gamma} \|\hat{y}_{s,t}\|_{\alpha-i\gamma} \\
&\lesssim \left\|y - \hat{y}, y' - \hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left([y]^2_{\gamma,\alpha-i\gamma} + [y]_{\gamma,\alpha-i\gamma} + [\hat{y}]_{\gamma,\alpha-i\gamma}\right)(t-s)^{2\gamma} \\
&\lesssim \left\|y - \hat{y}, y' - \hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\right)^2 \\
&\quad \times \left(\left\|y, y'\right\|^2_{\mathcal{D}^\gamma_{X,\alpha}} + \left\|y, y'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} + \left\|\hat{y}, \hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}}\right)(t-s)^{i\gamma},
\end{aligned}$$

for $i = 1, 2$. The first line can be rewritten as

$$\mathrm{I}_{s,t} = \int_0^1 \Big(\mathrm{D}_2 G\big(t, \hat{y}_t + r(y_t - \hat{y}_t)\big) - \mathrm{D}_2 G\big(s, \hat{y}_t + r(y_t - \hat{y}_t)\big)\Big)(y_t - \hat{y}_t) \,\mathrm{d}r, \tag{2-56}$$

which leads to

$$\|\mathrm{I}_{s,t}\|_{\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} \lesssim \left\|y-\hat{y},y'-\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{2\gamma},$$

for $i=1,2$ and therefore

$$\begin{aligned}[R^z]_{i\gamma,\widetilde{E}^d_{\alpha-i\gamma-\sigma_G}} &\lesssim \left\|y-\hat{y},y'-\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\right)^2 \\ &\qquad\times \left(\left\|y,y'\right\|^2_{\mathcal{D}^\gamma_{X,\alpha}} + \left\|y,y'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} + \left\|\hat{y},\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}}\right).\end{aligned}$$

Lastly, the Gubinelli derivative has to be rewritten similarly

$$\begin{aligned} z'_{s,t}-\hat{z}'_{s,t} = &\Big(\mathrm{D}_2G(t,y_t)-\mathrm{D}_2G(s,y_t)-\big(\mathrm{D}_2G(t,\hat{y}_t)-\mathrm{D}_2G(s,\hat{y}_t)\big)\Big)y'_t \\ &+\big(\mathrm{D}_2G(t,\hat{y}_t)-\mathrm{D}_2G(s,\hat{y}_t)\big)\big(y'_t-\hat{y}'_t\big) \\ &+\big(\mathrm{D}_2G(s,y_t)-\mathrm{D}_2G(s,y_s)\big)y'_t-\big(\mathrm{D}_2G(s,\hat{y}_t)-\mathrm{D}_2G(s,\hat{y}_s)\big)\hat{y}'_t \\ &+\big(\mathrm{D}_2G(s,y_s)-\mathrm{D}_2G(s,\hat{y}_s)\big)y'_{s,t}+\mathrm{D}_2G(s,\hat{y}_s)\left(y'_{s,t}-\hat{y}'_{s,t}\right) \\ =&\,\widetilde{\mathrm{I}}_{s,t}+\widetilde{\mathrm{II}}_{s,t}+\widetilde{\mathrm{III}}_{s,t}+\widetilde{\mathrm{IV}}_{s,t}.\end{aligned} \tag{2-57}$$

The second and last lines are straightforward to estimate

$$\begin{aligned}\left\|\widetilde{\mathrm{II}}_{s,t}\right\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} &\lesssim \left\|y-\hat{y},y'-\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{2\gamma}, \\ \left\|\widetilde{\mathrm{IV}}_{s,t}\right\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} &\lesssim \left\|y-\hat{y},y'-\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left(1+\left\|y,y'\right\|_{\mathcal{D}^\gamma_{X,\alpha}}\right)(t-s)^{\gamma}.\end{aligned}$$

For the first line, the same trick as in (2-56) can be used to obtain

$$\left\|\widetilde{\mathrm{I}}_{s,t}\right\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} \lesssim \left\|y-\hat{y},y'-\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left\|y,y'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{2\gamma}.$$

To estimate the last term, the difference has to be expressed once again as an integral

$$\begin{aligned}\widetilde{\mathrm{III}}_{s,t} = &\int_0^1 \left(\mathrm{D}^2_2G(s,y_s+ry_{s,t})-\mathrm{D}^2_2G(s,\hat{y}_s+r\hat{y}_{s,t})\right)\left[y_{s,t},y'_t\right]\,\mathrm{d}r \\ &+\int_0^1 \mathrm{D}^2_2G(s,\hat{y}_s+r\hat{y}_{s,t})\left[y_{s,t}-\hat{y}_{s,t},y'_t\right]\,\mathrm{d}r \\ &+\int_0^1 \mathrm{D}^2_2G(s,\hat{y}_s+r\hat{y}_{s,t})\left[\hat{y}_{s,t},y'_t-\hat{y}'_t\right]\,\mathrm{d}r,\end{aligned}$$

which leads to

$$\begin{aligned}\left\|\widetilde{\mathrm{III}}_{s,t}\right\|_{\widetilde{E}^d_{\alpha-2\gamma-\sigma_G}} &\lesssim \left\|y-\hat{y},y'-\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} \left(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\right)^2 \\ &\qquad\times\left(\left\|y,y'\right\|^2_{\mathcal{D}^\gamma_{X,\alpha}} + \left\|y,y'\right\|_{\mathcal{D}^\gamma_{X,\alpha}} + \left\|\hat{y},\hat{y}'\right\|_{\mathcal{D}^\gamma_{X,\alpha}}\right)(t-s)^{\gamma}.\end{aligned}$$

The elementary inequality

$$1+a+b+b^2 \lesssim (1+a+b)^2,$$

together with the above estimates, then leads to (2-53), which finishes the proof.

□

The metric $d^{\gamma',\gamma}_{\alpha,[0,T]}$ can also be estimated similar to Lemma 2.45.

Corollary 2.49. *Let $\alpha \in \mathbb{R}, \gamma \in (1/3, 1/2], \gamma' \in (0,\gamma]$ be such that*

$$\min\left\{\gamma' + 2\gamma, 2\gamma' + \gamma\right\} \geq 1,$$

*$\mathbf{X} \in \mathscr{C}^\gamma\big([0,T];\mathbb{R}^d\big)$, G satisfying **(G1)** and $(E_\beta)_{\beta\in\mathbb{R}}, \big(\widetilde{E}_\beta\big)_{\beta\in\mathbb{R}}$ two monotone families of interpolation spaces. For the two controlled rough paths $(y,y'), (\hat{y},\hat{y}') \in \mathcal{D}^{\gamma,\gamma',\gamma'}_{X,\alpha}$ define*

$$\big(z_t, z_t'\big) := \big(G(t,y_t), \mathrm{D}_2 G(t,y_t)y_t'\big) \quad \text{and} \quad \big(\hat{z}_t, \hat{z}_t'\big) := \big(G(t,\hat{y}_t), \mathrm{D}_2 G(t,\hat{y}_t)\hat{y}_t'\big)$$

for $t \in [0,T]$. Further, assume there exists a constant $M \geq 0$ such that

$$\varrho_{\gamma,[0,T]}(\mathbf{X}), \varrho_{\gamma,[0,T]}\big(\widehat{\mathbf{X}}\big), \left\|y,y'\right\|_{\mathcal{D}^{\gamma,\gamma',\gamma'}_{X,\alpha}}, \left\|\hat{y},\hat{y}'\right\|_{\mathcal{D}^{\gamma,\gamma',\gamma'}_{\widehat{X},\alpha}} \leq M.$$

Then we obtain

$$d^{\gamma',\gamma}_{[0,T],\widetilde{E}^d_{\alpha-\sigma_G}}(z,\hat{z}) \lesssim_M \|y_0 - \hat{y}_0\|_\alpha + \left\|y_0' - \hat{y}_0'\right\|_{\alpha-\gamma} + \varrho_{\gamma,[0,T]}\big(\mathbf{X},\widehat{\mathbf{X}}\big) + d^{\gamma',\gamma}_{[0,T],\alpha}(y,\hat{y}).$$

Proof. The proof follows the same strategy and estimates as in Lemma 2.48 and Lemma 2.45, so we omit it here. □

Now, we investigate the interaction of controlled rough paths with linear operators. At first glance, there are no major issues as long as the operator has the correct domain, as the following lemma shows.

Lemma 2.50. *Let $(E_\beta)_{\beta\in\mathbb{R}}$ and $\big(\widetilde{E}_\beta\big)_{\beta\in\mathbb{R}}$ be two monotone scales of interpolation spaces and $\mathcal{N} \in \bigcap_{i=0}^2 \mathcal{L}\big(\widetilde{E}_{\alpha_1-i\gamma}; E_{\alpha_2-i\gamma}\big)$ for $\alpha_1,\alpha_2 \in \mathbb{R}$. Then $\big(\mathcal{N}y,\mathcal{N}y'\big) \in \mathcal{D}^\gamma_{X,\alpha_2}$ holds for every $\big(y,y'\big) \in \widetilde{\mathcal{D}}^\gamma_{X,\alpha_1}$.*

Proof. That $\big(\mathcal{N}y,\mathcal{N}y'\big) \in \mathcal{C}(E_{\alpha_2}) \times \mathcal{C}(E_{\alpha_2-\gamma})$ holds, is a direct consequence of the assumption on $\mathcal{N}$. Define the remainder by $R^{\mathcal{N}y} := \mathcal{N}R^y$ and obtain

$$\begin{aligned}
\big[\mathcal{N}y'\big]_{\gamma,E_{\alpha_2-2\gamma}} &\leq \|\mathcal{N}\|_{\mathcal{L}(\widetilde{E}_{\alpha_1-2\gamma};E_{\alpha_2-2\gamma})} \big[y'\big]_{\gamma,\widetilde{E}_{\alpha_1-2\gamma}}, \\
\big[R^{\mathcal{N}y}\big]_{i\gamma,E_{\alpha_2-i\gamma}} &\leq \|\mathcal{N}\|_{\mathcal{L}(\widetilde{E}_{\alpha_1-i\gamma};E_{\alpha_2-i\gamma})} \big[R^y\big]_{i\gamma,\widetilde{E}_{\alpha_1-i\gamma}},
\end{aligned}$$

for $i = 1,2$, which concludes the proof. □

In particular, the controlled rough path norm in the situation of Lemma 2.50 can be bounded by

$$\left\|\mathcal{N}y,\mathcal{N}y'\right\|_{\mathcal{D}^\gamma_{X,\alpha_2}} \lesssim_{\mathcal{N}} \left\|y,y'\right\|_{\widetilde{\mathcal{D}}^\gamma_{X,\alpha_1}},$$

where the estimates depend only on the maximum of all operator norms of $\mathcal{N}$.

The reasoning behind this is to consider boundary value problems later on. At this point, the extrapolated operators introduced in Appendix C also appear, which is why the interaction between controlled rough paths and extrapolated operators is further examined in the following Corollary.

Corollary 2.51. *Let $A\colon D(A) \subset E \to E$ be a linear operator with bounded imaginary powers and $(E_\alpha, A_\alpha)_{\alpha\geq -m}$ be the corresponding interpolation-extrapolation scale of order $m \in \mathbb{N}$, and*

$(y, y') \in \mathcal{D}^{\gamma}_{X,\alpha+1}$ be a controlled rough path. Then $(A_{\alpha-i\gamma} y, A_{\alpha-i\gamma} y') \in \mathcal{D}^{\gamma}_{X,\alpha}$ holds for $i = 1, 2$ and even $(A_{\alpha} y, A_{\alpha-\gamma} y') \in \mathcal{D}^{\gamma}_{X,\alpha}$. In particular, the path components coincide

$$A_{\alpha} y = A_{\alpha-\gamma} y = A_{\alpha-2\gamma} y$$

as well as the Gubinelli derivatives $A_{\alpha-\gamma} y' = A_{\alpha-2\gamma} y'$.

Proof. That $(A_{\alpha-2\gamma} y, A_{\alpha-2\gamma} y') \in \mathcal{D}^{\gamma}_{X,\alpha}$ holds, is a direct consequence of Lemma 2.50, since $A_{\alpha-2\gamma} \in \mathcal{L}(E_{\alpha+1-2\gamma}; E_{\alpha-2\gamma})$ as well as

$$A_{\alpha-2\gamma}|_{E_{\alpha+1-\gamma}} = A_{\alpha-\gamma}, \qquad \text{and} \qquad A_{\alpha-2\gamma}|_{E_{\alpha+1}} = A_{\alpha}, \tag{2-58}$$

by definition. The other two statements are true since the path component and Gubinelli derivative have values in $E_{\alpha+1}$ or $E_{\alpha+1-\gamma}$ respectively, and we can use the same property of the abstract Banach scale as in (2-58). For the sake of completeness, we briefly sketch the proof of $(A_{\alpha} y, A_{\alpha-\gamma} y') \in \mathcal{D}^{\gamma}_{X,\alpha}$, the missing statement can then be proven analogously, see also [NS23, Lemma 3.9].

Let $(s, t) \in \Delta_{[0,T]}$. The fact that $y_t \in E_{\alpha+1}$ and $y'_t \in E_{\alpha+1-\gamma}$ yields $A_{\alpha} y_t \in E_{\alpha}$ and $A_{\alpha-\gamma} y'_t \in E_{\alpha-\gamma}$. Furthermore, due to (2-58)

$$\left\| A_{\alpha-\gamma} y'_{s,t} \right\|_{\alpha-2\gamma} = \left\| A_{\alpha-2\gamma} y'_{s,t} \right\|_{\alpha-2\gamma} \lesssim \left\| y'_{s,t} \right\|_{\alpha+1-2\gamma} \leq (t-s)^{\gamma} \left[y' \right]_{\gamma,\alpha+1-2\gamma},$$

holds. The remainder $A_{\alpha} y_{s,t} - A_{\alpha-\gamma} y'_s X_{s,t} = A_{\alpha-\gamma} R^y_{s,t} \in E_{\alpha-\gamma}$ can be bound by

$$\left\| A_{\alpha-\gamma} R^y_{s,t} \right\|_{\alpha-i\gamma} = \left\| A_{\alpha-i\gamma} R^y_{s,t} \right\|_{\alpha-i\gamma} \lesssim \left\| R^y_{s,t} \right\|_{\alpha+1-i\gamma} \lesssim (t-s)^{i\gamma} \left[R^y \right]_{i\gamma,\alpha+1-i\gamma},$$

for $i = 1, 2$, using again (2-58), which proves the claim. The equality of the path components and Gubinelli derivatives follows directly from Remark C.2. □

2.3.5 Alternative concepts of controlled rough paths

Similar to the variety of concepts associated with γ-Hölder rough paths, as discussed in Subsection 2.2.3, there are numerous interpretations of controlled rough paths tailored to different contexts. This section intends to introduce several alternative notions of controlled rough paths, highlighting the difference from the approach in [GHN21].

Controlled rough paths according to a semigroup

One of the first approaches to consider partial differential equations driven by rough paths is due to Gubinelli and Tindel [GT10]. Their idea is similar to the one introduced by [GHN21] with one significant difference: They incorporated the semigroup into the space of controlled rough paths. Recall, that a controlled rough path $(y, y') \in \mathcal{D}^{\gamma}_{X,\alpha}$ in the sense of Definition 2.37, fulfills

$$y_t - y_s = y'_s X_{s,t} + R^y_{s,t}.$$

In contrast to this equation, [GT10] treats

$$y_t - S_{t-s} y_s = S_{t-s} y'_s X_{s,t} + R^y_{s,t},$$

where $(S_t)_{t \geq 0}$ is an analytic semigroup. As a result, the space of controlled rough paths depends on the semigroup $(S_t)_{t \geq 0}$. Using this approach, it is possible to obtain local solutions [HN19], global solutions [HN20], and a Hörmander-type theorem [GH19] for semilinear rough partial differential equations.

Controlled rough paths for quasilinear equations

The controlled rough paths from [GHN21], as well as the approach of [GT10], can only be applied to semilinear equations. Quasilinear equations like

$$\mathrm{d}y_t = \big(A(y_t) + F(y_t)\big)\ \mathrm{d}t + G(y_t)\ \mathrm{d}\mathbf{X}_t, \tag{2-59}$$

cannot be considered with these approaches. A few changes are necessary to adapt the theory to this setting. First of all, the quasilinear case leads to the fact that not the whole range of function spaces $(E_\beta)_{\beta\in\mathbb{R}}$ can be used, but one is limited to the spaces $(E_\beta)_{\beta\in[0,1]}$. This means that the regularity losses in time and space of the Gubinelli derivative and the remainder must be tracked more carefully so as not to drop out of the range $[0,1]$. For that, two additional regularity parameters are introduced such that the Hölder regularity of the solution y, the Gubinelli derivative y', and the spatial regularity loss can be treated separately. This is quite similar to Definition 2.37. A second challenge, typical for quasilinear equations, is the dependence on the initial data y_0. To circumvent this, a time weight needs to be introduced. With this adapted version of a controlled rough path, it is then possible to prove the local existence of solutions for quasilinear parabolic problems like (2-59), see [HN24].

Chapter 3

Rough evolution equations

Following the definition of both the rough paths and controlled rough paths, rough partial differential equations can now be investigated. For that let E be a reflexive Banach space, $T \leq 1$, $\big(A(t)\big)_{t\in[0,T]}$ a family of linear operators $A(t)\colon D(A) \subset E \to E$ satisfying **(A)** and $\big(E_\beta, A_\beta\big)_{\beta\geq -m}$ the resulting interpolation-extrapolation scale of order $m \in \mathbb{N}$. More precisely, in this chapter, we investigate equations of the form

$$\mathrm{d}y_t = \big(A(t)y_t + F(t, y_t)\big)\ \mathrm{d}t + G(t, y_t)\ \mathrm{d}\mathbf{X}_t, \tag{RPDE}$$

where $\mathbf{X} \in \mathscr{C}^\gamma\big([0, T]; \mathbb{R}^d\big)$ is a γ-Hölder rough path for $\gamma \in (1/3, 1/2]$. For the rest of this thesis, we assume without loss of generality that the noise is scalar-valued. The extension to multidimensional noise can be done componentwise, as seen in the previous chapter.

The first two sections of this chapter are a summary of well-known facts extended to the non-autonomous case, which are also part of the article [BGVS25]. Section 3.3 was already published in [NS23] as well as Section 3.4 and 3.5 in [BGVS25], which contains an expansion of the earlier results presented in [BS24] and [GVR25]. Everything is based on a collaboration with Alexandra Blessing and Mazyar Ghani Varzaneh.

3.1 Local-in-time solutions

In this section, we examine the local-in-time solvability of (RPDE). To the best of our knowledge, there are only a few results on non-autonomous rough partial differential equations. In [GHN21] only the linear part is time-dependent, and in [HN24] the authors investigated quasilinear equations with a time-dependent drift term. Recently, Tappe [Tap25] investigated equations that are not parabolic. He uses a different approach for the space of controlled rough paths, which does not require an analytic semigroup but also allows time-dependent data.

Let us now state the conditions on the drift term F to specify the solution concept.

(F1) There exists a $\sigma_F \in [0, 1)$ such that $F(t, \cdot)\colon E_\alpha \to E_{\alpha-\sigma_F}$ is Lipschitz continuous, this means there exists a constant $L_{F,t} > 0$ such that $L_F := \sup_{t\in[0,T]} L_{F,t} < \infty$ and

$$\|F(t, x) - F(t, y)\|_{\alpha-\sigma_F} \leq L_F \|x - y\|_\alpha$$

for every $x, y \in E_\alpha$ and $t \in [0, T]$. Further, set $\overline{\sigma}_F := \max\{\sigma_F, 2\gamma\}$ and

$$C_F := \max\big\{L_F, \sup_{t\in[0,T]} \|F(t, 0)\|_{\alpha-\sigma_F}\big\}.$$

Definition 3.1. A mild solution of (RPDE) with initial value $y_s \in E_\alpha$ on an interval $[s, t]$ is a controlled rough path $\big(y, y'\big) \in \mathcal{D}^\gamma_{X,\alpha}([s, t])$, where $y' = G(\cdot, y)$ and the path component is given by

$$y_u := S_{u,s}y_s + \int_s^u S_{u,r}F(r,y_r)\,\mathrm{d}r + \int_s^u S_{u,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r \tag{3-1}$$

for $u \in [s,t]$.

Recall that Assumption **(A)** ensures that the resulting parabolic evolution family is smoothing, see Theorem D.3. In particular, the assumptions of Theorem 2.42 are fulfilled, and the rough convolution in (3-1) is well-defined.

For convenience, the local existence of a mild solution for (RPDE) is proven on the interval $[0,T]$ for $T \le 1$, but the statement holds for every arbitrary interval $[s,t]$.

Theorem 3.2. *Fix $\alpha \in \mathbb{R}$. Let $\big(A(t)\big)_{t\in[0,T]}$, F and G satisfy Assumption **(A)**,**(F1)** and **(G1)**. Then for every $y_0 \in E_\alpha$ there exists a time $T^* \le T$ and a unique controlled rough path $(y,y') \in \mathcal{D}^\gamma_{X,\alpha}([0,T^*))$ which is a mild solution to (RPDE).*

Proof. A mild solution for (RPDE) is a fixed point of the operator

$$P_T\big(y,y'\big) := \left(S_{\cdot,0}y_0 + \int_0^\cdot S_{\cdot,r}F(r,y_r)\,\mathrm{d}r + \int_0^\cdot S_{\cdot,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r, G(\cdot,y)\right).$$

Instead of proving the existence of a fixed point in $\mathcal{D}^\gamma_{X,\alpha}$, consider for $\gamma' < \gamma$ the space

$$\mathscr{B}_T(y_0) := \Big\{\big(y,y'\big) \in \mathcal{D}^{\gamma'}_{X,\alpha}([0,T]) : \big(y_0,y_0'\big) = \big(y_0, G(0,y_0)\big) \\ \text{and } \big\|y-\xi, y'-\xi'\big\|_{\mathcal{D}^{\gamma'}_{X,\alpha}([0,T])} < 1\Big\},$$

where $\xi_t := S_{t,0}y_0 + \int_0^t S_{t,r}G(r,y_0)\,\mathrm{d}\mathbf{X}_r$ and $\xi'_t := G(t,y_0)$ for $t \in [0,T]$. Since the diffusion coefficient G is time-dependent, ξ' is not constant, in contrast to [GHN21]. The outline of the fixed-point argument is briefly sketched to emphasize the difference from the autonomous situation. Let $(s,t) \in \Delta_{[0,T]}$.

i) First, show that $P_T\big(\mathscr{B}_T(y_0)\big) \subset \mathscr{B}_T(y_0)$ if T is small enough. Define $\big(z,z'\big) := P_T\big(y,y'\big)$ for some $\big(y,y'\big) \in \mathscr{B}_T(y_0)$. Then we have

$$z_t - \xi_t = \int_0^t S_{t,r}F(r,y_r)\,\mathrm{d}r + \int_0^t S_{t,r}\big(G(r,y_r)-G(r,y_0)\big)\,\mathrm{d}\mathbf{X}_r =: \mathscr{N}_t + \mathscr{M}_t.$$

Using the Lipschitz continuity and a similar representation for the integral as in (2-47) given by

$$\mathscr{N}_{s,t} = (S_{t,s}-\mathrm{Id}\,)\int_0^s S_{s,r}F(r,y_r)\,\mathrm{d}r + \int_s^t S_{t,r}F(r,y_r)\,\mathrm{d}r \tag{3-2}$$

entails

$$\|\mathscr{N},0\|_{\mathcal{D}^{\gamma'}_{X,\alpha}} \lesssim T^{\tilde{\kappa}}\big(1+\|y\|_{\infty,\alpha}\,\big) \lesssim T^{\tilde{\kappa}}\Big(1+\big\|\xi,\xi'\big\|_{\mathcal{D}^{\gamma'}_{X,\alpha}}\Big),$$

where $\tilde{\kappa} := \min\{1-2\gamma', 1-\sigma_F\}$ and we used $\big\|y,y'\big\|_{\mathcal{D}^{\gamma'}_{X,\alpha}} \le 1+\big\|\xi,\xi'\big\|_{\mathcal{D}^{\gamma'}_{X,\alpha}}$. Further, with (2-45) and (2-52) we obtain

$$\big\|\mathscr{M},\mathscr{M}'\big\|_{\mathcal{D}^{\gamma'}_{X,\alpha}} \lesssim \big(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\big)^3 T^\kappa\Big(1+\big\|y-y_0,y'\big\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)^2 \\ \le \big(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\big)^3 T^\kappa\Big(1+\big\|\xi-y_0,\xi'\big\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)^2.$$

where $\kappa := \min\left\{\gamma-\gamma', \gamma'-\sigma_G\right\}$. Since the controlled rough path norm $\left(\xi, \xi'\right)$ does only depend on G, y_0 and $\mathbf{X}$, this leads to

$$\left\|z-\xi, z'-\xi'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}} \lesssim_{F,G,\mathbf{X},y_0} T^{\min\{\kappa,\tilde{\kappa}\}},$$

which is smaller then 1 if T is small enough.

ii) For two different controlled rough paths $\left(y, y'\right), \left(\hat{y}, \hat{y}'\right) \in \mathcal{D}^{\gamma'}_{X,\alpha}$ define $\left(z, z'\right) := P_T\left(y, y'\right)$ and $\left(\hat{z}, \hat{z}'\right) := P_T\left(\hat{y}, \hat{y}'\right)$. Then we have

$$\begin{aligned} z_t - \hat{z}_t &= \int_0^t S_{t,r}\left(F(r, y_r) - F(r, \hat{y}_r)\right) \mathrm{d}r + \int_0^t S_{t,r}\left(G(r, y_r) - G(r, \hat{y}_r)\right) \mathrm{d}\mathbf{X}_r \\ &=: \widetilde{\mathscr{N}_t} + \widetilde{\mathscr{M}_t}. \end{aligned}$$

The Lipschitz continuity of F yields

$$\left\|\widetilde{\mathscr{N}}, 0\right\|_{\mathcal{D}^{\gamma'}_{X,\alpha}} \lesssim_{F,\mathbf{X},y_0,\alpha} T^{\tilde{\kappa}} \left\|y-\hat{y}, y'-\hat{y}'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}}.$$

For $\mathscr{M}$, we use again (2-45) and (2-53) to obtain

$$\begin{aligned} \left\|\widetilde{\mathscr{M}}, \widetilde{\mathscr{M}'}\right\|_{\mathcal{D}^{\gamma'}_{X,\alpha}} &\lesssim (1+\varrho_{\gamma,[0,T]}(\mathbf{X}))T^{\kappa} \left\|y-\hat{y}, y'-\hat{y}'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}} \left(1 + \left\|y, y'\right\|_{\mathcal{D}^{\gamma'}_{X,\alpha}} + \left\|\hat{y}, \hat{y}'\right\|_{\mathcal{D}^{\gamma'}_{X,\alpha}}\right) \\ &\lesssim (1+\varrho_{\gamma,[0,T]}(\mathbf{X}))T^{\kappa} \left\|y-\hat{y}, y'-\hat{y}'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}} \left(1 + \left\|\xi, \xi'\right\|_{\mathcal{D}^{\gamma'}_{X,\alpha}}\right). \end{aligned}$$

Since

$$\left\|z-\hat{z}, z'-\hat{z}'\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}} \leq \left\|\widetilde{\mathscr{N}}, 0\right\|_{\mathscr{D}^{\gamma'}_{X,\alpha}} + \left\|\widetilde{\mathscr{M}}, \widetilde{\mathscr{M}'}\right\|_{\mathcal{D}^{\gamma'}_{X,\alpha}},$$

this proves that P_{T^*} is a contraction for $T^* \leq T$ small enough.

Then, Banach's fixed-point theorem [vN22, Theorem 2.13] yields the existence of a unique controlled rough path $(y, y') \in \mathcal{D}^{\gamma'}_{X,\alpha}([0, T^*))$ such that $y' = G(\cdot, y)$ and y satisfies (3-1).

It remains to show, that $\left(y, y'\right) \in \mathcal{D}^{\gamma}_{X,\alpha}$. We sketch the proof in the case of $F \equiv 0$ and choose γ' such that $\sigma_G \leq \gamma' < \gamma$. First, $y' = G(\cdot, y) \in \mathcal{C}(E_{\alpha-\gamma})$ follows directly due to **(G1)** and $E_{\alpha-\sigma_G} \hookrightarrow E_{\alpha-\gamma}$. Using (2-46) and (2-47) further leads to

$$y_{s,t} = \mathscr{R}_{s,t} + S_{t,s}\mathrm{D}_2G(s, y_s)G(s, y_s)\mathbb{X}_{s,t} + S_{t,s}G(s, y_s)X_{s,t} + (S_{t,s} - \mathrm{Id}\,)\, y_s.$$

Since $X \in \mathcal{C}^{\gamma}([0, T]; \mathbb{R})$ and $\mathbb{X} \in \mathcal{C}^{2\gamma}_2([0, T]; \mathbb{R})$, this representation implies $y \in \mathcal{C}^{\gamma}(E_{\alpha-2\gamma})$ and $y' = G(\cdot, y) \in \mathcal{C}^{\gamma}(E_{\alpha-2\gamma})$ using the smoothing properties (2-41), (2-42) as well as (2-40) and (2-52). For the remainder, note that similarly

$$R^y_{s,t} = \mathscr{R}_{s,t} + S_{t,s}\mathrm{D}_2G(s, y_s)G(s, y_s)\mathbb{X}_{s,t} + (S_{t,s} - \mathrm{Id}\,)\, G(s, y_s)X_{s,t} + (S_{t,s} - \mathrm{Id}\,)\, y_s,$$

holds, which proves $R^{\gamma} \in \mathcal{C}^{\gamma}(E_{\alpha-\gamma}) \cap \mathcal{C}^{2\gamma}(E_{\alpha-2\gamma})$. □

Remark 3.3. Note, that the maximal interval $[0, T^*)$, depends on the initial state, more precisely on $\left\|y_0\right\|_{\alpha}$. The time T^* can be substituted by a time $\widetilde{T}^*(r)$ which depends on a constant $r > 0$, and then every solution with an initial datum satisfying $\left\|y_0\right\|_{\alpha} \leq r$ exists up to time $\widetilde{T}^*$. With a slight abuse of notation, we denote this time again by T^*.

A significant advantage of the rough path approach compared to the Itô calculus is the continuity of the Itô-Lyons map, providing the continuous dependence of the solution on the noise. For similar results see [GH19, Section 4.1] and [GHN21, Theorem 5.4].

Theorem 3.4. *Assume that $\big(A(t)\big)_{t\in[0,T]}$, F and G satisfy the Assumptions **(A)**, **(F1)** and **(G1)**. Let $y_0, \hat{y}_0 \in E_\alpha$, and $\big(y, G(\cdot,y)\big) \in \mathcal{D}^{\gamma}_{X,\alpha}$, $\big(\hat{y}, G(\cdot,\hat{y})\big) \in \mathcal{D}^{\gamma}_{\widehat{X},\alpha}$ be the mild solutions of (RPDE) driven by $\mathbf{X} \in \mathscr{C}^\gamma$ respectively $\widehat{\mathbf{X}} \in \mathscr{C}^\gamma$ with initial conditions y_0 and $\hat{y}_0$. If $\varrho_{\gamma,[0,T]}(\mathbf{X})$, $\varrho_{\gamma,[0,T]}\big(\widehat{\mathbf{X}}\big)$, $\|y_0\|_\alpha$, $\|\hat{y}_0\|_\alpha$ are bounded from above by the same constant, then for every $1/3 < \gamma' < \gamma$ we have*

$$d^{\gamma',\gamma}_{[0,T^*],\alpha}(y,\hat{y}) \lesssim \varrho_{\gamma,[0,T^*]}\big(\mathbf{X},\widehat{\mathbf{X}}\big) + \|y_0 - \hat{y}_0\|_\alpha\,, \tag{3-3}$$

for some $T^* < T$. If both solutions are globally defined, then (3-3) holds for $T^* := T$.

Proof. Using Lemma 2.45 and 2.49 infers that

$$d^{\gamma',\gamma}_{[0,T],\alpha}(y,\hat{y}) \lesssim_M \|y_0 - \hat{y}_0\|_\alpha + \varrho_{\gamma,[0,T]}\big(\mathbf{X},\widehat{\mathbf{X}}\big) + d^{\gamma',\gamma}_{[0,T],\alpha}(y,\hat{y})T^\kappa,$$

where $\kappa := \min\{\gamma-\gamma',\gamma'\}$. Choosing a small enough time T^*, this shows (3-3). If both solutions are globally defined, the process can be iterated up to $T^* = T$. ☐

3.2 Global-in-time solutions

The existence of a global-in-time solution to equations like (RPDE) was open for a long time. The first results on rough partial differential equations, as in [Gub04, GT10, GH19, GHN21], were limited to local-in-time existence. Then the works by Hesse and Neamţu [HN20, HN22] state the first condition for global solutions. It is worth noting that recently [Tap25] presents an alternative idea where the diffusion coefficient lifts the spatial regularity, and is assumed to map into $D\big(A^3\big)$. If the linear part is given by the Laplacian, so $A = \Delta$, on a bounded domain $\mathcal{O} \subset \mathbb{R}^n$, this would correspond to

$$D\big(A^3\big) = \big\{u \in \mathrm{H}^{6,2}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\big\}.$$

In the case of boundary noise, we encounter a similar assumption on the nonlinearity **($\mathbf{G_N}$)**. However, in the context of analytic semigroups, as is the case for parabolic equations, this condition is more restrictive than necessary, see [HN20, HN22]. Tappe's approach is therefore particularly interesting when non-parabolic equations are examined.

It turns out that the main obstacle to obtaining global solutions is the quadratic term on the right-hand side of (2-52). So, the goal is to establish a new bound to avoid these quadratic terms, which is why we need to impose additional assumptions on the nonlinearity.

(G2) For every $t \in [0,T]$, the derivative of

$$\mathrm{D}_2G(t,\cdot)G(t,\cdot)\colon E_{\alpha-\gamma} \to E_{\alpha-2\gamma-\sigma_G} \tag{3-4}$$

is bounded.

Remark 3.5. i) If G is either bounded or linear, then **(G2)** is obviously fulfilled. For concrete examples, see Subsection 3.2.1.

ii) That (3-4) has a bounded derivative, leads to the fact that

$$\begin{aligned}
&\big\|\big(\mathrm{D}_2G(t,y_t) - \mathrm{D}_2G(t,y_s)\big)G(t,y_t)\big\|_{\alpha-2\gamma-\sigma_G}\\
&\qquad\le \|\mathrm{D}_2G(t,y_t)G(t,y_t) - \mathrm{D}_2G(t,y_s)G(t,y_s)\|_{\alpha-2\gamma-\sigma_G}\\
&\qquad\quad + \big\|\mathrm{D}_2G(t,y_s)\big(G(t,y_t) - G(t,y_s)\big)\big\|_{\alpha-2\gamma-\sigma_G}\\
&\qquad\lesssim \|y_t - y_s\|_{\alpha-\gamma}\,,
\end{aligned} \tag{3-5}$$

for $(s,t) \in \Delta_{[0,T]}$.

Using the specific form of the mild solution $\big(y, G(\cdot, y)\big)$, the quadratic term in (2-52) can now be avoided.

Lemma 3.6. *Let G satisfy **(G1)-(G2)** and $\big(y, G(\cdot, y)\big) \in \mathcal{D}^{\gamma}_{X,\alpha}$. Then*

$$\big(G(\cdot, y), \mathrm{D}_2 G(\cdot, y) G(\cdot, y)\big) \in \mathcal{D}^{\gamma}_{X,\alpha-\sigma_G},$$

and we have the bound

$$\|G(\cdot, y), \mathrm{D}_2 G(\cdot, y) G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X,\alpha-\sigma_G}} \lesssim C_G + C_G\big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big)^2 \|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}} . \tag{3-6}$$

Proof. Let $(s, t) \in \Delta_{[0,T]}$. The supremum norms can be estimated directly, since $\mathrm{D}_2 G$ is bounded. Therefore, we have similar to Lemma 2.48

$$\begin{aligned} \|G(\cdot, y)\|_{\infty,\alpha-\sigma_G} &\lesssim 1 + \|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}} , \\ \|\mathrm{D}_2 G(\cdot, y) G(\cdot, y)\|_{\infty,\alpha-\sigma_G-\gamma} &\lesssim \|G(\cdot, y)\|_{\infty,\alpha-\gamma} \lesssim \|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}} . \end{aligned}$$

For the Hölder continuity, we consider the representation

$$\begin{aligned} \mathrm{D}_2 G(t, y_t) G(t, y_t) - \mathrm{D}_2 G(s, y_s) G(s, y_s) &= \big(\mathrm{D}_2 G(t, y_t) - \mathrm{D}_2 G(t, y_s)\big) G(t, y_t) \\ &\quad + \big(\mathrm{D}_2 G(t, y_s) - \mathrm{D}_2 G(s, y_s)\big) G(t, y_t) + \mathrm{D}_2 G(s, y_s)\big(G(t, y_t) - G(s, y_s)\big), \end{aligned}$$

and use **(G2)** as well as the embedding $E_{\alpha-\sigma_G-\gamma} \hookrightarrow E_{\alpha-2\gamma}$ to obtain

$$\begin{aligned} [\mathrm{D}_2 G(\cdot, y) G(\cdot, y)]_{\gamma,\alpha-\sigma_G-2\gamma} &\lesssim [y]_{\gamma,\alpha-\gamma} + [y']_{\infty,\alpha-\gamma} + [y']_{\gamma,\alpha-2\gamma} \\ &\lesssim \big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big) \|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}} . \end{aligned}$$

Consequently, only the Hölder semi-norms of the remainder are left to bound. This is possible using

$$\begin{aligned} R^{G(\cdot,y)}_{s,t} &= G(t, y_t) - G(s, y_s) - \mathrm{D}_2 G(s, y_s)\big(G(s, y_s) X_{s,t}\big) \\ &= G(t, y_t) - G(s, y_t) + G(s, y_t) - G(s, y_s) - \mathrm{D}_2 G(s, y_s)\big(G(s, y_s) X_{s,t}\big) \\ &= G(t, y_t) - G(s, y_t) \\ &\quad + \int_0^1 \big(\mathrm{D}_2 G(s, y_s + r y_{s,t}) - \mathrm{D}_2 G(s, y_s)\big)\big(G(s, y_s) X_{s,t}\big) \,\mathrm{d}r \\ &\quad + \int_0^1 \mathrm{D}_2 G(s, y_s + r y_{s,t}) \,\mathrm{d}r R^y_{s,t}, \end{aligned}$$

where we recall that $y_{s,t} = y_t - y_s$ denotes the difference. The first line can be estimated using $G(\cdot, y_t) \in \mathcal{C}^{i\gamma}(E_{\alpha-i\gamma-\sigma_G})$ for $i = 1, 2$. For the other two lines, the boundedness of $\mathrm{D}_2 G$ leads to

$$\begin{aligned} \Big[R^{G(\cdot,y)}\Big]_{\gamma,\alpha-\sigma_G-\gamma} &\lesssim 1 + \big(\, \|G(\cdot, y)\|_{\infty,\alpha-\gamma} [X]_\gamma + [R^y]_{\gamma,\alpha-\gamma} \big) \\ &\lesssim 1 + \big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big) \|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}} . \end{aligned}$$

Then (3-5) implies

$$\begin{aligned} \Big\|R^{G(\cdot,y)}_{s,t}\Big\|_{\alpha-\sigma_G-2\gamma} &\lesssim \int_0^1 \big\|\big(\mathrm{D}_2 G(s, y_s + r y_{s,t}) - \mathrm{D}_2 G(s, y_s)\big) G(s, y_s)\big\|_{\alpha-\sigma_G-2\gamma} \,\mathrm{d}r \,|X_{s,t}| \\ &\quad + \int_0^1 \big\|\mathrm{D}_2 G(s, y_s + r y_{s,t}) R^y_{s,t}\big\|_{\alpha-\sigma_G-2\gamma} \,\mathrm{d}r + (t-s)^{2\gamma} \\ &\lesssim \|y_{s,t}\|_{\alpha-\gamma} [X]_\gamma (t-s)^\gamma + [R^y]_{2\gamma,\alpha-2\gamma} (t-s)^{2\gamma} + (t-s)^{2\gamma} \end{aligned}$$

$$\lesssim [y]_{\gamma,\alpha-\gamma}\,[X]_{\gamma}\,(t-s)^{2\gamma} + \|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\,(t-s)^{2\gamma} + (t-s)^{2\gamma}$$
$$\lesssim (t-s)^{2\gamma}\left(1+\left(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\right)^2\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\right),$$

where (2-37) is used in the last inequality. This shows $R^{G(\cdot,y)} \in \mathcal{C}^{\gamma}(E_{\alpha-\gamma}) \cap \mathcal{C}^{2\gamma}(E_{\alpha-2\gamma})$ and collecting all the estimates proves the claim. □

To derive an a-priori estimate for the solution, we first need a bound on the terms involving the initial condition y_0 and the drift term F.

Lemma 3.7. *Let $y_0 \in E_\alpha$ and $(y,y') \in \mathcal{D}^{\gamma}_{X,\alpha}$. Assume that $\left(A(t)\right)_{t\in[0,T]}$, and F satisfy* ***(A)*** *and* ***(F1)****. Then we have $(S_{\cdot,0}y_0, 0), \left(\int_0^{\cdot} S_{\cdot,r}y_r\,\mathrm{d}r, 0\right) \in \mathcal{D}^{\gamma}_{X,\alpha}$ as well as the estimates*

$$\|S_{\cdot,0}y_0, 0\|_{\mathcal{D}^{\gamma}_{X,\alpha}} \lesssim \|y_0\|_{\alpha},$$
$$\left\|\int_0^{\cdot} S_{\cdot,r}F(r,y_r)\,\mathrm{d}r, 0\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}} \lesssim \left(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\right)T^{1-\overline{\sigma}_F},$$

where $\overline{\sigma}_F := \max\{2\gamma,\sigma_F\}$.

Proof. Let $(s,t) \in \Delta_{[0,T]}$. Note that if the Gubinelli derivative is 0, the remainder is given by $R^S_{s,t} = S_{t,0}y_0 - S_{s,0}y_0$. The same holds for the deterministic convolution. Recall that the evolution family is Hölder continuous with parameter $i\gamma$ for $i = 1,2$ if we assume the same amount of spatial regularity, see (D-2). This leads to

$$[S_{\cdot,\cdot}y_0]_{i\gamma,\alpha-i\gamma} \lesssim \|y_0\|_{\alpha},$$

and proves the first estimate since $\|S_{\cdot,0}y_0\|_{\infty,\alpha} \lesssim \|y_0\|_{\alpha}$.

The integral involving the drift can be estimated using the Lipschitz continuity

$$\left\|\int_0^{\cdot} S_{\cdot,r}F(r,y_r)\,\mathrm{d}r\right\|_{\infty,\alpha} \lesssim T^{1-\sigma_F}\left(1+\|y\|_{\infty,\alpha}\right).$$

The Hölder continuity of the corresponding remainder can be derived by using the representation (3-2) and the estimates

$$\left\|\int_s^t S_{t,r}F(r,y_r)\,\mathrm{d}r\right\|_{\alpha-i\gamma} \lesssim (t-s)^{1+\min\{i\gamma-\sigma_F,0\}}\left(1+\|y\|_{\infty,\alpha}\right),$$
$$\left\|\left(S_{t,s}-\mathrm{Id}\right)\int_0^s S_{t,r}F(r,y_r)\,\mathrm{d}r\right\|_{\alpha-i\gamma} \lesssim (t-s)^{i\gamma}s^{1-\sigma_F}\left(1+\|y\|_{\infty,\alpha}\right),$$

for $i = 1,2$, which proves the claim. □

Based on these inequalities, we can now state an a-priori estimate, which does not involve any term containing $\|y,y'\|^2_{\mathcal{D}^{\gamma}_{X,\alpha}}$.

Corollary 3.8. *Let $y_0 \in E_\alpha$. Assume $\left(A(t)\right)_{t\in[0,T]}$, F and G satisfy* ***(A)***,***(F1)*** *and* ***(G1)***-***(G2)****. Further, let $\left(y,G(\cdot,y)\right) \in \mathcal{D}^{\gamma}_{X,\alpha}$ be the local mild solution of* (RPDE)*. Then we have*

$$\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}} \lesssim 1+\left(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\right)\|y_0\|_{\alpha} + T^{\tilde{\kappa}}\left(1+\varrho_{\gamma,[0,T]}(\mathbf{X})\right)^3\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}}, \quad (3\text{-}7)$$

with $\tilde{\kappa} := \min\{1-\overline{\sigma}_F, \gamma-\sigma_G\}$.

Proof. Since $\big(y, G(\cdot, y)\big)$ solves (RPDE), we know in particular that the path component satisfies (3-1), so we have

$$\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}} \leq \|S_{t,0}y_0, 0\|_{\mathcal{D}^\gamma_{X,\alpha}} + \left\| \int_0^\cdot S_{t,r}F(r, y_r)\, \mathrm{d}r, 0 \right\|_{\mathcal{D}^\gamma_{X,\alpha}} + \left\| \int_0^\cdot S_{t,r}G(r, y_r)\, \mathrm{d}\mathbf{X}_r, G(\cdot, y) \right\|_{\mathcal{D}^\gamma_{X,\alpha}}.$$

Now a combination of the Lemmata 3.6, 3.7, and the estimate (2-45) leads to

$$\begin{aligned}
\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}} &\lesssim \|y_0\|_\alpha + \Big(1 + \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big) T^{1-\overline{\sigma}_F} \\
&\quad + \big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big) \Big(\|G(0, y_0)\|_{\alpha-\sigma_G} + \|\mathrm{D}_2G(0, y_0)G(0, y_0)\|_{\alpha-\sigma_G-\gamma} \\
&\quad + T^{\gamma-\sigma_G} \|G(\cdot, y), \mathrm{D}_2G(\cdot, y)G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha-\sigma_G}} \Big) \\
&\lesssim 1 + \big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big) \|y_0\|_\alpha + T^{1-\overline{\sigma}_F} \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}} \\
&\quad + T^{\gamma-\sigma_G} \big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big)^3 \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}} \\
&\leq 1 + \big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big) \|y_0\|_\alpha + T^{\tilde{\kappa}} \big(1 + \varrho_{\gamma,[0,T]}(\mathbf{X})\big)^3 \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}},
\end{aligned}$$

with $\tilde{\kappa} := \min\{1 - \overline{\sigma}_F, \gamma - \sigma_G\}$. □

With this essential estimate, it is now possible to state the existence of a global-in-time solution to (RPDE). The strategy is as follows: First, it is demonstrated that the endpoint y_{T^*} of the local solution is bounded by the same constant as the initial condition. Consequently, it is possible to employ this endpoint as a new initial condition, thereby generating a solution that persists for the same time as the original, starting at y_{T^*}. By gluing these solutions together, we obtain a solution on the interval $[0, 2T^*]$. Through repeated application, the solution can be extended to the entire time interval.

In the autonomous case, the proof of the following theorem can be found in [HN22, Theorem 3.9].

Theorem 3.9. *Fix $\alpha \in \mathbb{R}$. Let $\big(A(t)\big)_{t\in[0,T]}$, F and G satisfy Assumption* ***(A)**,**(F1)*** *and* ***(G1)**-**(G2)***. *Then there exists for every $y_0 \in E_\alpha$ a unique controlled rough path $\big(y, y'\big) \in \mathcal{D}^\gamma_{X,\alpha}([0,T])$ such that $y'_t = G(t, y_t)$ and*

$$y_t = S_{t,0}y_0 + \int_0^t S_{t,r}F(r, y_r)\, \mathrm{d}r + \int_0^t S_{t,r}G(r, y_r)\, \mathrm{d}\mathbf{X}_r,$$

for $t \in [0, T]$.

Proof. Without loss of generality, assume $\|y_0\|_\alpha \geq 1$. Otherwise we replace it by $\max\{1, \|y_0\|_\alpha\}$. Due to Theorem 3.2 there exists a time $T^* \leq T$ such that $\big(y, G(\cdot, y)\big)$ is a local solution on $[0, T^*)$.

i) We fix $T_0 < T^*$ and prove first that there exists a $\beta > 0$ such that the the path component of the solution satisfies

$$\|y\|_{\infty,[0,T_0],\alpha} \lesssim \|y_0\|_\alpha e^{\beta T_0} =: r. \tag{3-8}$$

Indeed, due to (3-7) there is a constant $C(\mathbf{X}) = C > 1$ such that

$$\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([0,T_0])} \leq C \left(1 + \|y_0\|_\alpha + T_0^{\tilde{\kappa}} \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([0,T_0])}\right),$$

for $\tilde{\kappa} = \min\{\gamma - \sigma_G, 1 - \overline{\sigma}_F\}$ and $T_0 < T^*$. Choosing a time $(2C)^{\tilde{\kappa}^{-1}} T_0^* < 1$ yields

$$\|y\|_{\infty,[0,T_0^*],\alpha} \leq \|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}([0,T_0^*])} \leq 4C \|y_0\|_\alpha .$$

If $T_0 \leq T_0^*$, then (3-8) holds for an arbitrary $a < 0$. Otherwise, choose some $N \in \mathbb{N}$ such that $T_0^*/2 < T_1 := T_0/N \leq T_0^*$, which is possible since $\tilde{\kappa} < 1$, and observe

$$\|y\|_{\infty,[0,T_1],\alpha} \leq 4C \|y_0\|_\alpha .$$

We define $y_t^{(k)} := y_{t+kT_1}$ for $k = 0, \ldots, N-1$ as the solution of (RPDE) on the interval $[0, T_1]$ with initial condition y_{kT_1} and therefore

$$\left\| y^{(k)}, G(\cdot, y^{(k)}) \right\|_{\mathcal{D}^{\gamma}_{X,\alpha}([0,T_1])} \leq C \left(1 + \|y_{kT_1}\|_\alpha + T_1^{\tilde{\kappa}} \left\| y^{(k)}, G(\cdot, y^{(k)}) \right\|_{\mathcal{D}^{\gamma}_{X,\alpha}([0,T_1])} \right).$$

Due to the definition of T_1, this implies

$$\left\| y^{(k)} \right\|_{\infty,[0,T_1],\alpha} \leq \left\| y^{(k)}, G(\cdot, y^{(k)}) \right\|_{\mathcal{D}^{\gamma}_{X,\alpha}([0,T_1])} \leq 2C \left(1 + \left\| y^{(k-1)} \right\|_{\infty,[0,T_1],\alpha} \right),$$

using $y_{kT_1} = y_{T_1}^{(k-1)}$. By induction, we obtain

$$\left\| y^{(k)} \right\|_{\infty,[0,T_1],\alpha} \leq (4C)^{k+1} \|y_0\|_\alpha ,$$

for $k = 0, \ldots, N-1$ and therefore

$$\|y\|_{\infty,[0,T_0],\alpha} \leq \max_{k=0,\ldots,N-1} \left\{ \left\| y^{(k)} \right\|_{\infty,[0,T],\alpha} \right\} \leq (4C)^N \|y_0\|_\alpha .$$

Recall that $T_0 = NT_1$ and $T_1 > T_0^*/2$, which leads to

$$\|y\|_{\infty,[0,T_0],\alpha} \leq \left((4C)^{\frac{2}{T_0^*}} \right)^{T_0} \|y_0\|_\alpha \lesssim e^{aT_0} \|y_0\|_\alpha ,$$

for some $a > 0$ large enough.

ii) Now we prove the global existence. Choose $N \in \mathbb{N}$ large enough, such that $T/N < T^*$. Due to (3-8), we know $\|y_t\|_\alpha \leq r$ for every $t \in [0, T/N]$, and in particular $\left\| y_{T/N} \right\| \leq r$. Therefore, we obtain a solution on $[0, T/N]$ with initial condition $y_{T/N}$ using Remark 3.3. Gluing these two solutions together results in a solution on the interval $[0, 2T/N]$. Iterating this process N times leads to a solution y, which is defined on the whole time interval $[0, T]$.

□

Remark 3.10. We briefly explain why the quadratic term can be problematic in the above proof. Starting from an initial datum satisfying $\|y_0\|_\alpha > 1$ still leads to a solution that we can bound as in (3-8). But, if we start with estimates like (2-52), then the a-priori estimate of the solution also contains a quadratic term, which leads to the fact that (3-8) has the form

$$\|y\|_{\infty,[0,T_0],\alpha} \lesssim \|y_0\|_\alpha^2 e^{aT_0} =: r.$$

Therefore, starting with an initial datum $\|y_0\|_\alpha \leq r$ results in $\left\| y_{T/N} \right\|_\alpha \leq r^2$. Consequently, the second interval of the solution may be smaller than the first one. In this case, it is not certain whether the entire interval $[0, T]$ can be filled by iterating this procedure, as the added interval becomes smaller with every new step.

3.2.1 Examples

We assume without loss of generality $F \equiv 0$. Since the conditions on F are very general, the focus here is on the diffusion coefficient G.

Example 3.11. *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open and bounded domain, $E = \mathrm{L}^p(\mathcal{O})$ for $p \geq 2$ and $\big(A(t)\big)_{t\in[0,\infty)}$ a family of second-order operators with Dirichlet boundary conditions. Then, under suitable assumptions on the coefficients, it can be shown that $\big(A(t)\big)_{t\in[0,\infty)}$ satisfies **(A)**. For more information, see Example D.8. We recall that the corresponding fractional power scale is then given by*

$$E_\alpha := \begin{cases} \mathrm{H}^{2\alpha,p}(\mathcal{O}), & 0 \leq \alpha < \frac{1}{2p}, \\ \big\{u \in \mathrm{H}^{2\alpha,p}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\big\}, & \frac{1}{2p} < \alpha \leq 1, \alpha \neq \frac{p+1}{2p}. \end{cases}$$

As a diffusion coefficient, we consider a nonlinear integral operator G, similar to [HN19, Section 7] and [GALS16, Section 7]

$$G(t,u) := a(t) \int_{\mathcal{O}} g\big(\cdot, u(x)\big)\, \mathrm{d}x,$$

with $a \in \mathcal{C}^{2\gamma}([0,\infty);\mathbb{R})$. The kernel $g\colon \overline{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is assumed to be bounded and three times continuously differentiable with bounded derivatives such that $\sup_{x\in\mathbb{R}} \big|\mathrm{D}_2^k g(\cdot,x)\big|$ is bounded for $k = 1,2,3$, $i = 0,1,2$ and $g|_{\partial\mathcal{O}\times\mathbb{R}} = 0$. Then $G(t,\cdot)\colon \mathrm{L}^p(\mathcal{O}) \to \mathrm{L}^p(\mathcal{O})$ is three times continuously differentiable with derivatives

$$\begin{aligned} \mathrm{D}_2 G(t,y)h_1 &= a(t) \int_{\mathcal{O}} \mathrm{D}_2 g\big(\cdot, y(x)\big) h_1(x)\, \mathrm{d}x, \\ \mathrm{D}_2^2 G(t,y)[h_1,h_2] &= a(t) \int_{\mathcal{O}} \mathrm{D}_2^2 g\big(\cdot, y(x)\big)[h_1(x),h_2(x)]\, \mathrm{d}x, \\ \mathrm{D}_2^3 G(t,y)[h_1,h_2,h_3] &= a(t) \int_{\mathcal{O}} \mathrm{D}_2^3 g\big(\cdot, y(x)\big)[h_1(x),h_2(x),h_3(x)]\, \mathrm{d}x, \end{aligned}$$

which is proven in [KA82, XVII.3] for $y, h_1, h_2, h_3 \in \mathrm{L}^p(\mathcal{O})$ using the mean value theorem and the fact that G is a parameter dependent integral. If we choose α big enough, such that $E_{\alpha-i\gamma} \hookrightarrow C\big(\overline{\mathcal{O}}\big)$ holds, i.e. $\alpha > {}^{n}/{2p} + 2\gamma$, using Sobolev inequalities, we obtain further that all the derivatives of G are bounded operators from $E^k_{\alpha-i\gamma}$ to $E_{\alpha-i\gamma-\sigma_G}$ for $k = 1,2,3$, $i = 0,1,2$, and $0 \leq \sigma_G < \gamma$ see [GALS16, Section 7]. Since every derivative is Hölder continuous in time, the resulting operator

$$G\colon [0,\infty) \times E_{\alpha-i\gamma} \to E_{\alpha-i\gamma-\sigma_G},$$

*fulfills **(G1)**.*

For $\mathrm{D}_2 G(\cdot,y)G(\cdot,y)$, we know

$$z \mapsto \Big(\mathrm{D}_2 G(t,y)G(t,y)\Big)(z) = a(t)^2 \int_{\mathcal{O}} \mathrm{D}_2 g\big(z, y(x)\big) \int_{\mathcal{O}} g\big(x, y(\tilde{x})\big)\, \mathrm{d}\tilde{x}\mathrm{d}x$$

is an element of $E_{\alpha-2\gamma-\sigma_G}$, which means that

$$\mathrm{D}_2 G(t,\cdot)G(t,\cdot)\colon E_{\alpha-i\gamma} \to E_{\alpha-i\gamma-\sigma_G}.$$

The derivative of this operator is then given by

$$\Big(\mathrm{D}_2\big(\mathrm{D}_2 G(t,y)G(t,y)\big)\Big)(h) = \mathrm{D}_2^2 G(t,y)[G(t,y),h] + \mathrm{D}_2 G(t,y)\big(\mathrm{D}_2 G(t,y)h\big)$$

$$= a(t)^2 \int_{\mathcal{O}} \int_{\mathcal{O}} \Big(\mathrm{D}_2^2 g(\cdot, y(x)) g(x, y(\tilde{x})) + \mathrm{D}_2 g(\cdot, y(x)) \mathrm{D}_2 g(x, y(\tilde{x})) \Big) h(\tilde{x}) \, \mathrm{d}\tilde{x} \mathrm{d}x,$$

*for $h, y \in E_{\alpha-\gamma}$. Due to the assumptions on g, this derivative is bounded. Therefore, **(G2)** is also satisfied.*

Example 3.12. *For this example, we consider the n-dimensional torus $\mathbb{T}^n$, $p \geq 1$, and the Laplacian $A = \Delta$. In this case, the resulting scale of function spaces is given by the Sobolev tower $E_\alpha := \mathrm{H}^{2\alpha,p}(\mathbb{T}^n)$. We let $\gamma \in (1/3, 1/2]$, $\sigma_G < \gamma$ and $k \in \mathbb{N}$ be large enough such that $\mathrm{H}^{2(\alpha - i\gamma - \sigma_G),p}(\mathbb{T}^n)$ is an algebra for $i = 0, 1, 2$, which is the case for $\alpha > n/2p + 2\gamma + \sigma_G$. Furthermore, choose $g \in \mathcal{C}^{2\gamma}\big([0,\infty); E_{\alpha-\sigma_G}\big)$ and define the diffusion coefficient by $G(t, y) := g(t, \cdot)(-\Delta)^{\sigma_G} y$ for $y \in E_\alpha$. Due to the definition of Bessel-potential spaces we obtain $(-\Delta)^{\sigma_G} \colon E_\alpha \to E_{\alpha-\sigma_G}$. Since $\mathrm{H}^{k+i\alpha-2\gamma-\sigma_G,p}(\mathbb{T}^n)$ is an algebra, for every $i = 0, 1, 2$, the spaces $E_{\alpha-i\gamma-\sigma_G}$ are closed under multiplication, which implies*

$$G \colon [0,\infty) \times E_{\alpha-i\gamma} \to E_{\alpha-i\gamma-\sigma_G}.$$

*Furthermore, G is linear, consequently the derivatives $\mathrm{D}_2^k G$ exist for every $k = 1, 2, 3$ and **(G1)** is fulfilled. The linearity also directly implies that **(G2)** holds.*

3.3 Evolution equations with rough boundary noise

In this section, we investigate a special type of evolution equations, where the noise acts on the boundary of an open subset $\mathcal{O} \subset \mathbb{R}^n$ with $\mathcal{C}^\infty$-boundary. To be precise, we investigate the semilinear parabolic evolution equation with nonlinear rough boundary noise given by

$$\begin{cases} \frac{\partial}{\partial t} y_t = \mathcal{A} y_t + F(t, y) & \text{in } \mathcal{O}, \\ \mathcal{B} y_t = G(t, y_t) \, \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{X}_t & \text{on } \partial\mathcal{O}, \end{cases} \tag{BN}$$

where $\mathbf{X}$ is a geometric γ-Hölder rough path with $\gamma \in (1/3, 1/2]$, F and G are nonlinear terms and $(\mathcal{A}, \mathcal{B})$ is a normally elliptic boundary value problem given by

$$\mathcal{A} := \sum_{i,j=1}^n \partial_i \left(a_{ij} \partial_j \right) + b, \quad \mathcal{B} := \sum_{i,j=1}^n \nu_i \gamma_\partial a_{ij} \partial_j,$$

for suitable coefficients $(a_{ij})_{i,j=1}^n$ and b. Further, let $A \colon D(A) \subset \mathrm{L}^p(\mathcal{O}) \to \mathrm{L}^p(\mathcal{O})$ be the L^p-realization associated to the boundary value problem $(\mathcal{A}, \mathcal{B})$. For more information on these boundary value problems, see Appendix D.2.

In this chapter, it is essential to note that the focus is not on a time-dependent family of linear operators. This is because, by incorporating boundary noise, the resulting domains are time-dependent, and the Kato–Tanabe conditions are not satisfied. Therefore, we consider here an analytic semigroup $(S_t)_{t \geq 0}$ generated by A. However, we expect that the theory presented here can be extended to time-dependent domains using more general assumptions, such as the Acquistapace–Terreni conditions [AT87], see also Chapter 7.

The content of this subsection is already published in [NS23] as the result of a collaboration with Alexandra Blessing. Although, the presentation has changed and the proofs are more streamlined now.

3.3.1 Solution concept

As mentioned in the introduction, there are various approaches to address noise on the boundary. The general strategy in this thesis is to rewrite the original equation into something that does not involve any boundary terms and use the results of Section 3.1 and 3.2.

The motivation behind this idea comes from the deterministic control theory setting, see [Bal81, Section 4.2] and [LT81]. Consider the formal boundary value problem

$$\begin{cases} \dot{y} &= \mathcal{A}y \quad \text{in} \quad \mathcal{O}, \\ \mathcal{B}y &= g \quad \text{on} \quad \partial\mathcal{O}, \end{cases} \tag{3-9}$$

where $(\mathcal{A}, \mathcal{B})$ forms a boundary value problem with L^p-realization A and g a time-dependent inhomogeneity. Assume that A generates the semigroup $(S_t)_{t\geq 0}$ and that

$$\mathcal{A}u = 0, \quad \mathcal{B}u = g, \tag{3-10}$$

is uniquely solvable with solution operator $\mathcal{N}g = u$. Then the difference $v := y - \mathcal{N}g$ satisfies

$$\dot{v} = Av - \mathcal{N}\dot{g},$$

and v solves a partial differential equation with homogeneous boundary conditions, since $\mathcal{B}v = 0$. Formally, the mild solution of this equation is given by

$$v_t = S_t v_0 + \int_0^t S_{t-r}\mathcal{N}\dot{g} \,\mathrm{d}r = S_t y_0 - \mathcal{N}g(t) + \int_0^t AS_{t-r}\mathcal{N}g(r) \,\mathrm{d}r$$

using $v_0 = y_0 - \mathcal{N}g(0)$, integration by parts and $\frac{\mathrm{d}}{\mathrm{d}t}S_t = AS_t$. This shows that we can express the mild solution y as

$$y_t = S_t y_0 + \int_0^t AS_{t-r}\mathcal{N}g(r) \,\mathrm{d}r.$$

Using Theorem C.11, we can swap the operator A and the semigroup by replacing A with its extrapolated operator A_{-1}. This leads to the observation that (3-9) is, in a mild sense, equivalent to

$$\begin{cases} \dot{y} &= Ay + A_{-1}\mathcal{N}g \quad \text{in} \quad \mathcal{O}, \\ \mathcal{B}y &= 0 \quad \text{on} \quad \partial\mathcal{O}. \end{cases}$$

We now replace g by a noise term, such as the derivative of a Brownian motion, see [DPZ93]. For that, consider a smooth basis $(e_l)_{l\in\mathbb{N}}$ of $\mathrm{L}^2([0,T];\mathbb{R})$ and a sequence of independent standard normal distributed random variables $(\xi_l)_{l\in\mathbb{N}}$. Then define $B_t^{(k)} := \sum_{l=1}^k \xi_l e_l(t)$ and $g = \dot{B}^{(k)}$. We obtain, using the same strategy as above, that

$$y_t^{(k)} = S_t y_0 + \int_0^t AS_{t-r}\mathcal{N}\dot{B}_r^{(k)} \,\mathrm{d}r$$

is the mild solution to the boundary value problem

$$\begin{cases} \dot{y^{(k)}} &= \mathcal{A}y^{(k)} \quad \text{in} \quad \mathcal{O}, \\ \mathcal{B}y^{(k)} &= \dot{B}^{(k)} \quad \text{on} \quad \partial\mathcal{O}. \end{cases} \tag{3-11}$$

Now, Da Prato and Zabczyk have shown in [DPZ93, Section 2] that

$$\int_0^t AS_{t-r}\mathcal{N}\dot{B}_r^{(k)} \,\mathrm{d}r \to \int_0^t AS_{t-r}\mathcal{N} \,\mathrm{d}B_r,$$

in L^2 for all $t \in [0,T]$ under certain conditions on the adjoint of the integrand $\mathcal{N}^* S_r^* A^*$. This shows that

$$y_t = S_t y_0 + \int_0^t S_{t,r} A_{-1} \mathcal{N} \, \mathrm{d}B_r,$$

is the mild solution to (3-11). In particular, this proves the equivalence of

$$\begin{cases} \dot{y} &= \mathcal{A} y \quad \text{in} \quad \mathcal{O}, \\ \mathcal{B} y &= \dot{B} \quad \text{on} \quad \partial\mathcal{O}, \end{cases}$$

to the stochastic partial differential equation

$$\mathrm{d}y = Ay + A_{-1}\mathcal{N} \, \mathrm{d}B.$$

The procedure in the stochastic case of Da Prato and Zabczyk suggests that such an equivalence can also be shown for rough partial differential equations as in (BN). More precisely, let $\mathbf{X}$ be a geometric rough path, so there exists a sequence $\left(\mathbf{X}^{(k)}\right)_{k\in\mathbb{N}} = \left(X^{(k)}, \mathbb{X}^{(k)}\right)_{k\in\mathbb{N}} \subset \mathcal{L}(\mathcal{C}^\infty)$ with $\mathbf{X}^{(k)} \to \mathbf{X}$ for $k \to \infty$, in the rough path metric. Consider, as in [DPZ93]

$$\begin{cases} \dot{y}_t &= \mathcal{A} y_t \quad \text{in} \quad \mathcal{O}, \\ \mathcal{B} y_t &= G(t, y_t) \dot{X}_t^{(k)} \quad \text{on} \quad \partial\mathcal{O}. \end{cases}$$

The above computation, and the fact that $X^{(k)} \in \mathcal{C}^\infty$, leads to the mild formulation

$$y_t = S_t y_0 + \int_0^t A S_{t-r} \mathcal{N} G(r, y_r) \dot{X}_r^{(k)} \, \mathrm{d}r = S_t y_0 + \int_0^t A S_{t-r} \mathcal{N} G(r, y_r) \, \mathrm{d}X_r^{(k)},$$

where the integral is defined as a Riemann–Stieltjes integral. In particular, since $X^{(k)}$ is smooth, the right-hand side is even equal to the respective rough integral if the integrand is a controlled rough path, which is proven in the following subsection. Since the rough integral depends continuously on the noise, recall Lemma 2.45, we obtain

$$S_t y_0 + \int_0^t A S_{t-r} \mathcal{N} G(r, y_r) \, \mathrm{d}\mathbf{X}_r^{(k)} \to S_t y_0 + \int_0^t A S_{t-r} \mathcal{N} G(r, y_r) \, \mathrm{d}\mathbf{X}_r.$$

To summarize, due to the stability of the rough integral, the mild formulation of (BN) is given by

$$y_t = S_t y_0 + \int_0^t A S_{t-r} \mathcal{N} G(r, y_r) \, \mathrm{d}\mathbf{X}_r, \tag{3-12}$$

as long as integrand of the rough convolution together with its Gubinelli derivative form a controlled rough path.

3.3.2 Existence and uniqueness

As we have seen in Subsection 3.3.1, we have to make sense of the stochastic convolution involving the Neumann operator. In particular,

$$\int_0^t S_{t-r} \mathcal{N} G(r, y_r) \, \mathrm{d}\mathbf{X}_r \in D(A),$$

has to be true, in order to investigate (BN) in the space of controlled rough paths.

The main result in this section is to prove this regularity statement and to determine a meaningful Gubinelli derivative. We have already laid the foundations for this in Subsection 2.3.4 by investigating the interaction between controlled rough paths and extrapolated operators. Furthermore, we continue to investigate the transformation of the evolution equation with boundary noise to a rough partial differential equation with homogeneous boundary conditions, which was already mentioned in Subsection 3.3.1 and is particularly important for the global existence of the solution. This all ends in an existence result for equations with rough boundary noise.

Therefore, let us first specify the scales of Banach spaces that are required in our framework. Let $p \geq 2$, $2 > 2\alpha > 1 + 1/p$ and set $E := \mathrm{L}^p(\mathcal{O})$. In this case, we define the spaces

$$\widetilde{E}_\beta := \mathrm{B}_{p,p}^{\beta-1-\frac{1}{p}}(\partial\mathcal{O})$$

for $\beta \in \mathbb{R}$ and let $(A_\beta, E_\beta)_{\beta\in[-2,\infty)}$ be the interpolation-extrapolation scale generated by $A\colon D(A) \subset E \to E$, see Example C.12. Further, we denote by $\mathcal{N}$ the solution operator, called the Neumann map of the boundary value problem associated to $(\mathcal{A}, \mathcal{B})$, see Appendix D.2 for more detailed information.

Corollary 3.13. *Let $p < \frac{1}{1-2\gamma}$, $0 < \epsilon < 1/2 + 1/2p$ and $(y, y') \in \widetilde{\mathcal{D}}^\gamma_{X,\alpha}$. Then*

$$\int_0^t S_{t-r}\mathcal{N}y_r \,\mathrm{d}\mathbf{X}_r,$$

is well-defined and belongs to $D(A)$ for every $t \in [0, T]$.

Proof. We know that $\mathcal{N} \in \mathcal{L}\big(\widetilde{E}_\beta; E_{\beta/2}\big)$ for every $\beta/2 \in [0, 1]$ such that $\beta \neq 1/p, 1 + 1/p$ due to Theorem D.12. In particular, $\mathcal{N}y$ is a strong solution of the boundary value problem associated to $\mathcal{N}$ for every $y \in E_\alpha$, due to the assumption on α. Since $\epsilon < 1/2 + 1/2p < \alpha$ we have $E_{\alpha-i\gamma/2} \hookrightarrow E_{\epsilon-i\gamma}$ for $i = 0, 1, 2$. So, $\mathcal{N}$ satisfies the assumptions of Lemma 2.50 and Corollary 2.44. Therefore, we obtain for every $\varsigma \in [0, \gamma)$ that $\big(\int_0^\cdot S_{\cdot-r}\mathcal{N}y_r \,\mathrm{d}\mathbf{X}_r, Ny\big) \in \mathcal{D}^\gamma_{X,\epsilon+\varsigma}$. If we choose $\varsigma = \gamma - \delta$, for some small $\delta > 0$, we have

$$\frac{1}{p} > 1 - 2\gamma = 1 - 2\varsigma + 2\delta \quad \iff \quad \frac{1}{2} + \frac{1}{2p} - \delta + \varsigma > 1,$$

using the assumption on p. Therefore, we obtain $\int_0^t S_{t-r}\mathcal{N}y_r \,\mathrm{d}\mathbf{X}_r \in D(A)$ choosing $\epsilon = 1/2 + 1/2p - \delta < 1/2 + 1/2p$. □

Remark 3.14. i) The essential step in the proof of Corollary 3.13 is that we can choose an ϵ such that $\epsilon > 1 - \gamma$. In the case of Neumann conditions, we have proven that this is possible. But for Dirichlet boundary conditions, we can choose ϵ only up to $1/2p$. This comes from the fact that the Dirichlet map is only bounded from $\mathrm{B}^{\beta-1/p}_{p,p}(\partial\mathcal{O})$ to $\mathrm{H}^{\beta,p}(\mathcal{O})$ and provides a strong solution for $\beta > 1/p$. Since $p \geq 2$ and $\gamma < 1/2$, it is not possible to find an ϵ such that $\epsilon > 1 - \gamma$ and $\epsilon < 1/2p$ to hold simultaneously.

ii) In the Young regime, this means $\widetilde{\gamma} \in (1/2, 1)$, we can incorporate Dirichlet boundary noise since the conditions $\epsilon > 1 - \widetilde{\gamma}$ and $\epsilon < 1/2p$ can simultaneously be fulfilled. For additive fractional noise, it is known that Dirichlet boundary conditions can be incorporated provided that $H \in (3/4, 1)$ as established in [DPDM02]. We provide further details on the well-posedness of (BN) with multiplicative Dirichlet boundary noise in Theorem 3.25.

From now on we assume that p, γ and ϵ are chosen as in Corollary 3.13, such that the rough convolution $\int_0^t S_{t-r}\mathcal{N}y_r \,\mathrm{d}\mathbf{X}_r$ is an element in the domain of A.

Before we proceed, note that the operator A being in front of the rough integral introduces some complications. It is much easier to handle if the integral and the operator are swapped, as in (3-12). In particular, it is not possible to show that (BN) has a global solution working with the controlled rough path where the path component is $A\int_0^t S_{t-r}\mathcal{N}y_r\,\mathrm{d}\mathbf{X}_r$, even though we could establish a local solution using a fixed-point argument. Therefore, we prove first that it is possible to swap the operator A and the rough integral, see [MP07] for an analogous result for additive fractional noise. As a consequence of Theorem 2.42, the integral exists as an element in $E_{\epsilon-2\gamma+\beta}$ for every $\beta \in [0, 3\gamma)$. So the equality

$$\widetilde{A}\int_0^t S_{t-r}\mathcal{N}y_r\,\mathrm{d}\mathbf{X}_r = \int_0^t \widetilde{A}S_{t-r}\mathcal{N}y_r\,\mathrm{d}\mathbf{X}_r, \tag{3-13}$$

holds for a bounded, and therefore continuous, operator $\widetilde{A}$ with domain $E_{\epsilon-2\gamma+\beta}$. In particular, (3-13) holds for $\beta := 2\gamma-\epsilon+1 < 3\gamma$ and for $A \in \mathcal{L}(E_1;E_0)$, due to our restriction on ϵ. Using Theorem C.11, we can swap A and the analytic semigroup to obtain

$$A\int_0^t S_{t-r}\mathcal{N}y_r\,\mathrm{d}\mathbf{X}_r = \int_0^t S_{t-r}A_{\epsilon-1}\mathcal{N}y_r\,\mathrm{d}\mathbf{X}_r, \tag{3-14}$$

using the extrapolated operator $A_{\epsilon-1}$.

Remark 3.15. To make sure that the right-hand side is well-defined as a controlled rough integral, we need to find a Gubinelli derivative for $A_{\epsilon-1}\mathcal{N}y$. A natural choice would be $A_{\epsilon-1}\mathcal{N}y'$, but since y' loses spatial regularity, this term is not well-defined. Therefore, we need to lift the extrapolated operator. In the end, one can show using Corollary 2.51 that

$$\big(A_{\epsilon-1}\mathcal{N}y, A_{\epsilon-1-\gamma}\mathcal{N}y'\big) \in \mathcal{D}^{\gamma}_{X,\epsilon-1}.$$

To avoid working with two different indices for the extrapolation operator in the path component and its Gubinelli derivative, we rely again on Corollary 2.51, which leads to the next result.

Corollary 3.16. *For every $\big(y,y'\big) \in \widetilde{\mathcal{D}}^{\gamma}_{X,\alpha}$ we have*

$$\big(A_{\epsilon-1-\gamma}\mathcal{N}y, A_{\epsilon-1-\gamma}\mathcal{N}y'\big) \in \mathcal{D}^{\gamma}_{X,\epsilon-1-\gamma}.$$

Consequently, this allows us to define the rough convolution in the sense of controlled rough paths.

Lemma 3.17. *For every $(y,y') \in \widetilde{\mathcal{D}}^{\gamma}_{X,\alpha}$ the right-hand side of (3-14) is well-defined and we have*

$$\left(\int_0^{\cdot} S_{\cdot-r}A_{\epsilon-1-\gamma}\mathcal{N}y_r\,\mathrm{d}\mathbf{X}_r, A_{\epsilon-1-\gamma}\mathcal{N}y\right) \in \mathcal{D}^{\gamma}_{X,\epsilon-1}.$$

Since the solution operator is applied on functions defined on the boundary, and the solution of (BN) itself is a function on the whole domain, the nonlinearity G needs to map from $\big(\widetilde{E}_\beta\big)_{\beta\in\mathbb{R}}$ to $(E_\beta)_{\beta\in\mathbb{R}}$ satisfying **(G1)**. Then Lemma 2.48 shows that the composition of G and a controlled rough path is again a controlled rough path. Therefore, we obtain that (BN) is equivalent to the semilinear evolution equation without boundary noise

$$\begin{cases} \mathrm{d}y = \big(Ay + F(t,y)\big)\,\mathrm{d}t + A_{\epsilon-1-\gamma}\mathcal{N}G(t,y)\,\mathrm{d}\mathbf{X}_t, \\ y_0 \in E_{\epsilon-1}, \end{cases} \tag{3-15}$$

using (3-14) and Lemma 3.17 as highlighted in Subsection 3.3.1. This leads to the following solution concept, which is used throughout this subsection.

Definition 3.18. We call $(y, y') \in \mathcal{D}^{\gamma}_{X,\epsilon-1}$ a mild solution to (BN) if $y' = A_{\epsilon-1-\gamma}\mathcal{N}G(\cdot, y)$ and the path component satisfies

$$y_t = S_t y_0 + \int_0^t S_{t-r}F(r, y_r)\ \mathrm{d}r + \int_0^t S_{t-r}A_{\epsilon-1-\gamma}\mathcal{N}G(r, y_r)\ \mathrm{d}\mathbf{X}_r, \tag{3-16}$$

for every $t \in [0, T]$.

Remark 3.19. The idea to rewrite (BN) as a semilinear problem without boundary noise as in (3-15) using an extrapolation operator was also applied in [SV11]. We note that the index of the extrapolation operator needed there is $\tilde{\alpha}/2 - 1$ where $\tilde{\alpha} \in (1, 1 + 1/p)$. Therefore, our result is consistent with the one in [SV11] for Brownian noise, since in both cases the extrapolation index satisfies

$$\epsilon - 1, \frac{\tilde{\alpha}}{2} - 1 \in \left(-\frac{1}{2}, -\frac{1}{2} + \frac{1}{2p}\right)$$

due to the restriction $\epsilon > 1 - \gamma$ and $\gamma \in (1/3, 1/2]$.

We can now state the local-in-time existence of a solution to (BN). Note that $\tilde{\sigma}_G \le 0$ is negative, which means that the diffusion coefficient has to lift the spatial regularity, which is necessary due to the presence of the extrapolation operator. In combination, the regularity loss due to $A_{\epsilon-1-\gamma}\mathcal{N}$ is then compensated by G.

Theorem 3.20. ([NS23, Theorem 3.16]) *Assume that there exists a $\tilde{\sigma}_G \le 0$ satisfying $-\tilde{\sigma}_G > 1 - \epsilon + 1 + 1/p$ such that **(G1)** holds and F satisfies Assumption **(F1)**. Then there exists for every initial condition $y_0 \in E_{\epsilon-1}$ a time $T^* \le T$ and a unique solution $(y, A_{\epsilon-1-\gamma}\mathcal{N}G(\cdot, y)) \in \mathcal{D}^{\gamma}_{X,\epsilon-1}([0, T^*))$ to (BN) such that the path component satisfies (3-16) for all $t < T^*$.*

Proof. We need to show that $\widetilde{G} := A_{\epsilon-1-\gamma}\mathcal{N}G$ satisfies **(G1)** for $\sigma_G = 0$. Then the claim follows from Theorem 3.2.

Let $t \in [0, T]$. Due to $\epsilon - 1 - \tilde{\sigma}_G > 1 + 1/p$ we obtain $\mathcal{N}G(t, \cdot)\colon E_{\epsilon-1} \to E_{\epsilon}$ using the definition of the Neumann map. Due to Remark C.2 we obtain $A_{\epsilon-1-\gamma}\mathcal{N}G(t, \cdot) = A_{\epsilon-1}\mathcal{N}G(t, \cdot)$, therefore, $\widetilde{G}(t, \cdot)\colon E_{\epsilon-1} \to E_{\epsilon-1}$ is well-defined. Since $\mathcal{N}G(t, \cdot)\colon E_{\epsilon-1-i\gamma} \to E_{\epsilon-i\gamma}$ is also valid, we conclude further that $\widetilde{G}\colon E_{\epsilon-1-i\gamma} \to E_{\epsilon-1-i\gamma}$ is well-defined for $i = 1, 2$. Note that $A_{\epsilon-1-\gamma}\mathcal{N}G(t, y_t) = A_{\epsilon-1-2\gamma}\mathcal{N}G(t, y_t)$ since $\mathcal{N}G(t, y_t) \in E_{\epsilon-\gamma}$, which follows from the key property of the abstract Banach scales, see Remark C.2. Therefore, we obtain by combining Lemma 2.50 and Corollary 2.51 that $\widetilde{G}$ satisfies **(G1)** for $\sigma_G = 0$. Then Lemma 2.48 yields

$$\left(\widetilde{G}(\cdot, y), \mathrm{D}_2 G(\cdot, y)\widetilde{G}(\cdot, y)\right) \in \mathcal{D}^{\gamma}_{X,\epsilon-1},$$

which proves the claim. □

Remark 3.21. Due to the rough path techniques, the assumptions on the diffusion coefficient G differ from those in [SV11], where G maps into E_{-1}, is Lipschitz continuous, and satisfies a linear growth condition. Therefore, we assume a higher degree of differentiability for the diffusion coefficient than is typically assumed. Such issues are common for rough convolutions, in particular for global-in-time existence, and have also been encountered in [HN19, GALS16, Tap25]. However, the advantage of our approach is that we can consider more general noise, including a non-Markovian process, in contrast to [SV11], where only the Brownian motion is investigated.

To prove the global-in-time existence of the solution, we need an additional assumption to avoid the quadratic terms appearing by the composition of a controlled rough path with a smooth function, similar to Lemma 3.6.

($\mathbf{G_N}$) There exists a $\tilde\sigma_G \le 0$ satisfying $-\tilde\sigma_G > 1-\epsilon+1+1/p$ such that **(G1)** holds. Further, the derivative of

$$\mathrm{D}_2 G(t,\cdot)\widetilde{G}(t,\cdot)\colon E_{\epsilon-1-\gamma} \to \widetilde{E}_{\epsilon-1-2\gamma-\tilde\sigma_G},$$

is bounded for every $t\in[0,T]$, where $\widetilde{G} \coloneqq A_{\epsilon-1-\gamma}\mathcal{N}G$.

Remark 3.22. The assumption stated in [NS23], demanded that $\mathrm{D}_2G(t,\cdot)\widetilde{G}(t,\cdot)$ maps to $\widetilde{E}_{\epsilon-1-\gamma-\tilde\sigma_G}$, which is a stronger assumption because of the embedding

$$\widetilde{E}_{\epsilon-1-\gamma-\tilde\sigma_G} \hookrightarrow \widetilde{E}_{\epsilon-1-2\gamma-\tilde\sigma_G}.$$

However, it turns out that the condition in **($\mathbf{G_N}$)** is sufficient for global existence, as also seen in [HN22].

Based on Assumption **($\mathbf{G_N}$)**, we can now state the global-in-time existence.

Theorem 3.23. ([NS23, Theorem 3.20]) *Assume that F and G satisfy* ***(F1)*** *and* ***($\mathbf{G_N}$)****. Then there exists for every initial condition $y_0\in E_{\epsilon-1}$ a unique solution*

$$\big(y, A_{\epsilon-1-\gamma}\mathcal{N}G(\cdot,y)\big) = \big(y,\widetilde{G}(\cdot,y)\big) \in \mathcal{D}^{\gamma}_{X,\epsilon-1}([0,T])$$

to (BN) *such that the path component satisfies* (3-16) *for all $t\le T$.*

Proof. Let $t\in[0,T]$. We only need to show that **($\mathbf{G_N}$)** implies **(G2)**. Indeed, note that $\mathrm{D}_2\widetilde{G}(t,\cdot) = A_{\epsilon-1-\gamma}\mathcal{N}\mathrm{D}_2G(t,\cdot)$ holds for every $t\in[0,T]$, since $\mathcal{N}$ and the extrapolated operator are linear. This leads to

$$\mathrm{D}_2\widetilde{G}(t,\cdot)\widetilde{G}(t,\cdot) = A_{\epsilon-1-\gamma}\mathcal{N}\big(\mathrm{D}_2G(t,\cdot)\widetilde{G}(t,\cdot)\big).$$

Due to **($\mathbf{G_N}$)**, we obtain $\mathrm{D}_2\widetilde{G}(t,\cdot)\widetilde{G}(t,\cdot)\colon E_{\epsilon-1-\gamma}\to E_{\epsilon-1-2\gamma}$, where the derivative is bounded since the derivative of $\mathrm{D}_2G(t,\cdot)\widetilde{G}(t,\cdot)$ is bounded. Therefore, G satisfies Assumption **($\mathbf{G_N}$)**. The claim follows from Theorem 3.9. □

Furthermore, we obtain a similar stability result of the solution for the initial condition and the noise term as in Theorem 3.4.

Corollary 3.24. *Assume that F and G satisfy* ***(F1)*** *and* ***($\mathbf{G_N}$)****. Let $y_0,\hat y_0 \in E_{\epsilon-1}$ be two initial conditions, and let $\big(y,\widetilde{G}(\cdot,y)\big)\in\mathcal{D}^{\gamma}_{X,\epsilon-1}$, $\big(\hat y,\widetilde{G}(\cdot,\hat y)\big)\in\mathcal{D}^{\gamma}_{\widehat{X},\epsilon-1}$ be the mild solutions of* (BN) *driven by $\mathbf{X}$ respectively $\widehat{\mathbf{X}}$ with initial conditions y_0 and $\hat y_0$. If $\varrho_{\gamma,[0,T]}(\mathbf{X})$, $\varrho_{\gamma,[0,T]}(\widehat{\mathbf{X}})$, $\|y_0\|_{\epsilon-1}$, $\|\hat y_0\|_{\epsilon-1}$ are bounded by the same constant, then for every $1/3<\gamma'<\gamma$ we have*

$$d^{\gamma',\gamma}_{[0,T],\epsilon-1}(y,\hat y) \lesssim \varrho_{\gamma,[0,T]}\big(\mathbf{X},\widehat{\mathbf{X}}\big) + \|y_0-\hat y_0\|_{\epsilon-1}.$$

The Young case and Dirichlet boundary conditions

As mentioned earlier in Remark 3.14, we are not able to treat Dirichlet boundary conditions using a noise that is γ-Hölder continuous for $\gamma\le 1/2$, similar to [DPZ93]. In the case of a fractional Brownian motion $\big(B^H_t\big)_{t\ge0}$ with Hurst parameter H on the boundary, it is known that the equation can be solved in the Dirichlet case if the Hurst parameter is big enough [DPDM02]. Therefore, we expect the same for more general noise terms. For paths $X\in\mathcal{C}^{\tilde\gamma}([0,T];\mathbb{R})$ with $\tilde\gamma\in(1/2,1)$ we do not need any rough path techniques, since all integrals are well-defined using the theory of Young. For the sake of completeness, we give the results in this regime without repeating the proofs, since everything can be handled similarly if we simply ignore the terms involving $\mathbb{X}$.

So, let $X \in \mathcal{C}^{\tilde{\gamma}}([0,T];\mathbb{R})$ for $\tilde{\gamma} \in (1/2, 1)$. We denote by $\mathfrak{D}$ the solution operator of the boundary value problem $(\mathcal{A}, \gamma_\partial)$. In this case, the interpolation-extrapolation scale generated by the L^p-realization A of $(\mathcal{A}, \gamma_\partial)$ is different, see (C-6), and is denoted by $\big(A_\beta, E^D_\beta\big)_{\beta \geq -m}$. Further, define $\widetilde{E}^D_\beta := B^{\beta - 1/p}_{p,p}(\partial\mathcal{O})$. Then, the Dirichlet map $\mathfrak{D}$ is bounded from $\widetilde{E}^D_\beta$ to $E^D_{\epsilon_D}$ for every $\epsilon_D < 1/2p$. Furthermore, the boundary value problem (3-10), with $\mathcal{B}$ substituted by γ_∂, has a strong solution for $2\beta > 1/p$. Then we can define the rough convolution as a Young integral

$$\int_0^t S_{t-r} y_r \,\mathrm{d}X_r := \lim_{|\pi| \to 0} \sum_{[u,v] \in \pi} S_{t-u} y_u X_{u,v},$$

for every $y \in \mathcal{C}\big(E^D_\beta\big) \cap \mathcal{C}^{\tilde{\gamma}}\big(E^D_{\beta - \tilde{\gamma}}\big)$ as in Theorem 2.2. Then the Dirichlet map satisfies $\mathfrak{D}\colon \widetilde{E}^D_{\beta - \tilde{\gamma}/2} \to E^D_{\epsilon_D - \tilde{\gamma}}$ for every $2\beta > 1 + 1/p$. It is not possible to simply assume $2\beta > 1/p$, as one would expect in the Dirichlet case, because then $\beta - \tilde{\gamma}/2$ can become negative. However, we need $\beta - \tilde{\gamma}/2 > 0$, because then Theorem D.12 provides us with a solution.

Furthermore, $\tilde{\gamma} > 1 - \epsilon_D$ needs to be satisfied in order to ensure $\int_0^t S_{t-r} y_r \,\mathrm{d}X_r \in D(A)$. In the Neumann case, this condition holds for rough noise $\gamma \in (1/3, 1/2]$ if $p \leq \frac{1}{1-2\gamma}$. In the Young regime, we need $1 - 1/2p < \tilde{\gamma} < 1$ to be fulfilled, compare [DPDM02]. Under these assumptions, we obtain the equivalent rough partial differential equation

$$\begin{cases} \mathrm{d}y = \big(Ay + F(t,y)\big)\,\mathrm{d}t + A_{\epsilon_D - \tilde{\gamma} - 1}\mathfrak{D}G(t,y)\,\mathrm{d}X_t, \\ y_0 \in E^D_{\epsilon_D - 1}. \end{cases} \tag{3-17}$$

To stick to the framework presented in this thesis, we impose the following conditions on G to solve (3-17).

($\mathbf{G_D}$) There exists a $\tilde{\sigma}^D_G \leq 0$ satisfying $-\tilde{\sigma}^D_G > 1 - \epsilon_D + 1 + 1/p$ such that

$$G\colon [0,T] \times E_{\epsilon_D - 1 - i\tilde{\gamma}} \to \widetilde{E}^D_{\epsilon_D - 1 - i\tilde{\gamma} - \tilde{\sigma}^D_G},$$

satisfies for every $i = 0, 1$ the following conditions:

i) For every $t \in [0,T]$, $G(t,\cdot)$ is twice continuously Fréchet differentiable, where the derivatives are bounded uniformly in time.

ii) For every $x \in E^D_{\epsilon_D - 1 - i\tilde{\gamma}}$, $G(\cdot, x)$, as well as all existing Fréchet derivatives, are $2\tilde{\gamma}$-Hölder continuous, and the Hölder constants are uniformly bounded in $E_{\epsilon_D - 1 - i\tilde{\gamma}}$.

The additional assumption to ensure global existence is not needed in the Young regime, since we do not have to investigate any remainder or Gubinelli derivative.

Theorem 3.25. ([NS23, Theorem 3.24]) *Let $X \in \mathcal{C}^{\tilde{\gamma}}([0,T];\mathbb{R})$ with $\tilde{\gamma} \in (1 - 1/2p, 1)$. Assume that F and G satisfy Assumption **(F1)** and **($\mathbf{G_D}$)**. Then there exists for every initial condition $y_0 \in E^D_{\epsilon_D - 1}$ a unique mild solution $y \in \mathcal{C}\big(E^D_{\epsilon_D - 1}\big) \cap \mathcal{C}^{\tilde{\gamma}}\big(E^D_{\epsilon - 1 - \tilde{\gamma}}\big)$ that satisfies*

$$y_t = S_t y_0 + \int_0^t S_{t-r} F(r, y_r)\,\mathrm{d}r + \int_0^t S_{t-r} A_{\epsilon_D - 1 - \tilde{\gamma}} \mathfrak{D}G(r, y_r)\,\mathrm{d}X_r,$$

for all $t \in [0,T]$, where the integral is defined in the sense of Young (2.2).

3.3.3 Examples

To conclude this section, we add possible diffusion coefficients, which satisfy **($\mathbf{G_N}$)** or **($\mathbf{G_D}$)**. For simplicity, assume that $p = 2$.

Young regime

We consider the Dirichlet case $(\mathcal{A}, \gamma_\partial)$ and the respective realization A. For the regularity of the noise, we take $\tilde{\gamma} \in (3/4, 1)$, such that the condition $\epsilon_D > 1 - \tilde{\gamma}$ is satisfied. To verify the Assumption **($\mathbf{G_D}$)**, we first investigate the extrapolation spaces. Recall that they are given by

$$E^D_\beta := \begin{cases} \left\{u \in \mathrm{H}^{2\beta,2}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\right\}, & 2\beta > \frac{1}{2}, \\ \mathrm{H}^{2\beta,2}(\mathcal{O}), & -\frac{3}{2} < 2\beta < \frac{1}{2}, \\ \left\{u \in \mathrm{H}^{-2\beta,2}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\right\}', & 2\beta < -\frac{3}{2}. \end{cases}$$

Since $\varepsilon_D - 1 - \tilde{\gamma} > -2\tilde{\gamma} > -2$, we are interested in E^D_{-2}. Now, it is known that E^D_{-2} can be characterized by the dual space of $D\big(A^2\big)$, see Theorem C.10. But due to Proposition A.11 we obtain $\big(\mathrm{H}^{4,2}_0(\mathcal{O})\big)' = \mathrm{H}^{-4,2}(\mathcal{O})$. Combined with

$$\mathrm{H}^{4,2}_0(\mathcal{O}) \hookrightarrow D\big(A^2\big) = E^D_2,$$

we can E^D_{-2} continuously embed into $H^{-4,2}(\mathcal{O})$. Based on these considerations, it is sufficient to find a linear continuous mapping from $\mathrm{H}^{-4,2}(\mathcal{O})$ to $\mathrm{H}^{\tilde{\delta}+1,2}(\partial\mathcal{O}) = \mathrm{B}^{\delta+1}_{2,2}(\partial\mathcal{O})$ for a small $\delta > 0$.

Define the operator

$$\Lambda^{\nu_\mathfrak{D}} : \mathrm{H}^{-4,2}(\mathbb{R}^n) \to \mathrm{H}^{\delta+1+\frac{1}{2},2}(\mathbb{R}^n), f \mapsto \mathscr{F}^{-1}(1 + |\cdot|^2)^{\frac{\nu_\mathfrak{D}}{2}} \mathscr{F} f,$$

where $\mathscr{F}$ is the Fourier transform and $\nu_\mathfrak{D} := -11/2 - \delta$. Such operators occur in the theory of pseudo-differential operators, see for example [Hö07, Theorem 18.1.13]. To extend this to a map on the bounded domain $\mathcal{O}$, we use a retraction/co-retraction argument. More precisely, consider the linear and continuous operators

$$e_\mathcal{O} \colon \mathrm{H}^{-4,2}(\mathcal{O}) \to \mathrm{H}^{-4,2}\big(\mathbb{R}^n\big), \quad r_\mathcal{O} \colon \mathrm{H}^{\tilde{\delta}+1+\frac{1}{2},2}\big(\mathbb{R}^n\big) \to \mathrm{H}^{\tilde{\delta}+1+\frac{1}{2},2}(\mathcal{O}),$$

see [Tri78, Theorem 4.2.2]. Since the trace operator

$$\gamma_\partial \colon \mathrm{H}^{\tilde{\delta}+1+\frac{1}{2},2}(\mathcal{O}) \to \mathrm{H}^{\tilde{\delta}+1,2}(\partial\mathcal{O})$$

is also linear and continuous due to Proposition A.12, we conclude that the operator $G := \gamma_\partial r_\mathcal{O} \Lambda^{\nu_\mathfrak{D}} e_\mathcal{O}$ satisfies **($\mathbf{G_N}$)**.

Rough regime

Now consider the original boundary value problem (BN), with Neumann boundary conditions. Let A be the L^2-realization of $(\mathcal{A}, \mathcal{B})$ and $\gamma \in (1/3, 1/2]$. We recall the characterization of the extrapolation spaces associated with the interpolation-extrapolation scale generated by A

$$E_\beta = \begin{cases} \left\{u \in \mathrm{H}^{2\beta,2}(\mathcal{O}) : \mathcal{B}u = 0\right\}, & 2\beta > \frac{3}{2}, \\ \mathrm{H}^{2\beta,2}(\mathcal{O}), & -\frac{1}{2} < 2\beta < \frac{3}{2}, \\ \big(\mathrm{H}^{-2\beta,2}(\mathcal{O})\big)', & -\frac{3}{2} < 2\beta \le -\frac{1}{2}, \\ \left\{u \in \mathrm{H}^{-2\beta,2}(\mathcal{O}) : \mathcal{B}u = 0\right\}', & 2\beta < -\frac{3}{2}. \end{cases} \tag{3-18}$$

In particular, we are interested in $E_{\epsilon-1-2\gamma}$. Since $\epsilon - 1 > -\gamma$ we have

$$\epsilon - 1 - 2\gamma > -3/2 > -2$$

and therefore $E_{\epsilon-1-2\gamma} \hookrightarrow E_{-2}$. Similar to the example in the Young regime, it holds that $E_{-2} \hookrightarrow \mathrm{H}^{-4,2}(\mathcal{O})$. Now we define $G := \gamma_\partial r_{\mathcal{O}} \Lambda^\nu e_{\mathcal{O}}$ in the same way as before, but set $\nu := -9/2 - \delta$ for some small $\delta > 0$. In this case, G maps $\mathrm{H}^{-4,2}(\mathcal{O})$ into $\mathrm{H}^{\tilde{\delta},2}(\partial\mathcal{O})$. Since G is linear and bounded, the same holds for $A_{\epsilon-1-\gamma}\mathcal{N}G$. This means that F satisfies the Assumption **($\mathbf{G_N}$)** and Theorem 3.23 entails a global-in-time solution.

Remark 3.26. Before we conclude this subsection, we compare our results to the ones for additive infinite-dimensional fractional noise in Hilbert spaces. Similar results have also been derived in Banach spaces, see [CMO22, Section 5.2].

To this aim, let U and V be two separable Hilbert spaces. For a U-cylindrical fractional Brownian motion $(B_t^H)_{t\geq 0}$ it is known that the stochastic convolution

$$\int_0^t S_{t-r}\Phi \, \mathrm{d}B_r^H,$$

is well-defined if $\|S(t)\Phi\|_{\mathcal{L}_2(U;V)} < t^{-a}$ holds for some $a < H$, where $\mathcal{L}_2(U;V)$ denotes the space of Hilbert-Schmidt operators from U to V and $\Phi \in \mathcal{L}(U;V)$. See [DPDM02, Corollary 3.1] for the case $H > 1/2$ and [DPDM06, Corollary 11.9] for $H < 1/2$. Consequently, to incorporate boundary noise given by an $U := \mathrm{L}^2(\partial\mathcal{O})$-cylindrical fractional Brownian motion with covariance operator $Q^{1/2} \in \mathcal{L}_2(U)$, one has to verify that

$$\left\| AS(t)\mathcal{N}Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(U;V)} \leq t^{-a},$$

where $V := \mathrm{L}^2(\mathcal{O})$ and $a < H$. For $E_\epsilon := D(A^\epsilon)$, we obtain

$$\left\| AS_t\mathcal{N}Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(U;V)} \leq \|AS_t\|_{\mathcal{L}(E_\epsilon;V)} \|\mathcal{N}\|_{\mathcal{L}(U;E_\epsilon)} \left\| Q^{\frac{1}{2}} \right\|_{\mathcal{L}_2(U;U)} \lesssim t^{\epsilon-1}.$$

This means that if we have $\epsilon > 1 - H$, then the stochastic partial differential equation driven by the fractional Brownian motion on the boundary admits a solution. Since $\epsilon < 3/4$, recall Corollary 3.13; this means that it is possible to deal with fractional Neumann boundary noise for $H > 1/4$ and even with Dirichlet boundary conditions if $H > 3/4$. The condition $\epsilon > 1 - H$ is consistent with the result of Corollary 3.13, which is established in the more general setting of rough path theory.

3.4 Integrable solution bounds

Up to now, the rough path $\mathbf{X}$ was mostly deterministic, apart from a few side notes. In this section, we will now investigate a stochastic process $\mathbf{X}$, and answer the question of under which conditions the solution of (RPDE) is integrable.

This issue is already addressed in [CLL13], where known a-priori bounds for the solutions of rough differential equations were adjusted to make use of the probabilistic properties of Gaussian rough paths to prove the integrability. Subsequently, the results in [CLL13] were refined and broadened to encompass a wider class of rough differential equations in [FR13]. However, for rough partial differential equations, the challenge of identifying an integrable a-priori bound has remained unresolved for a long time. Indeed, in [GH19, Page 51] it was stated:

"The [integrable] moment bounds for the rough path norms of solution and Jacobian (.....) [for the RPDE] might not be easy to obtain in general and require a closer look as a separate problem on its own."

Recently, this gap was filled by [GVR25] for autonomous rough partial differential equations, which was extended to the non-autonomous setting in [BGVS25] in a collaboration with

Alexandra Blessing and Mazyar Ghani Varzaneh. Some of the result were also use and published in [BS24]. This section aims to work through the whole proof in detail; the quick reader can also skip to the main result in Theorem 3.42.

Sewing lemma revisited

The main idea is to accordingly modify the sewing lemma, replacing the Hölder norms of the rough input by controls as in Definition 2.25, since they have better integrability properties compared to the Hölder norms. Then we have to make sure that we can estimate (2-40) properly, in a way that the terms on the right-hand side contain either the control defined in Definition 2.25 or no controlled rough path norm. To obtain this, we have to review the proof of the sewing lemma, essentially outlined in Theorem 2.42, and rewrite the stochastic convolution in a specific way.

Let $[s,t]\subset\mathbb{R}$ be an arbitrary interval. In order to prove the integrability of the solution, we need the following condition on the diffusion coefficient.

(G3) There exists a constant $\sigma_G<\gamma$ such that $G\colon [s,t]\times E_{\alpha-i\gamma}\to E_{\alpha-i\gamma-\sigma_G}$ is bounded for every $i=0,1,2$ and satisfies **(G1)**.

Lemma 3.27. ([GVR25, Lemma 2.3]) *Let $\alpha\in\mathbb{R}$, $(y,y')\in\mathcal{D}^{\gamma}_{X,\alpha}$ and $\big(A(r)\big)_{r\in[s,t]}$, G satisfying **(A)** and **(G1)**. Then we have for $i=0,1,2$*

$$\begin{aligned}\left\|\int_s^t S_{t,r}G_r(r,y_r)\,\mathrm{d}\mathbf{X}_r - S_{t,s}\big(G(s,y_s)X_{s,t}-\mathrm{D}_2G(s,y_s)G(s,y_s)\mathbb{X}_{s,t}\big)\right\|_{\alpha-i\gamma}\\ \leq\sum_{k\geq 0}\sum_{0\leq m<2^k}\left\|\Xi^{s_k^m,s_{k+1}^{2m+1}}_{s,t}+\Xi^{s_{k+1}^{2m+1},s_k^{m+1}}_{s,t}-\Xi^{s_k^m,s_k^{m+1}}_{s,t}\right\|_{\alpha-i\gamma},\end{aligned}\tag{3-19}$$

where $\pi^k:=\big\{s_k^m:=s+\frac{m}{2^k}(t-s):0\leq m\leq 2^k\big\}$ is the k-the dyadic partition of $[s,t]$ and

$$\Xi^{u,v}_{s,t}:=S_{t,u}\big(G(u,y_u)X_{u,v}+\mathrm{D}_2G(u,y_u)G(u,y_u)\mathbb{X}_{u,v}\big),$$

for $(u,v)\in\Delta_{[s,t]}$.

Proof. This was already shown in Theorem 2.42, see (2-43). □

To estimate the right-hand side of (3-19), we have to use a specific representation of the individual terms.

Lemma 3.28. *Let $\alpha \in \mathbb{R}$, $\big(y, G(\cdot, y)\big) \in \mathcal{D}^{\gamma}_{X,\alpha}$ and $\big(A(r)\big)_{r\in[s,t]}, G$ satisfying **(A)** and **(G1)**. Then we have*

$$
\begin{aligned}
\Xi^{u,v}_{s,t} + \Xi^{v,w}_{s,t} - \Xi^{u,w}_{s,t} &= S_{t,u}\big(G(v,y_v) - G(u,y_v)\big)X_{v,w} \\
&+ S_{t,u}\left(\int_0^1\int_0^1 \tilde{r}\mathrm{D}^2_2 G(u, y_u + r\tilde{r}y_{u,v})\,[G(u,y_u)X_{u,v}, G(u,y_u)X_{u,v}]\ \mathrm{d}r\mathrm{d}\tilde{r}\right)X_{v,w} \\
&+ S_{t,u}\left(\int_0^1\int_0^1 \tilde{r}\mathrm{D}^2_2 G(u, y_u + r\tilde{r}y_{u,v})\,\big[G(u,y_u)X_{u,v}, R^y_{u,v}\big]\ \mathrm{d}r\mathrm{d}\tilde{r}\right)X_{v,w} \\
&+ S_{t,u}\left(\int_0^1 \mathrm{D}_2 G(u, y_u + ry_{u,v})R^y_{u,v}\ \mathrm{d}r\right)X_{v,w} \\
&+ S_{t,u}\left(\mathrm{D}_2 G(v,y_v)\int_0^1 \mathrm{D}_2 G(v, y_u + ry_{u,v})G(u,y_u)X_{u,v}\ \mathrm{d}r)\right)\mathbb{X}_{v,w} \\
&+ S_{t,u}\left(\mathrm{D}_2 G(v,y_v)\int_0^1 \mathrm{D}_2 G(v, y_u + ry_{u,v})[R^y_{u,v}]\ \mathrm{d}r)\right)\mathbb{X}_{v,w} \\
&+ S_{t,u}\left(\int_0^1 \mathrm{D}^2_2 G(v, y_u + ry_{u,v})\,[G(u,y_u)X_{u,v}, G(v,y_u)]\ \mathrm{d}r\right)\mathbb{X}_{v,w} \\
&+ S_{t,u}\left(\int_0^1 \mathrm{D}^2_2 G(v, y_u + ry_{u,v})\,\big[R^y_{u,v}, G(v,y_u)\big]\ \mathrm{d}r\right)\mathbb{X}_{v,w} \\
&+ S_{t,u}\Big(\big(\mathrm{D}_2G(v,y_u) - \mathrm{D}_2G(u,y_u)\big)G(v,y_u) + \mathrm{D}_2G(u,y_u)\big(G(v,y_u) - G(u,y_u)\big)\Big)\mathbb{X}_{v,w} \\
&- S_{t,v}(S_{v,u} - \mathrm{Id}\,)\big(G(v,y_v)X_{v,w} + \mathrm{D}_2G(v,y_v)G(v,y_v)\mathbb{X}_{v,w}\big),
\end{aligned}
\tag{3-20}
$$

for $(u,v,w) \in \Delta^{(3)}_{[s,t]}$.

Proof. Let $(u,v,w) \in \Delta^{(3)}_{[s,t]}$. Using Chen's relation (2-14), we obtain

$$
\begin{aligned}
\Xi^{u,v}_{s,t} + \Xi^{v,w}_{s,t} - \Xi^{u,w}_{s,t} &= \big(-S_{t,u}G(u,y_u) + S_{t,v}G(v,y_v)\big)X_{v,w} \\
&\quad + S_{t,u}\mathrm{D}_2G(u,y_u)G(u,y_u)(\mathbb{X}_{u,v} - \mathbb{X}_{u,w}) \\
&\quad + S_{t,v}\big(G(v,y_v)X_{v,w} + \mathrm{D}_2G(v,y_v)G(v,y_v)\mathbb{X}_{v,w}\big) \\
&= S_{t,u}R^{G(\cdot,y)_{u,v}}X_{v,w} \\
&\quad - S_{t,v}(S_{v,u} - \mathrm{Id})\big(G(v,y_v)X_{v,w} - \mathrm{D}_2G(v,y_v)G(v,y_v)\mathbb{X}_{v,w}\big) \\
&\quad + S_{t,u}\big(\mathrm{D}_2G(v,y_v)G(v,y_v) - \mathrm{D}_2G(u,y_u)G(u,y_u)\big)\mathbb{X}_{v,w}.
\end{aligned}
$$

The last line can be rewritten as an integral

$$
\begin{aligned}
\mathrm{D}_2G(v,y_v)G(v,y_v) - \mathrm{D}_2G(u,y_u)G(u,y_u) &= \mathrm{D}_2G(v,y_v)\int_0^1 \mathrm{D}_2G(v, y_u + ry_{u,v})y_{u,v}\ \mathrm{d}r \\
&+ \int_0^1 \mathrm{D}^2_2G(v, y_u + ry_{u,v})\,[y_{u,v}, G(v,y_u)]\ \mathrm{d}r \\
&+ \big(\mathrm{D}_2G(v,y_u) - \mathrm{D}_2G(u,y_u)\big)G(v,y_u) + \mathrm{D}_2G(u,y_u)\big(G(v,y_u) - G(u,y_u)\big).
\end{aligned}
$$

Further, we can rewrite the remainder by using Taylor's theorem

$$
\begin{aligned}
R^{G(\cdot,y)}_{u,v} = G(v,y_v) - G(u,y_v) &+ \int_0^1\int_0^1 \tilde{r}\mathrm{D}^2_2G(u, y_u + r\tilde{r}y_{u,v})[G(u,y_u)X_{u,v}, y_{u,v}]\ \mathrm{d}r\mathrm{d}\tilde{r} \\
&+ \int_0^1 \mathrm{D}_2G(u, y_u + ry_{u,v})R^y_{u,v}\ \mathrm{d}r.
\end{aligned}
$$

Putting these representations together and using $y_{u,v} = G(u, y_u)X_{u,v} + R^y_{u,v}$ we finally obtain (3-20). $\square$

In order to estimate all terms in (3-20) properly, we do it in two steps: First, we estimate all lines involving the remainder, followed by the remaining terms. In order to clarify the procedure, we use I_i to denote the i-th line in (3-20), and omit the dependence on s_k^m, s_{k+1}^{2m+1} and s_k^{m+1}.

The estimates for each term utilize the same basic techniques. Before delving into the proofs, let us briefly emphasize them.

Remark 3.29. i) The first major ingredient is the smoothing property of the parabolic evolution family. Note that we have

$$\|S_{v,u}x\|_\beta \lesssim_{\alpha,\beta} (v-u)^{-(\beta-\alpha)} \|x\|_\alpha ,$$

for $(u, v) \in \Delta_{[s,t]}$ where the difference $\beta - \alpha \geq 0$ is positive and is allowed to exceed 1, see Theorem D.3. This is important when it comes to the next two lemmas, since in both proofs the case $i = 0$ would otherwise be problematic, since $2\gamma + \sigma_G$ is possibly greater than 1. This is the reason why [GVR25] and [BGVS25] have the restriction $\sigma_G < \frac{1-\gamma}{2}$.

ii) Due to the assumption (2-35), and (A-5) we know that

$$\mathrm{D}_2^k G(r, y_r) \in \mathcal{L}\big(E^k_{\alpha-\varsigma}; E_{\alpha-\sigma_G-\varsigma}\big),$$

holds for every $k = 1, 2, 3$, $r \in [s, t]$ and $\varsigma \in [0, 2\gamma]$ since we know

$$\mathrm{D}_2^k G(r, y_r) \in \mathcal{L}\big(E^k_\alpha; E_{\alpha-\sigma_G}\big) \cap \mathcal{L}\big(E^k_{\alpha-2\gamma}; E_{\alpha-\sigma_G-2\gamma}\big).$$

iii) Beside the two items above and Lemma 2.41 we also frequently use the interpolation inequality

$$\big\|R^y_{u,v}\big\|_{\alpha-\varsigma} \lesssim \big\|y, y'\big\|_{\mathcal{D}^\gamma_{X,\alpha}} (v-u)^\varsigma, \tag{3-21}$$

for every $\varsigma \in [\gamma, 2\gamma]$ and $(u, v) \in \Delta_{[s,t]}$, which can be shown, similar to (2-38), using (2-36).

In the next lemma, we start by estimating all terms involving the remainder.

Lemma 3.30. *Let $\alpha \in \mathbb{R}$, $\big(y, y'\big) \in \mathcal{D}^\gamma_{X,\alpha}([s, t])$ and $\big(A(r)\big)_{r\in[s,t]}$, G satisfying **(A)** and **(G3)**. Further, we choose an $\varepsilon > 0$ such that $\sigma_G + \varepsilon < \gamma$. Then we obtain*

$$\sum_{k\geq 0} \sum_{0\leq m<2^k} \|\mathrm{I}_3\|_{\alpha-i\gamma} + \|\mathrm{I}_4\|_{\alpha-i\gamma} + \|\mathrm{I}_6\|_{\alpha-i\gamma} + \|\mathrm{I}_8\|_{\alpha-i\gamma} \lesssim (t-s)^{i\gamma} \big\|y, y'\big\|_{\mathcal{D}^\gamma_{X,\alpha}}$$
$$\times \max\Big\{(t-s)^\varepsilon W_{\mathbf{X},\sigma_G+\varepsilon}(s, t)^{\gamma-\sigma_G-\varepsilon}, (t-s)^{2\varepsilon} W_{\mathbf{X},\sigma_G+\varepsilon}(s, t)^{2(\gamma-\sigma_G-\varepsilon)}\Big\},$$

for $i = 0, 1, 2$.

Proof. First, note that we have

$$\sum_{0\leq m<2^k} W_{\mathbf{X},\sigma_G+\varepsilon}\big(s_k^m, s_k^{m+1}\big) \leq W_{\mathbf{X},\sigma_G+\varepsilon}(s, t), \tag{3-22}$$

since $W_{\mathbf{X},\sigma_G+\varepsilon}$ is super-additive. Fix $u := s_k^m, v := s_{k+1}^{2m+1}, w := s_k^{m+1}$, let $i = 0, 1, 2$ be arbitrary, and recall $s_k^m = s + {}^m\!/_{2^k}(t-s)$. Therefore, we have

$$t - s_k^m = (t-s)\left(1 - \frac{2m}{2^{k+1}}\right), \quad s_k^{m+1} - s_{k+1}^{2m+1} = s_{k+1}^{2m+1} - s_m^k = (t-s)\frac{1}{2^{k+1}},$$

and $\frac{1}{2^{k+1}} \leq 1 - \frac{2m}{2^{k+1}}$ for $m < 2^k$ and $k \geq 0$. Then, using the smoothing properties of the parabolic evolution family, (3-21) and the monotone embeddings of $(E_\alpha)_{\alpha\in\mathbb{R}}$, we obtain

$$\begin{aligned}\|I_3\|_{\alpha-i\gamma} &\lesssim (t-u)^{(i-2)\gamma-\sigma_G} |X_{v,w}| |X_{u,v}| \|G(u,y_u)\|_{\alpha-2\gamma} \left\|R^y_{u,v}\right\|_{\alpha-2\gamma} \\ &\lesssim (t-u)^{(i-2)\gamma-\sigma_G} \|y, G(\cdot,y)\|_{\mathcal{D}^\gamma_{X,\alpha}} (w-v)^{\sigma_G+\varepsilon}(v-u)^{\sigma_G+\varepsilon}(v-u)^{2\gamma-\sigma_G} \\ &\quad \times W^{\gamma-\sigma_G-\varepsilon}_{\mathbf{X},\sigma_G+\varepsilon}(v,w) W^{\gamma-\sigma_G-\varepsilon}_{\mathbf{X},\sigma_G+\varepsilon}(u,v).\end{aligned}$$

In fact, we can estimate all four terms similarly, which leads to

$$\begin{aligned}\|I_3\|_{\alpha-i\gamma}, \|I_6\|_{\alpha-i\gamma}, \|I_8\|_{\alpha-i\gamma} &\lesssim \|y, G(\cdot,y)\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{i\gamma+2\varepsilon} \\ &\quad \times \left(1 - \frac{2m}{2^{k+1}}\right)^{(i-2)\gamma-\sigma_G} \left(2^{-(k+1)}\right)^{2\gamma+\sigma_G+2\varepsilon} W^{2(\gamma-\sigma_G-\varepsilon)}_{\mathbf{X},\sigma_G+\varepsilon}(u,w), \\ \|I_4\|_{\alpha-i\gamma} &\lesssim \|y, G(\cdot,y)\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{i\gamma+\varepsilon} \\ &\quad \times \left(1 - \frac{2m}{2^{k+1}}\right)^{(i-2)\gamma-\sigma_G} \left(2^{-(k+1)}\right)^{2\gamma+\sigma_G+\varepsilon} W^{\gamma-\sigma_G-\varepsilon}_{\mathbf{X},\sigma_G+\varepsilon}(u,w).\end{aligned}$$

The Hölder inequality and (3-22) yield

$$\begin{aligned}&\sum_{k\geq 0} \sum_{0\leq m<2^k} \|\mathrm{I}_3\|_{\alpha-i\gamma} \\ &\quad \lesssim W^{2(\gamma-\sigma_G-\varepsilon)}_{\mathbf{X},\sigma_G+\varepsilon}(s,t) \|y, G(\cdot,y)\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{i\gamma+2\varepsilon} \\ &\quad\quad \times \sum_{k\geq 0} \left(\sum_{0\leq m<2^k} \left(1 - \frac{2m}{2^{k+1}}\right)^{\frac{(i-2)\gamma-\sigma_G}{1-2(\gamma-\sigma_G-\varepsilon)}} \left(2^{-(k+1)}\right)^{\frac{2\gamma+\sigma_G+\varepsilon}{1-2(\gamma-\sigma_G-\varepsilon)}} \right)^{1-2(\gamma-\sigma_G-\varepsilon)} \\ &\quad \lesssim W^{2(\gamma-\sigma_G-\varepsilon)}_{\mathbf{X},\sigma_G+\varepsilon}(s,t) \|y, G(\cdot,y)\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{i\gamma+2\varepsilon},\end{aligned}$$

where we used Lemma 2.41 in the last line. The same applies to I_6 and I_8. For I_4 we obtain

$$\begin{aligned}&\sum_{k\geq 0} \sum_{0\leq m<2^k} \|\mathrm{I}_4\|_{\alpha-i\gamma} \\ &\quad \lesssim W^{\gamma-\sigma_G-\varepsilon}_{\mathbf{X},\sigma_G+\varepsilon}(s,t) \|y, G(\cdot,y)\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{i\gamma+\varepsilon} \\ &\quad\quad \times \sum_{k\geq 0} \left(\sum_{0\leq m<2^k} \left(1 - \frac{2m}{2^{k+1}}\right)^{\frac{(i-2)\gamma-\sigma_G}{1-(\gamma-\sigma_G-\varepsilon)}} \left(2^{-(k+1)}\right)^{\frac{2\gamma+\sigma_G+\varepsilon}{1-(\gamma-\sigma_G-\varepsilon)}} \right)^{1-(\gamma-\sigma_G-\varepsilon)} \\ &\quad \lesssim W^{\gamma-\sigma_G-\varepsilon}_{\mathbf{X},\sigma_G+\varepsilon}(s,t) \|y, G(\cdot,y)\|_{\mathcal{D}^\gamma_{X,\alpha}} (t-s)^{i\gamma+\varepsilon},\end{aligned}$$

which proves the claim. □

Lemma 3.31. *Let $\alpha \in \mathbb{R}$, $(y,y') \in \mathcal{D}^\gamma_{X,\alpha}([s,t])$ and $(A(r))_{r\in[s,t]}$, G satisfying **(A)** and **(G3)**. Then we obtain for $i = 0,1,2$*

$$\begin{aligned}&\sum_{k\geq 0} \sum_{0\leq m<2^k} \|\mathrm{I}_1\|_{\alpha-i\gamma} + \|\mathrm{I}_2\|_{\alpha-i\gamma} + \|\mathrm{I}_5\|_{\alpha-i\gamma} + \|\mathrm{I}_7\|_{\alpha-i\gamma} + \|\mathrm{I}_9\|_{\alpha-i\gamma} + \|\mathrm{I}_{10}\|_{\alpha-i\gamma} \\ &\quad \lesssim \max_{k=1,2,3} \left\{(t-s)^{i\gamma+k(\gamma-\sigma_G)}\right\} \widetilde{P}\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big),\end{aligned}$$

where $\widetilde{P}(x,y) = x^3 + xy + x + y$ is a polynomial in two variables.

Proof. Fix $u := s_k^m, v := s_{k+1}^{2m+1}, w := s_k^{m+1}$ and recall that $s_k^m = s + {}^{m}/_{2^k}(t-s)$. These terms are now easier to estimate than the ones in Lemma 3.30, since we do not need to bound them with the control.

For $i = 0, 1, 2$ we obtain as in Lemma 3.30

$$\begin{aligned} \|\mathrm{I}_1\|_{\alpha-i\gamma} &\lesssim [X]_{\gamma,[s,t]}\,(t-u)^{(i-2)\gamma-\sigma_G}(w-v)^{\gamma}(v-u)^{2\gamma},\\ \|\mathrm{I}_2\|_{\alpha-i\gamma} &\lesssim [X]^3_{\gamma,[s,t]}\,(t-u)^{(i-2)\gamma-\sigma_G}(w-v)^{\gamma}(v-u)^{2\gamma},\\ \|\mathrm{I}_5\|_{\alpha-i\gamma}, \|\mathrm{I}_7\|_{\alpha-i\gamma} &\lesssim [\mathbb{X}]_{\gamma,[s,t]}\,[X]_{\gamma,[s,t]}\,(t-u)^{(i-2)\gamma-\sigma_G}(w-v)^{2\gamma}(v-u)^{\gamma},\\ \|\mathrm{I}_9\|_{\alpha-i\gamma} &\lesssim [\mathbb{X}]_{\gamma,[s,t]}\,(t-u)^{(i-2)\gamma-\sigma_G}(w-v)^{2\gamma}(v-u)^{2\gamma}, \end{aligned}$$

where we several times use the boundedness of G. The only term that requires special consideration is I_{10} since this is the only term which includes $S_{t,v}(S_{v,u}-\mathrm{Id})$. For $i=2$, we can proceed with the first part as before

$$\|S_{t,v}(S_{v,u}-\mathrm{Id}\,)G(v,y_v)X_{v,w}\|_{\alpha-2\gamma} \lesssim [X]_{\gamma,[s,t]}\,(t-u)^{-\sigma_G}(w-v)^{\gamma}(v-u)^{2\gamma},$$

For $i = 0, 1$ we choose $\sigma_1 \in (1-\gamma+\sigma_G, 1), \sigma_2 \in (1-\gamma, 1)$ such that $\sigma_2 - \sigma_1 = -\sigma_G$ which implies

$$\|S_{t,v}(S_{v,u}-\mathrm{Id}\,)G(v,y_v)X_{v,w}\|_{\alpha-i\gamma} \lesssim [X]_{\gamma,[s,t]}\,(t-u)^{-\sigma_1}(w-v)^{\gamma}(v-u)^{\sigma_2}.$$

For the term consisting of $\mathbb{X}$, it is the other way around; here we can estimate $i = 1, 2$ as before

$$\|S_{t,v}(S_{v,u}-\mathrm{Id}\,)\mathrm{D}_2G(v,y_v)G(v,y_v)\mathbb{X}_{v,w}\|_{\alpha-i\gamma} \lesssim [\mathbb{X}]_{2\gamma,[s,t]}\,(t-v)^{-2\sigma_G}(v-u)^{i\gamma}(w-v)^{2\gamma},$$

and for $i = 0$ we choose $\sigma_1 \in (1-2\gamma+2\sigma_G, 1)$ and $\sigma_2 \in (1-2\gamma, 1)$ such that $\sigma_2 - \sigma_1 = -2\sigma_G$ which leads to

$$\|S_{t,v}(S_{v,u}-\mathrm{Id}\,)\mathrm{D}_2G(v,y_v)G(v,y_v)\mathbb{X}_{v,w}\|_{\alpha} \lesssim [\mathbb{X}]_{2\gamma,[s,t]}\,(t-v)^{-\sigma_1}(v-u)^{\sigma_2}(w-v)^{2\gamma}.$$

Using now Lemma 2.41 for every one of these terms, the claim follows. □

Combining now Lemma 3.28, Lemma 3.30, and Lemma 3.31, we obtain the following bound.

Corollary 3.32. ([GVR25, Proposition 2.7],[BGVS25, Lemma 3.7]) *Let $\alpha \in \mathbb{R}$, $(y, y') \in \mathcal{D}^{\gamma}_{X,\alpha}([s,t])$ and $\big(A(r)\big)_{r\in[s,t]}, G$ satisfying **(A)** and **(G3)**. Further, we choose an $\varepsilon > 0$ such that $\sigma_G + \varepsilon < \gamma$. Then there exists a constant $\widetilde{M} := \widetilde{M}(\sigma_G, \varepsilon, \gamma, \alpha) > 1$ with*

$$\begin{aligned} &\left\|\int_s^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r - S_{t,s}\big(G(s,y_s)X_{s,t} - \mathrm{D}_2G(s,y_s)G(s,y_s)\mathbb{X}_{s,t}\big)\right\|_{\alpha-i\gamma}\\ &\quad\le \widetilde{M}(t-s)^{i\gamma}\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\\ &\qquad\times\max\Big\{(t-s)^{\varepsilon}W_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(s,t)^{\gamma-\sigma_G-\varepsilon}, (t-s)^{2\epsilon}W_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(s,t)^{2(\gamma-\sigma_G-\varepsilon)}\Big\}\\ &\qquad+\widetilde{M}\max_{k=1,2,3}\Big\{(t-s)^{i\gamma+k(\gamma-\sigma_G)}\Big\}\,\widetilde{P}\big([X]_{\gamma},[\mathbb{X}]_{2\gamma}\big), \end{aligned} \tag{3-23}$$

for $i = 0, 1, 2$, where $\widetilde{P}(x,y) = x^3 + xy + x + y$ is a polynomial.

Remark 3.33. We want to highlight why we assume G to be bounded in this section. For example, for $u = s_k^{m+1}, v = s_{k+1}^{m+1}$ and $w = s_{k+2}^{m+1}$ we obtain

$$\|S_{t,v}(S_{v,u}-\mathrm{Id})G(v,y_v)X_{v,w}\|_{\alpha-i\gamma} \lesssim (t-v)^{-\sigma_1}(v-u)^{\sigma_2}\,\|G(v,y_v)\|_{\alpha-i\gamma+\sigma_2-\sigma_1}\,|X_{v,w}|$$

$$
\begin{aligned}
&\lesssim (t-v)^{-\sigma_1}(v-u)^{\sigma_2}(w-v)^{\sigma_G+\epsilon} W^{\gamma-\sigma_G-\varepsilon}_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(v,w)\, \|G(v,y_v)\|_{\alpha-i\gamma+\sigma_2-\sigma_1} \\
&\lesssim (t-s)^{\sigma_2-\sigma_1+\sigma_G+\varepsilon}\left(1-\frac{2n}{2^{m+1}}\right)^{-\sigma_1}\left(\frac{1}{2^{m+1}}\right)^{\sigma_2+\sigma_G+\varepsilon} \\
&\quad \times W^{\gamma-\sigma_G-\varepsilon}_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(v,w)\, \|G(v,y_v)\|_{\alpha-i\gamma+\sigma_2-\sigma_1},
\end{aligned}
$$

with suitable choices of σ_1, σ_2. Using that $\big(y, G(\cdot,y)\big) \in \mathcal{D}^{\gamma}_{X,\alpha}$ is a solution of (RPDE) together with a bound of the form

$$
\|G(v,y_v)\|_{\alpha-i\gamma+\sigma_2-\sigma_1} \leq \|G(v,y)\|_{\infty,\alpha-\gamma} \leq \|y, G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}},
$$

would lead to $-i\gamma+\sigma_2-\sigma_1 = -\gamma$, which entails $\sigma_2-\sigma_1+\sigma_G+\varepsilon = i\gamma+\sigma_G+\varepsilon-\gamma$. Since we assume $\sigma+\varepsilon<\gamma$, the exponent of $(t-s)$, is in this case less than $i\gamma$. On the other hand, one could try to bound $G(v,y_v)$ by its Hölder norm

$$
\|G(v,y_v)\|_{\alpha-i\gamma+\sigma_2-\sigma_1} \leq \|G(0,y_0)\|_{\alpha-i\gamma+\sigma_2-\sigma_1} + v[G(\cdot,y)]_{\gamma,\alpha-2\gamma},
$$

but such a bound is only helpful if $G(0,y_0)=0$. In conclusion, using the control defined in (2.25), we cannot drop the boundedness of G, since there would be terms that we cannot estimate by the right-hand side of (3-23) in a way that it contains $(t-s)^{i\gamma}$. This is because the control has lost a small amount of time regularity to achieve better integrability. This limitation has been removed recently in [BGV25] using different techniques.

Integrable moments of every order

After obtaining the essential estimate (3-23), we can proceed with proving the desired integrability. Therefore, fix a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and specify which stochastic processes we want to treat. In the probabilistic setting, we mostly omit the dependencies $W_{s,t}(\omega) := W_{\mathbf{X}(\omega),\gamma,\eta}(s,t)$ if there is no risk of confusion.

(N) Let $X\colon \mathbb{R}\times\Omega\to\mathbb{R}$ be a continuous, centered Gaussian process with the associated Cameron–Martin space $\mathcal{H}$ such that $(\mathcal{W},\mathcal{H},\mu)$ is the abstract Wiener space associated to X. Further, let $\gamma'>0$ be such that $\gamma+\gamma'-2(\sigma_G+\varepsilon)>1$ for some arbitrarily small $\varepsilon>0$ and assume the following properties to hold:

i) Every $h\in\mathcal{H}$ is continuous, $\mathcal{H}\subset\mathcal{C}^{1/\gamma'\text{-var}}(\mathbb{R})$ and we have

$$
W_{\mathbf{h},\gamma',\sigma_G+\varepsilon}(0,1) \lesssim \|h\|_{\mathcal{H}}^{\frac{1}{\gamma'-\sigma_G-\varepsilon}}.
$$

for all $h\in\mathcal{H}$.

ii) There is some $q\in[1,3/2)$ such that he covariance $R_X(s,t) := \mathbb{E}[X_sX_t]$ has finite q-variation with $[R_X]_{q\text{-var},[s,t]^2} \lesssim (t-s)^{1/q}$, for every $[s,t]\subset[0,\infty)$.

Remark 3.34. It is possible to choose the probability space in a way such that it coincides with the abstract Wiener space, i.e., $\Omega=\mathcal{W}$ and $\mathbb{P}=\mu$. For the Brownian motion, this would mean that Ω is given by the space of continuous functions and $\mathbb{P}$ is the Wiener measure, see also Example 4.5. In the following, we use the probability space $(\Omega,\mathcal{F},\mathbb{P})$, to keep the presentation consistent with the following sections, having in mind that $\mathbb{P}$ is actually the measure of the abstract Wiener space associated to $\mathbf{X}$.

In particular, Assumption **(N)** ii) entails that X can be enhanced on any interval $[s,t]$ to a geometric γ-Hölder rough path $\mathbf{X}(\omega) = \big(X(\omega),\mathbb{X}(\omega)\big) \in \mathscr{C}^{\gamma}_g([s,t];\mathbb{R})$, see Theorem 2.19. We have shown earlier that Volterra processes, under additional assumptions on the kernel, are examples of continuous paths that fulfill Assumption **(N)**, as stated in Lemma 2.27. In particular, this applies to the fractional Brownian motion.

Let us introduce the concept of greedy times, which is an essential element later on. In essence, greedy times are strategically chosen time points such that each interval is maximized under the constraint that the control remains bounded by a fixed constant.

Definition 3.35. Let $\eta \in [0,\gamma)$, $\chi > 0$, $\omega \in \Omega$ and $\mathbf{X}$ satisfies (N).

i) We define the sequence of greedy times $\left(\tau_{k,\eta,\omega}^{[s,t]}(\chi)\right)_{k\in\mathbb{N}_0}$ of the interval $[s,t]$ through $\tau_0(\omega) := s$ and recursively for $k \in \mathbb{N}_0$

$$\tau_{k+1}(\omega) := \tau_{k+1,\eta,\omega}^{[s,t]}(\chi) := \sup\left\{r \in [\tau_k(\omega),t] : W_{\tau_k(\omega),r}^{\gamma-\eta}(\omega) \leq \chi\right\}.$$

ii) Let

$$N_{s,t}(\omega) := N\big([s,t],\eta,\chi,\mathbf{X}(\omega)\big) := \inf\left\{k > 0 : \tau_{k,\eta,\omega}^{[s,t]}(\omega) = t\right\},$$

be the number of greedy time steps in the interval $[s,t]$.

The number of actual greedy time steps can be bounded by the control, which is again bounded by the Hölder norms of X and $\mathbb{X}$, see Lemma 2.26.

Lemma 3.36. *Let $\eta \in [0,\gamma)$, $\chi > 0$, $\omega \in \Omega$ and $\mathbf{X}$ satisfy (N). We can bound the number of greedy time steps by*

$$N_{s,t}(\omega) \leq W_{s,t}(\omega)\chi^{-\frac{1}{\gamma-\eta}} + 1.$$

Proof. Due to the super-additivity of $W_{s,t}(\omega)$ we obtain

$$N_{s,t}(\omega) - 1 \leq \sum_{j=0}^{N_{s,t}(\omega)-2} W_{\tau_j,\tau_{j+1}}(\omega)\chi^{-\frac{1}{\gamma-\eta}} \leq W_{s,\tau_{N_{s,t}(\omega)-1}}(\omega)\chi^{-\frac{1}{\gamma-\eta}} \leq W_{s,t}(\omega)\chi^{-\frac{1}{\gamma-\eta}},$$

which proves the claim. □

We now investigate the probability of having a large number of greedy time steps. In particular, we need to show that this probability decays exponentially. The key ingredient in this part is Borell's inequality, which essentially gives a lower exponential bound on the measure of specific subsets of the Cameron–Martin space. To this aim, we recall that the inner measure $\mathbb{P}^*$ induced by $\mathbb{P}$ is given by

$$\mathbb{P}^* \colon \mathcal{P}(\Omega) \to [0,\infty], O \mapsto \mathbb{P}^*(O) := \sup\left\{\mathbb{P}(\widetilde{O}) : \widetilde{O} \in \mathcal{F}, \widetilde{O} \subset O\right\}.$$

Theorem 3.37. ([Led96, Theorem 4.3]) *Let $O \subset \Omega$ be a measurable Borel set with $\mathbb{P}(O) > 0$. Choose an $a \in (-\infty,\infty]$ such that*

$$\mathbb{P}(O) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{x^2}{2}}\,\mathrm{d}x.$$

Then for every $r \geq 0$, we obtain

$$\mathbb{P}^*(O + r\mathfrak{B}) \geq \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a+r} e^{-\frac{x^2}{2}}\,\mathrm{d}x, \tag{3-24}$$

where $\mathfrak{B}$ denotes the unit ball in $\mathcal{H}$ and $O + r\mathfrak{B} := \{a + rk : a \in O, k \in \mathfrak{B}\}$.

We use Borell's inequality on the Cameron–Martin space associated with a rough path. For this we define the translation of a rough path $\mathbf{X} \in \mathscr{C}^\gamma$ by a continuous function of finite

$1/\gamma'$-variation $h \in \mathcal{C}^{1/\gamma'\text{-var}}$ by

$$(s,t) \mapsto \big(\mathcal{T}_h(\mathbf{X})\big)_{s,t} := \left(h_t + X_t, \int_s^t h_{r,t}\,\mathrm{d}h_r + \int_s^t h_{r,t}\,\mathrm{d}X_r + \int_s^t X_{s,t}\,\mathrm{d}h_r + \mathbb{X}_{r,t}\right), \tag{3-25}$$

providing all integrals are well-defined, which is, for example, the case if $\gamma + \gamma' > 1$. Then all appearing integrals can be defined in the sense of Young, as seen in Theorem 2.2. If Assumption **(N)** holds, then we know that every element in the Cameron–Martin space can be associated with a continuous function of finite $1/\gamma'$-variation. In particular, we have the following results, which will be stated without proof.

Lemma 3.38. ([CLL13, Lemma 5.4]) *Assume* $\mathbf{X}$ *satisfies* ***(N)****. Then there exists a measurable subset* $\widetilde{\Omega} \subset \Omega$ *of full measure such that*

$$\mathcal{T}_h\big(\mathbf{X}(\omega)\big) \equiv \mathbf{X}(\omega + h),$$

holds for any $\omega \in \widetilde{\Omega}$ *and* $h \in \mathcal{H}$.

Lemma 3.39. ([GVR25, Lemma 2.9]) *Let* $\eta > 0$ *be such that* $\eta < \gamma$ *and* $W_{\mathbf{X},\gamma,\eta}(s,t) < \infty$. *Further assume that* $W_{\mathbf{h},\gamma',\eta}(s,t) < \infty$ *for every* $\gamma' > 0$ *with* $\gamma + \gamma' > 1$ *and continuous* $h \in \mathcal{C}^{1/\gamma'\text{-var}}([s,t];\mathbb{R})$*. If* $\gamma + \gamma' > 1 + 2\eta$ *we obtain for such* h

$$W_{\mathcal{T}_h(\mathbf{X}),\gamma,\eta}(s,t) \lesssim_\eta W_{\mathbf{X},\gamma,\eta}(s,t) + W_{\mathbf{h},\gamma',\eta}(s,t)^{\frac{\gamma'-\eta}{\gamma-\eta}}. \tag{3-26}$$

Proof. Recall that the finite variation of h yields the existence of

$$\int_s^t h_{s,r}\,\mathrm{d}X_r = \lim_{|\pi|\to 0} \sum_{[u,v]\in\pi} h_{s,u}X_{u,v},$$

and analogously for the other integrals appearing in $\mathcal{T}_h(\mathbf{X})$. Removing one point s_k of the partition $\pi = \big\{[s_k, s_{k+1}] : s =: s_0 \leq \ldots \leq s_{|\pi|} := t\big\}$, denoted then by π', leads to

$$\begin{aligned}\left|\sum_{[u,v]\in\pi} h_{s,u}X_{u,v} - \sum_{[u,v]\in\pi'} h_{s,u}X_{u,v}\right| &\leq \big|h_{s_{k-1},s_k}\big|\,\big|X_{s_k,s_{k+1}}\big| \\ &\leq W^{\gamma'-\eta}_{\mathbf{h},\gamma',\eta}(s_{k-1},s_k)W^{\gamma-\eta}_{\mathbf{X},\gamma,\eta}(s_k,s_{k+1})(s_k-s_{k-1})^\eta(s_{k+1}-s_k)^\eta.\end{aligned}$$

Now there exists $k \leq |\pi|$ such that

$$W^{\frac{\gamma'-\eta}{\gamma+\gamma'-2\eta}}_{\mathbf{h},\gamma',\eta}(s_{k-1},s_k)W^{\frac{\gamma-\eta}{\gamma+\gamma'-2\eta}}_{\mathbf{X},\gamma,\eta}(s_k,s_{k+1}) \leq \frac{2}{|\pi|-1}W^{\frac{\gamma'-\eta}{\gamma+\gamma'-2\eta}}_{\mathbf{h},\gamma',\eta}(s,t)W^{\frac{\gamma-\eta}{\gamma+\gamma'-2\eta}}_{\mathbf{X},\gamma,\eta}(s,t),$$

since $W^{\frac{\gamma'-\eta}{\gamma+\gamma'-2\eta}}_{\mathbf{h},\gamma',\eta}W^{\frac{\gamma-\eta}{\gamma+\gamma'-2\eta}}_{\mathbf{X},\gamma,\eta}$ is a control due to $\frac{\gamma'-\eta}{\gamma+\gamma'-2\eta}+\frac{\gamma-\eta}{\gamma+\gamma'-2\eta} = 1$. Repeating this by replacing π by π' and removing another point, we conclude

$$\left|\int_s^t h_{s,r}\,\mathrm{d}X_r\right| \leq W^{\gamma'-\eta}_{\mathbf{h},\gamma',\eta}(s,t)W^{\gamma-\eta}_{\mathbf{X},\gamma,\eta}(s,t)(t-s)^{2\eta}\sum_{k\geq 1}\left(\frac{2}{k}\right)^{\gamma+\gamma'-2\eta}.$$

Therefore, Youngs inequality leads to

$$\frac{\left|\int_s^t h_{s,r}\,\mathrm{d}X_r\right|^{\frac{1}{2(\gamma-\eta)}}}{|t-s|^{\frac{\eta}{\gamma-\eta}}} \lesssim W^{\frac{\gamma'-\eta}{2(\gamma-\eta)}}_{\mathbf{h},\gamma',\eta}(s,t)W^{\frac{1}{2}}_{\mathbf{X},\gamma,\eta}(s,t) \lesssim W^{\frac{\gamma'-\eta}{\gamma-\eta}}_{\mathbf{h},\gamma',\eta}(s,t) + W_{\mathbf{X},\gamma,\eta}(s,t),$$

and with similar estimates, we can also bound the remaining integrals appearing in (3-25). □

Proposition 3.40. ([GVR25, Proposition 2.16]) *Assume that $\chi > 0$, $\mathbf{X}$ satisfies* **(N)** *and set $\eta := \sigma_G + \varepsilon$ for some $\varepsilon > 0$ such that $\sigma_G + \varepsilon < \gamma$. Then there exists a constant $M_0 := M_0(\eta, \chi) < \infty$ such that*

$$\mathbb{P}(N_{s,t} > k) \lesssim_{\eta,\chi} e^{-M_0 k^{2(\gamma'-\eta)}},$$

for every $k \in \mathbb{N}$.

Proof. Before we prove the actual inequality, we have to establish a bound on the number of greedy times $N_{s,t}$ first. Using Lemma 3.38 we obtain $\mathcal{T}_h\big(\mathbf{X}(\omega - h)\big) = \mathbf{X}(\omega)$. Therefore, we can estimate the control, applied to the greedy time steps $\big(\tau_k(\omega)\big)_{k\in\mathbb{N}}$, using (3-26)

$$\begin{aligned} W^{\gamma-\eta}_{\mathbf{X}(\omega),\gamma,\eta}\big(\tau_k(\omega), \tau_{k+1}(\omega)\big) \leq{}& C_\eta W^{\gamma'-\eta}_{\mathbf{h},\gamma',\eta}\big(\tau_k(\omega), \tau_{k+1}(\omega)\big) \\ &+ C_\eta W^{\gamma-\eta}_{\mathbf{X}(\omega-h),\gamma,\eta}\big(\tau_k(\omega), \tau_{k+1}(\omega)\big), \end{aligned}$$

for some constant $C_\eta > 0$. Due to the definition of greedy times, we have for $\tau_{k+1}(\omega) < t$

$$W_{\mathbf{X}(\omega),\gamma,\eta}\big(\tau_k(\omega), \tau_{k+1}(\omega)\big) = \chi.$$

Setting $\chi = 2^{\frac{1}{\gamma-\eta}} C_\eta W_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t)$ leads to

$$W^{\frac{\gamma-\eta}{\gamma'-\eta}}_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t) \leq W_{\mathbf{h},\gamma',\eta}\big(\tau_k(\omega), \tau_{k+1}(\omega)\big),$$

for $\tau_{k+1}(\omega) < t$. Summing over all k, entails

$$\begin{aligned} \Big(N\big([s,t], \eta, 2^{\frac{1}{\gamma-\eta}} C_\eta W_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t), \mathbf{X}(\omega)\big) - 1\Big) W^{\frac{\gamma-\eta}{\gamma'-\eta}}_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t) &\leq W_{\mathbf{h},\gamma',\eta}(s,t) \\ &\leq \widetilde{C}_\eta \, \|h\|^{\frac{1}{\gamma'-\eta}}_{\mathcal{H}}, \end{aligned} \tag{3-27}$$

for some constant $\widetilde{C}_\eta > 0$ using Assumption **(N)**.

Now Borell's inequality can be used. Therefore, define

$$O := \left\{\omega \in \Omega : \frac{\chi}{2^{\frac{1}{\gamma-\eta}+1} C_\eta} \leq W_{\mathbf{X}(\omega),\gamma,\eta}(s,t) \leq \frac{\chi}{2^{\frac{1}{\gamma-\eta}} C_\eta}\right\},$$

where $\mathbb{P}(O) > 0$ holds. Indeed, a support theorem for Gaussian rough paths, see [FV10, Theorem 15.64], states that the support of the law of $\mathbf{X}$, in our case $\mathbb{P}$, contains the rough path lift of every $h \in \mathcal{H}$. This lift is well-defined due to $\mathcal{H} \subset \mathcal{C}^{1/\gamma'\text{-var}}$. This yields $\mathbb{P}(O) > 0$.

Observe that for every $\omega - h \in O$ we have

$$\frac{\chi}{2} \leq 2^{\frac{1}{\gamma-\eta}} C_\eta W_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t) \leq \chi,$$

therefore, we can bound the number of greedy times for every $\omega - h \in O$ using (3-27) and the fact that $\chi \to N([s,t], \eta, \chi, \mathbf{X})$ is decreasing.

$$\begin{aligned} &\Big(N\big([s,t], \eta, \chi, \mathbf{X}(\omega)\big) - 1\Big) W^{\frac{\gamma-\eta}{\gamma'-\eta}}_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t) \\ &\qquad \leq \Big(N\big([s,t], \eta, 2^{\frac{1}{\gamma-\eta}} C_\eta W_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t), \mathbf{X}(\omega)\big) - 1\Big) W^{\frac{\gamma-\eta}{\gamma'-\eta}}_{\mathbf{X}(\omega-h),\gamma,\eta}(s,t) \end{aligned}$$

$$\leq \widetilde{C}_\eta \|h\|_{\mathcal{H}}^{\frac{1}{\gamma'-\eta}} .$$

If in addition

$$\|h\|_{\mathcal{H}} \leq r_k := \frac{(k-1)^{\gamma'-\eta}\chi^{\gamma-\eta}}{2^{\gamma-\eta+1}C_\eta^{\gamma-\eta}\widetilde{C}_\eta^{\gamma'-\eta}}$$

holds for $\chi > 0$ and $k \in \mathbb{N}$, we further obtain

$$\begin{aligned}\Big(N\big([s,t],\eta,\chi,\mathbf{X}(\omega)\big)-1\Big)\left(\frac{\chi}{2^{\frac{1}{\gamma-\eta}+1}C_\eta}\right)^{\frac{\gamma-\eta}{\gamma'-\eta}} &\leq \widetilde{C}_\eta \|h\|_{\mathcal{H}}^{\frac{1}{\gamma'-\eta}} \leq \widetilde{C}_\eta r_k^{\frac{1}{\gamma'-\eta}} \\ &\leq (k-1)\left(\frac{\chi}{2^{\frac{1}{\gamma-\eta}+1}C_\eta}\right)^{\frac{\gamma-\eta}{\gamma'-\eta}} .\end{aligned}$$

This implies in particular $\{N([s,t],\eta,\chi,\mathbf{X}) > k\} \subset \Omega \setminus \{O + r_k\mathfrak{B}\}$ for every $k \in \mathbb{N}$, where $\mathfrak{B}$ denotes the unit ball in $\mathcal{H}$. Then Borell's inequality (3-24) entails finally

$$\begin{aligned}\mathbb{P}\big(N([s,t],\eta,\chi,\mathbf{X}) > k\big) &\leq 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{r_k+a} e^{-\frac{x^2}{2}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{r_k+a}^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x \\ &\lesssim e^{\frac{-(r_k+a)^2}{2}} \lesssim e^{-M_0(k-1)^{2(\gamma'-\eta)}},\end{aligned}$$

for some $a \in (-\infty,\infty]$ and $M_0 = \frac{\chi^{\gamma-\eta}}{2^{\gamma-\eta+1}C_\eta^{\gamma-\eta}\widetilde{C}_\eta^{\gamma'-\eta}}$, which proves the claim. □

The last ingredient needed is the following simple concatenation argument for the controlled rough path norm.

Lemma 3.41. *Let $\big(y,y'\big) \in \mathcal{D}^\gamma_{X,\alpha}([s,t])$ and a possible random partition $(s_k)_{k=0}^{\widetilde{N}}$ of $[s,t]$ with $s_0 = s$ and $s_{\widetilde{N}} = t$. Then we have*

$$\|y,y'\|_{\mathcal{D}^\gamma_{X(\omega),\alpha}([s,t])} \leq \Big(1 + [X(\omega)]_{\gamma,[s,t]}\Big)\sum_{k=0}^{\widetilde{N}-1}\|y,y'\|_{\mathcal{D}^\gamma_{X(\omega),\alpha}([s_k,s_{k+1}])} . \tag{3-28}$$

Proof. The only non-trivial estimate is the Hölder semi-norm of the remainder. For this, we have

$$[R^y]_{i\gamma,[s,t],\alpha-i\gamma} \leq [R^y]_{i\gamma,[s,u],\alpha-i\gamma} + [R^y]_{i\gamma,[u,t],\alpha-i\gamma} + [X]_{\gamma,[u,t]}\,\big[y'\big]_{(i-1)\gamma,[s,u],\alpha-i\gamma}$$

for $(s,u,t) \in \Delta^{(3)}_{[0,T]}$. This leads to

$$\|y,y'\|_{\mathcal{D}^\gamma_{X(\omega),\alpha}([s,t])} \leq \Big(1 + [X(\omega)]_{\gamma,[s,t]}\Big)\|y,y'\|_{\mathcal{D}^\gamma_{X(\omega),\alpha}([s,u])} + \|y,y'\|_{\mathcal{D}^\gamma_{X(\omega),\alpha}([u,t])} .$$

Iterating this process for every time step of the partition $(s_k)_{k=0}^{\widetilde{N}}$ proves the claim. □

Putting all of the above statements together yields the main theorem of this section. First proven in [GVR25] and later slightly adjusted in [BS24]. Here, we extend the theorem to the non-autonomous case. The difference in the non-autonomous case lies in particular in the proof of Lemma 3.28. Due to the time-dependent coefficients, a different representation of $\Xi^{u,v}_{s,t} + \Xi^{v,w}_{s,t} - \Xi^{u,w}_{s,t}$ must be chosen in order to modify the Sewing Lemma accordingly.

Since we use the inequality for the analysis of the long-term behavior, we specify the constants as concretely as possible to maintain readability.

Theorem 3.42. ([BS24, Theorem 2.15], [GVR25, Theorem 2.17]) *Let $\chi \in (0,1)$ and $\eta \in (\sigma_G, \gamma)$ such that $\widetilde{M}\chi^{\gamma-\eta} \leq 1/4$, where $\widetilde{M}$ is the constant from Corollary 3.32. Assume $\big(A(r)\big)_{r\in[s,t]}, F, G$ and $\mathbf{X}$ satisfy **(A)**, **(F1)**, **(G3)** and **(N)**.*

i) For $t-s \leq \min\left\{\big(4\widetilde{M}\big)^{-\frac{1}{1-\overline{\sigma}_F}}, 1\right\} =: \min\{d,1\}$ and any fixed $\omega \in \Omega$ the solution $\big(y,y'\big) \in \mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])$ of (RPDE) satisfies

$$\sup_{r\in[s,t]} \|y_r\|_\alpha \leq \sup_{0\leq k\leq N_{s,t}(\omega)-1} \|y,y'\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([\tau_k,\tau_{k+1}])}$$
$$\leq e^{N_{s,t}(\omega)M}\|y_s\|_\alpha + \frac{e^{N_{s,t}(\omega)M+M}-1}{2\widetilde{M}-1} P\big([X(\omega)]_{\gamma,[s,t]}, [\mathbb{X}(\omega)]_{2\gamma,[s,t]}\big),$$

where $P(x,y) = 1 + x^3 + xy + x + y$ and $M := \log\big(2\widetilde{M}\big)$.

ii) For an interval $[s,t]$ with $d < t-s \leq 1$ and any fixed $\omega \in \Omega$, the solution $\big(y,y'\big) \in \mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])$ of (RPDE) satisfies

$$\begin{aligned}\|y,y'\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])} &\leq \widetilde{N}\left(N_{s,t}(\omega)\Big(1+[X(\omega)]_{\gamma,[s,t]}\Big)e^{N_{s,t}(\omega)M}\right)^{\widetilde{N}+1}\|y_s\|_\alpha \\ &+ \widetilde{N}N_{s,t}(\omega)\Big(1+[X(\omega)]_{\gamma,[s,t]}\Big)\frac{e^{N_{s,t}(\omega)M+M}-1}{2\widetilde{M}-1}P\big([X(\omega)]_{\gamma,[s,t]}, [\mathbb{X}(\omega)]_{2\gamma,[s,t]}\big) \\ &\times \sum_{k=1}^{\widetilde{N}}\left(N_{s,t}(\omega)\Big(1+[X(\omega)]_{\gamma,[s,t]}\Big)e^{N_{s,t}(\omega)M}\right)^k,\end{aligned} \tag{3-29}$$

for $\widetilde{N} := \lceil d^{-1}(t-s)\rceil$.

Proof. i) Recall that the mild solution of (RPDE) satisfies

$$y_t = S_{t,s}y_s + \int_s^t S_{t,r}F(r,y_r)\,\mathrm{d}r + \int_s^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r,$$

and $y' := G(\cdot,y)$. Using Corollary 3.32, we obtain

$$\begin{aligned}\left\|\int_s^\cdot S_{\cdot,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r, G(\cdot,y)\right\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])} &\leq \widetilde{M}P\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big) \\ &+ \widetilde{M}\max\left\{(t-s)^{\varepsilon}W_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(s,t)^{\gamma-\sigma_G-\varepsilon}, (t-s)^{2\varepsilon}W_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(s,t)^{2(\gamma-\sigma_G-\varepsilon)}\right\} \\ &\times\left(1+\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])}\right),\end{aligned}$$

which leads to

$$\begin{aligned}\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])} &\leq \widetilde{M}\left(1+\|y_s\|_\alpha + C_F(t-s)^{1-\overline{\sigma}_F}\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])}\right) \\ &+ \widetilde{M}\widetilde{P}\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big) \\ &+ \widetilde{M}\max\left\{(t-s)^{\varepsilon}W_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(s,t)^{\gamma-\sigma_G-\varepsilon}, (t-s)^{2\epsilon}W_{\mathbf{X},\gamma,\sigma_G+\varepsilon}(s,t)^{2(\gamma-\sigma_G-\varepsilon)}\right\} \\ &\times\left(1+\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])}\right),\end{aligned}$$

where we also used Lemma 3.7 and $\widetilde{M} > 1$. Using the assumption on χ, $\widetilde{M}$ and $t-s$, we obtain

$$\|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([\tau_k,\tau_{k+1}])} \leq 2\widetilde{M}\, \|y_{\tau_k}\|_{\alpha} + 2\widetilde{M} P\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big)$$

for $k < N_{s,t}(\omega)$, where $P(x,y) = 1 + \widetilde{P}(x,y)$ and $(\tau_k)_{k=0}^{N_{s,t}(\omega)} = \left(\tau^{[s,t]}_{k,\sigma_G+\varepsilon,\omega}(\chi)\right)_{k=0}^{N_{s,t}(\omega)}$ is the sequence of greedy times. Due to $\|y_{\tau_k}\|_{\alpha} \leq \|y, G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([\tau_{k-1},\tau_k])}$ for every $0 < k \leq N_{s,t}$, we can iterate this estimate, which leads to

$$\begin{aligned}
\|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([\tau_k,\tau_{k+1}])} &\leq 2\widetilde{M}\Big(\|y_{\tau_k}\|_{\alpha} + P\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big)\Big) \\
&\leq \big(2\widetilde{M}\big)^2 \Big(\|y, G(\cdot, y)\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([\tau_{k-2},\tau_{k-1}])} + P\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big)\Big) \\
&\quad + 2\widetilde{M} P\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big) \\
&\leq \big(2\widetilde{M}\big)^{k+1} \|y_s\|_{\alpha} + P\big([X]_{\gamma,[s,t]}, [\mathbb{X}]_{2\gamma,[s,t]}\big) \sum_{l=1}^{k+1} \big(2\widetilde{M}\big)^{l-1}.
\end{aligned}$$

Defining $M := \log\big(2\widetilde{M}\big)$ then proves the claim.

ii) For the second part, we use Lemma 3.41 for the sequence of greedy times $(\tau_k)_{k=0}^{N_{s,t}(\omega)}$ defined in Definition 3.35. The inequality (3-28) then leads to

$$\big\|y, y'\big\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s,t])} \leq p_1(\omega, [s,t])\, \|y_s\|_{\alpha} + p_2(\omega, [s,t]), \tag{3-30}$$

for $t - s \leq d$ and

$$\begin{aligned}
p_1(\omega, [s,t]) &:= N_{s,t}(\omega)\Big(1 + [X(\omega)]_{\gamma,[s,t]}\Big) e^{N_{s,t}(\omega) M}, \\
p_2(\omega, [s,t]) &:= N_{s,t}(\omega)\Big(1 + [X(\omega)]_{\gamma,[s,t]}\Big) \frac{e^{N_{s,t}(\omega)M + M} - 1}{2\widetilde{M} - 1} P\big(\, [X(\omega)]_{\gamma,[s,t]}\,, [\mathbb{X}(\omega)]_{2\gamma,[s,t]}\,\big).
\end{aligned}$$

To extend the statement for arbitrary intervals $t - s > d$, we define iteratively $s_0 = s$ and $s_k := \min\{d + s_{k-1}, t\}$ for $k = 1, \ldots, \widetilde{N} = \lceil d^{-1}(t-s)\rceil$. Then $s_{k+1} - s_k \leq d$, and we can use (3-30) to obtain iteratively

$$\begin{aligned}
\big\|y, y'\big\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s_k,s_{k+1}])} &\leq p_1(\omega, [s_k, s_{k+1}])\, \|y_{s_k}\|_{\alpha} + p_2(\omega, [s_k, s_{k+1}]) \\
&\leq p_1(\omega, [s_k, s_{k+1}])\, \big\|y, y'\big\|_{\mathcal{D}^{\gamma}_{X(\omega),\alpha}([s_{k-1},s_k])} + p_2(\omega, [s_k, s_{k+1}]) \\
&\leq p_1(\omega, [s_k, s_{k+1}])\, \big(p_1(\omega, [s_{k-1}, s_k])\, \big\|y_{s_{k-1}}\big\|_{\alpha} + p_2(\omega, [s_{k-1}, s_k])\big) + p_2(\omega, [s_k, s_{k+1}]) \\
&\leq \prod_{l=0}^{k} p_1(\omega, [s_l, s_{l+1}])\, \|y_s\|_{\alpha} + \sum_{n=0}^{k} p_2(\omega, [s_{k-n}, s_{k-n+1}]) \prod_{l=1}^{n} p_1(\omega, [s_{k-n+l}, s_{k-n+l+1}]) \\
&\leq p_1(\omega, [s,t])^{\widetilde{N}}\, \|y_s\|_{\alpha} + p_2(\omega, [s,t]) \sum_{l=0}^{\widetilde{N}-1} p_1(\omega, [s,t])^l,
\end{aligned}$$

since $p_1(\omega, [s_k, s_{k+1}]) \leq p_1(\omega, [s,t])$ and analogously $p_2(\omega, [s_k, s_{k+1}]) \leq p_2(\omega, [s,t])$, for $k < \widetilde{N} - 1$. Together with (3-28), this provides (3-29).

□

We do not highlight the dependence of $\widetilde{N}$ on the length of the interval $[s,t]$, since we later use these estimates only for intervals of length one. In this case, we have $\widetilde{N} = \lceil d^{-1}\rceil$, with $d = \big(4\widetilde{M}\big)^{-\frac{1}{1-\max\{\sigma_F, 2\gamma\}}} < 1$.

Since the inequality is lengthy, we define the abbreviations

$$\begin{aligned}
P_0(\omega,[s,t]) &:= N_{s,t}(\omega)\Big(1+[X(\omega)]_{\gamma,[s,t]}\Big)e^{N_{s,t}(\omega)M},\\
P_1(\omega,[s,t]) &:= \widetilde{N} P_0(\omega,[s,t])^{\widetilde{N}+1},\\
P_2(\omega,[s,t]) &:= \widetilde{N} N_{s,t}(\omega)\Big(1+[X(\omega)]_{\gamma,[s,t]}\Big)\frac{e^{N_{s,t}(\omega)M+M}-1}{2\widetilde{M}-1}\\
&\qquad \times P\big([X(\omega)]_{\gamma,[s,t]},[\mathbb{X}(\omega)]_{2\gamma,[s,t]}\big)\frac{P_0(\omega,[s,t])^{\widetilde{N}}-1}{P_0(\omega,[s,t])-1},
\end{aligned}$$

for better readability. With these abbreviations, the bound for the solution (y,y') of (RPDE) becomes

$$\left\|y,y'\right\|_{\mathcal{D}^{2\gamma}_{X(\omega),\alpha}([s,t])} \le \|y_s\|_\alpha\, P_1(\omega,[s,t]) + P_2(\omega,[s,t]). \tag{3-31}$$

Finally, we can prove that the solution of (RPDE) has bounded moments of all orders $p \ge 1$.

Lemma 3.43. *Under the same assumptions of Theorem 3.42 the bound of the solution* $\|y,G(\cdot,y)\|_{\mathcal{D}^{2\gamma}_{X,\alpha}([s,t])}$ *in* (3-29) *is integrable. In particular, we have*

$$N_{s,t}(\cdot)\Big(1+[X(\cdot)]_{\gamma,[s,t]}\Big)e^{N_{s,t}(\cdot)M} \in \bigcap_{p\ge1}\mathrm{L}^p(\Omega)$$

and therefore

$$P_1(\cdot,[s,t]), P_2(\cdot,[s,t]) \in \bigcap_{p\ge1}\mathrm{L}^p(\Omega).$$

Proof. Recall that Assumption **(N)** leads to

$$\mathbb{P}(N_{s,t}>k) \lesssim e^{-M_0 k^{2(\gamma'-\eta)}},$$

for every $k\in\mathbb{N}$, due to $\gamma+\gamma'>1+2\eta$ which implies $c:=2(\gamma'-\eta)>1$. Since the expected value can be written as

$$\begin{aligned}
\mathbb{E}\left[\left|e^{MN_{s,t}}\right|^p\right] &= \mathbb{E}\left[e^{pMN_{s,t}}\right] = \sum_{k\ge0} e^{pMk}\mathbb{P}(N_{s,t}=k)\\
&= \sum_{k\ge0}\left(e^{pM\cdot0}+\sum_{j=0}^{k-1}\left(e^{pM(j+1)}-e^{pMj}\right)\right)\mathbb{P}(N_{s,t}=k)\\
&= 1+\sum_{k\ge0}\left(e^{pM(k+1)}-e^{pMk}\right)\mathbb{P}(N_{s,t}>k) = 1+\left(e^{pM}-1\right)\sum_{k\ge0}e^{pMk}\mathbb{P}(N_{s,t}>k)\\
&\lesssim 1+\left(e^{pM}-1\right)\sum_{k\ge0}e^{pMk-M_0k^c},
\end{aligned}$$

where we use Tonelli's theorem and the fact that $\mathbb{P}(N_{s,t}>k)=\sum_{j=k+1}^{\infty}\mathbb{P}(N_{s,t}=k)$. Since $c>1$, the last sum converges and we obtain $e^{MN_{s,t}} \in \bigcap_{p\ge1}\mathrm{L}^p(\Omega)$. Since $\varrho_{\gamma,[s,t]}(\mathbf{X})$ is also L^p-integrable due to Corollary 2.20, we further have $N_{s,t}(\cdot)\big(1+[X(\cdot)]_{\gamma,[s,t]}\Big)e^{N_{s,t}(\cdot)M} \in \bigcap_{p\ge1}\mathrm{L}^p(\Omega)$. Since P is a polynomial, this leads in particular to the claimed integrability of P_1, P_2 and therefore also for $\|y,G(\cdot,y)\|_{\mathcal{D}^{2\gamma}_{X,\alpha}([s,t])}$ using the estimate (3-31). ☐

3.5 Mild rough Gronwall inequalities

In this section, we prove a Gronwall Lemma for the mild solution of rough differential equations on an arbitrary interval $[s,t]$. In particular, we present here the results published in [BGVS25] resulting from a collaboration with Alexandra Blessing and Mazyar Ghani Varzaneh.

As a reminder, the classical Gronwall inequality states that paths y that fulfill

$$y_t \leq a + \int_0^t b y_r \, \mathrm{d}r,$$

have maximum exponential growth $y_t \leq a e^{bt}$ for $t \geq 0$, see [Gro19]. This inequality has attracted considerable attention, and various Gronwall-type estimates have been developed over time. In particular, the singular Gronwall–Henry lemma is useful for partial differential equations. It is also employed in this thesis to prove the existence of a global attractor, see Theorem 5.2. More recently, stochastic versions of the Gronwall inequality have also been considered; see, for example, [Sch13, MS21, Gei24] and the references therein.

Before turning to the Gronwall lemma for the mild solution, we note that there is another version of a rough Gronwall inequality, developed in [Hof18, DGHT19] for weak solutions of rough partial differential equations using unbounded rough drivers. We sketch a simple version of the Gronwall-type lemma they used. To this aim, let g be a path and W_1, W_2 two controls such that

$$g_{s,t} \lesssim \sup_{r\in[0,t]} g_r W_1(s,t) + W_2(s,t), \tag{3-32}$$

holds for $s \leq t$, where W_1 is bounded. Then (3-32) implies

$$\sup_{t\in[0,T]} g_t \lesssim e^{c_1 W_1(0,T)} \left(g_0 + \sup_{t\in[0,T]} \left(W_2(0,t) e^{-c_1 W_1(0,t)} \right) \right)$$

for some $c_1 > 0$. In applications, the controls W_1 and W_2 are dependent on the rough path, or in their case, on the unbounded rough driver. The inequality (3-32) is derived through energy estimates of the variational/weak formulation of the partial differential equation. However, this is possible only for rough partial differential equations with additive noise; how such an inequality (3-32) can be derived for multiplicative noise remains unclear. This is the primary reason why we derive a mild Gronwall inequality in this section.

To be precise, our goal is to obtain a bound for the solution of (RPDE) given by $\big(y, G(\cdot,y)\big) \in \mathcal{D}^{\gamma}_{X,\alpha}$ of the form

$$\|y, G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])} \lesssim \big(\|y_s\|_{\alpha} + \|G(s,y_s)\|_{\alpha-\gamma} \big) e^{c_2(t-s)},$$

for a suitable constant $c_2 > 0$ and $(s,t) \in \Delta_{[s,t]}$. Furthermore, this inequality is applied to the linearization of the equation during this section.

Lemma 3.44. ([BGVS25, Lemma 4.2]) *Suppose $\big(A(r)\big)_{r\in[s,t]}$, F and G satisfy* ***(A)***, ***(F1)*** *and* ***(G3)***. *Then the solution $\big(y, G(\cdot,y)\big) \in \mathcal{D}^{\gamma}_{X,\alpha}([s,t])$ to (RPDE) satisfies*

$$\|y, G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])} \leq C_1 \big(1 + \|y_s\|_{\alpha} + \|G(s,y_s)\|_{\alpha-\gamma} \big) e^{C_2(t-s)}, \tag{3-33}$$

where the constants are given by

$$C_1 := e^{C_2} \max\left\{ \frac{1 - C\tau^{\tilde{\kappa}}\Phi_3}{2C\Phi_2 - 1 + C\tau^{\tilde{\kappa}}\Phi_3}, \frac{(1 - C\tau^{\tilde{\kappa}}\Phi_3) C\Phi_1}{(C\tau^{\tilde{\kappa}}\Phi_3 + 2C\Phi_2 - 1)^2} \right\} \big(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\big),$$

$$C_2 := \frac{1}{\tau} \ln\left(\frac{2C\Phi_2}{1 - C\tau^{\tilde{\kappa}}\Phi_3}\right),$$

with $C := C(S, \alpha, \sigma_G, \sigma_F, \gamma) > 0$, $\tilde{\kappa} := \min\{1 - 2\gamma, 1 - \sigma_F, \gamma - \sigma_G\}$, $\tau < 1$ *such that* $2C\Phi_2 > 1 - C\tau^{\tilde{\kappa}}\Phi_3 > 0$ *and*

$$\begin{aligned}
\Phi_1 &:= C_F + C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right)^3 + C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right), \\
\Phi_2 &:= \max\left\{1, C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right)\right\}, \\
\Phi_3 &:= C_F + C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right)^3.
\end{aligned}$$

Proof. Let $(u, v) \in \Delta_{[s,t]}$. As in Lemma 3.6 and 3.7 and we obtain for $(y, z) \in \Delta_{[s,t]}$ with $v - u < 1$:

$$\begin{aligned}
\|S_{\cdot,u}y_u, 0\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])} &\lesssim \|y_u\|_\alpha, \\
\left\|\int_u^\cdot S_{\cdot,r}F(r, y_r)\,\mathrm{d}r, 0\right\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])} &\leq C_F(v-u)^{1-\overline{\sigma}_F}\left(1 + \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])}\right), \\
\left\|G(\cdot, y), (G(\cdot, y))'\right\|_{\mathcal{D}^\gamma_{X,\alpha-\sigma_G}([u,v])} &\leq C_G\left(1 + \varrho_{\gamma,[u,v]}(\mathbf{X})\right)^2\left(1 + \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])}\right).
\end{aligned}$$

Combining these estimates with (2-45) we obtain

$$\begin{aligned}
\|z, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])} &\lesssim \|y_u\|_\alpha + C_F(v-u)^{1-\overline{\sigma}_F}\left(1 + \|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])}\right) \\
&\quad + \left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right)\Big(\|G(u, y_u)\|_{\alpha-\sigma_G} + \left\|(G(u, y_u))'\right\|_{\alpha-\sigma_G-\gamma} \\
&\quad + (v-u)^{\gamma-\sigma_G}\left\|G(\cdot, y), (G(\cdot, y))'\right\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])}\Big) \\
&\lesssim C_F + C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right)^3 + C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right) + \|y_u\|_\alpha \\
&\quad + C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right)\|G(u, y_u)\|_{\alpha-\gamma} \\
&\quad + \left(C_F + C_G\left(1 + \varrho_{\gamma,[s,t]}(\mathbf{X})\right)^3\right)(v-u)^{\tilde{\kappa}}\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])} \\
&=: \Phi_1 + \Phi_2\left(\|y_u\|_\alpha + \|G(u, y_u)\|_{\alpha-\gamma}\right) + \Phi_3(v-u)^{\tilde{\kappa}}\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}([u,v])}.
\end{aligned}$$

We now choose a sequence of intervals $I_n := [s_k, s_{k+1}]$ with $s_k := \min\{s + k\tau, t\}$ and

$$N(\tau) := \inf\{k \in \mathbb{N} : s_k = t\},$$

where $\tau < 1$ is fixed such that

$$2C\Phi_2 > 1 - C\tau^{\tilde{\kappa}}\Phi_3 > 0.$$

So we obtain for $k < N(\tau)$

$$\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}(I_k)} \leq C\Phi_1 + 2C\Phi_2\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}(I_{k-1})} + C\tau^{\tilde{\kappa}}\Phi_3\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}(I_k)},$$

which leads to

$$\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}(I_k)} < \frac{C\Phi_1}{1 - C\tau^{\tilde{\kappa}}\Phi_3} + \frac{2C\Phi_2}{1 - C\tau^{\tilde{\kappa}}\Phi_3}\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}(I_{k-1})}.$$

Iterating these estimates entails

$$\|y, G(\cdot, y)\|_{\mathcal{D}^\gamma_{X,\alpha}(I_k)} \leq \left(\frac{2C\Phi_2}{1 - C\tau^{\tilde{\kappa}}\Phi_3}\right)^{k+1}\left(\|y_s\|_\alpha + \|G(s, y_s)\|_{\alpha-\gamma}\right)$$

$$
\begin{aligned}
&+ \frac{C\Phi_1}{1-C\tau^{\tilde{\kappa}}\Phi_3}\sum_{j=0}^{k}\left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)^{j} \\
&= \left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)^{k+1}\big(\|y_s\|_\alpha + \|G(s,y_s)\|_{\alpha-\gamma}\big) + \frac{C\Phi_1}{1-C\tau^{\tilde{\kappa}}\Phi_3}\frac{1-\left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)^{k+1}}{1-\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}} \\
&= \left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)^{k+1}\big(\|y_s\|_\alpha + \|G(s,y_s)\|_{\alpha-\gamma}\big) \\
&\quad + \frac{C\Phi_1}{C\tau^{\tilde{\kappa}}\Phi_3+2C\Phi_2-1}\left(\left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)^{k+1}-1\right) \\
&\le \left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)^{k+1}\left(\|y_s\|_\alpha + \|G(s,y_s)\|_{\alpha-\gamma} + \frac{C\Phi_1}{C\tau^{\tilde{\kappa}}\Phi_3+2C\Phi_2-1}\right),
\end{aligned}
$$

since $2C\Phi_2 + Cr^{\tilde{\kappa}}\Phi_3 - 1 > 0$. Using now (3-28) we derive

$$
\begin{aligned}
\|y, G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])} &\le \big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\sum_{k=0}^{N(\tau)-1}\|y, G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X,\alpha}(I_k)} \\
&\le \big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\left(\|y_s\|_\alpha + \|G(s,y_s)\|_{\alpha-\gamma} + \frac{C\Phi_1}{C\tau^{\tilde{\kappa}}\Phi_3+2C\Phi_2-1}\right) \\
&\quad \times \frac{\left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)^{N(\tau)+1} - \frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}}{\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}-1} \\
&\le \big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\left(\|y_s\|_\alpha + \|G(s,y_s)\|_{\alpha-\gamma} + \frac{C\Phi_1}{C\tau^{\tilde{\kappa}}\Phi_3+2C\Phi_2-1}\right) \\
&\quad \times e^{(N(\tau)+1)\ln\left(\frac{2C\Phi_2}{1-C\tau^{\tilde{\kappa}}\Phi_3}\right)}\frac{1-C\tau^{\tilde{\kappa}}\Phi_3}{2C\Phi_2-1+C\tau^{\tilde{\kappa}}\Phi_3}.
\end{aligned}
$$

Finally, the bound $N(\tau) \le (t-s)\tau^{-1}$ entails (3-33). □

3.5.1 Linearization of rough partial differential equations

It is often advantageous to also examine the linearized equation of (RPDE), as the behavior of the linearization also provides insights about the original equation. Due to this, a Gronwall lemma for the linearized equation is established in this subsection.

Recall that the equation of interest is

$$
\begin{cases} \mathrm{d}y_t = \big(A(t)y_t + F(t,y_t)\big)\,\mathrm{d}t + G(t,y_t)\,\mathrm{d}\mathbf{X}_t, \\ y_0 \in E_\alpha. \end{cases} \tag{RPDE}
$$

The linearization $\mathrm{D}y_t^{y_0}$ of (RPDE) along an arbitrary solution $y_t^{y_0}$, with initial value y_0, is defined as the solution z_t of the following equation

$$
\begin{cases} \mathrm{d}z_t = \big(A(t)z_t + \mathrm{D}_2F\big(t,y_t^{y_0}\big)z_t\big)\,\mathrm{d}t + \mathrm{D}_2G\big(t,y_t^{y_0}\big)z_t\,\mathrm{d}\mathbf{X}_t, \\ z_0 \in E_\alpha, \end{cases} \tag{3-34}
$$

also called the first variation equation. Here, D_2F and D_2G denote the Fréchet derivatives of the nonlinear terms F and G with respect to the second variable. Suppressing the dependency of y on the initial condition y_0, the Gubinelli derivative of $H(y,z) := \mathrm{D}_2G(\cdot,y)z$ is given by

$$
\big(\mathrm{D}_2G(t,y_t)z_t\big)' = \mathrm{D}_2^2G(t,y_t)\big[y_t', z_t\big] + \mathrm{D}_2G(t,y_t)z_t',
$$

using the chain rule and the product rule for two controlled rough paths $(y,y'),(z,z') \in \mathcal{D}^{\gamma}_{X,\alpha}$. We first show that $(H(y,z),(H(y,z))') \in \mathcal{D}^{\gamma}_{X,\alpha-\sigma_G}$ together with an a-priori estimate. Based on this, we obtain a bound for the solution of the linearization (3-34) using the mild rough Gronwall lemma.

Lemma 3.45. *Let $(y,y'),(z,z') \in \mathcal{D}^{\gamma}_{X,\alpha}$ be the solution to (RPDE) with initial value $y_0 \in E_\alpha$ and the linearization along the solution given by (3-34). We have $(H(y,z),(H(y,z))') \in \mathcal{D}^{\gamma}_{X,\alpha-\sigma_G}$ and*

$$\left\| H(y,z),\big(H(y,z)\big)' \right\|_{\mathcal{D}^{\gamma}_{X,\alpha-\sigma_G}} \lesssim C_G\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)^2\Big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)^2 \|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}. \tag{3-35}$$

Proof. Let $(u,v) \in \Delta_{[s,t]}$. We directly obtain

$$\|\mathrm{D}_2G(\cdot,y)z\|_{\infty,\alpha-\sigma_G} \le C_G\|z\|_{\infty,\alpha},$$

as well as

$$\begin{aligned}\left\|\big(\mathrm{D}_2G(\cdot,y)z\big)'\right\|_{\infty,\alpha-\sigma_G-\gamma} &\lesssim C_G\big(\,\|y'\|_{\infty,\alpha-\gamma}\,\|z\|_{\infty,\alpha}+\|z'\|_{\infty,\alpha-\gamma}\,\big)\\ &\le C_G\Big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)\,\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}.\end{aligned}$$

The γ-Hölder regularity of $\big(H(y,z)\big)'$ in $E_{\alpha-2\gamma-\sigma_G}$ is straightforward using that

$$\begin{aligned}&\mathrm{D}_2^2G(v,y_v)\big[y'_v,z_v\big]-\mathrm{D}_2^2G(u,y_u)\big[y'_u,z_u\big]+\mathrm{D}_2G(v,y_v)z'_v-\mathrm{D}_2G(u,y_u)z'_u\\ &\quad=\big(\mathrm{D}_2^2G(v,y_v)-\mathrm{D}_2^2G(u,y_v)\big)\big[y'_v,z_v\big]+\big(\mathrm{D}_2^2G(u,y_v)-\mathrm{D}_2^2G(u,y_u)\big)\big[y'_v,z_v\big]\\ &\qquad+\mathrm{D}_2^2G(u,y_u)\big[(y'_v-y'_u),z_v\big]+\mathrm{D}_2^2G(u,y_u)\big[y'_u,(z_v-z_u)\big]\\ &\qquad+\big(\mathrm{D}_2G(v,y_v)-\mathrm{D}_2G(u,y_v)\big)z'_v\\ &\qquad+\big(\mathrm{D}_2G(u,y_v)-\mathrm{D}_2G(u,y_u)\big)z'_v+\mathrm{D}_2G(u,y_u)\big(z'_v-z'_u\big).\end{aligned}$$

For the first line we have, using (2-37) and the Hölder continuity of G

$$\begin{aligned}\big\|\big(\mathrm{D}_2^2G(v,y_v)-\mathrm{D}_2^2G(u,y_v)\big)\big[y'_v,z_v\big]\big\|_{\alpha-\sigma_G-2\gamma} &\lesssim C_G(v-u)^{2\gamma}\,\|y'\|_{\infty,\alpha-\gamma}\,\|z\|_{\infty,\alpha},\\ \big\|\big(\mathrm{D}^2G(u,y_v)-\mathrm{D}^2G(u,y_u)\big)\big[y'_v,z_v\big]\big\|_{\alpha-\sigma_G-2\gamma} &\lesssim C_G(v-u)^{\gamma}[y]_{\gamma,\alpha-2\gamma}\,\|y'\|_{\infty,\alpha-\gamma}\,\|z\|_{\infty,\alpha}\\ &\le C_G(v-u)^{\gamma}\,\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\,\|y,y'\|^2_{\mathcal{D}^{\gamma}_{X,\alpha}}\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}.\end{aligned}$$

The second line can be controlled using

$$\begin{aligned}\big\|(y'_v-y'_u)z_u\big\|_{\alpha-2\gamma} &\lesssim (v-u)^{\gamma}\,[y']_{\gamma,\alpha-2\gamma}\|z\|_{\infty,\alpha}\le (v-u)^{\gamma}\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}},\\ \big\|y'_v(z_v-z_u)\big\|_{\alpha-2\gamma} &\lesssim (v-u)^{\gamma}\|y'\|_{\infty,\alpha-\gamma}[z]_{\gamma,\alpha-2\gamma}\le (v-u)^{\gamma}\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}.\end{aligned}$$

The third line can be estimated as before due to the Hölder regularity

$$\big\|\big(\mathrm{D}_2G(v,y_v)-\mathrm{D}_2G(u,y_v)\big)z'_v\big\|_{\alpha-\sigma_G-2\gamma}\lesssim C_G(v-u)^{2\gamma}\,\|z'\|_{\infty,\alpha-\gamma}.$$

Based on the boundedness of G, we can estimate the last line by

$$\begin{aligned}\big\|\big(\mathrm{D}_2G(u,y_v)-\mathrm{D}_2G(u,y_u)\big)z'_v\big\|_{\alpha-\sigma_G-2\gamma} &\lesssim C_G(v-u)^{\gamma}[y]_{\gamma,\alpha-2\gamma}\|z'\|_{\infty,\alpha-\gamma}\\ &\le C_G(v-u)^{\gamma}\,\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\,\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}},\\ \big\|\mathrm{D}_2G(u,y_u)\big(z'_v-z'_u\big)\big\|_{\alpha-\sigma_G-2\gamma} &\le C_G(v-u)^{\gamma}\,[z']_{\gamma,\alpha-2\gamma}.\end{aligned}$$

For the remainder of $H(y,z)$, denoted by R^H, we obtain similar to (2-54)

$$
\begin{aligned}
R^H_{u,v} &= \mathrm{D}_2G(v,y_v)z_{u,v} + \big(\mathrm{D}_2G(v,y_v) - \mathrm{D}_2G(u,y_u)\big)z_u \\
&\quad - \big(\mathrm{D}_2^2G(u,y_u)\big[y'_u, z_u\big] + \mathrm{D}_2G(u,y_u)z'_u\big)X_{u,v} \\
&= \mathrm{D}_2G(v,y_v)R^z_{u,v} + \big(\mathrm{D}_2G(v,y_v) - \mathrm{D}_2G(v,y_u)\big)z'_uX_{u,v} \\
&\quad + \big(\mathrm{D}_2G(v,y_u) - \mathrm{D}_2G(u,y_u)\big)z'_uX_{u,v} + \big(\mathrm{D}_2G(v,y_v) - \mathrm{D}_2G(u,y_v)\big)z_u \\
&\quad + \int_0^1 \mathrm{D}_2^2G(u,y_u+ry_{u,v})[y_{u,v},z_u]\ \mathrm{d}r - \mathrm{D}_2^2G(u,y_u)\big[y'_u,z_u\big]X_{u,v} \\
&= \mathrm{D}_2G(v,y_v)R^z_{u,v} + \big(\mathrm{D}_2G(v,y_u) - \mathrm{D}_2G(u,y_u)\big)z'_uX_{u,v} \\
&\quad + \big(\mathrm{D}_2G(v,y_v) - \mathrm{D}_2G(v,y_u)\big)z'_uX_{u,v} + \big(\mathrm{D}_2G(v,y_v) - \mathrm{D}_2G(u,y_v)\big)z_u \\
&\quad + \int_0^1\int_0^1 \tilde{r}\mathrm{D}_2^3G\big(u,\tilde{r}(y_u+ry_{u,v}) + (1-\tilde{r})y_u\big)\big[y_{u,v},y'_u,z_u\big]\ \mathrm{d}r\mathrm{d}\tilde{r}X_{u,v} \\
&\quad + \int_0^1 \mathrm{D}_2^2G(u,y_u+ry_{u,v})[R^y_{u,v},z_u]\ \mathrm{d}r.
\end{aligned}
\tag{3-36}
$$

Using this representation we obtain that the remainder R^H is γ-Hölder continuous in $E_{\alpha-\sigma-\gamma}$ and 2γ-Hölder continuous in $E_{\alpha-\sigma-2\gamma}$. Indeed, let $i=1,2$, then for the first line it is straightforward to show that

$$
\begin{aligned}
\big\|\mathrm{D}_2G(v,y_v)R^z_{u,v}\big\|_{\alpha-\sigma_G-i\gamma} &\le C_G(v-u)^{i\gamma}[R^v]_{i\gamma,\alpha-i\gamma}, \\
\big\|\big(\mathrm{D}_2G(v,y_u) - \mathrm{D}_2G(u,y_u)\big)z'_uX_{u,v}\big\|_{\alpha-\sigma_G-i\gamma} &\le (v-u)^{3\gamma}\left\|z'\right\|_{\infty,\alpha-\gamma}\varrho_{\gamma,[s,t]}(\mathbf{X}).
\end{aligned}
$$

For the second line, we obtain

$$
\begin{aligned}
\big\|\big(\mathrm{D}_2G(v,y_v) - \mathrm{D}_2G(v,y_u)\big)z'_uX_{u,v}\big\|_{\alpha-\sigma_G-i\gamma} &\lesssim C_G\varrho_{\gamma,[s,t]}(\mathbf{X})(v-u)^{2\gamma}[y]_{\gamma,\alpha-i\gamma}\left\|z'\right\|_{\infty,\alpha-\gamma} \\
&\le C_G\left(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\right)(v-u)^{2\gamma}\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}}, \\
\big\|\big(\mathrm{D}_2G(v,y_v) - \mathrm{D}_2G(u,y_v)\big)z_u\big\|_{\alpha-\sigma_G-i\gamma} &\lesssim C_G(v-u)^{2\gamma}\left\|z\right\|_{\infty,\alpha}.
\end{aligned}
$$

The integral in the third line can be estimated as follows

$$
\begin{aligned}
&\left\|\int_0^1\int_0^1 \tilde{r}\mathrm{D}_2^3G\big(u,\tilde{r}(y_u+ry_{u,v}) + (1-\tilde{r})y_u\big)\big[y'_u,z_u,y_{u,v}\big]\ \mathrm{d}\tilde{r}\mathrm{d}rX_{u,v}\right\|_{\alpha-\sigma_G-i\gamma} \\
&\lesssim C_G\varrho_{\gamma,[s,t]}(\mathbf{X})(v-u)^{2\gamma}\left\|y'\right\|_{\infty,\alpha-\gamma}\|z\|_{\infty,\alpha}[y]_{\gamma,\alpha-i\gamma} \\
&\le C_G(v-u)^{2\gamma}\left(1-\varrho_{\gamma,[s,t]}(\mathbf{X})\right)^2\left\|y,y'\right\|^2_{\mathcal{D}^\gamma_{X,\alpha}}\left\|z,z'\right\|_{\mathcal{D}^\gamma_{X,\alpha}},
\end{aligned}
$$

whereas the last line finally entails

$$
\left\|\int_0^1 \mathrm{D}_2^2G(u,y_u+ry_{u,v})[R^y_{u,v},z_u]\ \mathrm{d}r\right\|_{\alpha-\sigma_G-i\gamma} \lesssim C_G(v-u)^{i\gamma}[R^y]_{i\gamma,\alpha-i\gamma}\|z\|_{\infty,\alpha}.
$$

Putting all these estimates together leads to (3-35). □

Now we can formulate a Gronwall inequality for the solution of the linearized equation. We recall that

$$
z_t = S_{t,s}z_s + \int_s^t S_{t,r}\mathrm{D}_2F(r,y_r)z_r\ \mathrm{d}r + \int_s^t S_{t,r}\mathrm{D}_2G(r,y_r)z_r\ \mathrm{d}\mathbf{X}_r,
$$

is the mild solution of (3-34). To handle the second integral, we need to impose more conditions on F.

(F2) In addition to **(F1)**, we assume that $F(r,\cdot)$ is Fréchet differentiable for every fixed $r \in [s,t]$ and that there exists a constant $L_{DF,r} > 0$ such that $\mathrm{D}_2 F(r,\cdot)$ is Lipschitz and

$$L_{DF} \coloneqq \sup_{r\in[s,t]} L_{DF,r} < \infty.$$

In particular, we have

$$\begin{aligned} \|\mathrm{D}_2F(v,x) - \mathrm{D}_2F(u,y)\|_{\mathcal{L}(E_\alpha;E_{\alpha-\sigma_F})} &\le L_{DF}\,\|x-y\|_\alpha\,, \\ \|\mathrm{D}_2F(v,x)\|_{\mathcal{L}(E_\alpha;E_{\alpha-\sigma_F})} &\le C_{DF}\big(1+\|x\|_\alpha\big), \end{aligned}$$

for $x,y \in E_\alpha$, $(u,v) \in \Delta_{[s,t]}$ and

$$C_{DF} \coloneqq \max\left\{L_{DF},\ \sup_{r\in[s,t]} \|\mathrm{D}_2F(r,0)\|_{\alpha-\sigma_F}\right\} < \infty.$$

Remark 3.46. It is possible to extend our results to the case where the Fréchet derivative of F satisfies a polynomial growth condition, for example

$$\|\mathrm{D}_2F(r,x)\|_{\mathcal{L}(E_\alpha;E_{\alpha-\sigma_F})} \lesssim Q(\|x\|_\alpha)$$

for some polynomial Q. For computational simplicity, we work with the linear growth assumption.

Corollary 3.47. *Suppose A, F and G satisfy the Assumptions **(A)**,**(F1)**-**(F2)** and **(G1)**-**(G2)**. Let $(y,y') \in \mathcal{D}^\gamma_{X,\alpha}$ be the solution to (RPDE) with initial value $y_0 \in E_\alpha$ and $(z,z') \in \mathcal{D}^\gamma_{X,\alpha}$ the linearization along this solution satisfying the equation (3-34). Then we have $z' = \mathrm{D}_2G(\cdot,y_\cdot)z$ and the estimate*

$$\|z, \mathrm{D}_2G(\cdot,y_\cdot)z\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])} \le \widetilde{C}_1\left(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\right)\Big(\|z_s\|_\alpha + \|\mathrm{D}_2G(s,y_s)z_s\|_{\alpha-\gamma}\Big)\, e^{\widetilde{C}_2(t-s)}, \tag{3-37}$$

where the constants are given by

$$\widetilde{C}_1 \coloneqq e^{\widetilde{C}_2}\frac{1-C\tau^{\tilde\kappa}\widetilde{\Phi}_3}{2C\widetilde{\Phi}_2 - 1 + C\tau^{\tilde\kappa}\widetilde{\Phi}_3}, \qquad \widetilde{C}_2 \coloneqq \frac{1}{\tau}\ln\left(\frac{2C\widetilde{\Phi}_2}{1-C\tau^{\tilde\kappa}\widetilde{\Phi}_3}\right), \tag{3-38}$$

with $C(S,\alpha,\sigma_G,\sigma_F,\gamma) > 0$, $\tilde\kappa = \min\{1-2\gamma, 1-\sigma_F, \gamma-\sigma\}$, $\tau < 1$ such that $2C\widetilde{\Phi}_2 > 1 - C\tau^{\tilde\kappa}\widetilde{\Phi}_3 > 0$ and

$$\begin{aligned} \widetilde{\Phi}_2 &\coloneqq \max\left\{1, C_G\left(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\right), C_G^2\left(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\right)\right\}, \\ \widetilde{\Phi}_3 &\coloneqq C_{DF}\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}\Big) + C_G\left(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\right)^3\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}\Big)^2. \end{aligned}$$

Proof. Using Lemma 3.45 we obtain for $(y,y'),\ (z,z') \in \mathcal{D}^\gamma_{X,\alpha}$ and $t-s<1$

$$\begin{aligned} \|S_{\cdot,s}z_s, 0\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])} &\lesssim \|z_s\|_\alpha\,, \\ \left\|\int_s^\cdot S_{\cdot,r}\mathrm{D}_2F(r,y_r)z_r\,\mathrm{d}r, 0\right\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])} &\lesssim C_{DF}(t-s)^{1-\overline{\sigma}_F}\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}\Big) \\ &\quad \times \|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}, \\ \|\mathrm{D}_2G(\cdot,y)z, (\mathrm{D}_2G(\cdot,y)z)'\|_{\mathcal{D}^\gamma_{X,\alpha-\sigma_G}([s,t])} &\lesssim C_G\left(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\right)^2\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}\Big)^2 \\ &\quad \times \|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}. \end{aligned}$$

Combining these estimates with (2-45) entails

$$\begin{aligned}\|z, \mathrm{D}_2G(\cdot,y)z\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])} &\lesssim \|z_s\|_\alpha + C_{DF}(t-s)^{1-\overline{\sigma}_F}\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}\Big)\|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}\\ &+ \big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\Big(\|\mathrm{D}_2G(s,y_s)z_s\|_{\alpha-\sigma_G} + \big\|(\mathrm{D}_2G(s,y_s)z_s)'\big\|_{\alpha-\sigma_G-\gamma}\Big)\\ &+ \big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)(t-s)^{\gamma-\sigma_G}\big\|\mathrm{D}_2G(\cdot,y)z, (\mathrm{D}_2G(\cdot,y.)z)'\big\|_{\mathcal{D}^\gamma_{X,\alpha-\sigma}([s,t])}\\ &\lesssim \widetilde{\Phi}_2\Big(\|z_s\|_\alpha + \|\mathrm{D}_2G(s,y_s)z_s\|_{\alpha-\gamma}\Big) + \widetilde{\Phi}_3(t-s)^{\tilde{\kappa}}\|v, \mathrm{D}_2G(\cdot,y)z\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}.\end{aligned}$$

Here, we used the fact that $y'_s = G(s,y_s)$ to obtain

$$\Big\|\big(\mathrm{D}_2G(s,y_s)z_s\big)'\Big\|_{\alpha-\sigma_G-\gamma} \le C_G\Big(\|y'_s\|_{\alpha-\gamma}\|z_s\|_{\alpha-\gamma} + \|z'_s\|_{\alpha-\gamma}\Big) \lesssim C_G^2\|z_s\|_\alpha + C_G\|z'_s\|_{\alpha-\gamma}.$$

The remaining proof is the same as in Lemma 3.44, with Φ_2 and Φ_3 substituted by $\widetilde{\Phi}_2$ and $\widetilde{\Phi}_3$. □

To obtain stability statements, for example, the existence of invariant manifolds, we further need an estimate of the difference between two linearizations for two different initial data points. Although we do not utilize this Gronwall for the two different linearizations in this thesis, it is still interesting in its own right and worth mentioning. For an application see [BGVS25, Subsection 5.4].

Therefore, we let $y_0, \tilde{y}_0 \in E_\alpha$ be two initial conditions and $y_t := y_t^{y_0}, \tilde{y}_t := y_t^{\tilde{y}_0}$ the corresponding solutions to (RPDE), with linearizations z_t and $\tilde{z}_t$. Then we are interested in the difference between the two solutions

$$\begin{aligned}z_t - \tilde{z}_t = S_{t,s}(z_s - \tilde{z}_s) &+ \int_s^t S_{t,r}\big(\mathrm{D}_2F(r,y_r)z_r - \mathrm{D}_2F(r,\tilde{y}_r)\tilde{z}_r\big)\,\mathrm{d}r\\ &+ \int_s^t S_{t,r}\big(\mathrm{D}_2G(r,y_r)z_r - \mathrm{D}_2G(r,\tilde{y}_r)\tilde{z}_r\big)\,\mathrm{d}\mathbf{X}_r.\end{aligned}$$

Similar to Lemma 3.45 we first investigate

$$\widetilde{H}(y_t,\tilde{y}_t,z_t,\tilde{z}_t) = \mathrm{D}_2G(t,y_t)z_t - \mathrm{D}_2G(t,\tilde{y}_t)\tilde{z}_t = H(y_t,z_t) - H(\tilde{y}_t,\tilde{z}_t),$$

with the Gubinelli derivative

$$\big(\widetilde{H}(y_t,\tilde{y}_t,z_t,\tilde{z}_t)\big)' = \mathrm{D}_2^2G(t,y_t)\big[y'_t,z_t\big] + \mathrm{D}G(t,y_t)z'_t - \big(\mathrm{D}_2^2G(t,\tilde{y}_t)\big[\tilde{y}'_t,\tilde{z}_t\big] + \mathrm{D}_2G(t,\tilde{y}_t)\tilde{z}'_t\big). \tag{3-39}$$

Now we derive a bound for $\tilde{H}$ depending on the difference between the controlled rough path norms of $\big(y-\tilde{y}, y'-\tilde{y}'\big)$ respectively $\big(z-\tilde{z}, z'-\tilde{z}'\big)$. As we have already seen, the estimates of the linearization include the third derivatives of G. Considering the difference between the two linearizations, we expect that another derivative is needed, which motivates the following condition.

(G4) There exists a constant $\sigma_G < \gamma$, such that G satisfies **(G3)** and is four times Fréchet differentiable with bounded derivative.

Lemma 3.48. *Assume G satisfied **(G4)**. Let $(y,y') \in \mathcal{D}^\gamma_{X,\alpha}$, $(\tilde{y},\tilde{y}') \in \mathcal{D}^\gamma_{X,\alpha}$ be two solutions of (RPDE) with initial data $y_0,\tilde{y}_0 \in E_\alpha$ and $(z,z'),(\tilde{z},\tilde{z}') \in \mathcal{D}^\gamma_{X,\alpha}$ be the corresponding linearizations.*

Then we have $\big(\widetilde{H}(y,\tilde{y},z,\tilde{z}),(\widetilde{H}(y,\tilde{y},z,\tilde{z}))'\big)\in\mathcal{D}^{\gamma}_{X,\alpha-\sigma_G}$ *and*

$$\begin{aligned}
&\Big\|\widetilde{H}(y,\tilde{y},z,\tilde{z}),(\widetilde{H}(y,\tilde{y},z,\tilde{z}))'\Big\|_{\mathcal{D}^{\gamma}_{X,\alpha-\sigma_G}}\\
&\quad\lesssim C_G\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)^2\\
&\qquad\times\bigg(\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big(\big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\big)\big(1+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\big)+\|\tilde{y},\tilde{y}'\|^2_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)\\
&\qquad+\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big(\big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}+\|y,y'\|^2_{\mathcal{D}^{\gamma}_{X,\alpha}}\big)\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\\
&\qquad+\big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\big)\|\tilde{z},\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)\bigg).
\end{aligned}\tag{3-40}$$

Proof. We have to derive estimates for the path component, Gubinelli derivative (3-39), and the remainder. The path component

$$\mathrm{D}_2G(t,y_t)z_t-\mathrm{D}_tG(t,\tilde{y}_t)\tilde{z}_t=\big(\mathrm{D}_2G(t,y_t)-\mathrm{D}_2G(t,\tilde{y}_t)\big)z_t+\mathrm{D}_2G(t,\tilde{y}_t)(z_t-\tilde{z}_t),$$

as well as the supremum norm of the Gubinelli derivative, are straightforward to estimate

$$\begin{aligned}
\Big\|\widetilde{H}(y,\tilde{y},z,\tilde{z})\Big\|_{\infty,\alpha-\sigma_G}&\lesssim C_G\Big(\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}+\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big),\\
\Big\|\big(\widetilde{H}(y,\tilde{y},z,\tilde{z})\big)'\Big\|_{\infty,\alpha-\sigma_G-\gamma}&\lesssim C_G\Big(\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\big)\\
&\qquad+\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\big(1+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\big)\Big).
\end{aligned}$$

The estimates for the Hölder continuity of the Gubinelli derivative and the remainder are more involved. The representations we are using are similar to (2-55) and (2-57). Especially, the part of the Gubinelli derivative, which depends only on the first derivative, can be rewritten as in (2-57)

$$\begin{aligned}
&\mathrm{D}_2G(v,y_v)z'_v-\mathrm{D}_2G(v,\tilde{y}_v)\tilde{z}'_v-\big(\mathrm{D}_2G(u,y_u)z'_u-\mathrm{D}_2G(u,\tilde{y}_u)\tilde{z}'_u\big)\\
&\quad=\Big(\mathrm{D}_2G(v,y_v)-\mathrm{D}_2G(u,y_v)-\big(\mathrm{D}_2G(v,\tilde{y}_v)-\mathrm{D}_2G(u,\tilde{y}_v)\big)\Big)z'_v\\
&\qquad+\big(\mathrm{D}_2G(v,\tilde{y}_v)-\mathrm{D}_2G(u,\tilde{y}_v)\big)\big(z'_v-\tilde{z}'_v\big)\\
&\qquad+\big(\mathrm{D}_2G(u,y_v)-\mathrm{D}_2G(u,y_u)-\big(\mathrm{D}_2G(u,\tilde{y}_v)-\mathrm{D}_2G(u,\tilde{y}_u)\big)\big)\tilde{z}'_v\\
&\qquad+\big(\mathrm{D}_2G(u,y_v)-\mathrm{D}_2G(u,y_u)\big)\big(z'_v-\tilde{z}'_v\big)\\
&\qquad+\big(\mathrm{D}_2G(u,y_u)-\mathrm{D}_2G(u,\tilde{y}_u)\big)z'_{u,v}+\mathrm{D}_2G(u,\tilde{y}_u)\left(z'_{u,v}-\tilde{z}'_{u,v}\right).
\end{aligned}\tag{3-41}$$

For the terms with second-order derivatives, we obtain in a similar way

$$\begin{aligned}
&\mathrm{D}^2_2G(v,y_v)\big[y'_v,z_v\big]-\mathrm{D}^2_2G(v,\tilde{y}_v)\big[\tilde{y}'_v,\tilde{z}_v\big]-\big(\mathrm{D}^2_2G(u,y_u)\big[y'_u,z_u\big]-\mathrm{D}^2_2G(u,\tilde{y}_u)\big[\tilde{y}'_u,\tilde{z}_u\big]\big)\\
&\quad=\big(\mathrm{D}^2_2G(v,y_v)-\mathrm{D}^2_2G(u,y_v)\big)\big[y'_v-\tilde{y}'_v,z_v\big]+\big(\mathrm{D}^2_2G(v,\tilde{y}_v)-\mathrm{D}^2_2G(u,\tilde{y}_v)\big)\big[\tilde{y}_v,z_v-\tilde{z}_v\big]\\
&\qquad+\Big(\big(\mathrm{D}^2_2G(v,y_v)-\mathrm{D}^2_2G(u,y_v)\big)-\big(\mathrm{D}^2_2G(v,\tilde{y}_v)-\mathrm{D}^2_2G(u,\tilde{y}_v)\big)\Big)\big[\tilde{y}'_v,\tilde{z}_v\big]\\
&\qquad+\Big(\big(\mathrm{D}^2_2G(u,y_v)-\mathrm{D}^2_2G(u,y_u)\big)-\big(\mathrm{D}^2_2G(u,\tilde{y}_v)-\mathrm{D}^2_2G(u,\tilde{y}_u)\big)\Big)\big[y'_v,z_v\big]\\
&\qquad+\big(\mathrm{D}^2_2G(u,\tilde{y}_v)-\mathrm{D}^2_2G(u,\tilde{y}_u)\big)\big[y'_v-\tilde{y}'_v,z_v\big]+\big(\mathrm{D}^2_2G(u,\tilde{y}_v)-\mathrm{D}^2_2G(u,\tilde{y}_u)\big)\big[\tilde{y}'_v,z_v-\tilde{z}_v\big]\\
&\qquad+\big(\mathrm{D}^2_2G(u,y_u)-\mathrm{D}^2_2G(u,\tilde{y}_u)\big)\big[y'_{u,v},z_v\big]+\big(\mathrm{D}^2_2G(u,y_u)-\mathrm{D}^2_2G(u,\tilde{y}_u)\big)\big[y'_u,z_{u,v}\big]\\
&\qquad+\mathrm{D}^2_2G(u,\tilde{y}_u)\big[y'_{u,v}-\tilde{y}'_{u,v},z_v\big]+\mathrm{D}^2_2G(u,\tilde{y}_u)\big[y'_u,z_{u,v}-\tilde{z}_{u,v}\big]\\
&\qquad+\mathrm{D}^2_2G(u,\tilde{y}_u)\big[y'_u-\tilde{y}'_u,\tilde{z}_{u,v}\big]+\mathrm{D}^2_2G(u,\tilde{y}_u)\big[\tilde{y}'_{u,v},z_v-\tilde{z}_v\big].
\end{aligned}\tag{3-42}$$

Most of the terms above can be easily estimated, as shown in Lemma 3.45. The only non-trivial ones are the first and third lines in (3-41) and the second and third lines of (3-42). In these four situations, we use a similar integral representation as in (2-56), which leads to

$$\begin{aligned}
&\Big(\big(\mathrm{D}_2 G(v,y_v)-\mathrm{D}_2 G(u,y_v)\big)-\big(\mathrm{D}_2 G(v,\tilde y_v)-\mathrm{D}_2 G(u,\tilde y_v)\big)\Big)z'_v\\
&\quad=\int_0^1\Big(\mathrm{D}_2^2 G\big(v,\tilde y_v+r(y_v-\tilde y_v)\big)-\mathrm{D}_2^2 G\big(u,\tilde y_v+r(y_v-\tilde y_v)\big)\Big)\big[y_v-\tilde y_v,z'_v\big]\,\mathrm{d}r,\\
&\Big(\big(\mathrm{D}_2 G(u,y_v)-\mathrm{D}_2 G(u,y_u)\big)-\big(\mathrm{D}_2 G(u,\tilde y_v)-\mathrm{D}_2 G(u,\tilde y_u)\big)\Big)\tilde z'_v\\
&\quad=\int_0^1\big(\mathrm{D}_2^2 G(u,y_u+ry_{u,v})-\mathrm{D}_2^2 G(u,\tilde y_u+r\tilde y_{u,v})\big)\big[y_{u,v},\tilde z'_v\big]\,\mathrm{d}r\\
&\qquad+\int_0^1\mathrm{D}_2^2 G(u,\tilde y_u+r\tilde y_{u,v})[y_{u,v}-\tilde y_{u,v},\tilde z'_v]\,\mathrm{d}r,\\
&\Big(\big(\mathrm{D}_2^2 G(v,y_v)-\mathrm{D}_2^2 G(u,y_v)\big)-\big(\mathrm{D}_2^2 G(v,\tilde y_v)-\mathrm{D}_2^2 G(u,\tilde y_v)\big)\Big)\big[y'_v,z_v\big]\\
&\quad=\int_0^1\Big(\mathrm{D}_2^3 G\big(v,\tilde y_v+r(y_v-\tilde y_v)\big)-\mathrm{D}_2^3 G\big(u,\tilde y_v+r(y_v-\tilde y_v)\big)\Big)\big[y_v-\tilde y_v,y'_v,z_v\big]\,\mathrm{d}r,\\
&\Big(\big(\mathrm{D}_2^2 G(u,y_v)-\mathrm{D}_2^2 G(u,y_u)\big)-\big(\mathrm{D}_2^2 G(u,\tilde y_v)-\mathrm{D}_2^2 G(u,\tilde y_u)\big)\Big)\big[y'_v,z_v\big]\\
&\quad=\int_0^1\big(\mathrm{D}_2^3 G(u,y_u+ry_{u,v})-\mathrm{D}_2^3 G(u,\tilde y_u+r\tilde y_{u,v})\big)\big[y_{u,v},y'_v,z_v\big]\,\mathrm{d}r\\
&\qquad+\int_0^1\mathrm{D}_2^3 G(u,\tilde y_u+r\tilde y_{u,v})\big[y_{u,v}-\tilde y_{u,v},y'_v,z_v\big]\,\mathrm{d}r.
\end{aligned}$$

This also shows why we need a fourth derivative of G, since we want to use the Lipschitz continuity of $\mathrm{D}_2^3 G$. Using similar estimates to Lemma 3.45 yields

$$\begin{aligned}
&\Big[\big(\widetilde H(y,\tilde y,v,\tilde z)\big)'\Big]_{\gamma,\alpha-\sigma_G-2\gamma}\\
&\lesssim C_G\left(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\right)\Big(\|z-\tilde z,z'-\tilde z'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}+\|\tilde y,\tilde y'\|^2_{\mathcal{D}^\gamma_{X,\alpha}}+\|\tilde y,\tilde y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big)\\
&\quad+\|y-\tilde y,y'-\tilde y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(\|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}}(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}+\|y,y'\|^2_{\mathcal{D}^\gamma_{X,\alpha}}\\
&\quad+(1+\varrho_{\gamma,[s,t]}(\mathbf{X}))\|\tilde y,\tilde y'\|_{\mathcal{D}^\gamma_{X,\alpha}})+(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}})\|\tilde z,\tilde z'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big)\Big).
\end{aligned}$$

Using the representation of the remainder (3-36) for both linearizations, we obtain here for the remainder of $\tilde H$ denoted by $R^{\tilde H}$

$$\begin{aligned}
R^{\widetilde H}_{u,v}&=\big(\mathrm{D}_2 G(v,y_v)-\mathrm{D}_2 G(u,y_v)\big)z_v-\big(\mathrm{D}_2 G(v,\tilde y_v)-\mathrm{D}_2 G(u,\tilde y_v)\big)\tilde z_v\\
&\quad+\big(\mathrm{D}_2 G(u,y_v)-\mathrm{D}_2 G(u,\tilde y_v)\big)R^z_{u,v}+\mathrm{D}_2 G(u,\tilde y_v)\big(R^z_{u,v}-R^{\tilde z}_{u,v}\big)\\
&\quad+\big(\mathrm{D}_2 G(u,\tilde y_v)-\mathrm{D}_2 G(u,\tilde y_u)\big)\big(z'_u-\tilde z'_u\big)X_{u,v}\\
&\quad+\int_0^1\Big(\mathrm{D}_2^2 G\big(u,\tilde y_v+r(y_v-\tilde y_v)\big)-\mathrm{D}_2^2 G\big(u,\tilde y_u+r(y_u-\tilde y_u)\big)\Big)\big[y_v-\tilde y_v,z'_u\big]\,\mathrm{d}rX_{u,v}\\
&\quad+\int_0^1\mathrm{D}_2^2 G\big(u,\tilde y_v+r(y_v-\tilde y_v)\big)\big[y_{u,v}-\tilde y_{u,v},z'_u\big]\,\mathrm{d}rX_{u,v}\\
&\quad+\int_0^1\Big(\mathrm{D}_2^2 G\big(u,ry_v+(1-r)y_u\big)-\mathrm{D}_2^2 G\big(u,r\tilde y_v+(1-r)\tilde y_u\big)\Big)\big[R^y_{u,v},z_u\big]\,\mathrm{d}r\\
&\quad+\int_0^1\mathrm{D}_2^2 G\big(u,r\tilde y_v+(1-r)\tilde y_u\big)\big[R^y_{u,v}-R^{\tilde y}_{u,v},z_u\big]\,\mathrm{d}r
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \mathrm{D}_2^2G\big(u, r\tilde{y}_v+(1-r)\tilde{y}_u\big)\big[R^{\tilde{y}}_{u,v}, z_u \;-\tilde{z}_u\big]\,\mathrm{d}r\\
&+\int_0^1\int_0^1 \tilde{r}\big(\mathrm{D}_2^3G(u,y_u+r\tilde{r}y_{u,v})-\mathrm{D}_2^3G(u,\tilde{y}_u+r\tilde{r}\tilde{y}_{u,v})\big)\big[y'_u,z_u,y_{u,v}\big]\,\mathrm{d}r\mathrm{d}\tilde{r}X_{u,v}\\
&+\int_0^1\int_0^1 \tilde{r}\mathrm{D}_2^3G(u,\tilde{y}_u+r\tilde{r}\tilde{y}_{u,v})\big[y'_u-\tilde{y}'_u,z_u,y_{u,v}\big]\,\mathrm{d}r\mathrm{d}\tilde{r}X_{u,v}\\
&+\int_0^1\int_0^1 \tilde{r}\mathrm{D}_2^3G(u,\tilde{y}_u+r\tilde{r}\tilde{y}_{u,v})\big[\tilde{y}'_u,z_u-\tilde{z}_u,y_{u,v}\big]\,\mathrm{d}r\mathrm{d}\tilde{r}X_{u,v}\\
&+\int_0^1\int_0^1 \tilde{r}\mathrm{D}_2^3G(u,\tilde{y}_u+r\tilde{r}\tilde{y}_{u,v})\big[\tilde{y}',\tilde{z}_u,y_{u,v}-\tilde{y}_{u,v}\big]\,\mathrm{d}r\mathrm{d}\tilde{r}X_{u,v}.
\end{aligned}
$$

In conclusion, the Hölder norm of the remainder can be estimated by

$$
\begin{aligned}
\big[R^{\widetilde{H}}\big]_{i\gamma,\alpha-\sigma-i\gamma} \lesssim C_G\Bigg(&\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(1+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big)\\
&+\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(\Big(1+\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}+\|y,y'\|^2_{\mathcal{D}^\gamma_{X,\alpha}}\Big)\|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}}\\
&+\|\tilde{z},\tilde{z}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(1+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big)\Big)\Bigg)\,\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)^2,
\end{aligned}
$$

for $i=1,2$, which leads to (3-40). □

Remark 3.49. The bound on the right-hand side of (3-40) naturally depends on $\|y,y'\|_{\mathcal{D}^\gamma_{X,\alpha}}$, $\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}$, $\|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}}$, and $\|\tilde{z},\tilde{z}'\|_{\mathcal{D}^\gamma_{X,\alpha}}$. For notational simplicity, we use the abbreviation

$$
\begin{aligned}
&\Big\|\widetilde{H}(y,\tilde{y},z,\tilde{z}),\big(\widetilde{H}(y,\tilde{y},z,\tilde{z})\big)'\Big\|_{\mathcal{D}^\gamma_{X,\alpha-\sigma_G}}\\
&\le CC_G\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)^2p(y,\tilde{y},z,\tilde{z})\Big(\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^\gamma_{X,\alpha}}+\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big),
\end{aligned}
\tag{3-43}
$$

for some polynomial $p(y,\tilde{y},z,\tilde{z})$.

Corollary 3.50. *Suppose that $(A(t))_{r\in[s,t]}$, F and G satisfy the Assumptions **(A)**,**(F1)**-**(F2)**, **(G4)**. Let $(y,y')\in\mathcal{D}^\gamma_{X,\alpha}$, $(\tilde{y},\tilde{y}')\in\mathcal{D}^\gamma_{X,\alpha}$ be two solutions of (RPDE) with initial data $y_0,\tilde{y}_0\in E_\alpha$ and $(z,z'),(\tilde{z},\tilde{z}')\in\mathcal{D}^\gamma_{\mathbf{X},\alpha}$ be the corresponding linearizations. Then we obtain*

$$
\|z-\tilde{z},\mathrm{D}_2G(\cdot,y)z-\mathrm{D}_2G(\cdot,\tilde{y})\tilde{z}\|_{\mathcal{D}^\gamma_{X,\alpha}([s,t])}\le\widehat{C}_1\Big(\|z_s-\tilde{z}_s\|_\alpha+\|z'_s-\tilde{z}'_s\|_{\alpha-\gamma}\Big)e^{\widehat{C}_2(t-s)},
\tag{3-44}
$$

where the constants are given by

$$
\begin{aligned}
\widehat{C}_1&:=e^{\widehat{C}_2}\max\left\{\frac{1-C\tau^{\tilde{\kappa}}\widehat{\Phi}_3}{2C\widehat{\Phi}_2-1+C\tau^{\tilde{\kappa}}\widehat{\Phi}_3},\frac{(1-C\tau^{\tilde{\kappa}}\widehat{\Phi}_3)C\widehat{\Phi}_1}{(C\tau^{\tilde{\kappa}}\widehat{\Phi}_3+2C\widehat{\Phi}_2-1)^2}\right\}\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\\
\widehat{C}_2&:=\frac{1}{\tau}\ln\left(\frac{2C\widehat{\Phi}_2}{1-C\tau^{\tilde{\kappa}}\widehat{\Phi}_3}\right),
\end{aligned}
$$

with $C(S,\alpha,\sigma_G,\sigma_F,\gamma)>0$, $\tilde{\kappa}=\min\{1-2\gamma,1-\sigma_F,\gamma-\sigma_G\}$, $\tau<1$ such that $2C\widehat{\Phi}_2>1-C\tau^{\tilde{\kappa}}\widehat{\Phi}_3>0$ and

$$
\widehat{\Phi}_1:=\|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}}+\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^\gamma_{X,\alpha}}\Big(C_{DF}(t-s)^{1-\overline{\sigma}_F}\|z,z'\|_{\mathcal{D}^\gamma_{X,\alpha}}
$$

$$
\begin{aligned}
&+ (1+\varrho_{\gamma,[s,t]}(\mathbf{X}))C_G\Big(\|\tilde{z},\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}} + \|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}} + \|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)\\
&+ (t-s)^{\gamma-\sigma_G}C_G\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)^3 p(y,\tilde{y},z,\tilde{z})\Big),\\
\widehat{\Phi}_2 &:= 1+\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\,C_G\Big(1+\|\tilde{y},\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)\\
\widehat{\Phi}_3 &:= C_{DF}(t-s)^{1-\overline{\sigma}_F}\Big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big) + (t-s)^{\gamma-\sigma_G}C_G\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)^3 p(y,\tilde{y},z,\tilde{z}).
\end{aligned}
$$

Proof. Similar to Corollary 3.47, we obtain $\|S_{\cdot,s}(z_s-\tilde{z}_s),0\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])} \lesssim \|z_s-\tilde{z}_s\|_\alpha$ and

$$
\begin{aligned}
&\left\|\int_s^{\cdot} S_{\cdot,r}\big(\mathrm{D}_2F_r(r,y_r)z_r - \mathrm{D}_2F_r(r,\tilde{y}_r)\tilde{z}_r\big)\,\mathrm{d}r, 0\right\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\\
&\lesssim C_{DF}(t-s)^{1-\overline{\sigma}_F}\Big(\Big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big)\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\\
&+\|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big(1+\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big)\Big),
\end{aligned}
$$

where $t-s<1$. Together with (3-43) and (2-45) we obtain

$$
\begin{aligned}
&\|z-\tilde{z},\mathrm{D}_2G(\cdot,y)z-\mathrm{D}_2G(\cdot,\tilde{y})\tilde{z}\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])} \lesssim \|z_s-\tilde{z}_s\|_\alpha\\
&\quad + C_{DF}(t-s)^{1-\overline{\sigma}_F}\Big(\Big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big)\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\\
&\quad + \|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big(1+\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big)\Big)\\
&\quad + \big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\Big(\|\mathrm{D}_2G(s,y_s)z_s-\mathrm{D}_2G(s,\tilde{y}_s)\tilde{z}_s\|_{\alpha-\sigma_G}\\
&\quad + \Big\|\big(\mathrm{D}_2G(s,y_s)z_s-\mathrm{D}_2G(s,\tilde{y}_s)\tilde{z}_s\big)'\Big\|_{\alpha-\sigma_G-\gamma}\\
&\quad + (t-s)^{\gamma-\sigma_G}\big\|\mathrm{D}_2G(\cdot,y_\cdot)z-\mathrm{D}_2G(\cdot,\tilde{y})\tilde{z},\big(\mathrm{D}_2G(\cdot,y_\cdot)z-\mathrm{D}_2G(\cdot,\tilde{y}_\cdot)\tilde{z}\big)'\big\|_{\mathcal{D}^{\gamma}_{X,\alpha-\sigma_G}([s,t])}\Big)\\
&\lesssim \|z_s-\tilde{z}_s\|_\alpha + C_{DF}(t-s)^{1-\overline{\sigma}_F}\Big(\Big(1+\|y,y'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big)\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\\
&\quad + \|z,z'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big(1+\|y-\tilde{y},y'-\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}([s,t])}\Big)\Big)\\
&\quad + \big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)\Big(C_G\Big(\|z_s-\tilde{z}_s\|_\alpha + \|y_s-\tilde{y}_s\|_\alpha\|\tilde{z}_s\|_\alpha\Big)\\
&\quad + C_G\Big(\|y_s-\tilde{y}_s\|_\alpha\|y_s'\|_{\alpha-\gamma}\|z_s\|_\alpha + \|y_s'-\tilde{y}_s'\|_{\alpha-\gamma}\|z_s\|_\alpha\\
&\quad + \|z_s-\tilde{z}_s\|_\alpha\|\tilde{y}_s\|_\alpha + \|y_s-\tilde{y}_s\|_\alpha\|z_s'\|_{\alpha-\gamma} + \|z_s'-\tilde{z}_s'\|_{\alpha-\gamma}\Big)\\
&\quad + (t-s)^{\gamma-\sigma_G}C_G\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X})\big)^2 p(y,\tilde{y},z,\tilde{z})\Big(\|z-\tilde{z},z^-\tilde{z}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}} + \|y-\tilde{y},y^-\tilde{y}'\|_{\mathcal{D}^{\gamma}_{X,\alpha}}\Big)\Big)\\
&\lesssim \widehat{\Phi}_1 + \widehat{\Phi}_2\Big(\|z_s-\tilde{z}_s\|_\alpha + \|z_s'-\tilde{z}_s'\|_{\alpha-\gamma}\Big) + \widehat{\Phi}_3(t-s)^{\tilde{\kappa}}\|z-\tilde{z},z'-\tilde{z}'\|_{\mathcal{D}^{\alpha}_{X,\alpha}}.
\end{aligned}
$$

As in the proof of Lemma 3.45, this yields the claim. □

Remark 3.51. Note that the constants $\widehat{C}_1$ and $\widehat{C}_2$ used in (3-44) depend on the controlled rough path norms of the linearizations z and $\tilde{z}$. It is possible to use (3-37) to bound these norms, resulting in a Gronwall inequality where the right-hand side only depends on $y,\tilde{y}$, and the initial conditions z_s and z_s'.

Part II

Dynamics of stochastic evolution equations

Chapter 4

Random dynamics

4.1 Motivation

In this second part of this thesis, we deal with the long-term behavior of rough partial differential equations. This means we analyze the evolution of solutions as time approaches infinity, which is one of the key reasons why we were interested in global solutions in the previous part. Several aspects are interesting and frequently investigated, such as asymptotic behavior, bifurcations, stability, or invariant subsets of the phase space.

Before we start explaining the theory and its definitions, let us briefly describe some of its key aspects.

(Exponential) stability

Stability can be investigated in various forms, but what they all have in common is that we want to examine how a solution behaves under small perturbations.

Consider a physical pendulum fastened to a rigid bar. First, note that there are two possible states where there is no motion: Namely, the pendulum is either in the lowest or the highest possible position, in each case without kinetic energy. The two states are also referred to as equilibrium points. In both cases, the pendulum remains stationary. Now, two important things are observable: If the pendulum is deflected from the lower position, it swings for a while and then returns to the same position. However, deflecting the pendulum from its upper position causes it to swing for some time, but it does not return to its original position; instead, it settles into the lower equilibrium position. This observation makes the lower position "stable" and the upper "unstable".

Of course, this can be formulated more generally. Take a solution y, associated with an initial value y_0 of an evolution equation. The solution is then stable if every solution corresponding to $\tilde{y}_0$ with $\|y_0 - \tilde{y}_0\| < \delta$ for some small $\delta > 0$ behaves similarly. More precise, there exists a small $\varepsilon(\delta) > 0$ such that

$$\|y(t) - \tilde{y}(t)\| \leq \varepsilon, \tag{4-1}$$

for $t > 0$. If this is the case, the corresponding trajectory is called Lyapunov stable and exponentially stable if (4-1) even goes to zero with an exponential rate for $t \to \infty$. Therefore, an unstable trajectory means that a slight change in the initial conditions results in a completely different behavior of the solution.

A fundamental element of these studies is the concept of Lyapunov exponents, which quantify the rate at which two trajectories, initially differing slightly in their values, diverge or approach each other. A positive Lyapunov exponent indicates instability, while a negative one corresponds to stability.

Attractors

Strongly connected to the notion of stability is the concept of an attractor in an evolution equation. Hereby, we mean a subset of the phase space, which “attracts” a wide variety of solutions. Ideally, such a set attracts every solution regardless of the initial state. Defining attraction precisely is part of the following sections, as it is particularly challenging in non-autonomous and random settings.

The attractor is interesting for several reasons. Firstly, in some sense, the attractor reflects the “typical behavior” of the equation under consideration. To illustrate this point, let us consider dissipative systems, which are frequently encountered in physical applications. In this context, the system loses energy to the environment through friction or similar effects, as illustrated again by the physical pendulum. In the presence of oscillation, the amplitude of it decreases in proportion to the decrease in energy in the system, as a result of the air resistance experienced by the pendulum. After some time, the oscillation ceases. The bottom equilibrium point is then a candidate for the attractor of this system. Conversely, if a pendulum is initiated from its resting position and is periodically driven by an external force, such as a motor, it begins to oscillate with increasing amplitude over time. This continues until the air resistance and the driving force balance each other, resulting in the pendulum swinging with a constant amplitude in each period. In this state, the system exhibits its “typical behavior”. And this is precisely the idea of an attractor: For dissipative systems, we assume that the motion decays slowly, which corresponds to the solution converging towards an equilibrium. Introducing a force, we expect the system to take time to balance out the dissipation and the forcing.

In contrast to the pendulum, the time required for a system to exhibit its typical behavior can be quite long. Therefore, the attractive property of the attractor also includes a convergence to effectively approximate and understand the dynamics.

Another significant advantage of attractors is their tendency to be a finite-dimensional subspace of the phase space, even when the underlying system is infinite-dimensional. This characteristic is particularly beneficial in practical applications, as it simplifies the analysis of the system dynamics by reducing the complexity involved.

Invariant manifolds

To obtain an attractor, some sort of dissipativity of the system is needed. Invariant manifolds are more versatile in this regard and can also deliver information about the dynamics of non-dissipative systems. The attractor itself is an example of an invariant set, satisfying forward invariance, where a trajectory remains within the invariant set once it enters, and backward invariance, where the past trajectory of a point within the set also lies within it. These properties ensure that an invariant set maintains its structure over time.

Now there are many different invariant manifolds, like stable, unstable, or center manifolds, to name the most common ones. Each of them captures the dynamical behavior of certain points. The stable manifold of an equilibrium position, for example, contains every initial state such that the solution starting in this point ends up in the equilibrium. Conversely, the unstable manifold contains all positions that are moving away from the equilibrium. For example, consider the pendulum again. We have already seen that there are two equilibrium points. For the lower equilibrium point, the stable manifold is given by every starting position except the upper equilibrium point. The stable manifold of the upper equilibrium would consist just containing of the equilibrium point itself. For the unstable manifold, it is the opposite. In this example, we also see an illustration of the difference to the attractor: The attractor of a system marks the “endpoint” of motion, where the invariant manifolds illustrate the road to this endpoint.

Our main object of interest is the global attractor of a rough evolution equation. But also the stability, Lyapunov exponents, and invariant manifolds are addressed in Chapter 5. To do so, a framework is needed in which long-term behavior and concepts such as attraction can be investigated. It can be difficult to define "the" notion of attraction, since there are several possibilities. To understand where the problems lie and what is important for long-term behavior, we begin by investigating the autonomous case, followed by the addition of time dependencies. In the end, these observations can be applied to treat stochastic systems. The strategy of presenting the topic in this way is inspired by many articles, for example [CSG11, KR11, CLR13, CH16].

4.1.1 Dynamical systems

We refrain from giving a more detailed introduction to autonomous dynamical systems, as our main focus in this thesis is on random equations, which is why we give the more general definitions in the later sections. For further information, see, for example, the monographs [KH95, Rob99, Tes12] for finite-dimensional systems and [BV92, Tem97, SY02] for the infinite-dimensional case.

Let us consider the scalar-valued autonomous equation

$$\dot{y}_t = f(y_t), \tag{4-2}$$

for some function $f\colon \mathbb{R} \to \mathbb{R}$ and assume that (4-2) has a unique global solution for any initial value $y_s \in \mathbb{R}$. Denote by $y(\cdot, s; y_s)$ the solution, and therefore the trajectory, which starts at time s in y_s. Then, the uniqueness leads to the fact that $y(t, s; y_s) = y\big(t, r; y(r, s; y_s)\big)$ for every, $s < r < t$. Further, note that since f is time-independent, the solution trajectory does not depend on the initial time but only on the elapsed time. Indeed, it is possible to show that $y(t, s; y_s) = y(t-s, 0; y_s) =: y(t-s; y_s)$, see [KR11, Theorem 1.1]. This map $(t, x) \mapsto y(t; x)$ is then called a dynamical system generated by (4-2).

Now, address the question of what properties we want to require of an attractor. The most crucial one is how to define attraction and which points in phase space should be attracted at all. Since the attractor should encapsulate the behavior "at infinity", it is reasonable to take $t \to \infty$. So we say a set $\widetilde{O} \subset \mathbb{R}$ is attracted by O, if

$$\lim_{t\to\infty} \operatorname{dist}\big(y\big(t; \widetilde{O}\big), O\big) = 0,$$

where $y\big(t; \widetilde{O}\big) \coloneqq \big\{y(t; x) : x \in \widetilde{O}\big\}$ and

$$\operatorname{dist}\big(y\big(t; \widetilde{O}\big), O\big) \coloneqq \sup_{x \in y(t;\widetilde{O})} \inf_{y\in O} |x - y|,$$

is the Hausdorff semi-distance of sets. So, $\widetilde{O} \subset \mathbb{R}$ is attracted by O if any trajectory starting in $\widetilde{O}$ converges to O.

The question of which sets should be attracted to form an attractor is a little more subtle. Consider therefore $f(y) \coloneqq y\big(1 - y^2\big)$. The solution of the equation corresponding to this nonlinearity is given by

$$y(t; y_0) = \begin{cases} \dfrac{e^t}{\sqrt{e^{2t} + \frac{1}{y_0^2} - 1}}, & y_0 \geq 0, \\ -\dfrac{e^t}{\sqrt{e^{2t} + \frac{1}{|y_0|^2} - 1}}, & y_0 < 0. \end{cases}$$

It is easy to see that $-1, 0$ and 1 are the only equilibrium points; this means $f(y^*) = 0$, and that every trajectory converges to one of these equilibria. The minimal set that attracts every point is $\{-1, 0, 1\}$, which is referred to as a point attractor of the equation. A downside of this definition is that we can not attract every bounded set. For example, we have

$$\lim_{t\to\infty} \operatorname{dist}\big(y\big(t; (0, \varepsilon)\big), \{-1, 0, 1\}\big) = \frac{1}{2},$$

for some small $\varepsilon > 0$. Indeed, for every $t > 0$, there exists a initial state $0 < y_0 \le \varepsilon$ such that $y(t; y_0) = 1/2$, which leads to

$$\operatorname{dist}\big(y\big(t; (0, \varepsilon)\big), \{-1, 0, 1\}\big) = \frac{1}{2}.$$

To enable uniform attraction, we have to require the attraction of all bounded subsets. In the example above, every set attracting all bounded subsets has to contain at least $[0, 1]$.

The last thing is the uniqueness of the attractor. To achieve this, we additionally impose that the attractor is a compact invariant subset. A set $O \subset \mathbb{R}$ is called invariant if and only if $y(t; O) = O$ for every $t \in \mathbb{R}$. This means that the attractor can only contain global trajectories, so if y_0 is already in the attractor, then all future and past values also belong to the attractor.
To summarize, we obtain the following definition.

Definition 4.1. A subset $\mathscr{A}$ of the phase space is called an attractor to the dynamical system $(t, x) \mapsto y(t; x)$ if

i) $\mathscr{A}$ is compact,
ii) $\mathscr{A}$ is invariant,
iii) $\mathscr{A}$ attracts every bounded subset.

4.1.2 Non-autonomous dynamical systems

Consider instead of (4-2), a non-autonomous equation given by

$$\dot{y}_t = f(t, y_t), \tag{4-3}$$

such that the equation admits a unique global solution for any initial value $y_s \in \mathbb{R}$. Since the right-hand side is now time-dependent, we expect the solution to depend not only on the elapsed time but also on the initial time s. Denote again by $y(\cdot, s; y_s)$ the solution of (4-3), which starts at time s in y_s. Then, the uniqueness leads to the fact that

$$y(t, s; y_s) = y\big(t, r; y(r, s; y_s)\big)$$

for every $s < r < t$. The map

$$\{(s, t) : s, t \in \mathbb{R}, s \le t\} \times \mathbb{R} \to \mathbb{R}, (t, s, x) \mapsto y(s, t; x),$$

is called a non-autonomous dynamical system.

To define a non-autonomous attractor, the first idea is to follow the same approach as in the autonomous case. However, since the solution depends on the initial time, the resulting attractor does not necessarily reflect the actual dynamics of the system. To see this, consider first the concrete nonlinearity $f(t, x) = -x + t$. The solution is then given by

$$y(t, s; y_s) = t - 1 + \big(y_s - (s - 1)\big)e^{-(t-s)},$$

which means that for fixed $s \in \mathbb{R}$, we obtain

$$\lim_{t\to\infty} y(t,s;y_s) = \infty.$$

In particular, we do not have an attractor in the sense of the previous section. This is surprising, since the dissipative term $-y$ indicates the existence of an attractor, due to the corresponding autonomous equation $\dot{x} = -x, x(s) = \tilde{x} \in \mathbb{R}$ satisfying

$$\lim_{t\to\infty} x(t,s;\tilde{x}) = \lim_{t\to\infty} x(t-s,0;\tilde{x}) = \lim_{t\to\infty} \tilde{x}e^{-(t-s)} = 0.$$

The dependence on time appears to change the asymptotic behavior, but we can still infer something about long-term behavior. Indeed, consider two initial values $y_s, \tilde{y}_s \in \mathbb{R}$, then we obtain for the two corresponding solutions

$$|y(t) - \tilde{y}(t)| = |y_s - \tilde{y}_s|\, e^{-(t-s)} \to 0, \tag{4-4}$$

for $t \to \infty$. This means that all solutions of the equation converge towards each other! In particular, each solution exhibits the same asymptotic behavior, which is a primary motivation for the attractor. This suggests that the definition of the attractor needs to be adapted to the time dependence. So, how can we generalize the principle of attraction to reflect the behavior we can see in (4-4)?

In the autonomous case, this was achieved by fixing the initial state and sending $t \to \infty$. But since the solution only depends on the elapsed time, this is equivalent to fixing the observing time t instead and sending the initial time s to $-\infty$. For example, reconsider $\dot{x} = -x, x(s) = \tilde{x} \in \mathbb{R}$, here the solution satisfies

$$\lim_{t\to\infty} x(t-s,0;\tilde{x}) = \lim_{t\to\infty} \tilde{x}e^{-(t-s)} = 0 = \lim_{s\to-\infty} \tilde{x}e^{-(t-s)} = \lim_{s\to-\infty} x(t-s,0;\tilde{x}),$$

so both approaches lead to the same behavior. For non-autonomous equations, the solution depends explicitly on s and t, which means there is a difference between these two concepts, as the resulting limit depends in general on s and t respectively. In the example above, this pullback approach leads to

$$\lim_{s\to-\infty} y(t,s;x) = t - 1,$$

which tells us that every solution behaves asymptotically like $t \mapsto t-1$, or more precisely every solution converges to $t \mapsto t - 1 =: \mathscr{A}(t)$. In this example, $\big(\mathscr{A}(t)\big)_{t\in\mathbb{R}}$ would then be the canonical candidate for the non-autonomous attractor. We refer to this as a pullback attraction, since we pull the initial value back into the past and consider the system at a given time t. In contrast to the attraction in the autonomous case, where we have a fixed initial time s from which we move forward in time, which is sometimes referred to as forward attraction.

However, it generally makes sense that the attractor in a non-autonomous system itself depends on time, which means that this pullback approach is more suitable in this case. Still, due to the resulting attractor being also time-dependent, the invariance property must be adapted accordingly.

Definition 4.2. A family of sets $\big(\mathscr{A}(t)\big)_{t\in\mathbb{R}}$ is called a pullback attractor to the non-autonomous dynamical system $(s,t,x) \mapsto y(t,s;x)$ if

i) $\mathscr{A}(t)$ is compact for every $t \in \mathbb{R}$,
ii) $\mathscr{A}$ is invariant, this means $y\big(t,s;\mathscr{A}(s)\big) = \mathscr{A}(t)$ for every $s \le t$, and

iii) $\mathscr{A}$ pullback attracts every bounded subset, this means that for every bounded set O we have
$$\lim_{s\to-\infty} \operatorname{dist}\big(y(t,s;O),\mathscr{A}(t)\big) = 0.$$

Remark 4.3. i) The pullback approach also has some significant limitations. It describes, to some extent, the dynamical behavior in the past and does not always provide much insight into future behavior. In particular, $\mathscr{A}(t)$ describes the state of the system at time t after it has been running for an infinite amount of time and does not look ahead in that sense.

On the other hand, it is also possible to define the so-called forward attractor, replacing condition iii) by
$$\lim_{t\to\infty} \operatorname{dist}\big(y(t,s;B),\mathscr{A}(s)\big) = 0,$$
for every bounded subset O. The forward dynamics are often different to the pullback dynamics, as we have already seen in the previous example, but in order get a full description of the dynamical behavior both concepts are needed; see, for example, recent publications [KY21, Chapter 11] and [HKP22, CFLLN24, GFKLS25].

ii) Another alternative concept is the uniform attractor. Here, we replace the condition iii) by
$$\lim_{t\to\infty} \sup_{s\in\mathbb{R}} \operatorname{dist}\big(y(t,s;B),\mathscr{A}\big) = 0,$$
where it does not matter whether we consider the limit in the forward or pullback sense. Note that in this case, the resulting attractor does not depend on time! Therefore, the notion of invariance assumed in ii) does not make sense, and the properties i) and ii) are substituted by demanding that $\mathscr{A}$ is the minimal closed subset which attracts all bounded sets uniformly. For more information see [CLR13, Chapter 16],[CV02, Section IV.3] and for applications for example [Wan08].

iii) Another example is the exponential attractor, where the key difference is that the attraction property, which can be again either of pullback or forward type, is assumed to be exponentially fast, and that the attractor needs to have a finite fractal dimension. However, only forward invariance is required for this concept. It turns out that this is more general than the pullback/forward attractor, but not necessarily unique. It is possible to show that if a non-autonomous dynamical system generates an exponential attractor, then it also possesses a pullback/forward attractor. Another advantage is that exponential attractors are stable under perturbation. For more information, see [EFNT94, CS13].

iv) A natural question is if it is possible to allow further time-dependent families to be attracted, instead of only bounded subsets. Generally, the set of attractable sets is referred to as the basin of attraction. Choosing this basin as the set of all bounded subsets, this has the consequence that the pullback attractor itself is, in general, not part of the basin, since $\mathscr{A}(t)$ is time-dependent. A different basin is given by the so-called tempered time-dependent sets $\big(O(t)\big)_{t\in\mathbb{R}}$, which satisfy
$$\sup_{x\in O(t)} |x|\, e^{at} \to 0,$$
for $t\to-\infty$ and every $a>0$. Since every bounded set $O\subset\mathbb{R}$ is tempered, the pullback attractor with the collection of all tempered sets as the basin of attraction may be bigger. It turns out that in many cases, it is enough to consider only all bounded sets. Indeed, if the attractor itself is bounded in the past, meaning there exists a time τ and a bounded set $O\subset\mathbb{R}$, such that $\mathscr{A}(t)\subset O$ for every $t\le\tau$, then every basin of attraction which contains all bounded subsets, leads to the same pullback attractor [MRR09].

4.2 Random dynamical systems and rough dynamics

The reason for the detailed considerations in the previous section is that the results obtained can be applied to the stochastic case. This is because the noise $\mathrm{d}\mathbf{X}_t$ can be understood as a non-autonomous term, since it is a stochastic process and therefore time-dependent. For this reason, our aim in this section is to transfer the idea of the pullback attractor to the random case. The resulting notion of a random dynamical system is also discussed in many books on non-autonomous dynamical systems, due to the aforementioned connection, for example, [CH16, Chapter 4], [KR11, Chapter 14], and [CLR13, Section 1.7]. For a detailed discussion on random dynamical systems, see the monograph [Arn98].

4.2.1 Basic definitions

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\mathcal{E}$ a separable Banach space. We first specify the model for the noise, which enables us to track changes over time. Therefore, we recall the definition of a metric dynamical system.

Definition 4.4. i) A family of mappings $(\theta_t)_{t\in\mathbb{R}}$ defined by $\theta_t:\Omega\to\Omega$, is called measure-preserving if for every $O\in\mathcal{F}$ and $t\in\mathbb{R}$ we have $\mathbb{P}\big(\theta_t^{-1}O\big)=\mathbb{P}(O)$.

ii) The quadrupel $\big(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\big)$, where $(\theta_t)_{t\in\mathbb{R}}$ is measure-preserving, is called a metric dynamical system if
   1) $\theta_0=\mathrm{Id}_\Omega$,
   2) $(t,\omega)\mapsto\theta_t\omega$ is $\mathcal{B}(\mathbb{R})\otimes\Sigma-\Sigma$ measurable,
   3) $\theta_{t+s}=\theta_t\circ\theta_s$ for all $t,s\in\mathbb{R}$.

   We call it an ergodic metric dynamical system if for any $(\theta_t)_{t\in\mathbb{R}}$-invariant set $O\in\mathcal{F}$, we have $\mathbb{P}(O)\in\{0,1\}$.

This definition can be interpreted as follows: If the noise $\omega\in\Omega$ is observed at the initial time, say 0, and the system evolves for time t, then the noise that is observed after that time is $\theta_t\omega$. Restarting the equation from time t requires now this updated noise term represented by $\theta_t\omega$. For example, it is possible to generate a metric dynamical system from Brownian motion, as illustrated in the following example.

Example 4.5. *We introduce a metric dynamical system associated with a two-sided $\mathcal{E}$-valued Brownian motion defined on a filtered probability space $\big(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\geq0}\big)$, where $(\mathcal{F}_t)_{t\geq0}$ is the natural filtration of the Brownian motion. To this end, let $\mathscr{H}$ be a Hilbert space and $\big(B_{\mathscr{H}}(t)\big)_{t\geq0}$, where $B_{\mathscr{H}}\colon\mathscr{H}\times[0,\infty)\to\mathrm{L}^2(\Omega)$ is an $\mathscr{H}$-cylindrical Brownian motion, which means that $(t,\omega)\mapsto\big(B_{\mathscr{H}}(t)h\big)(\omega)$ is a real-valued Brownian motion for every $h\in\mathscr{H}$ and*

$$\mathbb{E}[B_{\mathscr{H}}(t)h_1B_{\mathscr{H}}(s)h_2]=\langle h_1,h_2\rangle_{\mathscr{H}}\min\{s,t\},$$

holds for every $s,t\in[0,T]$ and $h_1,h_2\in\mathscr{H}$. Further, fix an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ for $\mathscr{H}$ and let $\mathcal{G}\colon\mathscr{H}\to\mathcal{E}$ be a γ-radonifying operator. This means that $\mathcal{G}$ is linear and for any sequence $(\gamma_k)_{k\in\mathbb{N}}$ of independent Gaussian random variables, the series

$$\mathbb{E}\left[\left\|\sum_{k\in\mathbb{N}}\gamma_k\mathcal{G}e_k\right\|_{\mathcal{E}}^2\right]<\infty, \tag{4-5}$$

is finite. If $\mathscr{H}$ is isomorphic to $\mathcal{E}$, then (4-5) implies that $\mathcal{G}\in\mathcal{L}_2(\mathscr{H};\mathscr{H})$ is a Hilbert-Schmidt operator. In this case we obtain $\|\mathcal{G}\|_{\mathcal{L}_2(\mathscr{H};\mathscr{H})}=\mathrm{Tr}(\mathcal{G}\mathcal{G}^*)$.

According to [vN08, Proposition 8.8], the series

$$\sum_{n\in\mathbb{N}}B_{\mathscr{H}}(t)\tilde{e}_n\mathcal{G}e_n,$$

then converges almost surely and the limit defines an $\mathcal{E}$-valued Brownian motion with covariance operator $t\mathcal{G}\mathcal{G}^*$, where $(\tilde{e}_n)_{n\in\mathbb{N}} \subset \left(ker(\mathcal{G})\right)^{\perp}$ is an orthonormal basis. For a more detailed treatment of γ-radonifying operators and Banach space-valued Brownian motions, see [vN08, HvNVW17].

Now, let $\left(B^1_t\right)_{t\geq 0}$ and $\left(B^2_t\right)_{t\geq 0}$ be two independent $\mathcal{E}$-valued Brownian motions and define the two-sided Brownian motion with values in $\mathcal{E}$ by

$$B_t := \begin{cases} B^1_t, & t \geq 0, \\ B^2_{-t}, & t < 0. \end{cases}$$

Denote by $\mathcal{C}_0(\mathbb{R};\mathcal{E})$ the subset of continuous functions which are zero at zero and equip it with the compact open topology generated by the metric

$$d_{\mathcal{C}_0}(f,g) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(f,g)}{1+d_k(f,g)},$$

for $f,g \in \mathcal{C}_0(\mathbb{R};\mathcal{E})$ and $d_k(f,g) := \sup_{-k\leq t\leq k} \|f(t)-g(t)\|_{\mathcal{E}}$. Then Kolmogorov's existence theorem yields the existence of a probability space $\left(\mathcal{C}_0(\mathbb{R};\mathcal{E}), \mathcal{B}(\mathcal{C}_0(\mathbb{R};\mathcal{E})), \mathbb{P}_W\right)$ such that the Brownian motion is the canonical process $(t,\omega) \mapsto B_t(\omega) := \omega_t$ for $\omega \in \mathcal{C}_0(\mathbb{R};\mathcal{E})$, where $\mathbb{P}_W$ is the Wiener measure, see [vN08, Theorem 6.9]. To obtain a metric dynamical system, all that remains is to define $(\theta_t)_{t\in\mathbb{R}}$ by

$$\theta_t\omega(\cdot) := \omega(t+\cdot) - \omega(t),$$

for $t \in \mathbb{R}$ and $\omega \in \mathcal{C}_0(\mathbb{R};\mathcal{E})$, which is sometimes called the Wiener shift. Since Brownian motion has stationary increments, it follows almost directly that $(\theta_t)_{t\in\mathbb{R}}$ is measure-preserving. The ergodicity can be proven along the lines of Mañé's [Mañ87, Chapter 1] proof for Bernoulli shifts, which is carried out in Boxler's doctoral thesis [Box88, Lemma 3.3], see also [Arn98, Appendix A].

Remark 4.6. It is also known that more general processes generate an ergodic metric dynamical system. For example, ergodicity is demonstrated in Kümmel's doctoral thesis [Kü90, Theorem 3.6] for Levy processes and in [GAS11] for the fractional Brownian motion.

The next Lemma collects some standard growth properties of the Brownian motion, which are used to prove the existence of a random attractor for the pathwise mild solution in Chapter 6.

Lemma 4.7. ([GLR11, Lemma 3.3]) *In the situation of Example 4.5, there exists a $(\theta_t)_{t\in\mathbb{R}}$-invariant subset $\Omega \subset C_0(\mathbb{R};\mathcal{E})$ of full measure, with the following properties:*

i) For all $\omega \in \Omega$ and any $\varepsilon > 0$ there exists a time $T_0(\varepsilon,\omega) > 0$ such that for any $|t| \geq T_0(\varepsilon,\omega)$ we have linear growth of ω, i.e.

$$\|\omega(t)\|_{\mathcal{E}} \leq \varepsilon\,|t|\,. \tag{4-6}$$

ii) For any $\gamma \in (0, 1/2)$ and any $[s,r] \subset \mathbb{R}$ there exist a constant $c_{s,r,\gamma}(\omega) > 0$ such that $c_{s,r,\gamma} \in \mathrm{L}^1(\Omega)$ and

$$\|\omega\|_{\gamma,\mathcal{E},[s,r]} \leq c_{s,r,\gamma}(\omega). \tag{4-7}$$

Remark 4.8. Condition (4-6) implies that $\omega \in \Omega$ has subexponential growth, meaning that for any $\varepsilon > 0$ there exists a time $t_0 = t_0(\varepsilon,\omega)$ and a constant $c_\varepsilon(\omega,t_0)$ such that for all $|t| \geq t_0$, we have

$$\|\omega(t)\|_{\mathcal{E}} \leq c_\varepsilon(\omega,t_0)e^{\varepsilon|t|}.$$

Given a model of the noise, we can now define a random dynamical system.

Definition 4.9. A continuous random dynamical system on a separable Banach space $\mathcal{E}$ over a metric dynamical system $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ is a mapping

$$\phi : [0,\infty) \times \Omega \times \mathcal{E} \to \mathcal{E}, (t, \omega, x) \mapsto \phi(t, \omega, x),$$

which is $\big(\mathcal{B}([0,\infty)) \otimes \Sigma \otimes \mathcal{B}(\mathcal{E}), \mathcal{B}(\mathcal{E})\big)$-measurable and satisfying

i) $\phi(0, \omega, \cdot) = \mathrm{Id}_{\mathcal{E}}$ for every $\omega \in \Omega$,
ii) for all $\omega \in \Omega$, $t, s \in [0,\infty)$ and $x \in \mathcal{E}$ we have

$$\phi(t + s, \omega, x) = \phi\big(t, \theta_s\omega, \phi(s, \omega, x)\big), \tag{4-8}$$

iii) the map $\phi(t, \omega, \cdot) \colon \mathcal{E} \to \mathcal{E}$ is continuous for every $t \in [0,\infty)$ and $\omega \in \Omega$.

Condition ii) is referred to as the (perfect) cocycle property and is illustrated in Figure 4.1, which can be found in a similar form in many monographs like [Arn98, Figure 1.2].

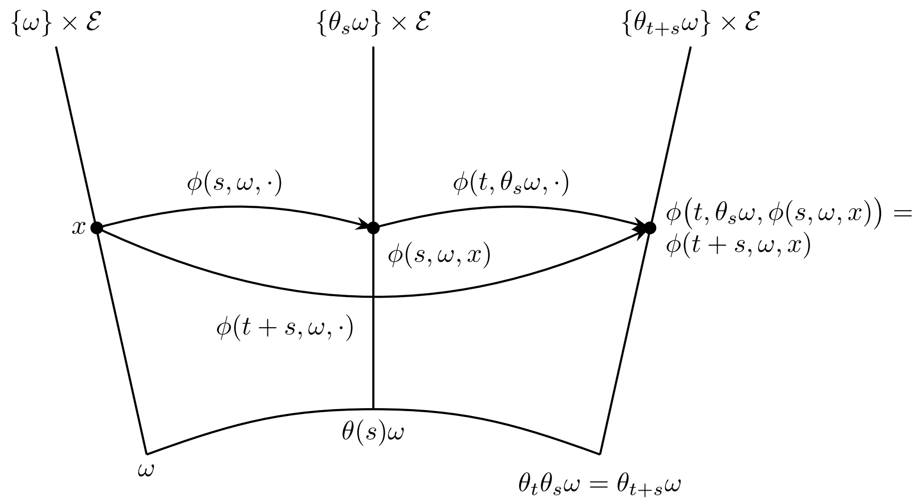

FIGURE 4.1: This diagram illustrates the cocycle property of a random dynamical system ϕ, where the evolution of a point $x \in \mathcal{E}$ under the flow $\phi(t, \omega, \cdot)$ corresponds to sequential applications along the metric dynamical system $(\theta_t)_{t\in\mathbb{R}}$.

Remark 4.10. The cocycle property must hold for every $\omega \in \Omega$, since we need to investigate the long-term behavior pathwise. However, it is possible to weaken this condition. If (4-8) holds for every fixed $s \in \mathbb{R}$ and all $t \in \mathbb{R}$ almost surely, so the exceptional set depends on s but not on t, (4-8) is called a crude cocycle property and if (4-8) holds for every fixed $s, t \in \mathbb{R}$ almost surely, therefore with an exceptional set depending on s and t, a very crude cocycle. It is possible to state a so-called perfection theorem [AS95], which constructs, starting from a crude or very crude cocycle ϕ, a map $\tilde{\phi}$ that satisfies the perfect co-cycle property and is indistinguishable from ϕ. The only thing not allowed is that the exceptional set depends on the initial datum x. This is a major reason why the existence of random dynamical systems generated by stochastic partial differential equations is tricky.

We now apply the same idea used for non-autonomous dynamical systems to define the notion of attraction. To do so, define the basin of attraction used in the random setting.

Definition 4.11. i) A family of non-empty closed sets $K := \{K(\omega)\}_{\omega\in\Omega}$ in $\mathcal{E}$ is called a random set if

$$\omega \mapsto \inf_{y \in K(\omega)} \|x - y\|_{\mathcal{E}},$$

is a random variable for all $x \in \mathcal{E}$. It is called a bounded, or compact, random set if $K(\omega)$ is bounded, respectively compact, for every $\omega \in \Omega$.

ii) A random variable $Y : \Omega \to [0, \infty)$ is called tempered with respect to $(\theta_t)_{t\in\mathbb{R}}$ if

$$\lim_{t\to\pm\infty} e^{-\beta|t|} Y(\theta_t \omega) = 0,$$

for all $\beta > 0$ and $\omega \in \Omega$. If $K := \{K(\omega)\}_{\omega\in\Omega}$ is a bounded random set and

$$\omega \mapsto \sup_{x\in K(\theta_{-t}\omega)} \|x\|_{\mathcal{E}},$$

is tempered, then K is called a tempered set.

Remark 4.12. A direct consequence of the definition of temperedness is that finite sums of tempered random variables, as well as positive powers, are again tempered.

The temperedness, defined in Definition 4.11, is equivalent to a subexponential growth condition, which means

$$\lim_{t\to\pm\infty} \frac{\log^+ \big(Y(\theta_t\omega)\big)}{|t|} = 0. \tag{4-9}$$

There are several ways to check the temperedness of random variables. In many cases, the following sufficient condition can be used.

Proposition 4.13. *Let $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ be an ergodic metric dynamical system. A random variable $Y : \Omega \to [0, \infty)$ satisfying*

$$\mathbb{E}\left[\sup_{r\in[s,t]} Y(\theta_r\omega)\right] < \infty, \tag{4-10}$$

for some interval $[s, t]$, is tempered.

Proof. This was proven for $[s, t] = [0, 1]$ in [Arn98, Theorem 4.1.3], see also [Tan74], using the ergodic properties of $(\theta_t)_{t\in\mathbb{R}}$ and a Borell–Cantelli argument. This argument is completely independent of the choice of the interval, so it is possible to do the same proof for an arbitrary interval $[s, t]$, see [BGAS14, Lemma 8]. □

Due to the subexponential growth of a tempered random variable, the following lemma is easy to verify.

Lemma 4.14. ([BGAS14, Lemma 3.7]). *Let Y be a tempered random variable and $\gamma > 0$. Then*

$$\omega \mapsto \int_{-\infty}^{0} e^{\gamma r} Y(\theta_r\omega) \, \mathrm{d}r$$

is tempered.

At this point, all the necessary elements are in place to define a random attractor. To translate the pullback attraction into the stochastic framework, consider first how to encode the starting time in the random dynamical system, since ϕ does not depend on the initial time as in the non-autonomous case. This is where the noise model comes into play, because the metric dynamical system tracks the temporal change by definition. So $\phi(t, \omega, x)$ means that the system that started at time 0 at x after the time t has elapsed, where ω corresponds to time 0. If instead $\theta_{-t}\omega$ is plugged in, this would mean that the initial time is now $-t$. Considering than $\phi(t, \theta_{-t}\omega, x)$ and sending $t \to \infty$, corresponds to sending the initial condition to $-\infty$ and evaluating the system at time 0.

Definition 4.15. A tempered random set $\mathscr{A} := \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ is called a random pullback attractor for the random dynamical system ϕ and ergodic metric dynamical system $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ if

i) $\mathscr{A}(\omega)$ is compact for every $\omega \in \Omega$,
ii) $\mathscr{A}$ is ϕ-invariant, this means for every $t \geq 0$ and $\omega \in \Omega$ we have

$$\phi\big(t, \omega, \mathscr{A}(\omega)\big) = \mathscr{A}(\theta_t\omega),$$

iii) $\mathscr{A}$ pullback attracts every tempered random set $K = \{K(\omega)\}_{\omega\in\Omega}$, this means

$$\lim_{t\to\infty} \operatorname{dist}\big(\phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega)), \mathscr{A}(\omega)\big) = 0, \tag{4-11}$$

holds for all $\omega \in \Omega$.

Remark 4.16. As in the deterministic case, the pullback attraction can be substituted to achieve different notions of an attractor, for example, the random forward attractor [CK18], the random exponential attractor [KNS21] and the random uniform attractor [ZLC21], where the difference to the random pullback attractor is the same as in the non-autonomous setting. What is different in the stochastic case is the possibility of including randomness in the definition, as happens with the weak attractor, or attractor in probability. Here, the pointwise convergence is substituted by convergence in probability. More precisely, replace (4-11) with

$$\begin{aligned}\lim_{t\to\infty} &\mathbb{P}\big(\{\omega \in \Omega : \operatorname{dist}(\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)), \mathscr{A}(\omega) > \delta\}\big)\\ &= \lim_{t\to\infty} \mathbb{P}\big(\{\omega \in \Omega : \operatorname{dist}(\phi(t, \omega, B(\omega)), \mathscr{A}(\theta_t\omega) > \delta\}\big),\end{aligned}$$

where the equality holds due to the θ_t-invariance of the probability measure. In particular, this leads to the fact that pullback and forward attraction yield the same result. For more information about this type of attractor, see [AO03].

In many applications, (4-11) is challenging to verify. A simpler option is to use the so-called absorbing sets. The existence of a compact absorbing set is a sufficient condition to ensure the existence of random pullback attractors.

Definition 4.17. A tempered random set $K = \{K(\omega)\}_{\omega\in\Omega}$ is called random pullback absorbing if for every tempered random set $\widetilde{K} = \{\widetilde{K}(\omega)\}_{\omega\in\Omega}$ and $\omega \in \Omega$ there exists an absorbing time $T_{\widetilde{K}}(\omega) > 0$ such that

$$\phi\big(t, \theta_{-t}\omega, \widetilde{K}(\theta_{-t}\omega)\big) \subset K(\omega),$$

for all $t \geq T_{\widetilde{K}}(\omega)$.

Lemma 4.18. *Let $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ be a metric dynamical system and ϕ a random dynamical system. If there exists a positive tempered random variable R, such that for any tempered random set $\{K(\omega)\}_{\omega\in\Omega}$, $\omega \in \Omega$ and $x \in K(\theta_{-t}\omega)$ the estimate*

$$\limsup_{t\to\infty} \big\|\phi\big(t, \theta_{-t}\omega, x\big)\big\|_{\mathcal{E}} \leq R(\omega) \tag{4-12}$$

holds, then the open ball $\{B(0, R(\omega) + \delta)\}_{\omega\in\Omega}$ is a random pullback absorbing set, for some constant $\delta > 0$.

Proof. For every $\delta > 0$ and $\omega \in \Omega$, there exists a $T_K(\omega) > 0$, due to (4-12), such that

$$\big\|\phi\big(t, \theta_{-t}\omega, x\big)\big\|_{\mathcal{E}} \leq R(\omega) + \delta,$$

for every $t \geq T_K(\omega)$. This proves, that $\{B(0, R(\omega) + \delta)\}_{\omega\in\Omega}$ is an absorbing set. □

Theorem 4.19. ([FS96, Theorem 3.5]) *Assume the existence of a compact tempered random set $K := \{K(\omega)\}_{\omega\in\Omega}$ which is random pullback absorbing with respect to the continuous random dynamical system ϕ. Then there exists a unique random pullback attractor $\mathscr{A} := \{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ for ϕ, given by*

$$\mathscr{A}(\omega) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \phi\big(t, \theta_{-t}\omega, K(\theta_{-t}\omega)\big)}.$$

Remark 4.20. Similar to the non-autonomous case, an intuitive question is, what happens if the collection of tempered random sets is substituted by a different basin of attraction? It turns out that the biggest possible attractor $\mathscr{A}_C$ is the one where the basin of attraction contains every compact deterministic set if it exists, see [Cra99]. Since every compact deterministic set is trivially tempered, the attractor achieved in Theorem 4.19 satisfies $\mathscr{A}_C = \mathscr{A}$ $\mathbb{P}$-almost surely if the metric dynamical system is ergodic.

4.2.2 Non-autonomous random dynamical systems

The random dynamical system, as defined so far, is only tailored to autonomous equations. Since the underlying idea behind the definition of random dynamical systems is to utilize results from non-autonomous deterministic theory, the noise can be viewed as a time-dependent force. Therefore, it is possible to incorporate additional time dependencies separately, which leads to the definition of a non-autonomous random dynamical system; see, for example, [Wan08, CL17]. For this approach, it is necessary to set up a model for the temporal change of the non-autonomous terms, similar to the metric dynamical system.

Definition 4.21. The tuple $\big(\Sigma, (\vartheta_t)_{t\in\mathbb{R}}\big)$ is a symbol space, if Σ is a Polish metric space and $\vartheta\colon \mathbb{R}\times\Sigma \to \Sigma$ satisfies

i) $\vartheta_0 = \mathrm{Id}_\Sigma$,
ii) $(t, \sigma) \mapsto \vartheta_t\sigma$ is continuous,
iii) $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ for all $t, s \in \mathbb{R}$.

With this at hand, it is possible to combine the non-autonomous dynamical systems and the random dynamical systems. Note that the classical definition of an autonomous random dynamical system can be recovered by setting $\Sigma = \emptyset$.

Definition 4.22. A continuous non-autonomous random dynamical system on a separable Banach space $\mathcal{E}$ over a metric dynamical system $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ and a symbol space $\big(\Sigma, (\vartheta_t)_{t\in\mathbb{R}}\big)$ is a mapping

$$\phi\colon [0,\infty)\times\Omega\times\Sigma\times\mathcal{E} \to \mathcal{E}, (t,\omega,\sigma,x) \mapsto \phi(t,\omega,\sigma,x),$$

which is $\big(\mathcal{B}([0,\infty))\otimes\mathcal{F}\otimes\mathcal{B}(\Sigma)\otimes\mathcal{B}(\mathcal{E}), \mathcal{B}(\mathcal{E})\big)$-measurable and satisfies

i) $\phi(0,\omega,\sigma,\cdot) = \mathrm{Id}_{\mathcal{E}}$ for every $\omega\in\Omega, \sigma\in\Sigma$,
ii) $\phi(t+s,\omega,\sigma,x) = \phi\big(t, \theta_s\omega, \vartheta_s\sigma, \phi(s,\omega,\sigma,x)\big)$ for all $\omega\in\Omega, \sigma\in\Sigma$, $t, s \in [0,\infty)$ and $x\in\mathcal{E}$,
iii) the map $\phi(t,\omega,\sigma,\cdot)\colon \mathcal{E}\to\mathcal{E}$ is continuous for every $t\in[0,\infty)$ and $\omega\in\Omega, \sigma\in\Sigma$.

It is now possible to adapt the notion of a pullback attractor in this non-autonomous framework [CL17]. However, the strategy presented in this thesis differs. Instead of using a non-autonomous random dynamical system, we construct a larger probability space that incorporates the non-autonomous parts through the nonlinearities. The only thing missing is a probability measure $\mathbb{P}_\Sigma$ acting on the time-dependencies. For a similar approach in the case of fractional Brownian motion and Young integrals, see [Hon25]. To be precise, the probability space is enlarged by the symbol space, which enables us to use results for autonomous random dynamical systems and makes the presentation clearer. This seems reasonable, since the symbol space and the metric dynamical system are defined similarly. Assuming that such a measure exists, define the extended probability space by

$$\big(\Omega \times \Sigma, \mathcal{F} \otimes \mathcal{B}(\Sigma), \mathbb{P} \otimes \mathbb{P}_\Sigma\big).$$

Construction of a compact symbol space

In this section, the construction of a symbol space for an equation of the form (RPDE) is discussed. In particular, it is important to obtain a compact symbol space, since this is needed for the theorem of Krylov–Bogolyubov, which states the existence of an invariant probability measure. This idea is also used in [BGVS25].

To do so, add the structural assumption $A(t) = A\big(\xi(t)\big)$ for some function ξ, to the Assumptions **(A)**. This means that ξ collects the time-dependence of the linear part of the equation, for example $A(t) = \xi(t)\Delta = A\big(\xi(t)\big)$. Most of the non-autonomous linear operators used in applications are of this form; see, for example, [CV02]. Together with the time-dependencies incorporated by the nonlinearities, the time symbol of the equation (RPDE) is then given by

$$\mathfrak{S}\colon \mathbb{R} \to \mathcal{X}, t \mapsto \mathfrak{S}(t) \coloneqq \big(\xi(t), F(t,\cdot), G(t,\cdot)\big)$$

for some topological Hausdorff function space $\mathcal{X}$, which we will specify later.

The symbol space Σ together with $(\vartheta_t)_{t\in\mathbb{R}}$ models the temporal change of the nonlinearities and the linear part. To achieve this, choose $(\vartheta_t)_{t\in\mathbb{R}}$ as the temporal shift $\vartheta_t y(\cdot) \coloneqq y(\cdot + t)$. Indeed, if the system is started at time s, which corresponds to $\mathfrak{S}(s)$, the symbol changes to $\vartheta_t\mathfrak{S}(s) = \mathfrak{S}(t+s)$ after the time t. The natural choice of Σ is then the collection of all time shifts of the original time symbol. Therefore, define the hull of $\mathfrak{S}$

$$\mathcal{H}(\mathfrak{S}) \coloneqq \overline{\{\mathfrak{S}(\cdot + s) : s \in \mathbb{R}\}}^{\mathcal{X}}$$

as the completion of the set of all time shifts with respect to the topology of $\mathcal{X}$. Indeed, $\mathcal{H}(\mathfrak{S})$ is invariant under $(\vartheta_t)_{t\in\mathbb{R}}$. In order to work with the hull as the symbol space, we make the following assumption:

(S) The hull $\mathcal{H}(\mathfrak{S})$ is a compact Polish metric space.

Then we define the symbol space as $\Sigma \coloneqq \mathcal{H}(\mathfrak{S})$ with the translation operator $\vartheta_t y \coloneqq y(\cdot + t)$ for every $y \in \Sigma$. Let us now state some conditions for the compactness of the hull such that Assumption **(S)** is satisfied. For further information and detailed proofs, see [CV02, Chapter V] and [CL17, Section 6].

Definition 4.23. A function $g \in \mathcal{X}$ is called translation compact if $\mathcal{H}(g)$ is compact.

The easiest way to obtain such translation compact functions is to consider periodic functions.

Example 4.24. *Let $\mathcal{X} = \mathcal{C}_b(\mathbb{R};\mathbb{R})$ be the space of bounded continuous functions, and assume that $g \in \mathcal{C}_b(\mathbb{R};\mathbb{R})$ is periodic with period T. Then it can be shown, using Arzelà–Ascoli's theorem, that $\mathcal{H}(g) = \{g(t+\cdot) : t \in [0,T]\}$ is compact. There are also generalizations of the periodicity, such as almost or quasi-periodic functions on $\mathcal{C}_b(\mathbb{R};\mathbb{R})$, which also entail compactness of the hull [CV02, Example IV.1.1 & IV.1.2].*

Some other sufficient and necessary conditions for translation compactness of a function hardly depend on the choice of $\mathcal{X}$. We mention three special cases here, which apply to our setting of semilinear parabolic evolution equations.

Proposition 4.25. ([CV02, Proposition 2.2, 3.3, 4.1])

i) Let $\big(\mathcal{M}, d_{\mathcal{M}}\big)$ be a complete metric space and define $\mathcal{X} := \mathcal{C}(\mathbb{R};\mathcal{M})$. Then a function $g \in \mathcal{X}$ is translation compact if and only if g is uniformly continuous, such that there exists a positive function k_g with $k_g(s) \to 0$ for $s \searrow 0$ and

$$d_{\mathcal{M}}\big(g(t), g(s)\big) \leq k_g\big(|t-s|\big),$$

for all $t, s \in \mathbb{R}$.

ii) Let $\big(\mathcal{M}, \|\cdot\|_{\mathcal{M}}\big)$ be a Banach space, $p \geq 1$ and define $\mathcal{X} := \mathrm{L}^p_{loc}(\mathbb{R};\mathcal{M})$, which is the space of locally L^p-integrable functions. Then a function $g \in \mathcal{X}$ is translation compact if and only if there exists a function k_g such that $k_g(s) \to 0$ for $s \searrow 0$ and

$$\int_t^{t+1} \|g(s) - g(s+t)\|^p_{\mathcal{M}} \; \mathrm{d}s \leq k_g\big(|t|\big),$$

for all $t \in \mathbb{R}$.

iii) Let $\big(\mathcal{M}, \|\cdot\|_{\mathcal{M}}\big)$ be a reflexive Banach space, $p \geq 1$ and define $\mathcal{X} := \mathrm{L}^p_{loc,w}(\mathbb{R};\mathcal{M})$, which is the space $\mathrm{L}^p_{loc}(\mathbb{R};\mathcal{M})$ endowed with the local weak convergence topology. Then a function $g \in \mathcal{X}$ is translation compact if and only if g is translation bounded in $\mathrm{L}^p_{loc}(\mathbb{R};\mathcal{M})$, which means

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(s)\|^p_{\mathcal{M}} \; \mathrm{d}s < \infty$$

In all three situations, the hull $\mathcal{H}(g)$ is a compact Polish space.

Lemma 4.26. *Let $(\mathcal{X}_i)_{i=1}^k$ be a collection of Hausdorff topological spaces and $(g_i)_{i=1}^k$ such that $g_i \in \mathcal{X}_i$ is translation compact. Then $g = (g_1, \dots, g_k) \in \mathcal{X} := \prod_{i=1}^k \mathcal{X}_i$ is translation compact and in particular $\mathcal{H}(g)$ is compact.*

Lemma 4.27. *Assume $\big(A(r)\big)_{r\in[s,t]}$, F and G satisfy **(A)**, **(F1)** and **(G1)**. Further let ξ be translation compact, so for example let $\xi \in \mathcal{C}_b(\mathbb{R};\mathbb{R})$ be periodic. Then Assumption **(S)** is satisfied.*

Proof. Due to Lemma 4.26, it is enough to consider each component of the time symbol separately. The first component is, by assumption, translation compact. Recall that the family of operators generates a Banach scale of order $m \in \mathbb{N}$ given by $(A_\beta, E_\beta)_{\beta\geq -m}$

The second component of the time symbol is the drift term F. Define the space $\mathcal{M}_2$ as the set of all continuous functions $f\colon E_\alpha \to E_{\alpha-\sigma_F}$ such that

$$\|f\|_{\mathcal{M}_2} := \sup_{x\in E_\alpha} \frac{\|f(x)\|_{\alpha-\sigma_F}}{1+\|x\|_\alpha}$$

is finite. Then $\big(\mathcal{M}_2, \|\cdot\|_{\mathcal{M}_2}\big)$ is a Banach space, see [CV02, Remark 2.10], and define $\mathcal{X}_2 = \mathrm{L}^p_{\mathrm{loc},w}(\mathbb{R};\mathcal{M}_2)$. Note that Assumption **(F1)** implies

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \left(\sup_{x\in E_\alpha} \frac{\|F(s,x)\|_{\alpha-\sigma_F}}{1+\|x\|_\alpha} \right)^p \; \mathrm{d}s \leq C_F^p < \infty.$$

The last component, the diffusion coefficient G, can be treated in a similar manner. Define $\mathcal{M}_3$ as the space of three times Fréchet differentiable functions $g\colon E_\alpha \to E_{\alpha-\sigma_G}$ such that

$$\|g\|_{\mathcal{M}_3} \coloneqq \sup_{x\in E_\alpha} \|g(x)\|_{\alpha-\sigma_G} + \sup_{x\in E_\alpha} \|\mathrm{D}g(x)\|_{\mathcal{L}(E_\alpha;E_{\alpha-\sigma_G})} + \sup_{x\in E_\alpha} \left\|\mathrm{D}^2 g(x)\right\|_{\mathcal{L}(E_\alpha^2;E_{\alpha-\sigma_G})} < \infty.$$

Then $\left(\mathcal{M}_3, \|\cdot\|_{\mathcal{M}_3}\right)$ is a Banach space and define $\mathcal{X}_3 = \mathcal{C}(\mathbb{R};\mathcal{M}_3)$. Additionally, due to Assumption **(G1)**, G is Hölder continuous in time. Define $s \mapsto k_G(s) \coloneqq s^{2\gamma}$, which satisfies $k_G(s) \to 0$ for $s \searrow 0$ and

$$\|G(t,x) - G(s,x)\|_{\mathcal{M}_3} \lesssim k_G\big(|t-s|\big),$$

due to the Hölder continuity. Then Assumption **(S)** is satisfied due to Proposition 4.25 and Lemma 4.26. ☐

Extended probability space

As the symbol space is now constructed, we can discuss how to enlarge the probability space to incorporate Σ. The main task is to equip $\big(\Sigma, \mathcal{B}(\Sigma)\big)$ with a probability measure $\mathbb{P}_\Sigma$, which leaves $(\vartheta_t)_{t\in\mathbb{R}}$ invariant. Afterward, we consider the extended metric dynamical system

$$\big(\Omega\times\Sigma, \mathcal{F}\otimes\mathcal{B}(\Sigma), \mathbb{P}\otimes\mathbb{P}_\Sigma, (\theta_t,\vartheta_t)_{t\in\mathbb{R}}\big). \tag{4-13}$$

Theorem 4.28. *Let Assumption* ***(S)*** *be satisfied. Then, there exists at least one probability measure $\mathbb{P}_\Sigma$ on $\big(\Sigma,\mathcal{B}(\Sigma)\big)$ such that $(\vartheta_t)_{t\in\mathbb{R}}$ is invariant under $\mathbb{P}_\Sigma$ and*

$$\mathbb{P}_\Sigma\big(\{\mathfrak{S}(\cdot+h) : h\geq 0\}\big) = 1.$$

Proof. Due to the compactness of Σ, a direct application of the Krylov-Bogolyubov theorem [BCD+89, Theorem 1.1] yields that

$$\nu \coloneqq \lim_{T\to\infty} \frac{1}{T}\int_0^T \delta_{\vartheta_t\mathfrak{S}(\cdot)}\,\mathrm{d}t,$$

is a probability measure on $\big(\Sigma,\mathcal{B}(\Sigma)\big)$. Since

$$\delta_{\vartheta_t\mathfrak{S}(\cdot)}\big(\{\mathfrak{S}(\cdot+h) : h\geq 0\}\big) = \delta_{\mathfrak{S}(\cdot+t)}\big(\{\mathfrak{S}(\cdot+h) : h\geq 0\}\big) = 1,$$

for every $t\in[0,T]$, we obtain $\nu\big(\{\mathfrak{S}(\cdot+h) : h\geq 0\}\big) = 1$, which proves the claim by setting $\mathbb{P}_\Sigma \coloneqq \nu$. ☐

The ergodicity of the resulting metric dynamical system (4-13) follows by the existence of an ergodic decomposition of $\mathbb{P}_\Sigma$, see [Arn98, Page 539].

Corollary 4.29. *The quadrupel defined in (4-13) is an ergodic metric dynamical system.*

Remark 4.30. In the following, the focus is on this extended metric dynamical system. Nevertheless, sometimes it will be necessary to refer to the original probability space. From now on, we denote the extended metric dynamical system as $\big(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}}\big)$ to stay as close as possible to the notation of random dynamical systems. If a reference must be made to the original metric dynamical system, it is referred to as $\big(\widetilde{\Omega},\widetilde{\Sigma},\widetilde{\mathbb{P}},(\tilde{\theta})_{t\in\mathbb{R}}\big)$. The integrability result Lemma 3.43 still holds in this extended probability space, as can be seen from Lemma 4.35.

4.2.3 Generation of random dynamical systems

What remains is the generation of random dynamical systems by partial differential equations. Since both rough and stochastic partial differential equations are considered in the following two chapters, this section is divided into two parts.

Random dynamical systems for rough partial differential equations

First, we prove that (RPDE) generates a random dynamic system. The critical point here is the cocycle property, which follows from the uniqueness of the solution, but which, untypically for stochastic equations, should apply pathwise. In this section, we focus on rough partial differential equations, where this problem does not occur because the solution is already pathwise. Therefore, the theory of rough paths is particularly suitable for investigating long-term behavior. The work [BRS17], which was the first to combine the theory of rough paths and that of random dynamical systems, paved the way for this.

Definition 4.31. ([BRS17, Definition 2]) We call a pair

$$\mathbf{X} = (X, \mathbb{X}) \colon \widetilde{\Omega} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R},$$

a (γ-Hölder) rough path cocycle if $\mathbf{X}|_{[s,t]}(\tilde{\omega})$ is a γ-Hölder rough path for every $[s,t] \subset \mathbb{R}$ and $\tilde{\omega} \in \widetilde{\Omega}$, and the cocycle property $X_{s,s+t}(\tilde{\omega}) = X_t\big(\tilde{\theta}_s\tilde{\omega}\big)$ as well as $\mathbb{X}_{s,s+t}(\tilde{\omega}) = \mathbb{X}_{0,t}\big(\tilde{\theta}_s\tilde{\omega}\big)$ holds true for every $s \in \mathbb{R}, t \in [0,\infty)$ and $\tilde{\omega} \in \widetilde{\Omega}$.

Remark 4.32. As a direct consequence of this definition, we obtain $\Big[X\big(\tilde{\theta}_r\tilde{\omega}\big)\Big]_{\gamma,[s,t]} = [X(\tilde{\omega})]_{\gamma,[s+r,t+r]}$ and $\Big[\mathbb{X}\big(\tilde{\theta}_r\tilde{\omega}\big)\Big]_{2\gamma,[s,t]} = [\mathbb{X}(\tilde{\omega})]_{2\gamma,[s+r,t+r]}$ for $r \in \mathbb{R}$. In particular, the same holds for the control $W_{s,t}\big(\tilde{\theta}_r\tilde{\omega}\big) = W_{s+r,t+r}(\tilde{\omega})$ and the number of greedy times $N_{s,t}\big(\tilde{\theta}_r\tilde{\omega}\big) = N_{s+r,t+r}(\tilde{\omega})$.

The central issue is determining which stochastic processes are rough path cocycles. To fulfill the cocycle property from Definition 4.31, the underlying path must exhibit stationary increments. Subsequently, the following result provides a rough path cocycle.

Theorem 4.33. ([BRS17, Thoerem 5]) *Let* $\mathbf{X}$ *be a stochastic process such that* $\mathbf{X}(\tilde{\omega}) \in \mathscr{C}^\gamma$ *for some* $\gamma \in (1/3, 1/2]$ *almost surely. Further, assume that* $\mathbf{X}$ *has stationary increments. Then there exists a modification of* $\mathbf{X}$*, denoted again by* $\mathbf{X}$*, and a metric dynamical system* $\big(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}}, (\tilde{\theta}_t)_{t\in\mathbb{R}}\big)$ *such that* $\mathbf{X}$ *is a rough path cocycle.*

Example 4.34. *We recall Example 2.24, where several stochastic processes that generate rough paths have been investigated. The Liouville fractional Brownian motion and the Ornstein–Uhlenbeck process do not have stationary increments, therefore do not lead to a rough path cocycle. However, both Brownian motion* B *and fractional Brownian motion* B^H *have stationary increments, so they are examples of a Markovian and a non-Markovian process that can be lifted to a rough cocycle, as seen in [BRS17, Example 10].*

With this at hand, it is easy to prove the existence of a random dynamical system; see, for example, [HN22, NS23].

Lemma 4.35. *Suppose* $\big(A(t)\big)_{r\in[s,t]}, F$ *and* G *satisfy the Assumptions **(A)**,**(F1)**, **(G3)** and **(S)**. Further, let* $\mathbf{X}$ *be a rough cocycle satisfying **(N)**. Then the solution of* (RPDE) *generates a continuous random dynamical system denoted by* φ.

Proof. For $\omega \in \Omega$, let $\Big\|y(\omega), \big(y(\omega)\big)'\Big\|_{\mathcal{D}^\gamma_{X(\tilde{\omega}),\alpha}([0,\infty))}$ be the global solution of (RPDE) and denote path component by $\varphi(t,\omega,x) \coloneqq y_t(\omega)$, where $x \in E_\alpha$ is the initial value. Using the fact that the path component satisfies (3-1), we obtain

$$\begin{aligned}\varphi(t+s,\omega,x) &= S_{t+s,0}x + \int_0^{t+s} S_{t+s,r}F\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}r\\ &\quad+ \int_0^{t+s} S_{t+s,r}G\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}\mathbf{X}_r(\tilde\omega)\\ &= S_{t+s,s}S_{s,0}x + S_{t+s,s}\int_0^{s} S_{s,r}F\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}r\\ &\quad+ S_{t+s,s}\int_0^{s} S_{s,r}G\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}\mathbf{X}_r(\tilde\omega)\\ &\quad+ \int_s^{t+s} S_{t+s,r}F\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}r + \int_s^{t+s} S_{t+s,r}G\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}\mathbf{X}_r(\tilde\omega)\\ &= S_{t+s,s}\varphi(t,\omega,x) + \int_s^{t+s} S_{t+s,r}F\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}r\\ &\quad+ \int_s^{t+s} S_{t+s,r}G\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}\mathbf{X}_r(\tilde\omega).\end{aligned}$$

Further, we emphasize the fact that the evolution family also depends on the symbol $\sigma\in\Sigma$, but this dependence is mostly omitted for notational simplicity. In particular, in this situation we have $S^{\sigma}_{t+s,r+s} = S^{\vartheta_s\sigma}_{t,r}$. Together with the shift property of the rough convolution, see [HN19, Corollary 4.5] and [HN20, Lemma 8], this yields

$$\begin{aligned}\varphi(t+s,\omega,x) &= S^{\sigma}_{t+s,s}\varphi(t,\omega,x) + \int_s^{t+s} S^{\sigma}_{t+s,r}F\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}r\\ &\quad+ \int_s^{t+s} S^{\sigma}_{t+s,r}G\big(r,\varphi(r,\omega,x)\big)\ \mathrm{d}\mathbf{X}_r(\tilde\omega)\\ &= S^{\sigma}_{t+s,s}\varphi(t,\omega,x) + \int_0^{t} S^{\sigma}_{t+s,r+s}F\big(r+s,\varphi(r+s,\omega,x)\big)\ \mathrm{d}r\\ &\quad+ \int_0^{t} S^{\sigma}_{t+s,r+s}G\big(r+s,\varphi(r+s,\omega,x)\big)\ \mathrm{d}\big(\tilde\theta_s\mathbf{X}_r\big)(\tilde\omega)\\ &= S^{\vartheta_s\sigma}_{t,0}\varphi(s,\omega,x) + \int_0^{t} S^{\vartheta_s\sigma}_{t,r}F\big(r+s,\varphi(r+s,\omega,x)\big)\ \mathrm{d}r\\ &\quad+ \int_0^{t} S^{\vartheta_s\sigma}_{t,r}G\big(r+s,\varphi(r+s,\omega,x)\big)\ \mathrm{d}\big(\tilde\theta_s\mathbf{X}_r\big)(\tilde\omega)\\ &= \varphi\big(t,\theta_s\omega,\varphi(s,\omega,x)\big),\end{aligned}$$

which verifies the cocycle property.

For the measurability, use a sequence of classical solutions to (RPDE) corresponding to a smooth approximation $\big(\mathbf{X}^k\big)_{k\in\mathbb{N}}$ of $\mathbf{X}$. Since the solution depends continuously on the rough input $\mathbf{X}$, see Theorem 3.4, the approximating sequence of solutions converges to the solution corresponding to $\mathbf{X}$. Using this, it is easy to see that $\varphi(t,\cdot,\cdot)\colon\Omega\times E_\alpha\to E_\alpha$ is measurable and $\varphi(\cdot,\omega,x)\colon[0,\infty)\to E_\alpha$ is continuous. Then [CV77, Lemma 3.14] yields the measurability of φ. □

Conjugated dynamical systems

As noted, stochastic partial differential equations do not directly generate a random dynamical system. The problem is that Kolmogorov's theorem fails in the infinite-dimensional setting. To circumvent this, we transform the original equation in a manner that generates a random dynamical system from the resulting equation. This conjugated random dynamical system is then transformed back to obtain a random dynamical system for the original equation.

This procedure is due to Imkeller and Schmalfuß [IS01]. A similar conjugation method is introduced by Imkeller and Lederer [IL01].

Definition 4.36. Let $\mathscr{T} : \Omega \times \mathcal{E} \to \mathcal{E}$ be such that for fixed $\omega \in \Omega$, $\mathscr{T}(\omega, \cdot)$ is a homeomorphism on $\mathcal{E}$ with inverse $\mathscr{T}^{-1}(\omega, \cdot)$ and $\mathscr{T}(\cdot, x)$, $\mathscr{T}^{-1}(\cdot, x)$ are measurable for every $x \in \mathcal{E}$. Then, given a random dynamical system ϕ, $\tilde{\phi}$ defined by

$$(t, \omega, x) \mapsto \tilde{\phi}(t, \omega, x) := \mathscr{T}\big(\theta_t\omega, \phi(t, \omega, \mathscr{T}^{-1}(\omega, x))\big),$$

is the conjugated random dynamical system to ϕ.

It is easy to see that $\tilde{\phi}$ is also a random dynamical system. Further, due to the conjugacy, ϕ and $\tilde{\phi}$ exhibit the same asymptotic behavior. In particular, if ϕ has a random attractor, the same holds for $\tilde{\phi}$.

Theorem 4.37. ([IS01, Theorem 2.1]) *Let $\mathscr{A}$ be a random pullback attractor of the random dynamical system ϕ and $\mathscr{T}$ satisfy Definition 4.36. Suppose that*

$$\big\{\{\mathscr{T}(\omega, K(\omega))\}_{\omega\in\Omega} : \{K(\omega)\}_{\omega\in\Omega} \text{ is tempered}\big\}$$

consists of tempered sets. Then $\big\{\widetilde{\mathscr{A}}(\omega)\big\}_{\omega\in\Omega} := \{\mathscr{T}(\omega, \mathscr{A}(\omega))\}_{\omega\in\Omega}$ is a random pullback attractor for the conjugated random dynamical system $\tilde{\phi}$.

Proof. The original statement was established for random dynamical systems in Hilbert spaces. But the proof is independent of the topology of the space, so it is directly applicable to Banach spaces. For the sake of completeness, we briefly sketch this proof.

Assume $\mathscr{A}$ is an attractor. Because $\mathscr{T}(\omega, \cdot)$ is continuous, the compactness of $\widetilde{\mathscr{A}}(\omega)$ follows. Due to the definition of $\mathscr{T}$ and $\widetilde{\mathscr{A}}$, we obtain

$$\tilde{\phi}\big(t, \omega, \widetilde{\mathscr{A}}(\omega)\big) = \mathscr{T}\Big(\theta_t\omega, \phi\big(t, \omega, \mathscr{T}^{-1}(\omega, \mathscr{T}(\omega, \mathscr{A}(\omega)))\big)\Big) = \widetilde{\mathscr{A}}(\omega).$$

Note that for every tempered set $\{K(\omega)\}_{\omega\in\Omega}$, $\big\{\mathscr{T}^{-1}(\omega, K(\omega))\big\}_{\omega\in\Omega}$ is also tempered. Then, the continuity of $\mathscr{T}(\omega, \cdot)$ yields

$$\lim_{t\to\infty} \operatorname{dist}\Big(\mathscr{T}\big(\omega, \phi(t, \theta_{-t}\omega, \mathscr{T}^{-1}(\theta_{-t}\omega, K(\theta_{-t}\omega)))\big), \mathscr{T}\big(\omega, \mathscr{A}(\omega)\big)\Big) = 0,$$

for every tempered set $\{K(\omega)\}_{\omega\in\Omega}$, where dist denotes the Hausdorff semi-distance. □

Remark 4.38. This technique is often used for stochastic partial differential equations with additive or linear multiplicative noise, where the underlying transformation is known as Doss–Sussmann transformation.

Chapter 5

Long-time behavior of rough evolution equations

Recall that the extended probability space $\Omega = \widetilde{\Omega} \times \Sigma$ consists of the symbol space Σ and $\widetilde{\Omega}$, which is the probability space of the noise term. We will highlight the original probability space by using $\tilde{\omega}$ and $\tilde{\Omega}$ every time it is needed. As before, E is a separable Banach space and the family of operators $\big(A(t)\big)_{t\geq 0}$ satisfies **(A)**. The respective interpolation-extrapolation scale of order $m \in \mathbb{N}_0$ is then denoted by $(A_\beta, E_\beta)_{\beta > -m}$.

5.1 Global attractors

In this section, we extend the results obtained in [BS24] to the non-autonomous setting. This was part of a collaboration with Alexandra Blessing.

5.1.1 Basic estimates

Before explaining the strategy for obtaining a random pullback attractor, it is necessary to collect several estimates to bound the solution. In particular, we need specific Gronwall-type estimates tailored to our equation (RPDE).

Remark 5.1. We believe that the existence of the random attractor can also be shown using the mild rough Gronwall inequality proven in Section 3.5. In this section, however, we rely on using the singular Gronwall–Henry lemma, since this approach was also used in [BS24]; the mild Gronwall lemma was developed after completing this project.

Theorem 5.2. ([Hen81, Lemma 7.1.1]) *Let $T \in (0, \infty]$ and $v, h \in \mathrm{L}^\infty_{\mathrm{loc}}\big([0,T);[0,\infty)\big)$ be non-negative functions satisfying*

$$v(t) \leq h(t) + a \int_0^t (t-r)^{\beta-1} v(r)\ \mathrm{d}r,$$

for $t \in [0,T)$ and $a, \beta > 0$. Then we have

$$v(t) \leq h(t) + \big(\Gamma(\beta)a\big)^{\beta^{-1}} \int_0^t h(r) \mathscr{E}'_{\beta,1}\Big((t-r)\big(\Gamma(\beta)a\big)^{\beta^{-1}}\Big)\ \mathrm{d}r,$$

for every $t \in [0,T)$ where $\Gamma(z) := \int_0^\infty e^{-r} r^{z-1}\ \mathrm{d}r$ is the Gamma function and $\mathscr{E}_{\beta,c}(z) := \sum_{k=0}^\infty \frac{z^{\beta k}}{\Gamma(k\beta+c)}$ is the Mittag–Leffler function.

This inequality is a generalization of the classical Gronwall lemma, which is a special case of the previous statement for $\beta = 1$ since $\mathscr{E}_{1,1}(z) = e^z$. To prove the existence of a random absorbing set later on, the asymptotic behavior of the derivative of the Mittag–Leffler function is crucial. Since the Mittag–Leffler function is a generalization of the exponential function, its asymptotic behavior is similar, see [Hen81, SY02].

Lemma 5.3. *For every $\beta \in (0,1)$, there exist a constant $M_\beta > 0$, such that for $z > 1$ large enough we have*

$$\mathscr{E}'_{\beta,1}(z) \leq M_\beta e^{2z}. \tag{5-1}$$

Proof. Due to [SY02, (94.19)] there exist two polynomials Q_0 and Q_1 such that

$$\mathscr{E}_{\beta,\beta}(z) \leq Q_0(z) + Q_1(z)e^z,$$

for $z > 1$. Since we can bound every polynomial by e^z, if z is large enough, this leads to

$$\mathscr{E}_{\beta,\beta}(z) \leq M_\beta e^{2z},$$

where M_β depends on the coefficients of Q_0 and Q_1. This further yields

$$\lim_{t\to\infty} \frac{1}{t} \log \mathscr{E}_{\beta,\beta}(\mu t) = \mu,$$

for all $\mu, \beta > 0$. A similar statement also holds for the derivative of the Mittag–Leffler's function. To obtain this, note that

$$\begin{aligned}\mathscr{E}'_{\beta,1}(z) &= \sum_{k=1}^{\infty} \frac{k\beta}{\Gamma(k\beta+1)} z^{k\beta-1} = \sum_{k=1}^{\infty} \frac{z^{k\beta-1}}{\Gamma(k\beta)} = \frac{1}{z}\sum_{k=1}^{\infty} \frac{z^{(k-1)\beta+\beta}}{\Gamma\big((k-1)\beta+\beta\big)} \\ &= \frac{z^\beta}{z}\sum_{k=0}^{\infty} \frac{z^{k\beta}}{\Gamma(k\beta+\beta)} = z^{\beta-1}\mathscr{E}_{\beta,\beta}(z).\end{aligned}$$

Therefore, the asymptotic behavior of $\mathscr{E}'_{\beta,1}$ is the same as that of $\mathscr{E}_{\beta,\beta}$, where the constant M_β has to be adjusted appropriately. □

Due to the fact, that $\mathbf{X}(\tilde{\omega}) = \big(X(\tilde{\omega}), \mathbb{X}(\tilde{\omega})\big)$ is only locally Hölder continuous, we need to discretize the rough integral to obtain infinitely many integrals over a finite time interval. This is also a known problem for Young integrals, see [DH23, GAMS10, KN23]. Therefore, we also need a discrete Gronwall inequality.

Lemma 5.4. ([DH23, Lemma 3.12]) *Let $(u_k)_{k\in\mathbb{N}_0}$, $(b_k)_{k\in\mathbb{N}_0}$ and $(c_k)_{\in\mathbb{N}_0}$ be non-negative sequences and $a \geq 0$, satisfying*

$$u_k \leq a + \sum_{l=0}^{k-1} b_l u_k + \sum_{l=0}^{k-1} c_l,$$

for all $k \in \mathbb{N}$. Then we have

$$u_k \leq \max\{a, u_0\} \prod_{j=0}^{k-1}(1+b_j) + \sum_{l=0}^{k-1} c_l \prod_{j=l+1}^{k-1}(1+b_j),$$

for all $n \in \mathbb{N}$.

Before proving the existence of a random pullback attractor, we also need estimates of the convolutions appearing in the mild formulation. Recall that $\mathbf{X} = (X, \mathbb{X})$ is a geometric rough path cocycle satisfying Assumption **(N)**.

From now on, let $\big(y(\omega), y'(\omega)\big) \in \mathcal{D}^\gamma_{X(\tilde{\omega}),\alpha}\big([0,\infty)\big)$ be the unique global solution of (RPDE) driven by $\mathbf{X}(\tilde{\omega})$. The semigroup is assumed to be exponentially stable, which means Assumption **($\mathbf{A_S}$)** is fulfilled. This is necessary to establish the existence of an attractor. Further, we use the notation $C_{\alpha-\beta}$ for the constants appearing in the smoothing property

(D-4), this means

$$\|S_{t,s}x\|_{\beta} \leq C_{\alpha-\beta}e^{-\lambda_A(t-s)}(t-s)^{\alpha-\beta}\|x\|_{\alpha}.$$

In the following, the lowercase c is used to denote the constants associated with the attractor, distinguishing them from the constants used in the mild Gronwall lemma; recall Section 3.5.

Lemma 5.5. *Let $\big(A(t)\big)_{t\geq 0}$, F and G satisfy Assumption* ***(A)***, ***(A$_S$)***, ***(F1)*** *and* ***(G3)*** *are satisfied. Furthermore, let $\mathbf{X}$ be a rough path cocycle satisfying* ***(N)*** *and fix $t\geq 0$. Then we have*

$$\left\|\int_0^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\tilde{\omega})\right\|_{\alpha} \leq C_G c_1 \sum_{l=0}^{\lfloor t\rfloor} e^{-\lambda_A(t-l-1)}P_3\big(\omega,[l,l+1]\big),$$
$$\left\|\int_0^t S_{t,r}F(r,y_r)\,\mathrm{d}r\right\|_{\alpha} \leq C_{-\sigma_F}C_F\int_0^t e^{-\lambda_A(t-r)}(t-r)^{-\sigma_F}\|y_r\|_{\alpha}\,\mathrm{d}r + c_2,$$

where the appearing constants are given by

$$\begin{aligned} c_1 &:= \max\{C_I C_A, C_I\},\\ c_2 &:= C_{-\sigma_F}C_F\lambda_A^{\sigma_F-1}\Gamma(1-\sigma_F),\\ P_3(\omega,[l,l+1]) &:= \big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3\Big(1+\big\|y(\omega),y'(\omega)\big\|_{\mathcal{D}^{\gamma}_{X,\alpha}([l,l+1])}\Big).\end{aligned}$$

Proof. Regarding Assumption **(A$_S$)**, (2-45) and (3-6) we obtain

$$\begin{aligned}
&\left\|\int_0^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\tilde{\omega})\right\|_{\alpha}\\
&\leq \sum_{l=0}^{\lfloor t\rfloor-1}\left\|\int_l^{l+1} S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\tilde{\omega})\right\|_{\alpha} + \left\|\int_{\lfloor t\rfloor}^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\tilde{\omega})\right\|_{\alpha}\\
&\leq C_A\sum_{l=0}^{\lfloor t\rfloor-1} e^{-\lambda_A(t-l-1)}\left\|\int_l^{l+1} S_{l+1,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\tilde{\omega})\right\|_{\alpha}\\
&\quad+\left\|\int_{\lfloor t\rfloor}^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\tilde{\omega})\right\|_{\alpha}\\
&\leq C_I C_A\sum_{l=0}^{\lfloor t\rfloor-1} e^{-\lambda_A(t-l-1)}\big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)\left\|G(\cdot,y),(G(\cdot,y))'\right\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha-\sigma_G}([l,l+1])}\\
&\quad+C_I\big(1+\varrho_{\gamma,[\lfloor t\rfloor,t]}(\mathbf{X}(\tilde{\omega}))\big)\left\|G(\cdot,y),(G(\cdot,y))'\right\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha-\sigma_G}([\lfloor t\rfloor,t])}(t-\lfloor t\rfloor)^{\gamma-\sigma_G}\\
&\leq C_G c_1\sum_{l=0}^{\lfloor t\rfloor} e^{-\lambda_A(t-l-1)}\big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3\Big(1+\|y,G(\cdot,y)\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}([l,l+1])}\Big),
\end{aligned}$$

which shows the first inequality. The second integral can be estimated using the Lipschitz continuity of F as follows

$$\begin{aligned}
\left\|\int_0^t S_{t,r}F(r,y_r)\,\mathrm{d}r\right\|_{\alpha} &\leq C_{-\sigma_F}C_F\int_0^t e^{-\lambda_A(t-r)}(t-r)^{-\sigma_F}\big(1+\|y_r\|_{\alpha}\big)\,\mathrm{d}r\\
&\leq C_{-\sigma_F}C_F\int_0^t e^{-\lambda_A(t-r)}(t-r)^{-\sigma_F}\|y_r\|_{\alpha}\,\mathrm{d}r\\
&\quad+C_{-\sigma_F}C_F\int_0^t e^{-\lambda_A(t-r)}(t-r)^{-\sigma_F}\,\mathrm{d}r,
\end{aligned}$$

where the last term is bounded by $\lambda_A^{\sigma_F-1}\Gamma(1-\sigma_F)$, which follows from a simple substitution and the definition of the Gamma function. □

5.1.2 Pullback attractor for rough partial differential equations

To show the existence of an attractor after all the preparations, we proceed as follows: First, the solution y_t is estimated using the singular Gronwall–Henry lemma, where the stochastic integral is split into integrals over intervals of length 1 as in Lemma 5.5. The right-hand side of the resulting estimate still depends on the solution itself, but always on an interval of length 1 due to the splitting. Using this estimate and the discrete Gronwall lemma, the solution can be estimated at discrete points in time, i.e., y_n, so that the right-hand side no longer depends on the solution. This estimate can then be used to check the condition in Lemma 4.18, which shows the existence of an absorbing set. Finally, a compactness argument in Theorem 5.10 provides the existence of an attractor.

Lemma 5.6. *Let $\big(A(t)\big)_{t\geq 0}$, F and G satisfy Assumption **(A)**, **(A$_S$)**, **(S)**, **(F1)** and **(G3)** are satisfied. Furthermore, let $\mathbf{X}$ be a rough path cocycle satisfying **(N)** and define*

$$\tilde{c}_1 = c_1 e^{\lambda_A} \max\left\{\widetilde{L}, \frac{M_{1-\sigma_F}}{2}\right\}, \quad \tilde{c}_2 = c_2\left(\widetilde{L} + \frac{LM_{1-\sigma_F}}{2\lambda}\right), \widetilde{C}_A = C_A \max\left\{\widetilde{L}, \frac{M_{1-\sigma_F}}{2}\right\},$$

where $L := 2\big(C_{-\sigma_F}C_F\Gamma(1-\sigma_F)\big)^{\frac{1}{1-\sigma_F}}$, $\widetilde{L} := \mathscr{E}_{1-\sigma_F,1}(t_0{}^L\!/2)+1$, $\lambda := \lambda_A - L$ and $t_0 > 0$ such that $t_0{}^L\!/2$ is large enough for (5-1) to hold. Then we obtain for $k \in \mathbb{N}_0$ with $k > t_0$ and $t \in [k, k+1]$ the estimate

$$\|y_t\|_\alpha e^{\lambda t} \leq \widetilde{C}_A \|y_0\|_\alpha + \tilde{c}_2 e^{\lambda t} + \tilde{c}_1 C_G \sum_{l=0}^{k} e^{\lambda l} P_3(\omega, [l, l+1]). \tag{5-2}$$

Proof. First assume $t \in [k, k+1)$. Due to (3-1), we can estimate the path component y_t by

$$\|y_t\|_\alpha \leq \|S_{t,0}y_0\|_\alpha + \left\|\int_0^t S_{t,r}F(r,y_r)\,\mathrm{d}r\right\|_\alpha + \left\|\int_0^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\tilde{\omega})\right\|_\alpha. \tag{5-3}$$

We now multiply (5-3) by $e^{\lambda_A t}$ and obtain together with (D-4) and Lemma 5.5

$$\begin{aligned}\|y_t\|_\alpha e^{\lambda_A t} \leq C_A \|y_0\|_\alpha + c_2 e^{\lambda_A t} + C_G c_1 \sum_{l=0}^{n} e^{\lambda_A (l+1)} P_3(\omega, [l, l+1])\\ + C_{-\sigma_F} C_F \int_0^t (t-r)^{-\sigma_F} e^{\lambda_A r} \|y_r\|_\alpha \,\mathrm{d}r,\end{aligned}$$

which fits in the setting of the singular Gronwall–Henry lemma. In conclusion, apply Theorem 5.2 to $v(r) := \|y_r\|_\alpha e^{\lambda_A r}$ and obtain

$$\begin{aligned}&\|y_t\|_\alpha e^{\lambda_A t} \leq C_A \|y_0\|_\alpha + c_2 e^{\lambda_A t} + C_G c_1 \sum_{l=0}^{k} e^{\lambda_A(l+1)} P_3(\omega, [l, l+1])\\ &+ \frac{L}{2}\int_0^t \left(C_A \|y_0\|_\alpha + c_2 e^{\lambda_A r} + C_G c_1 \sum_{l=0}^{\lfloor r \rfloor} e^{\lambda_A(l+1)} P_3(\omega, [l, l+1])\right) \mathscr{E}'_{1-\sigma_F,1}\left((t-r)\frac{L}{2}\right) \,\mathrm{d}r,\end{aligned} \tag{5-4}$$

with $L := 2\big(C_{-\sigma_F}C_F\Gamma(1-\sigma_F)\big)^{\frac{1}{1-\sigma_F}}$. Further, due to the time-continuity of y, the previous estimate also holds for $t = k+1$. Now we have to bound the derivative of the Mittag–Leffler function. Since the estimate (5-1) is only valid for large values, we consider $t_0 > 0$ such that

$t_0\frac{L}{2}$ is large enough for (5-1) to hold and introduce

$$h(r) := C_A \|y_0\|_\alpha + c_2 e^{\lambda_A r} + C_G c_1 \sum_{l=0}^{\lfloor r \rfloor} e^{\lambda_A (l+1)} P_3(\omega, [l, l+1]).$$

Since $t > t_0$, we can split the integral as follows and use that h is increasing

$$\begin{aligned}
\int_0^t h(r)\mathscr{E}'_{1-\sigma_F,1}\left((t-r)\frac{L}{2}\right)\,\mathrm{d}r &= \int_0^{t-t_0} h(r)\mathscr{E}'_{1-\sigma_F,1}\left((t-r)\frac{L}{2}\right)\,\mathrm{d}r \\
&\quad + \int_{t-t_0}^t h(r)\mathscr{E}'_{1-\sigma_F,1}\left((t-r)\frac{L}{2}\right)\,\mathrm{d}r \\
&\leq M_{1-\sigma_F}\int_0^{t-t_0} h(r)e^{(t-r)L}\,\mathrm{d}r + \int_0^{t_0} h(t-r)\mathscr{E}'_{1-\sigma_F,1}\left(r\frac{L}{2}\right)\,\mathrm{d}r \\
&\leq M_{1-\sigma_F}\int_0^{t} h(r)e^{(t-r)L}\,\mathrm{d}r + h(t)\int_0^{t_0} \mathscr{E}'_{1-\sigma_F,1}\left(r\frac{L}{2}\right)\,\mathrm{d}r \\
&= M_{1-\sigma_F}\int_0^{t} h(r)e^{(t-r)L}\,\mathrm{d}r + h(t)\frac{2\mathscr{E}'_{1-\sigma_F,1}\left(t_0\frac{L}{2}\right)}{L}.
\end{aligned}$$

We set $\widetilde{L} := \mathscr{E}_{1-\sigma_F,1}(t_0 L/2) + 1$ and multiply (5-4) by e^{-Lt} to obtain

$$\begin{aligned}
\|y_t\|_\alpha e^{\lambda t} &\leq \widetilde{L}\left(C_A\|y_0\|_\alpha e^{-Lt} + c_2 e^{\lambda t} + C_G c_1 \sum_{l=0}^{k} e^{-Lt}e^{\lambda_A(l+1)}P_3(\omega,[l,l+1])\right) \\
&\quad + \frac{LM_{1-\sigma_F}}{2}\int_0^t \left(C_A\|y_0\|_\alpha + c_2 e^{\lambda_A r} + C_G c_1 \sum_{l=0}^{\lfloor r\rfloor} e^{\lambda_A(l+1)}P_3(\omega,[l,l+1])\right)e^{-Lr}\,\mathrm{d}r,
\end{aligned}$$

where we need to evaluate the integrals in the second line. The only non-trivial term is the third one. By Fubini's theorem, we obtain

$$\begin{aligned}
\int_0^t \sum_{l=0}^{\lfloor r\rfloor} e^{\lambda_A(l+1)}P_3(\omega,[l,l+1])e^{-Lr}\,\mathrm{d}r &\leq \sum_{l=0}^{k} e^{\lambda l}e^{\lambda_A}P_3(\omega,[l,l+1])\int_l^t e^{-L(r-l)}\,\mathrm{d}r \\
&= \sum_{l=0}^{k} e^{\lambda l}e^{\lambda_A}P_3(\omega,[l,l+1])\left(\frac{1-e^{-L(t-l)}}{L}\right).
\end{aligned}$$

Putting all of these estimates together, this leads to

$$\begin{aligned}
\|y_t\|_\alpha e^{\lambda t} &\leq C_A\|y_0\|_\alpha\left(\widetilde{L}e^{-Lt} + \frac{M_{1-\sigma_F}}{2}\left(1-e^{-Lt}\right)\right) + c_2\left(\widetilde{L}e^{\lambda t} + \frac{LM_{1-\sigma_F}}{2\lambda}\left(e^{\lambda t}-1\right)\right) \\
&\quad + C_G c_1\sum_{l=0}^{k} e^{\lambda l}e^{\lambda_A}P_3(\omega,[l,l+1])\left(\widetilde{L}e^{-L(t-l)} + \frac{M_{1-\sigma_F}}{2}\left(1-e^{-L(t-l)}\right)\right) \\
&\leq \widetilde{C}_A\|y_0\|_\alpha + c_2 e^{\lambda t}\left(\widetilde{L} + \frac{LM_{1-\sigma_F}}{2\lambda}\left(1-e^{-\lambda t}\right)\right) + \tilde{c}_1 C_G\sum_{l=0}^{n} e^{\lambda l}P_3(\omega,[l,l+1]) \\
&\leq \widetilde{C}_A\|y_0\|_\alpha + \tilde{c}_2 e^{\lambda t} + \tilde{c}_1 C_G\sum_{l=0}^{k} e^{\lambda l}P_3(\omega,[l,l+1]),
\end{aligned}$$

with $\lambda, \tilde{c}_1, \tilde{c}_2$ and $\widetilde{C}_A$ defined as in the statement .

□

The right-hand side of (5-2) still depends on the solution y, via $P_3(\omega,[l,l+1])$. However, combining the discrete Gronwall lemma with (3-31), it is possible to get an estimate where the right-hand side is independent of the solution. First, recall that

$$P_3(\omega,[l,l+1]) = \big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3\Big(1+\big\|y(\omega),y'(\omega)\big\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}([l,l+1])}\Big),$$

and that the controlled rough path norm of the solution $\big(y,y'\big)=\big(y,G(y)\big)$ can be estimated due to Theorem 3.42 by

$$\big\|y,y'\big\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}([l,l+1])} \le \|y_l\|_\alpha P_1(\omega,[l,l+1]) + P_2(\omega,[l,l+1])$$

for $l\in\mathbb{N}_0$, where P_1 and P_2 are given by

$$\begin{aligned}
P_0(\omega,[l,l+1]) &\coloneqq N_{l,l+1}(\tilde{\omega})\big(1+[X(\tilde{\omega})]_{\gamma,[l,l+1]}\big)e^{N_{l,l+1}(\tilde{\omega})M},\\
P_1(\omega,[l,l+1]) &\coloneqq \widetilde{N}P_0(\omega,[l,l+1])^{\widetilde{N}+1},\\
P_2(\omega,[l,l+1]) &\coloneqq \widetilde{N}N_{l,l+1}(\tilde{\omega})\big(1+[X(\tilde{\omega})]_{\gamma,[l,l+1]}\big)\frac{e^{N_{l,l+1}(\tilde{\omega})M+M}-1}{2\widetilde{M}-1}\\
&\qquad\times P\big([X(\tilde{\omega})]_{\gamma,[l,l+1]},[\mathbb{X}(\tilde{\omega})]_{2\gamma,[l,l+1]}\big)\frac{P_0(\omega,[l,l+1])^{\widetilde{N}}-1}{P_0(\omega,[l,l+1])-1},
\end{aligned}$$

and P is a polynomial. In particular, these bounds only depend on $\tilde{\omega}$. This is essential in the following results.

Lemma 5.7. *Let the assumptions from Lemma 5.6 be satisfied. Let $k\in\mathbb{N}_0$ such that $k>t_0$. Then we obtain*

$$\begin{aligned}
\|y_k\|_\alpha &\le \widetilde{C}_A\|y_0\|_\alpha e^{-\lambda k}\prod_{j=0}^{k-1}\big(1+H_1(\omega,[j,j+1])\big)+\sum_{l=0}^{k-1}e^{-\lambda(k-l)}H_2(\omega,[l,l+1])\\
&\quad\times\prod_{j=l+1}^{k-1}\big(1+H_1(\omega,[j,j+1])\big),
\end{aligned}$$

where we define

$$\begin{aligned}
H_1(\omega,[l,l+1]) &\coloneqq \tilde{c}_1C_G\big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3P_1(\omega,[l,l+1]),\\
H_2(\omega,[l,l+1]) &\coloneqq \max\big\{\tilde{c}_2e^{\lambda},\tilde{c}_1C_G\big\}\big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3\big(1+P_2(\omega,[l,l+1])\big).
\end{aligned}\tag{5-5}$$

Proof. We apply Lemma 5.6 for $t=k$ and obtain

$$\begin{aligned}
\|y_k\|_\alpha e^{\lambda k} &\le \widetilde{C}_A\|y_0\|_\alpha+\tilde{c}_2e^{\lambda k}+\tilde{c}_1C_G\sum_{l=0}^{k-1}e^{\lambda l}P_3(\omega,[l,l+1])\\
&= \widetilde{C}_A\|y_0\|_\alpha+\tilde{c}_2e^{\lambda k}+\tilde{c}_1C_G\sum_{l=0}^{k-1}e^{\lambda l}\big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3\Big(1+\big\|y,y'\big\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}}\Big)\\
&\le \widetilde{C}_A\|y_0\|_\alpha+\tilde{c}_2e^{\lambda k}+\tilde{c}_1C_G\sum_{l=0}^{k-1}e^{\lambda l}\big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3\big(1+P_2(\omega,[l,l+1])\big)\\
&\quad+\tilde{c}_1C_G\sum_{l=0}^{k-1}e^{\lambda l}\big(1+\varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3\|y_l\|_\alpha P_1(\omega,[l,l+1])
\end{aligned}$$

$$\leq \widetilde{C}_A \|y_0\|_\alpha + \max\{\widetilde{c}_2 e^\lambda, \tilde{c}_1 C_G\} \sum_{l=0}^{k-1} e^{\lambda l}\big(1 + \varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3 \big(1 + P_2(\omega,[l,l+1])\big)$$
$$+ \tilde{c}_1 C_G \sum_{l=0}^{k-1} e^{\lambda l}\big(1 + \varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3 \|y_l\|_\alpha P_1(\omega,[l,l+1]).$$

With this we can now use the discrete Gronwall Lemma 5.4 for $u_k := \|y_k\|_\alpha e^{\lambda k}$ to obtain

$$\|y_k\|_\alpha e^{\lambda k} \leq \widetilde{C}_A \|y_0\|_\alpha \prod_{j=0}^{k-1} \Big(1 + \tilde{c}_1 C_G \big(1 + \varrho_{\gamma,[j,j+1]}(\mathbf{X}(\tilde{\omega}))\big)^3 P_1(\omega,[j,j+1])\Big)$$
$$+ \sum_{l=0}^{k-1} \max\{\widetilde{c}_2 e^\lambda, \tilde{c}_1 C_G\} e^{\lambda l}\big(1 + \varrho_{\gamma,[l,l+1]}(\mathbf{X}(\tilde{\omega}))\big)^3 \big(1 + P_2(\omega,[l,l+1])\big)$$
$$\times \prod_{j=l+1}^{k-1} \Big(1 + \tilde{c}_1 C_G \big(1 + \varrho_{\gamma,[j,j+1]}(\mathbf{X}(\tilde{\omega}))\big)^3 P_1(\omega,[j,j+1])\Big),$$

since $\widetilde{C}_A > 1$. □

The last ingredient required for the existence of an absorbing set is based on the ergodic properties of the noise. Therefore, we need Birkhoff's ergodic theorem [Bir31]. This essentially yields the equality of spatial and temporal averages under an ergodicity assumption.

Theorem 5.8. ([EW11, Theorem 2.30]) *If $\mathcal{T}\colon \Omega \to \Omega$ is an ergodic measure preserving function and $f \in \mathrm{L}^1(\Omega)$, then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(\mathcal{T}^i \omega) = \mathbb{E}[f],$$

almost surely.

Recall that due to Assumption (N), $\mathbf{X}$ is Gaussian and therefore all moments of X and $\mathbb{X}$ exist and in particular all moments of the respective Hölder semi-norms are bounded, see Corollary 2.20. Regarding the ergodicity of the metric dynamical system $(\theta_t)_{t\in\mathbb{R}}$, Birkhoff's ergodic theorem then yields

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} [X(\theta_{-j}\tilde{\omega})]^q_{\gamma,[-1,1]} = \mathbb{E}\left[[X]^q_{\gamma,[-1,1]}\right] =: K_q$$
$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} [\mathbb{X}(\theta_{-j}\tilde{\omega})]^q_{\gamma,[-1,1]} = \mathbb{E}\left[[\mathbb{X}]^q_{2\gamma,[-1,1]}\right] =: \mathbb{K}_q, \tag{5-6}$$

for $q \geq 1$.

For a better comprehension, recall that $\widetilde{N} = \left\lceil (4\widetilde{M})^{\frac{1}{1-\widetilde{\sigma}_F}} \right\rceil$ is the number of intervals with which we have to fill an interval of length 1 to obtain the integrable bounds from Theorem 3.42. Further define

$$C(\widetilde{N}) := \max\{1 + \widetilde{N}, 2^{4(2+\widetilde{N})} \widetilde{N} M^{1+\widetilde{N}}\}.$$

Note that $\widetilde{N}$ will determine how many moments of $(X, \mathbb{X})$ we have to control, but since the noise is assumed to be Gaussian, the concrete value of $\widetilde{N}$ is not important, since all moments are bounded. We further set $\mathbf{K}_q := K_q + \mathbb{K}_{2^{-1}q}$.

Lemma 5.9. *Let the assumptions from Lemma 5.6 be satisfied, and assume further that*

$$\lambda_A - 2\big(C_{-\sigma_F} C_F \Gamma(1-\sigma_F)\big)^{\frac{1}{1-\sigma_F}} > c\big(\mathbf{K}_q + 1\big), \tag{5-7}$$

holds, where q and c are given by

$$c := 2C\big(\widetilde{N}\big) \max\big\{\widetilde{M}, \tilde{c}_1 C_G\big\}, \quad q := \frac{4\big(1+\widetilde{N}\big)}{\gamma - \eta}.$$

Then the random dynamical system φ associated to (RPDE), possesses a random pullback absorbing set $\{K(\omega)\}_{\omega\in\Omega}$.

Proof. We fix $\{K(\omega)\}_{\omega\in\Omega}$ as a tempered random set and estimate $\varphi\big(t,\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)$ for $y_0(\omega) \in K(\omega)$. To do so we combine Lemma 5.7 and (3-31) to obtain

$$\begin{aligned}
\big\|\varphi\big(t,\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big\|_\alpha &\le \|y_k(\theta_{-t}\omega)\|_\alpha P_1(\theta_{-t}\omega,[k,k+1]) + P_2(\theta_{-t}\omega,[k,k+1]) \\
&\le \widetilde{C}_A \|y_0(\theta_{-t}\omega)\|_\alpha P_1(\theta_{-t}\omega,[k,k+1]) e^{-\lambda k} \prod_{j=0}^{k-1}\big(1+H_1(\theta_{-t}\omega,[j,j+1])\big) \\
&+ P_1(\theta_{-t}\omega,[k,k+1]) \sum_{l=0}^{k-1} e^{-\lambda(k-l)} H_2(\theta_{-t}\omega,[l,l+1]) \\
&\times \prod_{j=l+1}^{k-1}\big(1+H_1(\theta_{-t}\omega,[j,j+1])\big) + P_2(\theta_{-t}\omega,[k,k+1])
\end{aligned}$$

for $t_0 < t \in [k,k+1]$. Similar as in Remark 4.32 we can see that

$$P_i(\theta_r\omega,[s,t]) = P_i(\omega,[s+r,t+r]) \quad \text{and} \quad H_i(\theta_r\omega,[s,t]) = H_i(\omega,[s+r,t+r])$$

holds for $i = 1,2$ and $r \in \mathbb{R}$. Applying this to the previous estimate yields

$$\begin{aligned}
\big\|\varphi\big(t,\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big\|_\alpha \le \widetilde{C}_A & \|y_0(\theta_{-t}\omega)\|_\alpha P_1(\omega,[-1,1]) e^{-\lambda k} \\
&\times \sup_{\varepsilon\in[0,1]} \prod_{j=1}^{k}\big(1+H_1(\theta_{-j}\omega,[-\varepsilon,1-\varepsilon])\big) \\
&+ P_1(\omega,[-1,1]) \sup_{\varepsilon\in[0,1]} \sum_{l=1}^{\infty} e^{-\lambda l} H_2(\theta_{-l}\omega,[-\varepsilon,1-\varepsilon]) \\
&\times \prod_{j=1}^{l-1}\big(1+H_1(\theta_{-j}\omega,[-\varepsilon,1-\varepsilon])\big) + P_2(\omega,[-1,1]).
\end{aligned} \tag{5-8}$$

For the first term one can use the fact that y_0 is tempered, which means by definition that $e^{-\beta t}\|y_0(\theta_t\omega)\|_\alpha \to 0$ for $t\to\infty$ and every $\beta > 0$. To compute the needed exponent β, we note that $\log\big(1+ae^b\big) \le a+b$ holds, which leads to

$$\begin{aligned}
\log\big(1+H_1(\omega,[s,t])\big) &\le N_{s,t}(\tilde{\omega})\widetilde{M}\big(1+\widetilde{N}\big) \\
&+ \tilde{c}_1 C_G \widetilde{N}\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X}(\tilde{\omega}))\big)^3 \Big(M N_{s,t}(\tilde{\omega})\big(1+[X(\tilde{\omega})]_{\gamma,[s,t]}\big)\Big)^{1+\widetilde{N}}.
\end{aligned}$$

Using Lemma 3.36, Young's inequality and $(a+b)^p \le 2^{p-1}(a^p+b^p)$ multiple times we can bound the noise terms by

$$\big(1+\varrho_{\gamma,[s,t]}(\mathbf{X}(\tilde{\omega}))\big)^3 N_{s,t}(\tilde{\omega})^{1+\widetilde{N}}\big(1+[X(\tilde{\omega})]_{\gamma,[s,t]}\big)^{1+\widetilde{N}}$$

$$
\begin{aligned}
&\leq 2^{4+2\widetilde{N}}\left(1+[X(\tilde{\omega})]^3_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^3_{2\gamma,[s,t]}\right)\left(1+\left([X(\tilde{\omega})]^{\frac{1}{\gamma-\eta}}_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^{\frac{1}{2(\gamma-\eta)}}_{2\gamma,[s,t]}\right)^{1+\widetilde{N}}\right)\\
&\quad\times\left(1+[X(\tilde{\omega})]^{1+\widetilde{N}}_{\gamma,[s,t]}\right)\\
&\leq 2^{4+2\widetilde{N}}\left(1+[X(\tilde{\omega})]^3_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^3_{2\gamma,[s,t]}\right)\left(1+\left([X(\tilde{\omega})]^{\frac{1}{\gamma-\eta}}_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^{\frac{1}{2(\gamma-\eta)}}_{2\gamma,[s,t]}\right)^{1+\widetilde{N}}\right)^2\\
&\leq 2^{6+4\widetilde{N}}\left(1+[X(\tilde{\omega})]^3_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^3_{2\gamma,[s,t]}\right)\left(1+\left([X(\tilde{\omega})]^{\frac{2(1+\widetilde{N})}{\gamma-\eta}}_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^{\frac{1+\widetilde{N}}{\gamma-\eta}}_{2\gamma,[s,t]}\right)\right)\\
&\leq 2^{7+4\widetilde{N}}\left(1+[X(\tilde{\omega})]^6_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^6_{2\gamma,[s,t]}\right)\\
&\quad+2^{7+4\widetilde{N}}\left(1+\left([X(\tilde{\omega})]^{\frac{4(1+\widetilde{N})}{\gamma-\eta}}_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^{\frac{2(1+\widetilde{N})}{\gamma-\eta}}_{2\gamma,[s,t]}\right)\right)\\
&\leq 2^{7+4\widetilde{N}}\left(2+[X(\tilde{\omega})]^6_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^6_{2\gamma,[s,t]}+[X(\tilde{\omega})]^{\frac{4(1+\widetilde{N})}{\gamma-\eta}}_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^{\frac{2(1+\widetilde{N})}{\gamma-\eta}}_{2\gamma,[s,t]}\right)\\
&\leq 2^{4(2+\widetilde{N})}\left(1+[X(\tilde{\omega})]^{\frac{4(1+\widetilde{N})}{\gamma-\eta}}_{\gamma,[s,t]}+[\mathbb{X}(\tilde{\omega})]^{\frac{2(1+\widetilde{N})}{\gamma-\eta}}_{2\gamma,[s,t]}\right).
\end{aligned}
$$

Therefore we obtain

$$
\begin{aligned}
&\frac{1}{k}\log\left(\sup_{\varepsilon\in[0,1]}\prod_{j=1}^{k}\left(1+H_1(\theta_{-j}\omega,[-\varepsilon,1-\varepsilon])\right)\right)\\
&\qquad\leq\sup_{\varepsilon\in[0,1]}\frac{1}{k}\log\left(\prod_{j=1}^{k}\left(1+H_1(\theta_{-j}\omega,[-\varepsilon,1-\varepsilon])\right)\right)\\
&\qquad\leq\widetilde{M}\big(1+\widetilde{N}\big)\sup_{\varepsilon\in[0,1]}\frac{1}{k}\sum_{j=1}^{k}N_{-\varepsilon,1-\varepsilon}(\theta_{-j}\omega)\\
&\qquad\quad+2^{4\left(2+\widetilde{N}\right)}\widetilde{N}M^{1+\widetilde{N}}\tilde{c}_1C_G\sup_{\varepsilon\in[0,1]}\frac{1}{k}\sum_{j=1}^{k}\left(1+[X(\tilde{\omega})]^{\frac{4(1+\widetilde{N})}{\gamma-\eta}}_{\gamma,[-\varepsilon,1-\varepsilon]}+[\mathbb{X}(\tilde{\omega})]^{\frac{2(1+\widetilde{N})}{\gamma-\eta}}_{2\gamma,[-\varepsilon,1-\varepsilon]}\right)\\
&\qquad\leq\widetilde{M}\big(1+\widetilde{N}\big)+\widetilde{M}\big(1+\widetilde{N}\big)\chi^{-\frac{1}{\gamma-\eta}}\sup_{\varepsilon\in[0,1]}\frac{1}{k}\sum_{j=1}^{k}\left([X(\tilde{\omega})]^{\frac{1}{\gamma-\eta}}_{\gamma,[-\varepsilon,1-\varepsilon]}+[\mathbb{X}(\tilde{\omega})]^{\frac{1}{2(\gamma-\eta)}}_{2\gamma,[-\varepsilon,1-\varepsilon]}\right)\\
&\qquad\quad+2^{4\left(2+\widetilde{N}\right)}\widetilde{N}M^{1+\widetilde{N}}\tilde{c}_1C_G\sup_{\varepsilon\in[0,1]}\frac{1}{k}\sum_{j=1}^{k}\left(1+[X(\tilde{\omega})]^{\frac{4(1+\widetilde{N})}{\gamma-\eta}}_{\gamma,[-\varepsilon,1-\varepsilon]}+[\mathbb{X}(\tilde{\omega})]^{\frac{2(1+\widetilde{N})}{\gamma-\eta}}_{2\gamma,[-\varepsilon,1-\varepsilon]}\right).
\end{aligned}
$$

Using the ergodic properties of the noise (5-6), the limes superior can be bounded by

$$
\begin{aligned}
&\limsup_{k\to\infty}\frac{1}{k}\log\left(\sup_{\varepsilon\in[0,1]}\prod_{j=1}^{k}\left(1+H_1(\theta_{-j}\omega,[-\varepsilon,1-\varepsilon])\right)\right)\\
&\qquad\leq\widetilde{M}\big(1+\widetilde{N}\big)+\widetilde{M}\big(1+\widetilde{N}\big)\chi^{\frac{1}{\gamma-\eta}}\left(K_{\frac{1}{\gamma-\eta}}+\mathbb{K}_{\frac{1}{2(\gamma-\eta)}}\right)\\
&\qquad\quad+2^{4\left(2+\widetilde{N}\right)}\widetilde{N}M^{1+\widetilde{N}}\tilde{c}_1C_G\left(1+K_{\frac{4(1+\widetilde{N})}{\gamma-\eta}}+\mathbb{K}_{\frac{2(1+\widetilde{N})}{\gamma-\eta}}\right)\leq c\mathbf{K}_q+c,
\end{aligned}
$$

where we used that $\chi^{\frac{1}{\gamma-\eta}}<1$. Therefore, the right-hand side of (5-7) depends only on c and on a specific moment of the Gaussian rough path $\mathbf{X}(\tilde{\omega})$ of order q. Furthermore, there exists for any $\delta > 0$ some $k_0 \in \mathbb{N}_0$ such that

$$\sup_{\varepsilon\in[0,1]} \prod_{j=1}^{k} \big(1 + H_1(\theta_{-j}\omega, [-\varepsilon, 1-\varepsilon])\big) \leq e^{(c\mathbf{K}_q + c + \delta)k}, \tag{5-9}$$

holds for all $k \geq k_0$. Since the condition (5-7) leads to $\lambda - c\mathbf{K}_q - c - \delta > 0$ for some small $\delta > 0$, the temperedness of y_0 yields that the first term in (5-8) can be bounded by 1 for large times. So we obtain

$$\begin{aligned}\big\|\varphi\big(t, \theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big\|_{\alpha} \leq 1 + P_1(\omega, [-1,1]) &\sup_{\varepsilon\in[0,1]} \sum_{l=1}^{\infty} e^{-\lambda l} H_2(\theta_{-l}\omega, [-\varepsilon, 1-\varepsilon]) \\ &\times \prod_{j=1}^{k-1} \big(1 + H_1(\theta_{-j}\omega, [\varepsilon, 1-\varepsilon])\big) + P_2(\omega, [-1,1]),\end{aligned}$$

for t large. It remains to show that the expression on the right-hand side is a tempered random variable. First, note that $P_2(\cdot, [-1,1])$ is integrable due to Lemma 3.43, which is possible since P_2 in fact depends only on $\tilde{\omega}$. In particular, due to the special structure of P_2 it is easy to show that also $\log\big(P_2(\cdot, [-1,1])\big) \in \mathrm{L}^1(\Omega)$ and therefore

$$\limsup_{t\to\infty} \frac{\log\big(P_2(\theta_t\omega, [-1,1])\big)}{t} = 0,$$

due to [Arn98, Theorem 4.1.3 i)]. In conclusion (4-9) is fulfilled, and $P_2(\cdot, [-1,1])$ is tempered. The same argument can be used to show that also $H_2(\cdot, [-\varepsilon, 1-\varepsilon])$ is tempered, so for every $\delta > 0$ there exists a $l_0 \in \mathbb{N}_0$ such that for $l \geq l_0$ we obtain the bound

$$H_2(\theta_{-l}\omega, [-\varepsilon, 1-\varepsilon]) \leq e^{\delta l}.$$

Together with (5-9) and (5-7) this leads to

$$\begin{aligned}\widetilde{R}(\omega) &:= \sup_{\varepsilon\in[0,1]} \sum_{l=1}^{\infty} e^{-\lambda l} H_2(\theta_{-l}\omega, [-\varepsilon, 1-\varepsilon]) \prod_{j=1}^{l-1} \big(1 + H_1(\theta_{-j}\omega, [-\varepsilon, 1-\varepsilon])\big) \\ &\leq \sum_{l=l_0}^{\infty} e^{-(\lambda - c\mathbf{K}_q - c - \delta)l} + \sup_{\varepsilon\in[0,1]} \sum_{l=1}^{l_0-1} e^{-\lambda l} H_2(\theta_{-l}\omega, [-\varepsilon, 1-\varepsilon]) \\ &\quad \times \prod_{j=1}^{l-1} \big(1 + H_1(\theta_{-j}\omega, [-\varepsilon, 1-\varepsilon])\big) < \infty.\end{aligned}$$

To show the measurability, we recall that since $\mathbf{X}$ is a geometric rough path, the Hölder norms are continuous, according to Lemma 2.15. Therefore, $\varepsilon \mapsto H_i(\omega, [-\varepsilon, 1-\varepsilon])$ for $i = 1, 2$ is continuous. In conclusion, the supremum can be taken over $[0,1] \cap \mathbb{Q}$ instead of $[0,1]$, which yields measurability.

The temperedness of $\widetilde{R}$ follows by similar arguments as in [DH23, Proposition 3.5]. For this we need again that H_i, defined in (5-5), satisfies $H_i(\theta_\tau\omega, [s,t]) = H_i(\omega, [s+\tau, t+\tau])$ for $i = 1, 2$ and $\tau \in \mathbb{R}$. Further, we have

$$\begin{aligned}\limsup_{|t|\to\infty} \frac{\log\big(P_1(\theta_t\omega, [-1,1])\widetilde{R}(\theta_t\omega)\big)}{|t|} &\leq \limsup_{|t|\to\infty} \frac{\log\big(P_1(\theta_t\omega, [-1,1])\big)}{|t|} \\ &\quad + \limsup_{|t|\to\infty} \frac{\log\big(\widetilde{R}(\theta_t\omega)\big)}{|t|} = 0,\end{aligned}$$

and therefore the temperedness of $P_1(\cdot, [-1,1])\widetilde{R}(\cdot)$ follows again by [Arn98, Theorem 4.1.3 i)]. With the temperedness of $P_2(\cdot, [-1,1])$, also

$$R(\omega) := 1 + P_1(\omega, [-1,1])\widetilde{R}(\omega) + P_2(\omega, [-1,1])$$

is tempered. Hence, $K(\omega) := B(0, R(\omega) + \overline{\delta})$, for some small $\overline{\delta} > 0$ is a random absorbing set for φ in E_α, see Lemma 4.18. □

It remains to prove the existence of a compact absorbing set since $\{K(\omega)\}_{\omega\in\Omega}$, as constructed in Lemma 5.9, is not necessarily compact.

Theorem 5.10. *Let the assumptions from Lemma 5.9 be satisfied and assume that the embeddings $E_\beta \hookrightarrow E_\alpha$ are compact. Then the random dynamical system φ associated to (RPDE), possesses a random pullback attractor $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$.*

Proof. Since Lemma 5.9 ensures the existence of a tempered absorbing set $\{K(\omega)\}_{\omega\in\Omega}$ in E_α, we need a compactness argument such that Theorem 4.19 provides a global attractor. Therefore, we define

$$\widetilde{K}(\omega) := \overline{\varphi\big(T_1, \theta_{-T_1}\omega, K(\theta_{-T_1}\omega)\big)}^{E_\alpha} \subset K(\omega),$$

where $T_1 \geq T_K$ and T_K is the absorbing time of $K(\omega)$ where we omit the ω-dependence of the absorbing times. The fact that $\{\widetilde{K}(\omega)\}_{\omega\in\Omega}$ is indeed absorbing is a direct consequence of the cocycle property and the fact that $\{K(\omega)\}_{\omega\in\omega}$ is an absorbing set. The proof of the compactness of $\widetilde{K}(\omega)$ uses the compact embedding $E_{\alpha+\beta} \hookrightarrow E_\alpha$ for $0 < \beta < \min\{1-\sigma_F, \gamma - \sigma_G\}$. Indeed, we let $y_0 \in K(\theta_{-T_1}\omega)$ and observe that

$$\begin{aligned}\|\varphi(T_1, \theta_{-T_1}\omega, y_0)\|_{\alpha+\beta} \leq \|S_{T_1,0}y_0\|_{\alpha+\beta} &+ \left\|\int_0^{T_1} S_{T_1,r}F\big(r, \varphi(r, \theta_{-T_1}\omega, y_0)\big)\,\mathrm{d}r\right\|_{\alpha+\beta} \\ &+ \left\|\int_0^{T_1} S_{T_1,r}G\big(r, \varphi(r, \theta_{-T_1}\omega, y_0)\big)\,\mathrm{d}\mathbf{X}_r(\theta_{-T_1}\tilde{\omega})\right\|_{\alpha+\beta}.\end{aligned}$$

The first term can be bounded using (D-4), which leads to

$$\|S_{T_1,0}y_0\|_{\alpha+\beta} \leq C_{-\beta}e^{-\lambda_A T_1}T_1^{-\beta}\|y_0\|_\alpha \leq C_{-\beta}e^{-\lambda_A T_1}T_1^{-\beta}\big(R(\theta_{-T_1}\omega) + \overline{\delta}\big) < \infty,$$

due to $y_0 \in K(\theta_{-T_1}\omega) = K\big(0, R(\theta_{-T_1}\omega) + \overline{\delta}\big)$. Since F has linear growth, we can estimate the drift term by

$$\begin{aligned}&\left\|\int_0^{T_1} S_{T_1,r}F\big(r, \varphi(r, \theta_{-T_1}\omega, y_0)\big)\,\mathrm{d}r\right\|_{\alpha+\beta} \\ &\qquad \leq C_{-\beta-\sigma_F}\int_0^{T_1} e^{-\lambda_A(T_1-r)}(T_1-r)^{-\beta-\sigma_F}\left\|F\big(r, \varphi(r, \theta_{-T_1}\omega, y_0)\big)\right\|_{\alpha-\sigma_F}\,\mathrm{d}r \\ &\qquad \leq C_F C_{-\beta-\sigma_F}\int_0^{T_1} e^{-\lambda_A(T_1-r)}(T_1-r)^{-\beta-\sigma_F}\big(1 + \|\varphi(r, \theta_{-T_1}\omega, y_0)\|_\alpha\big)\,\mathrm{d}r \\ &\qquad \leq C_F C_{-\beta-\sigma_F}\int_{-T_1}^0 e^{\lambda_A r}(-r)^{-\beta-\sigma_F}\big(1 + \|\varphi(r+T_1, \theta_{-T_1}\omega, y_0)\|_\alpha\big)\,\mathrm{d}r.\end{aligned}$$

Since $r \mapsto \|\varphi(r+T_1, \theta_{-T_1}\omega, y_0)\|_\alpha$ is continuous, it is bounded on the compact interval $[-T_1, 0]$ by some constant C_φ which leads to

$$\left\|\int_0^{T_1} S_{T_1-r}F\big(r, \varphi(r, \theta_{-T_1}\omega, y_0)\big)\,\mathrm{d}r\right\|_{\alpha+\beta}$$

$$\leq C_F C_{-\beta-\sigma_F} \int_{-T_1}^{0} e^{\lambda_A r}(-r)^{-\beta-\sigma_F}\big(1+C_\varphi\big)\,\mathrm{d}r,$$

where the integral is bounded due to $\beta < 1-\sigma_F$. For the rough integral we combine (2-45), (3-6) and (3-29) to obtain

$$\begin{aligned}
&\left\| \int_0^{T_1} S_{T_1,r} G\big(r,\varphi(r,\theta_{-T_1}\omega,y_0)\big)\,\mathrm{d}\mathbf{X}_r(\theta_{-T_1}\tilde{\omega}) \right\|_{\alpha+\beta} \\
&\qquad \leq C_I C_G\big(1+\varrho_{\gamma,[0,T_1]}(\mathbf{X}(\theta_{-T_1}\tilde{\omega}))\big)^3 T_1^{-\beta+\gamma-\sigma_G} \\
&\qquad\quad \times \Big(1+\|y_0\|_\alpha P_1(\theta_{-T_1}\omega,[0,T_1]) + P_2(\theta_{-T_1}\omega,[0,T_1])\Big) \\
&\qquad \leq C_I C_G\big(1+\varrho_{\gamma,[0,T_1]}(\mathbf{X}(\theta_{-T_1}\tilde{\omega}))\big)^3 T_1^{-\beta+\gamma-\sigma_G} \\
&\qquad\quad \times \Big(1+\big(R(\theta_{-T_1}\omega)+\overline{\delta}\big)P_1(\theta_{-T_1}\omega,[0,T_1]) + P_2(\theta_{-T_1}\omega,[0,T_1])\Big),
\end{aligned}$$

which is finite due to $\beta+\sigma_G<\gamma$. This shows that

$$\|\varphi(T_1,\theta_{-T_1}\omega,y_0)\|_{\alpha+\beta} < \infty,$$

for arbitrary $y_0 \in K(\theta_{-T_1}\omega)$, which leads to $\widetilde{K}(\omega) \in E_{\alpha+\beta}$. Therefore, $\{\widetilde{K}(\omega)\}_{\omega\in\Omega}$ is a compact absorbing set in E_α. □

Remark 5.11. i) Generally, the condition (5-7) implies that (RPDE) has an attractor if the nonlinearities are sufficiently small in comparison to the spectral bound of the operator A. A similar condition to (5-7) was obtained in [DH23] for Young differential equations. We note that we have to control higher moments of the noise term $\mathbf{X}$, which is possible since $\mathbf{X}$ is assumed to be a Gaussian rough path.

ii) We also want to mention an alternative approach to obtaining an absorbing set. This was used in [GAMS10] for stochastic differential equations driven by a fractional Brownian motion with Hurst index $H>1/2$. This approach relies on stopping times required to control the Hölder norm of the random input. This technique was extended to the rough case in [YLZ23]. Here, the condition for the existence of a random attractor reads as follows. The constants C_F and C_G need to be sufficiently small such that

$$C_F < \frac{e^{\frac{\lambda_A}{2}}-1}{C_A(\frac{\lambda_A}{2}+1)} =: \mu, \quad C_G < \min\left\{1, \frac{\mu^{\lceil \beta^{-1}\rceil}}{Q(\gamma,\sigma_G,\mathbf{X})W^+(\lambda_A,\delta)} - \frac{\mu^{\lceil \beta^{-1}\rceil}}{Q(\gamma,\sigma_G,\mathbf{X})}\right\}$$

holds, where $0<\beta<\min\{1-\sigma_F,\gamma-\sigma_G\}$, Q depends on the moments of $\mathbb{X}$ up to the $\lceil\beta^{-1}\rceil$-order and $W^+<1$ depends on μ,λ_A,C_A and a small positive number δ. In particular, this means that C_G, and therefore the growth bound of G, has to be bounded by 1. In contrast, our technique also allows $C_G>1$ by selecting a sufficiently large λ_A.

iii) If the noise is additive, the existence of a random dynamical system can be established, transforming the stochastic partial differential equations into a random partial differential equation using the stationary Ornstein–Uhlenbeck process. In this case, the condition for the existence of the attractor (5-7) simplifies to

$$\lambda_A > 2\big(C_{-\sigma_F}C_F\Gamma(1-\sigma_F)\big)^{\frac{1}{1-\sigma_F}},$$

which is consistent with similar results for additive noise and the assumption $F\colon[0,\infty)\times E_\alpha \to E_\alpha$, consequently $\sigma_F=0$, compare [KNS21, Assumption 2].

5.1.3 Regularity of the random pullback attractor

A key feature of the mild formulation is that we directly obtain higher regularity of the attractor. If the smoothing of the evolution equation, or the analytic semigroup, is not usable, it is often quite complex to obtain similar results. The procedure depends then on the specific equation; see, for example, [Zha13, Tan15, Tan16, LSX25].

Corollary 5.12. *Let the same assumptions as in Theorem 5.10 be fulfilled, choose*

$$0<\beta<\min\{1-\sigma_F,\gamma-\sigma_G\}$$

and replace (5-7) by

$$\lambda_A-2\big(C_{-\sigma_F-\beta}C_F\Gamma(1-\sigma_F-\beta)\big)^{\frac{1}{1-\sigma_F-\beta}}>c_\beta\big(\mathbf{K}_q+1\big),$$

with $\widetilde{c}_{1,\beta}=\max\big\{C_I,C_{-\beta}C_I\big\}e^{\lambda_A}\min\Big\{\widetilde{L},\frac{M_{1-\sigma_F-\beta}}{2}\Big\}$ *and*

$$c_\beta:=2C\big(\widetilde{N}\big)\max\big\{\widetilde{M},\widetilde{c}_{1,\beta}C_G\big\}.$$

Then the random pullback attractor $\{\mathscr{A}(\omega)\}_{\omega\in\Omega}$ *obtained in Theorem 5.10 also belongs to* $E_{\alpha+\beta}$.

Proof. Let $y_0(\omega)\in K(\omega)$ where $\{K(\omega)\}_{\omega\in\Omega}$ is a tempered set in E_α. We need to estimate $\varphi\big(t,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)$ in the $E_{\alpha+\beta}$-norm, similar to the proof of Lemma 5.9. Therefore, the estimates obtained in Lemma 5.7 need to be adapted to the new Banach space. Based on the computations in Lemma 5.5, we observe that for $0<\beta<\min\{1-\sigma_F,\gamma-\sigma_G\}$ the estimates for the integral terms can be improved, exploiting the smoothing property (D-4)

$$\begin{aligned}\left\|\int_0^t S_{t,r}G(r,y_r)\,\mathrm{d}\mathbf{X}_r(\omega)\right\|_{\alpha+\beta}&\le C_Gc_{1,\beta}\sum_{l=0}^{\lfloor t\rfloor}e^{-\lambda_A(t-l-1)}P_3(\omega,[l,l+1]),\\ \left\|\int_0^t S_{t,r}F(r,y_r)\,\mathrm{d}r\right\|_{\alpha+\beta}&\le C_{-\sigma_F-\beta}C_F\int_0^t e^{-\lambda_A(t-r)}(t-r)^{-\sigma_F-\beta}\left\|y_r\right\|_\alpha\,\mathrm{d}r+c_{2,\beta},\end{aligned}\tag{5-10}$$

for all $t\geq 0$ where $c_{1,\beta}:=\max\{C_I,C_{-\beta},C_I\}$ and

$$c_{2,\beta}:=C_{-\sigma_F-\beta}C_F\lambda_A^{\sigma_F+\beta-1}\Gamma(1-\sigma_F+\beta).$$

Further, we obtain

$$\|S_{t,0}y_0(\omega)\|_{\alpha+\beta}\le C_{-\beta}e^{\lambda_At}t^{-\beta}\left\|y_0(\omega)\right\|_\alpha,$$

and combining this with (5-10) one gets an absorbing set, similar to Lemma 5.9. The compactness is shown in the same way as in Theorem 5.10. □

5.1.4 Applications

To end this section, we provide some examples for the nonlinear term G and indicate how the condition (5-7) on the existence of random attractors can be verified in concrete applications. Since the conditions on F are less restrictive than those on G, we can consider in both examples a global Lipschitz nonlinearity F. Therefore, we focus on G and A, and let $F\equiv 0$ without loss of generality.

PDEs with multiplicative rough boundary noise

As in Section 3.3, let $\mathcal{O}\subset\mathbb{R}^n$ be a bounded domain with $\mathcal{C}^\infty$-boundary and consider the semilinear parabolic evolution equation with multiplicative rough boundary noise in $E:=$

$L^p(\mathcal{O})$, for $2 \leq p < \infty$, given by

$$\begin{cases} \frac{\partial}{\partial t} y_t = \mathcal{A} y_t & \text{in } \mathcal{O}, \\ \mathcal{B} y_t = G(t, y_t) \, \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{X}_t & \text{on } \partial\mathcal{O}. \end{cases} \tag{5-11}$$

Here $\mathbf{X}$ is a γ-Hölder rough cocycle which satisfies Assumption **(N)** with $\gamma \in (1/3, 1/2]$. Further, $\mathcal{A}$ is a formal second-order differential operator in divergence form with Neumann boundary conditions $\mathcal{B}$ given by

$$\mathcal{A} u := \sum_{i,j=1}^{d} \partial_i \left(a_{ij} \partial_j\right) u - \tilde{\lambda}_A u, \quad \mathcal{B} u := \sum_{i,j=1}^{d} \nu_i \gamma_\partial a_{ij} \partial_j u, \tag{5-12}$$

where the coefficients $a_{ij} \colon \overline{\mathcal{O}} \to \mathbb{R}$ are smooth, $\tilde{\lambda}_A > 0$ is a constant, $(a_{ij})_{i,j=1}^d$ is symmetric and uniformly elliptic. Let $A \colon D(A) \subset E \to E$ be the E-realization of $(\mathcal{A}, \mathcal{B})$ with

$$D(A) := \left\{ u \in H^{2,p}(\mathcal{O}) : \mathcal{B} u = 0 \right\},$$

$(E_\alpha)_{\alpha \in \mathbb{R}}$ the respective fractional power scale and $\widetilde{E}_\alpha := H^{\alpha - 1 - 1/p, p}(\partial \mathcal{O})$ the scale for the boundary data. In Section 3.3, it was shown that one can transform (5-11) to a semilinear problem without boundary noise. This reads as

$$\begin{cases} \mathrm{d} y_t = A y_t \, \mathrm{d}t + A_{\epsilon - 1 - 2\gamma} \mathcal{N} G(t, y_t) \, \mathrm{d}\mathbf{X}_t, \\ y_0 \in E_{\epsilon - 1}, \end{cases} \tag{5-13}$$

where $\mathcal{N}$ is the Neumann map and $A_{\epsilon-1-2\gamma} \in \mathcal{L}(E_{\epsilon - 2\gamma}; E_{\epsilon - 1 - 2\gamma})$ an extrapolation operator for some $\epsilon < 1/2 + 1/2p$, see also Appendix C.

Theorem 5.13. ([BS24, Theorem 4.2]) *Let $\tilde{\lambda}_A > 0$ be large enough such that (5-7) holds for some $\lambda < \tilde{\lambda}_A$. Further assume that $G \colon [0, \infty) \times E_{\epsilon - 1 - i\gamma} \to \widetilde{E}_{\epsilon - 1 - i\gamma + \sigma_G}$ is bounded for $i = 0, 1, 2$ and satisfies **(G_N)**, A has a compact resolvent, and the principal part, i.e. $\widetilde{A} := \sum_{i,j=1}^d \partial_i (a_{ij} \partial_j)$, is dissipative and there exists a constant $a_0 > 0$ such that $\widetilde{A} - a_0$ is surjective. Then there exists a random dynamical system for (5-13) on $E_{\epsilon - 1}$, which possesses a global random pullback attractor.*

Proof. As in Theorem 3.20 it can be shown that $A_{\epsilon-1-2\gamma} \mathcal{N} G(y)$ satisfies the Assumptions **(G3)** with $\sigma_G = 0$. Consequently, there exists a global-in-time solution, see Theorem 3.23, which generates a random dynamical system on $E_{\epsilon - 1}$.

It is known that the operator A, constructed above, satisfies Assumption **(A)**, see Example D.8. Due to the assumptions on A, the Lumer–Phillips Theorem B.6 yields the existence of an analytic semigroup $\big(\widetilde{S}_t\big)_{t \geq 0}$ of contractions for the principal part $\widetilde{A}$. The semigroup generated by A is then given by $S_t = e^{-\tilde{\lambda}_A t} \widetilde{S}_t$ and is therefore exponential stable with parameter $\tilde{\lambda}_A$, since $\big(\widetilde{S}_t\big)_{t \geq 0}$ is contractive. The compactness of the semigroup follows from the fact that A has a compact resolvent, as stated in [Ama95, V.1.2.1]. The existence of the global attractor is finally ensured by Theorem 5.10. ☐

Remark 5.14. i) If the noise $X \in \mathcal{C}^{\widetilde{\gamma}}$ is Hölder continuous with parameter $\widetilde{\gamma} > 1/2$, we can use the Young integral instead of the rough integral. Then, the convolution can be defined for a path $y \in \mathcal{C}(E_\alpha) \cap \mathcal{C}^{\widetilde{\gamma}}(E_{\alpha - \widetilde{\gamma}})$ by

$$\int_s^t S_{t-r} y_r \, \mathrm{d}X_r = \lim_{|\pi| \to 0} \sum_{[u,v] \in \pi} S_{t-u} y_u X_{v,u},$$

as in Theorem 2.2. The techniques developed in this section also apply to the Young case, modifying (5-7) accordingly. More precisely, the constant c differs and $\mathbf{K}_q = K_q$, since $\mathbb{X}$ is not needed to define the integral. As a particular application of Theorem 5.10, we can also investigate Dirichlet boundary conditions if $\tilde{\gamma} > 1 - 1/2p$ and obtain a random pullback attractor in this situation.

ii) To the best of our knowledge, this is the first statement for attractors of equations with boundary noise. In general, there are only very few results on the long-term behavior of equations of the form (5-11). For example, stability criteria were investigated in [AB02b] and a stabilization effect by boundary noise was shown for the Chaffee–Infante equation in [FSTT19]. We further refer to [BDK24] for the analysis of warning signs for a Boussinesq model with boundary noise.

Parabolic PDEs with multiplicative rough noise

We consider the parabolic partial differential equation on $E \coloneqq \mathrm{L}^p(\mathcal{O})$ for $2 \leq p < \infty$ given by

$$\begin{cases} \mathrm{d}y_t = A(t)y_t \,\mathrm{d}t + G(t, y_t)\,\mathrm{d}\mathbf{X}_t, \\ y_0 \in E_\alpha. \end{cases} \tag{5-14}$$

Example 5.15. *The framework investigated in Example 3.11 can be adjusted to obtain a pullback attractor. Let $p \geq 3$ and $A = \Delta_D - \tilde{\lambda}_A$ with $\tilde{\lambda}_A > 0$. Then the corresponding fractional power scale is given by*

$$E_\beta \coloneqq \begin{cases} \mathrm{H}^{2\beta,p}(\mathcal{O}), & 0 \leq \beta < \frac{1}{2p}, \\ \left\{u \in \mathrm{H}^{2\beta,p}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\right\}, & \frac{1}{2p} < \beta \leq 1, \beta \neq \frac{p+1}{2p}. \end{cases}$$

We further choose $\alpha > n/2p + 2\gamma$ which leads to $E_{\alpha - i\gamma} \hookrightarrow C(\overline{\mathcal{O}})$. We introduce the operator G as

$$G\colon [0,\infty) \times E_{\alpha-i\gamma} \to E_{\alpha-i\gamma}, u \mapsto a(t) \int_{\mathcal{O}} g(\cdot, u(x))\,\mathrm{d}x,$$

*for $i = 0,1,2$ and $a \in \mathcal{C}^{2\gamma}([0,\infty);\mathbb{R})$. The kernel $g\colon \overline{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is assumed to be bounded and three times continuously differentiable with bounded derivates such that $\sup_{x\in\mathbb{R}} \left|\mathrm{D}_2^k g(\cdot, x)\right|$ is bounded for $k = 0,1,2,3$, $i = 0,1,2$ and $g|_{\partial\mathcal{O}\times\mathbb{R}} = 0$. Due to the boundedness assumption on g, the operator G fulfills **(G3)**, which can be proven as in Example 3.11.*

Theorem 5.16. *Let $\tilde{\lambda}_A$ be large enough such that (5-7) is satisfied for some $\lambda_A < \tilde{\lambda}_A$. Then there exists a random dynamical system φ for (5-14) on E_α that possesses a random pullback attractor.*

5.2 Existence of Lyapunov exponents

This section is meant to give a concrete application of the mild Gronwall lemma proven in Section 3.5. We published this result already as a part of [BGVS25], which was a collaboration with Alexandra Blessing and Mazyar Ghani Varzaneh. In particular, we use the mild Gronwall lemma to validate the integrability assumption of the multiplicative ergodic theorem, which ensures the existence of Lyapunov exponents. As this study focuses entirely on the application of the Gronwall Lemma, the proofs are kept brief. For a more detailed proof and other results in this direction, see [BGVS25].

Let us briefly explain what Lyapunov exponents are. In particular, they can measure chaos and instability of a system and quantify the extent to which nearby trajectories converge or diverge, as introduced by Lyapunov in his thesis [Lya92]. Consider a random dynamical system

ϕ in a Banach space $\mathcal{E}$, which is linear in its third variable, then the Lyapunov exponents are defined by

$$\lambda(\omega, x) \coloneqq \limsup_{t\to\infty} \frac{1}{t} \log\big(\, \|\phi(t,\omega,x)\|_{\mathcal{E}}\,\big).$$

In the simple case where the system satisfies $\dot{y} = My$, for a matrix M, the dynamical system is given by $(t,x) \mapsto e^{tM}x$ and the Lyapunov exponents are just the eigenvalues of M. It is well-known that the eigenvalues determine the stability of such a deterministic linear system. For example, if all eigenvalues are negative, the system is stable. This concept is extended to more complex systems where a top negative Lyapunov exponent indicates stability, whereas a positive one indicates chaotic behavior.

The demonstration of the existence of Lyapunov exponents typically involves the application of ergodic theorems, such as Kingman's subadditive ergodic theorem [Kin68] or the multiplicative ergodic theorem by Oseledets [Ose68]. More generally, Ruelle proved an ergodic theorem in Hilbert spaces [Rue82], which was extended to Banach spaces [LL10, GTQ15].

This section is now based on the more recent results by Varzaneh and Riedel [GVR23, GVR25, BGVS25]. The multiplicative ergodic theorem, proven in [GVR23], is more general than we need, as it is established for measurable fields of Banach spaces. In our case, we apply the theorem to a single Banach space.

5.2.1 Multiplicative ergodic theorem

Recall that we use the extended probability space $\Omega = \widetilde{\Omega}\times\Sigma$, established in (4-13), which also incorporates the symbol space to treat the non-autonomous part of the equation. $\widetilde{\Omega}$ represents the randomness described by the noise, and the symbol space Σ is constructed to incorporate the time dependencies.

Definition 5.17. A random point $Y\colon \Omega \to E_\alpha$ is referred to as a stationary point for the random dynamical system φ if the map $\omega \mapsto \|Y_\omega\|_\alpha$ is measurable and for every $t > 0, \omega \in \Omega$ we have

$$\varphi(t,\omega,Y_\omega) = Y_{\theta_t\omega}.$$

Now fix a stationary point Y. Then, the linearization of (RPDE) along $y_t(\omega) \coloneqq Y_{\theta_t\omega}$ is given by

$$\begin{cases} \mathrm{d}z_t = \big(A(t)z_t + \mathrm{D}_2F(t,Y_{\theta_t\omega})\big)z_t\,\mathrm{d}t + \mathrm{D}_2G(t,Y_{\theta_t\omega})z_t\,\mathrm{d}\mathbf{X}_t(\tilde{\omega}),\\ z_0 \in E_\alpha,\end{cases}$$

see also Subsection 3.5.1. Let ψ be the random dynamical system generated by this linearization.

Lemma 5.18. *Let $\big(A(t)\big)_{t\geq 0}, F$ and G satisfy Assumption **(A)**,**(S)**,**(F1)** and **(G3)**. Furthermore, let $\mathbf{X}$ be a rough path cocycle satisfying **(N)**. Then the solution operator φ of* (RPDE) *generates a continuous random dynamical system. If further **(F2)** is satisfied and $A(t)$ admits a compact inverse for every $t \in [0,T]$, then the solution operator ψ of the linearized equation along the stationary point $(Y_\omega)_{\omega\in\Omega}$ is a compact linear random dynamical system, meaning that $\psi(t,\omega,\cdot)$ is a compact operator for every $t \geq 0$ and $\omega \in \Omega$.*

Proof. That φ is a random dynamical system was shown in Lemma 4.35. Moreover, ψ is a random dynamical system; we only need to show the compactness. Since $A(t)$ has a compact inverse, we know that the Banach spaces $(E_\alpha)_{\alpha\in\mathbb{R}}$ are compactly embedded [Ama95, Theorem V.1.5.1]. Using the smoothing property of the parabolic evolution family, one can show that

$\psi(t,\omega,\cdot) \in \mathcal{L}(E_\alpha; E_{\alpha+\varepsilon})$ for some small $\varepsilon > 0$, similar to [GVR25, Proposition 3.7]. Then the compactness of the embedding $E_{\alpha+\varepsilon} \hookrightarrow E_\alpha$ yields the claim. □

Proposition 5.19. *Let the same assumptions as in Lemma 5.18 be fulfilled and fix a time $t_0 > 0$. We further assume that the stationary point fulfills*

$$\big(\omega \mapsto \|Y_\omega\|_\alpha\big) \in \bigcap_{p\geq 1} \mathrm{L}^p(\Omega), \tag{5-15}$$

for every $p \geq 1$. Then we have

$$\mathbb{E}\left[\sup_{0\leq t\leq t_0} \log^+\left(\|\psi(t,\cdot,\cdot)\|_{\mathcal{L}(E_\alpha;E_\alpha)}\right)\right] < \infty, \tag{5-16}$$

$$\mathbb{E}\left[\sup_{0\leq t\leq t_0} \log^+\left(\|\psi(t_0-t,\theta_t\cdot,)\|_{\mathcal{L}(E_\alpha;E_\alpha)}\right)\right] < \infty. \tag{5-17}$$

Proof. Using the mild Gronwall inequality in Corollary 3.47, we obtain

$$\|\psi(t,\omega,\cdot)\|_{\mathcal{L}(E_\alpha;E_\alpha)} = \sup_{\|x\|_\alpha=1} \|\psi(t,\omega,x)\|_\alpha \leq \widetilde{C}_1(\omega)\big(1+\varrho_{\gamma,[0,t]}(\mathbf{X}(\tilde{\omega}))\big)e^{t\widetilde{C}_2(\omega)}(1+C_G)$$

for $t \in [0,t_0]$. In particular, this leads to

$$\sup_{0\leq t\leq t_0} \log^+\left(\|\psi(t,\omega,\cdot)\|_{\mathcal{L}(E_\alpha;E_\alpha)}\right) \leq \log\Big(\widetilde{C}_1(\omega)\big(1+\varrho_{\gamma,[0,t_0]}(\mathbf{X}(\tilde{\omega}))\big)(1+C_G)\Big) + t_0\widetilde{C}_2(\omega). \tag{5-18}$$

To prove the integrability, we note that $\mathbf{X}(\tilde{\omega})$ satisfies Assumption **(N)** and we therefore obtain

$$\tilde{\omega} \mapsto \varrho_{\gamma,[0,t]}(\mathbf{X}(\tilde{\omega})) \in \bigcap_{p\geq 1} \mathrm{L}^p\big(\widetilde{\Omega}\big), \tag{5-19}$$

by Corollary 2.20. Further, since ψ is the linearization along the stationary solution Y, the random constants $\widetilde{C}_1(\omega)$ and $\widetilde{C}_2(\omega)$ depend on the solution $y_t(\omega) := Y_{\theta_t\omega}$. Then, Lemma 3.43 provides that

$$\tilde{\omega} \mapsto \big\|y(\tilde{\omega},\sigma),(y(\tilde{\omega},\sigma))'\big\|_{\mathcal{D}^\gamma_{X(\tilde{\omega}),\alpha}([0,\infty))} \in \bigcap_{p\geq 1} \mathrm{L}^p\big(\widetilde{\Omega}\big).$$

For the integrability with respect to $\mathbb{P}_\Sigma$ we note that

$$\begin{aligned}&\Big\|\big\|y(\tilde{\omega},\cdot),(y(\tilde{\omega},\cdot))'\big\|_{\mathcal{D}^\gamma_{X(\tilde{\omega}),\alpha}([0,\infty))}\Big\|_{\mathrm{L}^p(\Sigma)}\\ &\quad\leq \sup_{s\geq 0}\big\|y(\tilde{\omega},\mathfrak{S}(s+\cdot)),\big(y(\tilde{\omega},\mathfrak{S}(s+\cdot))\big)'\big\|_{\mathcal{D}^\gamma_{X(\tilde{\omega}),\alpha}([0,\infty))},\end{aligned}$$

holds, where we use that $\mathbb{P}_\Sigma\big(\{\mathfrak{S}(\cdot+s) : s\geq 0\}\big) = 1$. Further, recall that $\mathfrak{S}$ is the time-symbol associated with (RPDE) and the shift by time s is the same as choosing the starting time s. To be precise, consider the equations

$$\mathrm{d}v_t = \big(A(t)v_t + F(t,v_t)\big)\,\mathrm{d}t + G(t,v_t)\,\mathrm{d}\mathbf{X}_t, \tag{5-20}$$

$$\mathrm{d}\tilde{v}_t = \big(A(t+s)\tilde{v}_t + F(t+s,\tilde{v}_t)\big)\,\mathrm{d}t + G(t+s,\tilde{v}_t)\,\mathrm{d}\mathbf{X}_t, \tag{5-21}$$

with initial condition $\tilde{v}(0) = v(s) \in E_\alpha$. Then (5-20) corresponds to $\mathfrak{S}$ and (5-21) to $\mathfrak{S}(s+\cdot)$, which results into the fact that $\tilde{v}_t = v_{t+s}$. In particular, we have

$$y_t\big(\tilde{\omega}, \mathfrak{S}(s+\cdot)\big) = y_{t+s}(\tilde{\omega}, \mathfrak{S}),$$

which leads to

$$\begin{aligned}\Big\| \big\| y(\tilde{\omega},\cdot), (y(\tilde{\omega},\cdot))' \big\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}([0,\infty))} \Big\|_{\mathrm{L}^p(\Sigma)} &\leq \sup_{s\geq 0} \big\| y(\tilde{\omega},\mathfrak{S}), (y(\tilde{\omega},\mathfrak{S}))' \big\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}([s,\infty))} \\ &\leq \big\| y(\omega), (y(\omega))' \big\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}([0,\infty))}.\end{aligned}$$

Together with Fubini's theorem, we obtain

$$\omega \mapsto \big\| y(\omega), (y(\omega))' \big\|_{\mathcal{D}^{\gamma}_{X(\tilde{\omega}),\alpha}([0,\infty))} \in \bigcap_{p\geq 1} \mathrm{L}^p(\Omega). \tag{5-22}$$

Using (5-19), (5-22), and (5-15) it is easy to see that the random constants $\widetilde{C}_1$ and $\widetilde{C}_2$ defined in (3-38) are also integrable. Then (5-16) follows using this integrability and the inequality derived in (5-18).

The second integrability condition (5-17) can be shown similarly. Indeed, we obtain

$$\begin{aligned}\sup_{0\leq t\leq t_0} \log^+ \big(\|\psi(t_0-t,\theta_t\omega,\cdot)\|_{\mathcal{L}(E_\alpha;E_\alpha)} \big) &\leq \log \Big(\sup_{0\leq t\leq t_0} \widetilde{C}_1(\theta_t\omega) \varrho_{\gamma,[0,t_0]}\big(\mathbf{X}(\tilde{\theta}_t\tilde{\omega})\big)(1+C_G) \Big) \\ &\quad + t_0 \sup_{0\leq t\leq t_0} \widetilde{C}_2(\theta_t\omega).\end{aligned}$$

So, it is enough to prove that $\sup_{0\leq t\leq t_0} \big\| y(\theta_t\omega), y(\theta_t\omega))' \big\|_{\mathcal{D}^{\gamma}_{X,\alpha}}$ is in $\mathrm{L}^p(\Omega)$ for every $p\geq 1$. But again, due to $y_s(\theta_t\omega) = Y_{\theta_s\theta_t\omega} = y_{s+t}(\omega)$, we obtain (5-17) analogously to (5-16). □

In the next theorem, we use our previous results to obtain the existence of Lyapunov exponents for the random dynamical system constructed from the linearization of the non-autonomous rough partial differential equation (RPDE) along a stationary point.

Theorem 5.20. ([BGVS25, Theorem 5.10]) *We assume that the same conditions as in Proposition 5.19 are satisfied and define*

$$\mathscr{G}_\lambda(\omega) := \left\{ x \in E_\alpha : \limsup_{t\to\infty} \frac{1}{t} \log \|\psi(t,\omega,x)\|_\alpha \leq \lambda \right\},$$

for $\lambda \in \mathbb{R}\cup\{-\infty\}$. Then, on a θ_t-invariant subset of Ω with full measure, which is denoted again by Ω, there exists a decreasing sequence $(\lambda_i)_{i\geq 1}$, known as Lyapunov exponents with $\lambda_i \in [-\infty,\infty)$, such that $\lim_{i\to\infty}\lambda_i = -\infty$. Moreover, for each $i\geq 1$, either $\lambda_i > \lambda_{i+1}$ or $\lambda_i = \lambda_{i+1} = -\infty$ holds. For every $i \geq 1$ with $\lambda_i > -\infty$, there exist a finite-dimensional subspace $\mathscr{H}^i_\omega \subset E_\alpha$ with the following properties:

i) Every $\mathscr{H}^i_\omega$ is invariant, this means that $\psi\big(t,\omega,\mathscr{H}^i_\omega\big) = \mathscr{H}^i_{\theta_t\omega}$ holds for all $t\geq 0$.
ii) There exists a splitting of E_α. Indeed, we have $\mathscr{G}_{\lambda_1}(\omega) = E_\alpha$ and $\mathscr{H}^i_\omega \oplus \mathscr{G}_{\lambda_{i+1}}(\omega) = \mathscr{G}_{\lambda_i}(\omega)$ for each i. In particular, for every i we have

$$E_\alpha = \bigoplus_{1\leq j\leq i} \mathscr{H}^j_\omega \oplus \mathscr{G}_{\lambda_{i+1}}(\omega).$$

iii) The spaces $\mathscr{H}^i_\omega$ are fastly growing, in the sense that for each $h \in \mathscr{H}^j_\omega$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log \|\psi(t,\omega,h)\|_\alpha = \lambda_j$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log \left\| \psi(t, \theta_{-t}\omega, \cdot)^{-1}(h) \right\|_\alpha = -\lambda_j.$$

Proof. For every $t_0 > 0$ we can construct a discrete time random dynamical system $(\psi_\omega^{nt_0})_{n\in\mathbb{N},\omega\in\Omega}$. Due to the bounds (5-16) and (5-17), $\big(\psi(nt_0, \omega, \cdot)\big)_{n\in\mathbb{N},\omega\in\Omega}$ satisfies the integrability conditions of the multiplicative ergodic theorem obtained in [GVR23, Theorem 1.21], which proves the statement for the discrete time random dynamical system. The extension of this result to the continuous time setting, this means, for $\big(\psi(t, \omega, \cdot)\big)_{t\geq 0,\omega\in\Omega}$ follows by standard arguments, see [LL10, Theorem 3.3] for more details on this procedure. □

Remark 5.21. i) Further sign information on the Lyapunov exponents yields the existence of invariant manifolds for the random dynamical system generated by (RPDE). This contains, for example, stable manifolds [BGVS25, Theorem 5.17] and similarly unstable or center manifolds.

ii) A further result, which is not expanded further here, is the independence of the Lyapunov exponents from the index α. In particular, it can be demonstrated that the Lyapunov exponents from Theorem 5.20 remain unchanged when E_α is substituted for $E_{\alpha+\beta}$, as outlined in [BGVS25, Theorem 5.14].

5.2.2 Applications

We can now apply the above results to similar examples as we have already discussed in Section 3.3 and Subsection 5.1.4. Therefore, we do not repeat the examples of parabolic partial differential equations as in Subsection 3.2.1. Instead, we focus on an equation with boundary noise, where we prove the existence of a stationary solution and Lyapunov exponents.

Let $\mathcal{O} \subset \mathbb{R}^n$ be an open bounded domain with $\mathcal{C}^\infty$-boundary and consider the semilinear parabolic evolution equation with multiplicative rough boundary noise in $E := \mathrm{L}^2(\mathcal{O})$ given by

$$\begin{cases} \frac{\partial}{\partial t} y_t = \mathcal{A} y_t & \text{in } \mathcal{O}, \\ \mathcal{B} y_t = G(t, y_t)\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}_t & \text{on } \partial\mathcal{O}. \end{cases} \tag{5-23}$$

Here, $\mathbf{X}$ is a γ-Hölder rough cocycle which satisfies Assumption **(N)** with $\gamma \in (1/3, 1/2]$ and G a time-dependent nonlinearity. Furthermore, $\mathcal{A}$ is a formal second order differential operator in divergence form with corresponding Neumann boundary conditions $\mathcal{B}$ given by (5-12) and define the E-realization of $(\mathcal{A}, \mathcal{B})$ by $A \colon D(A) \subset E \to E$ with

$$D(A) := \left\{ u \in \mathrm{H}^{2,2}(\mathcal{O}) : \mathcal{B}u = 0 \right\}$$

and $(E_\alpha)_{\alpha\in\mathbb{R}}$ the respective fractional power scale as in (3-18). To keep the analysis as simple as possible, we work in $\mathrm{L}^2(\mathcal{O})$ although it is possible to treat (5-23) in $\mathrm{L}^p(\mathcal{O})$. In this case, it is possible to verify **(A1)**-**(A2)** for A and **(A3)** holds trivially. Let $(S_t)_{t\geq 0}$ be the analytic semigroup generated by A, which is exponential stable

$$\|S_t\|_{\mathcal{L}(E;E)} \leq C_S e^{-\tilde{\lambda}_A t}. \tag{5-24}$$

Then (5-23) is equivalent to the transformed equation

$$\begin{cases} \mathrm{d}y = Ay\ \mathrm{d}t + A_{\epsilon-1-\gamma}\mathcal{N}G(t,y)\ \mathrm{d}\mathbf{X}_t, \\ y_0 \in E_{\epsilon-1}, \end{cases} \tag{5-25}$$

as seen in Section 3.3. We further define the extended probability space $\Omega = \widetilde{\Omega} \times \Sigma$, where $\widetilde{\Omega}$ is the probability space associated to $\mathbf{X}(\tilde{\omega})$ and Σ the symbol space associated to the non-autonomoüous terms F and G. Similar to Subsection 3.5.1, we consider the linearized rough partial differential equation along the path component y given by

$$\mathrm{d}z_t = Az_t \,\mathrm{d}t + A_{\epsilon-1-\gamma}\mathcal{N}\mathrm{D}_2 G(t, y_t) z_t \,\mathrm{d}\mathbf{X}_t(\tilde{\omega}).$$

The solution operator of the linearization generates a random dynamical system ψ.

Lemma 5.22. *We assume that the diffusion coefficient G is bounded and satisfies **($\mathbf{G_N}$)**, and that A has a compact resolvent. Then ψ is a linear, compact random dynamical system.*

Proof. Since A has a compact resolvent we conclude that the embeddings $E_\beta \hookrightarrow E_\alpha$ are compact for $\beta > \alpha$, see [Ama95, V.1.5.1]. Then the claim follows using the smoothing properties of the semigroup and compactness of the embeddings $E_{\alpha+\epsilon} \hookrightarrow E_\alpha$ for $\epsilon > 0$, as in Lemma 5.18. □

To apply Theorem 5.20, we have to linearize (5-25) along a stationary solution. In the following, we establish such a solution for the rough evolution equation (5-25), where $\mathbf{X} := \mathbf{B} = \big(B, \mathbb{B}^{\text{Itô}}\big)$ is the Itô Brownian rough path, which satisfies assumption **(N)**, see Subsection 2.2.2. It is known that the stationary solution of the linear stochastic partial differential equation driven by a Brownian motion

$$\mathrm{d}Z_t = AZ_t \,\mathrm{d}t + \mathrm{d}B_t$$

is given by the stationary Ornstein–Uhlenbeck process

$$Z_t = \int_{-\infty}^t S_{t-r} \,\mathrm{d}B_r.$$

Consequently, we would expect that a stationary solution of (5-25) has the form

$$y_t = \int_{-\infty}^t S_{t-r} A_{\epsilon-1-\gamma}\mathcal{N} G(r, y_r) \,\mathrm{d}\mathbf{B}_r.$$

To prove this, we first note that the rough convolution coincides with the stochastic convolution defined in the Itô sense, which was shown in Lemma 2.46. In particular

$$\int_0^t S_{t-r} A_{\epsilon-1-\gamma}\mathcal{N} G(r, y_r) \,\mathrm{d}\mathbf{B}_r = \int_0^t S_{t-r} A_{\epsilon-1-\gamma}\mathcal{N} G(r, y_r) \,\mathrm{d}B_r, \tag{5-26}$$

holds almost surely.

Remark 5.23. The same statement as in (5-26) holds also, considering the Stratonovich lift $\big(B, \mathbb{B}^{\text{Strat}}\big)$ of the Brownian motion. Likewise, all the following statements remain true for $\big(B, \mathbb{B}^{\text{Strat}}\big)$ instead of $\big(B, \mathbb{B}^{\text{Itô}}\big)$.

The two sided Brownian $(B_t)_{t\in\mathbb{R}}$ is adapted to the two-parameter filtration $\big(\mathcal{F}_s^t\big)_{s\le t}$ given by $\mathcal{F}_s^t := \sigma(B_r : s \le r \le t)$ and set $\mathcal{F}_{-\infty}^t := \sigma\big(\bigcup_{s<t} \mathcal{F}_s^t\big)$.

Lemma 5.24. *Let the same Assumptions as in Lemma 5.22 be fulfilled, and assume further that*

$$\frac{C_S C_G}{\sqrt{2\tilde{\lambda}_A}} \|A_{\epsilon-1-\gamma}\|_{\mathcal{L}(E_\epsilon;E_{\epsilon-1})} \|\mathcal{N}\|_{\mathcal{L}(E_{\epsilon-1};E_\epsilon)} < 1$$

holds. Then there exists a stochastic process $y : \mathbb{R} \times \Omega \to E_{\epsilon-1}$ *adapted to* $(\mathcal{F}^t_{-\infty})_{t\in\mathbb{R}}$ *given by*

$$y_t = \int_{-\infty}^t S_{t-r} A_{\epsilon-1-\gamma} \mathcal{N} G(r, y_r) \, \mathrm{d}B_r.$$

Proof. For $t \in \mathbb{R}$ we define the map $\Gamma : \Lambda \to \Lambda$

$$\Gamma(y)(t) := \int_{-\infty}^t S_{t-r} A_{\epsilon-1-\gamma} \mathcal{N} G(r, y_r) \, \mathrm{d}B_r,$$

where

$$y \in \Lambda := \Big\{ y \colon \mathbb{R} \times \Omega \to E_{\epsilon-1} : y \text{ is continuous}, \, (\mathcal{F}^t_{-\infty})_{t\in\mathbb{R}} \text{ adapted}, \sup_{t\in\mathbb{R}} \mathbb{E}\big[\, \|y_t\|^2_{\epsilon-1} \,\big]^{\frac{1}{2}} < \infty \Big\}.$$

Due to an inequality similar to the Itô's isometry for UMD spaces [vNVW07, Corollary 3.10], which is applicable since every Hilbert space is a UMD space as well as **($\mathbf{G_N}$)** and (5-24) we obtain

$$\begin{aligned} \mathbb{E}\big[\, \|\Gamma(y)(t)\|^2_{\epsilon-1} \,\big] &\leq \mathbb{E}\left[\int_{-\infty}^t \|S_{t-r} A_{\epsilon-1-\gamma} \mathcal{N} G(r, y_r)\|^2_{\epsilon-1} \, \mathrm{d}r \right] \\ &\lesssim \int_{-\infty}^t e^{-2(t-r)\tilde{\lambda}_A} \, \mathrm{d}r = \int_{-\infty}^0 e^{2\tilde{\lambda}_A r} \, \mathrm{d}r, \end{aligned}$$

meaning that $\Gamma(y) \in \Lambda$ holds for every $y \in \Lambda$. In addition, we have

$$\begin{aligned} \mathbb{E}[\|\Gamma(y)(t) - \Gamma(\tilde{y})(t)\|^2_{\epsilon-1}] &\leq \mathbb{E}\Big[\int_{-\infty}^t \big\| S_{t-r} A_{\epsilon-1-\gamma} \mathcal{N}\big(G(r, y_r) - G(r, \tilde{y}_r)\big) \big\|^2_{\epsilon-1} \, \mathrm{d}r \Big] \\ &\leq \frac{C_S^2 C_G^2}{2\tilde{\lambda}_A} \|A_{\epsilon-1-\gamma}\|^2_{\mathcal{L}(E_\epsilon; E_{\epsilon-1})} \|\mathcal{N}\|^2_{\mathcal{L}(E_{\epsilon-1}; E_\epsilon)} \sup_{r\in\mathbb{R}} \mathbb{E}\big[\, \|y_r - \tilde{y}_r\|^2_{\epsilon-1} \,\big] \end{aligned}$$

for every $y, \tilde{y} \in \Lambda$. Applying Banach's fixed-point theorem [vN22, Theorem 2.13], we infer that there exists a $y \in \Lambda$ such that $\Gamma(y) = y$. □

It only remains to show that $(Y_\omega)_{\omega\in\Omega}$, defined by $Y_\omega := y_0(\omega)$, satisfies the integrability condition (5-15), where y is the fixed point obtain in Lemma 5.24.

Lemma 5.25. *Let the same Assumptions as in Lemma 5.24 be fulfilled. Then, the random variable* $(Y_\omega)_{\omega\in\Omega}$ *is stationary with respect to the random dynamical system* φ *generated by the solution of* (5-25) *and fulfills*

$$\big(\omega \mapsto \|Y_\omega\|_{\epsilon-1}\big) \in \bigcap_{p\geq 1} \mathrm{L}^p(\Omega).$$

Proof. It is easy to see that Y fulfills $\varphi(t, \omega, Y_\omega) = Y_{\theta_t \omega}$, which means that Y is a stationary solution of (5-25). Furthermore, we have

$$y_t - y_s = \int_{-\infty}^s S_{s-r}(S_{t-s} - \mathrm{Id}) A_{\epsilon-1-\gamma} \mathcal{N} G(r, y_r) \, \mathrm{d}B_r + \int_s^t S_{t-r} A_{\epsilon-1-\gamma} \mathcal{N} G(r, y_r) \, \mathrm{d}B_r,$$

for $s \leq t$. Using again [vNVW07, Corollary 3.10] and **($\mathbf{G_N}$)** we obtain

$$\mathbb{E}\big[\, \|y_t - y_s\|^{2m}_{\epsilon-1} \,\big] \lesssim (t-s)^m,$$

for $m \in \mathbb{N}$ and $s \leq t$. The exponential stability of the semigroup assumed in (5-24) further leads to $\mathbb{E}\big[\, \|y_0\|_{\epsilon-1} \big] < \infty$. Therefore, Kolmogorov's continuity theorem [Kun90, Theorem 1.4.1] entails that $y_0 \in \mathrm{L}^m(\Omega; E_{\epsilon-1})$ for all $m \in \mathbb{N}$, which proves the claim. □

Finally, the following theorem summarizes the above considerations.

Theorem 5.26. *Let the same assumptions as in Lemma 5.24 be satisfied. Then there exists a sequence of Lyapunov exponents $(\lambda_i)_{i\geq 1}$ for (5-23), which fulfill the properties from Theorem 5.20.*

Chapter 6

Random attractors using pathwise mild solutions

This chapter is based on a collaboration [BSST25] with Alexandra Blessing, Stefanie Sonner and Bao Quoc Tang.

In the final chapter of this thesis, we focus on a different solution concept for stochastic differential equations, which does not utilize the theory of rough paths. This is because we now consider equations where the linear part is also random, not just time-dependent. More precisely, the focus is on the following stochastic evolution equation

$$\mathrm{d}y_t = \big(A(t,\omega)y_t + F(y_t) + f\big)\ \mathrm{d}t + \sigma \mathrm{d}B_t, \tag{6-1}$$

where $(B_t)_{t\geq 0}$ is an infinite-dimensional Brownian motion on a filtered probability space $\big(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\big)$, $\sigma > 0$ denotes the intensity of the noise, $\big(A(t,\omega)\big)_{t\in\mathbb{R},\,\omega\in\Omega}$ is a family of random, time-dependent operators, F a nonlinear drift term, f a given external force and the initial datum y_0 is $\mathcal{F}_0$-measurable.

To capture the asymptotic behavior of (6-1), we again rely on the random dynamical system approach. Since the linear part is now also random, this requires the following structural assumption

$$A(t,\omega) \coloneqq A(\theta_t\omega),$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega$, where $\big(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}}\big)$ is an ergodic metric dynamical system. With a slight abuse of notation, we use the same notation as for the filtered probability space of the Brownian motion, except in the proof of Theorem 6.6, where we highlight the filtered probability space in particular. This property ensures that the random parabolic evolution family $\big(S_{t,s}(\omega)\big)_{t\geq s,\omega\in\Omega}$ generated by the family of operators $\big(A(t,\omega)\big)_{t\in\mathbb{R},\,\omega\in\Omega}$ forms a random dynamical system.

Formally, the mild solution of (6-1) is given by the variation of constants formula

$$y_t = S_{t,0}(\omega)y_0 + \int_0^t S_{t,r}(\omega)\big(F(y_r) + f\big)\ \mathrm{d}r + \sigma\left(\int_0^t S_{t,s}(\omega)\ \mathrm{d}B_s\right)(\omega).$$

However, the stochastic integral is not well-defined due to the randomness of the evolution family. In particular, to define the stochastic convolution as an Itô integral, the integrand has to be $\mathcal{F}_s$-measurable, which is not in general true for the random evolution family. This problem can be resolved by formally applying integration by parts, which leads to the following pathwise representation formula

$$\begin{aligned} y_t = S_{t,0}(\omega)y_0 + \sigma S_{t,0}(\omega)B_t(\omega) + \int_0^t S_{t,r}(\omega)\big(F(y_s) + f\big)\ \mathrm{d}r \\ - \sigma\int_0^t S_{t,r}(\omega)A(r,\omega)\big(B_t(\omega) - B_r(\omega)\big)\ \mathrm{d}r. \end{aligned} \tag{6-2}$$

The key advantage is that it is not necessary to define any stochastic integral to use this representation. A process satisfying (6-2) is called a pathwise mild solution of (6-1), which was introduced in [PV14]; more information on this subject is given in Subsection 6.1.1.

The existence and uniqueness of pathwise mild solutions for stochastic parabolic equations of the form (6-1) with globally Lipschitz continuous nonlinearities F, and more general noise terms is the content of [PV14]. These well-posedness results for pathwise mild solutions are applied in [KNS21] to investigate the existence of global and exponential random pullback attractors for the problem (6-1) with a globally Lipschitz nonlinearity F. Recently, this result was extended to locally Lipschitz continuous functions [BSST25], which is the main topic of this chapter.

The fact that F is only locally Lipschitz continuous causes some problems. To be precise, we cannot use a-priori estimates of the solution directly to obtain global well-posedness. To circumvent this, we will employ a bootstrapping method introduced in the deterministic case by [BCNS22] and extend it to a setting with random non-autonomous generators and infinite-dimensional additive noise. This approach heavily relies on the construction of an Ornstein–Uhlenbeck process, which is tempered in certain regular function spaces, see Subsection 6.1.2. This is the main difference from the approach used in [KNS21]. Due to the global Lipschitz continuity of F, the authors achieved global solvability without the need for a transformation. In addition to proving the existence of an attractor, the stochastic variant of the bootstrapping method in [BCNS22] is another important result and also an interesting insight in its own right.

To the best of our knowledge, [KNS21, BSST25] are the only works that address random attractors for stochastic parabolic evolution equations of the form (6-1) using the concept of pathwise mild solutions. On the other hand, a vast literature exists on random attractors for stochastic reaction-diffusion equations driven by real-valued additive or linear multiplicative noise; see, for example, [BLW09, GW20, Tan16, CSY15, LCK17] and the references therein. These works heavily rely on the standard approach to prove the existence of random attractors, where a finite-dimensional stationary Ornstein–Uhlenbeck process is used to transform the stochastic problem into a partial differential equation with random, non-autonomous coefficients as in Subsection 4.2.3. For infinite-dimensional additive noise, the existence of random attractors for reaction-diffusion equations has been obtained in [CW25] using mean random dynamical systems, instead of a pathwise approach as presented here.

Several dynamical aspects have been investigated for random evolution equations without noise, meaning (6-1) with $\sigma = 0$. For instance, principal Lyapunov exponents and Floquet theory were analyzed in [MS03, MS08, MS13], stable and unstable manifolds in [CDLS10], and the multiplicative ergodic theorem in [CDLS10, LNS18].

6.1 Abstract setting

In this chapter, we work with the metric dynamical system established in Example 4.5 restricted to the situation of Lemma 4.7. Therefore, $(\theta_t)_{t\in\mathbb{R}}$ is the Wiener shift and $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space associated to the $\mathcal{E}$-valued Brownian motion $(B_t)_{t\in\mathbb{R}}$, such that $B_t(\omega) = \omega_t$ for every $\omega \in \Omega$. Further note that all elements $\omega \in \Omega$ have sublinear and subexponential growth, and are locally Hölder continuous, as stated in Lemma 4.7.

6.1.1 Pathwise mild solutions

To treat the random linear part, Assumption **(A)** must be modified in a similar way to [PV14] and [KNS21]. Therefore, let $\mathcal{E}$ be a separable, reflexive Banach space of type 2, which means for example $\mathcal{E} = \mathrm{L}^2(\mathcal{O})$.

($\tilde{\mathbf{A}}$1) The operators $A(t,\omega)$ are closed and densely defined with fixed domains, i.e. $D_A := D\big(A(t,\omega)\big)$ for every $t \in \mathbb{R}, \omega \in \Omega$. Furthermore, they have bounded imaginary powers.

($\tilde{\mathbf{A}}$2) There exists a $\vartheta \in (\pi/2, \pi)$, such that $\Sigma_{0,\vartheta} \cup \{0\} \subset \rho\big(A(t,\omega)\big)$ holds for all $t \in \mathbb{R}, \omega \in \Omega$, where ρ denotes the resolvent, and there exists a constant $M > 0$ such that

$$\left\| \big(z - A(t,\omega)\big)^{-1} \right\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \leq \frac{M}{1+|z|}$$

holds for every $t \in \mathbb{R}, \omega \in \Omega$ and $z \in \Sigma_{0,\vartheta} \cup \{0\}$.

($\tilde{\mathbf{A}}$3) There exists a constant $\nu \in (0,1]$ such that

$$\|A(t,\omega) - A(s,\omega)\|_{\mathcal{L}(D_A;\mathcal{E})} \leq C(\omega)\, |t-s|^{\nu} \tag{6-3}$$

holds for every $t, s \in \mathbb{R}, \omega \in \Omega$ and some mapping $C\colon \Omega \to [0,\infty)$ which is uniformly bounded.

($\tilde{\mathbf{A}}$4) There exists a constant $\nu^* > 0$ such that the adjoint operators $\big(A^*(t,\omega)\big)_{t\in\mathbb{R},\,\omega\in\Omega}$ satisfy (6-3) with exponent ν^*.

($\tilde{\mathbf{A}}$5) The mapping $A\colon \mathbb{R} \times \Omega \to \mathcal{L}(D_A;\mathcal{E})$ is strongly measurable, adapted and for every $t \in \mathbb{R}, \omega \in \Omega$ the operator $A(t,\omega)$ has a compact inverse.

($\tilde{\mathbf{A}}$6) For all $t \in \mathbb{R}$ and $\omega \in \Omega$ the structural assumption $A(t,\omega) = A(\theta_t\omega)$ holds, with slight abuse of notation.

The Assumptions **($\tilde{\mathbf{A}}$1)**-**($\tilde{\mathbf{A}}$3)** are similar to the ones imposed in **(A)** but adapted to the random setting. The uniform boundedness of the random constant $C(\omega)$ in Assumption **($\tilde{\mathbf{A}}$3)** can be removed by a localization argument; we refer to [PV14, Section 5.3] for more details on this procedure. Assumption **($\tilde{\mathbf{A}}$4)** is used in Lemma 6.3 to provide refined smoothing inequalities. **($\tilde{\mathbf{A}}$5)**-**($\tilde{\mathbf{A}}$6)** are necessary because the generators depend on the random parameter ω. In particular, the structural dependence in **($\tilde{\mathbf{A}}$6)** is essential to show that the problem generates a random dynamical system. We say that $\big(A(t,\omega)\big)_{t\in\mathbb{R},\omega\in\Omega}$ satisfies Assumption **($\tilde{\mathbf{A}}$)** if it satisfies **($\tilde{\mathbf{A}}$1)**-**($\tilde{\mathbf{A}}$6)**.

Similar to the deterministic case in Theorem D.1, it is possible to show the existence of a parabolic evolution family, which is now random. Let $\mathcal{E}$ be an UMD space. For more details on UMD spaces see for example [HvNVW16, Chapter 4]. For our purposes it is enough to know that every Hilbert space is also an UMD space [HvNVW16, Proposition 4.2.14].

Theorem 6.1. ([PV14, Theorem 2.2, Proposition 2.4]) *Assume that $\big(A(t,\omega)\big)_{t\in\mathbb{R},\omega\in\Omega}$ satisfies Assumption **($\tilde{\mathbf{A}}$)**. Then there exists a unique parabolic evolution family $S\colon \Delta_{\mathbb{R}} \times \Omega \to \mathcal{L}(\mathcal{E};\mathcal{E})$ which is strongly measurable in the operator topology and satisfies the following properties:*

i) $S_{t,t}(\omega) = \mathrm{Id}_{\mathcal{E}}$ *for all* $t \geq 0, \omega \in \Omega$.

ii) For all $0 \leq r \leq s \leq t, \omega \in \Omega$ *we have*

$$S_{t,s}(\omega) S_{s,r}(\omega) = S_{t,r}(\omega). \tag{6-4}$$

iii) For all $s < t$ *the formula*

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{t,s}(\omega) = A(t,\omega) S_{t,s}(\omega)$$

holds pointwise in Ω.

iv) $S_{\cdot,\cdot}(\omega)$ *is strongly continuous for every* $\omega \in \Omega$, *and* $S_{t,s}(\cdot)$ *is strongly* $\mathcal{F}_t$*-measurable in the uniform operator topology for every* $t \geq s$.

For $\beta \geq 0$, we denote the fractional power spaces by $\mathcal{E}_\beta := D\big((-A(t,\omega))^\beta\big)$ endowed with the norm

$$\|x\|_\beta := \|x\|_{\mathcal{E}_\beta} := \left\| \big(-A(t,\omega)\big)^\beta x \right\|_{\mathcal{E}},$$

and by $\mathcal{E}_{-\beta}$ the completion of $\mathcal{E}$ with respect to the norm

$$\|x\|_{-\beta} := \|x\|_{\mathcal{E}_{-\beta}} := \left\| \big(-A(t,\omega)\big)^{-\beta} x \right\|_{\mathcal{E}}.$$

Since we assume that $A(t,\omega)$ is densely defined, all fractional powers $A(t,\omega)^{\beta}$ are also closed and densely defined; for more information, see the Appendices B and C.

Remark 6.2. In Assumption **($\tilde{\mathbf{A}}$1)**, we impose that the domain of $A(\theta_t\omega)$ is independent of t and ω. In general, this does not necessarily hold for the fractional power spaces. However, since we further assume that the operators $A(\theta_t\omega)$ have bounded imaginary powers, the fractional power spaces can be identified using complex interpolation. Therefore, we obtain $\mathcal{E}_\alpha = [\mathcal{E}, D_A]_\alpha = D\big((-A(\theta_t\omega))^\alpha\big)$ for any $\alpha \in (0,1)$ due to [Ama95, Theorem V.1.5.4]. In particular, the spaces $\mathcal{E}_\alpha$ do in fact not depend on t and ω. This is similar to Remark D.2.

Note that we can also transfer the results from Theorem D.3, Corollary D.4, and Remark D.6 to the random case. For this, we also assume an exponential stability of the parabolic evolution family.

($\tilde{\mathbf{A}}_{\mathbf{S}}$) The evolution family is exponentially stable. This means that there exists $\tilde{\lambda}_A > 0, C_S > 0$ such that

$$\|S_{t,s}(\omega)\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \leq C_S e^{-\tilde{\lambda}_A(t-s)}, \tag{6-5}$$

holds for all $t \geq s$ and $\omega \in \Omega$. We can fix a $\lambda_A < \tilde{\lambda}_A$ to obtain smoothing and exponential decay simultaneously as in Corollary D.4.

Lemma 6.3. ([PV14, Lemma 2.6, 2.7]) *Let the Assumptions **($\tilde{A}$)** and **($\tilde{A}_S$)** hold. Then for every $t > 0$ and $\omega \in \Omega$ the mapping $S_{t,\cdot}(\omega) \in \mathcal{C}^1\big([0,t);\mathcal{L}(\mathcal{E};\mathcal{E})\big)$ satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}s} S_{t,s}(\omega) = -S_{t,s}(\omega)A(s,\omega)x,$$

for all $x \in D_A$. Moreover, for $\alpha \in [0,1]$ and $\beta \in (0,1)$ we obtain

$$\left\| \big(-A(t,\omega)\big)^{\alpha} S_{t,s}(\omega)x \right\|_{\mathcal{E}} \leq \widetilde{C}_\alpha \frac{e^{-\lambda_A(t-s)}}{(t-s)^\alpha} \|x\|_{\mathcal{E}}, \qquad x \in \mathcal{E}, \tag{6-6}$$

$$\left\| S_{t,s}(\omega)\big(-A(s,\omega)\big)^{\alpha} x \right\|_{\mathcal{E}} \leq \widetilde{C}_\alpha \frac{e^{-\lambda_A(t-s)}}{(t-s)^\alpha} \|x\|_{\mathcal{E}}, \qquad x \in \mathcal{E}_\alpha,$$

$$\left\| \big(-A(t,\omega)\big)^{-\alpha} S_{t,s}(\omega)\big(-A(s,\omega)\big)^{\beta} x \right\|_{\mathcal{E}} \leq \widetilde{C}_{\alpha,\beta} \frac{e^{-\lambda_A(t-s)}}{(t-s)^{\beta-\alpha}} \|x\|_{\mathcal{E}}, \qquad x \in \mathcal{E}_\beta, \tag{6-7}$$

for $t > s$ and $\omega \in \Omega$. In particular, we have, similar to Remark D.6, an unique extension of

$$S_{t,s}(\omega)\big(-A(s,\omega)\big)^{\alpha}$$

to a bounded linear operator on $\mathcal{E}$ as well as an unique extension of $S_{t,s}(\omega)$ to a bounded linear operator on $\mathcal{E}_{-\alpha}$ for $\alpha \in [0,1]$.

Keeping this in mind, the mild formulation corresponding to the linear part of (6-1) is formally given $\mathbb{P}$-almost surely by

$$y_t = S_{t,0}(\omega)y_0(\omega) + \sigma \left(\int_0^t S_{t,s}(\omega)\, \mathrm{d}B_s \right)(\omega)$$

for every $t > 0$. As in (6-2), this is equal to

$$y_t = S_{t,0}(\omega)y_0 + \sigma S_{t,0}(\omega)B_t - \sigma \int_0^t S_{t,s}(\omega)A(s,\omega)\big(B_t - B_s\big)\,\mathrm{d}s, \tag{6-8}$$

using integration by parts. Note that the integral in (6-8) is well-defined due to the Hölder-regularity of the Brownian motion $(B_t)_{t\geq 0}$ which compensates for the singularity arising from the estimate $\|S_{t,s}(\omega)A(s,\omega)\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \lesssim (t-s)^{-1}$. To shorten notations, we set

$$h(t) := \sigma S_{t,0}(\omega)B_t - \sigma \int_0^t S_{t,s}(\omega)A(s,\omega)\big(B_t - B_s\big)\,\mathrm{d}s. \tag{6-9}$$

Definition 6.4. Let $\alpha \in [0,1)$. A local pathwise mild solution for (6-1) is an $\mathcal{E}_\alpha$-valued process y together with a stopping time τ such that $\mathbb{P}(\tau > 0) = 1$ and for every $t > 0$ the identity

$$\begin{aligned} y_t = S_{t,0}(\omega)y_0(\omega) + \sigma S_{t,0}(\omega)B_t + \int_0^t S_{t,s}(\omega)\big(F(y_s(\omega)) + f\big)\,\mathrm{d}s \\ - \sigma \int_0^t S_{t,s}(\omega)A(s,\omega)\big(B_t - B_s\big)\,\mathrm{d}s \end{aligned}$$

holds $\mathbb{P}$-almost surely on the event $\{t < \tau\}$. Moreover, if $\overline{\tau}$ is a stopping time such that $\overline{\tau} \leq \tau$ almost surely and the corresponding solution satisfies $\overline{y}(t) = y(t)$ almost surely on $\{t < \overline{\tau}\}$, then the pair $(\overline{y}, \overline{\tau})$ is also a local solution and (y,τ) extends $(\overline{y}, \overline{\tau})$. Finally, $(\overline{y}, \overline{\tau})$ is a maximal pathwise mild solution if on the set $\{\overline{\tau} < \infty\}$ one has $\lim\limits_{t\to\overline{\tau}} \|\overline{y}_t\|_\alpha = \infty$ almost surely and there exists a sequence (y_n, τ_n) of local mild solutions with increasing stopping times τ_n such that $\overline{\tau} = \sup\limits_n \tau_n$ almost surely and $(\overline{y}, \overline{\tau})$ extends each of the local mild solutions (y_n, τ_n). The pathwise mild solution is globally provided that $\tau = \infty$ almost surely.

The structural assumption $A(t,\omega) = A(\theta_t\omega)$ on the generators allows us to prove that the corresponding evolution family generates a random dynamical system.

Theorem 6.5. ([KNS21, Theorem 3.1]) *Let the Assumption **($\tilde{A}$)** hold and $S : \Delta_{\mathbb{R}} \times \Omega \to \mathcal{L}(\mathcal{E};\mathcal{E})$ be the evolution family generated by $\big(A(\theta_t\omega)\big)_{t\in\mathbb{R},\,\omega\in\Omega}$. Then*

$$\widetilde{S}\colon [0,\infty) \times \Omega \times \mathcal{E} \to \mathcal{E}, \quad (t,\omega,x) \mapsto S_{t,0}(\omega)x$$

is a random dynamical system.

Proof. We only give a brief argument to emphasize why the structural assumption on the random generator is required. First, observe that

$$S_{t+s,s}(\omega) = S_{t,0}(\theta_s\omega)$$

holds for $t,s \geq 0$ and $\omega \in \Omega$. Intuitively, this means that starting at time s on the ω-fiber, and letting the system evolve for time t, is the same as starting at time 0 on the shifted $\theta_s\omega$-fiber and letting t time units pass. On the level of generators, the evolution operator $S_{t+s,s}(\omega)$ is obtained from $A(\theta_t\omega)$ and the evolution operator $S_{t,0}(\theta_s\omega)$ from $A(\theta_{t-s} \circ \theta_s\omega)$. Due to the group property of the metric dynamical system θ, these generators are the same. Together with (6-4), this results in

$$\widetilde{S}_{t+s}(\omega) = \widetilde{S}_t(\theta_s\omega)\widetilde{S}_s(\omega),$$

for every $t,s \geq 0$ and $\omega \in \Omega$. Hence, $\widetilde{S}$ fulfills the cocycle property, and the measurability can be shown using the properties of Theorem 6.1. □

The local-in-time existence of a pathwise mild solution for (6-1) can be shown following the proof of [Hai23, Theorem 7.8], which relies on Banach's fixed-point theorem and subtracting the modified stochastic convolution h in (6-9). This argument is simpler than the setting in [PV14, Theorem 5.3] due to the additive structure of the noise. They can consider more general noise terms, but the nonlinearity is globally Lipschitz, which we do not assume in this case.

In order to distinguish the filtered probability space from the metric dynamical system, it is denoted by $\big(\overline{\Omega}, \overline{\Sigma}, \overline{\mathbb{P}}, (\mathcal{F}_t)_{t\geq 0}\big)$ for the following theorem.

Theorem 6.6. *Let $\alpha \in [0, 1/2)$, $f \in \mathcal{E}_\alpha$ and $\big(A(t,\omega)\big)_{t\in\mathbb{R},\omega\in\Omega}$ a family of operators fulfilling Assumption **($\widetilde{\mathbf{A}}$)**. Assume that $y_0(\omega) \in \mathcal{E}_\alpha$ is $\mathcal{F}_0$-measurable and that the modified stochastic convolution h in (6-9) has $\overline{\mathbb{P}}$-almost surely continuous sample paths in $\mathcal{E}_\alpha$. Furthermore, let $\beta \geq 0$ be such that $\alpha + \beta < 1$ and assume that the map*

$$F\colon \mathcal{E}_\alpha \to \mathcal{E}_{-\beta},$$

is Lipschitz continuous on bounded subsets of $\mathcal{E}_\alpha$ and grows at most polynomially, i.e. there are constants $R > 0, L = L_R > 0$, $C_F > 0$ and $n_\alpha \geq 1$, such that the

$$\begin{aligned} \|F(u) - F(v)\|_{-\beta} &\leq L\|u - v\|_\alpha, \quad && \|u\|_\alpha, \|v\|_\alpha \leq R,\\ \|F(u)\|_{-\beta} &\leq C_F\big(1 + \|u\|_\alpha^{n_\alpha}\big), && u \in \mathcal{E}_\alpha, \end{aligned}$$

holds. Then (6-1) has a unique maximal pathwise mild solution $y \in \mathcal{C}([0,\tau);\mathcal{E}_\alpha)$. Moreover, this solution exists globally provided that $\|y_t\|_\alpha \leq C$ for all $t \geq 0$.

Proof. We fix a sufficiently small terminal time $T > 0$ and apply Banach's fixed-point theorem to the mapping $\Phi\colon \mathcal{C}([0,T];\mathcal{E}_\alpha) \to \mathcal{C}([0,T];\mathcal{E}_\alpha)$ given by

$$\big(\Phi(u)\big)(t) := \int_0^t S_{t,s}(\omega)\big(F(u(s)) + f\big)\,\mathrm{d}s + g(t),$$

with $t \leq T$, where

$$\begin{aligned} g(t,\omega) &:= S_{t,0}(\omega)y_0(\omega) + \sigma S_{t,0}(\omega)B_t - \sigma\int_0^t S_{t,r}(\omega)A(r,\omega)\big(B_t - B_r\big)\,\mathrm{d}r\\ &= S_{t,0}(\omega)y_0(\omega) + h(t,\omega) \end{aligned}$$

has $\overline{\mathbb{P}}$-almost surely continuous paths in $\mathcal{E}_\alpha$ due to our assumptions. Therefore, $g(\cdot,\omega) \in \mathcal{C}([0,T];\mathcal{E}_\alpha)$ for all $\omega \in \overline{\Omega}$ up to a null set N_0. Let $c\colon \overline{\Omega} \times [0,T] \to [0,\infty)$ be defined by

$$c(t,\omega) := \begin{cases} \sup\limits_{s\in[0,t]} \|g(s,\omega)\|_\alpha, & \omega \in \overline{\Omega}\setminus N_0,\\ 0, & \omega \in N_0. \end{cases}$$

We observe that $c(\cdot,\omega)$ is continuous and increasing in t. Furthermore, we define $d\colon \overline{\Omega}\times[0,T] \to [0,\infty)$ by

$$d(t,\omega) := \max\left\{\widetilde{C}_{\alpha,\beta}L_{c(t,\omega)+1}, \widetilde{C}_{\alpha,\beta}C_F\big(1 + (1 + c(t,\omega))^{n_\alpha} + \|f\|_\alpha\big)\right\},$$

where $\|S_{t,s}(\omega)\|_{\mathcal{L}(\mathcal{E}_{-\beta};\mathcal{E}_\alpha)} \leq \widetilde{C}_{\alpha,\beta}(t-s)^{-(\alpha+\beta)}$ and the constant $\widetilde{C}_{\alpha,\beta}$ is uniformly bounded in ω by (6-7) and Assumption **($\widetilde{\mathbf{A}}$3)**. By definition, $d(t,\cdot)$ is $\mathcal{F}_t$-measurable for every $t \in [0,T]$. Therefore, $\tau\colon \overline{\Omega} \to [0,T]$ given by

$$\tau(\omega) = \inf\left\{t \in [0,T] : t^{1-(\alpha+\beta)}d(t,\omega) \geq 1/2\right\},$$

is a stopping time, where we set $\inf\emptyset := T$.

Let $\omega \in \overline{\Omega}\setminus N_0$ and $t \leq \tau(\omega)$ which implies that $t^{1-(\alpha+\beta)}d(t,\omega) < 1/2$. We further consider the set

$$\mathscr{B} := \left\{ u \in \mathcal{C}([0,t];\mathcal{E}_\alpha) : \sup_{s\in[0,t]} \|u(s)-g(s)\|_\alpha \leq 1 \right\}$$

and show that $\Phi : \mathcal{C}([0,t];\mathcal{E}_\alpha) \to \mathcal{C}([0,t];\mathcal{E}_\alpha)$ leaves $\mathscr{B}$ invariant and is a contraction. To verify the invariance, let $u \in \mathscr{B}$. Then we obtain $\sup\limits_{s\in[0,t]} \|u(s)\|_\alpha \leq 1 + c(t,\omega)$ as well as

$$\begin{aligned}
\sup_{s\in[0,t]} \|\big(\Phi(u)(s)\big) - g(s,\omega)\|_\alpha &\leq \sup_{s\in[0,t]} \int_0^s \|S_{s,r}(\omega)\|_{\mathcal{L}(\mathcal{E}_{-\beta};\mathcal{E}_\alpha)} \big\|F(u(r)) + f\big\|_{-\beta}\,\mathrm{d}r \\
&\leq C_F \widetilde{C}_{\alpha,\beta} \sup_{s\in[0,t]} \int_0^s (s-r)^{-(\alpha+\beta)} e^{-\lambda_A(s-r)} \big(1 + \|u(r)\|_\alpha^{n_\alpha} + \|f\|_\alpha\big)\,\mathrm{d}r \\
&\leq C_F \widetilde{C}_{\alpha,\beta} \sup_{s\in[0,t]} \int_0^s (s-r)^{-(\alpha+\beta)} \big(1 + \|u(r)\|_\alpha^{n_\alpha} + \|f\|_\alpha\big)\,\mathrm{d}r \\
&\leq C_F \widetilde{C}_{\alpha,\beta} t^{1-(\alpha+\beta)} \big(1 + (1+c(t,\omega))^{n_\alpha} + \|f\|_\alpha\big) < \frac{1}{2},
\end{aligned}$$

which shows the invariance of $\mathscr{B}$.

For $u, v \in \mathscr{B}$ we obtain

$$\begin{aligned}
\big\|\big(\Phi(u)\big)(t) - \big(\Phi(v)\big)(t)\big\|_\alpha &\leq \int_0^t \big\|S_{t,s}(\omega)\big(F(u(s)) - F(v(s)\big)\big\|_\alpha\,\mathrm{d}s \\
&\leq \int_0^t \|S_{t,s}(\omega)\|_{\mathcal{L}(\mathcal{E}_{-\beta};\mathcal{E}_\alpha)} \|F(u(s)) - F(v(s))\|_{-\beta}\,\mathrm{d}s \\
&\leq \widetilde{C}_{\alpha,\beta} \int_0^t (t-s)^{-(\alpha+\beta)} \|F(u(s)) - F(v(s))\|_{-\beta}\,\mathrm{d}s \\
&\leq \widetilde{C}_{\alpha,\beta} L_{1+c(t,\omega)} \int_0^t (t-s)^{-(\alpha+\beta)} \|u(s) - v(s)\|_\alpha\,\mathrm{d}s
\end{aligned}$$

by using that F is locally Lipschitz continuous. This implies that

$$\begin{aligned}
\sup_{s\in[0,t]} \big\|\big(\Phi(u)\big)(s) - \big(\Phi(v)\big)(s)\big\|_\alpha &\leq t^{1-(\alpha+\beta)} d(t,\omega) \sup_{s\in[0,t]} \|u(s) - v(s)\|_\alpha \\
&< \frac{1}{2} \sup_{s\in[0,t]} \|u(s) - v(s)\|_\alpha,
\end{aligned}$$

proving the contraction property. Therefore, Banach's fixed-point theorem [vN22, Theorem 2.13] yields the existence of a unique pathwise mild solution (y,τ) which belongs to $\mathcal{E}_\alpha$. Iterating this argument entails the existence of a maximal pathwise mild solution (y,τ), and obviously, if the solution does not explode in finite time, it exists globally. □

Remark 6.7. We apply the abstract setting of Theorem 6.6 to reaction-diffusion equations in Section 6.2 with $\mathcal{E} = \mathrm{L}^2(\mathcal{O})$, $\alpha = 1/2 - \varepsilon$ for $\varepsilon \in (0, 1/2)$ and $\beta = s/2$ for $s \in [0,1)$.

The global existence, i.e., $\tau = \infty$, holds almost surely and is shown under additional dissipativity assumptions on the nonlinearity F at the level of the random partial differential equations similar to [BLW09, Section 3] and [BCNS22]. Subsequently, we prove that the reaction-diffusion equation generates a random dynamical system and investigate its asymptotic behavior in Section 6.2.

6.1.2 Temperedness of an Ornstein–Uhlenbeck-type process

In this subsection, we investigate the properties of a stationary Ornstein–Uhlenbeck process. This process is defined as the solution of the following linear problem with random non-autonomous generators

$$\begin{cases} \mathrm{d}Z(t,\omega) = A(\theta_t\omega)Z(t,\omega)\ \mathrm{d}t + \mathrm{d}\omega_t, \\ Z(\omega) := Z(0,\omega) = \int_{-\infty}^{0} S_{0,r}(\omega)A(\theta_r\omega)\omega_r\ \mathrm{d}r, \end{cases} \tag{6-10}$$

where $(S_{t,s}(\omega))_{s\leq t,\omega\in\Omega}$ is the evolution family corresponding to the operators $\big(A(t,\omega)\big)_{t\in\mathbb{R},\,\omega\in\Omega}$. For the initial condition for Z we would expect $Z(\omega) = \int_{-\infty}^{0} S_{0,r}(\omega)\ \mathrm{d}\omega_r$. However, as in the previous subsection, the integral is not well-defined due to the ω-dependence of S. Hence, similar to the definition of pathwise mild solutions, we formally apply integration by parts and obtain

$$\int_{-\infty}^{0} S_{0,r}(\omega)\ \mathrm{d}\omega_r = \lim_{a\to\infty}\Big(- S_{0,a}(\omega)\omega_{-a} + \int_{-a}^{0} S_{0,r}(\omega)A(\theta_r\omega)\omega_r\ \mathrm{d}r\Big), \tag{6-11}$$

where the limit is taken in $\mathcal{E}$. The first term converges to 0 due to the exponential decay of the evolution family and the subexponential growth of ω, which means

$$\|S_{0,-a}(\omega)\omega_{-a}\|_{\mathcal{E}} \leq C_S e^{-\lambda_A a}\,\|\omega_{-a}\|_{\mathcal{E}} \leq C_S c_\varepsilon e^{-a(\lambda_A-\varepsilon)} \to 0$$

as $a\to\infty$ for ε small enough, see Remark 4.8. The second term in (6-11) is well-defined due to the sublinear growth (4-6) of ω which for $-r\geq T_0(\varepsilon,\omega)$ implies that

$$\|S_{0,r}(\omega)A(\theta_r\omega)\omega_r\|_{\mathcal{E}} \leq \widetilde{C}_1\frac{e^{\lambda_A r}}{-r}\,\|\omega_r\|_{\mathcal{E}} \leq \varepsilon e^{\lambda_A r}\widetilde{C}_1.$$

Hence, the right-hand side of (6-11) is well-defined and converges to the $Z(0,\omega)$ given in (6-10).

The pathwise mild solution of (6-10) is then given by

$$Z(t,\omega) = S_{t,0}(\omega)Z(\omega) + S_{t,0}(\omega)\omega_t - \int_0^t S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\ \mathrm{d}r \tag{6-12}$$

for $t\geq 0$. Further, we obtain

$$\begin{aligned} Z(t,\omega) &= \omega_t + \int_{-\infty}^{t} S_{t,r}(\omega)A(\theta_r\omega)\omega_r\ \mathrm{d}r = \omega_t + \int_{-\infty}^{0} S_{t,r+t}(\omega)A(\theta_{r+t}\omega)\omega_{r+t}\ \mathrm{d}r \\ &= \omega_t + \int_{-\infty}^{0} S_{0,r}(\theta_t\omega)A(\theta_r\circ\theta_t\omega)(\theta_t\omega_r+\omega_t)\ \mathrm{d}r \\ &= \int_{-\infty}^{0} S_{0,r}(\theta_t\omega)A(\theta_r\circ\theta_t\omega)\theta_t\omega_r\ \mathrm{d}r, \end{aligned}$$

which is well-defined for every $t\in\mathbb{R}$. Therefore, we define the Ornstein–Uhlenbeck process as $(t,\omega)\mapsto Z(\theta_t\omega) := Z(t,\omega)$.

Lemma 6.8. *The Ornstein–Uhlenbeck process Z is stationary and tempered in $\mathcal{E}$.*

Proof. For every $t,s\in\mathbb{R}$ and $\omega\in\Omega$ we have

$$Z(t+s,\omega) = \omega_{t+s} + \int_{-\infty}^{t+s} S_{t+s,r}(\omega)A(\theta_r\omega)\omega_r\ \mathrm{d}r$$

$$= \theta_s\omega_t + \omega_s + \int_{-\infty}^{t} S_{t+s,r+s}(\omega)A(\theta_{r+s}\omega)\omega_{r+s}\,\mathrm{d}r$$
$$= \theta_s\omega_t + \int_{-\infty}^{t} S_{t,r}(\theta_s\omega)A(\theta_r\circ\theta_s\omega)\theta_s\omega_r\,\mathrm{d}r = Z(t,\theta_s\omega).$$

For $k\in\mathbb{N}$ and $t,t_1,\dots,t_k\in\mathbb{R}$ set $Z_{t_1,\dots,t_k} := \big(Z(t_1,\cdot),\dots,Z(t_k,\cdot)\big)$. Then the $(\theta_t)_{t\in\mathbb{R}}$-invariance of $\mathbb{P}$ implies that

$$\mathbb{P}\big(Z_{t_1+t,\dots,t_k+t}\in O\big) = \mathbb{P}\big(Z_{t_1,\dots,t_k}(\theta_t\cdot)\in O\big) = \mathbb{P}\big(Z_{t_1,\dots,t_k}\in O\big),$$

holds for any $O\in\mathcal{B}(\mathcal{E}^k)$, which shows that Z is stationary.

To verify the temperedness, we use the sufficient condition (4-10). The representation of the Ornstein–Uhlenbeck process in (6-12), together with (6-5) leads to

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,1]}\|Z(\theta_t\omega)\|_{\mathcal{E}}\right] &\le C_S\mathbb{E}\big[\,\|Z(\omega)\|_{\mathcal{E}}\,\big] + C_S\mathbb{E}\left[\sup_{t\in[0,1]}\|\omega_t\|_{\mathcal{E}}\right]\\ &\quad + \mathbb{E}\left[\sup_{t\in[0,1]}\left\|\int_0^t S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\,\mathrm{d}r\right\|_{\mathcal{E}}\right].\end{aligned} \tag{6-13}$$

To estimate the first term, we use the sublinear growth of the noise. In particular, for $\omega\in\Omega,\varepsilon<\lambda_A, a>0$ and $r\le -T_0(\omega)$, we obtain

$$\|S_{0,-a}(\omega)\omega_{-a}\|_{\mathcal{E}} \le C_S e^{-\lambda_A a}\,\|\omega_{-a}\|_{\mathcal{E}} \le C_S c_\varepsilon e^{-a(\lambda_A-\varepsilon)},$$
$$\|S_{0,r}(\omega)A(\theta_r\omega)\omega_r\|_{\mathcal{E}} \le \widetilde{C}_1\frac{e^{\lambda_A r}}{-r}\,\|\omega_r\|_{\mathcal{E}} \le e^{\lambda_A r}\varepsilon\widetilde{C}_1.$$

These estimates allow us to bound the first term in (6-13)

$$\begin{aligned}\mathbb{E}\big[\,\|Z(\omega)\|_{\mathcal{E}}\,\big] &\le \lim_{a\to\infty}\left(\mathbb{E}\big[\,\|-S_{0,a}(\omega)\omega_{-a}\|_{\mathcal{E}}\,\big] + \mathbb{E}\left[\left\|\int_{-a}^0 S_{0,r}(\omega)A(\theta_r\omega)\omega_r\,\mathrm{d}r\right\|_{\mathcal{E}}\right]\right)\\ &\le \lim_{a\to\infty}\left(C_S c_\varepsilon e^{-a(\lambda_A-\varepsilon)} + \int_{-a}^0 e^{\lambda_A r}\varepsilon\widetilde{C}_1\,\mathrm{d}r\right)\\ &= \lim_{a\to\infty}\left(C_S c_\varepsilon e^{-a(\lambda_A-\varepsilon)} + \frac{1-e^{-\lambda_A a}}{\lambda_A}\widetilde{C}_1\varepsilon\right) = \frac{1}{\lambda_A}\widetilde{C}_1\varepsilon < \infty.\end{aligned}$$

The second term can be estimated by Doob's maximal inequality [RY99, Theorem II.1.7], since $\big(\,\|\omega_t\|_{\mathcal{E}}\,\big)_{t\in\mathbb{R}}$ is a positive submartingale. In fact, using that $(\omega_t)_{t\in\mathbb{R}}$ is an $\mathcal{E}$-valued martingale implies that

$$\mathbb{E}\left[\sup_{t\in[0,1]}\|\omega_t\|_{\mathcal{E}}\right] \lesssim \mathbb{E}\left[\left(\sup_{t\in[0,1]}\|\omega_t\|_{\mathcal{E}}\right)^2\right] \lesssim \mathbb{E}\left[\|\omega_1\|_{\mathcal{E}}^2\right],$$

due to the embedding $\mathrm{L}^2(\Omega)\hookrightarrow\mathrm{L}^1(\Omega)$. Hence, this term is finite due to (4-5). The last term in (6-13) can be estimated using the Hölder continuity of the noise, see Lemma 4.7 ii), which yields

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,1]}\left\|\int_0^t S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\,\mathrm{d}r\right\|_{\mathcal{E}}\right] \le \widetilde{C}_1\mathbb{E}\left[\sup_{t\in[0,1]}\int_0^t \frac{e^{-\lambda_A(t-r)}}{t-r}\,\|\omega_t-\omega_r\|_{\mathcal{E}}\,\mathrm{d}r\right]\\ &\le \widetilde{C}_1\mathbb{E}\left[\|\omega\|_{\gamma,\mathcal{E},[0,1]}\sup_{t\in[0,1]}\int_0^t e^{-\lambda_A(t-r)}(t-r)^{\gamma-1}\,\mathrm{d}r\right]\end{aligned}$$

$$\leq C\widetilde{C}_1 \mathbb{E}\big[\, \|\omega\|_{\gamma,\mathcal{E},[0,1]} \,\big] \leq C\widetilde{C}_1 \mathbb{E}[c_1(\omega,\gamma,0,1)] < \infty,$$

where the integral is finite due to $\gamma - 1 > -1$. □

This result can be generalized to fractional power spaces, which is needed in the following section to show the global solvability of (6-1) and the existence of an attractor.

Proposition 6.9. *For all $\beta \in [0, 1/2)$, the stationary Ornstein–Uhlenbeck process*

$$Z\colon \mathbb{R}\times\Omega \to \mathcal{E}_\beta, (t,\omega) \mapsto Z(\theta_t\omega)$$

is tempered.

Proof. In comparison to the proof of Lemma 6.8, we need to change the interval in the sufficient condition (4-10) to take into account the singularity of $e^{-\lambda_A t}t^{-\beta}$ in 0. Consequently, we obtain

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[1,2]} \|Z(\theta_t\omega)\|_\beta\right] \leq \mathbb{E}\left[\sup_{t\in[1,2]} \|S_{t,0}(\omega)Z(\omega)\|_\beta\right] + \mathbb{E}\left[\sup_{t\in[1,2]} \|S_{t,0}(\omega)\omega_t\|_\beta\right] \\ + \mathbb{E}\left[\sup_{t\in[1,2]} \left\|\int_0^t S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\,\mathrm{d}r\right\|_\beta\right].\end{aligned} \tag{6-14}$$

Since $\mathcal{E}_\beta$ is endowed with the norm $\left\|(-A(\theta_t\omega))^\beta\cdot\right\|_\mathcal{E}$, we can estimate the terms similarly to the proof of Lemma 6.8. In fact, (6-6) implies that

$$\left\|(-A(\theta_t\omega))^\beta S_{t,0}(\omega)Z(\omega)\right\|_\mathcal{E} \leq \widetilde{C}_\beta \frac{e^{-\lambda_A t}}{t^\beta}\|Z(\omega)\|_\mathcal{E},$$

and since $\sup_{t\in[1,2]} e^{-\lambda_A t}t^{-\beta} = e^{-\lambda_A}$, the first term in (6-14) is bounded by

$$\mathbb{E}\left[\sup_{t\in[1,2]} \|S_{t,0}(\omega)Z(\omega)\|_\beta\right] \leq \widetilde{C}_\beta e^{-\lambda_A}\mathbb{E}\big[\,\|Z(\omega)\|_\mathcal{E}\,\big] < \infty,$$

where $\mathbb{E}\big[\,\|Z(\omega)\|_\mathcal{E}\,\big] < \infty$ as in Lemma 6.8. For the second term in (6-14) we use again Doob's maximal inequality [RY99, Theorem II.1.7] and the γ-radonifying property (4-5) to conclude that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[1,2]} \|S_{t,0}(\omega)\omega_t\|_\beta\right] &\leq \widetilde{C}_\beta e^{-\lambda_A}\mathbb{E}\left[\sup_{t\in[1,2]} \|\omega_t\|_\mathcal{E}\right] \lesssim \widetilde{C}_\beta e^{-\lambda_A}\mathbb{E}\left[\left(\sup_{t\in[0,2]} \|\omega_t\|_\mathcal{E}\right)^2\right] \\ &\lesssim \widetilde{C}_\beta e^{-\lambda_A}\mathbb{E}\left[\|\omega_2\|_\mathcal{E}^2\right] < \infty.\end{aligned}$$

To estimate the last term in (6-14), we recall that we can extend the inequalities (6-6)-(6-7) to all $x \in \mathcal{E}$, see (D-6). Therefore, we obtain

$$\begin{aligned}\|S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\|_\beta &\leq \left\|S_{t,\frac{t+r}{2}}(\omega)\right\|_{\mathcal{L}(\mathcal{E};\mathcal{E}_\beta)} \left\|S_{\frac{t+r}{2},r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\right\|_\mathcal{E} \\ &\leq \widetilde{C}_\beta\widetilde{C}_1 e^{-\lambda_A\frac{t-r}{2}}\left(\frac{t-r}{2}\right)^{-\beta} e^{-\lambda_A\frac{t-r}{2}}\left(\frac{t-r}{2}\right)^{-1}\|\omega_t-\omega_r\|_\mathcal{E} \\ &\leq 2^{1+\beta}\|\omega\|_{\gamma,\mathcal{E},[0,t]}\widetilde{C}_\beta\widetilde{C}_1\frac{e^{-\lambda_A(t-r)}}{(t-r)^{1+\beta-\gamma}}\end{aligned}$$

for $0 \le r \le t$. The resulting integral is finite if and only if $\gamma - 1 - \beta > -1$, which means that for $\beta < \gamma < 1/2$ we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[1,2]} \left\| \int_0^t S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\,\mathrm{d}r \right\|_\beta\right] \\
&\qquad \le 2^{1+\beta}\widetilde{C}_\beta\widetilde{C}_1\mathbb{E}\left[\sup_{t\in[1,2]}\|\omega\|_{\gamma,\mathcal{E},[0,t]}\int_0^t \frac{e^{-\lambda_A(t-r)}}{(t-r)^{1+\beta-\gamma}}\,\mathrm{d}r\right] \\
&\qquad \le 2^{1+\beta}\widetilde{C}_\beta\widetilde{C}_1\mathbb{E}\left[\|\omega\|_{\gamma,\mathcal{E},[0,2]}\sup_{t\in[0,1]}\int_0^t \frac{e^{-\lambda_A(t-r)}}{(t-r)^{1+\beta-\gamma}}\,\mathrm{d}r\right] \\
&\qquad \lesssim 2^{1+\beta}\widetilde{C}_\beta\widetilde{C}_1\mathbb{E}\big[\,\|\omega\|_{\gamma,\mathcal{E},[0,2]}\,\big] \lesssim 2^{1+\beta}\widetilde{C}_\beta\widetilde{C}_1\mathbb{E}[c_1(\omega,\gamma,0,2)] < \infty,
\end{aligned}$$

where we used again Lemma 4.7 ii). □

Combining Proposition 6.9 with Sobolev embeddings from Proposition A.10, we finally obtain the following properties of the stationary Ornstein–Uhlenbeck process Z.

Corollary 6.10. *Assume* ***(E1)-(E2)****. The process $Z\colon \mathbb{R}\times\Omega \to E_\beta$ is tempered for any $\beta \in [0, 1/2)$. Consequently, Z is tempered in $\mathrm{L}^q(\mathcal{O})$ where $q=\infty$ if $n=1$, $1\le q<\infty$ arbitrary if $n=2$, and $1 \le q < \frac{2n}{n-2}$ arbitrary if $n \ge 3$.*

The temperedness of Z can be generalized even further if the underlying Brownian motion is more regular in space. To see this, assume that the noise is an $\mathcal{E}_\alpha$-valued Brownian motion using now a γ-radonifying operator $\mathcal{G}\colon \mathscr{H} \to \mathcal{E}_\alpha$ with values in $\mathcal{E}_\alpha$ instead of $\mathcal{E}$. Let $\Omega_\alpha \subset \mathcal{C}_0(\mathbb{R};\mathcal{E}_\alpha)$ be the set such that (4-6) and (4-7) are satisfied for an $\mathcal{E}_\alpha$-valued Brownian motion endowed with the respective σ-algebra $\Sigma_\alpha \coloneqq \mathcal{B}\big(C_0(\mathbb{R};\mathcal{E}_\alpha)\big)\cap\Omega_\alpha$ and let $\mathbb{P}_\alpha$ be the restriction of $\mathbb{P}_W$ to $\mathcal{F}_\alpha$.

Lemma 6.11. *For all $\alpha\in[0,1]$ and $\beta\in[\alpha,\alpha+\gamma)$, the Ornstein–Uhlenbeck process*

$$Z : \mathbb{R}\times\Omega_\alpha \to \mathcal{E}_\beta$$

is tempered.

Proof. We observe that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[1,2]}\|Z(\theta_t\omega)\|_\beta\right] &\le \mathbb{E}\left[\sup_{t\in[1,2]}\|S_{t,0}(\omega)Z(\omega)\|_\beta\right] + \mathbb{E}\left[\sup_{t\in[1,2]}\|S_{t,0}(\omega)\omega_t\|_\beta\right] \\
&\quad + \mathbb{E}\left[\sup_{t\in[1,2]}\left\|\int_0^t S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\,\mathrm{d}r\right\|_\beta\right].
\end{aligned} \tag{6-15}$$

The first two terms can be estimated exactly as in the proof of Corollary 6.9 using that $\mathcal{E}_\alpha \hookrightarrow \mathcal{E}$. Therefore, we only need to estimate the last term. We have

$$\begin{aligned}
&\|S_{t,r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\|_\beta \le \left\|S_{t,\frac{t+r}{2}}(\omega)\right\|_{\mathcal{L}(\mathcal{E};\mathcal{E}_\beta)}\left\|S_{\frac{t+r}{2},r}(\omega)A(\theta_r\omega)(\omega_t-\omega_r)\right\|_{\mathcal{E}} \\
&\le \widetilde{C}_\beta e^{-\lambda_A\frac{t-r}{2}}\left(\frac{t-r}{2}\right)^{-\beta}\left\|S_{\frac{t+r}{2},r}(\omega)\big(-A(\theta_r\omega)\big)^{1-\alpha}(-A(\theta_r\omega))^{\alpha}(\omega_t-\omega_r)\right\|_{\mathcal{E}} \\
&\le \widetilde{C}_\beta\widetilde{C}_{1-\alpha}e^{-\lambda_A\frac{t-r}{2}}\left(\frac{t-r}{2}\right)^{-\beta}e^{-\lambda_A\frac{t-r}{2}}\left(\frac{t-r}{2}\right)^{-1+\alpha}\|\omega_t-\omega_r\|_\alpha \\
&\le 2^{1+\beta}\,\|\omega\|_{C^\gamma([0,t];\mathcal{E}_\alpha)}\,\widetilde{C}_\beta\widetilde{C}_{1-\alpha}\frac{e^{-\lambda_A(t-r)}}{(t-r)^{1+\beta-\alpha-\gamma}},
\end{aligned}$$

for $t > r$, where we used that the norms in the fractional power spaces are equivalent for $t \in [1, 2]$, see Remark 6.2. Therefore, all terms on the right-hand side in (6-15) are finite, i.e., $Z\colon \mathbb{R} \times \Omega_\alpha \to \mathcal{E}_\beta$ is tempered. □

6.2 Random attractors for reaction-diffusion equations

In this subsection, we analyze a concrete reaction-diffusion equation. Before addressing the global well-posedness and the existence of an attractor, it is necessary to establish first the framework in which we operate.

Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded domain with $\mathcal{C}^\infty$-boundary $\partial\mathcal{O}$, $E := \mathrm{L}^2(\mathcal{O})$ with the inner product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_E$ and we consider the equation

$$\begin{cases} \mathrm{d}y_t(\omega) = \big(\mathcal{A}(t,\omega)y_t(\omega) + F(y_t(\omega)) + f\big)\mathrm{d}t + \sigma \; \mathrm{d}B_t(\omega), & \text{in } \mathcal{O}, \\ y_t(\omega) = 0, & \text{on } \partial\mathcal{O}, \end{cases} \tag{6-16}$$

with initial condition y_0 for $\omega \in \Omega, t > 0$, where $(B_t)_{t\in\mathbb{R}}$ is the E-valued Brownian motion introduced in Example 4.5 and $\mathcal{A}$ is a second-order differential operator in divergence form given by

$$\mathcal{A}(t,\omega,x) := \sum_{i,j=1}^{n} \partial_i a_{ij}(t,\omega,x)\partial_j := \nabla \cdot \big(a(t,\omega,x)\nabla\big),$$

with diffusion matrix $a\colon [0,\infty) \times \Omega \times \mathcal{O} \to \mathbb{R}^{n\times n}$. This random diffusion matrix a satisfies the following ellipticity and structural assumptions.

(E1) The random matrix $a\colon [0,\infty) \times \Omega \times \mathcal{O} \to \mathbb{R}^{n\times n}$ is symmetric, bounded, Hölder continuous in t with exponent $\nu \in (0, 1]$ and continuously differentiable in x, uniformly with respect to the remaining variables. Further, we assume uniformly ellipticity, which means there is a $c_a > 0$ such that

$$\xi^\top a(t,\omega,x)\xi \geq c_a|\xi|^2$$

holds for every $(t,\omega,x) \in [0,\infty) \times \Omega \times \mathcal{O}$ and $\xi \in \mathbb{R}^n$.

(E2) We have $a(t,\omega,\cdot) = a(\theta_t\omega,\cdot)$ for all $(t,\omega) \in \mathbb{R} \times \Omega$, with a slight abuse of notation.

The corresponding fractional power spaces are again defined through interpolation of E and

$$E_1 = \big\{u \in \mathrm{H}^{2,2}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\big\},$$

and denoted by $(E_\beta)_{\beta\in[-1,1]}$. Note that in this case we can identify $E_{1/2} = \mathrm{H}^1_0(\mathcal{O})$ and its dual space as $E_{-1/2} = \mathrm{H}^{-1}(\mathcal{O})$, see Proposition A.11. We now define the operator $A(t,\omega)\colon E_{1/2} \to E_{-1/2}$ by

$$\langle A(t,\omega)u, v\rangle = \int_{\mathcal{O}} \nabla v \cdot a(t,\omega,x)\nabla u \; \mathrm{d}x$$

for every $u, v \in E_{1/2}$. In Subsection 6.2.3, we will prove that the operator family satisfies **($\tilde{\mathbf{A}}$)**.

The growth and dissipativity assumptions on the nonlinearity F, depend on the dimension n of the domain $\mathcal{O}$.

($\tilde{\mathbf{F}}$1) The function $F\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous. Moreover, we assume that the dimension is either $n = 1$ or $n \geq 2$ and there exists a constant $C_F > 0$ such that

$$|F(u) - F(v)| \leq C_F|u - v| \left(|u|^{\rho-1} + |v|^{\rho-1}\right)$$

holds for all $u, v \in \mathbb{R}$, where the exponent ρ satisfies

$$\rho < \rho_{\text{critical}} := \begin{cases} \infty & \text{if } n = 2, \\ \dfrac{n}{n-2} & \text{if } n \geq 3. \end{cases} \tag{6-17}$$

($\tilde{\mathbf{F}}$2) There are positive constants C_0, C_1 such that

$$F(u)u \leq -C_0|u|^{1+\rho} + C_1 \quad \forall u \in \mathbb{R}$$

where $\rho < \infty$ is arbitrary if $n = 1$ and ρ satisfies (6-17) if $n \geq 2$.

The local Lipschitz continuity and growth assumption in **($\tilde{\mathbf{F}}$1)** imply the local existence of pathwise mild solutions, while the dissipativity assumption **($\tilde{\mathbf{F}}$2)** ensures global existence. Note that for $n \geq 2$, the assumption **($\tilde{\mathbf{F}}$1)** implies that

$$|F(u)| \leq C_F\big(1 + |u|^\rho\big)$$

for all $u \in \mathbb{R}$, where we use under slight abuse of notation the same constant as in **($\tilde{\mathbf{F}}$2)**.

Remark 6.12. Due to the noise, the global well-posedness of (6-1) and the existence of the random attractor are shown under the growth condition (6-17), which is more restrictive than for deterministic autonomous problems where the subcritical growth assumption is $\rho < \frac{2n}{n-2}$. This assumption only suffices for the local existence of solutions of (6-16). Moreover, we remark that in [LCK17] even for finite-dimensional noise, the more restrictive condition $\rho \leq 1 + 4/n$ was imposed.

Lemma 6.13. *Let the Assumption* **($\tilde{\mathbf{F}}$1)** *be satisfied. Then, for any* $s \in [0,1)$ *and* $\varepsilon \in (0, 1/2)$ *the Nemytskii operator* $\widetilde{F}$ *given by*

$$\widetilde{F}(u) := F\big(u(\cdot)\big)$$

is Lipschitz continuous from $E_{1/2-\varepsilon}$ *to* $E_{-s/2}$ *on bounded subsets of* $E_{1/2-\varepsilon}$.

Proof. The lemma follows from the proof of [CLR13, Theorem 12.1]; for the sake of completeness, we sketch it here.

$n = 1:$ Due to the Sobolev embeddings in Proposition A.10, we obtain

$$E_{1/2-\varepsilon} \hookrightarrow \mathcal{C}\big(\overline{\mathcal{O}}\big).$$

Since $\mathcal{C}\big(\overline{\mathcal{O}}\big) \hookrightarrow \mathrm{L}^2(\mathcal{O})$ is also true, we obtain $\widetilde{F}\colon E_{1/2-\varepsilon} \to E_{-s/2}$ for every $s \in [0,1)$. Due to the local Lipschitz continuity of F, this also holds for $\widetilde{F}$.

$n \geq 2:$ Due to **($\tilde{\mathbf{F}}$1)**, we obtain

$$\Big\|\widetilde{F}(u)\Big\|^2_{\mathrm{L}^2(\mathcal{O})} \lesssim 1 + \|u\|^{2\rho}_{\mathrm{L}^{2\rho}(\mathcal{O})}. \tag{6-18}$$

This means the nonlinearity satisfies $\widetilde{F} : \mathrm{L}^{2\rho}(\mathcal{O}) \to \mathrm{L}^2(\mathcal{O}) \hookrightarrow E_{-s/2}$. Now Proposition A.10 leads to

$$E_{1/2-\varepsilon} \hookrightarrow \mathrm{L}^q(\mathcal{O})$$

for every $q < \frac{2n}{n-2}$, and we end up with $\widetilde{F}\colon E_{1/2-\varepsilon} \to \mathrm{L}^2(\mathcal{O})$, which proves the claim due to $\mathrm{L}^2(\mathcal{O}) \hookrightarrow E_{-s/2}$. □

We set $\widetilde{B}_t(\omega) := B_t(\cdot,\omega)$, where $B_t(\cdot,\omega)$ is the two-sided Brownian motion defined in Example 4.5. With a slight abuse of notation, we use F and B instead of $\widetilde{F}$ and $\widetilde{B}$. The reaction-diffusion Equation (6-16) can then be rewritten in the abstract form

$$\mathrm{d}y_t(\omega) = \big(A(\theta_t\omega)y_t(\omega) + F(y_t(\omega)) + f\big)\ \mathrm{d}t + \sigma\ \mathrm{d}B_t(\omega), \tag{6-19}$$

which, according to Theorem 6.5, generates a random dynamical system ϕ.

To establish a global solution, we must also consider the transformed equation. Therefore recall that the stationary Ornstein–Uhlenbeck process $(t,\omega)\mapsto Z(t,\omega) = Z(\theta_t\omega)$, constructed in Subsection 6.1.2, is the stationary solution of the linear equation

$$\begin{cases} \mathrm{d}Z(\theta_t\omega) = A(\theta_t\omega)Z(\theta_t\omega)\ \mathrm{d}t + \mathrm{d}B_t(\omega), \\ Z(\omega) := Z(0,\omega) = \int_{-\infty}^0 S_{0,r}(\omega)A(\theta_r\omega)\omega_r\ \mathrm{d}r. \end{cases}$$

Then define $v_t(\omega) = y_t(\omega) - \sigma Z(\theta_t\omega)$ as the solution of

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}v_t(\omega) = A(\theta_t\omega)v_t(\omega) + F(v_t + \sigma Z(\theta_t\omega)) + f, \\ v_0(\omega) := y_0(\omega) + \sigma Z(\omega) = y_0(\omega) + \sigma\int_{-\infty}^0 S_{0,r}(\omega)A(\theta_r\omega)\omega_r\ \mathrm{d}r, \end{cases} \tag{6-20}$$

which means the solution to the original problem can be expressed by (6-19) $y_t(\omega) = v_t(\omega) + \sigma Z(\theta_t\omega)$. Due to the regularity of Z, we expect the solution of the transformed equation v to have higher regularity. The bootstrapping method introduced in [BCNS22] then yields the global well-posedness.

6.2.1 Global existence

Due to the time-dependent random differential operators and the only locally Lipschitz continuous nonlinearity, we cannot directly obtain a-priori estimates for the solution in more regular function spaces, more specifically, in the fractional power spaces corresponding to the random non-autonomous generators. Such estimates are essential for the global well-posedness of (6-19) and the compactness argument, which is needed for the existence of the random attractor. In order to overcome this issue, we derive a-priori bounds in suitable $\mathrm{L}^{\rho+1}$-spaces where ρ is determined by the subcritical growth of the nonlinearity F. These bounds, combined with a bootstrapping argument based on the regularizing property of parabolic evolution families, will entail a-priori estimates of the solution in more regular spaces, similar to [BCNS22] for deterministic reaction-diffusion equations with non-autonomous generators.

For the rest of this section, we consider the phase space E_α where α is chosen such that

$$\frac{n(\rho-1)}{4(\rho+1)} \le \alpha < \min\left\{\frac{n}{4},\frac{1}{2}\right\} \quad \text{and} \quad E_\alpha \hookrightarrow \mathrm{L}^{2\rho}(\mathcal{O}). \tag{6-21}$$

Due to the conditions on ρ specified in **($\tilde{\mathbf{F}}$1)** and **($\tilde{\mathbf{F}}$2)**, such a constant α always exists. Throughout this section, C denotes a universal constant that varies from line to line.

Lemma 6.14. *Assume that $\big(A(\theta_t\omega)\big)_{t\in\mathbb{R},\omega\in\Omega}$ and F fulfill the Assumptions **(E1)-(E2)** and **($\tilde{F}$1)**, and fix α satisfying (6-21) and let $f\in E_\alpha$. Then, there exists for any $\omega\in\Omega$ and initial datum $v_0(\omega)\in E_\alpha$ a local mild solution to (6-20) in E_α on the maximal interval of existence $\big(0,T_{\max}(\omega)\big)$ in the following sense: For all $t\in\big(0,T_{\max}(\omega)\big)$ the solution $v\in\mathcal{C}([0,t];E_\alpha)$ satisfies the variation of constants formula*

$$v_t(\omega) = S_{t,0}(\omega)v_0(\omega) + \int_0^t S_{t,s}(\omega)\big(F(v_s(\omega)+\sigma Z(\theta_s\omega)) + f\big)\ \mathrm{d}s.$$

Moreover, for each $t \in \big(0, T_{\max}(\omega)\big)$, we have the regularity $v_t(\omega) \in E_\eta$ for any $\eta \in [\alpha, 1)$.

Proof. The local existence of v is standard and follows by a fixed-point argument in $\mathcal{C}([0,t]; E_\alpha)$ for $t \in [0, T_{\max})$.

To show the regularity of v, we use the smoothing effect of the parabolic evolution family S in Lemma 6.3 and the growth assumption on F, which leads in particular to (6-18). This means, for $\eta \in [\alpha, 1)$ and any $t \in (0, T_{\max})$ we obtain

$$\begin{aligned}
\|v_t(\omega)\|_\eta &\leq \|S_{t,0}(\omega)v_0(\omega)\|_\eta + \int_0^t \Big\| S_{t,s}(\omega)\Big(F\big(v_s(\omega) + \sigma Z(\theta_s\omega)\big) + f\Big)\Big\|_\eta \, \mathrm{d}s \\
&\leq \widetilde{C}_{\eta,\alpha} e^{-\lambda_A t} t^{\alpha-\eta} \|v_0(\omega)\|_\alpha + C\|f\|_\alpha \int_0^t (t-s)^{-\eta+\alpha} \, \mathrm{d}s \\
&\quad + C \int_0^t (t-s)^{-\eta} \big\| F\big(v_s(\omega) + \sigma Z(\theta_s\omega)\big)\big\|_E \, \mathrm{d}s \\
&\leq \widetilde{C}_{\eta,\alpha} e^{-\lambda_A t} t^{\alpha-\eta} \|v_0(\omega)\|_\alpha + C\|f\|_\alpha \\
&\quad + CC_F \int_0^t (t-s)^{-\eta} \Big(1 + \|v_s(\omega)\|^\rho_{\mathrm{L}^{2\rho}(\mathcal{O})} + \sigma^\rho \|Z(\theta_s\omega)\|^\rho_{\mathrm{L}^{2\rho}(\mathcal{O})}\Big) \, \mathrm{d}s \\
&\leq \widetilde{C}_{\eta,\alpha} e^{-\lambda_A t} t^{\alpha-\eta} \|v_0(\omega)\|_\alpha \\
&\quad + CC_F \left(\sup_{s \in [0,t]} \|v_s(\omega)\|^\rho_\alpha + \sigma^\rho \|Z(\theta_s\omega)\|^\rho_\alpha \right) \int_0^t (t-s)^{-\eta} \, \mathrm{d}s + C\|f\|_\alpha < \infty,
\end{aligned}$$

due to the embedding $E_\alpha \hookrightarrow \mathrm{L}^{2\rho}(\mathcal{O})$, together with the fact that $v \in \mathcal{C}([0,t]; E_\alpha)$ and $t \mapsto Z(\theta_t\omega) \in \mathcal{C}([0,t]; E_\alpha)$. This computation also shows $v_t \in E_\eta$ for every $t \in \big(0, T_{\max}(\omega)\big)$. □

In the following, once ω is fixed we use $T_{\max}$ instead of $T_{\max}(\omega)$ for simplicity. Here, $T_{\max}(\omega)$ denotes the maximal existence time as constructed in the proof of Theorem 6.6.

Proposition 6.15. *Let the same assumptions as in Lemma 6.14 be fulfilled. Then for every $\omega \in \Omega$ and any initial value $y_0(\omega) \in E_\alpha$, there exists a unique local mild solution y of (6-19) with maximal interval of existence $(0, T_{\max})$.*

Proof. The statement immediately follows from Theorem 6.6 and Lemma 6.10. □

Recall that $E_\alpha \hookrightarrow \mathrm{L}^{2\rho}(\mathcal{O})$ holds, if α satisfies (6-21). Since $y_0(\omega) \in E_\alpha$, it follows that $v_0(\omega) \in E_\alpha$.

Lemma 6.16. *Assume that $\big(A(\theta_t\omega)\big)_{t\in\mathbb{R},\omega\in\Omega}$ and F fulfill the Assumptions **(E1)**-**(E2)**, **($\tilde{\mathbf{F}}$1)** and **($\tilde{\mathbf{F}}$2)**. Fix α satisfying (6-21) and let $f \in E_\alpha$. Then for every $\omega \in \Omega$ and initial data $v_0(\omega) \in E_\alpha$ we have*

$$\|v_t(\omega)\|_{\mathrm{L}^2(\mathcal{O})} \leq C(T_{\max}, \|f\|_\alpha, \|v_0(\omega)\|_{\mathrm{L}^2(\mathcal{O})})$$

for every $t \in (0, T_{\max})$, where $C(T_{\max}) := C(T_{\max}, \|f\|_\alpha, \|v_0(\omega)\|_{\mathrm{L}^2(\mathcal{O})})$ depends continuously on $T_{\max}$ and on the norm of the external force $\|f\|_\alpha$.

Proof. Due to the regularity of v established in Lemma 6.14, we can take the inner product in $\mathrm{L}^2(\mathcal{O})$ of (6-20) with v. Integrating by parts, and using the assumptions **(E1)** and **($\tilde{\mathbf{F}}$2)** then imply that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}\\
&\quad= \langle -A(\theta_t\omega)v_t(\omega), v_t(\omega)\rangle + \int_{\mathcal{O}} F\big(v_t(\omega)+\sigma Z(\theta_t\omega)\big)v_t(\omega)\ \mathrm{d}x + \int_{\mathcal{O}} f v_t(\omega)\ \mathrm{d}x\\
&\quad\le -c_a\|v_t(\omega)\|^2_{\mathrm{H}^1_0(\mathcal{O})} - C_0\int_{\mathcal{O}} |v_t(\omega)+\sigma Z(\theta_t\omega)|^{1+\rho}\ \mathrm{d}x + C_1|\mathcal{O}|\\
&\qquad - \sigma\int_{\mathcal{O}} F\big(v_t(\omega)+\sigma Z(\theta_t\omega)\big)Z(\theta_t\omega)\ \mathrm{d}x + \|v_t(\omega)\|_{\mathrm{L}^2(\mathcal{O})}\|f\|_{\mathrm{L}^2(\mathcal{O})} \qquad (6\text{-}22)\\
&\quad\le -c_a\|v_t(\omega)\|^2_{\mathrm{H}^1_0(\mathcal{O})} - C(\rho)\int_{\mathcal{O}} |v_t(\omega)|^{1+\rho}\ \mathrm{d}x + C(\rho,\sigma)\int_{\mathcal{O}} |Z(\theta_t\omega)|^{1+\rho}\ \mathrm{d}x\\
&\qquad + C_1|\mathcal{O}| - \sigma\int_{\mathcal{O}} F\big(v_t(\omega)+\sigma Z(\theta_t\omega)\big)Z(\theta_t\omega)\ \mathrm{d}x\\
&\qquad + \frac{\lambda_1}{2}\|v_t(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + \frac{1}{2\lambda_1}\|f\|^2_{\mathrm{L}^2(\mathcal{O})}.
\end{aligned}$$

Here, we used the norm $\|\cdot\|_{\mathrm{H}^1_0(\mathcal{O})} = \|\nabla\cdot\|_{\mathrm{L}^2(\mathcal{O})}$ which is equivalent to the $\mathrm{H}^1(\mathcal{O})$ norm due to the homogeneous Dirichlet boundary condition. To deal with the last term on the right-hand side, we use the growth condition **($\tilde{\mathbf{F}}$1)** and obtain

$$\begin{aligned}
\int_{\mathcal{O}} F\big(v_t(\omega)+\sigma Z(\theta_t\omega)\big)Z(\theta_t\omega)\ \mathrm{d}x &\le C_F\int_{\mathcal{O}} |Z(\theta_t\omega)|\big(|v_t(\omega)|^\rho + |Z(\theta_t\omega)|^\rho + 1\big)\ \mathrm{d}x\\
&\le C(\rho)\left(\frac{1}{2}\int_{\mathcal{O}} |v_t(\omega)|^{\rho+1}\ \mathrm{d}x + \int_{\mathcal{O}} |Z(\theta_t\omega)|^{\rho+1}\ \mathrm{d}x + 1\right).
\end{aligned}$$

We now use the Poincaré inequality from Theorem A.9, i.e. $\|v_t(\omega)\|_{\mathrm{H}^1_0(\mathcal{O})} \ge \lambda_1\|v_t(\omega)\|_{\mathrm{L}^2(\mathcal{O})}$, and the continuous embeddings $\|f\|_{\mathrm{L}^2(\mathcal{O})} \le C\|f\|_\alpha$ and $\|Z(\theta_t\omega)\|_{\mathrm{L}^{\rho+1}(\mathcal{O})} \le C\|Z(\theta_t\omega)\|_\alpha$ in (6-22) to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|v_t(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + c_a\lambda_1\|v_t(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C(\rho)\int_{\mathcal{O}} |v_t(\omega)|^{\rho+1}\ \mathrm{d}x\\
&\qquad\le C\left(\int_{\mathcal{O}} |Z(\theta_t\omega)|^{\rho+1}\ \mathrm{d}x + 1 + \|f\|_\alpha\right) \le C\left(\|Z(\theta_t\omega)\|^{\rho+1}_\alpha + 1\right).
\end{aligned} \tag{6-23}$$

This implies

$$\begin{aligned}
\|v_t(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} &\le e^{-c_a\lambda_1 t}\|v_0(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C\left(\int_0^t e^{-c_a\lambda_1(t-s)}\|Z(\theta_s\omega)\|^{\rho+1}_\alpha\ \mathrm{d}s + 1\right)\\
&\le \|v_0(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C\left(\int_0^{T_{\max}} \|Z(\theta_s\omega)\|^{\rho+1}_\alpha\ \mathrm{d}s + 1\right),
\end{aligned} \tag{6-24}$$

which shows the desired estimate, due to $\big(t\mapsto Z(\theta_t\omega)\big) \in \mathcal{C}([0,T_{\max}];E_\alpha)$, and the continuous dependence of the constant $C(T_{\max})$ on $T_{\max}$. □

Lemma 6.17. *Let the same assumptions as in Lemma 6.16 be fulfilled. Then for every $\omega\in\Omega$, $v_0(\omega)\in E_\alpha$ and any $\tau\in(0,T_{\max})$ we have*

$$\|v_{t+\tau}(\omega)\|_{\mathrm{L}^{\rho+1}(\mathcal{O})} \le C(\tau, T_{\max}),$$

for all $t\in(0,T_{\max}-\tau)$, where $C(\tau,T_{\max})$ depends continuously on τ and $T_{\max}$.

Proof. Fix $\tau > 0$ and consider $t \in (0, T_{\max} - \tau)$. Integrating (6-23) over $(t, t+\tau)$ yields

$$\int_t^{t+\tau} \|v_s(\omega)\|_{\mathrm{L}^{\rho+1}(\mathcal{O})}^{\rho+1} \, \mathrm{d}s \le \|v_t(\omega)\|_{\mathrm{L}^2(\mathcal{O})}^2 + C\left(\tau + \int_t^{t+\tau} \|Z(\theta_s\omega)\|_\alpha^{\rho+1} \, \mathrm{d}s\right). \tag{6-25}$$

Due to Lemma 6.14, we can multiply Equation (6-20) by $|v_t(\omega)|^{\rho-1} v_t(\omega)$. In particular

$$\begin{aligned}\int_{\mathcal{O}} \nabla\Big(v_t(\omega)\,|v_t(\omega)|^{\rho-1}\Big) \cdot \big(a(t,\omega,x)\nabla\big) v_t(\omega) \, \mathrm{d}x \\ = \rho \int_{\mathcal{O}} |v_t(\omega)|^{\rho-1} \nabla v_t(\omega) \cdot \big(a(t,\omega,x)\nabla\big) v_t(\omega) \, \mathrm{d}x,\end{aligned}$$

is well-defined due to the regularity of v_t. Indeed, if $n \le 4$, we have $v_t(\omega) \in \mathrm{L}^q(\mathcal{O})$ for an arbitrary $q \in [1, \infty)$ using the Sobolev embeddings from Proposition A.10 and Lemma 6.14, which yields $v_t(\omega) \in E_\eta$ for every $\eta \in [\alpha, 1)$. Therefore, the justification of the right-hand side is straightforward. Otherwise, if $n > 4$, the boundedness of a and the regularity of v give $|\nabla v_t(\omega)| \in E_{\eta - 1/2}$. Again, Proposition A.10 yields $|\nabla v_t(\omega)|^2 \in \mathrm{L}^{n/(n-2) - \delta_0}(\mathcal{O})$ for any $0 < \delta_0 \le \frac{2}{n-2}$. It remains to show that $|v_t(\omega)|^{\rho-1} \in \mathrm{L}^{n/2 + \delta_1}(\mathcal{O})$ for some $\delta_1 > 0$, which is achieved provided that $\rho < \frac{n}{n-4}$. But this always holds due to the assumption of ρ in (6-17). This implies that the right-hand side is well-defined. Further, using **(E1)**, it is also non-negative.

Integrating by parts then leads to

$$\begin{aligned}\frac{1}{\rho+1}\frac{\mathrm{d}}{\mathrm{d}t}\|v_t(\omega)\|_{\mathrm{L}^{\rho+1}}^{\rho+1} + \rho \int_{\mathcal{O}} |v_t(\omega)|^{\rho-1} \nabla v_t(\omega) \cdot \big(a(t,\omega,x)\nabla\big) v_t(\omega) \, \mathrm{d}x \\ \le \int_{\mathcal{O}} F\big(v_t(\omega) + \sigma Z(\theta_t\omega)\big) v_t(\omega) |v_t(\omega)|^{\rho-1} \, \mathrm{d}x + \int_{\mathcal{O}} |f||v_t(\omega)|^\rho \, \mathrm{d}x,\end{aligned} \tag{6-26}$$

where we also used that

$$\int_{\mathcal{O}} \frac{\mathrm{d}}{\mathrm{d}t} |v_t(\omega)|^{\rho+1} \, \mathrm{d}x = (\rho+1) \int_{\mathcal{O}} |v_t(\omega)|^\rho \frac{v_t(\omega)}{|v_t(\omega)|} \frac{\mathrm{d}}{\mathrm{d}t} v_t(\omega) \, \mathrm{d}x.$$

We further exploit the dissipativity of the nonlinearity F to obtain

$$\begin{aligned}&\int_{\mathcal{O}} F\big(v_t(\omega) + \sigma Z(\theta_t\omega)\big) v_t(\omega) |v_t(\omega)|^{\rho-1} \, \mathrm{d}x \\ &\quad = \int_{\mathcal{O}} F\big(v_t(\omega) + \sigma Z(\theta_t\omega)\big)\big(v_t(\omega) + \sigma Z(\theta_t\omega)\big) |v_t(\omega)|^{\rho-1} \, \mathrm{d}x \\ &\qquad - \sigma \int_{\mathcal{O}} F\big(v_t(\omega) + \sigma Z(\theta_t\omega)\big) Z(\theta_t\omega) |v_t(\omega)|^{\rho-1} \, \mathrm{d}x \\ &\quad \le -C_0 \int_{\mathcal{O}} |v_t(\omega) + \sigma Z(\theta_t\omega)|^{\rho+1} |v_t(\omega)|^{\rho-1} \, \mathrm{d}x + C_1 \int_{\mathcal{O}} |v_t(\omega)|^{\rho-1} \, \mathrm{d}x \\ &\qquad - \sigma \int_{\mathcal{O}} F\big(v_t(\omega) + \sigma Z(\theta_t\omega)\big) Z(\theta_t\omega) |v_t(\omega)|^{\rho-1} \, \mathrm{d}x \\ &\quad \le -C_0 \int_{\mathcal{O}} |v_t(\omega) + \sigma Z(\theta_t\omega)|^{\rho+1} |v_t(\omega)|^{\rho-1} \, \mathrm{d}x + C \int_{\mathcal{O}} |v_t(\omega)|^{\rho+1} \, \mathrm{d}x + C \\ &\qquad + C_F \int_{\mathcal{O}} \big(|v_t(\omega) + \sigma Z(\theta_t\omega)|^\rho + 1\big) |Z(\theta_t\omega)| |v_t(\omega)|^{\rho-1} \, \mathrm{d}x \\ &\quad \le -C_0 \int_{\mathcal{O}} |v_t(\omega)|^{2\rho} \, \mathrm{d}x + C_0 \sigma^{\rho+1} \int_{\mathcal{O}} |Z(\theta_t\omega)|^{\rho+1} |v_t(\omega)|^{\rho-1} + C \int_{\mathcal{O}} |v_t(\omega)|^{\rho+1} \, \mathrm{d}x + C \\ &\qquad + CC_F \int_{\mathcal{O}} |v_t(\omega)|^{2\rho-1} |Z(\theta_t\omega)| \, \mathrm{d}x + CC_F \int_{\mathcal{O}} |Z(\theta_t\omega)|^{\rho+1} |v_t(\omega)|^{\rho-1} \, \mathrm{d}x\end{aligned}$$

$$+ C_F \int_{\mathcal{O}} |Z(\theta_t\omega)||v_t(\omega)|^{\rho-1}\,\mathrm{d}x.$$

We use Young's inequality to estimate every term on the right-hand side

$$\begin{aligned}
C_0\sigma^{\rho+1}\int_{\mathcal{O}} |Z(\theta_t\omega)|^{\rho+1}|v_t(\omega)|^{\rho-1}\,\mathrm{d}x &\leq \frac{C_0}{5}\int_{\mathcal{O}}|v_t(\omega)|^{2\rho}\,\mathrm{d}x + C\int_{\mathcal{O}}|Z(\theta_t\omega)|^{2\rho}\,\mathrm{d}x,\\
C\int_{\mathcal{O}}|v_t(\omega)|^{\rho+1}\,\mathrm{d}x &\leq \frac{C_0}{6}\int_{\mathcal{O}}|v_t(\omega)|^{2\rho}\,\mathrm{d}x + C,\\
C\int_{\mathcal{O}}|v_t(\omega)|^{2\rho-1}|Z(\theta_t\omega)|\,\mathrm{d}x &\leq \frac{C_0}{6}\int_{\mathcal{O}}|v_t(\omega)|^{2\rho}\,\mathrm{d}x + C\int_{\mathcal{O}}|Z(\theta_t\omega)|^{2\rho}\,\mathrm{d}x,\\
C\int_{\mathcal{O}}|Z(\theta_t\omega)|^{\rho+1}|v_t(\omega)|^{\rho-1}\,\mathrm{d}x &\leq \frac{C_0}{6}\int_{\mathcal{O}}|v_t(\omega)|^{2\rho}\,\mathrm{d}x + C\int_{\mathcal{O}}|Z(\theta_t\omega)|^{2\rho}\,\mathrm{d}x,\\
C\int_{\mathcal{O}}|Z(\theta_t\omega)||v_t(\omega)|^{\rho-1}\,\mathrm{d}x &\leq \frac{C_0}{6}\int_{\mathcal{O}}|v_t(\omega)|^{2\rho}\,\mathrm{d}x + C\int_{\mathcal{O}}|Z(\theta_t\omega)|^{2\rho}\,\mathrm{d}x + C
\end{aligned}$$

and

$$\int_{\mathcal{O}}|f||v_t(\omega)|^{\rho}\mathrm{d}x \leq \frac{C_0}{6}\int_{\mathcal{O}}|v_t(\omega)|^{2\rho}\mathrm{d}x + C\|f\|^2_{\mathrm{L}^2(\mathcal{O})} \leq \frac{C_0}{6}\int_{\mathcal{O}}|v_t(\omega)|^{2\rho}\mathrm{d}x + C\|f\|^2_{\alpha}.$$

By plugging these estimates in (6-26) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v_t(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})} \leq C\left(\int_{\mathcal{O}}|Z(\theta_t\omega)|^{2\rho}\,\mathrm{d}x + 1 + \|f\|^2_{\alpha}\right) \leq C(\|Z(\theta_t\omega)\|^{2\rho}_{\alpha}+1). \tag{6-27}$$

We now integrate (6-27) over $(s, t+\tau)$ with $s\in(t,t+\tau)$, which yields

$$\|v_{t+\tau}(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})} \leq \|v_s(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})} + C\int_s^{t+\tau}\left(\|Z(\theta_r\omega)\|^{2\rho}_{\alpha}+1\right)\,\mathrm{d}r.$$

Integrating this inequality now with respect to s over $(t,t+\tau)$ and using (6-25) leads to

$$\begin{aligned}
\tau\|v_{t+\tau}(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})} &\leq \int_t^{t+\tau}\|v_s(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\,\mathrm{d}s + C\int_t^{t+\tau}\int_s^{t+\tau}\left(\|Z(\theta_r\omega)\|^{2\rho}_{\alpha}+1\right)\,\mathrm{d}r\mathrm{d}s\\
&\leq \|v_t(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C\left(\tau+\tau^2+\tau\int_t^{t+\tau}\left(\|Z(\theta_s\omega)\|^{2\rho}_{\alpha}+\|Z(\theta_s\omega)\|^{\rho+1}_{\alpha}\right)\,\mathrm{d}s\right)\\
&\leq \|v_t(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C(\tau)\left(1+\int_t^{t+\tau}\|Z(\theta_s\omega)\|^{2\rho}_{\alpha}\,\mathrm{d}s\right)\\
&\leq C(T_{\max}) + C(\tau)\left(1+\int_0^{T_{\max}}\|Z(\theta_s\omega)\|^{2\rho}_{\alpha}\,\mathrm{d}s\right),
\end{aligned} \tag{6-28}$$

where we use $\rho\geq 1$ and Young's inequality in the last step. □

For the global existence, we establish first a bound for v in E_η for some arbitrary $\eta\in[\alpha,1)$.

Lemma 6.18. *Let the same assumptions as in Lemma 6.16 be fulfilled. Then for every $\omega\in\Omega$, $v_0(\omega)\in E_\alpha$ and any $\eta>0$ such that $\eta+\alpha<1$, we obtain for every $\tau\in(0,T_{\max}/2), t\in(0,T_{\max}-\tau)$ the estimate*

$$\|v_{t+\tau}(\omega)\|_\eta \leq C(t+\tau, T_{\max})$$

for some constant $C(t+\tau,T_{\max})$ depending continuously on $T_{\max}$ such that

$$\lim_{t\to T_{\max}-\tau} C(t+\tau,T_{\max}) < \infty.$$

Proof. We use the mild formulation for v and the regularization properties of the parabolic evolution family $(S_{t,s})_{s\leq t}$. We first show that

$$\big\|F\big(v_t(\omega)+\sigma Z(\theta_t\omega)\big)\big\|_{-\alpha} \leq C\big(\|v_t(\omega)\|^{\rho}_{\mathrm{L}^{\rho+1}(\mathcal{O})} + \sigma^{\rho}\|Z(\theta_t\omega)\|^{\rho}_{\mathrm{L}^{\rho+1}(\mathcal{O})} + 1\big). \tag{6-29}$$

Indeed, for $u \in E_\alpha \hookrightarrow \mathrm{L}^{2n/(n-4\alpha)}(\mathcal{O})$ we have

$$\begin{aligned}
\big|\big\langle F\big(v_t(\omega)+\sigma Z(\theta_t\omega)\big), u\big\rangle\big| &\leq \int_{\mathcal{O}} \big|F\big(v_t(\omega)+\sigma Z(\theta_t\omega)\big)\big||u| \,\mathrm{d}x \\
&\leq C_F \int_{\mathcal{O}} \big(|v_t(\omega)+\sigma Z(\theta_t\omega)|^{\rho}+1\big)|u| \,\mathrm{d}x \\
&\leq CC_F\|u\|_{\mathrm{L}^{2n/(n-4\alpha)}(\mathcal{O})}\|v_t(\omega)+\sigma Z(\theta_t\omega)\|^{\rho}_{\mathrm{L}^{2n\rho/(n+4\alpha)}(\mathcal{O})} + C_F\|u\|_{\mathrm{L}^1(\mathcal{O})} \\
&\leq CC_F\|u\|_{\alpha}\left(\|v_t(\omega)+\sigma Z(\theta_t\omega)\|^{\rho}_{\mathrm{L}^{2n\rho/(n+4\alpha)}(\mathcal{O})}+1\right).
\end{aligned}$$

Due to the choice of α specified in (6-21), we obtain

$$\begin{aligned}
\|v_t(\omega)+\sigma Z(\theta_t\omega)\|_{\mathrm{L}^{2n\rho/(n+4\alpha)}(\mathcal{O})} &\leq C\|v_t(\omega)+\sigma Z(\theta_t\omega)\|_{\mathrm{L}^{\rho+1}(\mathcal{O})} \\
&\leq C\|v_t(\omega)\|_{\mathrm{L}^{\rho+1}(\mathcal{O})} + C\sigma\|Z(\theta_t\omega)\|_{\mathrm{L}^{\rho+1}(\mathcal{O})},
\end{aligned}$$

which yields (6-29). In the rest of this proof, we also highlight the dependence of v on the initial condition, i.e. $v_t(\omega, v_0(\omega))$. Then, we can estimate the solution v using the mild formulation. For any $t\in(\tau, T_{\max}-\tau)$ we have according to (6-29) that

$$\begin{aligned}
&\big\|v_{t+\tau}\big(\omega, v_0(\omega)\big)\big\|_{\eta} = \big\|v_{\tau}\big(\theta_t\omega, v_t(\omega, v_0(\omega))\big)\big\|_{\eta} \\
&\quad\leq \big\|S_{t+\tau,t}(\omega)v_t\big(\omega, v_0(\omega)\big)\big\|_{\eta} \\
&\qquad+\int_t^{t+\tau} \big\|S_{t+\tau,s}(\omega)\big(F(v_s(\omega,v_0(\omega))+\sigma Z(\theta_s\omega))+f\big)\big\|_{\eta}\,\mathrm{d}s \\
&\quad\leq \big\|S_{t+\tau,t}(\omega)v_t\big(\omega, v_0(\omega)\big)\big\|_{\eta} \\
&\qquad+\widetilde{C}_{\eta,\alpha}\int_t^{t+\tau}(t+\tau-s)^{-\eta-\alpha}e^{-\lambda_A(t+\tau-s)}\big\|F\big(v_s(\omega,v_0(\omega))+\sigma Z(\theta_s\omega)\big)\big\|_{-\alpha}\,\mathrm{d}s \\
&\qquad+\widetilde{C}_{\eta,\alpha}\int_t^{t+\tau}(t+\tau-s)^{-\eta+\alpha}e^{-\lambda_A(t+\tau-s)}\|f\|_{\alpha}\mathrm{d}s \\
&\quad\leq \big\|S_{t+\tau,t}(\omega)v_t\big(\omega, v_0(\omega)\big)\big\|_{\eta} \\
&\qquad+\widetilde{C}_{\eta,\alpha}C_F\int_t^{t+\tau}(t+\tau-s)^{-\eta-\alpha}e^{-\lambda_A(t+\tau-s)}\big\|v_s\big(\omega,v_0(\omega)\big)\big\|^{\rho}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\,\mathrm{d}s \\
&\qquad+\widetilde{C}_{\eta,\alpha}C_F\int_t^{t+\tau}(t+\tau-s)^{-\eta-\alpha}e^{-\lambda_A(t+\tau-s)}\Big(\sigma^{\rho}\|Z(\theta_s\omega)\|^{\rho}_{\mathrm{L}^{\rho+1}(\mathcal{O})}+1\Big)\,\mathrm{d}s \\
&\qquad+\widetilde{C}_{\eta,\alpha}\|f\|_{\alpha}\int_t^{t+\tau}(t+\tau-s)^{-\eta+\alpha}e^{-\lambda_A(t+\tau-s)}\mathrm{d}s.
\end{aligned}$$

We recall that S satisfies $S_{t+\tau,t}(\omega)=S_{\tau,0}(\theta_t\omega)$ as established in Theorem 6.5. Moreover, the constants $\widetilde{C}_\eta=\widetilde{C}_\eta(\omega)$, respectively $\widetilde{C}_{\eta,\alpha}=\widetilde{C}_{\eta,\alpha}(\omega)$ in the computation below are uniformly bounded with respect to $\omega\in\Omega$, due to Assumption **($\widetilde{\mathbf{A}}$3)**. To estimate the first term on the right-hand side, we use (6-6) to obtain

$$\big\|S_{t+\tau,t}(\omega)v_t\big(\omega,v_0(\omega)\big)\big\|_{\eta} \leq \widetilde{C}_\eta\frac{e^{-\lambda_A(t+\tau-t)}}{(t+\tau-t)^{\eta}}\big\|v_t\big(\omega,v_0(\omega)\big)\big\|_{\mathrm{L}^2(\mathcal{O})} \leq C_\eta(\tau)\big\|v_t\big(\omega,v_0(\omega)\big)\big\|_{\mathrm{L}^2(\mathcal{O})},$$

and the second term is bounded by

$$\begin{aligned}\widetilde{C}_{\eta,\alpha}C_F \sup_{s\in(\tau,T_{\max})}&\Big(\|Z(\theta_s\omega)\|^{\rho}_{\mathrm{L}^{\rho+1}(\mathcal{O})}+\big\|v_s\big(\omega,v_0(\omega)\big)\big\|^{\rho}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\Big)\\&\times\int_t^{t+\tau}(t+\tau-s)^{-\eta-\alpha}e^{-\lambda_A(t+\tau-s)}\,\mathrm{d}s\\&+\widetilde{C}_{\eta,\alpha}C_F\int_t^{t+\tau}(t+\tau-s)^{-\eta-\alpha}e^{-\lambda_A(t+\tau-s)}\,\mathrm{d}s\\&\le C_\eta(\tau)\Big(\sup_{s\in(\tau,T_{\max})}\big(\|Z(\theta_s\omega)\|^{\rho}_{\alpha}+\big\|v_s\big(\omega,v_0(\omega)\big)\big\|_{\mathrm{L}^{\rho+1}(\mathcal{O})}\big)+1\Big)\\&\le C_\eta(\tau)\left(C(t+\tau,T_{\max})+\sup_{s\in(\tau,T_{\max})}\|Z(\theta_s\omega)\|^{\rho}_{\alpha}\right)+1,\end{aligned}$$

where we used that $\eta+\alpha<1$, the estimate for v in Lemma 6.17 and the embedding $E_\alpha\hookrightarrow\mathrm{L}^{\rho+1}(\mathcal{O})$. Finally, the last term is bounded by

$$\widetilde{C}_{\eta,\alpha}\int_t^{t+\tau}(t+\tau-s)^{-\eta+\alpha}e^{-\lambda_A(t+\tau-s)}\,\mathrm{d}s\|f\|_\alpha\le C_{\eta,\alpha}(\tau)C(t+\tau,T_{\max})\|f\|_\alpha. \qquad\square$$

Theorem 6.19. *Let the same assumptions as in Lemma 6.16 be fulfilled. Then for every $\omega\in\Omega$ and initial data $v_0(\omega)=y_0(\omega)+\sigma Z(\omega)\in E_\alpha$, Equation (6-20) has a unique global solution in $\mathcal{C}([0,\infty);E_\alpha)$. Consequently, for every $\omega\in\Omega$ and initial data $y_0(\omega)\in E_\alpha$ Equation (6-19) has a unique global solution in $\mathcal{C}([0,\infty);E_\alpha)$.*

Proof. From Lemma 6.18, by choosing $\eta=\alpha$, we conclude the global solvability of (6-20). The global existence of (6-19) follows immediately due to the relation $y_t(\omega)=v_t(\omega)+\sigma Z(\theta_t\omega)$. $\square$

6.2.2 Random attractors

Due to the global existence of solutions proved in the previous section, (6-20) gives rise to a continuous random dynamical system ϕ on E_α defined by

$$\phi\colon[0,\infty)\times\Omega\times E_\alpha\to E_\alpha,\ \big(t,\omega,v_0(\omega)\big)\mapsto v_t\big(\omega,v_0(\omega)\big).$$

We now define a transformation in the sense of Definition 4.36 by

$$\mathscr{T}:\Omega\times E_\alpha,(\omega,x)\mapsto\mathscr{T}(\omega,x):=x+\sigma Z(\omega).$$

Applying this to the random dynamical system ϕ of (6-20), entails

$$\begin{aligned}&\varphi\colon[0,\infty)\times\Omega\times E_\alpha\to E_\alpha,\\&\qquad\big(t,\omega,v_0(\omega)\big)\mapsto\mathscr{T}\big(\theta_t\omega,\phi(t,\omega,\mathscr{T}^{-1}(\omega,y_0))\big)=v_t\big(\omega,v_0(\omega)\big)+\sigma Z(\theta_t\omega),\end{aligned}$$

where $v_0(\omega)=y_0(\omega)-\sigma Z(\omega)$ and $\mathscr{T}^{-1}(\omega,y_0)=v_0$ is the inverse with respect to the second variable. Therefore, ϕ is the conjugate random dynamical system to the original Equation (6-19).

We now investigate the long-time behavior of φ, establishing the existence of a random attractor. We use a similar bootstrapping method as for the global solution and prove first the existence of an absorbing set in L^2, then in $\mathrm{L}^{\rho+1}$, and finally in E_η for some $\eta\ge\alpha$. The latter regularity implies than also the compactness of the absorbing set, which leads to the existence of an attractor.

Lemma 6.20. *Let the same assumptions as in Lemma 6.16 be fulfilled. Then there exists an absorbing set in* $\mathrm{L}^2(\mathcal{O})$ *for the random dynamical system* φ.

Proof. In the Estimate (6-24) in Lemma 6.16 we can replace ω by $\theta_{-t}\omega$ and obtain

$$
\begin{aligned}
&\big\|\phi\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big\|^2_{\mathrm{L}^2(\mathcal{O})} \\
&\leq e^{-c_a\lambda_1 t}\|v_0(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C\left(\int_0^t e^{-c_a\lambda_1(t-s)}\|Z(\theta_{s-t}\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\,\mathrm{d}s + 1\right) \\
&\leq e^{-c_a\lambda_1 t}\|v_0(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C\left(\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|^{\rho+1}_{\alpha}\,\mathrm{d}\tau + 1\right),
\end{aligned}
\tag{6-30}
$$

where we used the change of variables $\tau = s-t$ and the embedding $E_\alpha \hookrightarrow \mathrm{L}^{\rho+1}(\mathcal{O})$ in the last step.

If $\{K(\omega)\}_{\omega\in\Omega}$ is a tempered random set with $K(\omega)\in E_\alpha$ and $y_0(\theta_{-t}\omega)\in K(\theta_{-t}\omega)$ we obtain

$$
\begin{aligned}
\big\|\varphi\big(t,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\big\|^2_{\mathrm{L}^2(\mathcal{O})} &\leq 2\big\|\phi\big(t,\theta_{-t}\omega,v_0(\theta_{-t}\omega)\big)\big\|^2_{\mathrm{L}^2(\mathcal{O})} + 2\sigma^2\|Z(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} \\
&\leq 2e^{-c_a\lambda_1 t}\|v_0(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C + C\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|^{\rho+1}_\alpha\,\mathrm{d}\tau + 2\sigma^2\|Z(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} \\
&\leq 2e^{-c_a\lambda_1 t}\|y_0(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + 2\sigma^2 e^{-c_a\lambda_1 t}\|Z(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} \\
&\quad + C\left(1+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|^{\rho+1}_\alpha\,\mathrm{d}\tau + \|Z(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}\right).
\end{aligned}
$$

Taking the limes superior of this expression leads to

$$
\limsup_{t\to\infty}\big\|\varphi\big(t,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\big\|_{\mathrm{L}^2(\mathcal{O})} \leq r_2(\omega),
$$

where

$$
r_2(\omega) := \left(2C\left(1+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|^{\rho+1}_\alpha\,\mathrm{d}\tau + \|Z(\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}\right)\right)^{1/2}.
$$

using $y_0(\theta_{-t}\omega)\in K(\theta_{-t}\omega)$. The temperedness of $\|Z(\omega)\|_\alpha$, which was established in Proposition 6.9, implies the temperedness of $\|Z(\omega)\|_{\mathrm{L}^2(\mathcal{O})}$. Therefore, r_2 is tempered due to Lemma 4.14 and Remark 4.12. We conclude that there exists $T_2(\omega)>0$ such that

$$
\big\|\varphi\big(t,\theta_{-t}\omega,y_0(\theta_{-t}\omega)\big)\big\|_{\mathrm{L}^2(\mathcal{O})} \leq r_2(\omega)
$$

holds for all $t\geq T_2(\omega)$, which completes the proof of this lemma. □

Lemma 6.21. *Let the same assumptions as in Lemma 6.16 be fulfilled. Then, there exists an absorbing set in* $\mathrm{L}^{\rho+1}(\mathcal{O})$ *for the random dynamical system* φ.

Proof. Taking $\tau = 1$ in (6-28) and using (6-30) leads to

$$\begin{aligned}
&\big\|\phi\big(t+1,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\\
&\qquad\le \big\|\phi\big(t,\theta_{-t-1}\omega, v_0(\theta_{-t-1}\omega)\big)\big\|^{2}_{\mathrm{L}^{2}(\mathcal{O})} + C\left(1+\int_t^{t+1}\|Z(\theta_{s-t-1}\omega)\|_\alpha^{2\rho}\,\mathrm{d}s\right)\\
&\qquad\le e^{-c_a\lambda_1 t}\|v_0(\theta_{-t-1}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C\left(\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|_\alpha^{\rho+1}\,\mathrm{d}\tau+1\right)\\
&\qquad\quad + C\left(1+e^{c_a\lambda_1}\int_t^{t+1} e^{-c_a\lambda_1(t+1-s)}\|Z(\theta_{s-t-1}\omega)\|_\alpha^{2\rho}\,\mathrm{d}s\right)\\
&\qquad\le e^{c_a\lambda_1}e^{-c_a\lambda_1(t+1)}\|v_0(\theta_{-t-1}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + C\left(1+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|_\alpha^{2\rho}\,\mathrm{d}\tau\right).
\end{aligned}$$

Hence, if $\{K(\omega)\}_{\omega\in\Omega}$ is a tempered random set with $K(\omega)\in E_\alpha$ and $y_0(\theta_{-t}\omega)\in K(\theta_{-t}\omega)$ we obtain

$$\begin{aligned}
\big\|\varphi\big(t,\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})} &\le C(\rho,\sigma)\left(\big\|\phi\big(t,\theta_{-t}\omega, v_0(\theta_{-t}\omega)\big)\big\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})} + \|Z(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\right)\\
&\le C(\rho,\sigma)e^{c_a\lambda_1}e^{-c_a\lambda_1 t}\|v_0(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}\\
&\quad + C\left(1+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|_\alpha^{2\rho}\mathrm{d}\tau\right) + \|Z(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\\
&\le C(\rho,\sigma)e^{c_a\lambda_1}\left(2e^{-c_a\lambda_1 t}\|y_0(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})} + 2\sigma^2 e^{-c_a\lambda_1 t}\|Z(\theta_{-t}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}\right)\\
&\quad + C\left(1+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|_\alpha^{2\rho}\mathrm{d}\tau + \|Z(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}(\mathcal{O})}\right),
\end{aligned}$$

for $t\ge 1$. Taking the limes superior yields

$$\limsup_{t\to\infty}\big\|\varphi\big(t,\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big\|_{\mathrm{L}^{\rho+1}(\mathcal{O})} \le r_\rho(\omega),$$

where

$$r_\rho(\omega) := C^{1/(\rho+1)}\left(1+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|_\alpha^{2\rho}\,\mathrm{d}\tau + \|Z(\omega)\|^{\rho+1}_{\mathrm{L}^{\rho+1}}(\mathcal{O})\right)^{1/(\rho+1)}.$$

using the temperedness of $\omega\mapsto\|Z(\omega)\|_\alpha$ which implies in particular the temperedness of $\omega\mapsto\|Z(\omega)\|_{\mathrm{L}^{\rho+1}(\mathcal{O})}$. Similar calculations to those for r_2 in the proof of Lemma 6.20 show that r_ρ is tempered, i.e. there exists a time $T_\rho(\omega)>0$ such that

$$\big\|\varphi\big(t,\theta_{-t}\omega, y_0(\theta_{-t}\omega)\big)\big\|_{\mathrm{L}^{\rho+1}(\mathcal{O})} \le r_\rho(\omega) \tag{6-31}$$

holds for every $t\ge T_\rho(\omega)$. Therefore, there exists an absorbing set for φ in $\mathrm{L}^{\rho+1}(\mathcal{O})$. □

Theorem 6.22. *Let the same assumptions as in Lemma 6.16 be fulfilled. Then the random dynamical system φ generated by (6-16) possesses a unique random attractor $\mathscr{A}$ in E_α.*

Proof. Let $\{K(\omega)\}_{\omega\in\Omega}$ be a tempered random set with $K(\omega)\in E_\alpha$ and fix $y_0(\omega)\in K(\omega)$. We first estimate ϕ similarly as in the proof of Lemma 6.18 for some $\eta>\alpha$ which yields

$$\begin{aligned}
&\big\|\phi\big(t+1,\omega,v_0(\omega)\big)\big\|_\eta = \big\|\phi\big(1,\theta_t\omega,\phi(t,\omega,v_0(\omega))\big)\big\|_\eta\\
&\quad\le \big\|S_{t+1,t}(\omega)\phi\big(t,\omega,v_0(\omega)\big)\big\|_\eta\\
&\qquad+\int_t^{t+1}\big\|S_{t+1,s}(\omega)\big(F(\phi(s,\omega,v_0(\omega))+\sigma Z(\theta_s\omega))+f\big)\big\|_\eta\,\mathrm{d}s\\
&\quad\le \big\|S_{t+1,t}(\omega)\phi\big(t,\omega,v_0(\omega)\big)\big\|_\eta\\
&\qquad+\widetilde{C}_{\eta,\alpha}\int_t^{t+1}(t+1-s)^{-\eta-\alpha}e^{-\lambda_A(t+1-s)}\big\|F\big(\phi(s,\omega,v_0(\omega))+\sigma Z(\theta_s\omega)\big)\big\|_{-\alpha}\,\mathrm{d}s\\
&\qquad+\widetilde{C}_{\eta,\alpha}\int_t^{t+1}(t+1-s)^{-\eta+\alpha}e^{-\lambda_A(t+1-s)}\|f\|_\alpha\mathrm{d}s\\
&\quad\le \big\|S_{t+1,t}(\omega)\phi\big(t,\omega,v_0(\omega)\big)\big\|_\eta+\widetilde{C}_{\eta,\alpha}C_F\int_t^{t+1}(t+1-s)^{-\eta-\alpha}e^{-\lambda_A(t+1-s)}\\
&\qquad\times\Big(\big\|\phi\big(s,\omega,v_0(\omega)\big)\big\|^\rho_{\mathrm{L}^{\rho+1}(\mathcal{O})}+\sigma^\rho\|Z(\theta_s\omega)\|^\rho_{\mathrm{L}^{\rho+1}(\mathcal{O})}+1\Big)\,\mathrm{d}s+C\|f\|_\alpha.
\end{aligned}\tag{6-32}$$

Recall that $S_{t+1,t}(\omega)=S_{1,0}(\theta_t\omega)$holds for every $t\in\mathbb{R}$ as stated in Theorem 6.5 and that the constants $\widetilde{C}_\eta=\widetilde{C}_\eta(\omega)$ as well as $\widetilde{C}_{\eta,\alpha}=\widetilde{C}_{\eta,\alpha}(\omega)$ in the estimates below are uniformly bounded with respect to $\omega\in\Omega$ due to Assumption **($\tilde{\mathbf{A}}$3)**. We now replace ω by $\theta_{-t-1}\omega$ and use (6-30) to obtain

$$\begin{aligned}
&\big\|S_{t+1,t}(\theta_{-t-1}\omega)\phi\big(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|_\eta\\
&\quad\le \widetilde{C}_\eta\frac{e^{-\lambda_A(t+1-t)}}{(t+1-t)^\eta}\big\|\phi\big(t,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|_{\mathrm{L}^2(\mathcal{O})}\\
&\quad\le C_\eta\left(e^{-c_a\lambda_1 t}\|v_0(\theta_{-t-1}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_{\tau-1}\omega)\|^{\rho+1}_\alpha\,\mathrm{d}\tau+1\right)^{1/2}\\
&\quad\le C_\eta\Big(2e^{-c_a\lambda_1 t}\|y_0(\theta_{-t-1}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}+2\sigma^2e^{-c_a\lambda_1 t}\|Z(\theta_{-1}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}\\
&\qquad+e^{c_a\lambda_1}\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|^{\rho+1}_\alpha\,\mathrm{d}\tau+1\Big)^{1/2}\\
&\quad\le C\left(e^{-c_a\lambda_1(t+1)}\|y_0(\theta_{-t-1}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}+\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|^{\rho+1}_\alpha\,\mathrm{d}\tau+1\right)^{1/2},
\end{aligned}\tag{6-33}$$

where we chose t large enough such that $2\sigma^2e^{-c_a\lambda_1 t}\|Z(\theta_{-1}\omega)\|^2_{\mathrm{L}^2(\mathcal{O})}\le 1$. For the second term on the right-hand side of (6-32), we also replace ω by $\theta_{-t-1}\omega$ and then use the absorbing property of φ in $\mathrm{L}^{\rho+1}(\mathcal{O})$ obtained in (6-31). Hence, we have

$$\begin{aligned}
&\widetilde{C}_{\eta,\alpha}\int_t^{t+1}(t+1-s)^{-\eta-\alpha}\Big(\big\|\phi\big(s,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)+\sigma Z(\theta_{-t-1})\big\|^\rho_{\mathrm{L}^{\rho+1}(\mathcal{O})}+1\Big)\,\mathrm{d}s\\
&\quad\le \widetilde{C}_{\eta,\alpha}\left(\int_t^{t+1}(t+1-s)^{-\eta-\alpha}r_\rho(\theta_{s-t-1}\omega)^\rho\,\mathrm{d}s+1\right)\\
&\quad\le C_\eta\left(\int_t^{t+1}(t+1-s)^{-\eta-\alpha}r_\rho(\theta_{s-t-1}\omega)^{\rho+1}\,\mathrm{d}s+\int_t^{t+1}(t+1-s)^{-\eta-\alpha}\,\mathrm{d}s+1\right)\\
&\quad\le C_\eta\left(1+\int_0^1\tau^{-\eta-\alpha}r_\rho(\theta_{-\tau}\omega)^{\rho+1}\,\mathrm{d}\tau\right)
\end{aligned}\tag{6-34}$$

for every $t \geq T_\rho(\omega)$. From (6-32), (6-33) and (6-34) we conclude

$$\begin{aligned}
&\big\|\varphi\big(t+1,\theta_{-t-1}\omega,y_0(\theta_{-t-1}\omega)\big)\big\|_\eta \\
&\qquad \leq \big\|\phi\big(t+1,\theta_{-t-1}\omega,v_0(\theta_{-t-1}\omega)\big)\big\|_\eta + \sigma\|Z(\omega)\|_\eta \\
&\qquad \leq C_\eta e^{-c_a\lambda_1(t+1)}\|y_0(\theta_{-t-1}\omega)\|_{\mathrm{L}^2(\mathcal{O})} + C + C\|f\|_\alpha + \sigma\|Z(\omega)\|_\eta \\
&\qquad\quad + C\left(\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|_\alpha^{\rho+1}\,\mathrm{d}\tau\right)^{1/2} + C_\eta\left(1+\int_0^1 \tau^{-\eta-\alpha}r_\rho(\theta_{-\tau}\omega)^{\rho+1}\,\mathrm{d}\tau\right) \\
&\qquad = C_\eta e^{-c_a\lambda_1(t+1)}\|y_0(\theta_{-t-1}\omega)\|_{\mathrm{L}^2(\mathcal{O})} + \frac{1}{2}r_\eta(\omega),
\end{aligned}$$

for all $t \geq T_\rho(\omega)$ where

$$\begin{aligned}
r_\eta(\omega) := C + C\left(\int_{-\infty}^0 e^{c_a\lambda_1\tau}\|Z(\theta_\tau\omega)\|_\alpha^{\rho+1}\,\mathrm{d}\tau\right)^{1/2} + \sigma\|Z(\omega)\|_\eta + C\|f\|_\alpha \\
+ C_\eta\left(1+\int_0^1 \tau^{-\eta-\alpha}r_\rho(\theta_{-\tau}\omega)^{\rho+1}\,\mathrm{d}\tau\right).
\end{aligned}$$

Note that $\eta < 1/2$, hence, using the temperedness of $\|Z(\omega)\|_\eta$ and of r_ρ it follows that r_η is tempered. Since $y_0(\theta_{-t-1}\omega) \in K(\theta_{-t-1}\omega)$, there exists $T_\eta(\omega) > 0$ such that

$$\|\varphi(t,\theta_{-t}\omega,y_0(\theta_{-t}\omega))\|_\eta \leq r_\eta(\omega),$$

for all $t \geq T_\eta(\omega)+1$. Consequently, the random dynamical system φ possesses an absorbing set in E_η. Since $\eta > \alpha$, the embedding $E_\eta \hookrightarrow E_\alpha$ is compact, and we conclude that φ possesses a unique random attractor $\mathscr{A}$ in E_α according to Theorem 4.19. ☐

6.2.3 Generalization for higher-order elliptic operators

Standard examples of operators satisfying Assumptions **($\tilde{\mathbf{A}}$)** are uniformly elliptic operators with Dirichlet, Neumann, or Robin boundary conditions. The following example was given in [KNS21], see also [MS03, MS08]. It generalizes the results of the previous section for higher-order uniformly elliptic random operators. More precisely, for $m \in \mathbb{N}$, we consider the following stochastic partial differential equation

$$\begin{cases} \mathrm{d}y_t(\omega) = \big(\mathcal{A}_m(t,\omega)y_t(\omega) + F(y_t(\omega)) + f\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t(\omega), & \text{in } \mathcal{O}, \\ y_t(\omega) = 0, & \text{on } \partial\mathcal{O}, \end{cases} \tag{6-35}$$

for $\omega \in \Omega, t > 0$ where $\mathcal{O} \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\mathcal{O}$, and the differential operator is defined as

$$\mathcal{A}_m(\theta_t\omega,x) := \sum_{|k_1|,|k_2|\leq m} \partial^{k_1}\big(a_{k_1,k_2}(\theta_t\omega,x)\partial^{k_2}\big),$$

for every $\omega \in \Omega,\ x \in \mathcal{O},\ t \in \mathbb{R}$. We assume that the coefficient functions satisfy the following properties:

($\hat{\mathbf{E}}$1) The coefficients a_{k_1,k_2} are bounded and symmetric. More precisely, there exists a constant $K \geq 1$ such that

$$|a_{k_1,k_2}(\theta_t\omega,x)| \leq K$$

for all $|k_1|,|k_2| \leq m,\ t \in \mathbb{R}, x \in \mathcal{O}, \omega \in \Omega$ and

$$a_{k_1,k_2}(\cdot,\cdot) = a_{k_2,k_1}(\cdot,\cdot),$$

for $|k_1|, |k_2| \leq m$.

(Ê2) The coefficients form a stochastic process $(t,\omega) \mapsto a_{k_1,k_2}(\theta_t\omega,\cdot) \in \mathcal{C}^m(\overline{\mathcal{O}})$ with Hölder continuous trajectories. This means that there exists $\nu \in (0,1]$ such that

$$|a_{k_1,k_2}(\theta_t\omega,x) - a_{k_1,k_2}(\theta_s\omega,x)| \leq \overline{c}_2|t-s|^\nu$$

for all $t \in \mathbb{R},\ x \in \overline{\mathcal{O}},\ |k_1|, |k_2| \leq m$ and some constant $\overline{c}_2 > 0$. Furthermore, the mapping $t \mapsto \partial^k a_{k_1,k_2}(\theta_t\omega,x)$ is continuous for $|k|, |k_1|, |k_2| \leq m$, $\omega \in \Omega$ and $x \in \mathcal{O}$.

(Ê3) The operator $\mathcal{A}_m$ is uniformly elliptic in $\mathcal{O}$, i.e. there exists a constant $\bar{c} > 0$ such that

$$\sum_{|k_1|=|k_2|=m} a_{k_1,k_2}(\theta_t\omega,x)\xi_{k_1}\xi_{k_2} \geq \bar{c}|\xi|^{2m},$$

for all $t \in \mathbb{R},\ x \in \overline{\mathcal{O}}$ and $\xi \in \mathbb{R}^n$.

We now introduce the random partial differential operator on the Banach space $E := \mathrm{L}^2(\mathcal{O})$. To this end we define the $\mathrm{L}^2(\mathcal{O})$-realization A of $\mathcal{A}_m(\cdot,\cdot,\mathrm{D})$ as

$$\begin{aligned} D_A &:= D\big(A(\theta_t\omega)\big) = \mathrm{H}^{2m,2}(\mathcal{O}) \cap \mathrm{H}^{m,2}_0(\mathcal{O}), \\ A(\theta_t\omega)u &:= \mathcal{A}_m(\theta_t\omega,x)u, \end{aligned}$$

for $u \in D_A$ and assume it has a compact inverse. Then it is known that, after a possible shift, every $A(\theta_t\omega)$, generates an analytic semigroup, see Theorem B.13, which means **(Ã1)** and **(Ã2)** are satisfied. The Assumptions **(Ã5)** and **(Ã6)** follow directly by definition, so we only have to verify **(Ã3)** and **(Ã4)**. The Hölder continuity of $A(\theta_t\omega)$ was verified in [KNS21, Example 4.2], however, for the sake of completeness, we will state the argument here. Therefore note, that for every $s,t \in \mathbb{R}$ we have

$$\begin{aligned} &\|A(\theta_t\omega) - A(\theta_s\omega)\|^2_{\mathcal{L}(D_A;E)} \\ &\quad = \sup_{v\in D_A, \|v\|_{D_A}=1} \left\| \sum_{|k_1|,|k_2|\leq m} \Big(\partial^{k_1}\big(a_{k_1,k_2}(\theta_t\omega,x) - a_{k_1,k_2}(\theta_s\omega,x)\partial^{k_2}\big)\Big)v \right\|_E^2 . \end{aligned}$$

Further, we obtain

$$\begin{aligned} &\left\| \sum_{|k_1|,|k_2|\leq m} \Big(\partial^{k_1}\big(a_{k_1,k_2}(\theta_t\omega,x) - a_{k_1,k_2}(\theta_s\omega,x)\partial^{k_2}\big)\Big)v \right\|_E^2 \\ &\quad \lesssim \sum_{|k_1|,|k_2|\leq m} \int_{\mathcal{O}} \left| \Big(\partial^{k_1}\big(a_{k_1,k_2}(\theta_t\omega,x) - a_{k_1,k_2}(\theta_s\omega,x)\partial^{k_2}\big)\Big)v(x)\right|^2 \,\mathrm{d}x \\ &\quad \lesssim \sum_{|k_1|,|k_2|\leq m} \sup_{x\in\mathcal{O}} \left|\partial^{k_1}\big(a_{k_1,k_2}(\theta_t\omega,x) - a_{k_1,k_2}(\theta_s\omega,x)\big)\right|_E^2 \int_{\mathcal{O}} \left|\partial^{k_2}v(x)\right|^2 \,\mathrm{d}x \\ &\quad \lesssim \|v\|_{\mathrm{W}^{2m,2}(\mathcal{O})} \sum_{|k_1|,|k_2|\leq m} \|a_{k_1,k_2}(\theta_t\omega,x) - a_{k_1,k_2}(\theta_s\omega,x)\|^2_{\mathcal{C}^m(\overline{\mathcal{O}})}, \end{aligned}$$

for every $v \in D_A$. This estimate and the Hölder continuity of a_{k_1,k_2} then yields **(Ã3)**. Since A is self adjoint in this particular case, we obtain also **(Ã4)**. Therefore $\big(A(\theta_t\omega)\big)_{t\in\mathbb{R},\omega\in\Omega}$ satisfies Assumption **(Ã)** and generates a parabolic evolution family on E as established in Theorem 6.1.

For the nonlinearity, we assume the following:

($\hat{\mathbf{F}}$1) $F:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz, and either $n\le 2m$ or $n\ge 2m+1$ and there is $C>0$ such that

$$|F(u)-F(v)|\le C_F|u-v|\left(|u|^{\mu-1}+|v|^{\mu-1}\right),$$

for all $u,v\in\mathbb{R}$, where μ satisfies

$$\mu<\mu_{\text{critical}}\coloneqq\begin{cases}\infty & \text{if } n\le 2m,\\ \frac{n}{n-2m} & \text{if } n\ge 2m+1.\end{cases}$$

($\hat{\mathbf{F}}$2) There are constants C_0, C_1 such that

$$F(u)u\le -C_0|u|^{1+\mu}+C_1,$$

for all $u\in\mathbb{R}$, with μ from (i).

Using the same arguments as in the previous section, one can show the existence of an attractor as in Theorem 6.22.

Theorem 6.23. *Suppose that the above assumptions on the differential operator $\mathcal{A}_m$ and the nonlinearity F hold. Fix $\alpha\in(0,1/2)$ such that*

$$\frac{n(\mu-1)}{4(\mu+1)}\le\alpha<\min\left\{\frac{n}{4},\frac{1}{2}\right\}\qquad\textit{and}\qquad E_\alpha\hookrightarrow \mathrm{L}^{2\mu}(\mathcal{O})$$

holds. Then for any initial data $y_0(\omega)\in E_\alpha$, and any $f\in E_\alpha$ the Equation (6-35) has a unique global solution in E_α. Moreover, the random dynamical system φ generated by (6-35) as

$$\varphi\colon[0,\infty)\times\Omega\times E_\alpha\to E_\alpha,\ \big(t,\omega,y_0(\omega)\big)\mapsto\varphi\big(t,\omega,y_0(\omega)\big)\coloneqq y_t\big(\omega,y_0(\omega)\big)$$

possesses a unique random attractor $\mathscr{A}$ in E_α.

Chapter 7

Summary and Outlook

To summarize, this thesis primarily investigates stochastic evolution equations with nonlinear multiplicative noise and their long-term behavior. Due to the stochastic influence, the solutions are, in general, not pathwise defined, which causes several problems when examining long-term behavior. In particular, it is challenging to consider equations with nonlinear multiplicative noise. For this reason, the focus of this work is on pathwise solution concepts. To be more precise, we consider mild solutions using the theory of rough paths and pathwise mild solutions. The results are applied to several examples, such as equations with noise on the boundary. We also highlight some open questions or potential areas for extension.

In Chapter 2, we introduce the theory of rough paths and controlled rough paths. A rough path enables the examination of solutions to stochastic partial differential equations in a pathwise sense. Through rough paths, non-Markovian processes can also be considered, which includes the fractional Brownian motion. Additionally, in Subsection 2.2.2, we investigate the Cameron–Martin space of a Gaussian rough path, as this information becomes important when it comes to the integrability of solutions. Based on the theory of rough paths, we introduce the rough integral in Section 2.3. First, the integral over so-called one-forms is discussed, which then leads to the definition of controlled rough paths in Subsection 2.3.3, as developed by Gerasimovičs, Hocquet, and Nilssen [GHN21]. Controlled rough paths form a class of functions that can be integrated against a rough path and are adapted to the setting of parabolic equations. This section extends the established results by including non-autonomous diffusion coefficients. Additionally, in Subsection 2.3.4, we examine how the controlled rough path is affected by various operations.

In Chapter 3, we deal with the solutions of rough evolution equations. In particular, Subsections 3.1 and 3.2 first examine the local and then the global solvability of semilinear parabolic evolution equations with nonlinear multiplicative noise. Furthermore, we investigate the solutions of these equations in more detail to obtain a-priori estimates. We can prove integrable bounds for arbitrary moments of the solution in Subsection 3.4 and Gronwall-type estimates in Subsection 3.5. In addition, in Subsection 3.3 we apply the results of this chapter to equations with boundary noise.

Everything so far is based on the assumption that the linear part of the semilinear equation $\big(A(t)\big)_{t\in\mathbb{R}}$ has a time-independent domain. It is an interesting question to consider how to extend the results from the first chapters to the situation where the domain is allowed to change over time. This extension requires the more general assumptions of Acquistapace–Terreni, leading to time-dependent fractional power spaces and, consequently, to time-dependent controlled rough path spaces. These assumptions do not fit in the framework of Gerasimovičs, Hocquet, and Nilssen [GHN21]. In the case of the Brownian motion, this extension has already been made in [SV11], and it would be interesting to investigate in future works whether this can be transferred to the rough path case. Furthermore, the conditions under which global solutions for quasi-linear equations exist remain open. Local solvability has only recently been investigated; see [HN24].

In the second part of this thesis, we use the insights from the first part to investigate the long-term behavior. In Chapter 4, we deal with the introduction of random dynamical systems, which provides the basis for examining the dynamics. More precisely, we introduce the notion of rough path cocycles, which in turn leads to the fact that a rough evolution equation generates a random dynamical system if the solution exists globally. Furthermore, we briefly introduce the concept of a conjugated random dynamical system, which we later apply in the context of pathwise mild solutions. To consider non-autonomous equations, we also construct the extended metric dynamical system in Subsection 4.2.2, which models both temporal and random change. This construction allows us to apply the theory of autonomous random dynamical systems to non-autonomous equations.

In Chapter 5, we prove the existence of a global pullback attractor in Section 5.1 and Lyapunov exponents in Section 5.2. Again, we apply these results to equations with rough boundary noise. Apart from studying attractors and Lyapunov exponents, many other asymptotic concepts could be investigated in more detail using the Gronwall inequality, the a-priori estimates, and the multiplicative ergodic theory. Another open question is the dimension of the global attractors. To date, no studies are addressing this issue in the context of rough paths. However, we aim to examine the dynamics of boundary noise in greater detail in future works. In contrast to regular stochastic partial differential equations, there is much less literature on these systems.

In the last Chapter 6, we consider an alternative pathwise solution theory, namely pathwise mild solutions. This solution concept can be used in particular to consider equations in which the linear part not only depends on time but is also random, i.e., $\big(A(t,\omega)\big)_{t\in[0,T],\omega\in\Omega}$. Using these pathwise mild solutions, we can show the existence of a pullback attractor for reaction-diffusion equations with additive noise. The pathwise mild solutions are even more generally defined for multiplicative noise. Nevertheless, it is still open whether it is possible to prove the existence of an attractor in this case. Here, we transform the equation into a random partial differential equation, which is not feasible for nonlinear multiplicative noise. Moreover, another open question is whether the two parts of this dissertation can be connected, i.e., whether the results on the pathwise mild solution can be linked to the theory of rough paths to prove, for example, the existence of attractors for random moving domains with a rough path approach.

Appendix A

Sobolev spaces and interpolation theory

Most of the calculations in this thesis use L^p-based Sobolev spaces for $p \geq 2$. In particular, the fractional power spaces introduced in Appendix B are often given by Bessel potential spaces. Since we aim to investigate boundary value problems, we also end up using Besov spaces, which are the canonical spaces for the range of trace operators. Therefore, we introduce in the first part of this appendix the important function spaces and their properties as far as we need them. The second part of this appendix is devoted to the key results of interpolation theory, which is used in the context of Banach scales and is strongly connected to function spaces.

For a more rigorous introduction to the following topics, we recommend [AF03, BG26] for function spaces and [BL76, Tri78, Lun18] for interpolation theory.

Let $\mathcal{E}$ be a Banach space. Recall the definitions of basic function and distribution spaces, and further fix an open subset $\mathcal{O} \subset \mathbb{R}^n$ which can be identical to $\mathbb{R}^n$ for some dimension $n \in \mathbb{N}$.

Definition A.1. i) The space of test functions $\mathscr{D}(\mathcal{O}) \coloneqq \mathcal{C}_c^\infty(\mathcal{O};\mathbb{C})$ consists of all smooth functions with compact support, endowed with the topology where a sequence $(f_k)_{k\in\mathbb{N}_0} \subset \mathscr{D}(\mathcal{O})$ converges to zero if and only if there exists a compact set $K \subset \mathcal{O}$ satisfying $\operatorname{supp}(f_k) \subset K$ for every $k \in \mathbb{N}_0$, and

$$\sup_{x\in K} |\partial^\alpha f_k(x)| \to 0$$

as $k \to \infty$ for all $\alpha \in \mathbb{N}_0^n$, with $\partial^\alpha \coloneqq \frac{\partial^{\alpha_1}}{\partial x_1} \cdot \ldots \cdot \frac{\partial^{\alpha_n}}{\partial x_n}$.

ii) The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ contains all smooth functions $f \in \mathcal{C}^\infty(\mathbb{R}^n;\mathbb{C})$ which are rapidly decaying, i.e. that

$$[f]_{N,\mathscr{S}} \coloneqq \max_{|\alpha|\leq N} \sup_{x\in\mathbb{R}^n} \left(1 + |x|^N\right) |\partial^\alpha f(x)| < \infty,$$

for every $N \in \mathbb{N}_0$, where $|\alpha| \coloneqq \sum_{k=0}^n \alpha_i$ for $\alpha \in \mathbb{N}_0^n$. $\mathscr{S}(\mathbb{R}^n)$ is endowed with the locally convex topology generated by the family of semi-norms $\left([\cdot]_{N,\mathscr{S}}\right)_{N\in\mathbb{N}_0}$.

iii) The space of distributions $\mathscr{D}'(\mathcal{O})$ is given by the topological dual space of $\mathscr{D}(\mathcal{O})$, and the space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual space of $\mathscr{S}(\mathbb{R}^n)$.

A.1 Sobolev spaces

We begin by recalling the definition of a Bochner space, which is the generalization of a Lebesgue space for functions with values in a Banach space.

Given a measure space $(I, \mathscr{I}, \mu)$, a function $f\colon I \to \mathcal{E}$ is strongly μ-measurable if and only if f is μ-almost surely the limit of simple functions of the form $\sum \mathbb{1}_{A_k} \otimes x_k$, where only finitely many terms are nonzero, $x_k \in \mathcal{E}$ and every $A_k \in \mathscr{I}$ has finite measure $\mu(A_k) < \infty$. Then the

Bochner space $\mathrm{L}^p(I;\mathcal{E})$ is given by all strongly μ-measurable functions such that

$$\int_I \|f(x)\|_{\mathcal{E}}^p \, \mathrm{d}\mu(x) < \infty,$$

for $p \in [1,\infty]$, see for example [HvNVW16, Chapter 1]. For measure spaces endowed with a Lebesgue measure λ, we use $\mathrm{d}x$ instead of $\mathrm{d}\lambda(x)$. Then the Sobolev space $\mathrm{W}^{k,p}(\mathcal{O};\mathcal{E})$ is defined by

$$\mathrm{W}^{k,p}(\mathcal{O};\mathcal{E}) \coloneqq \left\{ f \in \mathrm{L}^p(\mathcal{O};\mathcal{E}) : \partial^\alpha f \in \mathrm{L}^p(\mathcal{O};\mathcal{E}) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ such that } |\alpha| \le k \right\},$$

where in this case $\partial^\alpha f$ is the weak derivative of f. The canonical Sobolev norm is then given by

$$\|f\|_{\mathrm{W}^{k,p}(\mathcal{O};\mathcal{E})} \coloneqq \begin{cases} \left(\sum_{|\alpha|\le k} \|\partial^\alpha f\|_{\mathcal{E}}^p\right)^{\frac{1}{p}}, & p \in [1,\infty), \\ \sup_{|\alpha|\le k} \|\partial^\alpha f\|_{\mathcal{E}}^p, & p = \infty. \end{cases}$$

Remark A.2. For the following definitions, we fix $\mathcal{E} \coloneqq \mathbb{R}$, which simplifies the presentation. In this case, denote the function spaces by $\mathrm{L}^p(\mathcal{O}), \mathrm{W}^{k,p}(\mathcal{O})$, and so on. Note that it is possible to extend all of the following function spaces to an arbitrary Banach space $\mathcal{E}$. For more details, we refer to [HvNVW23, Chapter 14].

Definition A.3. For $s \in \mathbb{R}$ and $p \in (1,\infty)$ the Bessel potential space is given by

$$\mathrm{H}^{s,p}(\mathbb{R}^n) \coloneqq \left\{ f \in \mathscr{S}'(\mathbb{R}^n) : \|f\|_{\mathrm{H}^{s,p}(\mathbb{R}^d)} \coloneqq \left\| \mathscr{F}^{-1}(1+|\cdot|^2)^{\frac{s}{2}} \mathscr{F} f \right\|_{\mathrm{L}^p(\mathbb{R}^n)} < \infty \right\}.$$

It is possible to show that $\mathrm{H}^{k,p}(\mathbb{R}^d) = \mathrm{W}^{k,p}(\mathbb{R}^n)$ for $k \in \mathbb{N}$, see [Tri78, Theorem 2.3.3].

A different approach to fractional order Sobolev spaces is based on a generalization of the Hölder condition to the L^p-setting. So let $(\zeta_k)_{k\in\mathbb{N}_0} \subset \mathscr{D}(\mathbb{R}^n)$ be a sequence of test functions which have

- dyadic support, i.e. $\operatorname{supp}(\zeta_k) \subset \left\{ x \in \mathbb{R}^n : 2^{k-1} < |x| < 2^{k+1} \right\}$ for $k \in \mathbb{N}$ and $\operatorname{supp}(\zeta_0) \subset B_2(0)$,
- a strictly positive sum, so there exists a constant $C > 0$ such that $\sum_{k\in\mathbb{N}_0} \zeta(x) \ge C$ for every $x \in \mathbb{R}^n$,
- and satisfies the Michlin condition, i.e. for every $\alpha \in \mathbb{N}_0^d$ we have a constant $m_\alpha > 0$ such that

  $$|x|^{|\alpha|} \, |\partial^\alpha \zeta_k(x)| \le m_\alpha$$

  for every $x \in \mathbb{R}^n$.

The last property yields that the associated operators

$$\operatorname{op}(\zeta_k) \colon \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n), u \mapsto \mathscr{F}^{-1}\zeta_k \mathscr{F} u$$

are bounded in $\mathrm{L}^p(\mathbb{R}^n)$ due to the Michlin multiplier theorem, see [Gra14, Theorem 6.2.7].

Definition A.4. i) For $p, q \in [1,\infty]$ and $s \in \mathbb{R}$ define the Besov spaces by

$$\mathrm{B}^s_{p,q}(\mathbb{R}^n) \coloneqq \left\{ u \in \mathscr{S}'(\mathbb{R}^n) : \|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} \coloneqq \|(\operatorname{op}(\zeta_k)u)_{k\in\mathbb{N}_0}\|_{l^s_q(\mathrm{L}^p(\mathbb{R}^n))} < \infty \right\},$$

where $l_q^s(\mathrm{L}^p(\mathbb{R}^n))$ is the space of all dyadic weighted sequences $(x_k)_{k\in\mathbb{N}_0} \subset \mathrm{L}^p(\mathbb{R}^n)$ endowed with the norm

$$\|(x_k)_{k\in\mathbb{N}_0}\|_{l_q^s(\mathrm{L}^p(\mathbb{R}^n))} := \begin{cases} \left(\sum_{k\in\mathbb{N}_0} \left(2^{sk} \|x_k\|_{\mathrm{L}^p(\mathbb{R}^n)}\right)^q\right)^{\frac{1}{q}}, & q \in [1,\infty), \\ \sup_{k\in\mathbb{N}_0} 2^{sk} \|x_k\|_{\mathrm{L}^p(\mathbb{R}^n)}, & q = \infty. \end{cases}$$

ii) For $p \in [1,\infty]$ and $s \geq 0$ define the Sobolev–Slobodeckij spaces by

$$\mathrm{W}^{s,p}(\mathbb{R}^n) := \begin{cases} \mathrm{H}^{s,p}(\mathbb{R}^n), & s \in \mathbb{N}_0, \\ \mathrm{B}^s_{p,p}(\mathbb{R}^n), & s \in (0,\infty) \setminus \mathbb{N}. \end{cases}$$

Remark A.5. The function spaces can also be generalized to an open set $\mathcal{O} \subset \mathbb{R}^n$ with $\mathcal{C}^\infty$-boundary via restriction. For example, define

$$\mathrm{H}^{s,p}(\mathcal{O}) := \left\{ f \in \mathscr{D}'(\mathcal{O}) : \text{There exists a } \tilde{f} \in \mathrm{H}^{s,p}(\mathbb{R}^n) \text{ such that } f = \tilde{f}|_{\mathcal{O}} \right\},$$

with the canonical norm

$$\|f\|_{\mathrm{H}^{s,p}(\mathcal{O})} := \inf \left\{ \|\tilde{f}\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} : f = \tilde{f}|_{\mathcal{O}} \right\}.$$

The same way we can define $\mathrm{B}^s_{p,q}(\mathcal{O})$ and $\mathrm{W}^{s,p}(\mathcal{O})$.

To investigate the dual space of Sobolev spaces, it is useful to consider the closure of smooth functions with respect to the Sobolev norm.

Definition A.6. For $s \in \mathbb{R}, p > 1$ and $q \in [1,\infty]$ define

$$\mathrm{W}^{s,p}_0(\mathcal{O}) := \overline{C_0^\infty}^{\|\cdot\|_{\mathrm{W}^{s,p}(\mathcal{O})}}, \tag{A-1}$$

and similarly define $\mathrm{H}^{s,p}_0(\mathcal{O})$.

As mentioned before, the Sobolev–Slobodeckij space generalizes the Hölder condition. In particular, this can be observed using the following equivalent norm.

Proposition A.7. ([Tri78, Equation 2.5.1 (15)]) *For every $s = k + s_0 \geq 0$ with $k \in \mathbb{N}_0$ and $s_0 \in (0,1)$, we have that*

$$\|f\|_{\mathrm{W}^{k,p}(\mathcal{O})} + \sum_{|\alpha|=k} \underbrace{\left(\int_{\mathcal{O}} \int_{\mathcal{O}} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|^p}{|x-y|^{n+s_0 p}} \,\mathrm{d}x\mathrm{d}y \right)}_{:=[f]^p_{\mathrm{W}^{\alpha,p}(\mathcal{O})}}$$

is an equivalent norm on $\mathrm{W}^{s,p}(\mathcal{O})$.

Corollary A.8. ([FV10, Corollary A.2-A.3]) *Let $[u,v] \subset \mathbb{R}$ be an interval. Then for every $f \in \mathcal{C}([u,v];\mathbb{R})$ we have*

$$[f]_{\alpha-\frac{1}{p},[u,v]} \lesssim_{\alpha,p} [f]_{\mathrm{W}^{\alpha,p}((u,v))}, \tag{A-2}$$

$$[f]_{\frac{1}{\alpha}\text{-var},[u,v]} \lesssim_{\alpha,p} |v-u|^{\alpha-\frac{1}{p}} [f]_{\mathrm{W}^{\alpha,p}((u,v))}. \tag{A-3}$$

Next, we state the Poincaré inequality, which gives us an equivalent norm on $\mathrm{W}^{s,p}_0(\mathcal{O})$.

Theorem A.9. ([AF03, Theorem 6.30]) *If $\mathcal{O}$ is bounded in at least one direction, then we have for any $p \in [1,\infty)$ and $u \in \mathrm{W}^{1,p}(\mathcal{O})$ the inequality*

$$\|u\|_{\mathrm{L}^p(\mathcal{O})} \lesssim \|\nabla u\|_{\mathrm{L}^p(\mathcal{O})} \leq \|u\|_{\mathrm{W}^{1,p}(\mathcal{O})} . \tag{A-4}$$

Throughout this thesis, we also use several results concerning Sobolev embeddings, dual spaces, and trace operators.

Proposition A.10. ([AF03, Theorem 7.34]) *Let $p \geq 1$. Then we have for $s \geq 0$ satisfying $sp < n$ the embeddings*

$$\mathrm{W}^{s,p}(\mathcal{O}) \hookrightarrow \mathrm{L}^q(\mathcal{O}) \quad \text{for} \quad 1 \leq q \leq \frac{np}{n-sp},$$

and for $s \geq 0$ satisfying $sp > n$

$$\mathrm{W}^{s,p}(\mathcal{O}) \hookrightarrow \mathcal{C}(\overline{\mathcal{O}}).$$

Proposition A.11. ([Tri78, Theorem 4.8.2])

i) Let $p, q \geq 1$ and $s \in ({}^1\!/_p - 1, {}^1\!/_p)$, then the dual spaces are given by $\big(\mathrm{H}^{s,p}(\mathcal{O})\big)' = \mathrm{H}^{-s,p'}(\mathcal{O})$ and $\big(\mathrm{B}^s_{p,q}(\mathcal{O})\big)' = \mathrm{B}^{-s}_{p',q'}(\mathcal{O})$, where ${}^1\!/_p + {}^1\!/_{p'} = 1 = {}^1\!/_q + {}^1\!/_{q'}$. In particular, this means that $\big(\mathrm{W}^{s,p}(\mathcal{O})\big)' = \mathrm{W}^{-s,p'}(\mathcal{O})$.

ii) Let $p \in (1,\infty)$ and $s > {}^1\!/_p$ such that $s - {}^1\!/_p$ is not an integer, then we have $\big(\mathrm{H}^{s,p}_0(\mathcal{O})\big)' = \mathrm{H}^{-s,p'}(\mathcal{O})$ and $\big(\mathrm{W}^{s,p}_0(\mathcal{O})\big)' = \mathrm{W}^{-s,p'}(\mathcal{O})$, where ${}^1\!/_p + {}^1\!/_{p'} = 1$.

Proposition A.12. ([Tri92, Theorem 4.4.2 & Remark 5.1.3]) *Let $n \geq 2, p, q \in [2,\infty]$ and $s > {}^1\!/_p$. Then, the trace operator $\gamma_\partial \colon \mathrm{H}^{s,p}(\mathcal{O}) \to \mathrm{B}^{s-1/p}_{p,p}(\partial\mathcal{O}), u \mapsto u|_{\partial\mathcal{O}}$ is bounded.*

A.2 Interpolation theory

An interpolation space is, in some sense, the mathematical description of a space between two general Banach spaces. This specification can be realized in various ways. The most common approaches are the complex and real interpolation spaces. Before we briefly explain these concepts, let us introduce the general notion of an interpolation functor.

Definition A.13. We call $(\mathcal{E}_0, \mathcal{E}_1)$ an interpolation pair if $\mathcal{E}_0$ and $\mathcal{E}_1$ are Banach spaces and there exists a topological Hausdorff space $\mathcal{G}$ such that $\mathcal{E}_0, \mathcal{E}_1 \subset \mathcal{G}$. A map $\{\cdot,\cdot\}$ is called interpolation functor if it assigns to every interpolation pair $(\mathcal{E}_0, \mathcal{E}_1)$ an interpolation space $\{\mathcal{E}_0, \mathcal{E}_1\}$ such that for all interpolation pairs $(\mathcal{E}_0, \mathcal{E}_1)$ and $\big(\widetilde{\mathcal{E}}_0, \widetilde{\mathcal{E}}_1\big)$ we have

i) $\mathcal{E}_0 \cap \mathcal{E}_1 \hookrightarrow \{\mathcal{E}_0, \mathcal{E}_1\} \hookrightarrow \mathcal{E}_0 + \mathcal{E}_1$ and
ii) for every $A \in \mathcal{L}\big(\mathcal{E}_0; \widetilde{\mathcal{E}}_0\big) \cap \mathcal{L}\big(\mathcal{E}_1; \widetilde{\mathcal{E}}_1\big)$ the restriction is again a linear bounded operator, i.e.

$$A|_{\{\mathcal{E}_0,\mathcal{E}_1\}} \in \mathcal{L}\left(\{\mathcal{E}_0, \mathcal{E}_1\}; \{\mathcal{E}_0, \widetilde{\mathcal{E}}_1\}\right). \tag{A-5}$$

Definition A.14. ([Lun18, Definition 2.3]) Let $(\mathcal{E}_0, \mathcal{E}_1)$ be an interpolation pair. Then define for $\alpha \in (0,1)$ the complex interpolation space

$$[\mathcal{E}_0, \mathcal{E}_1]_\alpha \coloneqq \{f(\alpha) : f \in \mathcal{F}(\mathcal{E}_0; \mathcal{E}_1)\},$$

where $\mathcal{F}(\mathcal{E}_0; \mathcal{E}_1)$ is the space of all holomorphic functions

$$f \colon \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\} \to \mathcal{E}_0 + \mathcal{E}_1$$

which are continuously bounded up to the boundary, and for which the norm

$$\|f\|_{\mathcal{F}(\mathcal{E}_0;\mathcal{E}_1)} := \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{\mathcal{E}_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{\mathcal{E}_1}\right\} \tag{A-6}$$

is finite, where i denotes the imaginary unit. The norm on $[\mathcal{E}_0,\mathcal{E}_1]_\alpha$ is given by

$$\|x\|_{[\mathcal{E}_0,\mathcal{E}_1]_\alpha} := \inf_{\substack{f(\alpha)=x\\ f\in\mathcal{F}(\mathcal{E}_0;\mathcal{E}_1)}} \|f\|_{\mathcal{F}(\mathcal{E}_0;\mathcal{E}_1)}\,. \tag{A-7}$$

In particular, every complex interpolation space is also a Banach space and the interpolation inequality

$$\|x\|_{[\mathcal{E}_0,\mathcal{E}_1]_\alpha} \leq \|x\|_{\mathcal{E}_0}^{1-\alpha}\,\|x\|_{\mathcal{E}_1}^{\alpha}\,, \tag{A-8}$$

holds for every $x \in \mathcal{E}_0 \cap \mathcal{E}_1$, see [Lun18, Corollary 2.8].

Lemma A.15. ([Tri78, Theorem 4.3.1])

i) For every $s_0, s_1 \in \mathbb{R}, p_0, p_1 \in (1,\infty)$ and $\alpha \in (0,1)$ such that $s_0 \neq s_1$ we obtain

$$[\mathrm{H}^{s_0,p_0}(\mathcal{O}), \mathrm{H}^{s_1,p_1}(\mathcal{O})]_\alpha = \mathrm{H}^{s,p}(\mathcal{O}),$$

where $s := (1-\alpha)s_0 + \alpha s_1$ and $\frac{1}{p} := \frac{1-\alpha}{p_0} + \frac{\alpha}{p_1}$.

ii) For every $s_0, s_1 \in \mathbb{R}, p \in (1,\infty), q_1, q_2 \in [1,\infty]$ and $\alpha \in (0,1)$ such that $s_0 \neq s_1$ and $q_0 < \infty$ we obtain

$$\left[\mathrm{B}^{s_0}_{p,q_0}(\mathcal{O}), \mathrm{B}^{s_1}_{p,q_1}(\mathcal{O})\right]_\alpha = \mathrm{B}^{s}_{p,q}(\mathcal{O}),$$

where $s := (1-\alpha)s_0 + \alpha s_1$ and $\frac{1}{q} := \frac{1-\alpha}{q_0} + \frac{\alpha}{q_1}$.

Appendix B

Sectorial operators and analytic semigroups

In order to analyze mild solutions of stochastic partial differential equations, it is necessary to consider analytic semigroups and their generators. In this appendix, we collect the definition of sectorial operators, which essentially is the class of operators that generate an analytic semigroup, and provide some examples also used in this thesis. Furthermore, we highlight the scale of fractional power spaces, which is used in Appendix C to define the extrapolation-interpolation scale. The non-autonomous version, in the form of parabolic evolution families, is treated in Appendix D. Further information on analytic semigroups and their generators can be found, for example, in [Lun95], [Paz83, Section 2.5], [HvNVW17, Appendix G.5], [EN00, Section II.4] and [Vra03, Chapter 7].

B.1 Definition and basic properties

Let $\mathcal{E}$ be a complex Banach space. Note that every complex Banach space can be treated as a real one by restricting the scalars to $\mathbb{R}$. Further, the case of real Banach spaces can be treated by complexification $\mathcal{E}_{\mathbb{C}}$, which is the Cartesian product $\mathcal{E} \times \mathcal{E}$ equipped with the scalar product

$$(a + ib) \cdot (e, f) := (ax - by, ay + bx).$$

It is not in the scope of this thesis to discuss this topic further; for more information, see [HvNVW16, Appendix B.4].

Definition B.1. A densely defined operator $A\colon D(A) \subset \mathcal{E} \to \mathcal{E}$ is said to be sectorial if there exists an angle $\vartheta \in (\frac{\pi}{2}, \pi)$ such that

$$\Sigma_{w_A,\vartheta} := \left\{\mu \in \mathbb{C} \setminus \{w_A\} : |\arg(\mu - w_A)| < \vartheta\right\} \subset \rho(A),$$

and for every $\mu \in \Sigma_{w_A,\vartheta}$ the resolvent fulfills

$$\left\|(\mu - A)^{-1}\right\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \lesssim |\mu - w_A|^{-1}. \tag{B-1}$$

The maximal angle ϑ_A such that $\Sigma_{w_A,\vartheta} \subset \rho(A)$ and (B-1) holds is called the spectral angle and w_A spectral bound of A.

Remark B.2. i) If A is a sectorial operator with spectral bound w_A, then the shifted operator $A - w_A$ is again sectorial with spectral bound 0 and the same spectral angle.

ii) Since the resolvent set of a sectorial operator A is nonempty, A is, in particular, a closed operator. Further, this means that the graph norm $\|\cdot\|_{\mathcal{E}} + \|A\cdot\|_{\mathcal{E}}$ is an equivalent norm on $D(A)$.

iii) If $\mathcal{E}$ is a reflexive Banach space, we can drop the assumption that $D(A) \subset \mathcal{E}$ is dense. This is because it is possible to show that the resolvent inequality and the fact that A is closed are sufficient to prove that A is densely defined, as seen in [Kat59, Corollary 2].

Despite the focus on reflexive spaces in this thesis, we maintain the assumption that the sectorial operators must be densely defined.

Definition B.3. We say the family of operators $(S_t)_{t\geq 0}$ is a C_0-semigroup on $\mathcal{E}$ if $S_t \in \mathcal{L}(\mathcal{E};\mathcal{E})$, $S_{t+s} = S_t S_s$ for all $t, s \geq 0$ and

$$\lim_{t\to\infty} S_t x = x,$$

for every $x \in \mathcal{E}$. Further, $(S_t)_{t\geq 0}$ is an analytic semigroup, if there exists an angle $\vartheta \in (0,\pi]$, $w \in \mathbb{R}$ and an extension $\widetilde{S}_t \colon \overline{\Sigma}_{w,\vartheta} \to \mathcal{L}(\mathcal{E};\mathcal{E})$ of S_t for every $t \geq 0$, which is analytic in $\Sigma_{w,\vartheta}$. If there exists a constant $M_{\tilde{\vartheta}} > 0$ for every $\tilde{\vartheta} \in (0,\vartheta)$ such that $\|\widetilde{S}_z\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \leq M_{\tilde{\vartheta}}$ for all $z \in \Sigma_{w,\tilde{\vartheta}}$ we call $(S_t)_{t\geq 0}$ a bounded analytic semigroup.

These two definitions of analytic semigroups and sectorial operators are strongly connected, because for each sectorial operator A it can be shown that the operator family $(S_t)_{t\geq 0}$ given by the Dunford integral

$$S_t \coloneqq \frac{1}{2\pi i}\int_{w_A+\gamma} e^{t\lambda}(\lambda - A)^{-1}\,\mathrm{d}\lambda,$$

is an analytic semigroup. The integration is along the curve

$$\gamma \coloneqq (\infty,\varepsilon]e^{-i\eta} \cup \varepsilon e^{i[-\eta,\eta]} \cup e^{i\eta}[\varepsilon,\infty)$$

for some small $\varepsilon > 0$ and $\eta \in (\pi/2, \vartheta_A)$. For a proof, see for example [EN00, Proposition 4.3 and 4.4] and [Sin85, Proposition 1.1] for the case that A is not densely defined. Further, the following results hold for every analytic semigroup generated by A.

Lemma B.4. ([Lun95, Proposition 2.1.1 iii)]) *If A generates an analytic semigroup $(S_t)_{t\geq 0}$, we obtain $\|S_t\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \lesssim e^{tw_A}$ for every $t \geq 0$.*

Based on this lemma, we call semigroups generated by a sectorial operator with negative spectral bound $w_A < 0$ exponentially stable.

Remark B.5. It is easy to see that if A generates the analytic semigroup $(S_t)_{t\geq 0}$, the operator $A - w$ for every $w \in \mathbb{R}\setminus\{0\}$ also generates a semigroup given by $\left(e^{-wt}S_t\right)_{t\geq 0}$. In particular, for $w \coloneqq w_A$, the resulting analytic semigroup is bounded.

There are many theorems about the generation of (analytic) semigroups. In the following, we mention one of Lumer–Phillips [LP61], for contraction semigroups.

Theorem B.6. ([EN00, Theorem II.3.15]) *Let $A\colon D(A) \to \mathcal{E}$ be a densely defined operator which is dissipative, i.e. $\|(\lambda - A)x\|_{\mathcal{E}} \geq \lambda\|x\|_{\mathcal{E}}$ for every $\lambda > 0$ and $x \in D(A)$. Then, the closure of A generates a contractive C_0-semigroup, i.e. $\|S_t\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \leq 1$, if and only if there exists some $\lambda > 0$ such that the range of $\lambda - A$ is dense in $\mathcal{E}$.*

B.2 Fractional power spaces

Analytic semigroups possess the so-called smoothing property, which essentially means that they increase the spatial regularity of the functions to which they are applied. To capture this smoothing, we introduce the fractional power spaces of sectorial operators in this section.

Definition B.7. Let A be the generator of an analytic semigroup $(S_t)_{t\geq 0}$.

i) For $\alpha > 0$ the linear operator $(-A)^{-\alpha}$ is defined by

$$D\big((-A)^{-\alpha}\big) \coloneqq \left\{x \in \mathcal{E} : \int_0^\infty t^{\alpha-1}S_t x\ \mathrm{d}t \ \text{ is convergent}\right\},$$

$$(-A)^{-\alpha}x := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S_t x \, \mathrm{d}t$$

for all $x \in D\big((-A)^{-\alpha}\big)$, where Γ is the Gamma function.

ii) Assume in addition that the semigroup is exponentially stable. For $\alpha > 0$ define the linear operator $(-A)^{\alpha} := ((-A)^{-\alpha})^{-1}$ where the domain $D\big((-A)^{\alpha}\big)$ is given by the range of $(-A)^{-\alpha}$.

To see that the positive powers are well-defined, note that the exponential stability of $(S_t)_{t\geq 0}$ yields that $(-A)^{-\alpha}$ is injective, see [Vra03, Lemma 7.6.5], and therefore the inverse exists. It is further possible to show that the notation is meaningful in the sense that for $\alpha \in \mathbb{Z}$ we end up with the exact powers of the operator, see [Lun95, Subsection 2.2.2].

Proposition B.8. ([Vra03, Theorem 7.6.1]) *Let A be the generator of an exponentially stable analytic semigroup. Then the following statements are true.*

i) For every $\alpha \in \mathbb{R}$, $(-A)^{\alpha}$ is a closed and densely defined operator.
ii) For all $\alpha, \beta \geq$ we have $(-A)^{\alpha+\beta} = (-A)^{\alpha}(-A)^{\beta}$.
iii) For all $\alpha > \beta \geq 0$ we have $D\big((-A)^{\alpha}\big) \hookrightarrow D\big((-A)^{\beta}\big)$.

Since $(-A)^{\alpha}$ is a closed operator, $\|\cdot\|_{\alpha} := \|\cdot\|_{D((-A)^{\alpha})} := \|(-A)^{\alpha}\cdot\|_{\mathcal{E}}$ is an equivalent norm on $D\big((-A)^{\alpha}\big)$ provided $0 \in \rho(A)$. We can now state the aforementioned smoothing property of the semigroup.

Theorem B.9. ([Vra03, Theorem 7.7.2]) *Let A be the generator of an exponentially stable analytic semigroup. Then the following statements hold.*

i) For every $t \geq 0$ and $\alpha \geq 0$ we have $S_t x \in D\big((-A)^{\alpha}\big)$ for every $x \in \mathcal{E}$.
ii) For every $t \geq 0, \alpha \geq \beta \geq 0$ and $x \in D\big((-A)^{\alpha}\big)$ we have $S_t(-A)^{\alpha}x = (-A)^{\alpha}S_t x$.
iii) For every $\alpha \geq 0$ there exists an $a \in (0, -w_A)$ such that

$$\|S_t x\|_{\alpha} \lesssim_{\alpha,\beta} t^{-(\alpha-\beta)} e^{-at} \|x\|_{\beta} \tag{B-2}$$

holds for every $t > 0$ and $x \in D\big((-A)^{\beta}\big)$.
iv) For every $\alpha, \beta \in (0, 1]$ with $\alpha > \beta$ we have

$$\|S_t x - x\|_{\beta} \lesssim_{\alpha,\beta} t^{\alpha-\beta} \|x\|_{\alpha} \tag{B-3}$$

for every $t > 0$ and $x \in D\big((-A)^{\alpha}\big)$.

Remark B.10. i) The two properties (B-2) and (B-3) reflect the smoothing effect of an analytic semigroup. It is possible to trade between time- and spatial regularity, for times away from 0. This is particularly important, given that the semigroup is not Hölder continuous at 0, which is necessary in the context of mild solutions. However, if we allow a certain loss of regularity in space, we can circumvent this issue using these smoothing properties.

ii) As we have seen in Remark B.5, it is possible to consider the shift $A - w$ of an arbitrary sectorial operator by some $w \in \mathbb{R}$, and if $w \geq 0$ is big enough, we end up with an exponentially stable semigroup. Then the fractional powers of the shifted operator $D\big((w-A)^{\alpha}\big)$ are well-defined for all $\alpha \in \mathbb{R}$. Therefore, we can define fractional power spaces for every sectorial operator, not just for the exponentially stable ones. For convenience, we omit the shift in the notation $D\big((-A)^{\alpha}\big)$.

B.3 Bounded imaginary powers

A special case of sectorial operators is those that have bounded imaginary powers. This class of operators is particularly important for parabolic problems in the context of maximal regularity, as seen in [Ama95, DHP03, AV22] and the references therein.

Definition B.11. A sectorial operator $A\colon D(A) \subset \mathcal{E} \to \mathcal{E}$ is said to have bounded imaginary powers if A^{is} is bounded for every $s \in \mathbb{R}$, and there exists a constant $C > 0$ such that $\left\|A^{is}\right\|_{\mathcal{L}(\mathcal{E};\mathcal{E})} \leq C$ for every $|s| \leq 1$. Here, i denotes the imaginary unit.

In particular, for this class of operators, the fractional power spaces of A can be characterized using complex interpolation.

Theorem B.12. ([Tri78, Theorem 1.15.3]) *Suppose $A\colon D(A) \subset \mathcal{E} \to \mathcal{E}$ has bounded imaginary powers. Then, fractional power spaces are well-defined, and we obtain*

$$[D(A^{\alpha_0}), D(A^{\alpha_1})]_\alpha = D\big(A^{(1-\alpha)\alpha_0 + \alpha\alpha_1}\big), \tag{B-4}$$

with equivalent norms, for $0 \leq \alpha_1 < \alpha_2 \leq 1$ and $\alpha \in (0,1)$.

It is known that many differential operators have bounded imaginary powers. Let $\mathcal{O} \subset \mathbb{R}^n$ be an open bounded domain with $\mathcal{C}^\infty$-boundary and define the Banach space $E = \mathrm{L}^p(\mathcal{O})$ for some $p \geq 1$. Then consider the differential operators of order $2m$ given by

$$\mathcal{A} \coloneqq \sum_{|\beta| \leq 2m} a_\beta(x)\,(-i\mathrm{D})^\beta\,, \qquad \mathcal{B}_j \coloneqq \sum_{|\beta| \leq m_j} b_{j,\beta}(x)(-i\mathrm{D})^\beta \tag{B-5}$$

for $j = 1, \dots, m$ where i is the imaginary unit, $\mathrm{D} \coloneqq (\partial_1, \dots, \partial_n)$, and $m, m_1, \dots, m_m \in \mathbb{N}$ with $m_j < m$ for every $j = 1, \dots, m$. Assume that the pair $\big(\mathcal{A}, (\mathcal{B}_j)_{j=1}^m\big)$ is normally elliptic, which is examined in more detail in Section D.2. Consider the E-realization of $(\mathcal{A}, \mathcal{B})$ given by $A_B u \coloneqq \mathcal{A}u$ for $u \in D(A_B)$ where

$$D(A_B) \coloneqq \big\{u \in \mathrm{H}^{2m,p}(\mathcal{O}) : \mathcal{B}_j u = 0\big\}.$$

To ensure that A_B has bounded imaginary powers, further regularity assumptions on the coefficients are needed.

(R) There exists some $\mu \in (0,1]$, such that $a_\beta \in \mathcal{C}^\mu(\overline{\mathcal{O}})$ for all $|\beta| = 2m$, and $a_\beta \in \mathrm{L}^\infty(\mathcal{O})$ for $|\beta| < 2m$. Further, the coefficients of the boundary operator satisfies $b_{j,\beta} \in \mathcal{C}^{2m-m_j}(\partial\mathcal{O})$ for every $j = 1, \dots, m$ and $|\beta| \leq m_j$.

Theorem B.13. ([DDH$^+$04, Theorem 2.3]) *Assume the coefficients of the normally elliptic boundary value problem $(\mathcal{A}, \mathcal{B})$ satisfy the regularity Assumption **(R)**. Then, there exists a $w \geq 0$ such that $w + A_B$ has bounded imaginary powers. In particular, $w + A_B$ generates an analytic semigroup.*

Appendix C

Abstract Banach scales

A weak solution of parabolic equations is often an element in some space that is larger than the original one, which corresponds to a space of "negative" regularity, for example, $D\big((-A)^{-1}\big)$. Due to this fact, the weak solution is no longer an element of $D(A)$. This is particularly an issue if boundary value problems are the objects of interest. To get around this problem, one idea is to extend the operator A in a way that the "extrapolated" operator A_{-1} acts on $D\big((-A)^{-1}\big)$ instead of $D(A)$. The first mentions of this extrapolation technique were in [Nag, Wal86] and then in more detail in a series of articles by Amann [Ama86a, Ama87, Ama88a, Ama88b]. In particular, Amann went a step further and constructed a whole scale of operators, which act on the scale of function spaces.

This appendix is devoted to the abstract construction of these families that consist of function spaces and operators, as well as their application to the setting primarily dealt with in this thesis.

The initial step is to provide a general definition, as outlined by Amann [Ama95, Chapter V].

Definition C.1. Let $I \in \{\mathbb{N}, \mathbb{Z}, [0,\infty), \mathbb{R}\}$ and $(\mathcal{E}_\alpha, A_\alpha)_{\alpha\in I}$ be a family of Banach spaces $\big(\mathcal{E}_\alpha, \|\cdot\|_{\mathcal{E}_\alpha}\big)$ and closed linear operators $A_\alpha \in \mathcal{L}(\mathcal{E}_{\alpha+1};\mathcal{E}_\alpha)$. Then this family is called an (abstract) Banach scale if $\mathcal{E}_\alpha \hookrightarrow \mathcal{E}_\beta$ for $\alpha > \beta$, and for every $\alpha > \beta$ the equality

$$j_\alpha^\beta A_\alpha = A_\beta j_{\alpha+1}^{\beta+1}, \tag{C-1}$$

holds, where $j_\alpha^\beta \colon \mathcal{E}_\alpha \to \mathcal{E}_\beta$ denotes the embedding. If all embeddings are dense or compact, we refer to the Banach scale as densely or compactly injected.

Remark C.2. The key assumption in this definition is (C-1). In particular, (C-1) holds if and only if the diagram

$$\begin{array}{ccc} \mathcal{E}_{1+\alpha} & \hookrightarrow & \mathcal{E}_{1+\beta} \\ A_\alpha \big\downarrow & & \big\downarrow A_\beta \\ \mathcal{E}_\alpha & \hookrightarrow & \mathcal{E}_\beta \end{array}$$

is commutative for every $\alpha > \beta$. A direct consequence of this fact, is that $D(A_\alpha) = \mathcal{E}_{1+\alpha} \hookrightarrow \mathcal{E}_{1+\beta} = D(A_\beta)$ and $A_\beta x = A_\alpha x$ for every $x \in \mathcal{E}_{1+\alpha}$, which means $A_\alpha \subset A_\beta$.

Recall that the $\mathcal{E}_0$-realization $A|_{\mathcal{E}_0}$ of $A : D(A) \subset \mathcal{E} \to \mathcal{E}$ is given by

$$A|_{\mathcal{E}_0} \colon D\left(A|_{\mathcal{E}_0}\right) := \{x \in \mathcal{E}_0 \cap D(A) : Ax \in \mathcal{E}_0\} \to \mathcal{E}_0, x \mapsto Ax,$$

for two Banach spaces $\mathcal{E}_0 \hookrightarrow \mathcal{E}$.

Lemma C.3. ([Ama95, Remark V.1.1.2]) *A densely injected abstract Banach scale* $(\mathcal{E}_\alpha, A_\alpha)_{\alpha\in I}$ *is completely determined by* $A_0 \colon \mathcal{E}_1 \to \mathcal{E}_0$.

Proof. Fix $\alpha > \beta$. Since A_α is a closed operator, it is closable in $\mathcal{E}_\beta$. Due to $A_\alpha \subset A_\beta$ the closure equals the restriction of A_β to the closure of $\mathcal{E}_{\alpha+1}$ in $\mathcal{E}_{\beta+1}$. Since $\mathcal{E}_{\alpha+1} \hookrightarrow \mathcal{E}_{\beta+1}$ is densely embedded, this means that $\overline{A_\alpha}^{\mathcal{E}_\beta} = A_\beta$. On the other side, A_α is the $\mathcal{E}_{\alpha+1}$-realization of A_β. So every operator A_α can be written as a closure ($\alpha < 0$) or a realization ($\alpha > 0$) of A_0. □

Remark C.4. In particular, A_{-1} can be written as a closure of A_0, so it is an extension of the operator, acting on a larger space $\mathcal{E}_{-1}$. This is a key idea behind the definition of the abstract Banach scales, as mentioned at the beginning of this appendix.

C.1 Interpolation-extrapolation scales

Let $\mathcal{E}$ be a Banach space and $A\colon D(A) \subset \mathcal{E} \to \mathcal{E}$ some sectorial operator with negative spectral bound $w_A < 0$. Then A is, in particular, a closed operator, and the fractional power spaces are well-defined. So, define the spaces $\mathcal{E}_k := D\big((-A)^k\big)$ with norm $\|\cdot\|_k := \big\|(-A)^k\cdot\big\|_{\mathcal{E}}$ for $k > 0$ and set $\mathcal{E}_0 := \mathcal{E}$. Since these spaces are continuously and densely embedded, it is justified to define the $\mathcal{E}_k$-realization of A as A_k.

Theorem C.5. ([Ama95, Theorem V.1.2.1]) *Let A be a sectorial operator with a negative spectral bound. Then $(\mathcal{E}_k, A_k)_{k\in\mathbb{N}_0}$ is an densely injected Banach scale, called the power scale generated by A.*

To extend this (one-sided) power scale to negative values, note that $\mathcal{E}_0$ and $A_0 = A$ can be constructed using $\mathcal{E}_1$ and A_1. Indeed, $\mathcal{E}_1$ endowed with the norm $\big\|(-A_1)^{-1}\cdot\big\|$ is a dense linear subspace of $\mathcal{E}_0$ such that $\big\|(-A_1)^{-1}\cdot\big\|$ and $\|\cdot\|_0$ are equivalent. Therefore, the completion of $\mathcal{E}_1$ endowed with the norm $\big\|(-A_1)^{-1}\cdot\big\|$ is $\mathcal{E}_0$. The operator A_0 is then the continuous extension of A_1.

This motivates the definition of an extrapolation space and operator by

$$\mathcal{E}_{-1} := \overline{\mathcal{E}}^{\|\cdot\|_{-1}} \quad \text{and} \quad A_{-1} := \overline{A_0}^{\mathcal{E}_{-1}},$$

where $\|\cdot\|_{-1} := \big\|(-A_0)^{-1}\big\|_0$. Replacing A_0 here by A_{-1}, leads to the definition of $\mathcal{E}_{-2}$ and A_{-2}. Iterating this procedure leads to

$$\mathcal{E}_{-m} := \overline{\mathcal{E}}^{\|\cdot\|_{-m}}, \qquad A_{-m} := \overline{A_{-m+1}}^{\mathcal{E}_{-m}}, \tag{C-2}$$

for $m \in \mathbb{N}$, with $\|\cdot\|_{-m} := \big\|(-A_{-m+1})^{-1}\big\|_0$.

Remark C.6. Even if this construction can be carried out for all $m \in \mathbb{N}$, it should be treated with caution. As new steps E_k are incorporated, the embeddings in Definition C.1 can become incompatible. To address this, each existing extrapolation space must be substituted with an isomorphic copy. For more detailed information, see [Ama95, Page 263].

Theorem C.7. ([Ama95, Theorem V.1.3.2]) *Let A be a sectorial operator with negative spectral bound, $m \in \mathbb{N}$ and $(\mathcal{E}_k, A_k)_{k\in\mathbb{N}_0}$ the power scale generated by A. Let A_{-k} and $\mathcal{E}_{-k}$ for $k = 1, \ldots, m$ be defined as in (C-2). Then $(\mathcal{E}_k, A_k)_{k\in\mathbb{Z}, k\geq -m}$ is a densely injected Banach scale, called the extrapolated power scale of order m generated by A.*

An important consequence is that every operator A_k for $k = -m, \ldots$ inherits the properties of the generator A, see [Ama95, Lemma V.1.3.7]. That is, if A is sectorial, then so is A_k; if A has bounded imaginary powers, then so does A_k.

We still need to close the respective gaps between two integers using fractionally indexed spaces and operators. As the name of the subsection suggests, we use interpolation for this purpose. In the following, and also in the course of this thesis, we limit ourselves to complex interpolation. However, it is possible to define Banach scales using a general exact interpolation function; this also applies to real interpolation.

Theorem C.8. ([Ama95, Theorem 1.5.1]) *Let A be a sectorial operator with negative spectral bound, $m \in \mathbb{N}$ and $(\mathcal{E}_k, A_k)_{k\in\mathbb{Z},k\geq -m}$ the extrapolated power scale of order m generated by A. Then $(\mathcal{E}_\alpha, A_\alpha)_{\alpha\geq -m}$ is an abstract Banach scale, called the interpolation-extrapolation scale of order m generated by A, where*

$$\mathcal{E}_{\beta+k} := [\mathcal{E}_k, \mathcal{E}_{k+1}]_\beta \quad \text{and} \quad A_{\beta+k} := \mathcal{E}_{\beta+k}\text{-realization of } A_k,$$

for $k \in \mathbb{Z}, k \geq -m$ and $\beta \in (0,1)$.

If A is an operator with bounded imaginary powers, the Banach spaces can be characterized by the fractional power spaces.

Theorem C.9. ([Ama95, Theorem V.1.5.4 & Proposition V.1.5.5]) *Let A be a linear operator with bounded imaginary powers, $m \in \mathbb{N}$ and $(\mathcal{E}_\alpha, A_\alpha)_{\alpha\geq -m}$ the interpolation-extrapolation scale of order m, generated by A. Then for every $\alpha \geq -m$ the operator A_α has also bounded imaginary powers and satisfies $(A_\alpha)^{\alpha-\beta} \in \mathcal{L}(\mathcal{E}_\alpha; \mathcal{E}_\beta)$, which is also an isomorphism, for $\beta > \alpha$. Further, the spaces $\mathcal{E}_\alpha$ can be characterized using the fractional powers of A*

$$\mathcal{E}_\alpha = \begin{cases} D\big((-A)^\alpha\big), & \alpha \geq 0, \\ \overline{\mathcal{E}_0}^{\|\cdot\|_\alpha}, & -m \leq \alpha < 0, \end{cases} \tag{C-3}$$

with $\|\cdot\|_\alpha := \|(-A)^\alpha \cdot\|_{\mathcal{E}}$. Moreover, the spaces satisfying the reiteration property, which means that for every $-m \leq \alpha_0 < \alpha_1 < \infty$ and $\alpha \in (0,1)$, we have

$$[\mathcal{E}_{\alpha_1}, \mathcal{E}_{\alpha_2}]_\alpha = \mathcal{E}_{(1-\alpha)\alpha_1+\alpha\alpha_2},$$

with equivalent norms.

In particular, this shows that every interpolation-extrapolation scale generated by an operator A with bounded imaginary powers is a monotone family of interpolation spaces in the sense of Definition 2.35.

Characterizing an extrapolated space is often challenging. If $\mathcal{E}$ is reflexive, the extrapolated spaces can at least be calculated using duality arguments. Recall that $\mathcal{E}$ is reflexive if there exists an isometric isomorphism between $\mathcal{E}$ and its bidual $\mathcal{E}''$.

Theorem C.10. ([Ama95, Theorem V.1.5.12]) *Let $\mathcal{E}$ be a reflexive Banach space, A a linear operator with bounded imaginary powers, $m \in \mathbb{N}$ and $(\mathcal{E}_\alpha, A_\alpha)_{\alpha\geq -m}$ the interpolation-extrapolation scale of order m, generated by A. Then $\mathcal{E}_\alpha$ is reflexive for every $\alpha \geq -m$ and the adjoint operator*

$$B_0 := A^* \colon D\big(A^*\big) =: \mathcal{F}_1 \subset \mathcal{E}' \to \mathcal{E}' := \mathcal{F}_0$$

generates an interpolation-extrapolation scale $(\mathcal{F}_\alpha, B_\alpha)_{\alpha\geq -m}$ satisfying

$$(\mathcal{E}_\alpha)' = \mathcal{F}_{-\alpha} \quad \text{and} \quad (A_\alpha)^* = B_{-\alpha}$$

for every $\alpha \geq -m$.

Using Theorem C.10, the extrapolated space is then given by the dual space of $D(A^*)$.

As mentioned before, if A generates an analytic semigroup, then also A_α. To study mild solutions of equations with boundary noise, it is essential to know the connection between extrapolated semigroups and the original semigroup.

Theorem C.11. ([Ama95, Theorem V.2.1.3]) *Let $\mathcal{E}$ be a reflexive Banach space, A be a linear operator with bounded imaginary powers, $m \in \mathbb{N}$ and $(\mathcal{E}_\alpha, A_\alpha)_{\alpha\geq -m}$ the interpolation-extrapolation scale of order m, generated by A. Then A_α generates an analytic semigroup.*

Further, for $\alpha > \beta$, the semigroup generated by A_α is the same as the semigroup generated by A_β restricted to $\mathcal{E}_\alpha$.

Example C.12. *Of particular interest is the interpolation-extrapolation scale generated by a second-order differential equation with corresponding boundary conditions. This is a special case of (B-5) for $m = 1$. Let*

$$\mathcal{A} := \sum_{i,j=1}^{n} \partial_i (a_{ij}\partial_j) + b, \quad \mathcal{B}_N := \sum_{i,j=1}^{n} \nu_i \gamma_\partial a_{ij} \partial_j, \quad \mathcal{B}_D := \gamma_\partial \tag{C-4}$$

where ν is the outer normal vector of $\mathcal{O}$, γ_∂ is the trace and $(a_{ij})_{i,j=1}^n, b$ fulfill ***(R)*** *for $m = 1$.*

The boundary operator is therefore either of Neumann or Dirichlet type. Further, define the L^p-realization of the boundary value problem $(\mathcal{A}, \mathcal{B}_{N,D})$ as

$$A_{N,D} \colon D(A_{N,D}) \subset \mathrm{L}^p(\mathcal{O}) \to \mathrm{L}^p(\mathcal{O}), u \mapsto A_{N,D}u := \mathcal{A}u$$

with domain

$$D(A_{N,D}) := \left\{ u \in \mathrm{H}^{2,p}(\mathcal{O}) : \mathcal{B}_{N,D}u = 0 \right\},$$

where we will mostly neglect the index in the case of Neumann boundary conditions. If $(\mathcal{A}, \mathcal{B}_{N,D})$ is normally elliptic, which is true under certain ellipticity conditions on $(a_{ij})_{i,j=1}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi^i \xi^j \gtrsim |\xi|^2$$

for $x \in \mathcal{O}, \xi \in \mathbb{R}^n$, the realization $A \in \{A_N, A_D\}$ has bounded imaginary powers and generates an analytic semigroup, see B.13.

In that case, it is known that A generates, in particular, an interpolation-extrapolation scale, given by

$$\mathcal{E}_\beta = \begin{cases} \left\{ u \in \mathrm{H}^{2\beta,p}(\mathcal{O}) : \mathcal{B}u = 0 \right\}, & 1 + \frac{1}{p} < 2\beta, \\ \mathrm{H}^{2\beta,p}(\mathcal{O}), & -1 + \frac{1}{p} < 2\beta < 1 + \frac{1}{p}, \\ \left(\mathrm{H}^{-2\beta,p'}(\mathcal{O}) \right)', & -2 + \frac{1}{p} < 2\beta \leq -1 + \frac{1}{p}, \\ \left\{ u \in \mathrm{H}^{-2\beta,p'}(\mathcal{O}) : \mathcal{B}u = 0 \right\}', & -2 \leq 2\beta < -2 + \frac{1}{p}, \end{cases} \tag{C-5}$$

for Neumann boundary conditions and

$$\mathcal{E}_\beta^D = \begin{cases} \left\{ u \in \mathrm{H}^{2\beta,p}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0 \right\}, & \frac{1}{p} < 2\beta, \\ \mathrm{H}^{-2\beta,p}(\mathcal{O}), & -2 + \frac{1}{p} < 2\beta \leq -1 + \frac{1}{p}, \\ \left\{ u \in \mathrm{H}^{-2\beta,p'}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0 \right\}', & m - 2 \leq 2\beta < -2 + \frac{1}{p}, \end{cases} \tag{C-6}$$

for Dirichlet boundary conditions, see for example [Ama93, (7.4),(7.5)].

Appendix D

Parabolic problems

In this appendix, we provide a detailed treatment of deterministic non-autonomous parabolic problems. To be precise, first, we specify some conditions under which a family of non-autonomous operators generates a so-called parabolic evolution family, which is the non-autonomous counterpart of a semigroup, and then determine the resulting properties. We focus on the case where the domain is time-independent; however, we briefly discuss a possible extension at the end of the section.

In the second section, we address parabolic boundary value problems and provide a brief review of the solution theory, without delving into excessive detail. Let $\mathcal{E}_0$ be a reflexive Banach space and $T > 0$.

D.1 Non-autonomous evolution families

We want to investigate problems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} y_t = A(t) y_t, \tag{D-1}$$

where $(A(t))_{t\in[0,T]}$ is a time-dependent family of linear operators. A large number of articles and books are available for studying such time-dependent problems under various conditions; see [Tan79, Ama86b, AT87, Acq88, Yag10] and the references therein.

In the autonomous case, $A(t) = A$, it is known that the solution operator is a semigroup $(S_t)_{t\geq 0}$ and the solution is given by $y_t = S_{t-s} y_s$ for some initial value y_s. By introducing a time dependence, the situation becomes more complex, but it is also possible to specify an equivalent to the semigroup. The main difference is that the so-called evolution family in the non-autonomous case depends on the starting time s and the observed time t. In contrast, the semigroup only depends on one time parameter, namely the elapsed time.

The following conditions are known as the Kato–Tanabe assumptions; for more details see [Paz83, Page 150], [Ama86b] and the references therein. They ensure the existence of a parabolic evolution family.

(A1) The family $\big(A(t)\big)_{t\in[0,T]}$ consists of closed and densely defined operators $A(t)\colon \mathcal{E}_1 \to \mathcal{E}_0$ with time independent domain $D(A) \coloneqq \mathcal{E}_1 \subset \mathcal{E}_0$.

(A2) There exists a $\vartheta \in (\pi/2, \pi)$ such that $\Sigma_{0,\vartheta} \cup \{0\} \subset \rho\big(A(t)\big)$ and

$$\big\|\big(z - A(t)\big)^{-1}\big\|_{\mathcal{L}(\mathcal{E}_i;\mathcal{E}_i)} \lesssim \frac{1}{1+|z|},$$

as well as $\big\|\big(z - A(t)\big)^{-1}\big\|_{\mathcal{L}(\mathcal{E}_0;\mathcal{E}_1)} \lesssim 1$ for all $z \in \Sigma_{0,\vartheta} \cup \{0\}$, $i = 0, 1$ and $t \in [0, T]$.

(A3) There exists a $\nu \in (0, 1]$ such that

$$\|A(t) - A(s)\|_{\mathcal{L}(\mathcal{E}_1;\mathcal{E}_0)} \lesssim |t - s|^{\nu},$$

for all $s, t \in [0, T]$.

In particular, **(A2)** implies that the operator $A(t)$ is sectorial for every $t \in [0, T]$. Note that the boundedness of the resolvent in $\mathcal{L}(\mathcal{E}_1; \mathcal{E}_1)$ and $\mathcal{L}(\mathcal{E}_0; \mathcal{E}_1)$ is added to obtain the more general smoothing property obtained in Theorem D.3.

Theorem D.1. ([Acq88, Theorem 2.3]) *Let $\big(A(t)\big)_{t\in[0,T]}$ satisfy Assumption **(A1)-(A3)**. Then there exists a unique parabolic evolution family $(S_{t,s})_{(s,t)\in\Delta_{[0,T]}}$ of linear operators $S_{t,s} \in \mathcal{L}(\mathcal{E}_0; \mathcal{E}_0) \cap \mathcal{L}(\mathcal{E}_1; \mathcal{E}_1)$ such that the following properties are satisfied:*

i) For all $(s, u, t) \in \Delta^{(3)}_{[0,T]}$ we have

$$S_{t,s} = S_{t,u} S_{u,s}$$

as well as the identity $S_{t,t} = \operatorname{Id}_{\mathcal{E}_0}$.

ii) The mapping $(s, t) \mapsto S_{t,s}$ is strongly continuous, this means

$$\lim_{s\nearrow t} \|S_{t,s}x - x\|_{\mathcal{E}_0} = \lim_{t\searrow s} \|S_{t,s}x - x\|_{\mathcal{E}_0} = 0,$$

if and only if $x \in \overline{\mathcal{E}_1}$.

iii) For $s \leq t$ we have the identity $\frac{\mathrm{d}}{\mathrm{d}t} S_{t,s} = A(t)S_{t,s}$ as well as the estimate

$$\|A(t)S_{t,s}\|_{\mathcal{L}(\mathcal{E}_0;\mathcal{E}_0)} \lesssim (t - s)^{-1},$$

for every $(s, t) \in \Delta_{[0,T]}$.

Next, we want to establish smoothing properties similar to those of analytic semigroups. Due to **(A2)**, every operator $A(t)$ generates an analytic semigroup, so the fractional power spaces $\mathcal{E}_\alpha := D\big((-A(t))^\alpha\big)$ endowed with the norm $\|\cdot\|_\alpha := \big\|\big(-A(t)\big)^\alpha \cdot\big\|_{\mathcal{E}_0}$ are well-defined.

Remark D.2. Assumption **(A1)** requires that the domain of $A(t)$ is independent of t. However, this does not mean that the fractional power spaces are, in general, also independent of t. The easiest way to circumvent this problem is to suppose further that every $A(t)$ has bounded imaginary powers. In that case, the fractional power spaces can be identified using complex interpolation (C-3). This means that for any $\alpha \in (0, 1)$ we have

$$\mathcal{E}_\alpha = [\mathcal{E}_0, D(A)]_\alpha = D\big((-A(t))^\alpha\big)$$

and therefore $\mathcal{E}_\alpha$ does not depend on time.

Theorem D.3. ([GHN21, Theorem 3.9]) *Let $k_-, k_+ \in \mathbb{N}$ and $\big(A(t)\big)_{t\in[0,T]}$ be a family of operators with bounded imaginary powers satisfying Assumption **(A1)-(A3)** where $(\mathcal{E}_0, \mathcal{E}_1)$ is substituted by $(\mathcal{E}_\alpha, \mathcal{E}_{\alpha+1})$ for every $\alpha \in [k_-, k_+]$. Then we have for every $\alpha, \beta \in [k_-, k_+]$ with $\beta \geq \alpha$ and $\varsigma \in [0, 1]$*

$$\begin{aligned} \|(S_{t,s} - Id)x\|_\alpha &\lesssim_{\alpha,\varsigma} |t - s|^\varsigma \|x\|_{\alpha+\varsigma}, \\ \|S_{t,s}x\|_\beta &\lesssim_{\alpha,\beta} |t - s|^{-(\beta-\alpha)} \|x\|_\alpha, \end{aligned} \tag{D-2}$$

for $(s, t) \in \Delta_{[0,T]}$.

To investigate the long-term behavior of evolution equations, it is often useful to assume that the resulting evolution family has an exponential decay. This leads to the two main assumptions in this thesis.

(A) The operator family $\big(A(t)\big)_{t\in[0,T]}$ satisfies Assumption **(A1)-(A3)**, where $(\mathcal{E}_0, \mathcal{E}_1)$ is substituted by $(\mathcal{E}_\alpha, \mathcal{E}_{\alpha+1})$ for every $\alpha \in [k_-, k_+]$, where $k_-, k_+ \in \mathbb{N}$. Further, assume that $A(t)$ has bounded imaginary powers.

$(\mathbf{A_S})$ The evolution family $(S_{t,s})_{(s,t)\in\Delta_{[0,T]}}$ is exponentially stable. This means that there exists a $\tilde{\lambda}_A > 0$ and a constant $C_A \geq 0$ such that

$$\|S_{t,s}\|_{\mathcal{L}(\mathcal{E}_0)} \leq C_A e^{-\tilde{\lambda}_A(t-s)} \leq C_A e^{-\lambda_A(t-s)}, \tag{D-3}$$

for all $t \geq s$ and $0 \leq \lambda_A \leq \tilde{\lambda}_A$.

In particular, Assumption $(\mathbf{A_S})$ is satisfied if every $A(t)$ has a negative spectral bound. As a result, all estimates from Theorem D.3 now also have an exponential decay on the right-hand side.

Similar to (B-2), it is not possible to obtain the smoothing property and the full decay rate in the same inequality.

Corollary D.4. *Let the Assumptions* ***(A)*** *and* $(\mathbf{A_S})$ *be satisfied and fix $\lambda_A < \tilde{\lambda}_A$. Then we have for every $\alpha, \beta \in [k_-, k_+]$ with $\beta \geq \alpha$*

$$\|S_{t,s}x\|_\beta \lesssim_{\alpha,\beta,\lambda_A/\tilde{\lambda}_A} \frac{e^{-\lambda_A(t-s)}}{(t-s)^{\beta-\alpha}} \|x\|_\alpha\,, \tag{D-4}$$

where the right-hand side explodes for $\lambda_A \to \tilde{\lambda}_A$.

Proof. The simultaneous exponential decay and smoothing can be proven similarly to [Vra03, Lemma 7.7.1]. Set $a = \lambda_A/\tilde{\lambda}_A < 1$, then we obtain

$$\begin{aligned}\|S_{t,s}x\|_\beta &\leq \left\|S_{t,t-a(t-s)}\right\|_{\mathcal{L}(\mathcal{E}_\beta;\mathcal{E}_\beta)} \left\|S_{t-a(t-s),s}x\right\|_\beta \\ &\lesssim e^{-\lambda_A(t-s)} \frac{1}{(1-a)^{\beta-\alpha}} (t-s)^{-(\beta-\alpha)} \|x\|_\alpha\,.\end{aligned}$$

□

Lemma D.5. ([PV14, Lemma 2.7]) *Let $\big(A(t)\big)_{t\in[0,T]}$ and $\big(A^*(t)\big)_{t\in[0,T]}$ satisfy the Assumption* ***(A)***. *Then we have for every $\alpha, \beta \in [k_-, k_+]$ with $\beta \geq \alpha$*

$$\left\|S_{t,s}\big(-A(s)\big)^\beta x\right\|_\alpha \lesssim_{\alpha,\beta} (t-s)^{-(\beta-\alpha)} \|x\|_\alpha\,, \tag{D-5}$$

where $x \in \mathcal{E}_\beta$.

Remark D.6. i) Note that for $\beta \geq \alpha$, the space $\mathcal{E}_\beta$ is densely embedded into $\mathcal{E}_\alpha$. Therefore, (D-5) shows that there exists a unique extension of $S_{t,s}\big(-A(s)\big)^\beta$ as a bounded linear operator on $\mathcal{E}_\alpha$. We denote the extension again by $S_{t,s}(-A(s))^\beta$. In particular, we have for $x \in \mathcal{E}_\alpha$

$$\left\|S_{t,s}\big(-A(s)\big)^\beta x\right\|_\alpha \lesssim (t-s)^{-(\beta-\alpha)} \|x\|_\alpha\,. \tag{D-6}$$

ii) Similarly, we can extend $S_{t,s}$ to an operator in $\mathcal{E}_{-\alpha}$. Indeed, $A(t)^\alpha$ is an isomorphism between $\mathcal{E}_\alpha$ and $\mathcal{E}_0$. Hence, for $x \in \mathcal{E}$ there exists an $y \in \mathcal{E}_\alpha$ such that $x = A(t)^\alpha y$, see Theorem C.9. This implies that

$$\begin{aligned}\|S_{t,s}x\|_\mathcal{E} &= \|S_{t,s}\big(-A(s)\big)^\alpha y\|_\mathcal{E} \lesssim (t-s)^{-\alpha}\|y\|_\mathcal{E} \\ &= (t-s)^{-\alpha}\|\big(-A(t)\big)^{-\alpha}x\|_\mathcal{E} = (t-s)^{-\alpha}\|x\|_{-\alpha},\end{aligned}$$

and since $\mathcal{E}_0$ is densely embedded into $\mathcal{E}_{-\alpha}$, we can extend $S_{t,s}$ to an operator from $\mathcal{E}_{-\alpha}$ to $\mathcal{E}$.

Remark D.7. It is also possible to consider non-autonomous evolution equations in the context of time-dependent domains. In this setting, the stated Kato–Tanabe conditions **(A2)**-**(A3)** are not enough to ensure the existence of a parabolic evolution family. With stronger conditions, for example, under the assumptions of Acquistapace–Terreni [AT87, Hypothesis I-II], we can obtain the same results as before, but now with possible time-dependent fractional power spaces. For a detailed discussion on different assumptions for non-autonomous evolution equations, see [AT87, Section 7] and also [Acq88, Yag10].

Example D.8. *Let $\mathcal{O} \subset \mathbb{R}^n$ be an open domain. Consider the time-dependent version of second-order differential operators*

$$\mathcal{A}(t) \coloneqq \sum_{i,j=1}^{n} \partial_i\big(a_{ij}(t)\partial_j\big),$$

where $\big(a_{ij}(t,\cdot)\big)_{i,j=1}^{n} \in \mathcal{C}^\infty\big(\overline{\mathcal{O}};\mathbb{R}^{n\times n}\big)$ satisfies the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(t,x)\xi^i\xi^j \gtrsim |\xi|^2$$

for $x \in \mathcal{O}, \xi \in \mathbb{R}^n$ and $t \in [0,T]$. Further, assume that the coefficients are Hölder continuous $a_{ij}(\cdot,x) \in \mathcal{C}^\mu([0,T];\mathbb{R})$ uniformly in $\mathcal{O}$ for some $\mu \in (0,1)$. Then the Dirichlet realization $A(t)\colon D\big(A(t)\big) \to \mathrm{L}^p(\mathcal{O})$ with

$$D\big(A(t)\big) \coloneqq \big\{u \in \mathrm{H}^{2,p}(\mathcal{O}) : u|_{\partial\mathcal{O}} = 0\big\},$$

*satisfies **(A)**, see [GHN21, Example 2.12 & 3.10] and [CM06].*

D.2 Parabolic boundary value problems

In this section, the theory of parabolic boundary value problems is briefly introduced based on the survey of Amann [Ama93]. For more information, see also [LM72, DHP03] and the references therein. Consider the equation

$$\mathcal{A}u = 0, \quad \mathcal{B}u = g \tag{D-7}$$

where $\mathcal{A}$ and $\mathcal{B}$ are formal differential operators as in (B-5) of order $2m$ with $m_j < m$ for every $j = 1,\dots,m$.

Definition D.9. i) Associate to the operators $\mathcal{A}$ and $\mathcal{B}$ the principal values

$$a_\pi(x,\xi) \coloneqq \sum_{|\beta|=2m} a_\beta(x)\xi^\beta, \qquad b_\pi(\tilde{x},\xi) = \sum_{|\beta|=m_j} b_{j,\beta}(\tilde{x})\xi^\beta$$

for $x \in \overline{\mathcal{O}}, \tilde{x} \in \partial\mathcal{O}$ and $\xi \in \mathbb{R}^n$.

ii) The operator $\mathcal{A}$ is called normally elliptic if

$$\sigma\big(a_\pi(x,\xi)\big) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\},$$

holds for every $x \in \overline{\mathcal{O}}$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$.

iii) A boundary operator $\mathcal{B}$ satisfies the normal complementing condition with respect to $\mathcal{A}$ if $u \equiv 0$ is the only exponentially decaying solution of

$$\begin{aligned}\big(a_\pi(\tilde{x},\xi+\nu(\tilde{x})i\partial_t)) + \lambda\big)u(t) &= 0,\\ b_\pi\big(\tilde{x},\xi+\nu(\tilde{x})i\partial_t)\big)u(0) &= 0\end{aligned}$$

for every $\lambda \in \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, $\tilde{x} \in \partial\mathcal{O}$ and ξ is in the tangential space of $\tilde{x}$ with $\tilde{x}, \xi \neq 0$.

iv) The pair $(\mathcal{A}, \mathcal{B})$ is called a normally elliptic boundary problem if $\mathcal{A}$ is normally elliptic and $\mathcal{B}$ satisfies the normal complementing condition with respect to $\mathcal{A}$.

In the following discussion, the focus is on second-order operators, as in (C-4), even if the definition allows more general orders. If $(\mathcal{A}, \mathcal{B})$ is a normally elliptic boundary value problem, then there exists an interpolation extrapolation scale generated by the L^p-realization A of the boundary value problem. We denote the scales by $\left(\mathcal{E}_\alpha^N, A_{N,\alpha}\right)_{\alpha \geq -m}$ for Neumann boundary conditions and $\left(\mathcal{E}_\alpha^D, A_{D,\alpha}\right)_{\alpha \geq -m}$ for Dirichlet boundary conditions as in Example C.12.

Definition D.10. i) Let $1/p < 2\alpha \leq 2$, $g \in \mathrm{B}_{p,p}^{2\alpha - 1/p}(\partial\mathcal{O}; \mathbb{R})$ and $\mathcal{B} = \gamma_\partial$. Then $u \in \mathcal{E}_\alpha^D$ is called a strong solution of (D-7) if (D-7) holds in the sense of distributions.

ii) Let $1 + 1/p < 2\alpha \leq 2$, $g \in \mathrm{B}_{p,p}^{2\alpha - 1 - 1/p}(\partial\mathcal{O}; \mathbb{R})$ and $\mathcal{B} = \mathcal{B}^N$. Then $u \in \mathcal{E}_\alpha^N$ is called a strong solution of (D-7) if (D-7) holds in the sense of distributions.

Remark D.11. It is possible to identify a solution to equation (D-7) for lower values of α, which are no longer in the sense of Definition D.10, but weak or very weak solutions. The precise definition of these concepts is not essential for our purposes and is not addressed in further detail in this discussion. For more information see [Ama93, Section 9].

Theorem D.12. ([Ama93, Theorem 9.2]) *Let $\alpha \in [0, 1]$ such that $2\alpha \neq 1/p, 1 + 1/p$, $(\mathcal{A}, \mathcal{B})$ a normally elliptic boundary value problem with coefficients satisfying **(R)** with $\mu > 2\alpha - 1$, then there exists a unique solution to (D-7). In particular, the solution operator $\mathcal{N}\colon \mathrm{B}_{p,p}^{2\alpha-1-1/p} \to \mathcal{E}_\alpha^N$, respectively $\mathfrak{D}\colon \mathrm{B}_{p,p}^{2\alpha-1/p} \to \mathcal{E}_\alpha^D$, is bounded and linear for every $\alpha \in [0, 1]$ such that $2\alpha \neq 1/p, 1 + 1/p$. If $2\alpha > 1 + 1/p$, in the Neumann or $2\alpha > 1/p$ in the Dirichlet case, we obtain even a strong solution.*

German summary

In dieser Thesis werden vor allem stochastische Evolutionsgleichungen mit nichtlinearem multiplikativem Rauschen und deren Langzeitverhalten untersucht. Aufgrund dessen dass durch den stochastischen Einfluss die Lösungen nur außerhalb von Nullmengen definiert sind, eröffnen sich bei der Untersuchung des Langzeitverhaltens mehrere Probleme. Insbesondere ist es erstmal schwierig Gleichungen mit nichtlinearem multiplikativem Rauschen zu betrachten. Unter anderem aus diesem Grund liegt der Fokus in dieser Arbeit auf pfadweisen Lösungskonzepten, da diese es ermöglichen, allgemeineres Rauschen zuzulassen. Die Ergebnisse werden dabei speziell auch auf Gleichungen mit Rauschen auf dem Rand angewandt.

Die Arbeit ist in zwei Teile gegliedert: Im ersten Teil werden raue Differenzialgleichungen eingeführt und auf Existenz und Eindeutigkeit untersucht. Im zweiten Teil wird dann das angesprochene Langzeitverhalten von rauen Evolutionsgleichungen sowie von pfadweisen milden Lösungen betrachtet. Nach einer Einleitung in Kapitel 1, wird in Kapitel 2 die Theorie der rauen Pfade ausführlich eingeführt. Hauptsächlich werden viele aus der Literatur bekannten Fakten, wie die Erzeugung von rauen Pfaden durch gaußsche Prozesse, zitiert. Ein rauer Pfad ist hierbei eine Verallgemeinerung des stochastischen Rauschens, welche es schlussendlich ermöglicht, Lösungen von partiellen Differenzialgleichungen pfadweise zu untersuchen. Durch die rauen Pfade können dann insbesondere auch nicht-markovsche Prozesse betrachtet werden, was auch die fraktionelle Brownsche Bewegung mit einschließt. Außerdem wird im Unterabschnitt 2.2.2 untersucht, wie der Cameron–Martin Raum eines Gaußschen rauen Pfades aussieht, da diese Information wichtig wird, wenn es um die Integrierbarkeit von Lösungen geht. Anhand der Theorie der rauen Pfade führen wir das raue Integral in Abschnitt 2.3 ein. Danach wird zunächst das Integral über sogenannte $1-$Formen diskutiert, was schließlich in Unterabschnitt 2.3.3 zur Definition von kontrollierten rauen Pfaden führt, wie es von Gerasimovičs, Hocquet und Nilssen [GHN21] entwickelt wurde. Die kontrollierten rauen Pfade bilden eine Klasse von Funktionen, welche sich gegen einen rauen Pfad integrieren lassen und auf das Setting von parabolischen Gleichungen angepasst sind. Dieser Abschnitt erweitert die etablierten Ergebnisse durch die Einbeziehung von nicht-autonomen Diffusionskoeffizienten. Zusätzlich wird in Unterabschnitt 2.3.4 untersucht, wie sich verschiedene Operationen auf einen kontrollierten rauen Pfad angewandt auswirken.

Kapitel 3 beschäftigt sich mit den Lösungen von rauen Evolutionsgleichungen. Insbesondere werden in den Unterabschnitten 3.1 und 3.2 erst die lokale und dann die globale Lösbarkeit untersucht. Außerdem wird in Unterabschnitt 3.4 eine integrierbare Schranke zur Lösung konstruiert und in Unterabschnitt 3.5 eine Gronwall-artige Ungleichung für milde Lösungen von rauen Evolutionsgleichungen bewiesen.

Die im ersten Teil erlangten Erkenntnisse werden im zweiten Teil der Arbeit dann benutzt, um das Langzeitverhalten zu untersuchen. Hier beschäftigt sich Kapitel 4 zunächst mit der Einführung der zufälligen dynamischen Systeme, was die Grundlage zur Untersuchung der Dynamik liefert. Mithilfe dieser zufälligen dynamischen Systeme, der Existenz von globalen

Lösungen, der integrierbaren Schranken und des milden Gronwall Lemmas ist es dann in Kapitel 5 möglich, die Existenz von Attraktoren und Lyapunov Exponenten zu zeigen.

Im letzten Kapitel 6 werden allgemeinere Evolutionsgleichungen betrachtet, bei denen nicht nur das Rauschen zufällig sein darf, sondern auch der lineare Operator. Um diese Gleichungen zu untersuchen, wird deshalb das Konzept der pfadweisen milden Lösungen betrachtet, welches ein weiteres pfadweises Lösungskonzept zu stochastischen Gleichungen darstellt. Mithilfe dieser Theorie wird hier die Existenz von globalen Attraktoren für Reaktions-Diffusions-Gleichungen bewiesen.

List of Figures

List of Symbols

$T \in [0,\infty]$ is a positive constant, $\gamma \in (0,1)$, $\mathcal{E}, \mathcal{F}$ and $\mathcal{G}$ general Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(\mathcal{E}_\alpha)_{\alpha\in\mathbb{R}}$ a scale of Banach spaces, A some linear operator between $\mathcal{E}$ and $\mathcal{F}$.

Function spaces	
$\mathcal{E} \times \mathcal{F}$	product space of $\mathcal{E}$ and $\mathcal{F}$
$\mathcal{E} \otimes \mathcal{F}$	tensor product space of $\mathcal{E}$ and $\mathcal{F}$
$\mathcal{C} := \mathcal{C}(\mathcal{E}) := \mathcal{C}([0,T];\mathcal{E})$	space of continuous functions
$\mathcal{C}^\gamma := \mathcal{C}^\gamma(\mathcal{E}) := \mathcal{C}^\gamma([0,T];\mathcal{E})$	space of γ-Hölder continuous functions
$\mathcal{C}^{0,\gamma} := \mathcal{C}^\gamma(\mathcal{E}) := \mathcal{C}^\gamma([0,T];\mathcal{E})$	closure of smooth functions in $\mathcal{C}^\gamma([0,T];\mathcal{E})$
$\mathcal{C}^{q\text{-var}} := \mathcal{C}^{q\text{-var}}(\mathcal{E}) := \mathcal{C}^{q\text{-var}}([0,T];\mathcal{E})$	space functions with finite q-variation
$\mathcal{C}_2^\gamma := \mathcal{C}_2^\gamma(\mathcal{E}) := \mathcal{C}_2^\gamma([0,T];\mathcal{E})$	space of two-parameter γ-Hölder continuous functions
$\mathcal{C}^k := \mathcal{C}^k(\mathcal{E}) := \mathcal{C}^k([0,T];\mathcal{E})$	space of k-times Fréchet differentiable functions with $k \in \mathbb{N} \cup \{\infty\}$
$\mathcal{C}_c^k := \mathcal{C}_c^k(\mathcal{E}) := \mathcal{C}_c^k([0,T];\mathcal{E})$	space of k-times Fréchet differentiable functions with compact support and $k \in \mathbb{N} \cup \{\infty\}$
$\mathrm{L}^p(\mathcal{O};\mathcal{E})$	Lebesgue space for $p \in [1,\infty]$
$\mathrm{W}^{s,p}(\mathcal{O};\mathcal{E})$	Sobolev–Slobodeckij space for $s \geq 0, p \in [1,\infty]$
$\mathrm{W}_0^{s,p}(\mathcal{O};\mathcal{E})$	closure of smooth function with respect to the norm in $W^{s,p}(\mathcal{O};\mathcal{E})$
$\mathrm{H}^{s,p}(\mathcal{O};\mathcal{E})$	Bessel potential space for $s \in \mathbb{R}, p \in (1,\infty)$
$\mathrm{H}_0^{s,p}(\mathcal{O};\mathcal{E})$	closure of smooth function with respect to the norm in $H^{s,p}(\mathcal{O};\mathcal{E})$
$\mathrm{B}_{p,q}^s(\mathcal{O};\mathcal{E})$	Besov space for $s \in \mathbb{R}, p,q \in [1,\infty]$
$\mathscr{C}^\gamma := \mathscr{C}^\gamma([0,T];\mathcal{E})$	space of γ-Hölder rough paths
$\mathscr{C}_g^\gamma := \mathscr{C}_g^\gamma([0,T];\mathcal{E})$	space of weakly geometric γ-Hölder rough paths
$\mathscr{C}_g^{0,\gamma} := \mathscr{C}_g^{0,\gamma}([0,T];\mathcal{E})$	space of geometric γ-Hölder rough paths
$\mathcal{D}_{X,\alpha}^\gamma := \mathcal{D}_{X,\alpha}^\gamma([0,T])$	space of controlled rough paths according to the path X and the family $(\mathcal{E}_\alpha)_{\alpha\in\mathbb{R}}$
$\mathcal{L}(\mathcal{E};\mathcal{F})$	space of bounded linear operators
$\mathcal{L}^{(2)}(\mathcal{E}\times\mathcal{F};\mathcal{G})$	space of bilinear operators
$\mathcal{E}'$	the dual space of $\mathcal{E}$
$\mathscr{D}(\mathcal{O})$	space of test functions
$\mathscr{D}'(\mathcal{O})$	space of distributions
$\mathscr{S}(\mathbb{R}^d)$	Schwartz space
$\mathscr{S}'(\mathbb{R}^d)$	space of tempered distributions

(Semi-)norms	
$\lVert\cdot\rVert$	absolut value in $\mathbb{K}$ and the Euclidean norm in $\mathbb{R}^d$
$\lVert\cdot\rVert_{\mathcal{E}}$ or $\lVert\cdot\rVert_\alpha := \lVert\cdot\rVert_{\mathcal{E}_\alpha}$	norm in $\mathcal{E}$ respectively $\mathcal{E}_\alpha$
$\lVert\cdot\rVert_{\gamma,\alpha} := \lVert\cdot\rVert_{\gamma,\mathcal{E}_\alpha,[0,T]}$	Hölder norm with values in $\mathcal{E}_\alpha$ on the interval $[0,T]$

$[\cdot]_{\gamma,\alpha} \coloneqq [\cdot]_{\gamma,\mathcal{E}_\alpha,[0,T]}$	Hölder semi-norm with values in $\mathcal{E}_\alpha$ on the interval $[0,T]$
$[\cdot]_{\gamma,[0,T]} \coloneqq [\cdot]_{\gamma,\mathbb{R}^d,[0,T]}$	Hölder semi-norm with values in $\mathbb{R}^d$ on the interval $[0,T]$
$\|\cdot,\cdot\|_{\mathcal{D}^\gamma_{X,\alpha}} \coloneqq \|\cdot,\cdot\|_{\mathcal{D}^\gamma_{X,\alpha}([0,T])}$	controlled rough path norm

Miscellaneous

$x \lesssim y$	$x \le Cy$ for some positive constant C
$x \lesssim_a y$	as above, but $C = C(a)$ depends on a
$\mathcal{E} \hookrightarrow \mathcal{F}$	$\mathcal{E}$ is embedded in $\mathcal{F}$, i.e. $\mathcal{E} \subset \mathcal{F}$ and $\|x\|_{\mathcal{F}} \lesssim \|x\|_{\mathcal{E}}$
$\mathrm{Re}(x)$	the real part of a complex number $x \in \mathbb{C}$
$\mathrm{Im}(x)$	the imaginary part of a complex number $x \in \mathbb{C}$
$\partial\mathcal{O}$	the boundary of a bounded domain $\mathcal{O} \subset \mathbb{R}^n$
γ_∂	the trace operator
$D(A)$	the domain of A
$\rho(A)$	the resolvent set of A
$\sigma(A)$	the spectrum of A
$\mathcal{E} \simeq \mathcal{F}$	$\mathcal{E}$ is isomorphic to $\mathcal{F}$
A^*	the adjoint operator of A
$\Delta_J^{(k)}$	subset of $\mathbb{R}^k$ for an arbitrary interval $J \in \mathbb{R}$ defined in (2-10)
$\times$	used to highlight a multiplication when the equation goes over a line break
$O(f), o(f)$	Landau symbols
$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial^{\alpha_1} x_1} \cdot \ldots \cdot \frac{\partial^{\alpha_1}}{\partial^{\alpha_n} x_n}$ $\lvert\alpha\rvert = \alpha_1 + \ldots + \alpha_n$	Multiindex notation for $\alpha \in \mathbb{N}_0^n$

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Declaration of Authorship

Affidavit according to §6 of the doctoral degree regulations of the University of Konstanz (Annex 1 doctoral degree regulations)

1) The dissertation submitted on the topic of

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   is my own independent work.
2) I have only used the sources and aids indicated and have not utilized any unauthorized assistance from third parties. In particular, I have identified content taken literally or analogously from other works as such.
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