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Non-Trapping Conditions and Local Energy Decay for Hyperbolic Problems

Raimund Pauen

Zusammenfassung

Diese Arbeit beschäftigt sich mit dem zeitasymptotischen Verhalten der Energie von Lösungen hyperbolischer Probleme in Außengebieten am Beispiel der Wellengleichung und der Elastizitätsgleichungen isotroper Medien. Hierbei stellen sich die Fragen, ob die Energie lokal gegen Null strebt (für $t \longrightarrow \infty$) und ob dieses zeitliche Abklingen gleichmäßig bezüglich solcher Anfangsdaten ist, deren Träger in einer beliebigen aber festen kompakten Menge enthalten ist.

Es sollen zunächst die wesentlichen Probleme und Ergebnisse für die Wellengleichung vorgestellt werden. Sowohl im Falle des Dirichlet- als auch des Neumann-Problems strebt die Energie lokal gegen Null. Ob dieses Abklingen gleichmäßig ist, hängt allerdings entscheidend vom Hindernis und von den Eigenschaften des Mediums ab. Es gibt Situationen, in denen gleichmäßiges Abklingen der lokalen Energie unmöglich ist. Andererseits existieren zahlreiche Bedingungen, die sicherstellen, dass das Abklingen tatsächlich gleichmäßig ist. Einige von ihnen sind äußerst anschaulich, aber lediglich *hinreichend*. Andere wiederum sind zwar sehr abstrakt, dafür aber durch das Ziel motiviert, eine *notwendige und hinreichende Bedingung* für gleichmäßiges Abklingen der lokalen Energie zu präsentieren. Solche Bedingungen – und in der Literatur sind einige davon zu finden – heißen *Non-Trapping-Bedingungen*.

Das Ziel dieser Arbeit ist der Vergleich und die geometrische Interpretation dieser Bedingungen. Für die Wellengleichung homogener Medien wurde dies vom Autor bereits in [Pau 96] untersucht. Das Ergebnis war die Äquivalenz aller Non-Trapping-Bedingungen für Hindernisse, die einer gewissen Nichtdegeneriertheits-Bedingung genügen. Hierbei spielt es keine Rolle, ob die Dirichletsche oder die Neumannsche Randwertaufgabe betrachtet wird. Für inhomogene Medien ist das Resultat nun im wesentlichen dasselbe, mit dem entscheidenden Unterschied, dass nicht mehr länger allein das Hindernis Wellen oder Lichtstrahlen einfangen kann, sondern auch das Medium, wie wir an einigen Beispielen verdeutlichen. Dieser Unterschied hat einen wesentlichen Einfluss auf die geometrische Interpretation dieser Bedingungen: Im homogenen Fall genügen konvexe, sternförmige, schlangenförmige und von innen oder von außen beleuchtbare Hindernisse jedweder Non-Trapping-Bedingung. Darüber hinaus gibt es für Gebiete im $\mathbb{R}^2$ eine geometrische Bedingung, die einerseits leicht zu überprüfen ist, und die andererseits für eine grosse Klasse von Hindernissen äquivalent zu gleichmäßigem Abklingen der lokalen Energie ist, wie bereits in [Pau 96] festgestellt wurde. Ein weiteres Resultat der vorliegenden Arbeit ist, dass diese geometrischen Bedingungen nur für geeignete Medien gleichmäßiges Abklingen der lokalen Energie implizieren. Ähnliche

Schwierigkeiten ergeben sich bei der Suche nach Situationen, in denen gleichmäßiges Abklingen nicht möglich ist. Wir können dennoch einige konkrete Beispiele hierfür angeben.

Die entscheidenden neuen Ergebnisse auf dem Weg zum Beweis der angekündigten Resultate sind: Die Existenz von Escape-Funktionen für geeignete Medien mit Hindernissen, die den oben genannten geometrischen Bedingungen genügen; die Implikation von Melroses Non-Trapping-Bedingung durch die Existenz einer Escape-Funktion; die Existenz und Eindeutigkeit der Greenschen Funktion sowohl als Kern des Lösungsoperators als auch als distributionelle Lösung der Wellengleichung; schließlich der Beweis, dass Melroses Non-Trapping-Bedingung die Non-Trapping-Bedingung von Vainberg impliziert.

Im Falle der isotropen Elasitizität ist die Situation erheblich schwieriger. Dies liegt daran, dass es sich hier um ein *System* von Differentialgleichungen handelt. Es existieren zwei verschiedene Wellen, die sich mit unterschiedlichen Geschwindigkeiten ausbreiten und sich am Rand gegenseitig beeinflussen. Im Falle freier Ränder (i.e. des Neumann-Problems) gibt es sogar noch eine dritte Welle in der Oberfläche des Hindernisses, die gleichmäßige Abklingabschätzungen unmöglich macht. Hier verhalten sich also – im Gegensatz zur Wellengleichung – Lösungen des Dirichlet- und des Neumann-Problems völlig unterschiedlich.

Doch auch für das Dirichlet-Problem sind die Resultate schwächer als im Falle der Wellengleichung. Es ist nicht einmal klar, ob die dort bewiesenen Äquivalenzen hier überhaupt gelten können. Wir kommen zu den folgenden Resultaten: Für die obigen hinreichenden geometrischen Bedingungen sind die Ergebnisse im wesentlichen unverändert. *Geeignet* muss das Medium nun bezüglich beider auftretenden Wellen sein. Um dies zu beweisen, führen wir eine spezielle Klasse von Escape-Funktionen für die isotrope Elastizität ein. Es gelingt erneut der Nachweis, dass die Existenz einer solchen Funktion Melroses Non-Trapping-Bedingung impliziert, wobei wir auch für diese Bedingung eine Formulierung gewählt haben, die den Elastizitätsgleichungen angemessen erscheint (und die beispielsweise unmittelbar die Non-Trapping-Bedingung an die schwache Lösung impliziert).

Auch hier beweisen wir die Existenz und Eindeutigkeit der Greenschen *Matrix* als Distributionskern und als distributionelle Lösung, die Implikation von Vainbergs Non-Trapping-Bedingung durch Melroses Non-Trapping-Bedingung und die übrigen Beziehungen zwischen den verschiedenen Non-Trapping-Bedingungen bis hin zu gleichmäßigem Abklingen der lokalen Energie wie im Falle der Wellengleichung. Den Kreis dieser Folgerungen zu schließen, gelingt allerdings derzeit nur für homogene Medien im $\mathbb{R}^2$ unter einer zusätzlichen Einschränkung an die Klasse der Hindernisse. Darüberhinaus können wir – wenigstens in einigen speziellen Situationen – Ralstons Konstruktionen lokalisierter Lösungen dazu verwenden, einige Besipiele anzugeben, in denen gleichmäßiges Abklingen der lokalen Energie nicht möglich ist.

Abstract

In working with Evolution Equations one is interested in answers to questions concerning

- the existence of a solution
- the uniqueness of the solution
- continuous dependence on prescribed data

and closely connected with the last question

- the asymptotic behaviour as $t \longrightarrow \infty$.

An important characteristic feature for the last point is the behaviour of the energy and the question whether the energy tends locally to zero (for $t \longrightarrow \infty$)—the so-called local energy decay (LED)—and whether this decay in time is uniform with respect to initial disturbances concentrated in some finite region—the so-called uniform local energy decay. Answers to the latter questions are important for existence and uniqueness results, too. If one is interested in solutions existing globally in time of non-linear Evolution Equations for small initial data, the relation between these questions could be roughly described as follows: The better the solutions of the corresponding linear problem decay in time, the larger the class of non-linearities leading to global existence is.

This thesis is devoted to the study of hyperbolic problems in exterior domains, namely the wave equation and the equations of elasticity in isotropic media, that describe the propagation of sound waves and the deformations of elastic solids respectively. An important phenomenon which is described by the latter equations is given by earthquakes.

We will point out the essential problems and results first for the wave equation. For the Dirichlet and the Neumann problem the question whether the local energy decays to zero in time has an affirmative answer. But whether this decay is uniform depends essentially on the complement of the domain, the so-called obstacle, and on the medium itself. There are examples in which no uniform local energy decay holds. However there are many conditions which assure that the decay is in fact uniform.

Some of them are very concrete and illustrative, but unfortunately only *sufficient* for uniform decay. Some others seem to be very abstract and thus difficult to understand, but nevertheless they are motivated by the aim to present a *necessary and sufficient* condition for uniform LED. Conditions like the latter ones are called non-trapping conditions (NTCs) and the literature provides several of them.

The subject of this thesis is to compare these different non-trapping conditions and to give some concrete geometric interpretation of them. For sound waves in a homogeneous medium this has already been done in [Pau 96], leading to the result that under some non-degeneracy condition on the boundary all non-trapping conditions are equivalent, no matter whether the Dirichlet or the Neumann problem is considered. For inhomogeneous media the result is essentially the same, with the important difference that no longer the obstacle alone can trap rays but the medium, too, as we will illustrate by some examples. This difference mainly influences the geometric interpretation of these conditions: In the homogeneous case obstacles with the nice geometric properties of being convex, star-shaped, snake-shaped, or able to be illuminated from the interior or from the exterior satisfy any (reasonable) non-trapping condition. For homogeneous media in $\mathbb{R}^2$ there is in fact a geometric condition which is on the one hand very easy to verify in concrete situations and on the other hand necessary and sufficient for uniform LED for a large class of obstacles, as we have already pointed out in [Pau 96]. This result is included here as Theorem 3.1.7 in a survey on the development of this subject since the late 1950s. One further result of this thesis is that in the general situation these geometric conditions are sufficient for uniform LED for suitable media only. The condition a medium has to fulfil for this purpose is stated explicitly in Section 3.2. The same complications arise in giving counterexamples which do not allow uniform LED. But nevertheless we state concrete conditions for non-decay, partly in a more visualized way and partly by explicit calculation (cf. Sections 2.1, 3.2, and 4.1).

We proceed as follows: In Chapter 1 we introduce the class of problems under consideration. This is followed by a chapter dealing with the famous Lax-Phillips conjecture of a necessary and sufficient condition for uniform LED and with Ralston's proofs concerning one implication of this assumed equivalence. Here we point out situations in which no uniform decay is possible. The third chapter is concerned with the so-called energy method. Here the above sufficient geometric conditions are presented. This finally leads to the definition of an escape function, which is the decisive link between these geometric and the more or less abstract non-trapping conditions. Chapter 4 gives a precise definition of the terms of the Lax-Phillips conjecture, i.e. the generalized bicharacteristics, and presents the Generalized Huygens' principle, being the most powerful tool for the comparison of the non-trapping conditions. As already remarked, Section 4.1 includes the calculation of two counterexamples to uniform LED. In Chapter 5 we formulate Melrose's non-trapping condition and prove that it is implied by the existence of an escape function (in fact this proof is given in the more complicated situation of Section 9.5.2). Chapter 6 presents the Vainberg method. We prove the existence and uniqueness of the Green's function both as the kernel of the solution operator and as the distributional solution of a differential equation. This makes the statement of Vainberg's non-trapping condition possible, followed by a presentation of the essential ideas and results of the Vainberg method. Chapter 7 is then devoted to the comparison of the different non-trapping conditions. On the one hand a chain of inclusions is proven, on the other hand it is pointed out, how this circle of implications can be closed, leading to the equivalence of all non-trapping conditions for a large class of obstacles. Here the main result is the implication of Vainberg's NTC by the one of Melrose. Some improvements for special situations and a summary of the geometric aspects conclude this chapter.

The situation is worse with isotropic elasticity. This is essentially due to the fact that it is a *system* of differential equations. The result is that two different waves appear, travelling at different speeds and influencing each other at the boundary (generally at different angles of incidence). Concerning the propagation of singularities this means that two bicharacteristic families exist and that e.g. changes from one family to the other are possible. For a free boundary (i.e. Neumann problem) there is actually a third wave travelling in the boundary and making uniform decay estimates impossible. Thus here—in contrast to the wave equation—solutions of the Dirichlet problem and the Neumann problem behave completely differently.

But even for the Dirichlet problem where this third wave does not occur, the results are weaker than those for the wave equation. It is not even clear, whether the Lax-Phillips conjecture may be true for isotropic elasticity. We achieve the following results: For the above sufficient geometric conditions things are essentially the same. In the inhomogeneous case the medium has to be "suitable" with respect to both waves and the condition to be fulfilled is, for both waves, the analogue to the wave equation. To prove this result we wanted to make use of the connecting role between the geometric conditions and Melrose's NTC, played by escape functions in the case of the wave equation. This was our motivation for the introduction of a special class of escape functions for isotropic elasticity. And in fact these functions are as useful as their analogues for the wave equation (cf. Section 9.5).

In comparing the different non-trapping conditions we can prove the same chain of implications as in the case of the wave equation. But at the present state of the art it is possible to close the circle only for homogeneous media in $\mathbb{R}^2$ under an additional assumption on the class of obstacles.

On the way to prove these results we proceed similarly to the wave equation. Chapter 8 presents the problems under consideration and the fundamental results. Chapter 9 is devoted to uniform decay results but also presents the more complicated propagation of singularities, the formulation of a Melrose-type non-trapping condition for isotropic elasticity, and the proof of the existence and uniqueness of the Green's matrix—both as a distribution kernel and as a distributional solution of a system of differential equations. Afterwards we present examples for non-uniform decay according to the results of the Gaussian beam constructions of Ralston. Chapter 11 is then devoted to the comparison of the different NTCs and to a summary of their geometric properties.

According to the two different problems under consideration, this thesis consists of two major parts, each of them dealing with one problem. But because of the similarities of these phenomena there are many parallels. Thus we tried to follow a certain rule to avoid too many repetitions: The results quoted from the literature are presented in more detail for the easier case of the wave equation to make the decisive ideas as clear as possible. The results of the author, however, are presented chiefly for the system of isotropic elasticity to point out exactly all the problems that arose in these proofs.

Many other problems have been treated with the methods discussed in this thesis. There are for example numerous results on the propagation of singularities and on uniform LED for other equations and systems, for more general boundary conditions, and for moving obstacles. We added an—in fact very individual—choice out of these publications at the end of this thesis. A short discussion of a part of the literature quoted there can be found in Chapter 11 of [Pau 96].

I want to thank Prof. Dr. Reinhard Racke: For giving me the possibility of writing this thesis at his chair at the University of Konstanz, whose constructive atmosphere I enjoyed very much, and for many interesting discussions during the preparation of this thesis. In addition I am indebted to Prof. James V. Ralston from UCLA for the email discussions about his Gaussian beam construction and the Lax-Phillips conjecture in isotropic elasticity, to Prof. Kazuhiro Yamamoto, Nagoya Institute of Technology, for the answers to some questions concerning the propagation of singularities in isotropic elasticity, and to Mrs. Helen Walter for the careful reading of this thesis and her many advices concerning the English language. Last but not least I want to thank my parents for their great support.

The present monograph is based on my thesis.

Contents

Notations

The following notations are used:

$$\partial_t := \frac{\partial}{\partial t}\,,$$
$$\partial_i := \frac{\partial}{\partial x_i}\,,$$
$$f_t := \partial_t f\,,$$
$$f_{x_i} := \partial_i f\,,$$
$$\nabla := \nabla_x := (\partial_1, \ldots, \partial_n)\,,$$
$$\Delta := \Delta_x := \sum_{i=1}^{n} \partial_i^2\,,$$
$$\Box := \Box_{t,x} := \partial_t^2 - \Delta\,.$$

For $X = (X_1, \ldots, X_n)$, $\xi = (\xi_1, \ldots, \xi_n)$, and a function $U(T, \ldots)$, the operators ∇_X, Δ_X, ∇_ξ, and the derivative U_T are defined in the same way.

For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^{n+1}$ and for $N \in \mathbb{N}$ let

$$|\alpha| := \sum_{i=1}^{n} \alpha_i\,, \qquad |\gamma| := \sum_{i=0}^{n} \gamma_i\,,$$
$$D_x^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}\,, \qquad D_{t,x}^\gamma := \partial_t^{\gamma_0}\partial_1^{\gamma_1} \cdots \partial_n^{\gamma_n}\,,$$
$$D_x^N := (D_x^\alpha, \quad |\alpha| = N)\,, \qquad D_{t,x}^N := (D_{t,x}^\gamma, \quad |\gamma| = N)\,,$$
$$\overline{D}_x^N := (D_x^\alpha, \quad |\alpha| \leqslant N)\,, \qquad \overline{D}_{t,x}^N := (D_{t,x}^\gamma, \quad |\gamma| \leqslant N)\,.$$

$D_{t,x}$ will frequently be used instead of $D_{t,x}^1$. Radial and normal derivatives are defined by

$$f_r := \partial_r f := \nabla f \cdot \frac{x}{|x|}\,,$$
$$\frac{\partial}{\partial \mathbf{n}} f := \nabla f \cdot \mathbf{n}\,,$$
$$\frac{\partial}{\partial \mathbf{n}_e} f := \nabla f \cdot \mathbf{n}_e\,,$$

$\mathbf{n}$ being the exterior normal of a domain $\Omega \subset \mathbb{R}^n$ and $\mathbf{n}_e$ the exterior normal of $\mathbb{R}^n \setminus \Omega$.

For a function V with values in $\mathbb{R}^n$,

$$\mathrm{div}_x V(x, \ldots) := \sum_{i=1}^{n} \partial_i V_i(x, \ldots)$$

defines the divergence of V with respect to x. Analogously we define $\mathrm{div}_X V(X, \ldots)$ and the divergence with respect to (t, x) for a $\mathbb{R}^{n+1}$-valued function U, $\mathrm{div}_{t,x} U(t, x, \ldots)$. Writing the equations of elasticity, one traditionally uses

$$\nabla' := \mathrm{div}_x\,.$$

For functions of one real variable, the abbreviations

$$f'(t) := \partial_t f(t)\,, \qquad f''(t) := \partial_t^2 f(t)\,, \qquad \text{etc.}$$

are used.

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. For $a > 0$ and $\Omega \subset \mathbb{R}^n$ let

$$B_a := \left\{ x \in \mathbb{R}^n \;\middle|\; |x| < a \right\},$$
$$\Omega_a := \Omega \cap B_a\,.$$

For $D \subset \mathbb{R}^n$, $\overline{D}$ denotes the closure of D in $\mathbb{R}^n$ and ∂D the boundary of D. We use

$$\operatorname{supp} f\,, \qquad \operatorname{supp}(f, g)$$

for the support of a function f and the union of the supports of functions f and g respectively. For two subsets D_1 and D_2 of $\mathbb{R}^n$

$$D_1 \subset\subset D_2$$

means that there exists a compact $K \subset \mathbb{R}^n$ with $D_1 \subset K \subset D_2$.

The following spaces and norms are used (cf. [Ad 75]; all functions are real-valued, if not indicated otherwise):

For an open subset G of $\mathbb{R}^n$

$$\mathcal{C}(G)\,, \qquad \text{and} \qquad \mathcal{C}_k(G)\,, \quad k \in \mathbb{N} \cup \infty$$

denote the continuous and the k times continuously differentiable functions on G respectively. In contrast to [Ad 75] we use

$$\mathcal{C}(\overline{G})\,, \qquad \text{and} \qquad \mathcal{C}_k(\overline{G})\,, \quad k \in \mathbb{N} \cup \infty$$

for the subsets of all functions f of $\mathcal{C}(G)$ and $\mathcal{C}_k(G)$ respectively which together with all their derivatives up to the order k can be continuously extended to $\overline{G}$; $C_1([a,b] \setminus J)$, where J is at most countable, has to be understood in the same way in Lemma 9.5.2 and the text preceding it. We define the space $\mathcal{C}_B(\overline{G})$ to consist of all those functions $f \in \mathcal{C}(\overline{G})$ which are bounded and uniformly continuous on G. ($\mathcal{C}_B(\overline{G})$ is the space which Adams in [Ad 75] denotes by $\mathcal{C}(\overline{G})$).

$$\mathring{\mathcal{C}}(G)\,, \qquad \mathring{\mathcal{C}}_\infty(G) \qquad \text{and} \qquad \mathring{\mathcal{C}}_\infty(\overline{G})$$

are the subspaces of all functions f in $\mathcal{C}(G)$, $\mathcal{C}_\infty(G)$ and $\mathcal{C}_\infty(\overline{G})$ for which $\operatorname{supp} f \subset\subset G$ and $\operatorname{supp} f \subset\subset \overline{G}$ respectively. We denote by

$$\mathcal{D}'(G)$$

the space of all distributions on G. For $x \in G$ and $x_0 \in \overline{G}$

$$\delta(x - x_0)$$

denotes the Dirac δ-distribution. As an element of $\mathcal{D}'(G)$ it is defined by

$$\forall \varphi \in \mathring{\mathcal{C}}_\infty(G) \ : \ \delta(\cdot - x_0)(\varphi) := \varphi(x_0)$$

(with $\delta(\cdot - x_0)(\varphi) := 0$, if $x_0 \in \partial G$). As an element of $\mathcal{D}'(G \times G)$ it is given by

$$\forall \varphi \in \mathring{\mathcal{C}}_\infty(G \times G) \ : \ \delta(x - x_0)(\varphi(x, x_0)) := \int\limits_G \varphi(x_0)\, dx_0 \, .$$

For open subsets $\mathcal{O}$ of $\mathbb{R}^m$, the concept of convergence in $\mathcal{D}'(G)$ leads to the definition of

$$\mathcal{C}_\infty(\mathcal{O}, \mathcal{D}'(G)) \quad \text{and} \quad \mathcal{C}_\infty(\overline{\mathcal{O}}, \mathcal{D}'(G)) \, ,$$

the spaces of arbitrarily differentiable distribution-valued functions on $\mathcal{O}$ and $\overline{\mathcal{O}}$ respectively, given by

$$\begin{aligned} &f \in \mathcal{C}_\infty(\mathcal{O}, \mathcal{D}'(G)) \quad (f \in \mathcal{C}_\infty(\overline{\mathcal{O}}, \mathcal{D}'(G))\,) \ :\Longleftrightarrow \\ &\forall \varphi \in \mathring{\mathcal{C}}_\infty(G) \ : \ f(\varphi) \in \mathcal{C}_\infty(\mathcal{O}) \quad (f(\varphi) \in \mathcal{C}_\infty(\overline{\mathcal{O}})\,) \, . \end{aligned}$$

For a domain $\Omega \subset \mathbb{R}^n$ let $\mathcal{L}^p(\Omega)$ be the usual space of p-integrable functions, equipped with the norm $\|\cdot\|_{\mathcal{L}^p(\Omega)}$. We often use the abbreviations

$$\|\cdot\| := \|\cdot\|_\Omega := \|\cdot\|_{\mathcal{L}^2(\Omega)}$$

(the first one only, if the domain is clear from the context). For $m \in \mathbb{N}$

$$\mathcal{H}_m(\Omega) := \mathcal{H}^2_m(\Omega)$$

denotes the usual Sobolev space of all functions which, together with their derivatives up to order m, belong to $\mathcal{L}^2(\Omega)$. The norm is given by

$$\|\cdot\|_m := \|\cdot\|_{m,\Omega} \, .$$

Here, too, the abbreviation is used, if the domain of integration is clear from the context.

$\mathring{\mathcal{H}}_m(\Omega)$ denotes the closure of $\mathring{\mathcal{C}}_\infty(\Omega)$ in $\mathcal{H}_m(\Omega)$.

Furthermore we use the subspaces $\mathcal{L}^2_a(\Omega)$ and $\mathring{\mathcal{H}}_{m,a}(\Omega)$ of $\mathcal{L}^2(\Omega)$ and $\mathring{\mathcal{H}}_m(\Omega)$, consisting of the functions with support in $\overline{\Omega}_a$.

When working with functions of the above spaces on Ω and Ω_a, $a > 0$, at the same time, we add to the norms of the spaces defined on Ω_a an additional index a, to distinguish them from the norms of the spaces defined on Ω. That is, we use $\|\cdot\|_a$ for the $\mathcal{L}^2$-norm on Ω_a and $\|\cdot\|_{m,a}$ for the norms of the above Sobolev spaces on Ω_a. Sometimes these norms are used for the restrictions to Ω_a of elements of $\mathcal{L}^2_a(\Omega)$ and $\mathring{\mathcal{H}}_{m,a}(\Omega)$.

We use $\langle \cdot, \cdot \rangle$ without an additional subscript for the inner products of both $\mathbb{R}^n$ and $\mathcal{L}^2(\Omega)$, if it is clear from the context which one is meant.

For a normed space Y with norm $\|\cdot\|_Y$,

$$Y^{\sim\|\cdot\|_Y}$$

denotes the completion of Y with respect to $\|\cdot\|_Y$. In $\overset{\circ}{\mathcal{C}}_\infty(\Omega)$ there can be defined a norm $|\cdot|_1$ given by

$$\forall \varphi \in \overset{\circ}{\mathcal{C}}_\infty(G) \; : \; |\varphi|_1 := \|\nabla\varphi\| \; .$$

One now defines

$$\mathcal{H}_\nabla(\Omega) := \left(\overset{\circ}{\mathcal{C}}_\infty(\Omega)\right)^{\sim|\cdot|_1} \; .$$

For two Banach spaces X and Y we denote the space of all continuous linear maps from X to Y by $\mathcal{L}_b(X,Y)$ and its norm by $\|\cdot\|_{\mathcal{L}_b(X,Y)}$.

The definitions of the next paragraph's terms can be found in [Hö I, Hö III]. They will be used for the abstract derivation of the generalized Huygens' principle in Chapter 4.

For $x \in \mathbb{R}^n$ and a subspace Y of $\mathbb{R}^n$ we denote by $T_x^*(\mathbb{R}^n)$ the cotangent space of $\mathbb{R}^n$ at x, by $N_x^*(Y)$ the conormal space to the subspace Y of $\mathbb{R}^n$ at x, and by $T^*(\mathbb{R}^n)$ the cotangent bundle of $\mathbb{R}^n$. Let X be a C_∞-manifold with boundary (for the definition cf. [Hö III], p. 482), $\overset{\circ}{X}$ the interior of X. Using charts one can define $\mathcal{C}_\infty(X)$, $\overset{\circ}{\mathcal{C}}_\infty(\overset{\circ}{X})$, $\overset{\circ}{\mathcal{C}}_\infty(\partial X)$ and thus $\mathcal{D}'(\overset{\circ}{X})$ and $\mathcal{D}'(\partial X)$. According to Section 6.4 in [Hö I] one defines the cotangent spaces and bundles $T_x^*(X)$, $T^*(X)$ and $T^*(\overset{\circ}{X})$ as well as the conormal space to ∂X in $x \in \partial X$, $N_x^*(\partial X)$. $T^*(X)$ being a $\mathcal{C}_\infty$-manifold with boundary itself allows the definition of $\mathcal{C}_\infty(T^*(X))$.

We use the above symbols for function spaces no matter whether the functions are scalar-, vector-, or matrix-valued. This implies a corresponding use of the notations for spaces of distributions. However, if there is the danger of a mistake, we will use specific notations, e.g. $(\overset{\circ}{\mathcal{C}}_\infty(\Omega))^n$ for a vector whose n components are functions in $\overset{\circ}{\mathcal{C}}_\infty(\Omega)$.

We sometimes work with complex-valued functions. If this is clear from the context, we use for the function spaces the same notations as above. For a complex-valued function u we denote by $\overline{u}$ its complex conjugate function.

By $f(t \pm 0)$ we denote the left and right limits of a function f at the point t. Instead of $f(0 \pm 0)$ we write $f(\pm 0)$.

$[A, B]$ means the commutator of the operators A and B.

I_n stands for the $(n \times n)$ unit matrix in $\mathbb{R}^n$ and A^T for the transpose of a matrix A.

δ_{ij} denotes the Kronecker symbol, that is

$$\delta_{ij} := \begin{cases} 1 \, , \; i = j \, , \\ 0 \, , \; i \neq j \, . \end{cases}$$

In estimates, C and c are used for (changing) positive constants.

We work in domains with compact $\mathcal{C}_\infty$-boundary. By this we mean the uniform $\mathcal{C}_\infty$-regularity property of the boundary according to [Ad 75], 4.6, p. 67.

Finally it should be noted that we use the Einstein summation convention.

Part I

The Wave Equation

Chapter 1

Fundamentals

Let $\Omega \subset \mathbb{R}^n$ be an exterior domain with compact $\mathcal{C}_\infty$-boundary $\partial\Omega$ and $\mathrm{B} := \mathbb{R}^n \setminus \Omega$. We assume that an $R_0 > 0$ exists with $\mathrm{B} \subset B_{R_0}$. By $\mathbf{n}$ we denote the exterior (unit) normal of $\partial\Omega$ and by $\mathbf{n}_e$ the exterior (unit) normal of $\partial\mathrm{B}$. The space dimension n is assumed to be larger or equal to 2. In this first part we study the wave equation for inhomogeneous media, that means $u = u(t,x)$ is supposed to be a solution of

$$(W.\Lambda) \quad \begin{cases} (\partial_t^2 - \partial_i a_{ij}(x)\partial_j)u = 0 & \text{in} \quad \mathbb{R} \times \Omega\,, \\ \Lambda u = 0 & \text{on} \quad \mathbb{R} \times \partial\Omega\,, \\ u(0) = u_0\,, \quad u_t(0) = u_1 & \text{in} \quad \Omega\,. \end{cases}$$

Here Λ denotes a differential operator of at most first order on $\partial\Omega \times \mathbb{R}$. For the real-valued functions $a_{ij} \in \mathcal{C}_\infty(\overline{\Omega})$, $i,j = 1,\ldots,n$, we assume

$$\begin{gathered} a_{ij} = a_{ji} \\ \exists R > R_0 : a_{ij}(x) = \delta_{ij}\,, \quad |x| > R \\ \exists c > 0 : a_{ij}(x)\xi_i\xi_j \geqslant c|\xi|^2\,, \quad x \in \overline{\Omega}\,, \quad \xi \in \mathbb{R}^n\,. \end{gathered}$$

The special case of a homogeneous medium is described by $a_{ij} \equiv \delta_{ij}$, i.e. $\partial_i a_{ij}(x)\partial_j = \Delta$. We restrict our attention to two kinds of boundary value problems,

$$u\Big|_{\partial\Omega} = 0 \quad \text{(Dirichlet, (W.D))} \qquad \text{and} \qquad \mathbf{n}_i a_{ij}\partial_j u\Big|_{\partial\Omega} = 0 \quad \text{(Neumann, (W.N))}.$$

The energy of a solution u at time t in $D \subset \overline{\Omega}$ is then given by

$$E\left(u(t), D\right) := \int\limits_D a_{ij} u_{x_i}(t) u_{x_j}(t) + u_t^2(t)\, dx\,.$$

Many of the results presented here also hold for the Robin boundary condition

$$\left(\mathbf{n}_i a_{ij}\partial_j u + \sigma u\right)\Big|_{\partial\Omega} = 0\,, \quad \sigma(x) \geqslant 0.$$

Note that in this case the energy is given by

$$E\left(u(t), D\right) := \int\limits_D a_{ij} u_{x_i}(t) u_{x_j}(t) + u_t^2(t)\, dx + \int\limits_{\overline{D} \cap \partial\Omega} \sigma u(t)^2\, do\,.$$

We are chiefly interested in solutions of (W.D) and (W.N) for which this energy is defined. For this purpose one may use the notion of a *solution with finite energy* of [Le 86, Definition 3.2]. Then Theorems 3.6 and 6.1 of [Le 86] give:

Theorem 1.0.1 *Suppose* $u_0 \in \overset{\circ}{\mathcal{H}}_1(\Omega)$, $u_1 \in \mathcal{L}^2(\Omega)$. *Then a unique solution* u *with finite energy of (W.D) exists. For this solution the following holds:*

1. $E\left(u(t), \Omega\right) = const.$,
2. $\forall K \subset\subset \overline{\Omega}: \quad E\left(u(t), K\right) \longrightarrow 0\,, \quad (t \longrightarrow \infty)$.

Remarks:

1. An analogous theorem holds for (W.N) if $u_0 \in \mathcal{H}_1(\Omega)$, $u_1 \in \mathcal{L}^2(\Omega)$.
2. A more general ansatz using semigroup theory allows for example in case of (W.D) u_0 in $\overset{\circ}{\mathcal{H}}_\nabla(\Omega)$ instead of $\overset{\circ}{\mathcal{H}}_1(\Omega)$. This leads to a result similar to the above theorem. For initial data with compact support this is in fact nothing new because of Poincaré's estimate (cf. [Le 86, p. 25]). Nevertheless it is sometimes useful for our purposes to work with this semigroup approach.
3. Unless specified differently, we mean such solutions with finite energy, if we talk about solutions of (W.D) and of (W.N).

The last result of Theorem 1.0.1 is referred to as *local energy decay* (LED for short). But this decay is not *uniform* with respect to the initial data. What we are interested in is a result of the following form:

$$\text{(P)} \qquad \begin{gathered} \forall\, K, K_1 \subset\subset \overline{\Omega} \quad \exists\, f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \;\text{ with } f(t) \to 0 \;\; (t \to \infty) \;: \\ \operatorname{supp}(u_0, u_1) \subset K_1 \;\; \Longrightarrow \;\; E\left(u(t), K\right) \leqslant f(t)\, E\left(u(0), \Omega\right). \end{gathered}$$

Here we speak of *uniform* LED, meaning decay with a rate $f(t)$, independent of the initial data with support in K_1. It should be remarked that without this restriction on the support of u_0 and u_1 such an estimate is not possible, because there would be an eternal transport of energy to K from regions far away, otherwise. We want to mention that an equivalent formulation of (P) is achieved by replacing K and K_1 by e.g. $\overline{\Omega}_a$ and $\overline{\Omega}_b$ with $a, b \geqslant R$. For the Cauchy problem in a homogeneous medium there is always uniform LED. Explicit representation formulae for the solutions (cf. [Le 86], (5.2), p. 97 and p. 104f.) yield

$$\begin{aligned} &n \geqslant 3, \text{ odd:} && u\Big|_K \equiv 0 \quad \text{for } t > T(K, K_1)\,, \\ &n \text{ even:} && E\left(u(t), K\right) \leqslant c(K, K_1)\, t^{-2n} E\left(u(0), \mathbb{R}^n\right) \quad \text{for } t > T_1(K, K_1) \end{aligned}$$

(cf. [Pau 96], Chapter 1 and Appendix A). The restriction to initial data with compact support, however, is not sufficient for uniform LED in the presence of an obstacle, as we will see next. So it seems natural, to look for a condition as good as possible, which guarantees that both the medium and the obstacle do not "retain energy" or "trap waves". Such a condition—and the literature on this topic provides several of them—is called a *non-trapping condition* (NTC for short). To compare these conditions and to analyse their geometrical properties is the subject of this thesis. We will first focus our attention on the last point, starting with the presentation of the famous conjecture of Peter D. Lax and Ralph S. Phillips.

Chapter 2

The Lax-Phillips Conjecture

It is well known that by the methods of geometric optics one can construct approximate solutions to the Cauchy problem of the wave equation, whose energy stays arbitrarily long near a given characteristic. Dealing with the Dirichlet problem for a homogeneous medium in odd dimensions $n \geqslant 3$, Lax and Phillips therefore made in [La/Ph 67], p. 155, (see also [La/Ph 89], p. 278) their famous conjecture, which will be presented here in a way appropriate for our problem (P). Starting at a point x in $\overline{\Omega}_R$ draw a ray in some direction ω and reflect it according to the classical law of reflection every time it hits the obstacle. Denote by $l(x, \omega)$ the total length (possibly infinity) of that ray within $\overline{\Omega}_R$ and by $l(R)$ the supremum of $l(x, \omega)$ for all such x, ω.

Lax-Phillips Conjecture: $l(R) < \infty \iff$ there is uniform LED.

Because of the formulation we have chosen here, this conjecture is also meaningful in all even dimensions.

It should be remarked that the behaviour of a ray at the boundary is more complicated than Lax and Phillips may have imagined when they presented their conjecture. If such a ray has a tangency of infinite order with the boundary it may not have a unique continuation; this has been proved by Taylor in [Tay 76]. On the other hand one should observe that in the case of inhomogeneous media the optical rays are in general not straight but curved lines. Taking these facts into account, the above conjecture states an assertion which has not been proved completely ever since. Making essential use of the so-called *generalized Huygens' principle*, the "if" part "$\Longrightarrow$" is proven by the results of [Me 79], [Ral 79], and [Ya 92b], if $n \geqslant 3$, and Section 6.3 of this thesis, if $n = 2$. We will discuss these results in more detail later.

For the other half of the conjecture there are the results of Ralston ([Ral 69, Ral 82]), which are subject of the next section. It is still an open problem, whether they include *all* obstacles.

2.1 Ralston's Proofs

In [Ral 69] Ralston considered *admissible paths* Γ in $\overline{\Omega} \subset \mathbb{R}^3$ for the wave equation in a homogeneous medium. These are generalized rays that are nowhere tangent to $\partial\Omega$. Furthermore they consist of only a finite number of straight line segments. Denote by $l_R(\Omega)$ the supremum of the length of the admissible paths contained in $\overline{\Omega}_R$. Ralston proved the following result:

Theorem 2.1.1 *Suppose $l_R(\Omega) = \infty$. Then, given any $\mu, t > 0$, there exist initial data $u_0, u_1 \in \mathring{\mathcal{C}}_\infty(\Omega_R)$ with*

$$E\,(u(0), \Omega_R) = 1\,,$$

such that the solution u of (W.D) satisfies

$$E\,(u(t), \Omega_R) > 1 - \mu\,.$$

This result was proved by the method of geometric optics, where near a caustic the geometrical optics solution is continued by a free space solution.

In [Ral 82] Ralston improved this result considerably. The Gaussian beam construction he used there holds for $n \geqslant 2$, arbitrary strictly hyperbolic differential equations and a large class of boundary conditions, especially the Dirichlet and the Neumann problem for the wave equation. The advantage of Gaussian beams over geometric optics is the use of a complex phase function avoiding the development of caustics. As remarked above, the admissible paths of an inhomogeneous medium do not consist of pure straight line segments any more. Nevertheless the only assumption of importance is that they do not "graze" i.e. that they are not tangent to $\partial\Omega$. For a precise definition of these rays cf. Section 4.1. For the reason of future reference we state this generalization of the above result as a theorem.

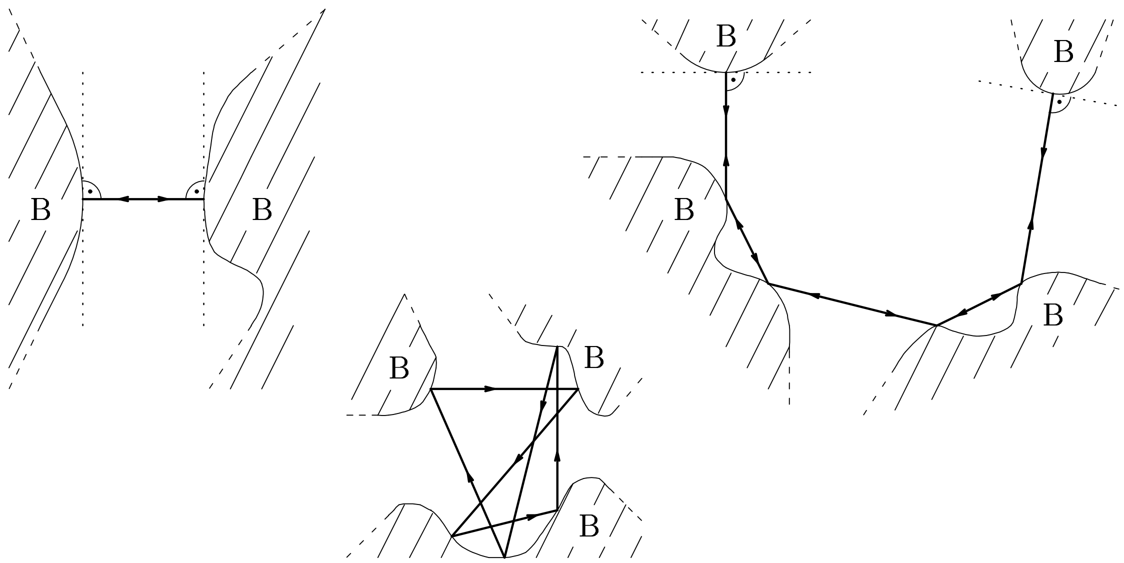
Theorem 2.1.2 *The result of Theorem 2.1.1 holds in arbitrary dimensions $n \geqslant 2$ for (W.D) and (W.N) in both homogeneous and inhomogeneous media.*

This is the possibly incomplete proof of the "only if" part of the Lax-Phillips conjecture. That this result might not be optimal is because it is not known up to now, whether all generalized rays can be approximated by non-grazing ones. This lack of evidence is due to the non-unique continuation of some rays at points of infinite order tangency (cf. Section 4.1).

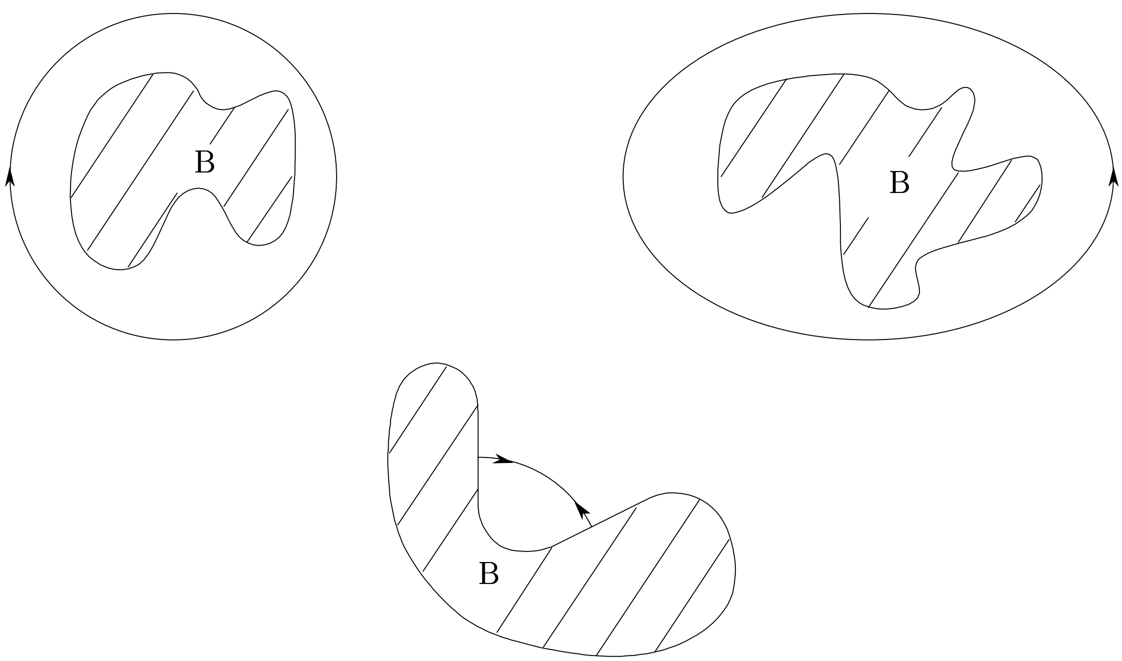
But these results imply the existence of obstacles and media for which uniform LED is impossible. For the homogeneous case there are

- Obstacles with admissible paths that hit the boundary perpendicularly at two points.
- Obstacles with admissible paths that form closed polygons.

Some characteristic examples:



Things are a little bit more complicated for general media. Now the medium itself may trap rays. We will present examples of media that possess circular and elliptic rays respectively (cf. Sections 3.2 and 4.1). So here, too, as well as for the curvilinear analogues of the special admissible paths mentioned above, no uniform LED is possible, because of Theorem 2.1.2. Again, we give some examples:



All these paths are referred to as *periodic* or *closed* rays.

As a short intermezzo before the investigation of the other half of the Lax-Phillips conjecture, we will discuss some interesting results of Walker, Ikawa, and Burq respectively. They proved that in some way slightly different from (P), there is uniform "local energy decay" even for the situations described above.

2.2 Uniform LED for Initial Data with Higher Regularity

According to Ralston's theorems, uniform LED as demanded in (P) is a property that cannot be expected in general. Using Theorem 1.0.1 and the Rellich selection theorem, allows, however, other uniform estimates of the local energy, if the initial data are sufficiently regular as was first proven by Walker in [Wa 77]. Because his proof can be literally applied to arbitrary space dimensions n, we formulate this result here without Walker's original restriction to $\mathbb{R}^3$.

For a compact set $K \subset \overline{\Omega}$ and $\beta \in \mathbb{R}^+$ we denote by $\mathring{\mathcal{H}}_\beta(K)$ the space of functions supported in K which possess square integrable derivatives of all orders up to β. $\|\cdot\|_{\mathring{\mathcal{H}}_\beta(K)}$ denotes the norm of $\mathring{\mathcal{H}}_\beta(K)$ (there a different possibilities for the definition of such spaces, cf. [Ad 75]). Walker proved the following result for a homogeneous medium:

Theorem 2.2.1 *Let $\alpha > 0$, $K \subset \overline{\Omega}$ compact, $u_0 \in \mathring{\mathcal{H}}_{1+\alpha}(K)$, $u_1 \in \mathring{\mathcal{H}}_\alpha(K)$, and*

$$\|(u_0, u_1)\|^2_{\mathring{H}_\alpha(K)} := \|u_0\|^2_{\mathring{\mathcal{H}}_{1+\alpha}(K)} + \|u_1\|^2_{\mathring{\mathcal{H}}_\alpha(K)} .$$

Then for a solution u of (W.D) with such initial data u_0 and u_1, it holds that

$$\int_K (|\nabla u(t)|^2 + u_t(t)^2)\, dx \leqslant P_{K,\alpha}(t)\, \|(u_0, u_1)\|^2_{\mathring{H}_\alpha(K)} \quad \text{with} \quad \lim_{t\to\infty} P_{K,\alpha}(t) = 0 .$$

A corresponding result holds for the Neumann problem and for more general domains, as long as a (non-uniform) decay result like Theorem 1.0.1 is available.

At first sight Theorem 2.2.1 looks surprising, because Ralston proved that in general uniform LED is not possible. Walker's result shows that it is important which norm appears on the right-hand side of the estimate. Our problem (P) demands the energy norm on both sides. And whereas the initial data of the localized solutions of Ralston have an energy equal to 1, they are highly oscillating functions, leading to large norms in their higher derivatives. These norms become larger, the better the approximation is, because the frequency of the oscillations tends towards infinity if the energy is concentrated along a single admissible path.

Walker's result is only of a qualitative nature, because it does not state precise decay rates. Explicit algebraic rates are given in only two special cases in [Wa 77].

Stronger results were proven by Ikawa in 1982 and 1988, finally leading to the following exponential decay for the Dirichlet problem in a homogeneous medium outside several strictly convex obstacles in [Ik 88] (with some assumptions on the positions of the obstacles relative to each other):

$$\int\limits_{\Omega_R} (|\nabla u(t)|^2 + u_t(t)^2)\, dx \leqslant C(R)\, e^{-\delta t} \left(\|u_0\|_3^2 + \|u_1\|_2^2\right) ,$$

$\delta > 0$ being independent of R.

The strict convexity is important for this result according to a counterexample of [Ik 82]: For two balls flattened on the sides facing each other, given by

$$\mathcal{O}_1 \subset \{x_1 \geqslant 1\} , \quad \left\{x_1 = 1 , \; x_2^2 + x_3^2 \leqslant 1\right\} \subset \partial\mathcal{O}_1 ,$$

$$\mathcal{O}_2 \subset \{x_1 \leqslant 0\} , \quad \left\{x_1 = 0 , \; x_2^2 + x_3^2 \leqslant 1\right\} \subset \partial\mathcal{O}_2 ,$$

Ikawa proved that the decay rates of Theorem 2.2.1 are at most algebraic.

While Ikawa dealt with some special geometric situations, it was Burq who gave explicit rates of decay for all exterior domains and both homogeneous and inhomogeneous media. According to his result [Bu 98, Théorème 1] the decay rates of Walker are at least logarithmic:

Theorem 2.2.2 *Let the assumptions of Theorem 2.2.1 be fulfilled. Then for the decay rates $P_{K,\alpha}(t)$ the following estimates hold:*

$$P_{K,\alpha}(t) \leqslant C(K,\alpha)(\ln(2+t))^{-\alpha} .$$

We want to conclude this section with a remark on a paper of Liu and Kazarinoff. For both the Dirichlet and the Neumann problem of a homogeneous medium filling the exterior of two strictly convex obstacles, they proved

$$\exists Q > 0\, \forall u_0, u_1 \in \mathring{\mathcal{C}}_\infty(\Omega)\, \exists T_0 > 0\, \forall t > T_0 \ : \ E\left(u(t), \Omega_R\right) \leqslant Q t^{-1} E\left(u(0), \Omega\right)$$

(cf. [Liu/Kaz 88, Theorem 2]). Although T_0 depends on u_0, u_1, seeming to be already in contrast to uniform LED, the crucial point here is that this result only holds for initial data in a dense subspace of the energy space $H = \mathring{\mathcal{H}}_1(\Omega) \times \mathcal{L}^2(\Omega)$ (or $\mathcal{H}_1(\Omega) \times \mathcal{L}^2(\Omega)$ for the Neumann problem). Since such an estimate for all $(u_0, u_1) \in H$ with compact support in $\overline{\Omega}$ would immediately imply uniform LED by the uniform boundedness principle in contradiction to Theorems 2.1.1 and 2.1.2 (cf. [Rau 78b, p. 150]). This leaves only one possibility for non-smooth initial data in H: during the process of approximation by smooth and compactly supported functions, the time T_0 after which the solution falls under the above regime tends to infinity at least for some data u_0, u_1 compactly supported in $\overline{\Omega}$.

Chapter 3

The Energy Method (Friedrichs' a-b-c Method)

The idea of the energy method is to multiply the differential equation with a factor Nu, N being a linear first-order differential operator, to integrate the result, and to derive estimates out of this identity.

Friedrichs himself used this method to prove the existence of weak solutions. The choice of suitable multipliers led to results even for non-hyperbolic problems (cf. [Fr 54, Fr 58]).

The crucial point for our purposes is to find a good multiplier and to express the product $Nu(\partial_t^2 - \partial_i a_{ij} \partial_j)u$ by suitable so-called *divergence* or *energy* identities. The general case has the form

$$(Au + (B \cdot \nabla u) + Cu_t)(\partial_t^2 - \partial_i a_{ij} \partial_j)u = \operatorname{div}_{t,x}(\cdots) + \text{quadratic terms}. \tag{3.1}$$

After integrating this expression over a domain in $\mathbb{R}^{n+1}$ this leads to certain surface and volume integrals. With the use of this method several uniform decay results were proved in the 1960s and 70s.

Before we discuss them in more detail, it should be mentioned that the first to prove uniform LED for some class of obstacles was Calvin H. Wilcox (cf. [Wil 59]). For the exterior of a spherical body $\mathrm{B} \subset \mathbb{R}^3$ he showed that the decay is at least exponential.

3.1 The Homogeneous Case

We start our considerations with the discussion of the wave equation for a homogeneous medium. This is on the one hand due to historical reasons, for the first results were achieved for the d'Alembert operator $\partial_t^2 - \Delta$. On the other hand the medium is fixed in this case. Therefore, only conditions on the obstacle have to be imposed. Considering inhomogeneous media in general, additional restrictions have to be made, for the medium itself may trap rays, as we will see in some detail later. The aim of this section is to give a short survey of the results with the main attention lying on the geometric conditions and their relations to each other. For more details cf. the quoted literature or the extended survey in [Pau 96, Chapters 3–6].

3.1.1 Star-Shaped Obstacles

Cathleen S. Morawetz was the first to succeed in proving uniform LED by the energy method. We want to give some motivation for her choice of the multipliers. If one intends to estimate the energy at time t by the initial energy, it is useful to integrate the divergence identity over the domain $(0,t) \times \Omega$. For initial data with compact support the finite propagation speed leads to surface integrals over Ω at times 0 and t and over $(0,t) \times \partial\Omega$. But in general a volume integral of the quadratic forms appears, too. A first choice of a multiplier may try to avoid these quadratic forms and, therefore, the volume integral. Using a certain symmetry argument this leaves only three linear independent multipliers (cf. [Pau 96] or [Mor 62] for a similar approach):

$A,\ B,\ C$	multiplier Nu	used in
$A = 0,\ B = 0,\ C = 1$	u_t	standard energy identity
$A = \frac{n-1}{2},\ B = x,\ C = t$	$\frac{n-1}{2} u + x \cdot \nabla u + t u_t$	[Mor 61]
$A = (n-1)t,\ B = 2tx,\ C = \lvert x\rvert^2 + t^2$	$(n-1)tu + 2\,t(x \cdot \nabla u) + (\lvert x\rvert^2 + t^2)u_t$	[Mor 62, Mor 67]

Whereas the standard energy identity

$$0 = u_t \,\Box u = \frac{1}{2}\left(|\nabla u|^2 + u_t^2\right)_t - \operatorname{div}_x(u_t \nabla u)$$

yields the conservation of the total energy, the other multipliers can be successfully used for decay estimates. But for the boundary integrals to have the "right" sign in the estimates a condition on the obstacle has to be imposed:

Definition 3.1.1 *An obstacle* B *is called* star-shaped, *if* $x \cdot \mathbf{n} \leqslant 0$ *for* $x \in \partial\mathrm{B}$. *It is called* strictly star-shaped *if* $x \cdot \mathbf{n} < 0$ *for* $x \in \partial\mathrm{B}$.

Remark: This is actually the definition of star-shapedness with respect to the origin. If the obstacle is star-shaped with respect to some $x_0 \in \mathbb{R}^n \setminus \{0\}$, one simply transforms the whole system to $x_0 = 0$. That may, for an inhomogeneous medium, change the radii R_0 and R from the beginning of Chapter 1, but not the class of problems under consideration here. This justifies dealing with star-shapedness with respect to the origin, only.

For star-shaped obstacles Morawetz derived a decay rate of ct^{-1} for $n \geqslant 2$ in [Mor 61]. She later improved this result for $n = 3$ to ct^{-2} in [Mor 62, Mor 67], the latter publication making an interesting ansatz using the Kelvin transformation.

A closer look at the proofs shows that the star-shapedness of the obstacle is in fact necessary and sufficient for the boundary integrals to have the "right" sign. Thus the above multipliers are insufficient for more general obstacles. On the other hand other linear combinations of these multipliers will not lead to better decay rates, for these rates are correlated to the highest exponent of t in these multipliers, a number that is not larger for linear combinations. Since 1962 the results of Morawetz have been considerably improved in several directions: The class of obstacles under consideration has been enlarged, better (optimal?) decay rates have been achieved, and the energy method has been applied to a wider class of equations. Whereas the latter will be discussed in Section 3.2 and Part II, a survey of the development of the first two points will be given next.

3.1.2 Exponential LED

There is an essential difference between odd ($\geqslant 3$) and even space dimensions. Whereas Huygens' principle holds in the former case, it does not in the latter. A consequence of this principle is that in free space a wave originating from an initial disturbance of compact support, has only a finite time of influence on any point in space. The situation is completely different, if the space dimension is even. Then there are persistent reverberations even for initial data of compact support. This has a remarkable effect on the decay of the local energy. While algebraic decay rates are the best one can achieve in even space dimensions, a much better result holds for odd n. This has been proved by means of the scattering theory of Lax and Phillips in [L/M/P 62, L/M/P 63] and by repeated subtraction of free space solutions in [Mor 66]. This last result of Morawetz was proved for homogeneous media and arbitrary *energy conserving* boundary conditions (for the precise Definition cf. [Mor 66]), including the Dirichlet, the Neumann, and the Robin problem. Because the crucial point in the proof is the validity of Huygens' principle, this result holds for all media treated in this thesis. Thus we formulate it here for the general situation:

Theorem 3.1.2 *Let $n \geqslant 3$ be odd and let the boundary condition $\Lambda u|_{\partial\Omega} = 0$ be energy conserving. Suppose uniform LED according to* (P) *holds for solutions of (W.Λ). Then the decay is in fact exponential.*

Because of its elegance, we just want to sketch the idea of the proof of Lax, Morawetz, and Phillips in [L/M/P 62, L/M/P 63]: The assumed uniform local energy decay yields that the operator norm of the Lax-Phillips semigroup $Z(t)$ becomes smaller than one at some time $T > 0$. But then the semigroup property implies that this operator norm decays at an exponential rate. Because $Z(t)$ fits exactly to the situation of a solution to compactly supported initial data, studied in a compact subset of $\overline{\Omega}$, this is equivalent to exponential LED.

From now on it is sufficient to derive any decay rate in odd dimensions $n \geqslant 3$; for then Theorem 3.1.2 shows that this decay is actually exponential. The results of [Mor 61, Mor 62, Mor 67], therefore, already yield exponential LED for a homogeneous medium outside a star-shaped obstacle.

We want to conclude this short section by a somewhat exotic application of Theorem 3.1.2 that has already been remarked in [Mor 75b, p. 26]: In Section 2.2 we introduced the decay results of Walker and of Burq for initial data with a higher regularity. Here the iterative procedure of [Mor 66] can be applied, too. But in each step of the iteration the assumed decay result has to be applied. Therefore in each step more regularity of the initial data is demanded. For smooth initial disturbances supported in $K \subset \overline{\Omega}$ this leads for all exterior domains Ω to the estimate

$$E\left(u(t), K\right) \leqslant c\, e^{-\delta t} \left\|(u_0, u_1)\right\|_{\mathring{H}_{t/T}(K)}$$

with positive constants c, δ, and T.

3.1.3 More General Obstacles

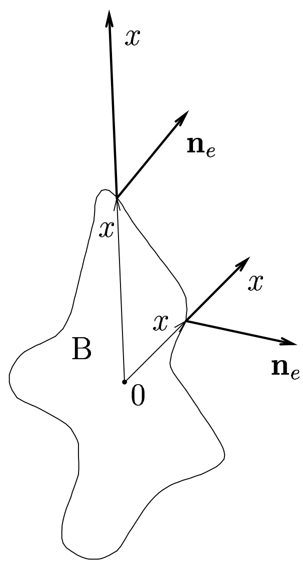
The ansatz to avoid the quadratic terms in (3.1) excluded all but three multipliers and made the star-shapedness necessary for decay estimates. Using other multipliers will lead to volume integrals, which we tried to avoid in Section 3.1.1. But possibly these integrals may be useful in estimating the energy. This in fact allows, as we will see next, larger classes of obstacles.

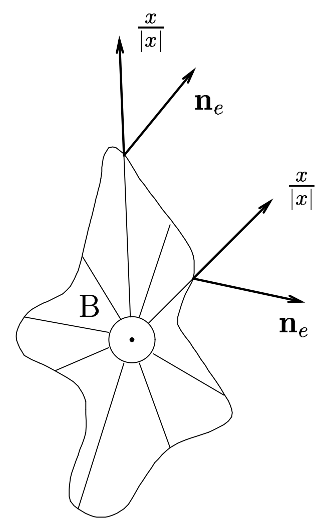
The first extension is an illustrative generalization of the strict star-shapedness. The condition $x \cdot \mathbf{n} < 0$ (i.e. $x \cdot \mathbf{n}_e > 0$) implies that each ray beginning at the origin intersects $\partial\Omega$ exactly once. This fact could be described this way: the interior of the obstacle can be illuminated by a source of light situated at the origin. Since the origin is an interior point of the obstacle, a small ball centred at the origin is contained in the interior of B. Thus the above rays of light are perpendicular to the surface of this ball. The idea of Bloom and Kazarinoff was to allow strictly convex bodies instead of balls for the illumination. Thus they define in [Bl/Kaz 74b]:

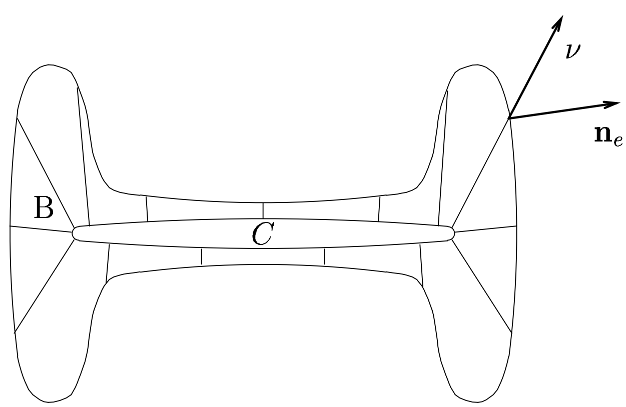
Definition 3.1.3 *An obstacle* B *can be* illuminated from the interior *if there exists a smooth strictly convex body C inside* B *such that $\mathbb{R}^n \setminus C$ is filled by a family of rays normal to ∂C and such that each ray intersects ∂*B *exactly once.*

Remark: In this form the definition is a generalization of the strict star-shapedness only. A closer look at [Bl/Kaz 74b] shows that it is sufficient to demand that an illuminating ray never enters the interior of the obstacle again after having once reached its boundary. This would include the possibility of illuminating rays that follow a straight part of the boundary for a certain time, before leaving the obstacle. The latter extension of the above defintion 3.1.3 would include all star-shaped obstacles.

The following images show how Definition 3.1.3 generalizes the strict star-shapedness. The last obstacle (due to Bloom and Kazarinoff) is, obviously, not strictly star-shaped.







“dog bone”

Now let B be illuminated from the interior by a strictly convex body C. Thus every point $x \in \mathbb{R}^n \setminus C$ is situated on one and only one ray starting at $c(x) \in \partial C$ and being orthogonal to ∂C with direction $\nu(c(x))$. Thus an $s(x) \in \mathbb{R}^+$ exists, such that

$$x = c(x) + s(x)\, \nu(c(x))\,.$$

Bloom and Kazarinoff define the following vector field:

$$\alpha^{(n)}(x) := s(x)\,\nu(c(x))\,.$$

Because B can be illuminated from the interior, for all $x \in \partial\mathrm{B}$ the following holds

$$\mathbf{n}_e(x)\cdot\alpha^{(n)}(x) = s(x)(\mathbf{n}_e(x)\cdot\nu(c(x))) \geqslant 0\,. \tag{3.2}$$

Now

$$Nu := \frac{n-1}{2}u + \alpha^{(n)}(x)\cdot\nabla u + tu_t \tag{3.3}$$

is chosen as multiplier in [Bl/Kaz 74b]. Integration of the resulting divergence identity yields a surface integral over $[t_0,t]\times\partial\Omega$. The integrand

$$\frac{1}{2}\left(\mathbf{n}_e(x)\cdot\alpha^{(n)}(x)\right)\left|\frac{\partial u}{\partial\mathbf{n}}\right|^2$$

has the desired positive semidefiniteness because of (3.2). Without going into details we want to mention that the resulting volume integrals, as announced at the beginning of this section, are very useful in the estimates. Thus Bloom and Kazarinoff derive the following decay rates for (W.D) and a homogeneous medium:

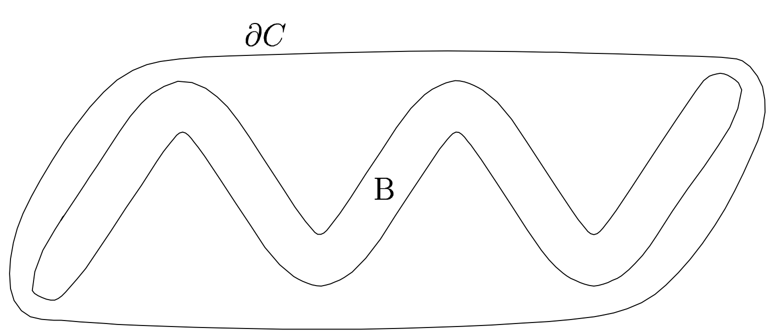
- $n=2$: $ct^{-1+\delta}$ with some $\delta\in(0,1)$,
- $n=3$: $ce^{-\delta t}$ with some $\delta>0$,

the latter using Theorem 3.1.2 after having established a polynomial rate. It should be remarked that the result for $n=3$ was proven under an additional geometric restriction on the obstacle. But we will see that this condition was of purely technical nature, for the above $\alpha^{(n)}$ satisfies (3.2), as we have already seen, and

$$\partial_i\alpha_j^{(n)}(x)\xi_i\xi_j \geqslant c\,|\xi|^2\,,\qquad x\in\overline{\Omega},\;\;\xi\in\mathbb{R}^n\,,$$

with a $c>0$, because of the strict convexity of C (cf. Sections 4 and 8 of [Bl/Kaz 74b]). Thus $\alpha^{(n)}$ is a special case of the vector fields of Walter A. Strauss, which we will discuss later.

An obstacle B that cannot be illuminated from the interior may have the following "snake-shape":



There is no strictly convex body in the interior of B possible that illuminates the obstacle. But uniform LED still holds, because there obviously exists a strictly convex body C whose boundary is able to illuminate ∂B from the *exterior* in the following way:

Definition 3.1.4 *We say that the boundary of an exterior domain Ω can be* illuminated from the exterior *iff there exists a convex body C containing $\partial\Omega$ with smooth boundary ∂C such that $\partial\Omega$ is filled by a family of non-intersecting rays normal to ∂C. Each ray is completely contained in Ω in the following sense: for each $x_0 \in \partial\Omega$ there exists a unique $x_1 \in \partial C$ and a number $s_0(x_1) \leqslant 0$ such that*

$$x_0 = s_0(x_1)\nu(x_1) + x_1 \,,$$

where ν is the outward normal to $\partial\Omega$ at x_1, and

$$x = t\nu(x_1) + x_1 \in \Omega \,, \qquad \text{for } t \in (s_0, \infty) \,.$$

Here we have chosen the definition of Liu De-Fu [Liu 87, Definition 1, p. 314], which is more general than the one of [Bl/Kaz 76, Definition 2.1, p. 24]. In relation to the star-shapedness of an obstacle the same difficulty arises here that was already remarked after Definition 3.1.3. We will briefly state the consequences of Definition 3.1.4: If $\partial\Omega$ (or B) can be illuminated from the exterior, then for all $x \in \overline{\Omega}$ there exist unique $s(x) \in \mathbb{R}$ and $c(x) \in \partial C$ such that

$$x = c(x) + s(x)\,\nu(c(x)) \,.$$

Because of the boundedness of C, there exists a $q \in \mathbb{R}^+$ such that $s(x) + q > 0$ in $\overline{\Omega}$. The vector field

$$\beta^{(n)}(x) = (s(x) + q)\nu(c(x))$$

satisfies for $n = 2, 3$

$$\begin{aligned} &(i) \quad \beta^{(n)}(x) \cdot \mathbf{n}_e(x) \geqslant 0 \,, && x \in \partial\Omega \,, \\ &(ii) \quad \partial_i \beta_j^{(n)}(x)\xi_i\xi_j \geqslant c\,|\xi|^2 \,, && x \in \overline{\Omega}, \ \ \xi \in \mathbb{R}^n \,, \end{aligned}$$

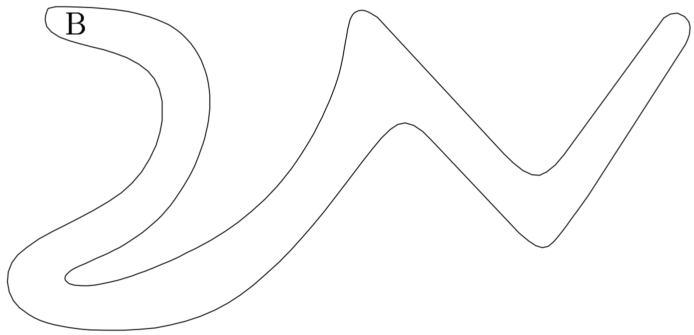
with a $c > 0$, (i) being an immediate consequence of Definition 3.1.4 and (ii) following from the calculations of [Liu 87, Section 1] (or [Bl/Kaz 76, Lemma 4.4]). Thus $\beta^{(n)}(x)$, too, provides one of those Straussian vector fields we discuss next. Definition 3.1.4 actually generalizes Definition 3.1.3, for every body that can be illuminated from the interior can also be illuminated from its exterior: One just takes a level surface of equidistant points to the (interior) illuminating body C that contains the obstacle. This enlargement of C then illuminates B from the exterior (cf. [Liu 87], remark 3, p. 330).

For the sake of completeness we want to mention that Liu proved an algebraic decay for a class of hyperbolic systems of second-order equations in the exterior of an obstacle in $\mathbb{R}^3$ that can be illuminated from the exterior. He used the multiplier

$$2tu + 2t\beta^{(n)}(x)\nabla u + (t^2 + (s(x) + q)^2)u \,,$$

which is a generalization of the third multiplier we presented at the beginning of Section 3.1.1, in contrast to the choice (3.3) of [Bl/Kaz 74b], which obviously generalizes the second.

Let us assume that the above snake is in a less pleasant position:



For such a 2-d snake there is no chance for a convex body to illuminate it, neither from the interior nor from the exterior. But this obstacle still allows uniform decay of the local energy, as the next results will show.

In [Str 75], Strauss proved uniform LED for (W.D) in exterior domains in $\mathbb{R}^n$, $n \geqslant 3$, provided a strictly expansive $\mathcal{C}_3$ vector field exists, that leaves $\overline{\Omega}$ strictly invariant. This means the following: There exists a $c > 0$, such that

$$\begin{array}{lll} (i) & l(x) \cdot \mathbf{n}_e(x) > 0\,, & x \in \partial\Omega\,, \\ (ii) & \partial_i l_j(x)\xi_i\xi_j \geqslant c\,|\xi|^2\,, & x \in \overline{\Omega},\ \ \xi \in \mathbb{R}^n\,. \end{array}$$

Strauss in fact demanded that l is the gradient of a function of $|x|$ outside $\overline{\Omega}_R$, but this can always be achieved, if (i) and (ii) hold, as Strauss has already conjectured in [Str 75] (a proof was given in [M/R/S 77]). Assuming that such a vector field exists Strauss used the multiplier

$$l \cdot \nabla u + \frac{1}{2}\,(\mathrm{div}_{\,x})\,u$$

and proved the following results:

- $n \geqslant$ odd: $\quad E\,(u(t), K) \leqslant c\,e^{-\delta t} E_0\,, \quad t \geqslant 0$
- $n \geqslant 4$ even: $\quad \int_0^\infty E\,(u(t), K)\,dt \leqslant c\,E_0\,,$

with some positive constants c and δ. This last uniform LED in an "integrated sense" implies a decay rate of at least ct^{-1} as was later shown in [M/R/S 77].

The above requirement on l can be somewhat relaxed, for Strauss proved that strictness in (i) or (ii) is sufficient for the existence of a vector field that is strict in both estimates. Non-strictness in (i) and (ii) is of course insufficient, because this is always fulfilled by $l \equiv 0$ (the asymptotic condition on l still causes no problem). And this cannot generally imply a uniform decay according to Theorems 2.1.1 and 2.1.2.

At the same time Morawetz proved a similar result for domains in $\mathbb{R}^2$ and $\mathbb{R}^3$ assuming the existence of some convex function χ (cf. [Mor 75a]). But then $\nabla\chi$ provides exactly such a Straussian vector field l. Morawetz proved for (W.D) uniform LED with the rates

- ct^{-2} for $n = 2$,
- $ce^{-\delta t}$ for $n = 3$.

Because she solely worked with $\nabla\chi$ instead of χ—without making use of the fact that it is a *gradient* field—her results actually hold under the more general conditions of Strauss. We want to add that Morawetz studied not only the Dirichlet problem in [Mor 75a]. For *convex* obstacles she established the above rates even for the Neumann boundary condition.

As we have already remarked above, the existence of a Straussian vector field is a generalization of the property that an obstacle can be illuminated from the interior. This includes the case of *strictly* star-shaped obstacles. But the choice of $l(x) = x$ provides a vector field which is strict in (ii) and non-strict in (i) for all star-shaped obstacles. Therefore, the existence of a Straussian vector field generalizes all the geometric conditions we have dealt with so far. And with some restriction on the behaviour of the curvature of the boundary, the existence of a Straussian vector field is an optimal requirement on obstacles in R^2. This is a consequence of the following theorem, proven in [M/R/S 77, Section 4]:

Theorem 3.1.5 *Let $\Omega \subset \mathbb{R}^2$. Suppose the curvature of $\partial\Omega$ does not change sign infinitely often and no straight line segment in $\overline{\Omega}$ is perpendicular to $\partial\Omega$ at both ends. Then a Straussian vector field exists.*

For later convenience, we will introduce a notation for obstacles that fulfil the assumptions of the above theorem:

Definition 3.1.6 *We say that* $\mathrm{B} \in \mathbb{R}^2$ *is* snake-shaped *if the curvature of* $\partial\mathrm{B}$ *does not change sign infinitely often and no straight line segment in $\overline{\Omega}$ is perpendicular to $\partial\Omega$ at both ends.*

Thus the non-illuminable obstacle on page 20 is snake-shaped, implying the existence of a Straussian vector field according to Theorem 3.1.5 and, therefore, uniform LED for (W.D) in a homogeneous medium.

If on the other hand one of the segments excluded in Theorem 3.1.5 exists, Theorem 2.1.2 can be applied, as long as only the endpoints lie on $\partial\Omega$. Otherwise the segment would be tangent to $\partial\Omega$, a situation explicitly excluded in Ralston's proof. But nevertheless Ralston's theorem can be used even here, since the boundary is locally convex near a point where such a segment is tangent to it. And, therefore, the results of [Me/Sj I, Me/Sj II] (they will be discussed in Chapter 4) together with an application in [Ya 86] show that a ray propagating along this segment can be approximated by the admissible paths of Ralston. This finally proves the following theorem:

Theorem 3.1.7 *Let $\Omega \subset \mathbb{R}^2$ and let the medium be homogeneous. Suppose the curvature of $\partial\Omega$ does not change sign infinitely often. Then the following properties are equivalent:*

(i) No straight line segment in $\overline{\Omega}$ is perpendicular to $\partial\Omega$ at both ends.

(ii) The supremum of the lengths of all admissible paths in $\overline{\Omega}_R$ is finite.

(iii) A Straussian vector field exists.

(iv) For (W.D) uniform LED holds.

Remark: The condition on the boundary curvature is a (not very strong) non-degeneracy condition. If it is violated, the obstacle must, because of its compactness, have an oscillating boundary. Thus Theorem 3.1.7 seems to be optimal for most of the obstacles that may appear in applications.

Whereas the existence of a Straussian vector field is a good criterion for uniform LED in $\mathbb{R}^2$, it seems to be too special in higher dimensions as obstacles like the one due to P. Ungar in [Mor 75a, Section 3.2, p. 259f.] show. This motivates the introduction of a generalization of these vector fields in [M/R/S 77]—the crowning publication for the energy method we discuss next.

3.1.4 Escape Functions

The aim of Morawetz, Ralston, and Strauss was to introduce a sufficiently general class of smooth functions which are strictly increasing along generalized bicharacteristics (heuristically: generalized rays $\times$ direction ξ). These functions would yield an upper bound of the lengths of generalized rays in $\overline{\Omega}_R$. Thus the authors of [M/R/S 77] define so-called *escape functions* (we use the notations $\frac{\partial\wp}{\partial x}(x,\xi) := \nabla_x \wp(x,\xi)$ etc. of [M/R/S 77]).

Definition 3.1.8 *A function $\wp(x,\xi) \in \mathcal{C}_\infty(\overline{\Omega} \times S^{n-1})$ is called* escape function $:\Longleftrightarrow$

$$(i) \quad \xi \cdot \frac{\partial\wp}{\partial x}(x,\xi) > 0 \qquad \text{in } \overline{\Omega} \times S^{n-1},$$

$$(ii) \quad \left.\begin{array}{ll} \dfrac{\wp(x,\xi) - \wp(x,\check{\xi})}{(\xi - \check{\xi}) \cdot \mathbf{n}_e(x)} > 0, & \text{if } \xi \cdot \mathbf{n}_e(x) > 0 \\ \mathbf{n}_e(x) \cdot \dfrac{\partial\wp}{\partial\xi}(x,\xi) > 0, & \text{if } \xi \cdot \mathbf{n}_e(x) = 0 \end{array}\right\} \text{ on } \partial\Omega \times S^{n-1}.$$

Here $\check{\xi} := \xi - 2(\xi \cdot \mathbf{n}_e(x))\mathbf{n}_e(x)$.

Remarks:

1. $\breve{\xi}$ and ξ are, as we will see later, the directions of an incoming ray and its reflection at $x \in \partial\Omega$ (or vice versa). $\xi \cdot \mathbf{n}_e(x) = 0$ corresponds to a tangential ray.

2. (i) means that $\wp$ is strictly increasing along bicharacteristics and according to (ii) this is still true at the boundary, especially at points of reflection.

3. The class of escape functions remains unchanged, if $\xi \cdot \mathbf{n}_e(x) > 0$ is replaced by $\xi \cdot \mathbf{n}_e(x) \neq 0$ in (ii).

4. In the special geometric situations discussed in Sections 3.1.1 and 3.1.3
$$x \cdot \xi\,, \quad \alpha^{(n)}(x) \cdot \xi\,, \quad \beta^{(n)}(x) \cdot \xi\,, \quad \nabla\chi(x) \cdot \xi\,, \quad \text{and} \quad l(x) \cdot \xi$$
provide escape functions that are linear in ξ. This is due to the fact that smoothing is possible for every Straussian vector field without changing the other properties. The result may be strict only in (i) or (ii) of Definition 3.1.8, but this causes no problem:

5. For an escape function, too, strictness in (i) or (ii) is sufficient. The proof is essentially the same as the one for a Straussian vector field.

6. It is sufficient to have an escape function for $x \in \overline{\Omega}_R$. This already yields the existence of an escape function for all $x \in \overline{\Omega}$, which, furthermore, satisfies
$$\wp(x,\xi) = c\left(1 - \frac{1}{|x|}\right)\frac{x}{|x|} \cdot \xi \qquad \text{for } |x| > 2R$$
(for details cf. [M/R/S 77]).

That this definition implies in fact that escape functions are strictly increasing along generalized bicharacteristics has been proved [Pau 96, Section 7.2]. In this thesis a corresponding result for isotropic elasticity is proved in Theorem 9.5.3.

Assuming the existence of an escape function, Morawetz, Ralston, and Strauss used $\wp$ for the construction of a pseudodifferential operator $P(x,D)$ and applied Pu as multiplier for the energy method. They proved the following result:

Theorem 3.1.9 *Let $n \geqslant 3$. Let u be a solution of (W.D) in a homogeneous medium for initial data u_0, u_1 with* $\operatorname{supp}(u_0,u_1) \subset \overline{\Omega}_R$. *Suppose an escape function $\wp$ exists. Then for all $a > 0$ there exist constants $c, \delta > 0$, such that*

$$\begin{aligned} &(i)\ n\ \textit{even:} && E\left(u(t),\Omega_a\right) \leqslant c\,t^{-1} E\left(u(0),\Omega\right), && t > 0\,,\\ &(ii)\ n\ \textit{odd:} && E\left(u(t),\Omega_a\right) \leqslant c\,e^{-\delta t} E\left(u(0),\Omega\right), && t \geqslant 0\,. \end{aligned}$$

Remark: The methods of [M/R/S 77] cannot be applied to domains in $\mathbb{R}^2$, but here the existence of a Straussian vector field is almost equivalent to uniform LED according to Theorem 3.1.7. Nevertheless generalizations are possible (cf. [M/R/S 77, M/R/S 78]): On the one hand all the inhomogeneous media subject of this thesis can be treated, as we will discuss in the next sections. On the other hand various boundary conditions are possible, for example the Robin condition, but *not* the Neumann condition.

3.2 The Inhomogeneous Case

The energy method was successfully applied to the case of inhomogeneous media, as well. But here conditions on the obstacle *and* the medium have to be imposed, as we have already remarked in Section 2.1. Ralston mentioned in [Ral 71] that operators of the form

$$\partial_t^2 - c(r)\Delta \qquad \text{with } c(r) = r^2 \text{ for } r_1 \leqslant r \leqslant r_2 \tag{3.4}$$

have circular rays for all r between r_1 and r_2, if they are studied in free space or in an exterior domain whose complement is contained in B_{r_1}. We calculated an operator in $\mathbb{R}^2$ that leads to elliptic rays: Denote by $A(x)$ the coefficient matrix of the space operator. Let $a, b > 0$ and

$$\rho(x) := \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}\,. \tag{3.5}$$

The choice of

$$A(x) = \begin{pmatrix} a^2\rho(x) & 0 \\ 0 & b^2\rho(x) \end{pmatrix} \qquad \text{in a neighbourhood of } \rho^{-1}(1) \tag{3.6}$$

yields the elliptic rays

$$x(s) = \begin{pmatrix} a\cos s \\ b\sin s \end{pmatrix}$$

in free space and in the case of obstacles B, which do not intersect this ray (we will give a proof in Section 4.1). These situations are illustrated in the figures of Section 2.1. There we presented a star-shaped obstacle with a circular segment making perpendicular incidence at both ends, too. For this situation it is even sufficient that the differential operator has the form (3.4) only locally and not in a whole annular region.

In the literature there are decay results for inhomogeneous media, proven by the energy method, too. The first of them treat star-shaped obstacles. Zachmanoglou considers in [Za 66] a large class of equations containing the problems studied here. The same is true for [Bl/Kaz 73, Bl/Kaz 74a] by Bloom and Kazarinoff. In contrast to Zachmanoglou they allow time-dependent coefficients such that the total energy of the system grows algebraically in time. If one reduces the numerous conditions of all these authors to the case under consideration here, it is essentially one restriction that remains: They suppose an $\alpha_0 \in [0, 1)$ exists such that

$$r\partial_r a_{ij}(x)\xi_i\xi_j \leqslant \alpha_0 a_{ij}(x)\xi_i\xi_j\,, \qquad \xi \in \mathbb{R}_0 \setminus \{0\},\ \ x \in \overline{\Omega}\,. \tag{3.7}$$

For (W.D) in the exterior of a star-shaped obstacle they established the following decay rates:

- $ct^{-(1-\alpha_1)}$, $\quad n \geqslant 3,\quad$ in [Za 66]
- $ct^{-(2-2\alpha_1)}$, $\quad n = 3,\quad$ in [Bl/Kaz 73, Bl/Kaz 74a],

with $\alpha_1 = \alpha_0 + \varepsilon$ for an arbitrary $\varepsilon > 0$. The decay rates in odd dimensions are in fact exponential by Theorem 3.1.2. One interesting feature of Zachmanoglou is, as we want to add shortly, that he proved the above result for *unbounded* star-shaped obstacles, too, as he had done before in [Za 63] for a homogeneous medium.

We have already mentioned that inhomogeneous media are discussed in [M/R/S 77], as well. But this requires a generalization of the above definition of an escape function. With $q(x,\xi) := \langle \xi, A(x)\xi \rangle$ and

$$K_R := \left\{ (x,\xi) \;\middle|\; x \in \overline{\Omega}_R,\ q(x,\xi) = 1 \right\}$$

this is done by

Definition 3.2.1 *A function* $\wp(x,\xi) \in \mathcal{C}_\infty(K_R)$ *is called* escape function $:\Longleftrightarrow$

$$(i)\quad \frac{\partial q}{\partial \xi} \cdot \frac{\partial \wp}{\partial x} - \frac{\partial q}{\partial x} \cdot \frac{\partial \wp}{\partial \xi} > 0 \qquad in \quad K_R\,,$$

$$(ii)\quad \left.\begin{array}{ll} \dfrac{\wp(x,\xi) - \wp(x,\breve{\xi}\,)}{(\xi - \breve{\xi}\,)\cdot \mathbf{n}_e(x)} > 0\,, & if\ \ A(x)\xi \cdot \mathbf{n}_e(x) > 0 \\ \mathbf{n}_e(x) \cdot \dfrac{\partial \wp}{\partial \xi}(x,\xi) > 0\,, & if\ \ A(x)\xi \cdot \mathbf{n}_e(x) = 0 \end{array}\right\} \ on\ \left\{ (x,\xi) \in K_R \;\middle|\; x \in \partial\Omega \right\}.$$

Here $\breve{\xi} := \xi - 2\frac{A(x)\xi \cdot \mathbf{n}_e(x)}{A(x)\mathbf{n}_e(x) \cdot \mathbf{n}_e(x)} \mathbf{n}_e(x)$.

Remarks:

1. If $a_{ij} = \delta_{ij}$ this is exactly a restatement of Definition 3.1.8.
2. In dealing with this general class of media, only the strictness in (ii) can be relaxed, as a counterexample at the end of this section shows.

Provided such an escape function exists, a slight modification of the proof of Theorem 3.1.9 yields the same result for (W.D) in an inhomogeneous medium (cf. [M/R/S 77, Section 6]).

In the homogeneous case, we discussed in some detail geometric conditions which imply the existence of special escape functions, i.e. of Straussian vector fields. We want to analyse now which additional conditions have to be imposed on the medium when the general case is studied. So let us assume that a Straussian vector field l exists. If we denote by $A_l(x)$ the derivative of $A(x)$ in direction l, then $l(x) \cdot \xi$ being an escape function requires

$$\begin{array}{ll} (i) & 2\langle A(x)\xi, \nabla_x (l(x)\cdot\xi)\rangle - |l(x)| \langle A_l(x)\xi, \xi\rangle > 0 \qquad \text{in} \quad K_R\,, \\ (ii) & \mathbf{n}_e(x) \cdot l(x) > 0 \qquad \text{on} \left\{ (x,\xi) \in K_R \;\middle|\; x \in \partial\Omega \right\}. \end{array} \tag{3.8}$$

Of course (ii) is automatically fulfilled by Straussian vector fields, but (i) is an additional condition if $A(x) \neq I_n$. If for instance the obstacle is star-shaped or snake-shaped or can be illuminated from the interior, i.e. if $l(x)$ is assumed to exist, then (i) is in fact a condition on the medium only. With this additional requirement these obstacles still possess escape functions. This especially yields uniform LED if the space dimension is larger than two. Because we will see later that the existence of an escape function implies uniform decay even for $n = 2$, we should bear in mind that snake-shaped obstacles have escape functions in appropriate media.

Condition (i) may be a little abstract. But it becomes very clear, if star-shaped obstacles are considered. Here the Straussian vector field is of the simple form $l(x) = x$. This simplifies (i) considerably. The result is the requirement

$$2\langle \xi, A(x)\xi \rangle - r\langle \xi, A_r(x)\xi \rangle > 0 \qquad \text{in} \quad K_R\,. \tag{3.9}$$

This condition is obviously weaker than (3.7), the assumption of [Za 66, Bl/Kaz 73, Bl/Kaz 74a]. So [M/R/S 77] generalized these results even for star-shaped obstacles. But can condition (3.9) be relaxed? At the beginning of this section we gave an example of a differential operator which makes uniform LED impossible for general star-shaped obstacles (the operator is locally of the form of (3.4)). For this special medium, locally

$$2\langle \xi, A(x)\xi \rangle = 2r^2|\xi|^2 = r\partial_r r^2|\xi|^2 = r\langle \xi, A_r(x)\xi \rangle$$

holds, violating (3.9). Thus condition (3.9) is sharp. We want to add that this proves the remark concerning the strictness in Definition 3.2.1, as well: If (3.4) is valid and a strictly star-shaped obstacle is lying within the interior sphere, then $x \cdot \xi$ is strict in (ii) and non-strict in (i), whereas uniform decay is not possible because of the circular rays.

The condition (3.9) was first used by Bloom in 1973 for an analysis of the resolvent of the stationary problem (his formulation [Bl 73, p. 311] is somewhat stronger, but what he really makes use of is (3.9)). We, therefore, refer to it as *Bloom's condition*. This requirement is also used by Rustenbach ([Ru 93], (2.12), p. 27) and in the publications [Rac 90a, Rac 90b, Rac 97] of Racke on damped systems and generalized Fourier transforms. The shape of Bloom's condition in these papers is always a little bit stronger than (3.9)—essentially depending on the matrix norms which are used—but it would be sufficient in all these cases to use the form presented here. The common aim of the authors is to derive good high-frequency estimates of the resolvent. For this purpose a non-trapping condition has to be imposed, as we will see in Chapter 6, when we consider the Vainberg method. And a star-shaped obstacle together with (3.9) is a situation in which such a condition is satisfied, as will be one of the results of this thesis.

3.3 Summary

We want to conclude our discussion of the energy method by summarizing the results: Starting with spherical obstacles the conditions on B could be repeatedly relaxed. For the non-homogeneous case however, additional requirements on the medium like Bloom's condition had to be imposed. All this culminated in the Definition 3.2.1 of a general escape function. All relations are expressed by the following diagram a non-trapping condition:

$$\text{B spherical} \Rightarrow \left\{ \begin{array}{c} \text{B strictly star-shaped} \Rightarrow \left\{ \begin{array}{c} \text{B star-shaped} \\ \text{B illum. int.} \Rightarrow \text{B illum. ext.} \end{array} \right\} \\ \\ \text{B snake-shaped} \end{array} \right\} \Rightarrow$$

$$\Rightarrow \qquad \exists\, l \qquad \overset{(*)}{\Rightarrow} \qquad \exists\, \wp\,,$$

where the $(*)$ means that condition (i) of (3.8) has to be fulfilled, which is an additional requirement for non-homogeneous media only.

In the following situations we are equipped with uniform decay results:

- For (W.D) and $n \geqslant 3$, if an escape function exists.
- For (W.D) and $n = 2$, if the medium is homogeneous and a Straussian vector field exists.
- For (W.N) and $n = 2, 3$, if the medium is homogeneous and the obstacle is convex.

The decay rates are algebraic in even dimensions and exponential—because of Huygens' principle—in odd dimensions.

For a homogeneous medium in $\mathbb{R}^2$ and the Dirichlet boundary condition we have the equivalences of Theorem 3.1.7 for a large class of obstacles, namely for those with non-oscillating boundaries.

Our aim in the rest of Part I is to establish such equivalence theorems for all dimensions. In proceeding this way we will improve the above list of decay results, as well. For this we have to give first the exact definition of a generalized bicharacteristic. This will lead to the *generalized Huygens' principle* and to the precise formulation of a first non-trapping condition.

Chapter 4

Propagation of Singularities

The notion of a generalized bicharacteristic has up to now been used in a heuristical way only. The aim of this chapter is to give first a precise definition of this term. Then we present the generalized Huygens' principle, which states the relation between the generalized bicharacteristic flow and the propagation of singularities according to [Me/Sj I, Me/Sj II]. As [Pau 96, Section 7.1], this chapter follows the presentations of [Tay 81] and especially [Hö III, Chapter XXIV], because Hörmander gives an elegant description of the results of Melrose and Sjöstrand for the Dirichlet problem. In addition to [Pau 96] we will prove here the existence of elliptic closed rays for the example (3.5)–(3.6).

4.1 Generalized Bicharacteristics

The Fourier transform gives an interesting representation of a differential operator $P(x, D)$ in free space:

$$P(x,D)u(x) = \left(\frac{1}{2\pi}\right)^{n/2} \int\limits_{\mathbb{R}^n} p(x,\xi)\hat{u}(\xi)e^{\mathrm{i}\,x\cdot\xi}\,d\xi$$

$$\text{with} \qquad \hat{u}(\xi) := \left(\frac{1}{2\pi}\right)^{n/2} \int\limits_{\mathbb{R}^n} u(y)e^{-\mathrm{i}\,y\cdot\xi}\,dy\,.$$

This especially transforms linear differential operators with constant coefficients into polynomials with respect to $\xi \in \mathbb{R}^n$. $p(x, \xi)$ is called the *symbol* of $P(x, D)$. The equivalence classes of such symbols modulo lower order terms in ξ and the members of these classes are called *principal symbols*. In the case of a homogeneous differential operator of order m it is, therefore, justified to call its symbol a principal symbol.

Definition 4.1.1 *Let $P(x,D)$ be a differential operator in $\mathbb{R}^n$ with real principal symbol $p(x,\xi)$. The* Hamilton vector field *of p is given by:*

$$H_p := \sum_{i=1}^{n} \left(\frac{\partial p}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial p}{\partial x_i} \frac{\partial}{\partial \xi_i} \right) .$$

A bicharacteristic *of p is an integral curve of H_p, that is a curve*

$$\gamma(s) = \begin{pmatrix} x(s) \\ \xi(s) \end{pmatrix} \qquad \text{with} \quad \gamma'(s) = H_p\, \gamma(s) .$$

The integral curves on which $p(x(s),\xi(s)) \equiv 0$, are called null bicharacteristics.

H_p is constant on bicharacteristics, because $H_p p = 0$. Thus an integral curve is already a null bicharacteristic, if p vanishes at one point of γ. It should be noted that $\gamma(s)$ is sometimes written as $\exp\{sH_p\}\gamma(0)$.

$x(s)$ and $\xi(s)$ are solutions of the system of ordinary differential equations

$$\begin{aligned} x'(s) &= \nabla_\xi\, p(x(s),\xi(s)) , \\ \xi'(s) &= -\nabla_x\, p(x(s),\xi(s)) . \end{aligned}$$

We want to give a few examples for the wave equation (here x has to be replaced by (t,x) and ξ by (τ,ξ)):

In the case of a homogeneous medium we have

$$p(t,x,\tau,\xi) = |\xi|^2 - \tau^2 \quad \text{and} \quad H_p = 2\left(\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i} - \tau \frac{\partial}{\partial t} \right) ,$$

yielding after a change of parameter the differential equations

$$\begin{aligned} x'(s) &= \xi(s) , \qquad & \xi'(s) &= 0 , \\ t'(s) &= -\tau(s) , \qquad & \tau'(s) &= 0 . \end{aligned}$$

The null bicharacteristics are, therefore, given by

$$\gamma(s) = (t_0 \pm s\,|\xi_0|, x_0 + s\,\xi_0, \mp|\xi_0|, \xi_0) .$$

It is assumed that $\tau_0, |\xi_0| \neq 0$, for the curve would degenerate to a single point, otherwise. The space-time projection of null bicharacteristics are the well-known characteristics $(t_0 \pm s\,|\xi_0|, x_0 + s\,\xi_0)$ of the wave equation. For fixed (t_0, x_0) they form a so-called *wave cone* or *light cone.*

As a second example we want to verify the elliptic closed rays of the medium presented at the beginning of Section 3.2. For the wave equation in an inhomogeneous medium we have

$$p(t,x,\tau,\xi) = \langle \xi, A(x)\xi \rangle - \tau^2$$

yielding the differential equations

$$x' = 2A(x)\xi\,, \qquad \xi' = -\nabla_x\langle \xi, A(x)\xi\rangle\,,$$
$$t' = -2\tau\,, \qquad \tau' = 0$$

for the bicharacteristics of p. For (3.5) and (3.6) this means

$$x' = 2\rho(x)\begin{pmatrix} a^2\xi_1 \\ b^2\xi_2 \end{pmatrix}\,, \qquad \xi' = -2(a^2\xi_1^2 + b^2\xi_2^2)\begin{pmatrix} \frac{1}{a^2}x_1 \\ \frac{1}{b^2}x_2 \end{pmatrix}$$
$$t' = -2\tau\,, \qquad \tau' = 0\,.$$

Then the curve $\gamma(s)$ given by

$$x(s) = \begin{pmatrix} a\cos s \\ b\sin s \end{pmatrix}\,, \quad \xi(s) = \frac{1}{2}\begin{pmatrix} -\frac{1}{a}\sin s \\ \frac{1}{b}\cos s \end{pmatrix}\,, \quad t(s) = \mp s\,, \quad \tau(s) = \pm\frac{1}{2}$$

is a null bicharacteristic of p. This is implied by the following calculations: Obviously $\rho(x(s)) \equiv 1$. This yields

$$\langle \xi(s), A(x(s))\xi(s)\rangle = (a^2\xi_1^2 + b^2\xi_2^2) \equiv \frac{1}{4} \equiv \tau(s)^2\,,$$

thus $p(\gamma(s)) \equiv 0$. Now

$$2\rho(x(s))\begin{pmatrix} a^2\xi_1(s) \\ b^2\xi_2(s) \end{pmatrix} = \begin{pmatrix} -a\sin s \\ b\cos s \end{pmatrix} = x'(s)\,,$$

$$-2(a^2\xi_1(s)^2 + b^2\xi_2(s)^2)\begin{pmatrix} \frac{x_1(s)}{a^2} \\ \frac{x_2(s)}{b^2} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \frac{1}{a}\cos s \\ \frac{1}{b}\sin s \end{pmatrix} = \xi'(s)\,,$$
$$t'(s) = \mp 1 = -2\tau(s)\,, \quad \text{and} \quad \tau'(s) = 0\,.$$

Similar calculations yield the null bicharacteristic

$$x(s) = r\begin{pmatrix} \cos rs \\ \sin rs \end{pmatrix}\,, \quad \xi(s) = \frac{1}{2}\begin{pmatrix} -\sin rs \\ \cos rs \end{pmatrix}\,, \quad t(s) = \mp rs\,, \quad \tau(s) = \pm\frac{r}{2}$$

of Ralston's example (3.4).

The following considerations on the propagation of singularities are not restricted to solutions with finite energy. It is the much wider class of distributional solutions, for which these results hold.

We have already mentioned Huygens' principle. It states that signals are propagated sharply, i.e. without reverberations. This principle holds for wave equations in $\mathbb{R}^n$ for all odd dimensions $n \geqslant 3$. Furthermore a *general Huygens' principle* exists, saying that a singularity in x_0 at time t_0 is propagated along the wave cone originating from (t_0, x_0). This principle holds for the free space problem in any dimension (the reverberations in even dimensions and in $\mathbb{R}^1$ are smooth). Considering the *wavefront set* of a solution instead of the singular support leads to a microlocal version of the general Huygens' principle. For this we have to introduce:

Definition 4.1.2

(*i*) *A set* $N \subset T^*(\mathbb{R}^n)$ *is called* conical, *if* $(x,\xi) \in N$ *implies* $(x, s\xi) \in N$ *for all* $s > 0$.

(*ii*) *Let* $f \in \mathcal{D}'(\mathbb{R}^n)$. *The* wavefront set $WF(f) \subset T^*(\mathbb{R}^n) \setminus 0$ *of* f *is defined as follows:*

Let $(x_0, \xi_0) \in T^*(\mathbb{R}^n) \setminus 0$. *Then* $(x_0, \xi_0) \notin WF(f) \quad :\Longleftrightarrow$
A $\rho \in \mathring{\mathcal{C}}_\infty(\mathbb{R}^n)$ *with* $\rho(x_0) = 1$ *and a conical neighbourhood* N *of* ξ_0 *exist such that*

$$\forall\, k \in \mathbb{N}\ \exists\, C_k > 0\ \forall\, \xi \in N : \ |\widehat{\rho f}(\xi)| \leqslant C_k(1 + |\xi|)^{-k},$$

with $\widehat{\rho f}$ *denoting the Fourier transform of* ρf.

Remarks:

1. The subset 0 of $T^*(\mathbb{R}^n)$ denotes the set of all (x,ξ) with $\xi = 0$.
2. $WF(f)$ is a conical set.
3. This definition is due to [Ral 82, p. 227]. It is equivalent to the one of [Tay 81, p. 126], according to Theorem VI.1.8 of Taylor's book.
4. Proposition VI.1.1 of [Tay 81] shows that singsupp f is the projection to x-space of $WF(f)$.

Now we can state the general Huygens' principle (for a proof cf. [Tay 81, Section VI.2]):

Theorem 4.1.3 *Let* P *be a differential operator with real principal symbol* p *and let* $f \in \mathcal{C}_\infty(\mathbb{R}^n)$. *Then for a solution* u *of* $P\,u = f$ *the following holds:*

(*i*) $\gamma \cap WF(u) = \gamma$ *or* $\gamma \cap WF(u) = \emptyset$ *for all null bicharacteristics* γ *of* p.

(*ii*) $WF(u)$ *is a union of null bicharacteristics.*

Remark: Theorem 4.1.3 means that $WF(u)$ is invariant under the flow of H_p. It holds in fact under more general assumptions on P (cf. [Tay 81]).

The situation becomes more complicated in the presence of an obstacle. Now reflections, finite order tangencies, and especially infinite order tangencies of bicharacteristics have to be considered. This will be done by the Definition 4.1.5 of a *generalized bicharacteristic*. Using modified *b-wavefront sets* WF_b (this is nothing new in the interior of the domain, for details cf. [Hö III, p. 135f.]), this will lead to the *generalized Huygens' principle* of Theorem 4.2.1, which holds for various boundary conditions. We will present this for the special case of the Dirichlet problem following [Hö III].

We start with the description of the general setting and the introduction of the special distributions to be used here. Let X be a closed $\mathcal{C}_\infty$-manifold with boundary

∂X. The b-wavefront sets mentioned above are actually not subsets of $T^*(X)$ but of the compressed cotangent bundle $\widetilde{T}^*(X)$ (for details cf. [Hö III, Section 18.3]). Nevertheless, a natural map

$$T^*(X) \longrightarrow \widetilde{T}^*(X) \tag{4.1}$$

exists that maps $T^*_x(X)$ one-to-one onto $\widetilde{T}^*_x(X)$, if $x \in \mathring{X}$. If $x \in \partial X$ the conormal space $N^*_x(\partial X)$ forms the kernel. In this case the image of $T^*_x(X)$ can be identified with $T^*_x(\partial X)$. Thus the image of $T^*(X)$ under the map (4.1) is isomorphic to $T^*(\mathring{X}) \cup T^*(\partial X)$ and it will, therefore, from now on not be distinguished from this set. While we deal in the sequel with distributions whose b-wavefront sets are contained in this image, we omit a forthgoing discussion of the properties of $\widetilde{T}^*(X)$.

Let ϕ be a defining function for X, i.e. $\phi \in \mathcal{C}_\infty(X)$ with

$$\phi > 0 \text{ in } \mathring{X}\,, \quad \phi = 0 \text{ on } \partial X\,, \quad \text{and} \quad \frac{\partial \phi}{\partial \mathbf{n}} \neq 0 \text{ on } \partial X\,.$$

Let P be a second-order differential operator with real principal symbol $p \in \mathcal{C}_\infty(T^*(X))$. Suppose

$$H^2_\phi p \neq 0, \quad \text{if } \phi = 0\,.$$

Because of

$$H^2_\phi p = 2\left|\frac{\partial \phi}{\partial \mathbf{n}}(x)\right|^2 p(x, \mathbf{n}(x)), \qquad \text{for } x \in \partial X,$$

this means that the boundary is non-characteristic with respect to P.

We study distributional solutions of

$$Pu = f \quad \text{in } \mathring{X}\,, \qquad u\Big|_{\partial X} = g\,, \tag{4.2}$$

i.e. $u \in \mathcal{D}'(\mathring{X})$ fulfils for given $f \in \mathcal{D}'(\mathring{X})$ and $g \in \mathcal{D}'(\partial X)$ the equation $Pu = f$ in $\mathcal{D}'(\mathring{X})$ and has trace g on ∂X. For the latter to make sense special distributions f and u are considered. For this we denote by $\overline{\mathcal{D}}'(\mathring{X})$ the space of extendable distributions on $\mathring{X}$. If $\mathring{X} \subset \mathbb{R}^n$, then $f \in \overline{\mathcal{D}}'(\mathring{X})$ iff

$$\exists F \in \mathcal{D}'(\mathbb{R}^n): \ F\Big|_{\mathring{X}} = f\,.$$

For an arbitrary $\mathcal{C}_\infty$-manifold with boundary extendable distributions are defined using charts (cf. [Hö III, Appendix B.2]). The space $\mathcal{N}(X)$ of those distributions we will use here, is characterized essentially by the following properties:

1. The restriction
$$\mathcal{N}(X) \ni u \longmapsto u\Big|_{\mathring{X}} \in \overline{\mathcal{D}}'(\mathring{X})$$
is injective ([Hö III], p. 133).

2. There is a unique continuous restriction map
$$\mathcal{N}(X) \longrightarrow \mathcal{D}'(\partial X)\,,$$
which agrees with the standard restriction on $\mathcal{C}_\infty(X)$ (cf. [Hö III], Proposition 18.3.21).

3. As announced we have
$$u \in \mathcal{N}(X) \implies WF_b(u)\Big|_{\partial X} \subset T^*(\partial X) \subsetneq \widetilde{T}^*(X)\Big|_{\partial X}$$
(cf. [Hö III], Definition 18.3.30).

4. Let $u \in \mathcal{N}(X)$ and $WF_b(u) = \emptyset$. Then $u \in \mathcal{C}_\infty(X)$ (cf. [Hö III], Theorem 18.3.27.(iii)).

The following lemma holds (for a proof cf. [Hö III]):

Lemma 4.1.4 ([Hö III], Corollary 18.3.31) *Let P be a differential operator with $\mathcal{C}_\infty$-coefficients in X such that ∂X is non-characteristic. If $f \in \mathcal{N}(X)$, and $u \in \overline{\mathcal{D}}'(\mathring{X})$ satisfies the equation $Pu = f$ in $\mathring{X}$, then u has a unique extension $u_0 \in \mathcal{N}(X)$.*

Remarks:

1. This u_0 has, therefore, a unique trace in $\mathcal{D}'(\partial X)$.

2. For the wave equation in $X = \mathbb{R}\times\overline{\Omega}$ the boundary ∂X is non-characteristic. This is still true for $X = \mathbb{R} \times \overline{\Omega}(t)$, if $\Omega(t)$ is the exterior of an obstacle that is moved or deformed at a sufficiently slow speed.

3. Weak solutions of the wave equation in an exterior domain Ω generate distributions that can be extended by zero to the obstacle. This, therefore, holds especially for solutions with finite energy. Then such a solution for all $t \in \mathbb{R}$ fulfils $(\partial_t^2 - \partial_i a_{ij} \partial_j)u = 0$ in $\mathcal{D}'(\mathbb{R} \times \Omega)$. Thus it is an element of $\mathcal{N}(\mathbb{R} \times \overline{\Omega})$, too, according to Lemma 4.1.4.

After having introduced a suitable class of distributions, we now want to discuss the propagation of singularities. In $\mathring{X}$, there is no difference to the Cauchy problem. However, we have to take care at the boundary of the manifold. When projecting $T^*(X)|_{\partial X}$ onto $T^*(\partial X)$, the conormal bundle forms the kernel. If we consider now the inverse image under this projection and compare the number of preimages in the *characteristic set* of P, char P, (i.e. the set of the zeros of p in $T^*(X) \setminus 0)$ $T^*(\partial X)$ can be divided into three disjoint subsets:

1. The subset of $T^*(\partial X)$ without inverse images in char P is called *elliptic set*. For smooth f and g this set is of no importance for the propagation of singularities in the case of the Dirichlet problem. For boundary conditions like

$$\frac{\partial u}{\partial \mathbf{n}} = \mathrm{i}a(x)\frac{\partial u}{\partial t} \quad \text{on } \partial\Omega$$

with a real-valued a not vanishing identically on the boundary, however, so-called *Rayleigh waves* propagating in the elliptic set occur according to [M/R/S 78] and [Me/Sj II, Proposition 5.13, p. 154].

2. The set of points with two preimages is called *hyperbolic set*. A bicharacteristic making incidence in such a preimage shall be continued (reflected) by the bicharacteristic starting at the second preimage. Hörmander denoted curves γ constructed by this procedure and having a discrete set of "reflection points" by *broken bicharacteristics* (cf. [Hö III, Definition 24.2.2]. The projections of broken bicharacteristics to $\overline{\Omega}$ are just the admissible paths of [Ral 69]. The image of γ in $\widetilde{T}^*(X)$ under the map (4.1) is called a *compressed broken bicharacteristic* and it is continuous even at points of reflection. Here for the propagation of singularities the following holds: Suppose f and g are smooth. Then a compressed broken bicharacteristic is completely contained in or disjoint to $WF_b(u)$. This is a microlocal version of the reflection law of optics (for a proof cf. [Hö III, p. 424f.]).

3. There remains the case of one preimage. This is the most difficult one. It corresponds to tangential bicharacteristics.

We will study this last point in some detail. This makes the introduction of a few terms and sets necessary:

The *glancing set* G or G^2 is the set of all points in char P in which H_p is tangent to $T^*(X)|_{\partial X}$, i.e.

$$G^2 := \left\{(x,\xi) \in T^*(X) \setminus 0 \;\middle|\; \phi(x) = p(x,\xi) = H_p\phi(x,\xi) = 0\right\}. \tag{4.3}$$

The *glancing set* G^k *of order at least* k is defined by the equations

$$p = 0\,, \quad H_p^j\phi = 0\,, \qquad 0 \leqslant j < k\,, \quad k \geqslant 3\,.$$

Thus G^k is the set of all points where H_p is tangent of order at least k to $T^*(X)|_{\partial X}$. These definitions imply:

$$G^2 \supset G^3 \supset \cdots \supset G^\infty\,.$$

$G^2 \setminus G^3$, the glancing set of order precisely 2, is further divided into

$$\begin{array}{lll} \text{the } \textit{diffractive set } G_d, & \text{where } H_p^2\phi > 0 & \text{and} \\ \text{the } \textit{gliding set } G_g, & \text{where } H_p^2\phi < 0\,. & \end{array}$$

Analysing the behaviour of non-tangential bicharacteristics near G_d and G_g (for details cf. [Hö III]) suggests the following definitions:

Firstly the *gliding vector field* H_p^G tangent to G^2 is given by

$$H_p^G := H_p + \left(H_p^2\phi / H_\phi^2 p\right) H_\phi . \tag{4.4}$$

Note that $H_p^G = H_p$ in G^3, because there $H_p^2\phi \equiv 0$ holds. Secondly the integral curves of H_p^G are called *gliding rays*. Finally, continuity considerations between H_p and H_p^G lead to the definition of generalized bicharacteristics ([Hö III], Definition 24.3.7):

Definition 4.1.5 *A* generalized bicharacteristic *of p is a map*

$$I \setminus B \ni s \longmapsto \gamma(s) \in T^*(\mathring{X}) \cup G$$

where I is an interval on $\mathbb{R}$ and B a subset of I, such that $p(\gamma(s)) = 0$ and

(i) *$\gamma(s)$ is differentiable and $\gamma'(s) = H_p(\gamma(s))$, if $\gamma(s) \in T^*(\mathring{X})$ or $\gamma(s) \in G_d$,*

(ii) *$\gamma(s)$ is differentiable and $\gamma'(s) = H_p^G(\gamma(s))$, if $\gamma(s) \in G^2 \setminus G_d$,*

(iii) *every $t \in B$ is isolated, $\gamma(s) \in T^*(\mathring{X})$, if $s \neq t$ and $|s - t|$ is small enough, the limits $\gamma(t \pm 0)$ exist and are different and their projections to $T^*(\partial X)$ yield the same hyperbolic point.*

The continuous curve $\widetilde{\gamma}$ obtained by mapping γ to $\widetilde{T}^*(X)$ by (4.1) is called a compressed generalized bicharacteristic.

Thus γ is differentiable in $I \setminus B$. In B right and left limits exist as well as right and left derivatives. Furthermore, γ and $\widetilde{\gamma}$ have the same image under the projection to X. The following results hold (for proofs cf. [Hö III, p. 438–441]):

Theorem 4.1.6 *Let $\gamma(s)$ be a generalized bicharacteristic, let $\gamma(s_0) \in G^2 \setminus G^\infty$ and denote by $\gamma_g(s)$ the gliding ray with $\gamma_g(s_0) = \gamma(s_0)$. Then it follows that in a one-sided deleted neighbourhood of t_0 we have*

$$\begin{array}{ll} \textit{either} & \gamma_g(s) \in G_g \textit{ and } \gamma(s) = \gamma_g(s) \\ \textit{or} & \gamma_g(s) \in G_d \textit{ and } \gamma(s) = \exp\{sH_p\}\gamma(s_0) . \end{array}$$

Remarks:

1. Thus points of G^3 can accumulate at points of G^∞ only.
2. The same is true for points of reflection: If they have a limit point it must lie over ∂X; it cannot be a point of reflection because these are isolated according to Definition 4.1.5. Because of Theorem 4.1.6 it does not belong to $G^2 \setminus G^\infty$ either. Thus it has to be in G^∞.

Corollary 4.1.7 *A generalized bicharacteristic γ with no point in G^∞ is uniquely determined by any one of its points.*

This statement may fail if γ contains a point of G^∞. An example was first given by Taylor in 1976. He could show that a generalized bicharacteristic may have a non-unique continuation in G^∞ (cf. [Tay 76], p. 28f. and [Hö III], example 24.3.11, p. 438).

Proposition 4.1.8 *Assume that—using local coordinates—a direction ξ_i which is tangent to the boundary exists such that $\frac{\partial p}{\partial \xi_i} > 0$. Then for all $\gamma_0 \in T^*(X) \cap p^{-1}(0)$ a generalized bicharacteristic γ with $\gamma(0) = \gamma_0$ exists which is a limit of broken bicharacteristics. If γ has a reflection point for $t = 0$ this means that $\gamma(+0) = \gamma_0$ or $\gamma(-0) = \gamma_0$.*

Remarks:

1. In view of Corollary 4.1.7 it follows that every generalized bicharacteristic which does not intersect G^∞ is a limit of broken bicharacteristics. It is, however, an open problem, whether every bicharacteristic can be obtained as such a limit, namely in the case of non-unique continuations at a point $\gamma_0 \in G^\infty$.

2. The assumption of this proposition is fulfilled by the choice $\xi_i = t$ for the wave equations in $X = \mathbb{R} \times \overline{\Omega}$ considered here.

3. In [Ya 86, p. 290], Yamamoto proved for a homogeneous medium that generalized bicharacteristics cannot branch at a convex part of the boundary. This holds especially for those generalized bicharacteristics whose projections to $\overline{\Omega}$ are the straight lines of Theorems 3.1.5 and 3.1.7. Proposition 4.1.8 shows, therefore, that these generalized bicharacteristics can be approximated by broken ones. Thus the above straight lines can be approximated by Ralston's admissible paths. This finally proves Theorem 3.1.7.

4.2 The Generalized Huygens' Principle

This short section is devoted to the presentation of the *generalized Huygens' principle* for the propagation of b-wavefront sets and a few related comments. A first result for the Dirichlet problem of a homogeneous medium was proven by Morawetz and Ludwig for strictly convex obstacles in [Mor/Lu 69]. But they had to add an extra time factor to be able to guarantee smoothness of solutions for initial data of compact support. It was Taylor who proved the generalized Huygens' principle for strictly convex obstacles in [Tay 76]. His results hold for a large class of boundary value problems and second-order differential operators, provided the obstacle is strictly convex *with respect to the bicharacteristics.* Results for the gliding set, i.e. boundary regions which are strictly concave with respect to the bicharacteristics, can be found in [An/Me 77]. While, therefore, some results for tangential bicharacteristics were available, it was due to Melrose and Sjöstrand to avoid assumptions like local convexity or concavity with respect to the bicharacteristics in [Me/Sj I, Me/Sj II] and to handle even the case of infinite order tangencies. This completely proved the generalized Huygens' principle:

Theorem 4.2.1 ([Hö III], Theorem 24.5.3, p. 458f.)
Let P be a second-order differential operator with real principal symbol p and $\mathcal{C}_\infty$-coefficients defined in a $\mathcal{C}_\infty$-manifold X with non-characteristic boundary ∂X. Let $u \in \mathcal{N}(X)$ be a solution of the boundary value problem (4.2) with $f \in \mathcal{N}(X)$ and $g \in \mathcal{D}'(\partial X)$. Every

$$\gamma_0 \in WF_b(u) \setminus (WF_b(f) \cup WF(g)),$$

is then either a characteristic of P in the interior of X or else a point in the hyperbolic or the glancing set in $T^*(\partial X)$. An open interval $(-T, T) \ni t \longmapsto \widetilde{\gamma}(t)$ with $\widetilde{\gamma}(0) = \gamma_0$ on a compressed generalized bicharacteristic is contained in $WF_b(u)$.

For a proof cf. [Hö III].

Remarks:

1. For the wave equation this means that a singularity is propagated on a compressed generalized bicharacteristic at least until it reaches $WF_b(f)$ or $WF(g)$. For our problem (W.D), $WF_b(u)$ is, therefore, a union of maximally continued compressed generalized bicharacteristics, i.e. $WF_b(u)$ is invariant under the compressed generalized bicharacteristic flow.

2. These results still hold for other problems (cf. [Me/Sj II], Section 6, especially Theorem 6.14), for example (W.N). Here we have the same generalized bicharacteristics as for (W.D). Furthermore, non-stationary obstacles are included as long as the boundary remains non-characteristic. This is the case for sufficiently slow moving or deforming obstacles. For the operator $\partial_t^2 - \Delta$, "slow" means a speed less than one. We will, therefore, use Theorem 4.2.1 as a reference for the generalized Huygens' principle in these general cases, too.

3. This theorem is—in contrast to Theorem 4.1.3, which still holds in $\mathring{X}$—not optimal. The behaviour of wavefronts at the branch points is in general not (yet?) predictable. Likewise—in contrast to the Cauchy problem—there is no answer to the related question, whether for any compressed generalized bicharacteristic $\widetilde{\gamma}$ a solution exists whose b-wavefront set is exactly the conical set generated by $\widetilde{\gamma}$ such that f is singular at the endpoints of $\widetilde{\gamma}$ only and $g \equiv 0$. This result is only proven for those compressed generalized bicharacteristics that are limits of compressed *broken* bicharacteristics (cf. [Hö III, Theorem 24.5.4]). Hence there is no way of predicting the future path of a singularity in general. Here the same gap remains as in Ralston's Theorem 2.1.1. This is due to the fact that the proof is closely related to the one of Ralston.

Chapter 5

Melrose's Non-Trapping Condition

The definition of a generalized bicharacteristic makes a precise formulation of the non-trapping condition of the Lax-Phillips conjecture possible. This has been done by Melrose in [Me 79]. After having presented this condition, we will study its relation to escape functions. We will conclude this chapter by giving two uniform decay results which have been derived from Melrose's non-trapping condition.

5.1 Melrose's NTC and Escape Functions

In [Me 79, p. 44] Melrose defines:

Definition 5.1.1

(*i*) *The projection of a generalized bicharacteristic to* $\overline{\Omega}$ *is called a (generalized)* geodesic.

(*ii*) **(Melrose's Non-Trapping Condition)**
Ω *is said to be* non-trapping
$:\Longleftrightarrow$ *for some (any)* $R_1 \geqslant R$ *there exists* T_{R_1} *such that no geodesic of length* T_{R_1} *lies completely in* $\overline{\Omega}_{R_1}$.

Although Melrose works in [Me 79] with a homogeneous medium only, this non-trapping condition can be used for all media considered here.

For a convex obstacle in a homogeneous medium maximally extended geodesics are either straight lines or consist of two rays by a single reflection at the boundary (cf. [Ya 86, p. 290]). Here convex obstacles fulfil, therefore, Melrose's non-trapping condition with $T_{R_1} = 4R_1$. In the sequel we will prove a more general result: Whenever an escape function exists, Melrose's NTC holds.

For this purpose we need to have a more concrete look at the objects of the last chapter in the case of the wave equation. As defining function for $\mathbb{R} \times \overline{\Omega}$ one can use locally the distance to the obstacle, for this is according to [Gi/Tr 83], Lemma 14.16, p. 355, in a $\overline{\Omega}$-neighbourhood $U(\partial\Omega)$ of $\partial\Omega$ as regular as the boundary itself. Thus we define

$$\phi(t,x) = \phi(x) := d(x, \partial\Omega) =: d(x) \quad \text{in } U(\partial\Omega) \subset \overline{\Omega}\,.$$

It is possible to extend d to $\overline{\Omega}$ in such a way that $d \in \mathcal{C}_\infty(\overline{\Omega})$ and $d|_\Omega > 0$. Furthermore

$$\nabla d(x) \;=\; \mathbf{n}_e(x) \qquad \text{on } \partial\Omega\,, \tag{5.1}$$

holds. Now we can calculate the different vector fields:

The Hamilton vector field of p:

$$\begin{aligned} P(t,x,D_{t,x}) &= \partial_t^2 - \partial_i a_{ij}(x)\partial_j \\ \Longrightarrow \quad p(t,x,\tau,\xi) &= \langle \xi, A(x)\xi\rangle - \tau^2 \\ \Longrightarrow \quad H_p &= \sum_{i=1}^n \left(2(A(x)\xi)_i \frac{\partial}{\partial x_i} - (\nabla\langle \xi, A(x)\xi\rangle)_i \frac{\partial}{\partial \xi_i}\right) - 2\tau\frac{\partial}{\partial t}\,. \end{aligned}$$

The Hamilton vector field of d:

$$H_d = -\sum_{i=1}^n \frac{\partial d}{\partial x_i}\frac{\partial}{\partial \xi_i}\,.$$

This yields

$$H_d p = -2\langle \nabla d, A(x)\xi\rangle \tag{5.2}$$

and because of (5.1) for $x \in \partial\Omega$

$$\begin{aligned} H_d\, p &= -2\langle \mathbf{n}_e(x), A(x)\xi\rangle\,, \\ H_d^2\, p &= 2\langle \mathbf{n}_e(x), A(x)\mathbf{n}_e(x)\rangle \end{aligned}$$

follow. According to (4.3), (5.1), and (5.2) the glancing set is given by

$$x \in \partial\Omega\,, \quad |\xi|^2 = \tau^2 \quad \text{and} \quad \mathbf{n}_e(x)\cdot A(x)\xi = 0\,,$$

i.e. here $A(x)\xi$ is a tangential direction to $\partial\Omega$ in x. For the gliding vector field (4.4) we have

$$H_p^G = H_p + \Big(H_p^2 d / \big(2\langle \mathbf{n}_e(x), A(x)\mathbf{n}_e(x)\rangle\big)\Big) H_d\,.$$

Thus gliding rays are solutions of the ordinary differential equations

$$\begin{aligned} &x'(s) = 2A(x)\xi\,, \qquad t'(s) = -2\tau\,, \qquad \tau'(s) = 0\,, \\ &\xi'(s) = -(\nabla\langle \xi, A(x)\xi\rangle) - \Big(H_p^2 d / \big(2\langle \mathbf{n}_e(x), A(x)\mathbf{n}_e(x)\rangle\big)\Big)\mathbf{n}_e(x)\,. \end{aligned} \tag{5.3}$$

At points of reflection (i.e. $\mathbf{n}_e \cdot A(x)\xi \neq 0$) one has to look for the other preimage in char P with respect to the natural projection to $T^*(\partial\Omega)$. This means looking for real α with

$$\langle \xi + \alpha \mathbf{n}_e, A(x)(\xi + \alpha \mathbf{n}_e) \rangle - \tau^2 = 0,$$

one solution, namely $\alpha = 0$, already being known. This equation is equivalent to

$$\underbrace{\langle \xi, A(x)\xi \rangle - \tau^2}_{=0} + 2\alpha \langle \mathbf{n}_e, A(x)\xi \rangle + \alpha^2 \langle \mathbf{n}_e, A(x)\mathbf{n}_e \rangle = 0$$

yielding

$$\alpha = 0 \quad \vee \quad \alpha = -2\frac{\langle \mathbf{n}_e, A(x)\xi \rangle}{\langle \mathbf{n}_e, A(x)\mathbf{n}_e \rangle}.$$

Thus the second direction is given by

$$\breve{\xi} := \xi - 2\frac{\langle \mathbf{n}_e, A(x)\xi \rangle}{\langle \mathbf{n}_e, A(x)\mathbf{n}_e \rangle}\mathbf{n}_e \tag{5.4}$$

as in the definition of an escape function.

Therefore, generalized bicharacteristics of the wave equation consist of integral curves of H_p, which, if necessary, have to be reflected according to (5.4), and of gliding rays, i.e. solutions of (5.3). In any case τ is constant along generalized bicharacteristics. Because

$$p(t, x, \tau, \xi) = 0$$

has to be fulfilled, $\langle \xi, A(x)\xi \rangle$ is constant on generalized bicharacteristics, too. While b-wavefront sets are conical and

$$(\tau, \xi) \longmapsto (\alpha\tau, \alpha\xi), \quad \alpha > 0,$$

changes only the parametrization of (x, t), but not the curve in $\mathbb{R} \times \overline{\Omega}$, we can assume w.l.o.g. that $\langle \xi, A(x)\xi \rangle = 1$. Thus we have reduced the problem to the study of generalized bicharacteristics with (x, ξ) in the compact set K_R of Definition 3.2.1. The following theorem holds:

Theorem 5.1.2 *Assume that an escape function exists. Then Ω fulfils Melrose's non-trapping condition.*

Remark: Because all obstacles and media with uniform LED discussed so far possess escape functions, they are non-trapping in the sense of Melrose.

A complete proof of this theorem for a homogeneous medium was given in [Pau 96, Section 7.2]. The proof of the general case is essentially the same, because a more implicit use of the different Hamilton vector fields than in [Pau 96] is sufficient for this purpose, as can be seen in the proof of the related Theorem 9.5.3 for isotropic elasticity we present in Part II.

5.2 Existence of Escape Functions for Homogeneous Media in $\mathbb{R}^3$

Besides the proof of Theorem 3.1.9 for both homogeneous and inhomogeneous media and the verification of the existence of a Straussian vector field for a large class of obstacles in a homogeneous medium in $\mathbb{R}^2$, a great part of [M/R/S 77] deals with the question, whether an escape function exists for homogeneous media in $\mathbb{R}^3$. Morawetz, Ralston, and Strauss proved that this is true for obstacles with non-degenerated boundaries that fulfil a non-trapping hypothesis. In this section we will discuss these two conditions in some detail. We begin with the

Non-Degeneracy Hypothesis:
Given any straight line l in $\mathbb{R}^3$, the function $d(s)$, giving the distance from points on l to $\partial\Omega$ as a function of arc length on l, only has zeros of finite order.

In [Pau 96] we proved by a simple calculation that:

$$\text{non-degeneracy hypothesis} \quad \Longleftrightarrow \quad G^\infty = \emptyset\,.$$

Things seem—at first sight—to be a little bit more complicated with the non-trapping hypothesis of [M/R/S 77]. We first continue the calculation of the last section for the special case of a homogeneous medium. Now

$$H_d\, p = -2\,\mathbf{n}_e(x)\cdot\xi$$

on the boundary, yielding $\mathbf{n}_e(x)\cdot\xi = 0$ in G and $\mathbf{n}_e(x)\cdot\xi \neq 0$ for the preimages of a hyperbolic point. Because

$$H_p^2 d = 4\nabla(\nabla d\cdot\xi)\cdot\xi$$

and

$$H_d^2\, p = 2\,\mathbf{n}_e(x)\cdot\mathbf{n}_e(x) \;=\; 2$$

in G, the gliding vector field becomes

$$H_p^G = 2\left(\sum_{i=1}^{n}\xi_i\,\frac{\partial}{\partial x_i} - \tau\frac{\partial}{\partial t} - \left(\xi\cdot\frac{\partial\mathbf{n}_e}{\partial x}\xi\right)\sum_{i=1}^{n}(\mathbf{n}_e)_i\,\frac{\partial}{\partial\xi_i}\right)\,.$$

Thus a gliding ray is a solution of

$$\begin{aligned} x'(s) &= \xi\,, & \xi'(s) &= -\left(\xi\cdot\frac{\partial\mathbf{n}_e}{\partial x}\xi\right)\mathbf{n}_e(x)\,,\\ t'(s) &= -\tau\,, & \tau'(s) &= 0\,. \end{aligned} \tag{5.5}$$

Thus the "generalized rays" of [M/R/S 77, p. 478] are exactly the (x,ξ)-components of generalized bicharacteristics. Denoting by a "gliding segment" the (x,ξ)-components of a gliding ray, the non-trapping hypothesis of [M/R/S 77] reads as follows:

Definition 5.2.1
(Non-Trapping Hypothesis of Morawetz, Ralston, and Strauss:)
For any $(x_0, \xi_0) \in \partial\mathrm{B} \times S^2$*, the generalized ray from* (x_0, ξ_0) *consists of a finite number of finite straight line segments, plus a finite number of finite gliding segments, plus a final segment which passes out of* $|x| < R$*.*

Remarks: It is no restriction that the generalized ray starts on $\partial\mathrm{B}$, for straight line segments in $\overline{\Omega}_R$ have at least length $2R$. But two questions arise:

1. Is the requirement that a generalized ray may consist of at most a finite number of segments an additional restriction?
2. Does this non-trapping hypothesis imply an upper limit for the length of generalized rays with $|x| < R$?

We can give an immediate answer to the first question:

Lemma 5.2.2 *Let* $\Omega \subset \mathbb{R}^3$ *and* $G^\infty = \emptyset$*. Then any generalized ray in* $\overline{\Omega}_R \times S^2$ *with a finite length consists of a finite number of straight line and gliding segments.*

We proved this lemma in [Pau 96, p. 49].

The answer to the second question is a consequence of the following theorem of Morawetz, Ralston, and Strauss.

Theorem 5.2.3 *Let* $\Omega \subset \mathbb{R}^3$*. Let the non-degeneracy and non-trapping hypotheses be fulfilled. Then an escape function for a homogeneous medium exists.*

For a proof cf. [M/R/S 77, Section 5].

Now we can answer the second question:

Lemma 5.2.4 *Let* $\Omega \subset \mathbb{R}^3$ *and* $G^\infty = \emptyset$*. Then the following conditions are equivalent:*

(i) *the lengths of all generalized rays in* $\overline{\Omega}_R \times S^2$ *are finite,*
(ii) *the supremum of the lengths of all generalized rays in* $\overline{\Omega}_R \times S^2$ *is finite.*

Proof: Obviously (ii) implies (i).
On the other hand Theorems 5.2.3 and 5.1.2 yield

$$(i) \implies \exists \text{ escape function} \implies (ii)\,. \qquad \square$$

This furthermore proves:

Corollary 5.2.5 *For obstacles in a homogeneous medium in* $\mathbb{R}^3$ *with non-degenerated boundaries Melrose's non-trapping condition is equivalent to the non-trapping hypothesis of Morawetz, Ralston, and Strauss.*

5.3 Uniform LED for Obstacles that Fulfil Melrose's Non-Trapping Condition

Up to now, uniform LED holds only under assumptions that are special cases of Melrose's non-trapping condition. Provided the medium is homogeneous, there are large classes of obstacles in $\mathbb{R}^2$ and $\mathbb{R}^3$ for which the existence of an escape function is equivalent to Melrose's NTC. But for this result additional requirements on the boundary curvature have to be imposed. In 1979, however, two papers were published which derived uniform LED for a homogeneous medium from Melrose's NTC without any further restriction. In addition, both publications treated the Neumann problem:

The first result was due to Melrose ([Me 79]). For the Neumann problem, however, he did not use the usual energy, for he included second derivatives of the solution. Thus he needed a special class of initial data, which has this additional regularity. For the Neumann problem the energy integral of Melrose has the following form:

$$E_2(u(t), D) := \int_D ((\Delta u)^2 + |\nabla u|^2 + |\nabla u_t|^2 + u_t^2)\, dx\,, \quad D \subset \Omega\,.$$

For the initial data he assumed

$$(u_0, u_1) \in \left\{ \phi \in (\mathring{\mathcal{C}}_\infty(\overline{\Omega}))^2 \;\middle|\; \frac{\partial \phi_1}{\partial \mathbf{n}} = \frac{\partial \phi_2}{\partial \mathbf{n}} = 0 \;\text{ on } \partial\Omega \right\}^{\sim \|\cdot\|_{E_2}},$$

with

$$\|\phi\|_{E_2} := \left(\int_\Omega ((\Delta \phi_1)^2 + |\nabla \phi_1|^2 + |\nabla \phi_2|^2 + \phi_2^2)\, dx \right)^{1/2}.$$

We denote *this* Neumann problem by (W.N)*. Melrose proved:

Theorem 5.3.1 *Let $n \geqslant 3$ and let the medium be homogeneous. Suppose Ω fulfils Melrose's NTC.*

1. *Let $K \subset\subset \overline{\Omega}$. Then there exist positive constants c and δ such that for all solutions u of (W.D) with initial data having support in $\overline{\Omega}_R$*

$$(i)\ n \text{ even: } E\,(u(t), K) \leqslant c\, t^{-n}\, E\,(u(0), \Omega)\,, \qquad t > 0\,,$$
$$(ii)\ n \text{ odd: } E\,(u(t), K) \leqslant c\, e^{-\delta t}\, E\,(u(0), \Omega)\,, \qquad t \geqslant 0\,.$$

2. *Let n be odd. Let $K \subset\subset \overline{\Omega}$. Then there exist positive constants c and δ such that for all solutions u of (W.N)* with initial data having support in $\overline{\Omega}_R$*

$$E_2(u(t), K) \leqslant c\, e^{-\delta t}\, E_2(u(0), \Omega)\,, \qquad t \geqslant 0\,.$$

Melrose proved this result by methods of the scattering theory of Lax and Phillips. He uses especially properties of the Lax-Phillips semigroup $Z(t)$. While [La/Ph 67] is restricted to odd dimensions larger than one, the scattering theory could be transferred to even dimension in [La/Ph 72]. But this made a modification of $Z(t)$ for even n necessary—Melrose used the one given in [Me 79, Proposition 5.1]—which is finally reflected by the non-exponential decay rate of Theorem 5.3.1.

A more general result was given in [Ral 79]. Ralston used the usual energy integral for both the Dirichlet and the Neumann problem. In both cases he established uniform LED for even space dimensions and here his polynomial rates are even better than those of Melrose.

Theorem 5.3.2 *Let $n \geqslant 3$ and let the medium be homogeneous. Suppose Ω fulfils Melrose's non-trapping condition. Let $K \subset\subset \overline{\Omega}$. Then there exist positive constants c and δ such that for all solutions u of (W.D) and (W.N) respectively with initial data having support in $\overline{\Omega}_R$ the following hold:*

$$(i)\ n\ \textit{even:}\ E\left(u(t),K\right) \leqslant c\, t^{-2n+2}\, E\left(u(0),\Omega\right), \qquad t>0\,,$$
$$(ii)\ n\ \textit{odd:}\ E\left(u(t),K\right) \leqslant c\, e^{-\delta t}\, E\left(u(0),\Omega\right), \qquad t\geqslant 0\,.$$

Ralston's proof is based on the Vainberg method. For this, Vainberg's non-trapping condition has to be verified first. While the next chapter deals with the Vainberg method for general media, we will compare the different non-trapping conditions afterwards. The key results for this purpose have already been presented, namely the generalized Huygens' principle and the constructions of localized solutions by Ralston.

Chapter 6

The Vainberg Method

The Vainberg method is essentially based on an analysis of the resolvent

$$R(k) := (\partial_i a_{ij}(x)\partial_j + k^2)^{-1}.$$

It can be roughly described as follows: One applies the Laplace transform to the differential equation. This transforms the differential operator $\partial_t^2 - \partial_i a_{ij}(x)\partial_j$ to $-(\partial_i a_{ij}(x)\partial_j + k^2)$. By inversion of this last operator and application of the inverse Laplace transform a representation formula for the solution of the original differential equation is achieved. But to derive estimates from this formula one has to extend the resolvent across its continuous spectrum and to shift the integral path in the inverse transform into the upper half-plane. The latter is only possible under a suitable non-trapping condition. Vainberg uses a condition that is different from Melrose's, and this condition even varies in his publications [Va 75] and [Va 89]. In the first section of this chapter we will present the non-trapping condition of [Va 89]. The relation to the older condition of [Va 75] will be discussed in Chapter 7. The other sections give a more detailed description of the Vainberg method and the decay results achieved by it. Some special attention is focussed on the two-dimensional case in Section 6.3, because to our knowledge there is no proof of uniform LED with the rates $c\,t^{-2}\ln^{-4} t$ (Dirichlet) and $c\,t^{-2}$ (Neumann) in the literature.

6.1 Green's Function and Vainberg's Non-Trapping Conditions

In order to understand and interpret Vainberg's non-trapping conditions, a few important results and definitions have to be introduced. The former will be presented without proofs. This is because of two reasons. For a homogeneous medium they have been proved in [Pau 96]. The proofs for the situation here can essentially be achieved by just exchanging the space operator. The second reason is that detailed proofs for the more complicated situation of isotropic elasticity are given in Section 9.6 of this thesis. Obvious simplifications of the arguments presented there yield the results of the present section.

Let $X_j \subset \mathbb{R}^n$ be open, $j = 1, 2$. Every function $K \in \mathcal{C}(X_1 \times X_2)$ defines an integral operator

$$\mathcal{K} : \overset{\circ}{\mathcal{C}}(X_2) \longrightarrow \mathcal{C}(X_1)$$

by the formula

$$(\mathcal{K}\varphi)(x_1) := \int_{X_2} K(x_1, x_2)\varphi(x_2)\, dx_2\,, \qquad \varphi \in \overset{\circ}{\mathcal{C}}(X_2)\,.$$

The following theorem shows that this definition can be extended to arbitrary distributions $K \in \mathcal{D}'(X_1 \times X_2)$, if $\mathcal{K}$ is restricted to $\overset{\circ}{\mathcal{C}}_\infty(X_2)$ and $(\mathcal{K}\varphi)$ is allowed to be a distribution. If we denote by

$$(\psi \otimes \varphi)(x_1, x_2) := \psi(x_1) \cdot \varphi(x_2)\,, \qquad \psi \in \overset{\circ}{\mathcal{C}}_\infty(X_1)\,,\ \ \varphi \in \overset{\circ}{\mathcal{C}}_\infty(X_2)\,,$$

the tensor product of ψ and φ in $\overset{\circ}{\mathcal{C}}_\infty(X_1 \times X_2)$, and interpret K and $(\mathcal{K}\varphi)$ even for $K \in \mathcal{C}(X_1 \times X_2)$ as distributions, we get:

$$(\mathcal{K}\varphi)(\psi) = K(\psi \otimes \varphi)\,, \qquad \psi \in \overset{\circ}{\mathcal{C}}_\infty(X_1)\,,\ \ \varphi \in \overset{\circ}{\mathcal{C}}_\infty(X_2)\,. \tag{6.1}$$

The *Schwartz kernel theorem* states that the correspondence between K and $\mathcal{K}$ is one-to-one (cf. [Hö I], Theorem 5.2.1):

Theorem 6.1.1 *Every $K \in \mathcal{D}'(X_1 \times X_2)$ defines according to* (6.1) *a linear map*

$$\mathcal{K} : \overset{\circ}{\mathcal{C}}_\infty(X_2) \longrightarrow \mathcal{D}'(X_1)$$

which is continuous in the sense that $\mathcal{K}\varphi_j \longrightarrow 0$ in $\mathcal{D}'(X_1)$, if $\varphi_j \longrightarrow 0$ in $\overset{\circ}{\mathcal{C}}_\infty(X_2)$. Conversely, to every such linear map $\mathcal{K}$ there is one and only one distribution K such that (6.1) *is valid. One calls K the* (distribution) kernel *of $\mathcal{K}$.*

For a proof cf. [Hö I], p. 129f.

Remark: The restriction of K to those functions $\psi \in \overset{\circ}{\mathcal{C}}_\infty(X_1)$ which can be represented as tensor products of functions $\psi \in \overset{\circ}{\mathcal{C}}_\infty(X_1)$ and $\varphi \in \overset{\circ}{\mathcal{C}}_\infty(X_2)$ already determines K completely as an element of $\mathcal{D}'(X_1 \times X_2)$.

Let $P\varphi$ denote the unique solution with finite energy for $\varphi \in \overset{\circ}{\mathcal{C}}_\infty(\Omega)$ of

$$\left\{\begin{array}{lll} (\partial_t^2 - \partial_i a_{ij}(x)\partial_j)u = 0 & & \text{in } \ \mathbb{R} \times \Omega\,, \\ & u = 0 & \text{on } \ \mathbb{R} \times \partial\Omega\,, \\ u(0) = 0\,, & u_t(0) = \varphi & \text{in } \ \Omega\,. \end{array}\right\} \tag{6.2}$$

$P\varphi$ obviously generates a distribution in $\mathcal{D}'(\mathbb{R} \times \Omega)$ which is, according to [Hö I], Theorem 4.4.8 (and the remark at the beginning of its proof there), even an element of $\mathcal{C}_\infty(\mathbb{R}, \mathcal{D}'(\Omega))$ and solves (6.2) in the distributional sense (i.e. all four equations hold in the distributional sense). The solution operator P now regarded as a map

$$P : \overset{\circ}{\mathcal{C}}_\infty(\Omega) \longrightarrow \mathcal{D}'(\mathbb{R} \times \Omega)$$

fulfils the assumptions of the Schwartz kernel theorem. This yields:

Theorem 6.1.2 *The operator P has a unique distribution kernel $G \in \mathcal{D}'(\mathbb{R} \times \Omega \times \Omega)$ with*

$$(P\varphi)(\psi) = G(\psi \otimes \varphi)\,, \qquad \forall \psi \in \mathring{\mathcal{C}}_\infty(X_1)\ \forall \varphi \in \mathring{\mathcal{C}}_\infty(X_2)\,.$$

Definition 6.1.3 *The distribution kernel G of P is called the* Green's function *of the problem* (6.2).

Remark: Vainberg defines the Green's function as the kernel of the solution operator of the restriction of problem (6.2) to $\mathbb{R}_0^+ \times \Omega$. Because of the uniqueness of the Schwartz kernel this is exactly the restriction of G to $\mathring{\mathcal{C}}_\infty(\mathbb{R}^+ \times \Omega \times \Omega)$.

We now formally add to G the variables $t, x,$ and x_0 to make it clear, how differential operators have to be applied. The above-mentioned properties of P and $P\varphi$ imply together with the remark following Theorem 6.1.1 that

$$G \in \mathcal{C}_\infty(\mathbb{R}, \mathcal{D}'(\Omega \times \Omega))$$

and that G has boundary values in $\mathcal{D}'(\mathbb{R} \times \partial\Omega \times \Omega)$. Therefore G is a solution of the following problem:

$$\left\{\begin{array}{lr} (\partial_t^2 - \partial_{x_i} a_{ij}(x)\partial_{x_j})\, G(t,x,x_0) = 0 & \text{in } \mathcal{D}'(\mathbb{R} \times \Omega \times \Omega)\,, \\ G(t,x,x_0) = 0 & \text{in } \mathcal{D}'(\mathbb{R} \times \partial\Omega \times \Omega)\,, \\ G(0,x,x_0) = 0\,,\ \partial_t G(0,x,x_0) = \delta(x - x_0) & \text{in } \mathcal{D}'(\Omega \times \Omega) \end{array}\right\} \tag{6.3}$$

(for the initial values cf. Section 9.6). Furthermore G is extendable to $\mathbb{R}^n$ with respect to x. This is implied by the Schwartz kernel theorem, because extending solutions of (6.2) by 0 to B generates distributions in $\mathbb{R} \times \mathbb{R}^n$.

A consequence of the generalized Huygens' principle is the following uniqueness result:

Theorem 6.1.4 *Let $I = \mathbb{R}$ or $I = \mathbb{R}^+$ and assume that $u_0, u_1 \in \mathcal{D}'(\Omega)$, $g \in \mathcal{D}'(I \times \partial\Omega)$ and $f \in \mathcal{C}(\overline{I}, \mathcal{D}'(\Omega))$. Then the problem*

$$\left\{\begin{array}{ll} (\partial_t^2 - \partial_i a_{ij}(x)\partial_j) u = f & \text{in } \mathcal{D}'(I \times \Omega)\,, \\ u = g & \text{in } \mathcal{D}'(I \times \partial\Omega)\,, \\ u(0) = u_0\,, \quad u_t(0) = u_1 & \text{in } \mathcal{D}'(\Omega)\,, \end{array}\right.$$

has at most one solution in $\overline{\mathcal{D}}'(I \times \Omega)$.

Theorem 6.1.4 together with the Schwartz kernel theorem yields uniqueness even for problem (6.3). This finally proves the following existence and uniqueness result:

Theorem 6.1.5 *Problem* (6.3) *has for all domains Ω a unique solution in the space of distributions which are extendable with respect to x, namely the Green's function.*

Remark: The Schwartz kernel theorem and the generalized Huygens' principle are valid for the Neumann problem, too. Thus the above results hold in this case also.

Now we have the results which allow us to state Vainberg's non-trapping condition. Furthermore they are important for the comparison of the different non-trapping conditions, which will be done in the next chapter. In [Va 89], p. 341, Vainberg defines:

Definition 6.1.6 (Vainberg's Non-Trapping Condition)
Let G be the Green's function of (6.2). Ω *is called* non-trapping
$:\Longleftrightarrow \ \forall a, b > R \ \exists \tau(\Omega, a, b) > 0 : \quad G \in \mathcal{C}_\infty([\tau, \infty) \times \overline{\Omega}_b \times \overline{\Omega}_a)\,.$

One will get the same decay results with nearly the same proofs, if the following non-trapping condition is used (cf. [Sh/Ts 84, Sh/Ts 86], [Iw/Sh 88], [Di 92]):

Definition 6.1.7 (Non-Trapping Condition for the Weak Solution)
Let u be the weak solution of (6.2) *for* $\varphi \in \mathcal{L}^2_a(\Omega)$. Ω *is called* non-trapping
$:\Longleftrightarrow \ \forall a > R \ \exists \tau(\Omega, a) > 0 \ \forall \varphi \in \mathcal{L}^2_a(\Omega) : \quad u \in \mathcal{C}_\infty([\tau, \infty) \times \overline{\Omega}_a)\,.$

Remarks:

1. Obviously $u(0) \in \mathring{\mathcal{H}}_1(\Omega)$ in (6.2). Therefore u is even a solution with finite energy. On the other hand the distribution generated by u solves the problem in the distributional sense.

2. For the relation between these two conditions cf. Chapter 7.

6.2 Uniform LED for Domains in $\mathbb{R}^n$ with $n \geqslant 3$

The presentation of the Vainberg method in this section follows [Sh/Ts 84] and [Sh/Ts 86], with the exception that we consider general media according to the results of [Ya 92b]. We explicitly treat the Dirichlet condition and indicate afterwards the changes necessary for the Neumann condition. We first introduce three Hilbert spaces

$$
\begin{aligned}
H &:= \left\{ (u,v) \;\middle|\; u \in \mathcal{H}_\nabla(\Omega), v \in \mathcal{L}^2(\Omega) \right\},\\
H_b &:= \left\{ (u,v) \;\middle|\; u \in \mathcal{H}_1(\Omega_b), v \in \mathcal{L}^2(\Omega_b), u = 0 \text{ on } \partial\Omega \right\},\\
\mathring{H}_a &:= \left\{ (u,v) \in H \;\middle|\; \operatorname{supp}(u,v) \subset \overline{\Omega}_a \right\} \;=\; \left\{ (u,v) \;\middle|\; u \in \mathring{\mathcal{H}}_{1,a}(\Omega), v \in \mathcal{L}^2_a(\Omega) \right\},
\end{aligned}
$$

with $a, b > R$. They shall be equipped with the following norms:

$$
\begin{aligned}
\|(u,v)\|_H &:= \|(u,v)\|_{\mathring{H}_a} := (\langle \nabla u, A(x) \nabla u \rangle + \|v\|^2)^{1/2}\,,\\
\|(u,v)\|_{H_b} &:= (\|u\|^2_{1,b} + \|v\|^2_b)^{1/2}\,.
\end{aligned}
$$

Let $\|\cdot\|_{a\to b}$ denote the operator norm from $\mathring{H}_a$ to H_b.

Denoting the space operator $\partial_i a_{ij}(x)\partial_j$ of the wave equation by $A(x,D)$, we next define the operator

$$\mathcal{A} := \begin{pmatrix} 0 & 1 \\ A(x,D) & 0 \end{pmatrix}$$

with

$$\begin{aligned} D(\mathcal{A}) &:= \left\{ (u,v) \in H \;\middle|\; \mathcal{A}(u,v) \in H \right\} \\ &= \left\{ (u,v) \in H \;\middle|\; D_x^2 u \in \mathcal{L}^2(\Omega), v \in \mathring{\mathcal{H}}_1(\Omega) \right\}. \end{aligned}$$

We consider the system

$$\left\{ \begin{aligned} V_t - \mathcal{A}V &= 0 && \text{in } \mathbb{R} \times \Omega\,, \\ V &= 0 && \text{on } \mathbb{R} \times \partial\Omega\,, \\ V(0) &= f && \text{in } \Omega\,. \end{aligned} \right\} \tag{6.4}$$

It is related to the wave equation by

$$V = \begin{pmatrix} u \\ u_t \end{pmatrix} \qquad \text{and} \qquad f = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

$\mathcal{A}$ is skew-adjoint according to [Ya 91]. Thus by Stone's theorem (cf. [Paz 83, Theorem 1.10.8]) $\mathcal{A}$ generates a one-parameter unitary group $\{U(t); t \in \mathbb{R}\}$. $U(t)f$ is for all $f \in H$ the solution of system (6.4).

Shibata, Tsutsumi, and Yamamoto restricted themselves to domains in $\mathbb{R}^n$ with $n \geqslant 3$ because of Lemma 6.2.4 below. They proved the following theorem:

Theorem 6.2.1 ([Sh/Ts 84], Theorem 2.3, [Ya 92b], Theorem 1.2) *Let $n \geqslant 3$ and assume that Ω fulfils Vainberg's non-trapping condition. Then, for any $a, b > R$ there exist positive constants C and δ depending on a, b, n, Ω such that*

$$\|U(t)\|_{a \to b} \leqslant C\, g(t)\,, \qquad t > 0\,,$$

$$\textit{with} \qquad g(t) = \begin{cases} e^{-\delta t}\,, & \textit{for odd } n, \\ t^{-n+1}\,, & \textit{for even } n. \end{cases}$$

Remarks:

1. This proves uniform LED with the rate $c\,g(t)^2$, because

$$\begin{aligned} E(u(t), \Omega_b) &\leqslant c(\|u(t)\|_{1,b}^2 + \|u_t(t)\|_b^2) \\ &= c\,\|U(t)f\|_{H_b}^2 \\ &\leqslant c\,g(t)^2\;\|f\|_{\mathring{H}_a}^2 \\ &= c\,g(t)^2\, E\,(u(0), \Omega)\,. \end{aligned}$$

2. One may compare this result with Theorem 5.3.2, which yields the same decay rates provided Melrose's non-trapping condition is fulfilled.

Proof: We will just give the outline of the proof according to [Sh/Ts 84].

Let

$$D^- := \left\{ k \in \mathbb{C} \;\middle|\; \operatorname{Im} k < 0 \right\},$$
$$D \;:= \begin{cases} \mathbb{C}, & n \geqslant 3 \text{ odd}, \\ \left\{ k \in \mathbb{C} \;\middle|\; -\dfrac{3\pi}{2} < \arg k < \dfrac{\pi}{2} \right\}, & n \geqslant 4 \text{ even}. \end{cases}$$

Because $U(t)$ is a unitary group, for all $k \in D^-$ and $f \in H$ the Laplace transform

$$\widetilde{U}(k)f := \int_0^\infty e^{-\mathrm{i}kt}\, U(t) f\, dt \qquad \text{in } H$$

exists. Partial integration yields

$$\mathrm{i}k\widetilde{U}(k)f = \int_0^\infty e^{-\mathrm{i}kt} (U(t)f)_t\, dt + f$$

and, therefore, because of

$$(\partial_t - \mathcal{A})U(t)f = 0$$

even

$$(\mathrm{i}k - \mathcal{A})\widetilde{U}(k)f = f$$

holds. Let $(\mathrm{i}k - \mathcal{A})^{-1}f$ denote for $f \in H$ and $k \in D^-$ the solution of $(\mathrm{i}k - \mathcal{A})W = f$ in Ω, $W = 0$ on $\partial\Omega$. This yields

$$\begin{aligned} \widetilde{U}(k)f &= (\mathrm{i}k - \mathcal{A})^{-1} f \\ &= \begin{pmatrix} -\mathrm{i}kR(k) & -R(k) \\ -R(k)\,A(x,D) & -\mathrm{i}kR(k) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \end{aligned} \tag{6.5}$$

Here $R(k) := (A(x,D) + k^2)^{-1}$ denotes the operator that maps g to a solution of

$$(A(x,D) + k^2)w = g \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega\,.$$

(6.5) shows that $R(k)$ plays the main part when working with $(\mathrm{i}k - \mathcal{A})^{-1}$. For an arbitrary $\sigma > 0$ and all $t \geqslant 0$ the inverse Laplace transform

$$U(t)f = \frac{1}{2\pi} \int_{-\infty-\mathrm{i}\sigma}^{+\infty-\mathrm{i}\sigma} e^{\mathrm{i}kt} (\mathrm{i}k - \mathcal{A})^{-1} f\, dk \tag{6.6}$$

as an identity in H is defined for all $f \in H$. Here the integral is to be understood as

$$\lim_{M\to\infty} \int_{-M-\mathrm{i}\sigma}^{M-\mathrm{i}\sigma} e^{\mathrm{i}kt} (\mathrm{i}k - \mathcal{A})^{-1} f\, dk\,,$$

where convergence in H is uniform for t in bounded intervals. Unfortunately the imaginary part of k seems to have the wrong sign to derive good estimates of the solution out of (6.6). The path of integration has to be moved (partly) into the upper half-plane. Therefore, a continuation of the resolvent $R(k)$ across the continuous spectrum is needed and its analytical properties together with the behaviour for high and low frequencies have to be investigated. This is done by the following three lemmas:

Lemma 6.2.2 ([Va 68], Theorem 3) *Suppose $a, b > R$. Then the resolvent $R(k)$, $k \in D^-$, can be continued meromorphically to D as a $\mathcal{L}_b(\mathcal{L}^2_a(\Omega), H_2(\Omega_b))$-valued function. The set of poles of $R(k)$ is discrete and its intersection with D^- is empty.*

The continuation of the resolvent will still be denoted by $R(k)$.

Remark: Because k^2 belongs, for $k \in \mathbb{R}$, to the spectrum of $-A(x, D)$, it is necessary to restrict the domain of the resolvent and to extend its range in order to achieve such a continuation. Ladyzhenskaya was the first to study the analytic continuation of the resolvent across the continuous spectrum in [Lad 57] (even if this is sometimes, e.g. [Rau 78a, p. 439], attributed to Dolph, McLeod, and Thoe, who were the first to derive this result in the American literature in [D/McL/T 66]).

Lemma 6.2.3 ([Va 75], Theorem 7)

Let $a, b > R$ and let Ω fulfil Vainberg's non-trapping condition. Then there exist positive constants α, β, C, T such that for integers $0 \leqslant s \leqslant 1$ and $0 \leqslant j \leqslant 2$

$$\|R(k)\|_{\mathcal{L}_b(\mathring{\mathcal{H}}_{s,a}(\Omega), \mathcal{H}_{s+2-j}(\Omega_b))} \leqslant C\, |k|^{1-j} e^{T|\mathrm{Im}\, k|}$$

for any k out of

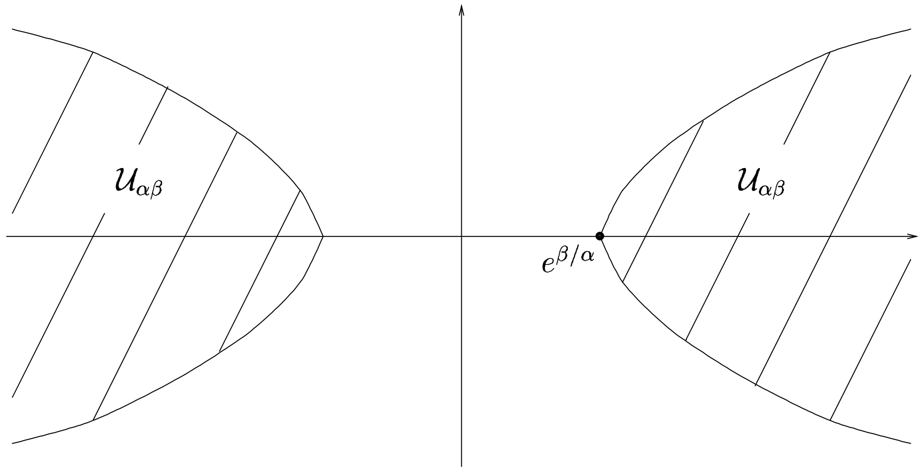
$$\mathcal{U}_{\alpha\beta} := \left\{ k \in D \;\middle|\; |\mathrm{Im}\, k| \leqslant \alpha \ln |\mathrm{Re}\, k| - \beta \right\}.$$

Remarks:

1. This especially means that $\mathcal{U}_{\alpha\beta}$ is a pole-free region of $R(k)$.

2. This result remains unchanged if the non-trapping condition for the weak solution is imposed (cf. [Di 92] and [Ya 92b]) .

We give a sketch of the regions $\mathcal{U}_{\alpha\beta}$ in D:



To be able to shift the path of integration into the upper half-plane there is still some information needed about the behaviour of the resolvent near $k = 0$.

Lemma 6.2.4 ([Va 73], Th. 2, [Ya 92a], Th. 4.6, [Ya 92b], Th. 1.1) *Suppose $n \geqslant 3$ and $a, b > R$. Then there exists a $\gamma > 0$ such that:*

1. *For odd n*

$$R(k) \quad \text{is holomorphic in} \quad W := \left\{ k \in D \;\middle|\; |k| < \gamma \right\}.$$

2. *For even n*

$$R(k) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} B_{mj} (k^{n-2} \ln k)^m k^j \quad \text{in} \quad W' := \left\{ k \in D \;\middle|\; |k| < \gamma \right\}.$$

Here the $B_{mj} \in \mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$ and the double series converge absolutely and uniformly in the operator norm.

Remarks:

1. This means that near $k = 0$ $R(k)$ has no pole and is bounded.

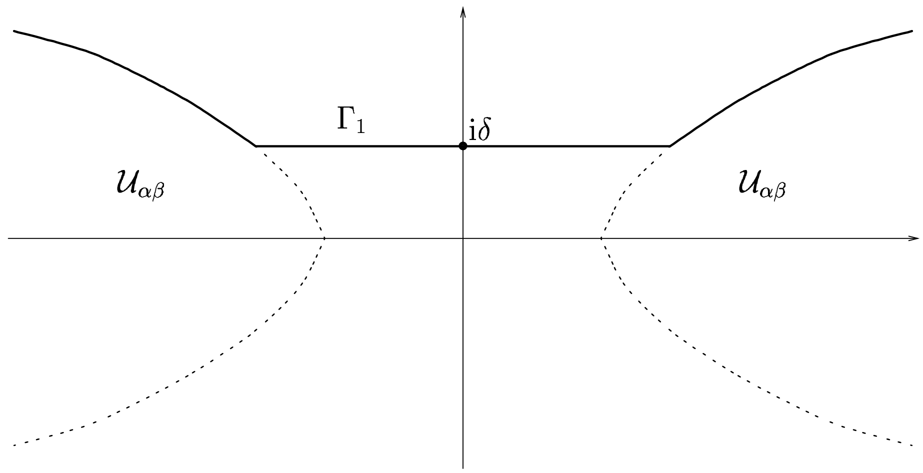
2. For even n Lemma 6.2.4 yields in W':

$$R(k) = B_1(k) + k^{n-2} \ln k \, B_2 + k^{n-2} B_3(k) \,,$$

   where $B_1(k)$ is holomorphic in W and $B_3(k)$ is bounded and continuous in W', both as $\mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$-valued functions, and $B_2 \in \mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$.

For a proof of these lemmas we refer to the quoted literature. Here we will continue with the sketch of the proof of Theorem 6.2.1: These lemmas make it possible to shift the contour of the integral in (6.6) and to estimate $\|U(t)f\|_{H_b}$. The following integral paths Γ_1 are chosen:

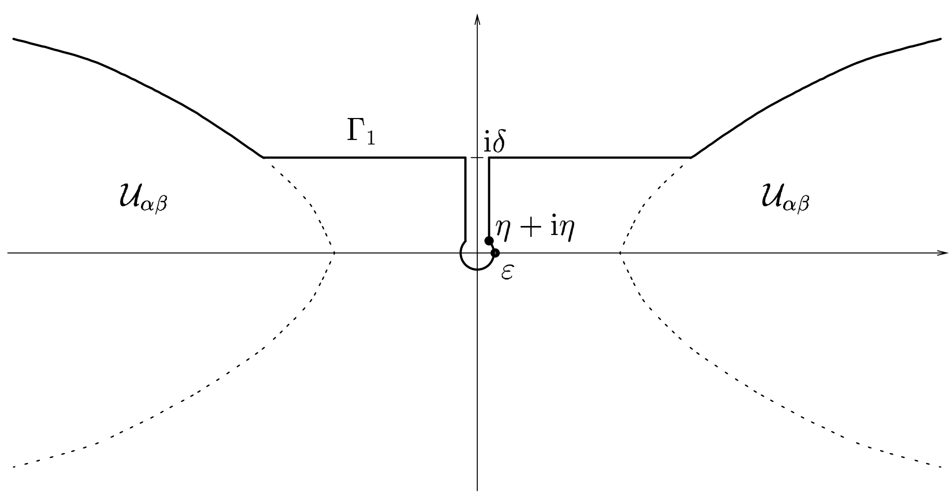
n odd: $\quad \Gamma_1 := \left(\left\{ k \in D \;\middle|\; \operatorname{Im} k = \delta \right\} \setminus \mathcal{U}_{\alpha\beta} \right) \cup \left(\left\{ k \in D \;\middle|\; \operatorname{Im} k \geqslant \delta \right\} \cap \partial \mathcal{U}_{\alpha\beta} \right).$



n even: Let $\varepsilon \in (0, \min(\delta, \sigma))$ and $\eta := \varepsilon/\sqrt{2}$. Because of the slit plane, here one chooses Γ_1 for $|\operatorname{Re} k| > \eta$ and $\operatorname{Im} k \geqslant \delta$ as above, adding the "thermometer"-shaped contour

$$\left\{k \in D \;\middle|\; \operatorname{Re} k = \pm\eta \;\wedge\; \eta \leqslant \operatorname{Im} k \leqslant \delta\right\} \cup \left\{k \in D \;\middle|\; |k| = \varepsilon \;\wedge\; \operatorname{Im} k < \eta\right\},$$

finally yielding:



Because of the above lemmas there exists a $\delta > 0$ with $\delta < \gamma$, such that no pole of $R(k)$ is situated between the old and the new contour. Let such a δ and an $\varepsilon \in (0, \min(\delta, \sigma))$ be fixed. In applying the Cauchy integral theorem some convergence problems have to be analysed to justify the shift of the contour. This is first possible for initial data with a higher regularity only. But because of continuity, the resulting identities still hold for initial data with finite energy. So the high-frequency behaviour—i.e. the large pole-free regions near the real axis—decides whether there is uniform LED or not. But the explicit rate is calculated from the low-frequency asymptotics, and here the resolvent behaves much better in odd space dimensions. With these remarks we conclude the proof of Theorem 6.2.1. The detailed application of the above lemmas can be found, for example, in [Pau 96]. □

Remark: The influence of the high- and low-frequency behaviour on the decay of the energy is closely related to the following facts. Singularities essentially result from the superposition of high-frequency waves and the localized solutions of Ralston have high-frequency initial data. The reverberations of the fundamental solution of the Cauchy problem in even dimensions, however, are smooth. So they essentially result from the superposition of low-frequency waves.

These results still hold for the Neumann condition with the following choice of H, $\mathring{H}_a$, and $D(\mathcal{A})$:

$$\begin{aligned} H &:= \left\{(u,v) \;\middle|\; u \in \mathcal{H}_1(\Omega), v \in \mathcal{L}^2(\Omega)\right\}, \\ \mathring{H}_a &:= \left\{(u,v) \in H \;\middle|\; \operatorname{supp}(u,v) \subset \overline{\Omega}_a\right\} \;=\; \left\{(u,v) \;\middle|\; u \in \mathcal{H}_{1,a}(\Omega), v \in \mathcal{L}^2_a(\Omega)\right\}, \end{aligned}$$

the norms remaining unchanged, and

$$D(\mathcal{A}) := \Big\{(u,v) \in H \;\Big|\; \mathcal{A}(u,v) \in H \text{ and } \langle \nabla u, A(x)\nabla f\rangle = -\langle A(x,D)u, f\rangle \;\; \forall f \in \mathcal{H}_1(\Omega)\Big\}.$$

Exchanging the boundary condition in the non-trapping condition then implies LED with the same decay rates as above (cf. [Sh/Ts 85], Lemma Ap. 1, p. 222; the exponential decay for odd n can be derived as usual from Theorem 3.1.2).

6.3 Uniform LED for Domains in $\mathbb{R}^2$

It is not due to any deficiency of the Vainberg method that Shibata and Tsutsumi did not establish results for domains in $\mathbb{R}^2$. What was missing was an adequate low-frequency asymptotic for this case. Theorem 2, p. 227, of [Va 73] includes such a result for a general class of problems:

$$\exists A, \alpha \in \mathbb{Z} \;\; \exists \gamma > 0 \;\; \forall k \in W' : \quad R(k) = k^{-A} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} B_{mj} (k \ln^{\alpha} k)^m (\ln k)^{-j},$$

where $B_{mj} \in \mathcal{L}_b(\mathcal{L}^2_a(\Omega), \mathcal{H}_2(\Omega_b))$ and the series converges absolutely and uniformly in the operator norm. But a more precise result for (W.D) and (W.N) was still needed. There are numerous publications dealing with this topic. We want to mention only a few of them. Whereas Muraveĭ imposed the convexity of the obstacle, when dealing with a homogeneous medium in [Mu 73, Mu 79], Weck and Witsch succeeded in giving the first terms of the series for general obstacles (even without a $\mathcal{C}_\infty$-boundary) and inhomogeneous media in [We/Wit 92], Theorem 2, p. 317. The complete low-frequency asymptotics for exterior domains with $\mathcal{C}_\infty$-boundary were finally proven by Kleinman and Vainberg in [Kl/Va 94], Theorems 1 and 2, p. 992f:

Lemma 6.3.1 *Let* $n = 2$ *and* $a, b > R$. *Then there exists a* $\gamma > 0$ *such that the following holds in* $W' = \left\{ k \in D \;\middle|\; |k| < \gamma \right\}$:

(1) *For (W.D):*

$$R(k) = \sum_{j=0}^{\infty} \sum_{m=0}^{j} \sum_{p=0}^{j-m+1} B_{pmj} \ln^m k \; k^{2j} (\ln k - z_0)^{-p} ,$$

$z_0 \in \mathbb{C}$ *being independent of* a, b *und* γ.

(2) *For (W.N):*

$$R(k) = \sum_{j=0}^{\infty} \sum_{m=0}^{j+1} B_{mj} \ln^m k \; k^{2j} .$$

Here $B_{pmj}, B_{mj} \in \mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$ *and both series converge absolutely and uniformly with respect to the operator norm.*

For a proof the reader is referred to [Kl/Va 94].

Remarks:

1. Thus near $k = 0$ the resolvent of the Dirichlet problem has no pole and is uniformly continuous and, therefore, bounded. In W' the lemma yields

$$R(k) = B_1(k) + (\ln k - z_0)^{-1} B_2 + k\, B_3(k) \,, \tag{6.7}$$

where $B_1(k)$ is holomorphic in W and $B_3(k)$ is bounded and continuous in W', both as $\mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$-valued functions, and $B_2 \in \mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$.

2. In the case of the Neumann problem the resolvent still has no pole in W', but is bounded as a $\mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$-valued function in the sets

$$\left\{ k \in D \;\middle|\; \varepsilon \leqslant |k| < \gamma \right\} \quad \text{with} \quad 0 < \varepsilon < \gamma,$$

only. In W'

$$R(k) = B_1(k) + \ln k \, B_2 + B_3(k) \tag{6.8}$$

holds, where $B_1(k)$, B_2, and $B_3(k)$ have the same properties as above.

3. For the calculation of z_0 cf. [Kl/Va 94] and [We/Wit 92].

With Lemma 6.3.1 uniform LED in two space dimensions can be established. The present author does not know a publication that proves this result. Thus he gave a proof for a homogeneous medium in [Pau 96, Section 8.3]. The proof of the general case is literally the same, so we do not repeat it here. We just want to mention that a skilful analytic procedure according to Muraveĭ makes it possible to derive from the slightly "better" representation (6.7) (in contrast to (6.8)) a stronger decay rate for the Dirichlet problem ($ct^{-2} \ln^{-4} t$) than for the Neumann problem (ct^{-2}).

6.4 Summarizing Theorem for the Vainberg Method

We want to conclude the considerations on the Vainberg method by summarizing the decay results of the preceding two sections in a theorem (the non-trapping conditions shall be the ones for the particular boundary value problem):

Theorem 6.4.1 *Let $n \geqslant 2$ and assume that Ω fulfils Vainberg's non-trapping condition or the non-trapping condition for the weak solution. Let $K \subset\subset \overline{\Omega}$. Then there exist positive constants c and δ such that for all solutions u of (W.D) and (W.N) respectively with initial data having support in $\overline{\Omega}_R$ the following hold:*

1. *For (W.D) with $n \geqslant 3$ and (W.N) with $n \geqslant 2$:*

$$(i)\ n\ \textit{even:}\ E\,(u(t),K) \leqslant c\,t^{-2n+2}\,E\,(u(0),\Omega)\,, \qquad t > 0\,,$$
$$(ii)\ n\ \textit{odd:}\ E\,(u(t),K) \leqslant c\,e^{-\delta t}\,E\,(u(0),\Omega)\,, \qquad t \geqslant 0\,.$$

2. *For (W.D) with $n = 2$:*

$$E\,(u(t),K) \leqslant c\,t^{-2}\ln^{-4} t\;E\,(u(0),\Omega)\,, \qquad t > 0\,.$$

Using the Vainberg method one can prove uniform LED for many other boundary value problems (cf. [Va 75, Va 89]). We just want to mention explicitly that Yamamoto proved the above result in [Ya 92b] even for the Robin condition, provided $n \geqslant 3$.

The similarity of Theorem 6.4.1 and Theorem 5.3.2 for $n \geqslant 3$ is obvious. This is due to the fact that Ralston proved Theorem 6.4.1 for $n \geqslant 3$ after having derived Vainberg's non-trapping condition from the one of Melrose. Results like the latter are subject of the following chapter.

Chapter 7

Comparison of the Different Non-Trapping Conditions

In the first chapters we have presented different methods for the proof of uniform LED together with their specific non-trapping conditions. We now want to compare these requirements. While we do this, some further conditions that appear in the literature will be included, which are closely related to those we have already discussed. As a reminder we list the latter ones.

1. Melrose's NTC (Definition 5.1.1).
2. The non-trapping hypothesis of Morawetz, Ralston, and Strauss (Definition 5.2.1).
3. Vainberg's NTC (Definition 6.1.6).
4. The NTC for the weak solution (Definition 6.1.7).
5. The existence of an escape function (Definition 3.2.1).

Remark: Generalized bicharacteristics were introduced without referring to a special boundary value problem. Thus there exists only *one* NTC of Melrose. Things are the same with Definitions 5.2.1 and 3.2.1 (even if the methods of [M/R/S 77] did not suffice for a proof of uniform LED for (W.N)). The conditions 3. and 4., however, both emphasize the particular boundary value problem. In the sequel we will, therefore, add a supplement to them which makes it clear which boundary condition is meant.

The non-trapping hypothesis of Morawetz, Ralston, and Strauss is formulated for "non-degenerated" obstacles in a homogeneous medium in $\mathbb{R}^3$. As we have seen, exactly those obstacles are non-degenerated for which $G^\infty = \emptyset$ holds. According to Corollary 5.2.5 this yields the equivalence of Melrose's non-trapping condition and the non-trapping hypothesis of Morawetz, Ralston, and Strauss. In addition we already know by Theorem 5.1.2 that the existence of an escape function implies Melrose's non-trapping condition.

The rest of this chapter is structured as follows: Starting with Melrose's NTC we prove a chain of implications leading finally to uniform LED for (W.D) or (W.N). At several stages we will include further conditions that appear in the literature. For obstacles without points in G^∞, uniform LED implies Melrose's NTC by the theorems of Ralston. For these obstacles, therefore, the equivalence of all non-trapping conditions under consideration is proven. We will add some results for both special dimensions and homogeneous media that will include the existence of escape functions and the eventual compactness of the Lax-Phillips semigroup $Z(t)$. Finally—as a reminiscence of the first chapters—we will present some geometrical annotations on non-trapping conditions in general.

Because the second part of this thesis deals with the more complicated situation of isotropic elasticity, we present most of the results of the present chapter without proofs. They are achieved by the obvious simplifications of the proofs of Chapter 11. Only the results concerning those conditions which are special features of the literature devoted to the wave equation will be proved here.

7.1 The General Theorems

Before we proceed in the announced chronology, we will state a result which shows the most immediate application of the generalized Huygens' principle in this context:

Theorem 7.1.1 *Melrose's non-trapping condition implies the non-trapping condition for the weak solution for both (W.D) and (W.N).*

Proof: The generalized Huygens' principle yields that the weak solution of (W.D) or (W.N) can only be singular along generalized bicharacteristics which at time $t = 0$ meet the support of the initial data. For the wave equation a maximal length of generalized bicharacteristics in $\overline{\Omega}_a$ implies a maximal length of the time interval by which this curve may w.l.o.g. be parametrized. For initial data with support in $\overline{\Omega}_a$, Melrose's NTC implies, therefore, according to Theorem 4.2.1 the existence of a $\tau(\Omega, a)$ such that

$$u \in \mathcal{C}_\infty([\tau, \infty) \times \overline{\Omega}_a) .$$

This yields in particular the NTC for the weak solution for both (W.D) and (W.N). □

In [Ral 79] there is the assertion that for a homogeneous medium Melrose's NTC implies—as a corollary of the generalized Huygens' principle—Vainberg's NTC (Dirichlet or Neumann). But one cannot apply Theorem 4.2.1 in this immediate way for two reasons: On the one hand, the Schwartz kernel theorem yields $G(t, x, x_0)$ as distributional solution of a differential equation in $\mathbb{R} \times \Omega \times \Omega$ and the principal symbol $p(t, x, x_0, \tau, \xi, \xi_0) = |\xi|^2 - \tau^2$, regarded as a function in $\mathcal{C}_\infty(T^*(\mathbb{R} \times \Omega \times \Omega))$, violates the assumptions of the generalized Huygens' principle. On the other hand, $\mathbb{R} \times \Omega \times \Omega$ is not a $\mathcal{C}_\infty$-manifold because of the "corner" in $\partial\Omega \times \partial\Omega$. What one needs is some information on the regularity of the Green's function with respect to x_0. Professor Ralston was so kind as to answer the author's question on this point with a refined proof of the above implication. He essentially showed that the dependence on x_0 is in fact $\mathcal{C}_\infty$, which makes Theorem 4.2.1 applicable. We presented a detailed version of this proof in [Pau 96, Chapter 9]. With some slight modifications this proof applies also to inhomogeneous media, yielding the following result:

Theorem 7.1.2 ([Ral 79]) *Melrose's non-trapping condition implies Vainberg's non-trapping condition for both (W.D) and (W.N).*

The author succeeded in proving the corresponding result for isotropic elasticity using the ideas of Ralston. Because we present this proof with all details in Chapter 11, we omit it here.

A condition very similar to Vainberg's non-trapping condition was used by Rauch in his extension [Rau 78a] of the results [Va 68, Va 73, Va 75] of Vainberg. For the Cauchy problem he demands a scattering of singularities in terms of Melrose's condition ([Rau 78a, (1.5)]). In the case of exterior boundary value problems he uses another condition, not least because at that time there was no complete understanding of the propagation of singularities for exterior boundary value problems even for the wave equation. So he postulates his "propagation of singularities hypothesis" [Rau 78a, (9.3)], which is obviously influenced by Vainberg's non-trapping condition. For the wave equation this condition reads (G being the Green's function of the Dirichlet problem or the Neumann poblem).

Definition 7.1.3 (The Propagation of Singularities Hypothesis of Rauch)
Ω *is called* non-trapping

$$:\Longleftrightarrow \quad \forall \chi \in \mathring{\mathcal{C}}_\infty(\overline{\Omega})\ \exists T > 0\ \forall t \geqslant T \ : \quad \chi(x)G(t,x,x_0)\chi(x_0) \in \mathring{\mathcal{C}}_\infty(\overline{\Omega} \times \overline{\Omega}) \ .$$

Remark: In Rauch's hypothesis there is a somewhat weaker condition on the support of $\chi G \chi$. But according to the definition of χ this support is actually compact for any $t \geqslant 0$. The essential point is, therefore, the regularity of the distribution for $t \geqslant T$.

Because Definition 7.1.3 involves *all* cut-off functions χ, the main importance lies not in the special shape of χ, but in its support. This is not least expressed by the following theorem:

Theorem 7.1.4 *Rauch's propagation of singularities hypothesis is equivalent to the non-trapping condition of Vainberg.*

We will present a proof for the related result of isotropic elasticity in Chapter 11.

In [Va 75] Vainberg used a non-trapping condition different from the one of [Va 89] (Definition 6.1.6 above). It was his condition D′, [Va 75, (p. 11)], and this is still the name it has in the literature:

Definition 7.1.5 (Vainberg's Condition D′)
For some $N \geqslant 2$ there is an $E_N(t,x,x_0) \in \mathcal{D}'(\Omega)$ depending continuously on parameters $x_0 \in \overline{\Omega}$, $x_0 \leqslant a$, and $t \geqslant 0$, where E_N satisfies the following two conditions:

(a) *E_N is a distributional solution of:*

$$\begin{cases} (\partial_t^2 - \partial_{x_i} a_{ij}(x)\, \partial_{x_j})\, E_N(t,x,x_0) = f_N(t,x,x_0)\,, & (t,x) \in (0,\infty) \times \Omega\,, \\ B_x\, E_N(t,x,x_0) = 0\,, & (t,x) \in (0,\infty) \times \partial\Omega\,, \\ E_N(0,x,x_0) = 0\,,\ \partial_t E_N(0,x,x_0) = \delta(x - x_0)\,, & x \in \Omega\,, \end{cases}$$

where f_N has N continuous derivatives with respect to t, x, x_0.

(b) *For any $r < \infty$ there is a $T(r) > 0$ such that $E_N = 0$ for $|x| \leqslant r$, $|x_0| \leqslant a$, and $t \geqslant T(r)$.*

Here B belongs to a very general class of boundary operators which includes

$$B = Id \qquad \text{and} \qquad B = \frac{\partial}{\partial \mathbf{n}}\,.$$

In (a) all equations are meant distributionally with respect to the given variables. The following result holds:

Theorem 7.1.6 *Vainberg's non-trapping condition implies Vainberg's condition* D′.

Remark: Vainberg avoided his condition D′ in [Va 89] because there, in contrast to [Va 75], he did not treat any boundary condition that involves time derivatives. This allowed him to use ξG with a suitable cut-off function $\xi(t,x)$ instead of E_N (cf. the proof preceding Theorem 11.2.2). This is not possible in case of the general boundary operators of [Va 75], because then ξG may not satisfy the original boundary conditions anymore. But this is of crucial importance in Vainberg's proof of the above Lemma 6.2.3. Thus it seems useful to demand condition D′ in [Va 75], that is to postulate at once the existence of a parametrix with good asymptotic properties. So with this condition D′, Vainberg was able to prove uniform local energy decay in [Va 75] for a very general class of boundary value problems. If $G^\infty = \emptyset$, condition D′ therefore implies Melrose's NTC and all conditions that follow from it.

One easily verifies the following theorem (cf. the proof of Theorem 11.1.8):

Theorem 7.1.7 *For (W.D) and (W.N) respectively the following holds:*
Vainberg's non-trapping condition implies the non-trapping condition for the weak solution.

The last chapter's proof of uniform LED using the Vainberg method followed the outline of Shibata and Tsutsumi. But they used the following NTC (actually for homogeneous media only, but the following discussion does not depend on this fact):

Definition 7.1.8 (Non-Trapping Condition of Shibata and Tsutsumi) *Let $a, b > R$. Let $G(t, x, x_0)$ denote the Green's function of (W.D) and (W.N) respectively. For $v \in \mathcal{L}^2_a(\Omega)$ put*

$$(\mathcal{G}v)(t,x) := \int_\Omega G(t,x,x_0)v(x_0)\,dx_0\,. \tag{7.1}$$

Ω *is called* non-trapping $:\Longleftrightarrow \exists\, T_0(\Omega, a, b) > 0 \;\; \forall v \in \mathcal{L}^2_a(\Omega): \;\; \mathcal{G}v \in \mathcal{C}_\infty([T_0,\infty)\times\overline{\Omega}_b)\,.$

Equation (7.1) needs an interpretation: A locally integrable function f on an open set $\mathcal{O} \subset \mathbb{R}^m$ generates a distribution $[f] \in \mathcal{D}'(\mathcal{O})$ by

$$\phi \in \mathring{\mathcal{C}}_\infty(\mathcal{O}) \longmapsto \int_{\mathcal{O}} f(x)\phi(x)\,dx \tag{7.2}$$

but (7.2) is often used formally for arbitrary distributions. If P denotes again the solution operator of (6.2) for smooth φ, one could return from the Schwartz kernel theorem to the beginning of Section 6.1 and write formally the distribution kernel as an integral kernel:

$$(P\varphi)(t,x) = \int_\Omega G(t,x,x_0)\varphi(x_0)\,dx_0\,. \tag{7.3}$$

P extends to a solution operator on $\mathcal{L}^2(\Omega)$, and in this spirit one has to interpret the right-hand side of (7.3) for these φ. Thus in (7.1) $(\mathcal{G}v)$ stands for the weak solution of

$$\begin{cases} (\partial_t^2 - \partial_i a_{ij}(x)\partial_j)u = 0 & \text{in} \;\; \mathbb{R}\times\Omega\,, \\ u = 0 \quad (\text{or } \dfrac{\partial u}{\partial \mathbf{n}} = 0) & \text{on} \;\; \mathbb{R}\times\partial\Omega\,, \\ u(0) = 0\,, \quad u_t(0) = v & \text{in} \;\; \Omega \end{cases}$$

for $v \in \mathcal{L}^2_a(\Omega)$, and so Shibata and Tsutsumi only use a slightly different way to write the non-trapping condition for the weak solution. Whether one uses two different radii $a, b > R$—with or without the restriction $a \geqslant b$—or only one radius a, does not matter, because this leads to equivalent conditions. It is also possible to assume

$$\forall \psi \in \mathring{\mathcal{C}}_\infty(\mathbb{R}^n) \;\; \exists\, T_0(a,\psi,\Omega) \;\; \forall v \in \mathcal{L}^2_a(\Omega): \;\; \psi(x)(\mathcal{G}v)(t,x) \in \mathcal{C}_\infty([T_0,\infty)\times\overline{\Omega}) \tag{7.4}$$

as Tsutsumi does in [Ts 83]. This requirement is equivalent to the non-trapping condition for the weak solution, too, as the proof of Lemma 11.2.1 will show: What has real importance in (7.4) is not ψ itself, but the support of ψ, and this has the same effect as the restriction of $\mathcal{G}v$ to $([T_0,\infty)\times\overline{\Omega}_b)$. This proves the following lemma:

Lemma 7.1.9 *For (W.D) and (W.N) respectively the following conditions are equivalent:*

(*i*) *The non-trapping condition for the weak solution.*

(*ii*) *The non-trapping condition of Shibata and Tsutsumi.*

(*iii*) *The non-trapping condition* (7.4) *of Tsutsumi.*

The non-trapping condition for the weak solution implies uniform LED for solution of (W.D) and (W.N) respectively by the Vainberg method (Theorem 6.4.1). Ralston's Theorems 2.1.1 and 2.1.2, therefore, yield for (W.D) and (W.N) an upper limit of the lengths of all geodesics in $\overline{\Omega}_R$ that can be approximated by non-tangential ones. This includes according to Corollary 4.1.7 and Proposition 4.1.8 all geodesics that give rise to generalized bicharacteristics which do not meet G^∞. As we have already remarked after Theorem 7.1.6, we can draw the same conclusions for domains and media which fulfil Vainberg's condition D′, because of the results of [Va 75]. This yields together with the other results of this section for all dimensions $n \geqslant 2$:

Theorem 7.1.10 *Let $G^\infty = \emptyset$. This yields the equivalence of*

(i) *The non-trapping condition of Melrose.*

(ii) *The non-trapping condition of Vainberg (Dirichlet).*

(iii) *The non-trapping condition of Vainberg (Neumann).*

(iv) *The propagation of singularities hypothesis of Rauch (Dirichlet).*

(v) *The propagation of singularities hypothesis of Rauch (Neumann).*

(vi) *Vainberg's condition* D′ *(Dirichlet).*

(vii) *Vainberg's condition* D′ *(Neumann).*

(viii) *The non-trapping condition for the weak solution (Dirichlet).*

(ix) *The non-trapping condition for the weak solution (Neumann).*

(x) *The non-trapping condition of Shibata and Tsutsumi (Dirichlet).*

(xi) *The non-trapping condition of Shibata and Tsutsumi (Neumann).*

(xii) *The non-trapping condition* (7.4) *of Tsutsumi (Dirichlet).*

(xiii) *The non-trapping condition* (7.4) *of Tsutsumi (Neumann).*

(xiv) *Uniform LED for solutions of (W.D) with the rates of Theorem 6.4.1.*

(xv) *Uniform LED for solutions of (W.N) with the rates of Theorem 6.4.1.*

Remarks:

1. If the medium is homogeneous, a class of obstacles with $G^\infty = \emptyset$ is given by those obstacles with analytic boundaries. This is due to the fact that at a boundary point which is the projection of an element of G^∞, the curvature must vanish of infinite order in at least one direction as we have seen by the explicit calculations of the Sections 5.1 and 5.2. In this direction an analytical boundary must, therefore, consist of a straight line in contradiction to the boundedness of the obstacle.

2. There is no general splitting of bicharacteristics in G^∞. We have already mentioned that the continuation of a bicharacteristic is unique at bicharacteristically convex parts of the boundary. Convexity with respect to the boundary, however, does not exclude points of G^∞. But obviously they do not cause any difficulties here. So the assumption of Theorem 7.1.10 can be considerably weakened. One possibility is to assume just that all generalized bicharacteristics can be approximated by broken ones. The discussion at the end of Section 4.1 shows that it is still an open problem whether this is a restriction at all.

7.2 Results for Special Cases

If the medium is homogeneous, the list of Theorem 7.1.10 can be extended under some additional restrictions on the space dimension n. This will be done in this section.

The original version of the Lax-Phillips conjecture for odd n contained the equivalence of Melrose's NTC and the eventual compactness of the Lax-Phillips semigroup $Z(t)$. But denoting by T_R the supremum of the sojourn times of geodesics in $\overline{\Omega}_R$, the following implications hold for the Dirichlet problem, provided the conjecture is true:

$$\begin{aligned} T_R < \infty \quad &\overset{\text{L.-Ph. conj.}}{\Longrightarrow} \quad Z(t) \text{ is eventually compact} \\ &\overset{\text{[La/Ph 89], p. 277}}{\Longrightarrow} \quad \|Z(t)\| < 1 \quad \text{for large } t \\ &\overset{\text{[L/M/P 63]}}{\Longrightarrow} \quad \text{uniform LED} \\ &\overset{\text{[L/M/P 63]}}{\Longrightarrow} \quad \|Z(t)\| < 1 \quad \text{for large } t \\ &\overset{\text{L.-Ph. conj.}}{\Longrightarrow} \quad T_R < \infty\,. \end{aligned}$$

Thus we have equivalence in all these implications, justifying the choice of the presentation of the Lax-Phillips conjecture in Chapter 2. With a modification of the operators $Z(t)$ for even $n \geqslant 4$, Melrose derived in his proof of Theorem 5.3.1 first for all dimension $n \geqslant 3$ the eventual compactness of $Z(t)$ from his non-trapping condition. Then he used this result to establish uniform LED. This yields for the operators $Z(t)$ in the case of the Dirichlet condition (the Neumann condition is not included because of the special ansatz of Melrose):

Theorem 7.2.1 *Let $n \geqslant 3$ and $G^\infty = \emptyset$. Let the medium be homogeneous. Then to the list of Theorem 7.1.10 the following equivalent condition can be added:*

(*xvi*) *The eventual compactness of $Z(t)$ for the Dirichlet problem.*

For exterior domains in $\mathbb{R}^3$ we are, furthermore, equipped with Theorems 3.1.9 and 5.2.3 of Morawetz, Ralston, and Strauss on uniform LED for solutions of (W.D) in domains that possess an escape function and on the existence of these functions. For $n = 3$ this therefore proves

Theorem 7.2.2 *Let $n = 3$ and $G^\infty = \emptyset$. Let the medium be homogeneous. Then to the lists of Theorems* (7.1.10) *and* (7.2.1) *the following equivalent conditions can be added:*

$(xvii)$ *The non-trapping condition of Morawetz, Ralston, and Strauss.*

$(xviii)$ *The existence of an escape function.*

Remark: Morawetz, Ralston, and Strauss make explicit use of their non-degeneracy condition, i.e. of $G^\infty = \emptyset$, both in the formulation of their non-trapping condition and in the construction of an escape function. Thus Remark 2 following Theorem 7.1.10 does not apply in this case.

For two space dimensions we proved Theorem 3.1.7. On the other hand a trivial consequence of Melrose's non-trapping condition is that no straight line segment in $\overline{\Omega}$ is perpendicular to $\partial\Omega$ at both ends. Furthermore, the assumption of only a finite number of points at which the boundary curvature changes its sign is fulfilled by all obstacles with $G^\infty = \emptyset$: On a smooth and compact boundary with infinitely many points of inflection, these points have to accumulate. This would contradict the regularity of the boundary, if the curvature in such a limit point did not vanish of infinite order. This, however, implies, as we already know, that the limit point is the projection of an element of G^∞. Thus, Theorem 3.1.7 implies:

Theorem 7.2.3 *Let $n = 2$ and $G^\infty = \emptyset$. Let the medium be homogeneous. Then to the list of Theorem* (7.1.10) *the following equivalent conditions can be added:*

$(xvi)'$ *No straight line segment in $\overline{\Omega}$ is perpendicular to $\partial\Omega$ at both ends.*

$(xvii)'$ *The existence of a Straussian vector field.*

$(xviii)$ *The existence of an escape function.*

Proof: What remains is to justify the inclusion of the existence of an escape function in this list. But this can be done as follows:

$$(xvii)' \Longrightarrow (xviii) \Longrightarrow (i) \Longrightarrow (xvi)' \Longrightarrow (xvii)'. \qquad \square$$

Remark: Here, too, Remark 2 following Theorem 7.1.10 does not apply, because the construction of l in [M/R/S 77] makes essential use of the at most finite number of inflection points.

7.3 Geometric Aspects

Point $(xvi)'$ of the last theorem is a very clear geometric condition. In this section we want to continue the geometric analysis of non-trapping conditions of Chapter 3 and Section 5.1. The connecting tool between the geometric properties sufficient for uniform LED and the various non-trapping conditions is given by escape functions, as the next theorem shows:

Theorem 7.3.1 *Let $n \geqslant 2$. Assume the existence of an escape function. Then the conditions (i) to (xv) of Theorem 7.1.10 are fulfilled. If in addition $n \geqslant 3$ and the medium is homogeneous, $Z(t)$ for the Dirichlet problem is eventually compact.*

Proof: The existence of an escape function implies Melrose's NTC according to Theorem 5.1.2. This implies all the other non-trapping conditions by the results of the first section of this chapter, which is a consequence of the generalized Huygens' principle. The eventual compactness of $Z(t)$ is implied by Theorem 5.1.2 and [Me 79]. □

We want to emphasize again the geometrical background of this theorem. All the geometric conditions of Chapter 3 imply the existence of an escape function, if the medium is homogeneous. For general media an additional requirement has to be imposed for this purpose. Let therefore $l(x)$ always denote the special Straussian vector field we obtained in each of the geometric situations discussed in Sections 3.1.1 and 3.1.3. Thus Theorem 7.3.1 implies:

Corollary 7.3.2 *Let the obstacle have one of the geometric properties spherical, convex, (strictly) star-shaped, illuminable from the interior ($n = 2, 3$), illuminable from the exterior ($n = 2, 3$), or snake-shaped ($n = 2$). Then the following hold:*

1. *If the medium is homogeneous, then all the non-trapping conditions (i) to (xiii) of Theorem 7.1.10 are fulfilled. If in addition $n \geqslant 3$, $Z(t)$ for the Dirichlet problem is eventually compact.*
2. *If the medium is inhomogeneous, but*

$$2\langle A(x)\xi, \nabla_x(l(x)\cdot\xi)\rangle - |l(x)|\langle A_l(x)\xi, \xi\rangle > 0\,, \qquad x \in \overline{\Omega}_R\,,\ \xi \in \mathbb{R}^n \setminus \{0\}$$

holds, then all the non-trapping conditions (i) to (xiii) of Theorem 7.1.10 are fulfilled.

We want to add one final geometric property for all space dimensions $n \geqslant 2$. It is not (yet) known, whether one of the non-trapping conditions discussed in this thesis is a *necessary* condition for uniform LED, if general domains with $\mathcal{C}_\infty$-boundary are considered, but *sufficient* are all of them. By the results on the existence of localized solutions of Ralston this implies that a domain which satisfies any of the above non-trapping conditions does not admit non-tangential closed rays. If the medium is homogeneous, this especially excludes the existence of a straight line segment in Ω that is perpendicular to $\partial\Omega$ at both ends. This is even true, if the straight line segment is tangent to the obstacle between its endpoints, as we have discussed after Proposition 4.1.8.

We conclude our discussion of the wave equation with a résumé: For a large class of obstacles the aim to impose an optimal condition for uniform LED has been reached. For this purpose in all space dimensions $n \geqslant 2$ various equivalent conditions for both the Dirichlet and the Neumann problem are available. If the restriction $G^\infty = \emptyset$, however, is violated, the necessity of these conditions could be established only for a few special situations (cf. e.g. the second remark following Theorem 7.1.10). What we can say is that Melrose's non-trapping condition is—besides the simpler geometric conditions of Chapter 3—the strongest requirement, whereas Vainberg's condition D′, the non-trapping condition for the weak solution, and—for the Dirichlet problem and $n \geqslant 3$—the eventual compactness of $Z(t)$ are the most general ones. We have no result on the relation between the latter conditions in the general case. The reason for these problems lies in the unanswered question, whether all generalized bicharacteristics can be approximated by broken ones. This is the point that remained open in the work [Me/Sj II] of Melrose and Sjöstrand in 1982 and there is to our knowledge no proof up till now. A positive answer to this question would finally prove the general equivalence of all non-trapping conditions of Part I of this thesis.

Part II

The Equations of Elasticity for Isotropic Media

Chapter 8

Fundamentals

In this chapter we state the equations of elasticity for isotropic media—for a rigorous derivation cf. [Ca 72]. The energy is defined and conditions on the Lamé moduli are given. In the last section we present the basic theorem on existence, uniqueness, and local energy decay.

8.1 Formulation of the Equations

Let Ω fulfil the assumptions stated in Chapter 1. Let $U = (U^1, \dots, U^n) = U(t,x)$ be the displacement vector of an isotropic elastic medium filling the whole of Ω. Then the equations of elasticity are

$$(\mathrm{El}.\Lambda) \quad \begin{cases} \partial_t^2 U - \operatorname{div}\big(\mu(\nabla U + (\nabla U)^T) + (\lambda \operatorname{div} U) I_n\big) = 0 & \text{in } \mathbb{R} \times \Omega\,, \\ \Lambda U = 0 & \text{on } \mathbb{R} \times \partial\Omega\,, \\ U(0) = U_0\,, \quad U_t(0) = U_1 & \text{in } \Omega\,, \end{cases}$$

$\lambda, \mu \in \mathcal{C}_\infty(\overline{\Omega})$ being the Lamé moduli of the medium, which are required here to become constant outside $\overline{\Omega}_R$ for some $R \geqslant R_0$. Some further restrictions on λ and μ will follow from Section 8.2. For later convenience, we introduce the operator $A(x, D)$ by the identity

$$A(x,D)U := \operatorname{div}\big(\mu(\nabla U + (\nabla U)^T) + (\lambda \operatorname{div} U) I_n\big)\,.$$

We study the following boundary value problems: The *Dirichlet* or *displacement problem* (El.D) with

$$\Lambda U := U$$

and the *Neumann* or *traction problem* (El.N) with

$$\Lambda U := \begin{pmatrix} \sum_{i=1}^n \mathbf{n}_i \sigma_{i1} \\ \vdots \\ \sum_{i=1}^n \mathbf{n}_i \sigma_{in} \end{pmatrix},$$

the σ_{ij} given by

$$\sigma_{ij} := \lambda \nabla' U \delta_{ij} + \mu(\partial_j U^i + \partial_i U^j)\,.$$

Because of

$$\operatorname{div}\big((\lambda \operatorname{div} U) I_n\big) = \nabla(\lambda \operatorname{div} U),$$

the differential equation can be written as

$$\partial_t^2 U - \nabla' \left(\mu(\nabla U + (\nabla U)^T)\right) - \nabla(\lambda \nabla' U) = 0$$

or as

$$\underbrace{\partial_t^2 U - (\lambda+\mu)\nabla\nabla' U - \mu \Delta U}_{\text{principal part}} - (\nabla U + (\nabla U)^T)\nabla\mu - (\nabla' U)\nabla\lambda = 0\,.$$

In case of a homogeneous medium there remains only the principal part yielding

$$\partial_t^2 U - (\lambda+\mu)\nabla\nabla' U - \mu \Delta U = 0\,.$$

The principal part is emphasized here not only for the case of homogeneous media, but also for the examination of the propagation of singularities, because there again the principal symbol of the differential operator plays the central role.

We now want to define the energy of solutions of (El.Λ). This will lead to some restrictions on the Lamé moduli, for an energy is still required to be a positive definite form in $D_{t,x}U$.

8.2 Definition of the Energy

We motivate the later definition of the energy by applying the energy method with the multiplier U_t to the system (El.Λ) leading in fact to the correct physical energy. We use the following scalar product for $(n \times n)$ matrices $A = (a_{ij})$ and $B = (b_{ij})$:

$$A \cdot B := a_{ij} b_{ij}\,.$$

After integrating over Ω we get for solutions of (El.D) or (El.N)

$$\begin{aligned} 0 &= \int_\Omega \Big(U_{tt} - \operatorname{div}\big(\mu(\nabla U + (\nabla U)^T) + (\lambda \operatorname{div} U) I_n\big)\Big) U_t \, dx \\ &= \int_\Omega U_{tt} U_t + \big(\mu(\nabla U + (\nabla U)^T) + (\lambda \operatorname{div} U) I_n\big) \cdot \nabla U_t \, dx \\ &= \int_\Omega U_{tt} U_t + \mu(\nabla U \cdot \nabla U_t) + \mu\, \partial_i U^j \partial_j U_t^i + \lambda \nabla' U \nabla' U_t \, dx \\ &= \frac{1}{2}\frac{d}{dt} \int_\Omega |U_t|^2 + \mu |\nabla U|^2 + \mu\, \partial_i U^j \partial_j U^i + \lambda |\nabla' U|^2 \, dx\,. \end{aligned}$$

We now define the energy $E(U(t), D)$ of a solution U of (El.D) and (El.N) respectively at time t in $D \subset \overline{\Omega}$ by

$$E(U(t), D) := \int_D |U_t(t)|^2 + \mu|\nabla U(t)|^2 + \mu\, \partial_i U^j(t) \partial_j U^i(t) + \lambda |\nabla' U(t)|^2 \, dx \,.$$

Then $E(U(t), \Omega)$ is obviously constant. However, for the positive definiteness of $E(U(t), \Omega)$, some restrictions on the Lamé moduli have to be made: For the special case of (El.D) with μ being constant we get by partial integration

$$E(U(t), \Omega) = \int_\Omega |U_t(t)|^2 + \mu|\nabla U(t)|^2 + (\lambda + \mu)|\nabla' U(t)|^2 \, dx \,. \tag{8.1}$$

This is the form of the energy in most of the literature dealing with homogeneous isotropic media. (8.1) yields at once

$$\lambda + \mu, \mu > 0 \quad \Longrightarrow \quad E(U(t), \Omega) \text{ is positive definite.}$$

But this can be improved a little bit, because

$$E(U(t), \Omega) = \int_\Omega |U_t(t)|^2 + \mu \underbrace{\left(|\nabla U(t)|^2 - \frac{1}{n} |\nabla' U(t)|^2 \right)}_{\geqslant 0} + \left(\lambda + \frac{n+1}{n} \mu \right) |\nabla' U(t)|^2 \, dx \,.$$

Therefore

$$\lambda + \frac{n+1}{n}\mu \,, \mu > 0 \quad \Longrightarrow \quad E(U(t), \Omega) \text{ is positive definite.}$$

Because of

$$\lambda + \frac{n+1}{n} \mu > \lambda + \mu,$$

this last condition on the Lamé moduli is weaker.

For inhomogeneous media and for the Neumann problem respectively, the above partial integration leads to a more complicated expression, which doesn't allow the above conclusions. But pointwise estimates of the integrand are helpful. Note first that

$$\begin{aligned}
&\mu|\nabla U|^2 + \mu \partial_i U^j \partial_j U^i + \lambda |\nabla' U|^2 \\
&= 2\mu |\partial_i U^i|^2 + \mu \sum_{1 \leqslant i < j \leqslant n} \left(|\partial_i U^j|^2 + |\partial_j U^i|^2 + 2 \partial_i U^j \partial_j U^i \right) + \lambda |\nabla' U|^2 \\
&= 2\mu \left(|\partial_i U^i|^2 - \frac{1}{n} |\nabla' U|^2 \right) + \mu \sum_{1 \leqslant i < j \leqslant n} \left(\partial_i U^j + \partial_j U^i \right)^2 + \left(\lambda + \frac{2}{n} \mu \right) |\nabla' U|^2 \,.
\end{aligned}$$

For the integrand of the energy integral to be a positive definite form in $D_{t,x} U$, it is therefore necessary and sufficient that $n\lambda + 2\mu, \mu > 0$. Thus we assume from now on:

Assumption 8.2.1 *The Lamé moduli λ and μ fulfil*

$$n\lambda + 2\mu, \mu > 0\,. \tag{8.2}$$

For the special case of (El.D) with $\mu = const.$ let

$$\lambda + \frac{n+1}{n}\mu, \mu > 0 \tag{8.3}$$

instead.

Note that for the energy we have the simpler expression (8.1) in the last case.

Remark: The above assumption implies, as we have seen, the existence of some positive constants $c_1, c_2 > 0$ such that

$$c_1 \left(\|\nabla U(t)\|^2 + \|\nabla U_t(t)\|^2\right) \leqslant E\left(U(t), \Omega\right) \leqslant c_2 \left(\|\nabla U(t)\|^2 + \|\nabla U_t(t)\|^2\right)\,. \tag{8.4}$$

The first inequality is often referred to as *Korn's (first) inequality.*

For the sake of completeness it should be mentioned that more general assumptions on λ and μ are possible in the case of the Cauchy problem: According to the remark in [Rac 92, p. 119], there exists an orthogonal decomposition of $\mathcal{L}^2 := (\mathcal{L}^2(\mathbb{R}^n))^n$ in potential and solenoidal fields given by

$$\mathcal{L}^2 = \overline{\nabla \mathcal{H}_1(\mathbb{R}^n)} \oplus \mathcal{D}_0\,,$$

with $\overline{\nabla \mathcal{H}_1(\mathbb{R}^n)}$ denoting the closure in $\mathcal{L}^2$ of

$$\nabla \mathcal{H}_1(\mathbb{R}^n) := \left\{\nabla\varphi \;\middle|\; \varphi \in \mathcal{H}_1(\mathbb{R}^n)\right\} = \left\{V \in \mathcal{L}^2 \;\middle|\; \exists\, \varphi \in \mathcal{H}_1(\mathbb{R}^n) \;:\; V = \nabla\varphi\right\}$$

and

$$\mathcal{D}_0 := \left\{W \in \mathcal{L}^2 \;\middle|\; \forall\, \varphi \in \mathring{\mathcal{C}}_\infty(\Omega) \;:\; \langle \nabla\varphi, W\rangle = 0\right\}$$

being the space of all weakly divergence-free vector fields. Such a decomposition of the initial values leads to a decomposition

$$U = U^{\mathrm{po}} + U^{\mathrm{so}}$$

of the solution, the so-called pressure and shear waves. In the distributional sense

$$\nabla\nabla' V = \nabla\nabla'\nabla\varphi = \nabla\Delta\varphi = \Delta\nabla\varphi = \Delta V \qquad \text{for} \quad V \in \nabla\mathcal{H}_1(\mathbb{R}^n)$$

holds. Thus U^{po} and U^{so} solve

$$\partial_t^2 U^{\mathrm{po}} - (\lambda + 2\mu)\Delta U^{\mathrm{po}} = 0\,, \qquad U^{\mathrm{po}}(0) = U_0^{\mathrm{po}}\,, \quad U_t^{\mathrm{po}}(0) = U_1^{\mathrm{po}}$$

and

$$\partial_t^2 U^{\mathrm{so}} - \mu\Delta U^{\mathrm{so}} = 0\,, \qquad U^{\mathrm{so}}(0) = U_0^{\mathrm{so}}\,, \quad U_t^{\mathrm{so}}(0) = U_1^{\mathrm{so}}\,.$$

These problems are hyperbolic iff $\lambda + 2\mu, \mu > 0$, which is an even more general condition on the Lamé moduli than the one of (8.3). This decomposition still holds in exterior domains, but here the potential and the solenoidal part are still coupled by the boundary condition, i.e. U^{po} and U^{so} are not solutions of (El.D) or (El.N) respectively any more. This coupling will lead to a mutual influence of the singular behaviour of each part at the boundary, as we will see in Section 4.

8.3 Unique Solvability

In this section we state the existence and uniqueness of *solutions with finite energy* and the local energy decay of such solutions. For more details cf. [Sh/Sog 89] and [Le 86].

The best possible *energy space* for the Dirichlet problem is given by

$$H := \left\{ (U,V) \;\middle|\; U \in \mathcal{H}_\nabla(\Omega)\,, V \in \mathcal{L}^2(\Omega) \right\}$$

(for the Neumann problem U has to be in $(\mathring{\mathcal{C}}_\infty(\overline{\Omega}))^{\sim|\cdot|_1}$). Transferring the ansatz of [La/Ph 67, La/Ph 72] to the case of a system and using essentially Stone's theorem and the scattering theory of Lax and Phillips, Shibata and Soga proved in [Sh/Sog 89] the following theorem:

Theorem 8.3.1 *Suppose that* $(U_0, U_1) \in H$. *Then (El.D) has a unique solution* U *with*

1. $(U, U_t) \in \mathcal{C}(\mathbb{R}, H)\,,$
2. $E\,(U(t), \Omega) = const.$

If furthermore the medium is homogeneous, then

3. $\forall K \subset\subset \overline{\Omega}: \quad E\,(U(t), K) \longrightarrow 0\,, \quad (t \longrightarrow \infty)$

holds.

Remarks:

1. An analogous theorem holds for (El.N).
2. Shibata and Soga assume $n \geqslant 3$. However, this is not necessary for the above results, cf. [Dan 96].
3. Shibata and Soga assume (8.2) throughout their work (cf. the Stability Assumption (A.2), p. 862, in [Sh/Sog 89], which requires (8.2) automatically). What they really need is Korn's inequality of (8.4) and for this purpose it is sufficient to make our Assumption 8.2.1 (cf. the proof of Theorem 1.5, [Sh/Sog 89]).
4. At first sight the energy spaces of Shibata and Soga may seem to be different from ours, because they use the completion with respect to $|\cdot|_1$ of

$$\left\{ U \in \mathring{\mathcal{C}}_\infty(\overline{\Omega}) \;\middle|\; \Lambda U = 0 \text{ on } \partial\Omega \right\}$$

as the first component of H. This leads in fact to the same result, for

$$\overset{\circ}{\mathcal{C}}_\infty(\Omega) \quad \text{is a dense subspace of} \quad \left\{U \in \overset{\circ}{\mathcal{C}}_\infty(\overline{\Omega}) \;\middle|\; u = 0 \text{ on } \partial\Omega\right\}$$

and

$$\left\{U \in \overset{\circ}{\mathcal{C}}_\infty(\overline{\Omega}) \;\middle|\; \Big(\sum_{i=1}^{n} \mathbf{n}_i \sigma_{ij}\Big)_j = 0 \text{ on } \partial\Omega\right\} \quad \text{is a dense subspace of} \quad \overset{\circ}{\mathcal{C}}_\infty(\overline{\Omega})$$

with respect to $|\cdot|_1$.

A similar result can be found in [Le 86, Chapter 11]. Here again Leis assumes that $U_0 \in \overset{\circ}{\mathcal{H}}_1(\Omega)$ instead of $U_0 \in \mathcal{H}_\nabla(\Omega)$ ($U_0 \in \mathcal{H}_1(\Omega)$ in case of (El.N)) with the consequence that $U(t)$ remains in this space for all times. This is his notion of a *solution with finite energy*. As for the wave equation Leis uses the spectral theorem for self-adjoint operators to prove his analogue to the above results. It should be noted again that this restriction of the energy space is of no importance for initial data with compact support in $\overline{\Omega}$, because

$$U_0 \in \mathcal{H}_\nabla(\Omega) \;\wedge\; \operatorname{supp} U_0 \subset\subset \overline{\Omega} \qquad \Longrightarrow \qquad U_0 \in \overset{\circ}{\mathcal{H}}_1(\Omega)$$

(a corresponding result holds for the Neumann problem). As in the case of the wave equation we mean such solutions in suitable energy spaces, if we talk about solutions of (El.D) and of (El.N) without further specification.

Chapter 9

Uniform Local Energy Decay

We now present various results on uniform local energy decay. The first section deals with the energy method. Then we introduce a few results of Kawashita guaranteeing optimal decay rates and therefore ruling out the Neumann problem from most considerations of this chapter. In the third section we study the propagation of singularities and formulate an analogue to Melrose's NTC. In Section 9.4 we describe physical phenomena that are closely related to the results on the propagation of singularities. In Part I we saw how useful escape functions are for relating geometric properties of the obstacle to non-trapping conditions. This was the reason for us to look for a suitable definition of escape functions for isotropic elasticity. We present the result in Section 9.5.1. In fact the existence of such an escape function implies the above analogue to Melrose's NTC (Theorem 9.5.3). And as in the case of the wave equation the nice geometric conditions of Chapter 3 yield for suitable media the existence of an escape function (Section 9.5.3). After having established these results, we define the Green's matrix of the equations of elasticity, prove some existence and uniqueness results for distributional solutions, and state two different non-trapping conditions for the Vainberg method. The last section of this chapter contains the decay results obtained by this method.

9.1 The Energy Method

There are only few publications concerning decay results for the equations of elasticity achieved by the energy method, namely those of Dassios ([Das 83]) and Kapitonov ([Kap 86, Kap 87]) for (El.D). They deal with homogeneous isotropic media with $\lambda, \mu > 0$ (Dassios makes a more implicit use of this last assumption). For their divergence identities it is important that some terms have the right sign which can be achieved at least by the more general assumption $\lambda+\mu, \mu > 0$. The two authors proved the existence of the following decay rates for star-shaped obstacles:

Dassios:	$n = 3$,	$f(t) = c\,t^{-1}$.
Kapitonov:	n arbitrary,	$f(t) = c\,t^{-1}$.

Remarks:

1. Both authors essentially used the multiplier $\frac{n-1}{2}U + rU_r + tU_t$, which is closely related to the one Morawetz used in [Mor 61] for the wave equation. But because of the two waves of isotropic elasticity which propagate with different phase speeds, the energy identities of [Das 83, Kap 87] are more complicated and more difficult to analyse.

2. For odd n Kapitonov used the Huygens' principle to improve the decay rate to $c\,e^{-\delta t}$ with some positve constant δ.

9.2 Improvement of Decay Rates and the Neumann Problem

After the presentation of the first decay results we want to have a closer look at two results of Kawashita for homogeneous media. On the one hand they improve the above decay rate and on the other hand they show why no uniform decay can be expected for (El.N).

In [Kaw 93] Kawashita proves—by a comparision with the Cauchy problem—the following result for (El.D) and (El.N) in case of homogeneous media:

Theorem 9.2.1 *Suppose $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$. If there is local energy decay with some rate, then this decay rate can be improved to*

$$\begin{array}{ll} c\,t^{-2n+2}\,, & n \geqslant 4 \text{ even,} \\ c\,e^{-\delta t}\,, & n \geqslant 3 \text{ odd.} \end{array}$$

This leads to a corresponding improvement of Kapitonov's decay results, if the space dimension is even.

Remarks:

1. Kawashita restricted his investigation to $n \geqslant 3$, becauses he used the results of [Sh/Sog 89]. As we have already remarked, this restriction is not necessary. On the other hand Vainberg's method will lead to an optimal rate of decay for $n = 2$ in Section 9.7.

2. In [Kaw 93] there is again the assumption (8.2) on the Lamé moduli, because the author refers to [Sh/Sog 89] and uses therefore the stability condition. As remarked above, assumption (8.3) is sufficient for the results of Shibata and Soga.

The Neumann problem allows so-called *Rayleigh waves*, which are surface waves on the boundary of the obstacle. Even here local energy decay holds, as we have seen in Theorem 8.3.1. But Kawashita proved that this is all that can be expected: In

[Kaw 92] he showed that for (El.N) there exists no decay "of strong type". On the other hand any decay rate implies, according to Theorem 9.2.1, such a decay "of strong type". Therefore there exists no decay rate for the Neumann problem at all. As a matter of fact this shows that non-trapping conditions do not make sense for (El.N).

9.3 The Propagation of Singularities

In Chapter 4 we studied the propagation of singularities for boundary value problems of the wave equation. The crowning result was the generalized Huygens' principle, stating that b-wavefronts of solutions of (W.D) or (W.N) propagate along generalized bicharacteristics. We now want to present the corresponding results for the equations of elasticity. What is the situation here? Now the determinant of the principal symbol has to be studied. It is given by

$$p(t,x,\tau,\xi) = (\mu|\xi|^2 - \tau^2)^{n-1}((\lambda+2\mu)|\xi|^2 - \tau^2)$$

(cf. [Ya 88], Lemma 1.1). This leads to two bicharacteristic families in $\operatorname{char} p$, namely those of

$$p_{\rho_i}(t,x,\tau,\xi) = \rho_i|\xi|^2 - \tau^2\,, \qquad \rho_1 = \mu\,,\ \ \rho_2 = \lambda + 2\mu\,.$$

We make from now on an additional assumption on the Lamé moduli, because otherwise the singular behaviour of elastic waves is much more complicated even in the interior $\mathbb{R} \times \Omega$ (cf. [Ral 76, Ral 78]):

Assumption 9.3.1 *The Lamè moduli λ and μ fulfil*

$$\mu \neq \lambda + 2\mu\,. \tag{9.1}$$

Remark: This is in fact an additional assumption for (El.D) with μ being constant only, for (8.2) already implies (9.1).

Let $X := \mathbb{R} \times \overline{\Omega}$. In the interior, i.e. in $T^*(\mathring{X})$, wavefronts propagate independently along the null bicharacteristics of each of the two families (cf. [Tay 81], Corollary VIII.2.3, p. 154f.). At the boundary, however, one has to consider the inverse images with respect to the projection

$$T^*(X)|_{\partial X} \longrightarrow T^*(\partial X) \tag{9.2}$$

in $\operatorname{char} p_\mu \cup \operatorname{char} p_{\lambda+2\mu}$. We want to illustrate what happens at a boundary point $(t,x) \in \partial X$. Let $(\tau,\xi) \neq (0,0)$ be a tangential direction to ∂X in (t,x), i.e. $\xi \cdot \mathbf{n}(x) = 0$. Then we can regard (τ,ξ) as an element of $T^*_{(t,x)}(\partial X)$. To look for inverse images in both characteristic sets means here to look for $\alpha \in \mathbb{R}$ with

$$p_{\rho_i}(t,x,\tau,\xi + \alpha\mathbf{n}(x)) = 0\,, \quad i = 1,2\,.$$

Because of

$$p_{\rho_i}(t,x,\tau,\xi + \alpha\mathbf{n}(x)) = \rho_i\left(|\xi|^2 + \alpha^2\right) - \tau^2,$$

this requires

$$\alpha = \pm\sqrt{\frac{\tau^2}{\rho_i} - |\xi|^2}\,, \qquad i = 1 \text{ or } i = 2\,.$$

For the existence of such real α

$$r_{\rho_i} := \rho_i|\xi|^2 - \tau^2 \leqslant 0\,, \qquad i = 1 \text{ or } i = 2$$

is needed. For fixed $(t,x) \in \partial X$, the characteristic sets of p_μ and $p_{\lambda+2\mu}$ form two double cones with common origin $(\tau_0, \xi_0) = (0,0)$. From the geometrical point of view there exist five possibilities for a tangential direction $\zeta := (\tau, \xi) \neq (0,0)$ with different numbers of inverse images with respect to the projection (9.2):

0. ζ is situated in the exterior of both cones $\Longrightarrow$ no inverse image.
1. ζ is situated on the boundary of the exterior cone $\Longrightarrow$ one inverse image.
2. ζ is situated between the two cones $\Longrightarrow$ two inverse images.
3. ζ is situated on the boundary of the interior cone $\Longrightarrow$ three inverse images.
4. ζ is situated within both cones $\Longrightarrow$ four inverse images.

Suppose without restriction $\lambda + \mu > 0$, that is $\lambda + 2\mu > \mu$ (otherwise one has to interchange μ and $\lambda + 2\mu$ in the next statements). Then these five cases correspond to

0. $0 < r_\mu \leqslant r_{\lambda+2\mu}\,,$ elliptic set.
1. $0 = r_\mu < r_{\lambda+2\mu}\,,$ tangential bicharacteristics of p_μ.
2. $r_\mu < 0 < r_{\lambda+2\mu}\,,$ one incoming and one outgoing non-tangential bicharacteristic of p_μ.
3. $r_\mu < r_{\lambda+2\mu} = 0\,,$ one incoming and one outgoing non-tangential bicharacteristic of p_μ, tangential bicharacteristics of $p_{\lambda+2\mu}$.
4. $r_\mu < r_{\lambda+2\mu} < 0\,,$ one incoming and one outgoing non-tangential bicharacteristic of p_μ as well as of $p_{\lambda+2\mu}$.

If we look for example at case 4., we get with

$$\alpha_i := \sqrt{\frac{\tau^2}{\rho_i} - |\xi|^2}\,, \qquad i = 1, 2,$$

in the special situation of homogeneous media the following four incoming and outgoing bicharacteristics:

$$(t - 2\tau s, x + 2\rho_i(\xi + \alpha_i \mathbf{n}(x))s, \tau, \xi + \alpha_i \mathbf{n}(x))\;, \quad s < 0\,,\;\; i = 1,2\,,$$

$$(t - 2\tau s, x + 2\rho_i(\xi - \alpha_i \mathbf{n}(x))s, \tau, \xi - \alpha_i \mathbf{n}(x))\;, \quad s > 0\,,\;\; i = 1,2\,.$$

Yamamoto was able to establish a generalized Huygens principle for isotropic elasticity in [Ya 88, Ya 89]. His results were formulated for homogeneous media, but his proofs actually apply to the general case also. This is due to the fact that he used local coordinates, since the corresponding transformation of the coordinates leads actually to a general isotropic medium. A distributional solution of (El.Λ) is again a distribution that satisfies all the identities of (El.Λ) in the distributional sense, and $(t, x, \tau, \xi) \in T^*(\mathbb{R} \times \partial\Omega)$ means that $x \in \partial\Omega$ and $\xi \cdot \mathbf{n}(x) = 0$. For the displacement problem there is no propagation of singularities in the elliptic set. Yamamoto proved the following result:

Theorem 9.3.2 (Generalized Huygens' Principle)
Suppose $U \in \overline{\mathcal{D}}'(\mathbb{R}\times\Omega)$ *is a distributional solution of* (El.D). *Then the following hold:*

1. *In* $T^*(\mathbb{R} \times \Omega)$, $WF_b(U)$ *is propagated along the bicharacteristics of* p_μ *and* $p_{\lambda+2\mu}$.
2. *Let* $(t, x, \tau, \xi) \in T^*(\mathbb{R} \times \partial\Omega) \setminus 0$. *If* $WF_b(U)$ *has an empty intersection with all incoming generalized bicharacteristics in* (t, x, τ, ξ), *so it has with all outgoing generalized bicharacteristics.*

Remarks:

1. The second statement includes the possibility that wavefronts ramify. So a solution may be singular along the incoming generalized bicharacteristic of one variety only, but along all outgoing generalized bicharacteristics and vice versa. That this happens indeed was proven by Yamamoto in [Ya 90]. Another possibility is even a change of the variety at the boundary.

2. In [Ya 89], Yamamoto assumes that $n = 3$. For his results, he does not need this restriction at all, because he always refers to the proofs of [Ya 88], where $n \geqslant 2$ as in our considerations.

3. For the traction problem the same results hold with one important difference (cf. [Ya 88]): The elliptic set is now involved, too. The so-called Rayleigh waves propagate in this set at a speed slower than $\mu^{1/2}$. In case that

$$\operatorname{singsupp}(U(0), U_t(0)) \cap (\mathbb{R} \times \partial\Omega) \neq \emptyset,$$

these Rayleigh waves occur. As remarked above, because of these surface waves, there is no uniform LED for (El.N).

The above considerations suggest the definition of special curves which reflect the existence of more than two inverse images in $p^{-1}(0)$ with respect to the projection (9.2). We therefore denote by a *generalized bicharacteristic path* a curve which consists of generalized bicharacteristics with the possibility of changing the bicharacteristic variety at each point of $T^*(\partial X)$ in the way indicated above. The projection of such a generalized bicharacteristic path to $\overline{\Omega}$ will be called a (*generalized*) *geodesic path*. Theorem 9.3.2 and the Gaussian beam construction of Ralston motivate now the definition of a Melrose-type non-trapping condition for (El.D):

Definition 9.3.3 (Melrose's NTC for (El.D))
Ω *is called* non-trapping
$:\Longleftrightarrow$ *for some* $R_1 > R$ *there exists a* T_{R_1} *such that no geodesic path of length* T_{R_1} *lies completely within* $\overline{\Omega}_{R_1}$.

Remark: This is not the non-trapping condition of Yamamoto in [Ya 89]. He used Melrose's NTC for the wave equation. His results on the propagation of singularities, however, show that one needs control of the length of *geodesic paths* near the obstacle if, for example, the eventual smoothness of a weak solution in $\overline{\Omega}_R$ should be implied. It is not clear from [Ya 89], how to derive an upper bound of the length of geodesic paths in $\overline{\Omega}_R$ from Melrose's original NTC. That is why the present author uses Definition 9.3.3, which seems to suit better to the quoted results of Ralston and Yamamoto.

9.4 Ramification of Singularities and Rayleigh Waves

In this short section we want to present some further details on the results of the last section and point out the relation to phenomena that occur in the applications (e.g. earthquakes).

The generalized Huygens' principle of Yamamoto allows the ramification of singularities to both bicharacteristic varieties at the boundary. We want to mention some results which show that this happens in fact. In Sommerfeld's book [Som 64], p. 301–304, for the traction problem of homogeneous media in a half-space in $\mathbb{R}^3$ the following reflection phenomena are described:

(a) one incident longitudinal wave
$\Longrightarrow$ one reflected longitudinal wave and one reflected transversal wave, polarized in the direction of the incident plane,

(b1) one incident transversal wave, polarized in the direction of the incident plane
$\Longrightarrow$ one reflected transversal wave with the same polarization and one reflected longitudinal wave,

(b2) one incident transversal wave, polarized orthogonal to the incident plane
$\Longrightarrow$ one reflected transversal wave with the same polarization and *no* reflected longitudinal wave.

Transversal waves and longitudinal waves propagate with velocities $\mu^{1/2}$ and $(\lambda+2\mu)^{1/2}$ respectively. These are exactly the shear and pressure waves of Section 8.2.

Motivated among other things by a seismogram, these three reflection phenomena of the free boundary value problem served as a starting point in [Ya 90]. Yamamoto studied this behaviour for $\lambda, \mu > 0$ in terms of the propagation of singularities. Transferring (a), (b1), and (b2) to generalized bicharacteristics with the same projection to $T^*(\mathbb{R} \times \partial\Omega)$, Yamamoto proved (a) for bicharacteristics of $p_{\lambda+2\mu}$ which make nearly tangential or nearly orthogonal incidence at $\partial\Omega$. Furthermore he showed that singularities must propagate according to (b1) and (b2) under some assumptions on λ, μ and the angle of incidence. All these results hold independently of this angle, if $\lambda \geqslant 4\mu$. It should be remarked that this last assumption excludes the possibility that singularities change the bicharacteristic variety at the boundary.

Finally we want to stress the importance of Rayleigh waves. In seismology the shear and pressure waves propagating with the above speeds can really be observed. But in case of a discontinuous impulse on the surface of the earth, caused for example by an earthquake, another wave appears, travelling along the boundary at a third, slower speed. This is a Rayleigh wave. Shear and pressure waves dissipate rapidly and decay inversely as the square of the distance from the epicentre. Rayleigh waves, however, travel slower and decay, as surface waves, only like the inverse of the distance. These waves were first analysed by Lord Rayleigh in [Ray 1885] for a planar boundary. He has already assumed that they could be of great significance in seismology (p. 11, ibid.):

> *It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids. Diverging in two dimensions only, they must acquire at a great distance from the source a continually increasing preponderance.*

While Rayleigh and others studied flat boundaries only, a rigorous treatment of the related singular behaviour for arbitrary smooth boundaries was first given by Taylor in [Tay 79].

9.5 Escape Functions

In this section we introduce escape functions for isotropic elasticity. This is an adaption of the definition of Morawetz, Ralston, and Strauss for the special situation of two bicharacteristic families that occurs here. The reason is not that we want to apply the energy method similar to [M/R/S 77] for a proof of uniform LED. What we want to make use of is the special link between some nice geometric conditions on the obstacle and the NTC of Melrose for (El.D), given by an escape function, as we have already seen with the wave equation. So after some work that may seem a little bit "academic", we prove that the existence of an escape function implies Melrose's NTC for (El.D). This finally enables us to conclude that under a few restrictions on the medium star-shaped obstacles, obstacles that can be illuminated from the interior or from the exterior, and snake-shaped obstacles are in fact non-trapping in the sense of Melrose.

9.5.1 Definition

In Section 9.3 we saw how to calculate the incoming and outgoing bicharacteristic directions of a point in $T^*(\partial X)$. For the definition of an escape function it is more useful to calculate the outgoing directions from an incoming one: Let there be an incoming bicharacteristic in $(t,x,\tau,\xi) \in T^*(X)|_{\partial X}$, i.e. $x \in \partial\Omega$ and $\xi \cdot \mathbf{n} \geqslant 0$, then

$$(t,x,\tau,\xi - (\xi \cdot \mathbf{n})\mathbf{n}) \in T^*(\partial X)\,.$$

Looking for the inverse images under the projection (9.2) means here to calculate the solutions α of

$$\rho_i|(\xi - (\xi \cdot \mathbf{n})\mathbf{n}) + \alpha\mathbf{n}|^2 - \tau^2 = 0\,, \qquad i = 1,2,$$

i.e. real α with

$$\alpha = \pm\sqrt{\frac{\tau^2}{\rho_i} - |\xi - (\xi \cdot \mathbf{n})\mathbf{n}|^2}\,,$$

one solution, namely $\alpha = \xi \cdot \mathbf{n}$, being already known from the starting point of our considerations. Here one should note that the related outgoing "directions" ξ_{out} to a given incoming ξ_{in} have the property that

$$\xi_{\text{out}} = \xi_{\text{in}} + \widetilde{\alpha}\mathbf{n} \qquad \text{with} \qquad \widetilde{\alpha} \leqslant 0\,.$$

We finally get the following result: If there is an incoming bicharacteristic of $p_{\rho_{\text{in}}}$ in $(t,x,\tau,\xi_{\text{in}})$, one calculates the starting point $(t,x,\tau,\xi_{\text{out}})$ of a related outgoing bicharacteristic of $p_{\rho_{\text{out}}}$ by

$$\xi_{\text{out}} = \xi_{\text{in}} - (\xi_{\text{in}} \cdot \mathbf{n})\mathbf{n} - \sqrt{\tau^2\left(\frac{1}{\rho_{\text{out}}} - \frac{1}{\rho_{\text{in}}}\right) + (\xi_{\text{in}} \cdot \mathbf{n})^2}\;\mathbf{n}\,,$$

where at least for $\rho_{\text{out}} = \rho_{\text{in}}$ a solution exists.

Before we define an escape function for isotropic elasticity, a few motivating remarks should be made: First there exist constants C_1 and C_2 such that

$$0 < C_1 \leqslant \lambda + 2\mu, \mu \leqslant C_2 \quad \text{in } \overline{\Omega}\,.$$

Secondly τ is constant along generalized bicharacteristic paths. Because the b-wavefront set of a distribution is conical, it therefore suffices to look at generalized bicharacteristic paths with $|\tau| = 1$. With this restriction $|\xi| = 1/\rho_i^{1/2}$ on $p_{\rho_i}^{-1}(0)$ holds, $i = 1,2$. Thus our escape functions will be defined on

$$K_R := \left\{(x,\xi) \;\middle|\; x \in \overline{\Omega}_R\,, \quad |\xi| = \frac{1}{\mu^{1/2}(x)} \;\vee\; |\xi| = \frac{1}{(\lambda + 2\mu)^{1/2}(x)}\right\}.$$

This means in particular that K_R is compact. We divide K_R into two compact subsets,

$$K_{\rho_i} := \left\{(x,\xi) \;\middle|\; x \in \overline{\Omega}_R\,,\, |\xi| = \frac{1}{\rho_i^{1/2}(x)}\right\}, \quad i = 1,2\,.$$

The intention of introducing escape functions is: They shall be strictly increasing along generalized bicharacteristic paths. So they shall behave like the original escape functions of [M/R/S 77] with the addition that the possibility of changing the bicharacteristic family at the boundary is taken into account. Thus we define

Definition 9.5.1 *A function* $\wp(x,\xi) \in \mathcal{C}_\infty(K_R)$ *is called* escape function (El) $:\Longleftrightarrow$

$$(i) \quad 2\rho_i(x)\xi \cdot \frac{\partial \wp}{\partial x}(x,\xi) - |\xi|^2 \nabla \rho_i(x) \cdot \frac{\partial \wp}{\partial \xi}(x,\xi) > 0 \qquad in \quad K_{\rho_i}\,, \qquad i = 1,2\,,$$

$$(ii) \quad \left.\begin{array}{ll} \wp(x,\xi_{\mathrm{out}}) - \wp(x,\xi_{\mathrm{in}}) > 0\,, & for \;\; \xi_{\mathrm{out}} \neq \xi_{\mathrm{in}} \\ \mathbf{n}_e(x) \cdot \dfrac{\partial \wp}{\partial \xi}(x,\xi) > 0\,, & for \;\; \xi \cdot \mathbf{n}_e(x) = 0 \end{array}\right\} \; on \; \Big\{(x,\xi) \in K_R \;\Big|\; x \in \partial\Omega\Big\}\,.$$

Remark: Here we used the above definitions of $\xi_{\mathrm{in}}, \xi_{\mathrm{out}}$. For the sake of completeness we repeat them using now $\mathbf{n}_e$ instead of $\mathbf{n}$: For ξ_{in}, $\xi_{\mathrm{in}} \cdot \mathbf{n}_e \leqslant 0$ holds and

$$\xi_{\mathrm{out}} = \xi_{\mathrm{in}} - (\xi_{\mathrm{in}} \cdot \mathbf{n}_e)\mathbf{n}_e + \sqrt{\tau^2 \left(\frac{1}{\rho_{\mathrm{out}}} - \frac{1}{\rho_{\mathrm{in}}}\right) + (\xi_{\mathrm{in}} \cdot \mathbf{n}_e)^2}\; \mathbf{n}_e\,.$$

9.5.2 Escape Functions and Melrose's NTC

Before we prove that the existence of an escape function (El) implies Melrose's NTC for (El.D), we need to have a closer look at the regularity of generalized bicharacteristic paths. Each of the two families comes from the principal symbol of a wave equation. There differentiability of generalized bicharacteristics holds with the exception of the points of reflection. We have already seen that these points are isolated and that right and left limits and derivatives exist at a reflection point. Thus there remain those points at which the bicharacteristic family is changed. According to [Ya 88, Ya 89] to each such $\gamma(s_0) \in T^*(X)|_{\partial X}$ there exists a neighbourhood $(s_0 - \varepsilon, s_0 + \varepsilon)$ of s_0 in which a change of the family is possible only at s_0. Therefore the set of those *changing points* is a discrete subset of the interval I on which γ is parametrized. In a changing point right and left limits and derivatives of γ exist, too. The only difference with respect to regularity to a generalized bicharacteristic of the wave equation is now that the set B of all singular points, i.e. the union of reflection points and changing points of γ, is no longer a discrete set, because reflection points could accumulate at a point of G^∞, where a change of the family may occur.

Now let $\gamma(s) := (t(s), x(s), \tau(s), \xi(s))$ be a generalized bicharacteristic path. Then $x(s)$ is continuously extendable at every singular point s_0 of γ, because there right and left limits of $x(s)$ coincide. That means: Whereas

$$\gamma : I \setminus B \longrightarrow T^*(\mathring{X}) \cup G^2_\mu \cup G^2_{\lambda+2\mu}$$

is not defined at the countable set B, one can regard $x(s)$ as a continuous curve

$$x : I \longrightarrow \overline{\Omega}\,.$$

One calculates the gliding vector field of p_{ρ_i} as in the case of the wave equation. This yields

$$H^G_{p_{\rho_i}} = H_{p_{\rho_i}} - \left(\frac{1}{2\rho_i(x)} H^2_{p_{\rho_i}} d\right) \langle \mathbf{n}_e(x), \nabla_\xi \cdot \rangle\,, \qquad i = 1,2\,.$$

As usual

$$H^2_{p_{\rho_i}} d \leqslant 0 \text{ in } G_{g_{\rho_i}} \cup G^3_{\rho_i} \ \ (= G^2_{\rho_i} \setminus G_{d_{\rho_i}}) , \qquad i = 1, 2 ,$$

holds.

Suppose now that an escape function exists. Let

$$\gamma : I \setminus B \longrightarrow T^*(\mathring{X}) \cup G^2_\mu \cup G^2_{\lambda+2\mu}$$

be a generalized bicharacteristic path with $|\tau| = 1$ and $x(s) \in \overline{\Omega}_R$ for all $x \in I$. Then $\widetilde{\wp}(s) := \wp(x(s), \xi(s))$ is differentiable in $I \setminus B$ and in B right and left limits of $\wp$ and $\wp'$ exist. In $I \setminus B$, γ is locally a generalized bicharacteristic of one family, namely a generalized bicharacteristic of p_{ρ_i}. Therefore in $I \setminus B$ the following holds

- for $x(s) \in \Omega$ or $\gamma(s) \in G_{d_{\rho_i}}$:

$$\begin{aligned} \frac{d\widetilde{\wp}}{ds}(s) &= \left(H_{p_{\rho_i}} \wp\right)(s) \\ &= \left(2\rho_i \xi \frac{\partial \wp}{\partial x} - |\xi|^2 \nabla \rho_i(x) \cdot \frac{\partial \wp}{\partial \xi}\right)(s) , \end{aligned}$$

- for $\gamma(s) \in G^2_{\rho_i} \setminus G_{d_{\rho_i}}$:

$$\begin{aligned} \frac{d\widetilde{\wp}}{ds}(s) &= \left(H^G_{p_{\rho_i}} \wp\right)(s) \\ &= \left(2\rho_i \xi \frac{\partial \wp}{\partial x} - |\xi|^2 \nabla \rho_i(x) \cdot \frac{\partial \wp}{\partial \xi}\right)(s) - \left(\left(\frac{1}{2\rho_i(x)} H^2_{p_{\rho_i}} d\right) \mathbf{n}_e(x) \cdot \frac{\partial \wp}{\partial \xi}\right)(s) . \end{aligned}$$

Because K_R is compact and $\wp$ is smooth, in Definition 9.5.1.(i) actually

$$2\rho_i \xi \cdot \frac{\partial \wp}{\partial x} - |\xi|^2 \nabla \rho_i \cdot \frac{\partial \wp}{\partial \xi} \geqslant c > 0$$

holds. In addition

$$\mathbf{n}_e(x) \cdot \frac{\partial \wp}{\partial \xi} > 0 \quad \text{ and } \quad H^2_{p_{\rho_i}} d \leqslant 0 \quad \text{ in } G^2_{\rho_i} \setminus G_{d_{\rho_i}} .$$

This yields

$$\frac{d\widetilde{\wp}}{ds}(s) \geqslant c > 0 \quad \text{ in } I \setminus B .$$

In B, on the other hand, an escape function is just so defined that

$$\begin{aligned} \widetilde{\wp}(s+0) - \widetilde{\wp}(s-0) &= \wp(x(s), \xi(s+0)) - \wp(x(s), \xi(s-0)) \\ &= \wp(x(s), \xi_{\text{out}}) - \wp(x(s), \xi_{\text{in}}) \\ &> 0 . \end{aligned}$$

Thus all assumptions of the following lemma have been verified:

Lemma 9.5.2 *Let $f \in \mathcal{C}_1([a,b] \setminus J)$ be real valued, where J is almost countable. Suppose the following holds:*

(i) $\forall\, t \in [a,b] \setminus J :\ f'(t) \geqslant c > 0\,,$

(ii) for all $t \in J$ there exist $f(t \pm 0), f'(t \pm 0)$ and $f(t+0) > f(t-0)$.

Then

$$f(b-0) - f(a+0) \geqslant c(b-a)\,.$$

The lemma is proved in essentially the same way as Lemma 7.11 of [Pau 96].

Since $\wp$ is bounded by some constant M on K_R, Lemma 9.5.2 implies that the length of an interval on which a generalized bicharacteristic path over K_R is parametrized is at most $2M/c$. Because of

$$|x'(s)| = |2\rho_i(x)\xi(s)| = 2|\rho_i(x)|\frac{1}{|\rho_i(x)|^{1/2}} = 2|\rho_i(x)|^{1/2} \leqslant 2C_2^{1/2} =: C_3\,,$$

for the length l of a geodesic path over K_R it follows that

$$l = \int_a^b |x'(s)|\, ds \leqslant C_3(b-a) \leqslant C_3 \frac{2M}{c} =: T_R < \infty\,.$$

This proves the desired implication:

Theorem 9.5.3

$$\exists \text{ escape function (El)} \implies \text{Melrose's NTC for (El.D) holds.}$$

9.5.3 Applications

We now want to point out a few situations in which escape functions for isotropic elasticity exist. As always in case of inhomogeneous media this leads to conditions on *both* the obstacle *and* the medium.

Suppose a Straussian vector field l exists, i.e. a smooth vector field l with a $c > 0$, such that

$$\begin{aligned} &(i) \quad l(x) \cdot \mathbf{n}_e(x) > 0\,, && x \in \partial\Omega\,, \\ &(ii) \quad \partial_i l_j(x)\xi_i\xi_j \geqslant c\,|\xi|^2\,, && x \in \overline{\Omega},\ \ \xi \in \mathbb{R}^n\,. \end{aligned}$$

When is $\wp(x,\xi) := l(x) \cdot \xi$ an escape function (El)? Because of

$$\frac{\partial \wp}{\partial \xi} = l(x) \quad \text{and} \quad \xi_{\text{out}} = \xi_{\text{in}} + \alpha \mathbf{n}_e \text{ with } \alpha > 0$$

point (ii) of Definition 9.5.1 obviously holds. Because of

$$\xi \cdot \frac{\partial \wp}{\partial x} = \partial_i l_j(x)\xi_i\xi_j$$

point (i) of Definition 9.5.1 means

$$2\rho_i(x)\partial_i l_j(x)\xi_i\xi_j - |\xi|^2\nabla\rho_i(x)\cdot l(x) > 0 \text{ in } K_{\rho_i}\,, \quad i=1,2\,. \tag{9.3}$$

For homogeneous media this inequality reduces to

$$2\rho_i(x)\partial_i l_j(x)\xi_i\xi_j > 0\,,$$

a condition which is fulfilled in K_{ρ_i}, $i = 1, 2$, by every Straussian vector field l. A direct application of Theorem 9.5.3, therefore, yields:

Theorem 9.5.4 *Assume that a Straussian vector field l exists. Then the following holds:*

1. *If the medium is homogeneous, then Melrose's NTC for* (El.D) *is fulfilled. This especially means that star-shaped obstacles ($n \geqslant 2$), obstacles that can be illuminated from the interior ($n = 2, 3$), obstacles that can be illuminated from the exterior ($n = 2, 3$), and snake-shaped obstacles ($n = 2$) are non-trapping.*
2. *If the medium is inhomogeneous and in addition condition* (9.3) *holds, then Melrose's NTC for* (El.D) *is fulfilled.*

As remarked earlier, strictness in (i) or (ii) is sufficient for a Straussian vector field. Therefore star-shapedness instead of strict star-shapedness is strong enough a condition on the obstacle.

We already know that for star-shaped obstacles $l(x) := x$ is a Straussian vector field. Because $\partial_i l_j(x) = \delta_{ij}$, (9.3) becomes here

$$|\xi|^2\left(2\rho_i(x) - r\partial_r\rho_i(x)\right) > 0 \text{ in } K_{\rho_i}\,, \quad i=1,2\,. \tag{9.4}$$

This corresponds exactly to the condition (3.9) of Bloom in the case of the wave equation for inhomogeneous media. And after the remarks on the existence of two different waves for the displacement problem of isotropic elasticity this should be actually no surprise: (9.4) means that the medium shall fulfil Bloom's condition with respect to both the shear and the pressure waves.

The preceding theorem shows no real difference to the wave equation. In the situations studied there, the existence of a usual escape function already implies the existence of an escape function (El). So what about general escape functions, as constructed in Section 5 of [M/R/S 77] for non-degenerated non-trapping obstacles in $\mathbb{R}^3$? The problem is that Definition 9.5.1 requires some additional structure of $\wp$ with respect to the variable ξ in order to take into account the possible changes of the bicharacteristic family. This structure is respected especially by those usual escape functions that are linear with respect to ξ. But this does not hold in the general case. What remains an open question is, whether the existence of a usual escape function gives rise to the construction of an escape function (El)—a problem which seems to be closely related to the question discussed above, whether the original NTC of Melrose is sufficient for our NTC of Melrose for (El.D).

9.6 Green's Matrix and the Non-Trapping Conditions of Vainberg

Similar to Section 6.1 we want to introduce the Green's matrix of the Dirichlet problem in isotropic elasticity and combine this with an existence and uniqueness result for distributional solutions. The situation here is again more difficult than with the wave equation because of the system of equations. Since the Schwartz kernel theorem is proven in [Hö I] for a "scalar" problem we have to introduce a certain formalism and to reduce the solution operator to a family of scalar operators first. Their distribution kernels will build the Green's matrix. This finally leads as in Section 6.1 to Vainberg's NTC.

Let $P\varphi$ denote the unique solution with finite energy for $\varphi \in \mathring{\mathcal{C}}_\infty(\Omega)$ of

$$\left.\begin{cases} \partial_t^2 U - \operatorname{div}\big(\mu(\nabla U + (\nabla U)^T) + (\lambda \operatorname{div} U) I_n\big) = 0 & \text{in } \mathbb{R}\times\Omega\,, \\ U = 0 & \text{on } \mathbb{R}\times\partial\Omega\,, \\ U(0) = 0\,, \qquad U_t(0) = \varphi & \text{in } \Omega\,. \end{cases}\right\} \tag{9.5}$$

Let $X_1 := \mathbb{R}\times\Omega$ and $X_2 := \Omega$. We want to apply the Schwartz kernel theorem to prove the existence of a distribution kernel of the operator P. The situation here is more complicated than in the case of wave equations, because now φ and $P\varphi$ are vector-valued and their components are coupled by the above system of differential equations. So we have to formalize the situation a little bit in order to apply Theorem 6.1.1. At the end we will be able to write again

$$(\mathcal{K}\varphi)(\psi) = K(\psi\otimes\varphi)\,, \qquad \psi \in \mathring{\mathcal{C}}_\infty(X_1)\,,\ \ \varphi \in \mathring{\mathcal{C}}_\infty(X_2) \tag{9.6}$$

with $\mathcal{K}\varphi := P\varphi$ and $K := G$, but now with vector-valued functions ψ and φ and with G being the Green's *matrix* of the problem. To avoid some confusion, we will—up to the beginning of Theorem 9.6.1—formally distinguish the dual of $(\mathring{\mathcal{C}}_\infty(Y))^n$ by the notation $(\mathcal{D}'(Y))^n$ from the dual of $\mathring{\mathcal{C}}_\infty(Y)$. (9.6) means that K is a $(n\times n)$ matrix whose components are distributions,

$$\psi\otimes\varphi := \begin{pmatrix} \psi_1\otimes\varphi_1 & \cdots & \psi_1\otimes\varphi_n \\ \vdots & & \vdots \\ \psi_n\otimes\varphi_1 & \cdots & \psi_n\otimes\varphi_n \end{pmatrix} \qquad \text{for } \psi \in (\mathring{\mathcal{C}}_\infty(X_1))^n \text{ and } \varphi \in (\mathring{\mathcal{C}}_\infty(X_2))^n\,,$$

and K and $\mathcal{K}\varphi$ have to be applied to $\psi\otimes\varphi$ and ψ respectively in the following way:

$$(\mathcal{K}\varphi)(\psi) := \sum_{i=1}^n (\mathcal{K}\varphi)_i(\psi_i) \tag{9.7}$$

and

$$K(\psi\otimes\varphi) := \sum_{i,j=1}^n K_{ij}(\psi\otimes\varphi)_{ij} = \sum_{i,j=1}^n K_{ij}(\psi_i\otimes\varphi_j)\,. \tag{9.8}$$

With this in mind we see that $P\varphi$ obviously generates a distribution in $(\mathcal{D}'(\mathbb{R}\times\Omega))^n$ which is, according to [Hö I], Theorem 4.4.8 (and the remark at the beginning of its proof there), even an element of $\mathcal{C}_\infty(\mathbb{R},(\mathcal{D}'(\Omega))^n)$ and solves (9.5) in the distributional sense (i.e. all four identities of (9.5) are valid in the distributional sense).

We now regard the solution operator P as a map

$$P:(\mathring{\mathcal{C}}_\infty(\Omega))^n \longrightarrow (\mathcal{D}'(\mathbb{R}\times\Omega))^n$$

and study

$$(P\varphi)(\psi)=\sum_{i=1}^n (P\varphi)_i(\psi_i)\,.$$

Define $\widetilde{\varphi}_j := \varphi_j e_j$, e_j being the j-th unit vector of $\mathbb{R}^n$. Because of

$$P\varphi=\sum_{j=1}^n P\widetilde{\varphi}_j$$

we get

$$\begin{aligned}(P\varphi)(\psi) &= \sum_{i=1}^n (P\varphi)_i(\psi_i)\\ &= \sum_{i,j=1}^n (P\widetilde{\varphi}_j)_i(\psi_i)\\ &=: \sum_{i,j=1}^n (P_{ij}\varphi_j)(\psi_i)\,.\end{aligned}$$

This defines linear mappings

$$P_{ij}:\mathring{\mathcal{C}}_\infty(\Omega)\longrightarrow \mathcal{D}'(\mathbb{R}\times\Omega)\,,\qquad i,j=1,\ldots,n\,.$$

The P_{ij} fulfil all the conditions of the Schwartz kernel theorem: Let $\chi\in\mathring{\mathcal{C}}_\infty(\mathbb{R}\times\Omega)$. Then there exists a $T>0$ such that $\operatorname{supp}\chi\subset(-T,T)\times\Omega$. With

$$\overline{\rho}:=\sup_{x\in\overline{\Omega}}\max(\lambda(x)+2\mu,\mu)$$

we have

$$\begin{aligned}&\phi_m\longrightarrow 0 \quad\text{in}\quad \mathring{\mathcal{C}}_\infty(\Omega)\quad (m\longrightarrow\infty)\\ \Longrightarrow\ &\phi_m\longrightarrow 0 \quad\text{in}\quad \mathcal{L}^2(\Omega)\quad (m\longrightarrow\infty)\qquad\wedge\qquad \exists a>0\,\forall m:\ \operatorname{supp}\varphi_m\subset\Omega_a\end{aligned}$$

and with $\widetilde{\varphi}_{j,m} = \phi_m e_j$ follows

$$
\begin{aligned}
|(P_{ij}\phi_m)(\chi)| &= \left| \int\limits_{(-T,T)\times\Omega} (P\widetilde{\varphi}_{j,m})_i(t,x)\chi(t,x)\,dxdt \right| \\
&\leqslant \|(P\widetilde{\varphi}_{j,m})_i\|_{(-T,T)\times\Omega_{a+\overline{\rho}T}} \|\chi\|_{(-T,T)\times\Omega} \\
&\leqslant C\left(\int\limits_{-T}^{T} \|\nabla(P\widetilde{\varphi}_{j,m})_i(t)\|^2_{\Omega_{a+\overline{\rho}T}}\,dt \right)^{1/2} \|\chi\|_{(-T,T)\times\Omega} \\
&\leqslant C\left(\int\limits_{-T}^{T} E((P\widetilde{\varphi}_{j,m})(t),\Omega)\,dt \right)^{1/2} \|\chi\|_{(-T,T)\times\Omega} \\
&= C\,(2T)^{1/2}\,E((P\widetilde{\varphi}_{j,m})(0),\Omega)^{1/2}\,\|\chi\|_{(-T,T)\times\Omega} \\
&= C\,(2T)^{1/2}\,\|\phi_m\|_\Omega\,\|\chi\|_{(-T,T)\times\Omega} \longrightarrow 0 \qquad (m\longrightarrow\infty)
\end{aligned}
$$

$$
\Longrightarrow (P_{ij}\phi_m) \longrightarrow 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}\times\Omega) \qquad (m\longrightarrow\infty)\,,
$$

where we used the finite propagation speed of solutions of (9.5) for the first estimate and Poincaré's inequality for the second. Thus the assumptions of Theorem 6.1.1 have been verified. Therefore a unique kernel $G_{ij} \in \mathcal{D}'(\mathbb{R}\times\Omega\times\Omega)$ of the operator P_{ij} exists with

$$
(P_{ij}\varphi_j)(\psi_i) = G_{ij}(\psi_i\otimes\varphi_j)\,, \qquad i,j = 1,\ldots,n\,.
$$

With $G := ((G_{ij}))_{ij}$ and the conventions (9.7) and (9.8) for the application of vectors matrices of distributions this finally proves the following theorem:

Theorem 9.6.1 *There exists a unique distribution $G \in \mathcal{D}'(\mathbb{R}\times\Omega\times\Omega)$ with*

$$
(P\varphi)(\psi) = G(\psi\otimes\varphi)\,, \qquad \forall\psi\in(\mathring{\mathcal{C}}_\infty(X_1))^n\ \forall\varphi\in(\mathring{\mathcal{C}}_\infty(X_2))^n\,.
$$

Thus G is still called the distribution kernel of P.

Definition 9.6.2 *The distribution kernel G of P is called the* Green's matrix *of the problem* (9.5).

Remark: Vainberg defines the Green's matrix as the kernel of the solution operator of the restriction of problem (9.5) to $\mathbb{R}_0^+\times\Omega$. Because of the uniqueness of the Schwartz kernel this is exactly the restriction of G to $\mathring{\mathcal{C}}_\infty(\mathbb{R}^+\times\Omega\times\Omega)$.

The differential operator of the equations of isotropic elasticity can be written as a matrix operator:

$$
L(x,D_{t,x}) := \big(\big(\delta_{ij}(\partial_t^2 - \partial_k\mu\partial_k) - \partial_j\mu\partial_i - \partial_i\lambda\partial_j\big)\big)_{ij}
$$

and thus

$$L(x,D_{t,x})U=\partial_t^2U-\operatorname{div}\left(\mu(\nabla U+(\nabla U)^T)+(\lambda\operatorname{div}U)I_n\right).$$

We again add formally to $G,\varphi,$ and ψ the variables $t,x,$ and x_0 to make it clear, how differential operators have to be applied. We have

$$\begin{aligned}\big(L(x,D_{t,x})G(t,x,x_0)\big)\big(\psi(t,x)\otimes\varphi(x_0)\big)&=G(t,x,x_0)\big(L(x,D_{t,x})(\psi(t,x)\otimes\varphi(x_0))\big)\\&=G(t,x,x_0)\big((L(x,D_{t,x})\psi(t,x))\otimes\varphi(x_0)\big)\\&=\big(P\varphi(t,x)\big)\big(L(x,D_{t,x})\psi(t,x)\big)\\&=\big(L(x,D_{t,x})(P\varphi)(t,x)\big)\big(\psi(t,x)\big)\\&=0\,,\qquad\psi\in\mathring{\mathcal{C}}_\infty(\mathbb{R}\times\Omega)\,,\ \ \varphi\in\mathring{\mathcal{C}}_\infty(\Omega)\,.\end{aligned}$$

Here L has to be applied to G and $\psi\otimes\varphi$ by the usual matrix multiplication. As in Section 6.1 it follows that

$$G\in\mathcal{C}_\infty(\mathbb{R},\mathcal{D}'(\Omega\times\Omega))\tag{9.9}$$

and that G has boundary values in $\mathcal{D}'(\mathbb{R}\times\partial\Omega\times\Omega)$. Therefore G is a solution of the following problem:

$$\left\{\begin{array}{lll}L(x,D_{t,x})G(t,x,x_0)=0 & & \text{in }\mathcal{D}'(\mathbb{R}\times\Omega\times\Omega)\,,\\ G(t,x,x_0)=0 & & \text{in }\mathcal{D}'(\mathbb{R}\times\partial\Omega\times\Omega)\,,\\ G(0,x,x_0)=0\,, & \partial_tG(0,x,x_0)=\delta(x-x_0)I_n & \text{in }\mathcal{D}'(\Omega\times\Omega)\,.\end{array}\right\}\tag{9.10}$$

That G has the above initial values can be seen by the following arguments: The Schwartz kernel theorem gives

$$(\partial_tG)(\psi\otimes\varphi)=-G(\partial_t\psi\otimes\varphi)=-(P\varphi)(\partial_t\psi)=(\partial_t(P\varphi))(\psi)$$

for all $\psi\in\mathring{\mathcal{C}}_\infty(\mathbb{R}\times\Omega)$ and $\varphi\in\mathring{\mathcal{C}}_\infty(\Omega)$. The right-hand side is continuous with respect to $t\in\mathbb{R}$ and known for $t=0$. With (9.9) we get

$$(\partial_tG(0))(\chi\otimes\varphi)=(\partial_t(P\varphi)(0))(\chi)=\int\limits_\Omega\varphi(x_0)\chi(x_0)\,dx_0\,,\qquad\chi,\varphi\in\mathring{\mathcal{C}}_\infty(\Omega)\,.$$

The remark following Theorem 6.1.1 leads to

$$\partial_tG(0,x,x_0)=\delta(x-x_0)\qquad\text{in }\mathcal{D}'(\Omega\times\Omega)\,.$$

Again G is extendable to $\mathbb{R}^n$ with respect to x.

Before we state Vainbergs non-trapping condition, a few remarks concerning distributional solutions of (9.5) and (9.10) should be added. Assigning boundary values to a distribution seems only possible for extendable distributions, i.e. elements of $\overline{\mathcal{D}}'$ (cf. Section 4.1). In the distributional sense the system (9.5) has at most one solution in the space $\overline{\mathcal{D}}'(\mathbb{R}\times\Omega)$, even if $\varphi\in\mathcal{D}'(\Omega)$. Because the author has no reference for this result, a proof follows, which makes essential use of the generalized Huygens' principle:

Let v be the difference of two solutions of (9.5) belonging to $\overline{\mathcal{D}}'(\mathbb{R}\times\Omega)$. Therefore v solves the same problem with $\varphi = 0$. Theorem 4.4.8, [Hö I], yields $v \in \mathcal{C}_\infty(\mathbb{R}, \mathcal{D}'(\Omega))$. Define $V \in \mathcal{C}(\mathbb{R}, \mathcal{D}'(\Omega))$ by

$$V(t) = \begin{cases} v(t)\,, & t \geqslant 0\,, \\ 0\,, & t < 0\,. \end{cases}$$

Clearly $V \in \overline{\mathcal{D}}'(\mathbb{R}\times\Omega)$. Because all right and left derivatives of V in $t = 0$ vanish identically, we even have $V \in \mathcal{C}_\infty(\mathbb{R}, \mathcal{D}'(\Omega))$ and all distributional time derivatives of V coincide with the distributions generated by the classical time derivatives. Thus V is a distributional solution of (9.5) with $\varphi = 0$, too. Therefore the generalized Huygens' principle, Theorem 9.3.2, can be applied yielding

$$V \in \mathcal{C}_\infty(\mathbb{R}\times\overline{\Omega})\,,$$

because

$$LV \in \mathcal{C}_\infty(\mathbb{R}\times\overline{\Omega})\,, \qquad V\Big|_{(\mathbb{R}\times\partial\Omega)} \in \mathcal{C}_\infty(\mathbb{R}\times\partial\Omega), \qquad \text{and} \qquad V \in \mathcal{C}_\infty(\mathbb{R}^-\times\overline{\Omega})\,.$$

This especially means $v \in \mathcal{C}_\infty(\mathbb{R}^+\times\overline{\Omega})$. So the same argument can be used for v proving that

$$v \in \mathcal{C}_\infty(\mathbb{R}\times\overline{\Omega})\,.$$

According to this, v is the unique smooth solution of the problem and therefore it vanishes identically for $(t,x) \in \mathbb{R}\times\overline{\Omega}$. This proves the uniqueness of a solution of (9.5) even in the space $\overline{\mathcal{D}}'(\mathbb{R}\times\Omega)$.

This uniqueness result evidently remains true, if one allows in (9.5) non-vanishing initial data $U(0)$, inhomogeneous boundary values, and any right-hand side in the first identity that is continuous in t (without this last assumption Theorem 4.4.8 of [Hö I] could not be applied and therefore it would not be possible to assign initial values in this "strong" formulation to a distributional solution).

The proof can be transferred to the restriction of (9.5) to $\mathbb{R}_0^+\times\Omega$. In this case one has to use Theorem 4.4.8' of [Hö I]. One still works with an extension of v by 0. This proves the following theorem:

Theorem 9.6.3 *Let $I = \mathbb{R}$ or $I = \mathbb{R}^+$ and assume that $U_0, U_1 \in \mathcal{D}'(\Omega)$, $g \in \mathcal{D}'(I\times\partial\Omega)$, and $f \in \mathcal{C}(\overline{I}, \mathcal{D}'(\Omega))$. Then the problem*

$$\begin{cases} \partial_t^2 U - \operatorname{div}\big(\mu(\nabla U + (\nabla U)^T) + (\lambda \operatorname{div} U) I_n\big) = f & \text{in } \mathbb{R}\times\Omega\,, \\ U = g & \text{on } \mathbb{R}\times\partial\Omega\,, \\ U(0) = U_0\,, \qquad U_t(0) = U_1 & \text{in } \Omega \end{cases}$$

has at most one solution in $\overline{\mathcal{D}}'(I\times\Omega)$.

Uniqueness holds for problem (9.10), too: A solution F shall still be extendable with respect to x. Then this is also true for $\mathcal{F}\varphi$ with $\varphi \in \mathring{\mathcal{C}}_\infty(\Omega)$ and $\mathcal{F}$ being the operator related to F by the Schwartz kernel theorem. One easily verifies that $\mathcal{F}\varphi$ is a distributional solution of (9.5). The above uniqueness result gives

$$\mathcal{F} = P$$

yielding

$$F = G$$

by the uniqueness of the distribution kernel. This finally proves the following existence and uniqueness result:

Theorem 9.6.4 *Problem* (9.10) *has for all domains* Ω *a unique solution in the space of distributions which are extendable with respect to* x*, namely the Green's matrix.*

It should be noted again that the above results are important for the next chapter's comparison of the different non-trapping conditions. But first they enable us to state Vainbergs NTC:

Definition 9.6.5 (Vainberg's non-trapping condition)
Let G *be the Green's matrix of* (9.5). Ω *is called* non-trapping
$$:\Longleftrightarrow \forall a, b > R \ \exists \tau(\Omega, a, b) > 0 : \quad G \in \mathcal{C}_\infty([\tau, \infty) \times \overline{\Omega}_b \times \overline{\Omega}_a)\,.$$

One will still get the same results with essentially the same proofs, if one uses the following non-trapping condition (cf. [Iw/Sh 88], [Di 92]):

Definition 9.6.6 (Non-trapping condition for the weak solution)
Let U *be the weak solution to* $\varphi \in \mathcal{L}^2_a(\Omega)$ *of* (9.5). Ω *is called* non-trapping
$$:\Longleftrightarrow \forall a > R \ \exists \tau(\Omega, a) > 0 \ \forall \varphi \in \mathcal{L}^2_a(\Omega) : \quad U \in \mathcal{C}_\infty([\tau, \infty) \times \overline{\Omega}_a)\,.$$

As in Section 6.1 we conclude that U is a solution with finite energy. The two non-trapping conditions will be compared in next chapter.

9.7 The Vainberg Method

Now we want to establish uniform local energy decay for solutions with finite energy of (El.D). The procedure here is essentially the same as for the wave equation: With

$$\mathcal{A} := \begin{pmatrix} 0 & I_n \\ A(x, D) & 0 \end{pmatrix},$$

and with $(\mathrm{i}k - \mathcal{A})^{-1} V$ denoting the solution of $(\mathrm{i}k - \mathcal{A}) W = V$ in Ω, $W = 0$ on $\partial\Omega$ for $V \in H$ and $k \in D^-$, we have

$$(\mathrm{i}k - \mathcal{A})^{-1} = \begin{pmatrix} -\mathrm{i}kR(k) & -R(k) \\ -R(k)\, A(x, D) & -\mathrm{i}kR(k) \end{pmatrix}$$

where $R(k) := (A(x,D)+k^2)^{-1}$ denotes the operator that maps f to a solution of

$$(A(x,D)+k^2)g = f \text{ in } \Omega\,, \quad g = 0 \text{ on } \partial\Omega\,.$$

We use the representation of the solution of (El.D) in terms of the Laplace transform:

$$\begin{pmatrix} U(t) \\ U_t(t) \end{pmatrix} = \frac{1}{2\pi} \int\limits_{-\infty-\mathrm{i}\sigma}^{+\infty-\mathrm{i}\sigma} e^{\mathrm{i}kt} (\mathrm{i}k - \mathcal{A})^{-1} \begin{pmatrix} U_0 \\ U_1 \end{pmatrix} dk\,. \tag{9.11}$$

The path of integration has to be moved into the upper half-plane. Therefore, a continuation of the resolvent $R(k)$ across the continuous spectrum is needed and its analytical properties together with the behaviour for high and low frequencies have to be investigated. As before the high-frequency asymptotics determine whether there is uniform local energy decay at all—this is the place where a non-trapping condition is needed—whereas the low-frequency asymptotics give the precise decay rates.

We first state the analogues of the Lemmata 6.2.2, 6.2.3, and 6.2.4. Afterwards we present the decay results for solutions of (El.D).

Lemma 9.7.1 ([Va 68], Th. 3, [Iw/Sh 88], Th. 1.3, [Dan 96], Th.1.4) *Suppose $a, b > R$. Then the resolvent $R(k)$, $k \in D^-$, can be continued meromorphically to D as a $\mathcal{L}_b(\mathcal{L}^2_a(\Omega), H_2(\Omega_b))$-valued function. The set of poles of $R(k)$ is discrete and its intersection with D^- is empty.*

The continuation of the resolvent will still be denoted by $R(k)$.

Lemma 9.7.2 *Suppose $a, b > R$. Then $R(k)$ has no pole in $\mathbb{R} \setminus \{0\}$.*

Remark: Iwishita and Shibata stated this last assertion for homogeneous media only (cf. Theorem 1.4 of [Iw/Sh 88]). The reason seems to be that the authors were not aware of the unique continuation principle for the inhomogeneous case. But such a result exists in [We 69a, We 69b, We 00, De/Ro 93, A/I/T/Y 98]. Therefore Lemma 9.7.2 is even true in the general situation.

Lemma 9.7.3 ([Va 75], Theorem 7) *Let $a, b > R$ and let Ω be non-trapping according to Definition 9.6.5. Then there exist positive constants α, β, C, T such that for integers $0 \leqslant s \leqslant 1$ and $0 \leqslant j \leqslant 2$*

$$\|R(k)\|_{\mathcal{L}_b(\mathring{\mathcal{H}}_{s,a}(\Omega), \mathcal{H}_{s+2-j}(\Omega_b))} \leqslant C\, |k|^{1-j} e^{T|\mathrm{Im}\, k|}$$

for any k out of

$$\mathcal{U}_{\alpha\beta} := \Big\{ k \in D \;\Big|\; |\mathrm{Im}\, k| \leqslant \alpha \ln |\mathrm{Re}\, k| - \beta \Big\}\,.$$

Remark: Like Lemma 6.2.3 this result remains true, if the non-trapping condition for the weak solution (Definition 9.6.6) is used instead of Vainberg's non-trapping condition.

The low-frequency asymptotics are due to [Va 73], Theorem 2, [Iw/Sh 88], Theorem 1.2, and [Dan 96], Theorem 1.2:

Lemma 9.7.4 *Suppose $a, b > R$. Then there exists a $\gamma > 0$ such that:*

1. *For odd n*
$$R(k) \quad \text{is holomorphic in} \quad W := \Big\{ k \in D \;\Big|\; |k| < \gamma \Big\}.$$
2. *For even $n \geqslant 4$*
$$R(k) = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} B_{mj} (k^{n-2} \ln k)^m k^j \qquad \text{in} \qquad W' := \Big\{ k \in D \;\Big|\; |k| < \gamma \Big\}.$$
3. *For $n = 2$*
$$R(k) = \sum_{m=0}^{\infty} \sum_{j=-m}^{\infty} B_{mj} k^m (\ln k)^{-j} \qquad \text{in} \qquad W'.$$

Here the $B_{mj} \in \mathcal{L}_b(\mathcal{L}_a^2(\Omega), \mathcal{H}_2(\Omega_b))$ and the double series converge absolutely and uniformly in the operator norm.

Remark: According to [Iw/Sh 88] and [Dan 96] similar expansions hold for the Neumann problem (for $n = 2$ there is a slight difference comparable to the one of the wave equation). That these results do not lead to uniform LED for solutions of (El.N) is due to the fact that no obstacle satisfies Vainberg's NTC or the NTC for the weak solution in case of homogeneous Neumann boundary values. So Lemma 9.7.3 is an empty statement for (El.N)!

Now the contour of the integral in (9.11) can be shifted into the upper half-plane as in Chapter 6. The application of the above lemmata leads then to the following decay results:

Theorem 9.7.5 *Assume that $a \geqslant R$ and that Ω fulfils Vainberg's non-trapping condition or the non-trapping condition for the weak solution. Let $K \subset\subset \overline{\Omega}$ and let U be a solution of* (El.D) *with $(U_0, U_1) \in H$ and* $\operatorname{supp}(U_0, U_1) \subset \overline{\Omega}_R$. *Then positive constants c and δ exist such that the following decay estimates hold:*

1. *For $n \geqslant 3$ odd*
$$E\left(U(t), K\right) \leqslant c\, e^{-\delta t}\, E\left(U(0), \Omega\right).$$
2. *For $n \geqslant 4$ even*
$$E\left(U(t), K\right) \leqslant c\, t^{-2n+2}\, E\left(U(0), \Omega\right), \qquad t > 0.$$
3. *For $n = 2$:*
$$E\left(U(t), K\right) \leqslant c\, t^{-2} \ln^{-4} t\; E\left(U(0), \Omega\right), \qquad t > 0.$$

Remarks: For homogeneous media this result is due to Iwashita and Shibata ($n \geqslant 3$) and Dan ($n = 2$), cf. [Iw/Sh 88], Theorem 4.3, and [Dan 96], Theorem 5.2. The only reason for their restriction to homogeneous media is the unique continuation principle which is needed for the proof of Lemma 9.7.2. As remarked above this principle holds for all isotropic media such that the results of [Iw/Sh 88, Dan 96] remain true for the inhomogeneous case also (compare with [Di 92]). For the technical details of the proof of Theorem 9.7.5 cf. for example [Di 92] and [Pau 96].

Chapter 10

Localized Solutions

In Section 2.1 we presented Ralston's proofs of the "only if" part of the Lax-Phillips conjecture. This result was achieved by the construction of so-called "localized solutions" which stay near a given ray for a long time. In [Ral 82] Ralston actually treated equations of higher order, too, with the effect that ramifications of geodesics and bicharacteristics similar to our situation in isotropic elasticity may occur. Thus a bicharacteristic starting at a single point will in general give rise to a whole shower that gets bigger at each point of reflection where several outgoing bicharacteristics exist. With the Gaussian beam construction Ralston used here, he proved the existence of solutions which are localized near the whole shower.

This result does not hold for equations only. In [Ral 99] Ralston showed how Gaussian beams could be used for the Cauchy problem of isotropic elasticity, the main difference being that the amplitudes of the beams are now vector-valued functions. With the methods indicated in [Ral 82] this construction works for boundary value problems, too. This yields the existence of solutions of (El.D) that are localized near a shower generated by the two bicharacteristic families. But is it possible to prove the above direction of the Lax-Phillips conjecture for isotropic elasticity with Gaussian beams? This question, which the author discussed in some detail with Prof. Ralston, is difficult to answer. On the one hand it is not clear that this conjecture (i.e. arbitrarily long trapped rays imply that there is no uniform local energy decay) is true in this case. This may be illustrated by an example of Ralston, which he introduced to us during our correspondence:

Let us consider a homogeneous isotropic elastic medium filling all of three space outside three small balls whose centres are the vertices of an equilateral triangle. Then there would be a periodic ray path which is an equilateral triangle with a vertex on the surface of each ball. Assume the Dirichlet boundary condition on the surfaces of the balls. Then one can build a Gaussian beam which follows the triangle at the high speed but reflects off a slow speed ray at each reflection. If the balls are small enough, these slow speed rays have no further contact with the obstacles (this can always be achieved because of our Assumption 9.3.1, for here the slow and high speed rays under consideration diverge at a fixed angle different from zero). This seems to be all that could be constructed in this situation, and the energy in the high speed part of the beam *does* decay (exponentially) because the slow speed parts carry a fixed proportion of the energy away at each reflection. This example does not show that the direction of the Lax-Phillips conjecture that we are interested in is false (there are other trapped rays—the shortest connections between pairs of balls—where Gaussian beam constructions would show that energy does not decay uniformly with respect to the energy of initial data), but it does indicate that one cannot prove the conjecture by Gaussian beam constructions along a shower containing arbitrarily long trapped rays.

That was Ralston's example. On the other hand, in Section 9.4 we have already mentioned Yamamoto's following result: If $\lambda \geqslant 4\mu$, then singularities cannot change the bicharacteristic family at the boundary. Only ramifications to both families are possible. Because of the strong connection between the propagation of singularities and the existence of localized solutions, the above detail of [Ya 88], too, points to a problem that seems to arise in proving the Lax-Phillips conjecture: It may in general not be possible to concentrate most of the energy near a single geodesic path of the shower.

Thus, in general, a proof of the Lax-Phillips conjecture using Gaussian beams seems almost possible with a refined analysis of the geometrical properties of the obstacle. But there are some results useful for our purposes that can be achieved by this construction as has already been indicated in Ralston's above example. If there is a closed ray of one family that makes no contact with the boundary, there is no influence from the other family at all. Then the Gaussian beam construction proves that there is no uniform local energy decay. Here is no difference to the wave equation. The more interesting case is the one where $\partial\Omega$ is involved. And for a few special situations, we have good results, too: Let us suppose the ratio λ/μ be constant throughout the medium—a requirement that is automatically fulfilled in the homogeneous case. Furthermore, assume the existence of a ray γ in $\overline{\Omega}$ that makes perpendicular incidence with $\partial\Omega$ at both endpoints and has no further contact with the boundary (otherwise the phenomenon described in Ralston's example would occur). Our assumption on the medium yields the differential equations

$$\dot{x} = 2c_i\mu(x)\xi, \quad \dot{\xi} = -c_i|\xi|^2\nabla\mu(x), \qquad i = 1, 2,$$

for the two families. Thus γ is a ray of both speeds. Because of the perpendicular incidence, a shower starting with any portion of γ would only consist of segments of $\pm\gamma$ of both speeds. Therefore, Gaussian beams exist which concentrate the energy arbitrarily long in a small neighbourhood of γ, showing that uniform LED for (El.D) does not hold in this case.

It should be remarked that our assumption on the medium can be considerably weakened: For our arguments to hold, it is sufficient that the ratio λ/μ is constant in some small neighbourhood of γ. We summarize these immediate consequences of Ralston's results in the following theorem:

Theorem 10.0.6 *Let one of the following two assumptions be fulfilled:*

(i) In Ω, a periodic ray of one family exists.

(ii) In $\overline{\Omega}$, a ray γ of one family exists that makes perpendicular incidence with $\partial\Omega$ at both endpoints and has no further contact with the boundary. The ratio λ/μ is constant in some $\overline{\Omega}$-neighbourhood of γ.

Then there is no uniform LED for (El.D).

If $\lambda + 2\mu$ or μ have the form cr^2 in a neighbourhood of a whole circle in Ω with centre at the origin, we have the situation of case (i), as we have already seen with the wave equation.

We can also give an easy example for case (ii), similar to one that occurred in Part I: If locally $\lambda = c_1 r^2$ and $\mu = c_2 r^2$, then star-shaped obstacles exist such that a circular segment gives rise to a periodic ray. The last image on page 9 provides an example for this situation. This shows that the Bloom-type condition (9.4) is sharp at least in the following sense: if it is violated by both propagation speeds then star-shaped obstacles may exist which don't allow uniform local energy decay.

Chapter 11

Comparison of the Different Non-Trapping Conditions

As in Chapter 7 of Part I we now want to compare the different non-trapping conditions that appeared in Part II. The conditions to be discussed for the displacement problem of isotropic elasticity are:

1. Melrose's NTC for (El.D) (Definition 9.3.3).
2. Vainberg's NTC (Definition 9.6.5).
3. The NTC for the weak solution (Definition 9.6.6).
4. The existence of an escape function (El) (Definition 9.5.1).

We already know that the existence of an escape function implies Melrose's NTC (Theorem 9.5.3). In this chapter we will prove the following implications between the above conditions:

$$\text{Melrose's NTC} \implies \text{Vainberg's NTC} \implies \text{the NTC for the weak solution} \qquad (11.1)$$

As announced in the first part of this thesis we furthermore include two other non-trapping conditions, namely Rauch's *propagation of singularities hypothesis* and Vainberg's former *condition* D'.

Before we start with the proof of (11.1), it should be noted that again Melrose's NTC at once implies the NTC for the weak solution. This consequence of the generalized Huygens' principle has already been remarked by Yamamoto in [Ya 89, (4.3), p. 217] as one step of his proof of uniform LED (Yamamoto used Melrose's original non-trapping condition, a problem we discussed above).

11.1 The Main Theorems

We begin this section with the proof of the first implication of (11.1). We use here, as already mentioned in Chapter 7, the ideas that Prof. Ralston told the author in 1996. We start with some preliminaries:

In this proof Sobolev spaces of negative order will be used. These are the conjugate spaces of the usual Sobolev spaces of positive order. One defines

$$\mathcal{H}_{-m}(\Omega) := (\mathring{\mathcal{H}}_m(\Omega))',$$

whereas the dual of $\mathcal{H}_m(\Omega)$ is just denoted by $(\mathcal{H}_m(\Omega))'$. These spaces are equipped with the norms $\|\cdot\|_{\mathcal{H}_{-m}(\Omega)}$ and $\|\cdot\|_{(\mathcal{H}_m(\Omega))'}$ respectively (for details cf. [Ad 75], p. 47ff.). $\mathcal{H}_m(\Omega)$ is continuously imbedded into $\mathcal{C}_B(\overline{\Omega})$, if $m > n/2$, according to Sobolev's imbedding theorem (cf. [Ad 75, Theorem 5.4]). For these m and fixed $x_0 \in \overline{\Omega}$ the distribution $\delta(\cdot - x_0)$ can, therefore, be extended to an element of $(\mathcal{H}_m(\Omega))'$.

Note that all elements of $\mathcal{H}_{-m}(\Omega)$ are extensions of distributions in $\mathcal{D}'(\Omega)$ to $\mathring{\mathcal{H}}_m(\Omega)$ (cf. [Ad 75], 3.9, p. 49). Now convergence in $\mathcal{H}_{-m}(\Omega)$ implies convergence in $\mathcal{D}'(\Omega)$:

Lemma 11.1.1 *Let the sequence $(g_j)_j \subset \mathcal{H}_{-m}(\Omega)$ converge to $g \in \mathcal{H}_{-m}(\Omega)$. Then the sequence of restrictions of the g_j to $\mathring{\mathcal{C}}_\infty(\Omega)$ converges in $\mathcal{D}'(\Omega)$ to the restriction of g.*

Proof: We have

$$\begin{aligned}
& g_j \longrightarrow g \qquad \text{in } \mathcal{H}_{-m}(\Omega) \qquad (j \longrightarrow \infty)\\
\Longrightarrow\ & \forall\, u \in \mathring{\mathcal{H}}_m(\Omega)\ :\ |g_j u - g u| \leqslant \|g_j - g\|_{\mathcal{H}_{-m}(\Omega)}\, \|u\|_{\mathring{\mathcal{H}}_m(\Omega)} \longrightarrow 0 \qquad (j \longrightarrow \infty)\\
\Longrightarrow\ & \forall\, \phi \in \mathring{\mathcal{C}}_\infty(\Omega)\ :\ |g_j\phi - g\phi| \longrightarrow 0 \qquad (j \longrightarrow \infty)\\
\Longrightarrow\ & g_j \longrightarrow g \qquad \text{in } \mathcal{D}'(\Omega) \qquad (j \longrightarrow \infty)\,.
\end{aligned}$$

This proves the lemma. □

For the proof of (11.1) we need to have a closer look at the ansatz of the scattering theory of [Sh/Sog 89]. We denote by $\langle\cdot,\cdot\rangle$ the inner product in $\mathcal{L}^2(\Omega)$ for both scalar-valued and vector-valued functions. We furthermore use for functions $U, \widetilde{U}$ with values in $\mathbb{R}^n$

$$\langle U, \widetilde{U}\rangle_D := \int_\Omega \mu\nabla U\cdot\nabla\widetilde{U} + \mu\sum_{i,j=1}^n \partial_i U^j \partial_j \widetilde{U}^i + \lambda\nabla' U \nabla' \widetilde{U}\, dx\,,$$

provided this integral makes sense. We have already introduced the energy space H for the Dirichlet problem,

$$H = \left\{ (U,V)\ \middle|\ U \in \mathcal{H}_\nabla(\Omega)\,, V \in \mathcal{L}^2(\Omega) \right\},$$

which is equipped with the so-called *energy norm* $\|(U,V)\|_E$, induced by the inner product

$$\langle (U,V),(\widetilde{U},\widetilde{V})\rangle_E := \langle U,\widetilde{U}\rangle_D + \langle V,\widetilde{V}\rangle\,, \qquad (U,V),(\widetilde{U},\widetilde{V}) \in H\,.$$

Although $(U,V) \in H$ is a function with values in $\mathbb{R}^{2n}$, U will be referred to as the *first* component of (U,V) and V as the *second*. f_1 and f_2 have to be understood in the same way, if $f \in H$. We will use the abbreviation A for $A(x,D)$ in this chapter. The operator $\mathcal{A}$ has already been defined by

$$\mathcal{A} = \begin{pmatrix} 0 & I_n \\ A(x,D) & 0 \end{pmatrix}.$$

For $D(\mathcal{A})$ Shibata and Soga choose

$$\begin{aligned} D(\mathcal{A}) &:= \left\{ (U,V) \in H \;\middle|\; \mathcal{A}(U,V)^T \in H \right\} \\ &= \left\{ (U,V) \in H \;\middle|\; D_x^2 U \in \mathcal{L}^2(\Omega), V \in \mathcal{H}_\nabla(\Omega) \right\} \end{aligned}$$

(this is indeed the same set as their $D(\mathcal{A})$ in [Sh/Sog 89, p. 863]). They prove that $\mathcal{A}$ generates a one-parameter group of unitary operators $\{\mathcal{U}(t); t \in \mathbb{R}\}$ on H. As usual for initial data $(U_0,U_1) \in H$ the first component of $\mathcal{U}(t)(U_0,U_1)^T$ provides a solution with finite energy of (El.D), whereas the second component is just the derivative with respect to t of the first. The decisive idea in the proof of the first implication of (11.1) is now to use properties of the group $\{\mathcal{U}(t); t \in \mathbb{R}\}$ and its infinitesimal generator $\mathcal{A}$. We need the following *high energy estimate*:

Lemma 11.1.2 *Let $U_0 \in \mathring{\mathcal{H}}_m(\Omega)$ and $U_1 \in \mathring{\mathcal{H}}_{m-1}(\Omega)$. Then for solutions U of (El.D) the following estimate holds:*

$$\left(\|U(t)\|_m^2 + \|U_t(t)\|_{m-1}^2\right)^{1/2} \leqslant C(t)\left(\|U_0\|_m^2 + \|U_1\|_{m-1}^2\right)^{1/2}, \qquad t \in \mathbb{R}. \tag{11.2}$$

Proof: For all $V \in D(A) := \{V \in \mathcal{H}_\nabla(\Omega) \mid D_x^2 V \in \mathcal{L}^2(\Omega)\}$ with $AV \in \mathcal{H}_k(\Omega)$ Jiang and Racke proved the elliptic regularity estimate ([Ji/Rac 90, Theorem A2.1])

$$\|V\|_{k+2} \leqslant c\big(\|AV\|_k + \|V\|\big)\,. \tag{11.3}$$

Now let U be a solution of (El.D) with initial data (U_0,U_1) as above. Because $\mathcal{A}$ and $\mathcal{U}(t)$ commute, one sees that $\mathcal{U}(t)$ preserves the domain of any power of $\mathcal{A}$. Thus (11.3) can be applied to $U(t)$ for all $t \in \mathbb{R}$ and $k \leqslant m$. Repeated application of this estimate and the differential equations together with the conservation of the total energy of

$\partial_t^l U,\ l = 0, \ldots, m-1$, yields

$$\|U(t)\|_m \leqslant C \begin{cases} \sum_{j=0}^{(m-2)/2} \|A^j U(t)\|_1 + \|A^{m/2} U(t)\|\,, & m \text{ even}, \\ \sum_{j=0}^{(m-1)/2} \|A^j U(t)\|_1\,, & m \text{ odd}, \end{cases}$$

$$\leqslant C \begin{cases} \sum_{j=0}^{(m-2)/2} \|\partial_t^{2j} U(t)\|_1 + \|\partial_t^m U(t)\|\,, & m \text{ even}, \\ \sum_{j=0}^{(m-1)/2} \|\partial_t^{2j} U(t)\|_1\,, & m \text{ odd}, \end{cases}$$

$$\leqslant C \left(\sum_{l=0}^{m-1} E(\partial_t^l U(0), \Omega)^{1/2} + \|U(t)\| \right) .$$

$\partial_t^l U$ is a solution of (El.D) with initial values

$$A^{l/2} U_0\,, \quad A^{l/2} U_1\,, \qquad \text{for even } l\,,$$

and

$$A^{(l-1)/2} U_1\,, \quad A^{(l+1)/2} U_0\,, \qquad \text{for odd } l\,.$$

Therefore

$$E(\partial_t^l U(0), \Omega) \leqslant C \left(\|U_0\|_{l+1}^2 + \|U_1\|_l^2 \right) .$$

On the other hand we have

$$\begin{aligned} \|U(t)\| &\leqslant \left\| \int_0^t U_t(\tau)\, d\tau \right\| + \|U_0\| \\ &\leqslant |t|\, E\left(U(0), \Omega\right)^{1/2} + \|U_0\| \\ &\leqslant C(1+|t|) \left(\|U_0\|_1^2 + \|U_1\|^2 \right)^{1/2} , \qquad t \in \mathbb{R}\,. \end{aligned}$$

This yields

$$\|U(t)\|_m^2 \leqslant c(t) \left(\|U_0\|_m^2 + \|U_1\|_{m-1}^2 \right) . \tag{11.4}$$

By similar methods we estimate

$$\|U_t(t)\|_{m-1}^2 \leqslant c \left(\|U_0\|_m^2 + \|U_1\|_{m-1}^2 \right) ,$$

which together with (11.4) proves (11.2). □

The first step in the proof of the first implication in (11.1) is the following lemma:

Lemma 11.1.3 *$\mathcal{U}(t)$ can be extended to a mapping of $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$.*

Proof: The commutativity of $\mathcal{A}$ and $\mathcal{U}(t)$ in particular yields: If the initial data $f = (f_1, f_2)$ are elements of $\mathring{\mathcal{C}}_\infty(\Omega)$, then both components of $\mathcal{U}(t)f$ are in $\mathcal{C}_\infty(\Omega)$ belonging to the domain of any power of $\mathcal{A}$. Thus one can extend $\mathcal{U}(t)$ to a mapping of $\mathcal{E}'(\Omega)$ to $\mathcal{D}'(\Omega)$ by duality making use of the unitarity of $\mathcal{U}(t)$ in the energy norm: for $f \in H \cap \mathcal{L}^2(\Omega)$ and $g \in D(\mathcal{A})$

$$\begin{aligned}\langle(\mathcal{U}(t)f)_1, Ag_1\rangle - \langle(\mathcal{U}(t)f)_2, g_2\rangle &= -\langle\mathcal{U}(t)f, g\rangle_E \\ &= -\langle\mathcal{U}(t)f, \mathcal{U}(t)\mathcal{U}(-t)g\rangle_E \\ &= -\langle f, \mathcal{U}(-t)g\rangle_E \\ &= \langle f_1, A(\mathcal{U}(-t)g)_1\rangle - \langle f_2, (\mathcal{U}(-t)g)_2\rangle\end{aligned} \tag{11.5}$$

holds. We want to get rid of the operator A on the right and left-hand side of this line of equations: Denote by H_1 the first component of H. Because $(H_1, \langle\cdot,\cdot\rangle_D)$ is a Hilbert space, the Riesz representation theorem yields:

$$\forall h \in \mathring{\mathcal{C}}_\infty(\Omega) \stackrel{1}{\exists} g \in H_1 \ \forall f \in H_1 : \ \langle g, f\rangle_D = \langle -h, f\rangle .$$

This identity especially holds for all $f \in \mathring{\mathcal{C}}_\infty(\Omega)$. By partial integration we get

$$\forall f \in \mathring{\mathcal{C}}_\infty(\Omega) : \ \langle g, Af\rangle = \langle h, f\rangle .$$

Therefore $Ag = h$ in the distributional sense. On the other hand $h \in \mathcal{L}^2(\Omega)$. This yields

$$\begin{pmatrix} g \\ 0 \end{pmatrix} \in D(\mathcal{A}) ,$$

for

$$\begin{pmatrix} g \\ 0 \end{pmatrix} \in H \quad \text{and} \quad \mathcal{A}\begin{pmatrix} g \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ Ag \end{pmatrix} \in H .$$

This proves

$$\forall \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathring{\mathcal{C}}_\infty(\Omega) \ \exists \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(\mathcal{A}) : \quad Ag_1 = h_1 \wedge g_2 = -h_2 ,$$

or, using the operator $\mathcal{A}$,

$$\forall \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathring{\mathcal{C}}_\infty(\Omega) \ \exists \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(\mathcal{A}) : \quad \mathcal{A}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} -h_2 \\ h_1 \end{pmatrix} .$$

Let $f = (f_1, f_2) \in H \cap \mathcal{L}^2(\Omega)$ and $h = (h_1, h_2) \in \mathring{\mathcal{C}}_\infty(\Omega)$. To h we choose $g = (g_1, g_2)$ as above. With (11.5) the following holds:

$$\begin{aligned}
\langle (\mathcal{U}(t)f)_1, h_1\rangle \; + \; \langle (\mathcal{U}(t)f)_2, h_2\rangle \qquad\qquad (11.6)\\
&= \langle (\mathcal{U}(t)f)_1, Ag_1\rangle \; - \; \langle (\mathcal{U}(t)f)_2, g_2\rangle \\
&= \langle f_1, A(\mathcal{U}(-t)g)_1\rangle \; - \; \langle f_2, (\mathcal{U}(-t)g)_2\rangle \\
&= \langle f_1, (\mathcal{A}\mathcal{U}(-t)g)_2\rangle \; - \; \langle f_2, (\mathcal{A}\mathcal{U}(-t)g)_1\rangle \\
&= \langle f_1, (\mathcal{U}(-t)\mathcal{A}g)_2\rangle \; - \; \langle f_2, (\mathcal{U}(-t)\mathcal{A}g)_1\rangle \\
&= \langle f_1, (\mathcal{U}(-t)(-h_2, h_1)^T)_2\rangle \; - \; \langle f_2, (\mathcal{U}(-t)(-h_2, h_1)^T)_1\rangle \, .
\end{aligned}$$

This yields the desired extension of $\mathcal{U}(t)$: For any $f = (f_1, f_2) \in \mathcal{E}'(\Omega)$ we define

$$\begin{aligned}
(\mathcal{U}(t)f)_1 \,(h_1) + (\mathcal{U}(t)f)_2 \,(h_2) \qquad\qquad (11.7)\\
:= f_1 \, \big((\mathcal{U}(-t)(-h_2, h_1)^T)_2\big) - f_2 \, \big((\mathcal{U}(-t)(-h_2, h_1)^T)_1\big)
\end{aligned}$$

for all $h = (h_1, h_2) \in \mathring{\mathcal{C}}_\infty(\Omega)$. Now the right-hand side obviously provides an element of $\mathcal{D}'(\Omega)$. □

(11.7) yields an extension of $\mathcal{U}(t)$ to a bounded map between Sobolev spaces of negative order, as well:

Lemma 11.1.4 (11.7) *extends* $\mathcal{U}(t)$ *to a bounded map from* $((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))')$ *to* $(\mathcal{H}_{-(m-1)}(\Omega) \times \mathcal{H}_{-m}(\Omega))$.

Proof: Let $m \geqslant 1$ and $h \in (\mathring{\mathcal{H}}_{m-1}(\Omega) \times \mathring{\mathcal{H}}_m(\Omega))$. Then

$$\begin{pmatrix} h_2 \\ h_1 \end{pmatrix} \in \Big(\mathring{\mathcal{H}}_m(\Omega) \times \mathring{\mathcal{H}}_{m-1}(\Omega)\Big)$$

and, therefore,

$$\mathcal{U}(-t)\begin{pmatrix} -h_2 \\ h_1 \end{pmatrix} \in (\mathcal{H}_m(\Omega) \times \mathcal{H}_{m-1}(\Omega)) \; , \qquad t \in \mathbb{R}.$$

Because of the high energy estimate (11.2)

$$\left\| \mathcal{U}(-t)\begin{pmatrix} -h_2 \\ h_1 \end{pmatrix} \right\|_{(\mathcal{H}_m(\Omega)\times\mathcal{H}_{m-1}(\Omega))} \leqslant C(t) \left\| \begin{pmatrix} -h_2 \\ h_1 \end{pmatrix} \right\|_{(\mathcal{H}_m(\Omega)\times\mathcal{H}_{m-1}(\Omega))}$$

holds for all $t \in \mathbb{R}$. Let furthermore

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \big((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))'\big) \; ,$$

so (11.7) defines an element $\mathcal{U}(t)f$ of $(\mathcal{H}_{-(m-1)}(\Omega) \times \mathcal{H}_{-m}(\Omega))$ and we have

$$\begin{aligned}
|(\mathcal{U}(t)f)(h)| &= |(\mathcal{U}(t)f)_1\,(h_1) + (\mathcal{U}(t)f)_2\,(h_2)| \\
&\leqslant \|f\|_{((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))')} \left\| \begin{pmatrix} \left(\mathcal{U}(-t)\binom{-h_2}{h_1}\right)_2 \\ \left(\mathcal{U}(-t)\binom{-h_2}{h_1}\right)_1 \end{pmatrix} \right\|_{(\mathcal{H}_{m-1}(\Omega) \times \mathcal{H}_m(\Omega))} \\
&= \|f\|_{((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))')} \left\| \mathcal{U}(-t) \begin{pmatrix} -h_2 \\ h_1 \end{pmatrix} \right\|_{(\mathcal{H}_m(\Omega) \times \mathcal{H}_{m-1}(\Omega))} \\
&\leqslant C(t)\, \|f\|_{((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))')} \left\| \begin{pmatrix} -h_2 \\ h_1 \end{pmatrix} \right\|_{(\mathcal{H}_m(\Omega) \times \mathcal{H}_{m-1}(\Omega))} \\
&= C(t)\, \|f\|_{((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))')} \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{(\mathcal{H}_{m-1}(\Omega) \times \mathcal{H}_m(\Omega))}, \qquad t \in \mathbb{R}.
\end{aligned}$$

This yields for all $f \in \big((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))'\big)$ and $t \in \mathbb{R}$

$$\|\mathcal{U}(t)f\|_{(\mathcal{H}_{-(m-1)}(\Omega) \times \mathcal{H}_{-m}(\Omega))} \leqslant C(t)\, \|f\|_{((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))')},$$

that is for $m \geqslant 1$ the continuity of $\mathcal{U}(t)$ as a map from $((\mathcal{H}_{m-1}(\Omega))' \times (\mathcal{H}_m(\Omega))')$ to $(\mathcal{H}_{-(m-1)}(\Omega) \times \mathcal{H}_{-m}(\Omega))$. □

Denote by $f_{\delta,j}$ the distribution $(0, \delta(\cdot - x_0)e_j)$. Then the following lemma holds (to make clear which variable is involved, it will be specified as a subscript):

Lemma 11.1.5

$$\mathcal{U}(t)f_{\delta,j} \in \mathcal{C}_\infty((\mathbb{R} \times \overline{\Omega}_{x_0}), \mathcal{D}'(\Omega_x)).$$

Proof: Lemma 11.1.4 especially means that $\mathcal{U}(t)f_{\delta,j}$ depends continuously on $x_0 \in \overline{\Omega}$ in the corresponding Sobolev norm. On the other hand, for any differential operator D acting in the x_0-variable one has $D\mathcal{U}(t)f_{\delta,j} = \mathcal{U}(t)Df_{\delta,j}$. That means here $Df_{\delta,j} = (0, D\,\delta(\cdot - x_0)e_j)$. Thus we conclude that $\mathcal{U}(t)f_{\delta,j}$ has continuous derivatives in x_0 of all orders considered as a function with values in a Sobolev space of sufficiently high negative order. One can also take derivatives with respect to t, since $\mathcal{U}(t)$ is strongly continuous and $\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t)$. This regularity of $\mathcal{U}(t)f_{\delta,j}$ as a function with values in Sobolev spaces of negative order transfers to $\mathcal{U}(t)f_{\delta,j}$ even as a distribution-valued function by Lemma 11.1.1. □

We are now able to prove that an analogue to Lemma 11.1.5 holds for the Green's matrix:

Lemma 11.1.6 *For the Green's matrix $G(t, x, x_0)$ of* (El.D) *the following holds:*

$$G \in \mathcal{C}_\infty((\mathbb{R} \times \overline{\Omega}_{x_0}), \mathcal{D}'(\Omega_x)).$$

Proof: Denoting by $(\mathcal{U}(t)f_\delta)_k$, for $k = 1, 2$, the matrix consisting of the n columns $(\mathcal{U}(t)f_{\delta,j})_k, j = 1, \ldots, n$, Lemma 11.1.5 yields

$$(\mathcal{U}(t)f_\delta)_k \in \mathcal{C}_\infty((\mathbb{R} \times \overline{\Omega}_{x_0}), \mathcal{D}'(\Omega_x)), \quad k = 1, 2. \tag{11.8}$$

$(\mathcal{U}(t)f_\delta)_1$, however, is the Green's matrix $G(t, x, x_0)$: To see this let $P\varphi$ denote the solution of (El.D) for initial values $U(0) = 0$ and $U_t(0) = \varphi$. Thus for all $\varphi \in \mathring{\mathcal{C}}_\infty(\Omega)$

$$P\varphi = \left(\mathcal{U}(t)\begin{pmatrix}0\\ \varphi\end{pmatrix}\right)_1 . \tag{11.9}$$

It is sufficient to show that

$$\begin{aligned}&(\mathcal{U}(t)f_\delta)_1(x)\ (\psi(t,x) \otimes \varphi(x_0)) \\ &\quad = \int_{\mathbb{R}}\int_{\Omega_x} (P\varphi)(t,x)\psi(t,x)\,dxdt \qquad \forall\, \psi \in \mathring{\mathcal{C}}_\infty(\mathbb{R} \times \Omega),\ \varphi \in \mathring{\mathcal{C}}_\infty(\Omega)\end{aligned} \tag{11.10}$$

(if necessary, we still formally add variables to distributions). Then the uniqueness statement of the Schwartz kernel theorem yields the assertion. For this purpose we first introduce the following notation:

$$\big(f(y,x)\,(\phi(z,x))\big)_x$$

means that the distribution f, depending continuously on y, is applied for fixed y and z to ϕ as a function of x. Because of (11.6), (11.7), (11.8), (11.9), and the conventions (9.7) and (9.8) for the application of vectors and matrices of distributions, one gets equation (11.10) as follows:

$$\begin{aligned}&(\mathcal{U}(t)f_\delta)_1(x)\ (\psi(t,x) \otimes \varphi(x_0)) \\ &\quad = \sum_{j=1}^{n}\int_{\mathbb{R}}\int_{\Omega_{x_0}} \big((\mathcal{U}(t)f_{\delta,j})_1(x)\ (\psi(t,x))\big)_x \varphi_j(x_0)\,dx_0dt \\ &\quad = -\int_{\mathbb{R}}\int_{\Omega_{x_0}} \sum_{j=1}^{n}\left(\delta(x-x_0)e_j\left(\left(\mathcal{U}(-t)\begin{pmatrix}0\\ \psi(t,\cdot)\end{pmatrix}\right)_1(x)\right)\right)_x \varphi_j(x_0)\,dx_0dt \\ &\quad = -\int_{\mathbb{R}}\int_{\Omega_{x_0}} \left(\mathcal{U}(-t)\begin{pmatrix}0\\ \psi(t,\cdot)\end{pmatrix}\right)_1(x_0)\ \varphi(x_0)\,dx_0dt \\ &\quad = \int_{\mathbb{R}}\int_{\Omega_{x_0}} \left(\mathcal{U}(t)\begin{pmatrix}0\\ \varphi(\cdot)\end{pmatrix}\right)_1(x_0)\ \psi(t,x_0)\,dx_0dt \\ &\quad = \int_{\mathbb{R}}\int_{\Omega_x} (P\varphi)(t,x)\psi(t,x)\,dxdt\,.\end{aligned}$$

Therefore $(\mathcal{U}(t)f_\delta)_1$ as the unique distribution kernel of P is identical with the Green's function. Because of (11.8) this finally proves the lemma. □

Now we know that G is a smooth distribution-valued function of t and x_0. Thus it especially solves the equations of problem (9.10) in the distributional sense for each fixed x_0. As a second step, therefore, the generalized Huygens' principle can be applied and we conclude

$$\forall a, b > R \ \exists \tau(a,b) > 0 \ \forall x_0 \in \overline{\Omega}_a \ : \ G(\cdot,\cdot,x_0) \in \mathcal{C}_\infty([\tau,\infty) \times \overline{\Omega}_b) \,. \tag{11.11}$$

To derive from this the regularity of the Green's matrix as a $\mathbb{R}^{n^2}$-valued function of $(t, x, x_0) \in [\tau, \infty) \times \overline{\Omega}_b \times \overline{\Omega}_a$, one should first note that the proof of Lemma 11.1.6 yields (11.11) with the same values of $a, b,$ and $\tau(a, b)$ for any x_0-derivative of the distribution G. Then one makes use of the commutativity of $U(t)$ and differential operators acting in x_0 again, especially of the identity

$$D^\alpha_{x_0} U(t)(0, \delta(\cdot - x_0)e_j) = U(t)(0, D^\alpha_{x_0}\delta(\cdot - x_0)e_j) \,, \quad \alpha \in \mathbb{N}_0^n \,,$$

now applied to $U(t)(0, \delta(\cdot - x_0)e_j)$ as a function on $[\tau, \infty) \times \overline{\Omega}_b \times \overline{\Omega}_a$ with values in $\mathbb{R}^{2n}$, finally leading to:

Theorem 11.1.7 *Let Ω fulfil Melrose's non-trapping condition for* (El.D).
Then Ω satisfies Vainberg's non-trapping condition.

The second implication of (11.1) is easy to verify:

Theorem 11.1.8 *Let Ω fulfil Vainberg's non-trapping condition.*
Then Ω satisfies the non-trapping condition for the weak solution.

Proof: For $\tau(\Omega, a)$ of Definition 9.6.6 take $\tau(\Omega, a, a)$ of Definition 9.6.5. Then for the weak solution U of (El.D) for initial data $U(0) = 0$ and $U_t(0) = U_1$ with $U_1 \in \mathcal{L}^2_a(\Omega)$ the following holds:

$$U(t,x) = \int_{\Omega_a} G(t,x,x_0) U_1(x_0)\, dx_0 \,, \qquad (t,x) \in ([\tau,\infty) \times \overline{\Omega}_a) \,.$$

Now the dominated convergence theorem of Lebesgue yields

$$U \in \mathcal{C}_\infty([\tau,\infty) \times \overline{\Omega}_a) \,.$$ □

Thus (11.1) has been proved completely. The next section will include some more conditions.

11.2 Other Non-Trapping Conditions

So far we have dealt with the non-trapping conditions of Melrose, Vainberg, and the one for the weak solution only, including the earlier decay results of Kapitonov and Dassios for star-shaped obstacles. Now, as announced in the introduction of this chapter, we want to include the propagation of singularities hypothesis of Rauch and Vainberg's former condition D′ into our considerations.

Rauch's propagation of singularities hypothesis reads for (El.D) like Defintion 7.1.3 with G now being the Green's matrix of (El.D). We want to prove the following theorem:

Theorem 11.2.1 *Rauch's propagation of singularities hypothesis is equivalent to the non-trapping condition of Vainberg.*

Proof: Suppose Ω fulfils Vainberg's non-trapping condition. For given $\chi \in \mathring{\mathcal{C}}_\infty(\overline{\Omega})$ choose $a > R$ such that $\operatorname{supp}\chi \subset \overline{\Omega}_a$. Then $\chi(x)G(t,x,x_0)\chi(x_0) \in \mathring{\mathcal{C}}_\infty(\overline{\Omega}\times\overline{\Omega})$, if $t \geqslant \tau(\Omega,a,a)$.

Conversely, let Ω be non-trapping in the sense of Rauch. Let $\chi \in \mathring{\mathcal{C}}_\infty(\overline{\Omega})$ and the related $T > 0$ be given. Choose now a $\check{\chi} \in \mathring{\mathcal{C}}_\infty(\overline{\Omega})$ with $\check{\chi} \equiv 1$ on $\operatorname{supp}\chi$. Then a $\check{T} > 0$ exists with

$$\check{\chi}(x)G(t,x,x_0)\check{\chi}(x_0) \in \mathring{\mathcal{C}}_\infty(\overline{\Omega}\times\overline{\Omega})\,, \quad t \geqslant \check{T}\,.$$

Thus we have

$$\chi(x)G(t,x,x_0)\chi(x_0) = \chi(x)\check{\chi}(x)G(t,x,x_0)\check{\chi}(x_0)\chi(x_0)$$

and for $t \geqslant T_0 := \max(T,\check{T})$

$$\chi(x)A_xG(t,x,x_0)\chi(x_0) = \chi(x)A_x(\check{\chi}(x)G(t,x,x_0)\check{\chi}(x_0))\chi(x_0) \in \mathring{\mathcal{C}}_\infty(\overline{\Omega}\times\overline{\Omega})\,.$$

As in the proof of Lemma 11.1.5 we now conclude that for $t \geqslant T_0$ one can also take arbitrary derivatives with respect to t of $\chi(x)G(t,x,x_0)\chi(x_0)$, because of the strong continuity of $\mathcal{U}(t)$ and the identity $\mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t)$. This leads to

$$\chi(x)G(t,x,x_0)\chi(x_0) \in \mathcal{C}_\infty([T_0,\infty)\times\overline{\Omega}\times\overline{\Omega})\,.$$

Now let a und b be given. With $c := \max(a,b)$ we choose $\chi \in \mathring{\mathcal{C}}_\infty(\overline{\Omega})$ such that $\chi \equiv 1$ on $\overline{\Omega}_c$. We get a $T_0 > 0$ as above. Put $\tau(\Omega,a,b) := T_0$. Then

$$G \in \mathcal{C}_\infty([\tau,\infty)\times\overline{\Omega}_b\times\overline{\Omega}_a)$$

and Ω, therefore, satisfies Vainberg's non-trapping condition. □

Condition D′ reads for (El.D):

Vainberg's condition D′:
For some $N \geqslant 2$ there is a matrix $E_N(t,x,x_0)$ whose elements are generalized functions in $\mathcal{D}'(\Omega)$ depending continuously on parameters $x_0 \in \overline{\Omega}$, $x_0 \leqslant a$, and $t \geqslant 0$, where E_N satisfies the following two conditions:

(a) *E_N is a distributional solution of:*

$$\begin{cases} L_{t,x}\, E_N(t,x,x_0) = f_N(t,x,x_0)\,, & (t,x) \in (0,\infty)\times\Omega\,,\\ \quad E_N(t,x,x_0) = 0\,, & (t,x)\in(0,\infty)\times\partial\Omega\,,\\ \quad E_N(0,x,x_0) = 0\,, \quad \partial_t E_N(0,x,x_0) = \delta(x-x_0) I_n\,, & x\in\Omega\,,\end{cases}$$

where the matrix f_N has N continuous derivatives with respect to t, x, x_0.

(b) *For any $r < \infty$ there is a $T(r) > 0$ such that $E_N = 0$ for $|x| \leqslant r$, $|x_0| \leqslant a$, and $t \geqslant T(r)$.*

Again, in (a) all equations are meant distributionally with respect to the given variables. In [Ral 79] it is shown how to derive condition D′ from Vainberg's non-trapping condition in the case of the wave equation for homogeneous media. With similar arguments we want to show the corresponding implication for the system of isotropic elasticity (the proof holds essentially unchanged for the wave equations of Part I also): For an arbitrary but fixed a one can regard the $\tau(\Omega,a,b)$ of Definition 9.6.5 as a function of the second radius b. We define $\tau(b) := \tau(\Omega,a,b)$. We can assume without restriction that τ is a non-decreasing positive function in $\mathcal{C}_\infty(\mathbb{R}_0^+)$, because otherwise it could be replaced by a larger function with these properties. Let $\tau_1 \in \mathcal{C}_\infty(\mathbb{R}_0^+)$ be non-decreasing, too, with

$$\tau_1(b) > \tau(b) \qquad \text{for } b \geqslant 0\,,$$

and let the distance between these two curves in $\mathbb{R}^2$ be not less than unity. Then a $\xi = \xi(t,x) \in \mathcal{C}_\infty(\mathbb{R}^{n+1})$ exists such that

$$\begin{aligned} \xi &= 1\,, & t &< \tau(|x|)\,,\\ \xi &= 0\,, & t &> \tau_1(|x|)\,,\\ \xi &= \xi(t)\,, & |x| &< a\,,\end{aligned}$$

and for all $p = (p_0, p_1, \ldots, p_n) \in \mathbb{N}_0^{n+1}$

$$|D^p_{t,x}\,\xi| < C_p\,.$$

Let such a ξ be fixed. Because of Lemma 11.1.6, the Green's matrix G is a distribution for each fixed x_0. Thus $E_\infty := \xi G$ satisfies (a): The definition of ξ implies that E_∞ has the same initial and boundary values as G. Furthermore for the function F defined by

$$L_{t,x}E_\infty = L_{t,x}(\xi G) = [L_{t,x}, \xi]G =: F$$

we have $F \in \mathcal{C}_\infty(\mathbb{R}_0^+ \times \overline{\Omega} \times \overline{\Omega}_a)$, because $[L_{t,x}, \xi]$ cuts off all the singular behaviour of G. For the validity of (b) one has just to choose $T(r) := \tau_1(r)$. Finally the desired continuous dependence of E_∞ on t and x_0 follows by Lemma 11.1.6 from the regularity of G. This finally proves:

Theorem 11.2.2 *Vainberg's non-trapping condition implies Vainberg's condition* D′.

11.3 Further Results for Domains in $\mathbb{R}^2$

As we have already mentioned, there is no proof of the Lax-Phillips conjecture for isotropic elasticity at the present state of the art—it may be possible that it is not even true in this situation, as was discussed in Chapter 10. Thus we are not able to prove the equivalence of the different non-trapping conditions for a large class of obstacles as in the case of the wave equation. Nevertheless, the existence of a Straussian vector field for a large class of obstacles in $\mathbb{R}^2$ according to [M/R/S 77] and the Gaussian beam construction allow the proof of such an equivalence, though under a stronger restriction than the one for the wave equation. To derive this result we are going to start with the geometric assumptions of [M/R/S 77] and list a number of implications. At some steps, we have to impose some additional conditions on the problems under consideration. Summing up all the restrictions we will have made so far, we will achieve an equivalence for (El.D) in the case of homogeneous media.

According to [M/R/S 77] we make the assumption

(i) The obstacle B is snake-shaped.

Therefore, a Straussian vector field l exists. We now assume in addition

(ii) The estimates (9.3) hold.

According to our Theorem 9.5.3 this implies Melrose's NTC for (El.D) ($l(x) \cdot \xi$ provides an escape function (El)) and, therefore, all the other non-trapping conditions discussed in Sections 11.1 and 11.2. All these conditions imply uniform LED by the Vainberg method. We assume next

(iii) The ratio λ/μ is constant throughout Ω.

As we have seen in Chapter 10 uniform LED guarantees now the absence of rays in $\overline{\Omega}$ which are perpendicular to $\partial\Omega$ at two points, and have no further contact with $\partial\Omega$. To close the circle of these implications we need to have control about straight line segments. But for this purpose we have to assume

(iv) The medium is homogeneous.

Thus the above rays are straight line segments. Unfortunately this does not rule out the existence of such segments that are in addition tangent to $\partial\Omega$ at interior points, as was discussed in Chapter 10. If these cases are excluded, we achieve the announced equivalence for homogeneous media, because the homogeneity already implies (ii) and (iii). This proves

Theorem 11.3.1 *Assume that $\Omega \subset \mathbb{R}^2$, the curvature of $\partial\Omega$ does not change sign infinitely often, and no straight line segment in $\overline{\Omega}$ which is tangent to the boundary in at least one point makes perpendicular incidence with $\partial\Omega$ at both ends. Then for the Dirichlet problem of homogeneous isotropic elasticity the following properties are equivalent:*

(i) There are no straight line segments in $\overline{\Omega}$ which are perpendicular to $\partial\Omega$ at two points.

(ii) The existence of a Straussian vector field.

(iii) The existence of an escape function (El).

(iv) Melrose's NTC for (El.D).

(v) Vainberg's NTC.

(vi) The NTC for the weak solution.

(vii) The propagation of singularities hypothesis of Rauch.

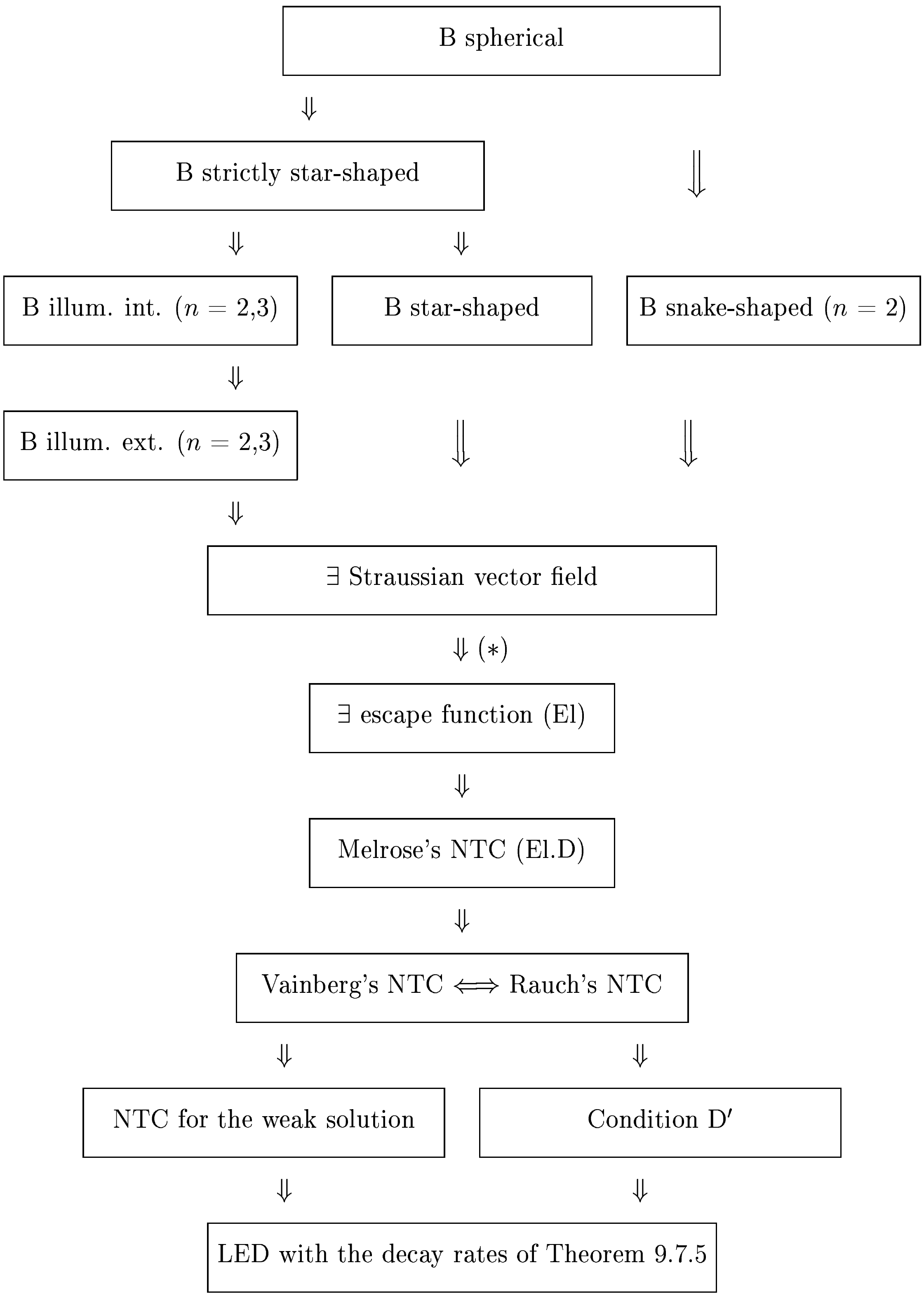
(viii) Vainberg's condition D′.

(ix) The existence of uniform LED.

11.4 Summary

In this section we just want to summarize graphically the results of Part II, as far as the relations between the different conditions on the obstacle and on the medium are concerned. We include the relations between the explicit geometric conditions already summarized in Section 3.3. So far we know:

The Relations between the Conditions for Uniform LED



Remember that the implication (∗) only holds for suitable media, i.e. if (9.3) is fulfilled.

We have, therefore, many conditions at hand which are sufficient for uniform local energy decay. On the other hand Ralston showed that localized solutions can be constructed using Gaussian beams. This leads at least to the necessary conditions of Theorem 10.0.6. But this result is too weak to achieve equivalences similar to Part I. However, as we have seen in Section 11.3, for a homogeneous medium in $\mathbb{R}^2$ this suffices to prove the equivalence of all non-trapping conditions—including the existence of escape functions—for a large class of obstacles.

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