On the Relationship of Information Processes and Asset Price Processes[1]

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Abstract

Asset price processes are completely described by information processes and investors' preferences. In this paper we derive the relationship between the process of investors' expectations of the terminal stock price and asset prices in a general continuous time pricing kernel framework. To derive the asset price process we make use of the modern technique of forward-backward stochastic differential equations. With this approach it is possible to show the driving factors for stochastic volatility of asset prices and to give theoretical arguments for empirically well documented facts. We show that stylized facts that look at first hand like financial market anomalies may be explained by an information process with stochastic volatility.

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1 Introduction

In the last decades much empirical work has been done on the time series of asset prices. Many studies report mean reversion in stock returns [see Fama and French [7]; Poterba, Summers [23]], predictability of the equity premium and other "anomalies". Empirical research on options suggests significant mispricing compared to theoretical option prices, especially compared to the Black-Scholes model [see Canina, Figlewski [4]; Ghysels, Harvey, Renault [10]; Buraschi, Jackwerth [3]]. Most of these well documented facts still lack a sound theoretical explanation. While the smile effect can be explained with stochastic volatility models there is, for example, no model which derives the randomness of volatility. Thus, usually a somewhat arbitrary volatility process is introduced.

Many theoretical papers have already investigated the viability of stochastic processes for asset prices, i.e. the consistency with an equilibrium. The usual approach is to start with a stochastic process for stock prices and to check whether this stochastic process can be an equilibrium process [see Bick [1]; Bick [2]; He, Leland [11]; Pham, Touzi [22]]. Franke, Stapleton, Subrahmanyam [9] choose a slightly different approach to investigate the viability of asset price processes. Instead of starting with the stochastic process of asset prices, they take the process of investors' expectations of the terminal asset price as given. Since an asset price is completely described by the distribution of its cash flow and by investors' preferences it is possible to construct any viable asset price process from the characteristics of information processes and preferences. By the assumption of rational investors it is possible to impose restrictions on the process representing investors' expectations, i.e. the information process. Hence, with the information process the distribution of the cash flow is given and from the assumptions on investors' preferences the characteristics of the pricing kernel are given, too. Thus, the asset price process can be derived from the underlying assumptions.

In this paper we follow the approach of Franke, Stapleton, Subrahmanyam. While Franke, Stapleton, Subrahmanyam emphasize the importance of the utility function or more precisely the elasticity of the pricing kernel our task is to show the influence of the variations in expectations, i.e. the influence of the volatility of the information process on the asset price process. We extend their approach in that we allow for a second risk factor driving the process of investors' expectations, i.e. the volatility of the information process may be stochastic and we give an economic justification for the generalization. We are arguing that introducing stochastic volatility of the information process is a sensible assumption. To see this consider a stochastic process with only one risk factor, e.g. the geometric Brownian motion. In this case, the uncertainty about the stock price in T is an only time dependent deterministic function. It is sensible to assume, that this uncertainty may also be a stochastic function since this uncertainty is driven by exogenous shocks. Unexpected news announcements may be seen as one of these exogenous shocks. We will turn to this point again in section 3.

With our approach we are able to link explicitly financial markets phenomena to the process of investors' expectations. We will show that many properties of asset price processes and especially empirically documented properties of the risk premia can be explained by the characteristics of the volatility of the information process. Further, we give an economic justification for stochastic volatility asset models and we discuss the justification of specifications of stochastic volatility by relating them to the process of investors' expectations.

The organization of this paper is as follows. The next section gives a short review on related papers. In section 3 we discuss the viability of information processes under the assumption of rational expectations. In section 4 we derive viable asset price processes with the modern technique of forward-backward stochastic differential equations (FBSDE). In section 5 we give some characterizations of the pricing kernel. Section 6 summarizes the main results.

2 A Short Survey on Related Papers

Before the seminal paper of Huang [15] continuous time models in finance were already prevalent. The usual assumption was that the equilibrium asset prices can be represented by Itô integrals. Huang was the first to give a sound theoretical justification for this assumption. One main result in his paper is that if equilibrium asset prices are adapted to a filtration generated by a Brownian motion, then equilibrium asset prices are Itô integrals. Thus Huang provided a justification for continuous sample paths of equilibrium asset prices by linking them to the information flow.

The studies in the 90's on the foundation of equilibrium asset price processes addressed the question which characteristics of the state price density can be supported by sensible assumptions on the utility function of a representative agent. Connected to this was the question, which utility functions are implied by equilibrium asset prices which are governed by specific stochastic differential equations. Bick [2] characterizes processes as viable by the "no-trade criteria", i.e. an asset price process is a possible equilibrium if there exists a von Neumann-Morgenstern utility function such that it is optimal for the representative agent to buy the market portfolio in $t = 0$ and hold it until T. Bick requires path-independence of the pricing kernel for viability. Ensuing papers of He, Leland [11], Hodges, Carverhill [13] and Hodges, Selby [12] generalize this analysis further.

Pham and Touzi [22] tackle the case of stochastic volatility. They provide utility-theoretic foundations for common assumptions on the risk premia in stochastic volatility models. Their analysis is similar to the previously mentioned, as they start with stochastic differential equations for the asset prices. The main results of their paper are necessary and sufficient conditions for the viability of the risk premia. Of special interest may be their analysis of the classical stochastic volatility model of Hull and White [14] and the concept of a minimal martingale measure introduced by Föllmer and Schweizer [8]. Hull and White were the first to derive an explicit formula for the price of an European option written on an asset with stochastic volatility. Yet their result crucially depends on the assumption that the volatility risk premium is independent of the underlying asset. Since in an incomplete market the equivalent martingale measure and, thus, the risk premia are not uniquely determined by arbitrage arguments, some restriction has to be imposed on the risk premia. The analysis of Pham and Touzi establishes that this kind of volatility risk premium is consistent with constant relative risk aversion. For the specification of the equivalent martingale measure in incomplete markets Föllmer and Schweizer [8] propose the concept of a minimal martingale measure. Loosely stated the minimal martingale measure is defined such that only traded risk is priced, hence, risk that is uncorrelated with traded assets has a price of zero. As intuition suggests this kind of equilibrium is supported by logarithmic preferences.

The analysis of Franke, Stapleton and Subrahmanyam [9] differs in various ways from the former papers. First, they do not assume the existence of a representative investor, instead they simply assume that markets do not admit arbitrage possibilities and hence, a pricing kernel exists. Second, they do not take the asset price process as given. Their approach is more fundamental as the basis of the model is a process for conditional expectations of the exogenously given asset price at some terminal date. From the assumption of rational investors they deduce the martingale property of the process of conditional expectations. With the assumption that the process of conditional expectations is governed by a geometric Brownian motion without drift their analysis establishes a strong relationship between the process of conditional expectations and the asset price process. In particular, they show that the asset price process follows a geometric Brownian motion if conditional expectations follow a geometric Brownian motion without drift and the pricing kernel has constant elasticity. They also derive properties of the price process for a pricing kernel with declining elasticity. In this case asset returns are autocorrelated and the variance of the asset price is higher than with constant elasticity of the pricing kernel.

3 Characterization and Viability of Information Processes

In the market under consideration we have a given time horizon $T > 0$ and the two dimensional standard Brownian motion $W = \left\{ \left(W_t^I, W_t^V \right) : t \in [0,T] \right\}$ on a given probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ where $(\mathcal{F}_t)_{t\ \in[0,T]}$ is the usual filtration generated by W with $\mathcal{F} = \mathcal{F}_T$. We define an information process I_t on the probability space. This process is assumed to represent investors' expectations about the exogenously given square integrable random value of an asset (which may be the market portfolio) at some terminal date T. Since investors are assumed to act totally rational, I_t is the process of conditional expectations of the value of the asset at date T. Hence, I_t is a P-martingale. Further, since the value of the asset at any time is strictly positive, the process I_t is strictly positive, too. Hence, I_t is a positive P-martingale and admits the following representation (see for example Karatzas and Shreve [17]).

$$I_t = I_0 + \int_0^t I_s \left[\sigma_s^I \, dW_s^I + \sigma_s^{I,V} \, dW_s^V \right] \; , \quad 0 \le t \le T. \tag{1}$$

The martingale representation theorem provides that there exist two processes σ_t^I and $\sigma_t^{I,V}$. By this theorem one only knows that these processes are adapted to $\mathcal{F}_t$ and that $P\left(\int_0^T \left(\sigma_s^I\right)^2 ds < \infty\right) = 1$ and $P\left(\int_0^T \left(\sigma_s^{I,V}\right)^2 ds < \infty\right) = 1$. In the following we make the assumption $\sigma_t^{I,V} = 0$ for all $t \in [0,T]$ and we require a special characterization of σ_t^I. Of course, with these assumptions we assume a special representation of I_t.

In the remainder of this paper we assume that the volatility σ_t^I of the information process is governed by the following stochastic differential equation:[1]

$$\sigma_t^I = \sigma_0^I + \int_0^t b(s, I_s, \sigma_s^I)\, ds + \int_0^t \sigma^V(s, I_s, \sigma_s^I)\, dW_s^V \ , \quad 0 \le t \le T, \qquad (2)$$

where b is the drift and σ^V describes the volatility of σ_t^I (these two functions are assumed to be deterministic). Since I_t represents investors' expectations in t about the value of the asset in T, the process I_t and the price process of the asset must be equal at time T. Thus, by definition $F_T = I_T$. Hence, with the information process I_t the distribution of the asset at date T is given. Equation (1) is a generalization of the information process considered in Franke, Stapleton, Subrahmanyam [9]. While their information process is modelled as a geometric Brownian motion, equation (1) admits constant, time varying, deterministic or stochastic volatility.

In the remainder of this paper $\sigma\left\{I_u \middle| 0 \le u \le t\right\}$ represents the filtration generated by I_t which by assumption represents all the information available to investors in t. We will now turn to the economic meaning of different volatility models. First take the case of constant volatility, i.e. $\sigma_t^I = \sigma_0^I$ for all $t \in [0, T]$. Hence, the logarithm of I_T is normally distributed with expectation $E^P\left[\ln I_T | \sigma\left\{I_u | \, 0 \le u \le t\right\}\right] = \ln(I_t) - \frac{\left(\sigma_0^I\right)^2}{2}(T-t)$ and variance[2] $var\left[\ln I_T | \sigma\left\{I_u | \, 0 \le u \le t\right\}\right] = \left(\sigma_0^I\right)^2 (T-t)$ for $0 \le t \le T$. Since the price of the asset in T is equal to I_T this implies that the uncertainty about the final value of the asset is a linearly decreasing function of time.

This constant rate of uncertainty resolution over time implies some special information flow. The intensity of information arrival must be constant over time. Deterministic but time varying volatility would allow for periods with a more intense information flow, but uncertainty resolution is still a deterministic function of time. Such deterministic time patterns might be explained by some sort of clustering of the information flow; companies announcing their results in certain periods, e.g. at the end of a year, many macroeconomic announcements such as monthly economic information releases occur at certain week-days (see Ederington and Lee [5] for a related

[1] The special characterization of the volatility relies on the following economic arguments.

[2] This is equal to $\int_t^T \left(\sigma_0^I\right)^2 ds$

econometric study). These facts may explain to some degree a deterministic time pattern of the volatility of the information process. Hence, under the assumption of time varying volatility the conditional variance of I_T is no more linear in t. But the volatility and the process of conditional variances of I_T can still be perfectly forecasted.

Since the information process is governed by such scheduled information releases but also by unforeseen information events we consider some randomness in the volatility of the information process to get a more realistic model for the conditional expectations. First assume that volatility is a borel function of t, I_t and σ_t^I, hence it is a deterministic function of t, I_t and σ_t^I:

$$\sigma_t^I = \sigma_0^I + \int_0^t b(s, I_s, \sigma_s^I)\, ds\ , \quad 0 \leq t \leq T. \tag{3}$$

This formulation of the volatility is, for example, consistent with a leverage effect, i.e. volatility increases with decreasing asset value. The easiest way to model the leverage effect, is to assume a constant elasticity of variance model (CEV)

$$\sigma_t^I = \overline{\sigma}\ I_t^{-\alpha}\ , 0 \leq t \leq T, \text{ for some } \alpha \text{ with } 0 \leq \alpha \leq 1 \text{ and } \overline{\sigma} > 0 \text{ constant.}$$

The model [equation (3)] includes all variations of the volatility of the information process which can be described by deterministic functions of t, I_t and σ_t^I. Notice that with our model [equation (3)] σ_t^I is random since it is a function of I_t, but it is $\sigma\{I_u|\, 0 \leq u \leq t\}$-measurable, thus the current volatility is known. Economically this means that the current (short-term or myopic) risk is known but the long-term risk evolves stochastically over time.

Even this more general model neglects some kind of uncertainty. Many news about the economy or politics as well as about markets and companies are published completely erratically so that stochastic terms have to be considered explicitly in the volatility process. Therefore to include these facts in the model, the volatility is governed by a separate stochastic differential equation with a stochastic term[equation (2)]. It will be obvious from Theorem 1 that modeling the volatility of the information process by a separate stochastic differential equation with a stochastic term has an important effect on the asset price process. With this model, the volatility risk of the information process is priced, i.e. a risk premium is paid. Hence, all other variations in the volatility of the information process are not priced.

We can conclude that the volatility of the information process is stochastic and exhibits some time pattern. The quantification of these facts is an empirical task and is closely related to the estimation of the volatility of asset prices. In the next section we show the close relationship between information processes and asset price processes.

4 Derivation of Asset Price Processes

In this section we derive the forward price process of the asset. Assuming that the market admits no arbitrage possibilities, it is well known that the forward price of any asset is given by

$$F_{t,T} = E^{P}\left(F_{T,T}\ \Phi_{t,T} | \sigma\left\{I_u | \, 0 \leq u \leq t\right\}\right) \ , \quad 0 \leqslant t \leqslant T, \tag{4}$$

where $F_{t,T}$ is the forward price at date $t \in [0,T]$ with delivery at date T and $\Phi_{t,T}$ is the pricing kernel which is just another representation of the fact that the absence of arbitrage opportunities implies the existence of a probability measure $\widetilde{P}$, equivalent to the objective probability measure P, under which the forward price process is a martingale. Because of the equality of $F_{T,T}$ and I_T the following relationship holds[3]

$$F_t = E^{\widetilde{P}}[F_T \, | \sigma\left\{I_u | \, 0 \leq u \leq t\right\}] = E^{\widetilde{P}}[I_T \, | \sigma\left\{I_u | \, 0 \leq u \leq t\right\}] \ , \quad 0 \leqslant t \leqslant T.$$

We assume that the transformation from P to $\widetilde{P}$ is given by a Girsanov-functional. More precisely we assume that there is an adapted $\mathbf{R}^2$-valued process $\lambda_s = (\lambda_s^1, \lambda_s^2)$ which defines the martingale[4]

$$\begin{aligned} \Phi_{0,t} \ &= \ \exp\left(-\int_0^t \lambda_s^1 dW_s^I - \int_0^t \lambda_s^2 dW_s^V - \frac{1}{2}\int_0^t \|\lambda_s\|^2 \, ds\right), \\ & \quad 0 \leqslant t \leqslant T, \end{aligned} \tag{5}$$

and the transformed probability measure

$$\widetilde{P}(A) = E[\Phi_{0,T} 1_A] \ , \qquad A \in \mathcal{F}_T.$$

With this definition P and $\widetilde{P}$ are mutually absolutely continuous on $\mathcal{F}_T$ and the process

$$\begin{pmatrix} \widetilde{W_t^I} \\ \widetilde{W_t^V} \end{pmatrix} := \begin{pmatrix} W_t^I + \int_0^t \lambda_s^1 ds \\ W_t^V + \int_0^t \lambda_s^2 ds \end{pmatrix}, \qquad 0 \leqslant t \leqslant T,$$

[3] In the remainder of this paper the second index (T) of the forward price ($F_{t,T}$) is omitted for notational convenience.

[4] $\| \cdot \|$ is the euclidean $\mathbf{R}^2$-norm.

is a 2-dimensional Brownian motion under $\widetilde{P}$. Hence, we have the representation for F under the probability measure P

$$F_t = I_T - \int_t^T \lambda_s^1 Z_s^1 ds - \int_t^T \lambda_s^2 Z_s^2 ds - \int_t^T Z_s^1 \, dW_s^I - \int_t^T Z_s^2 \, dW_s^V$$

for $0 \leqslant t \leqslant T$, where $Z = (Z^1, Z^2)$ is the process given by application of the martingale representation theorem on F. We assume that λ is a smooth deterministic function that may depend on t, I_t and σ_t^I: $\lambda\left(t, I_t, \sigma_t^I\right)$. The process $\lambda\left(t, I_t, \sigma_t^I\right)$ is the market price of risk (see Musiela and Rutkowski [20] or Karatzas and Shreve [16]).

In the following theorem we give a formula for the forward price F_t in terms of the information process and the market price of risk $\lambda\left(t, I_t, \sigma_t^I\right)$.

Theorem 1 *Assume that the information process I_t is governed by the stochastic differential equation*

$$I_t = I_0 + \int_0^t I_s \sigma_s^I \, dW_s^I \ , \quad 0 \leqslant t \leqslant T,$$

where the volatility process σ_t^I is given by

$$\sigma_t^I = \sigma_0^I + \int_0^t b(s, I_s, \sigma_s^I) \, ds + \int_0^t \sigma^V(s, I_s, \sigma_s^I) \, dW_s^V \ , \quad 0 \leqslant t \leqslant T,$$

with deterministic smooth functions b and σ^V. Then the forward price F_t of the asset admits the following representation under the probability measure P

$$\begin{aligned} F_t = I_T - \int_t^T \lambda^1(s, I_s, \sigma_s^I) Z_s^1 ds - \int_t^T \lambda^2(s, I_s, \sigma_s^I) Z_s^2 ds \\ - \int_t^T Z_s^1 \, dW_s^I - \int_t^T Z_s^2 \, dW_s^V, \\ 0 \leqslant t \leqslant T, \end{aligned}$$

with

$$Z_t = \begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix} = \begin{pmatrix} I_t \sigma_t^I & 0 \\ 0 & \sigma^V(t, I_t, \sigma_t^I) \end{pmatrix} \nabla u(t, I_t, \sigma_t^I) \quad , 0 \leqslant t \leqslant T,$$

and

$$\bigtriangledown u(t,x) = (u_{x_1}(t,x), u_{x_2}(t,x))^T$$

where $u(t,x)$ is the solution of the partial differential equation for $0 \leqslant t \leqslant T$

$$\begin{aligned} 0 &= u_t(t,x_1,x_2) + \lambda^1(t,x_1,x_2)\; x_1 x_2\, u_{x_1}(t,x_1,x_2) \qquad (6)\\ &\quad +\lambda^2(t,x_1,x_2)\,\sigma^V(t,x_1,x_2)\,u_{x_2}(t,x_1,x_2)\\ &\quad +\frac{1}{2}\left(x_1^2 x_2^2\, u_{x_1x_1}(t,x_1,x_2) + \left(\sigma^V(t,x_1,x_2)\right)^2 u_{x_2x_2}(t,x_1,x_2)\right)\\ &\quad +b(t,x_1,x_2)\,u_{x_2}(t,x_1,x_2) \qquad (7)\\ u(T,x) &= x_1 \end{aligned}$$

with indices on the function u indicating partial derivatives.
Moreover F_t is given by

$$F_t = u(t, I_t, \sigma_t^I) \quad , 0 \leqslant t \leqslant T.$$

Proof. We have the following system:
the forward stochastic differential equation $(FSDE)$ for the information process I_t and its volatility process σ_t^I

$$\begin{aligned} \begin{pmatrix} I_t \\ \sigma_t^I \end{pmatrix} &= \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_0^1 \\ X_0^2 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ b(s, X_s^1, X_s^2) \end{pmatrix} ds \qquad (8)\\ &\quad + \int_0^t \underbrace{\begin{pmatrix} X_s^1 X_s^2 & 0 \\ 0 & \sigma^V(s, X_s^1, X_s^2) \end{pmatrix}}_{\sigma} \begin{pmatrix} dW_s^I \\ dW_s^V \end{pmatrix} \quad , \quad 0 \leqslant t \leqslant T, \end{aligned}$$

and the backward stochastic differential equation $(BSDE)$ for the forward price F_t of the asset

$$\begin{aligned} F_t &= Y_t = X_T^1 - \int_t^T \left(\lambda^1\left(s, X_s^1, X_s^2\right) Z_s^1 + \lambda^2\left(s, X_s^1, X_s^2\right) Z_s^2\right) ds \qquad (9)\\ &\quad - \int_t^T \begin{pmatrix} Z_s^1 \\ Z_s^2 \end{pmatrix}^T \begin{pmatrix} dW_s^I \\ dW_s^V \end{pmatrix} \quad , \quad 0 \leqslant t \leqslant T. \end{aligned}$$

The coupled system (8) and (9) is a forward-backward stochastic differential equation. We use the Four-Step-Scheme given in Ma, Protter, Yong [18] to find the solution.[5]

[5] More precisely we apply the version given in Ma, Yong [19]. An outstanding overview on BSDE and FBSDE is given in El Karoui et al. [6].

Step 1: We define the function $z(t,x,y,w) = \sigma^T(t,x,y)w$ for $(t,x,y,w) \in \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R} \times \mathbf{R}^2$. With this definition we have the 2-dimensional function

$$z(t,x,y,w) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} (t,x,y,w) = \begin{pmatrix} x_1 x_2 \, w_1 \\ \sigma^V(t,x_1,x_2) \, w_2 \end{pmatrix}.$$

Step 2: With the function z we solve the partial differential equation (PDE)[6]

$$\begin{aligned} u(t,x) &= x_1 + \int_t^T \lambda^1 (s,x_1,x_2) \;\; z_1 (s,x,u(s,x), \bigtriangledown u(s,x)) \\ &\quad + \lambda^2 (s,x_1,x_2) \;\; z_2 (s,x,u(s,x), \bigtriangledown u(s,x)) \\ &\quad + \frac{1}{2} tr \left\{ \begin{pmatrix} x_1 x_2 & 0 \\ 0 & \sigma^V(s,x) \end{pmatrix}^2 \begin{pmatrix} u_{x_1 x_1} & u_{x_1 x_2} \\ u_{x_2 x_1} & u_{x_2 x_2} \end{pmatrix} (s,x) \right\} \\ &\quad + \left\langle \begin{pmatrix} 0 \\ b(s,x) \end{pmatrix}, \begin{pmatrix} u_{x_1}(s,x) \\ u_{x_2}(s,x) \end{pmatrix} \right\rangle ds \;, \quad 0 \leqslant t \leqslant T. \end{aligned}$$

Hence, we have for $0 \leqslant t \leqslant T$

$$\begin{aligned} 0 &= u_t(t,x) + \lambda^1 (t,x) \; x_1 x_2 \, u_{x_1}(t,x) + \lambda^2 (t,x) \, \sigma^V(t,x) \, u_{x_2}(t,x) \\ &\quad + \frac{1}{2} \left(x_1^2 x_2^2 \, u_{x_1 x_1}(t,x) + \left(\sigma^V(t,x)\right)^2 \, u_{x_2 x_2}(t,x) \right) + b(t,x) \, u_{x_2}(t,x) \\ u(T,x) &= x_1. \end{aligned}$$

We jump directly to *Step 4,* omitting *Step 3*: We define $Y_t = u(t,X_t)$ and $Z_t = \sigma^T(t,X_t,u(t,X_t)) \bigtriangledown u(t,X_t)$, then (X,Y,Z) is an adapted solution of (8) and (9). ■

Theorem 1 shows the close relationship between asset price processes and information processes. Given the information process and the pricing kernel Theorem 1 establishes the representation of the asset price process as a function of the information process and the pricing kernel. The drift of the asset price process is governed by the market price of risk λ_t and the diffusion of the asset price process Z_t. The diffusion Z_t depends on the information process I_t itself, on the volatility process of I_t, i.e. σ_t^I, and on the first derivatives of the asset price w.r.t. I_t and σ_t^I. Thus, with Theorem 1 we have an explicit representation of the asset price process in terms of the

[6] By $tr\{\cdot\}$ we denote the trace of a 2×2-matrix and by $\langle \cdot, \cdot \rangle$ the inner product of the Euclidean space $\mathbf{R}^2$.

information process and the pricing kernel for a 2-dimensional market model. Theorem 1 is also applicable to n-dimensional market models.

It is obvious from Theorem 1 that the drift of the asset price process depends on the volatility of the information process. Thus, empirical studies implicitly assuming non stochastic volatility of the information process may find unexplainable variations in the drift. Further, neglecting stochastic volatility of the information process leads to only one risk premium in the asset price process, i.e. $\sigma_t^V = 0$ provides $Z_t^2 = 0$.

To gain some better understanding of the implications of Theorem 1 in the remainder of this section we discuss the case when all coefficients are only functions in t. Thus, with this assumption equation (6) simplifies to

$$\begin{aligned} 0 &= u_t(t,x_1,x_2) + \lambda^1(t)\; x_1 x_2\, u_{x_1}(t,x_1,x_2) \\ &\quad +\lambda^2(t)\,\sigma^V(t)\,u_{x_2}(t,x_1,x_2) \\ &\quad +\frac{1}{2}\left(x_1^2 x_2^2\, u_{x_1x_1}(t,x_1,x_2) + \left(\sigma^V(t)\right)^2\, u_{x_2x_2}(t,x_1,x_2)\right) \\ &\quad +b(t)\,u_{x_2}(t,x_1,x_2)\;, \quad 0 \leqslant t \leqslant T, \\ u(T,x) &= x_1 \end{aligned} \tag{10}$$

and the solution of equation (10) for $u(t,x)$ is (see Appendix)

$$\begin{aligned} u(t,x_1,x_2) &= x_1 \exp\left(x_2 \int_t^T \lambda^1(s)ds\right) \\ &\quad \exp\left(\int_t^T \left(\lambda^2(r)\,\sigma^V(r) + b(r)\right)\left(\int_r^T \lambda^1(s)ds\right) dr\right) \\ &\quad \exp\left(\int_t^T \frac{1}{2}\left(\sigma^V(r)\right)^2 \left(\int_r^T \lambda^1(s)ds\right)^2 dr\right)\;, \quad 0 \leqslant t \leqslant T. \end{aligned}$$

Hence, F_t is

$$\begin{aligned} F_t &= I_t \exp\left(\sigma_t^I \int_t^T \lambda^1(s)ds\right) \\ &\quad \exp\left(\int_t^T \left(\lambda^2(r)\,\sigma^V(r) + b(r)\right)\left(\int_r^T \lambda^1(s)ds\right) dr\right) \\ &\quad \exp\left(\int_t^T \frac{1}{2}\left(\sigma^V(r)\right)^2 \left(\int_r^T \lambda^1(s)ds\right)^2 dr\right)\;, \quad 0 \leqslant t \leqslant T, \end{aligned}$$

and

$$Z_t = \begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix} = \begin{pmatrix} I_t \sigma_t^I & 0 \\ 0 & \sigma^V(t) \end{pmatrix} \triangledown u(t, I_t, \sigma_t^I) \quad , 0 \leqslant t \leqslant T,$$

with

$$\triangledown u(t, I_t, \sigma_t^I) = F_t \begin{pmatrix} 1/I_t \\ \int_t^T \lambda^1(s) ds \end{pmatrix} \quad , 0 \leqslant t \leqslant T.$$

Then the forward price of the asset is governed by

$$\begin{aligned} dF_t &= F_t \sigma_t^F \left[\sqrt{1-\rho_t^2} \lambda^1(t) + \lambda^2(t) \rho_t \right] dt \\ &\quad + F_t \sigma_t^F \left[\sqrt{1-\rho_t^2} dW_t^I + \rho_t \, dW_t^V \right] , \quad 0 \leqslant t \leqslant T, \qquad (11) \\ F_T &= I_T \end{aligned}$$

with

$$F_t \, \sigma_t^F := \sqrt{(Z_t^1)^2 + (Z_t^2)^2}$$

$$\rho_t := \frac{Z_t^2}{\sqrt{(Z_t^1)^2 + (Z_t^2)^2}}.$$

ρ is the instantaneous correlation between the asset price and its volatility. Equation (11) can then be rewritten in the usual notation[7] as

$$dF_t = F_t \sigma_t^F \left[\sqrt{1-\rho_t^2} \lambda^1(t) + \lambda^2(t) \rho_t \right] dt + F_t \sigma_t^F \; dW_t^F \;, \quad 0 \leqslant t \leqslant T, \quad (12)$$

where W^F and W^V have correlation ρ.

It is important to notice that even though the information process and its volatility process are uncorrelated, the asset price process and its volatility process are correlated. This is in contrast to a usual assumption in stochastic volatility models (see for example Hull, White [14] or Stein, Stein [24]) and has already been criticized by Pham and Touzi [22]. Corollary 1 establishes conditions for $\rho = 0$.

Corollary 1 *The correlation ρ is zero if and only if*
(i) $\sigma_t^V = 0$
or
(ii) $\lambda^1 = 0$ *a.s.*

[7] We make use of $dW_t^F = \sqrt{1-\rho_t^2} dW_t^I + \rho_t \, dW_t^V$.

Condition (i) is trivial since it implies that the volatility of the information process is not governed by a Brownian motion. Condition (ii) implies that the correlation ρ is zero, if λ^1, the risk premium relative to the source of uncertainty W^I, is zero.

With equation (12) we can give an explanation for the well documented time pattern of the adjusted drift $\frac{\mu_t}{\sigma_t^F}$:

$$\frac{\mu_t}{\sigma_t^F} := \sqrt{1-\rho_t^2}\lambda^1(t) + \lambda^2(t)\rho_t \ , \quad 0 \leqslant t \leqslant T. \tag{13}$$

This equation shows that the adjusted drift may not be constant even if the risk premia are constant, i.e. $\lambda^1(t) = const.$, $\lambda^2(t) = const.$ in equation (13). With risk premia being constant the variation in the adjusted drift solely stems from the variations of the correlation process ρ. Neglecting these effects by assuming a constant correlation ρ between W^F and W^I may be one reason for seemingly unexplainable variations of the risk premia.

To illustrate our results we now consider the case of time-dependent volatility of the information process. Hence, all coefficients depend on time only and the volatility of the information process is a function of time only, too.

Example 1 ***Time-dependent volatility***
Let σ_t^I depend on time only. Then the volatility of the information process satisfies

$$\sigma_t^I = \sigma_0^I + \int_0^t b(s)\, ds \ , \quad 0 \leq t \leq T.$$

Then, by Theorem 1, the forward price of the asset satisfies the stochastic differential equation

$$dF_t = F_t\lambda^1(t)\ \sigma_t^I\, dt + F_t\sigma_t^I\ dW_t^F \ , \quad 0 \leqslant t \leqslant T.$$

With this simplified model it is easily seen, that the properties of the volatility of the information process transfer to the properties of the drift and the volatility of the asset price process. The drift of the asset price process is equal to the volatility of the information process multiplied by the market price of risk. The volatility of the information process and the volatility of the asset price process are identical. Further simplifying the model by assuming *that the volatility of the information process is constant, i.e. the information process is governed by a geometric Brownian motion and assuming* λ^1 *constant, we get*

$$dF_t = F_t \lambda^1 \sigma^I \, dt + F_t \sigma^I \, dW_t^F \tag{14}$$

From equation (14) *it is obvious that the asset price follows a geometric Brownian motion process if the information process has constant volatility and the risk premium is constant.*

Remark 1 *As was shown in Franke, Stapleton, Subrahmanyam [9] equation* (14) *implies constant elasticity of the pricing kernel which implies constant proportional risk aversion in a representative agent economy. We will discuss this point in the following section.*

5 Characterization of the Pricing Kernel and the Risk Premia

In this section we give some characterizations of the pricing kernel in terms of the risk premia and volatilities. First notice that the pricing kernel given in equation (5) is governed by the stochastic differential equation:

$$\begin{aligned} d\Phi_{0,t} &= -\Phi_{0,t}\lambda^1(t, I_{t,}\sigma_t^I)dW_t^I - \Phi_{0,t}\lambda^2(t, I_{t,}\sigma_t^I)dW_t^V \ , \ 0 \le t \le T, \\ \Phi_{0,0} &= 1. \end{aligned}$$

Hence, we can compute the quadratic variation[8] of $\ln F_t$ and the cross-variation process of $\ln \Phi_{0,t}$ and $\ln F_t$

$$\begin{aligned} \langle \ln F \rangle_t &= \frac{(Z_t^1)^2 + (Z_t^2)^2}{F_t^2} \ , \ 0 \le t \le T, \\ \langle \ln \Phi_{0,\cdot}, \ln F \rangle_t &= \frac{-\lambda^1(t, I_{t,}\sigma_t^I)Z_t^1 - \lambda^2(t, I_{t,}\sigma_t^I)Z_t^2}{F_t} \ , \ 0 \le t \le T. \end{aligned}$$

Thus, the instantaneous η defined as $\eta_t := -\langle \ln \Phi_{0,\cdot}, \ln F \rangle_t / \langle \ln F \rangle_t$ is

$$\eta_t = \frac{\lambda^1(t, I_{t,}\sigma_t^I)Z_t^1 + \lambda^2(t, I_{t,}\sigma_t^I)Z_t^2}{(Z_t^1)^2 + (Z_t^2)^2} F_t \ , \ 0 \le t \le T. \tag{15}$$

[8] In the following $\langle \cdot \rangle_t$ denotes the quadratic variation which in economics is known as the instantaneous variance, similar for the cross variation $\langle \cdot, \cdot \rangle_t$ and the instantaneous covariance.

η defined in this way is the beta of a standard linear regression model. Since η is the ratio of the relative change of the pricing kernel Φ and the relative change of the asset price F with respect to different states at the same time t it is an approximation of the (instantaneous) elasticity of the pricing kernel with respect to the asset price. From equation (15) it is easily seen, that η is equal to the instantaneous drift divided by the instantaneous variance. First, let us turn back to Example 1. In Example 1 we derived the asset price process for constant volatility of the information process. We concluded, that the asset price follows a geometric Brownian motion if and only if the risk premia is constant. In this case η is

$$\eta = \frac{\lambda^1 \sigma^I}{\left(\sigma^I\right)^2}. \tag{16}$$

Example 1 combined with equation (16) is just another representation of Theorem 2 in Franke, Stapleton, Subrahmanyam [9].

The elasticity of the pricing kernel can also be derived in a mathematical way. This will be done under the assumptions of Example 1. It follows from representative agent models (see for example Bick [2]) that the pricing kernel Φ is a smooth function in t and F. Hence, by the Itô formula (the notation of the dependency of the processes on t and F is omitted)

$$d\Phi = \left(\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial F}\lambda\sigma^I F + \frac{1}{2}\frac{\partial^2\Phi}{\partial F^2}\left(\sigma^I\right)^2 F^2\right) dt + \frac{\partial\Phi}{\partial F}\sigma^I F\, dW^F.$$

On the other hand the pricing kernel is governed by $d\Phi = -\lambda^1\Phi\, dW^I$, and from above it is known that $W^I = W^F$. Comparing coefficients one derives the solvable system of differential equations

$$\frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial F}\lambda\sigma^I F + \frac{1}{2}\frac{\partial^2\Phi}{\partial F^2}\left(\sigma^I\right)^2 F^2 = 0 \tag{17}$$

$$-\lambda^1\Phi = \frac{\partial\Phi}{\partial F}\sigma^I F. \tag{18}$$

With equation (18) one easily gets the elasticity of the pricing kernel

$$-\frac{\partial\Phi(t,F)}{\partial F}\frac{F}{\Phi(t,F)} = \frac{\lambda^1}{\sigma^I}.$$

6 Conclusions

Different approaches to examine asset price processes exist. On the one hand, due to the availability of financial markets data, an enormous amount of time series properties have been well documented by recent empirical studies. On the other hand many theoretical papers establish necessary characteristics of asset price processes to be consistent with an equilibrium. Unfortunately the gap between empirically well documented stylized facts and theoretically explainable facts is still vast. The purpose of this paper was to show that this discrepancy between empirical and theoretical findings may result from the fact that one important source of risk is neglected. In this paper we have considered this risk, i.e. that even the risk of an asset is unknown and therefore risky, too. In this case the information process has stochastic volatility. With this generalization of usual information processes a foundation for stochastic volatility models of asset prices has been established. Further it has been shown that the still prevalent assumption of zero correlation between asset prices and their volatility is not sensible. Finally we have shown the close relationship between the volatility process of the information process and the risk premia of an asset. Because of this dependence of the risk premia on the volatility process a theoretical foundation can be given for financial market phenomena. Moreover we have pointed out in section 5 that these results hold even under the assumption of a path-independent pricing kernel, since path-independence of the pricing kernel has been assumed everywhere.

Our approach offers numerous avenues for future research. More research should be devoted to the information process and its volatility. Because of the established coherence between information processes and asset price processes this is no more a purely theoretical task. Characteristics of information processes can be deduced from asset price processes. Hence, it is possible to investigate empirically whether asset price processes are consistent with the strong assumptions usually made on information processes.

A Appendix

The PDE, with indices on the functions indicating partial derivatives,

$$
\begin{aligned}
0 &= u_t(t,x_1,x_2) + \lambda^1(t,x_1,x_2)\; x_1 x_2\, u_{x_1}(t,x_1,x_2) \\
&\quad + \lambda^2(t,x_1,x_2)\,\sigma^V(t,x_1,x_2)\,u_{x_2}(t,x_1,x_2) \\
&\quad + \frac{1}{2}\left(x_1^2 x_2^2\, u_{x_1 x_1}(t,x_1,x_2) + \left(\sigma^V(t,x_1,x_2)\right)^2 u_{x_2 x_2}(t,x_1,x_2)\right) \\
&\quad + b(t,x_1,x_2)\,u_{x_2}(t,x_1,x_2) \\
u(T,x) &= x_1
\end{aligned}
$$

can be solved for only-time-dependent coefficients.

For notational simplicity define the functions: $\alpha := \lambda^1$, $\beta := \lambda^2\sigma^V + b$, $\gamma := \sigma^V = \frac{\beta - b}{\lambda^2}$. Hence, the PDE can be written as

$$
\begin{aligned}
0 &= u_t(t,x_1,x_2) + \alpha(t,x_1,x_2)\; x_1 x_2\, u_{x_1}(t,x_1,x_2) + \beta(t,x_1,x_2)\,u_{x_2}(t,x_1,x_2) \\
&\quad + \frac{1}{2} x_1^2 x_2^2\, u_{x_1 x_1}(t,x_1,x_2) + \frac{1}{2}\gamma^2(t,x_1,x_2)\,u_{x_2 x_2}(t,x_1,x_2)
\end{aligned}
$$

with the boundary condition $u(T,x) = x_1$.

We have to consider the two cases $x_1 = 0$ and $x_1 \neq 0$. If $x_1 = 0$ we have the trivial solution. Thus, in the following we choose $x_1 \neq 0$.

Assume that the coefficients of the PDE do not depend on x_1. Then u can be separated as follows: $u(t,x_1,x_2) = \varphi(x_1,x_2)\psi(t,x_2)$. With the boundary condition $u(T,x_1,x_2) = x_1$ it follows $\varphi(x_1,x_2)\psi(T,x_2) = x_1$ and because of $x_1 \neq 0 : \psi(T,x_2) \neq 0$ for all x_2. Hence, the following relationship holds: $\varphi(x_1,x_2) = \frac{x_1}{\psi(T,x_2)}$.

After some computation the partial differential equation can be written as (without noting the variables of the functions α, β and γ)

$$
\begin{aligned}
0 &= \psi_t(t,x_2) + \psi(t,x_2)\left(\alpha x_2 - \beta\frac{\psi_{x_2}(T,x_2)}{\psi(T,x_2)} + \gamma^2\frac{\psi_{x_2}^2(T,x_2)}{\psi^2(T,x_2)} - \frac{\gamma^2}{2}\frac{\psi_{x_2 x_2}(T,x_2)}{\psi(T,x_2)}\right) \\
&\quad + \psi_{x_2}(t,x_2)\left(\beta - \gamma^2\frac{\psi_{x_2}(T,x_2)}{\psi(T,x_2)}\right) + \frac{\gamma^2}{2}\psi_{x_2 x_2}(t,x_2),
\end{aligned}
$$

without loss of generality choose the boundary condition $\psi(T,x_2) = 1$.

Now assume that all coefficients are only time-dependent.

In the case of constant volatility, that is $\sigma^V \equiv 0$ and $b \equiv 0$ and hence $\beta \equiv 0$ and $\gamma \equiv 0$, we have the following solution of the PDE

$$\psi(t, x_2) = \exp\left(x_2 \int_t^T \alpha(s)ds\right).$$

With this knowledge, for the general PDE with time-dependent coefficients we try the ansatz

$$\psi(t, x_2) = \exp\left(-x_2 \int_0^t \alpha(s)ds\right) A(t).$$

with $A(T) = 1$. This leads to an ordinary first-order differential equation for A which has the solution

$$A(t) = \exp\left(\frac{1}{2}\int_t^T 2\beta(r)\left(\int_r^T \alpha(s)ds\right) + \gamma^2(r)\left(\int_r^T \alpha(s)ds\right)^2 dr\right).$$

Hence, the solution of the PDE for u is

$$\begin{aligned} u(t, x_1, x_2) &= x_1 \exp\left(x_2 \int_t^T \alpha(s)ds\right) \\ &\quad \exp\left(\frac{1}{2}\int_t^T 2\beta(r)\left(\int_r^T \alpha(s)ds\right) + \gamma^2(r)\left(\int_r^T \alpha(s)ds\right)^2 dr\right) \end{aligned}$$

and in terms of the original coefficients

$$\begin{aligned} u(t, x_1, x_2) &= x_1 \exp\left(x_2 \int_t^T \lambda^1(s)ds\right) \\ &\quad \exp\left(\int_t^T \left(\lambda^2(r)\,\sigma^V(r) + b(r)\right)\left(\int_r^T \lambda^1(s)ds\right) dr\right) \\ &\quad \exp\left(\int_t^T \frac{1}{2}\left(\sigma^V(r)\right)^2 \left(\int_r^T \lambda^1(s)ds\right)^2 dr\right). \end{aligned}$$

References

[1] Bick, Avi (1987), On the Consistency of the Black-Scholes Model with a General Equilibrium Framework, Journal of Financial and Quantitative Analysis 22, 259-275.

[2] Bick, Avi (1990), On Viable Diffusion Price Processes of the Market Portfolio; Journal of Finance 45, 673-689.

[3] Buraschi, Andrea and Jackwerth, Jens (1999), Is Volatility Risk Priced in the Option Market? Empirical Evidence and Implications for Deterministic and Stochastic Option Pricing Models, working paper, London Business School.

[4] Canina, Linda and Figlewski, Stephen (1993), The Informational Content of Implied Volatility, Review of Financial Studies, Vol. 6, No. 3, 659-681.

[5] Ederington, Louis H. and Lee, Jae Ha (1993), How Markets Process Information: News Releases and Volatility, Journal of Finance 48 (4).

[6] El Karoui, N.; Peng, S. and Quenez, M. C. (1997), Backward stochastic differential equations in finance, Mathematical Finance, Vol.7, No.1, 1-71.

[7] Fama, Eugene F. and French, Kenneth R. (1988), Permanent and Temporary Components of Stock Prices, Journal of Political Economy 96 (2), 246-73.

[8] Föllmer, Hans and Schweizer, Martin (1991), Hedging of Contingent Claims under Incomplete Information, Applied Stochastic Analysis, Vol. 5, 389-414

[9] Franke, Günter; Stapleton, Richard C. and Subrahmanyam, Marti G. (1999), When are Options Overpriced? The Black-Scholes Model and Alternative Characterisations of the Pricing Kernel, European Finance Review 3(1), 79-102.

[10] Ghysels, Eric; Harvey, Andrew C. and Renault, Eric (1996), Stochastic Volatility, in: Handbook of Statistics, Vol. 14, 119-191.

[11] He, Hua and Leland, Hayne (1993), On Equilibrium Asset Price Processes, Review of Financial Studies, Vol. 6, No. 3, 593-617.

[12] Hodges, Stewart and Selby, Michael (1997), The Risk Premium in Trading Equilibria which Support Black-Scholes Option Pricing, in Michael Dempster, Stanley Pliska (ed.) Mathematics of Derivative Securities, Cambridge University Press.

[13] Hodges, Stewart and Carverhill Andrew (1993), Quasi Mean Reversion in an Efficient Stock Market: The Characterisation of Economic Equilibria which Support Black-Scholes Option Pricing, Economic Journal 103, 395-405.

[14] Hull, John C. and White, Alan (1987), The pricing of Options with Stochastic Volatilities, Journal of Finance 42, 281-300.

[15] Huang, Chi-Fu (1985), Information Structure and Equilibrium Asset Prices, Journal of Economic Theory 35, 33-71.

[16] Karatzas, Ioannis and Shreve, Steven E. (1999), Brownian Motion and Stochastic Calculus, Second Edition, Springer.

[17] Karatzas, Ioannis and Shreve, Steven E. (1998), Methods of Mathematical Finance, Springer.

[18] Ma, Jin ; Protter, Philip and Yong, Jiongmin (1994), Solving forward-backward stochastic differential equations explicitly - a four step scheme, Probab. Theory Relat. Fields 98, 339-359.

[19] Ma, Jin and Yong, Jiongmin (1997), Adapted solution of a degenerate backward spde, with applications, Stochastic Processes and their Applications 70, 59-84.

[20] Musiela, Marek and Rutkowski, Marek (1997), Martingale Methods in Financial Modelling, Springer.

[21] Pardoux, E. and Peng, S. (1992), Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations. In: Rozovskii, B. L. and Sowers, R. S. (eds), Stochastic partial differential equations and their applications (Lect. Notes Control inf. Sci. 176, pp. 200-217) Springer.

[22] Pham, Huyên and Touzi, Nizar (1996), Equilibrium State Prices in a Stochastic Volatility Model, Mathematical Finance, Vol.6, No.2 , 215-236.

[23] Poterba, James and Summmers, Lawrence (1988), Mean Reversion in Stock Prices, Journal of Financial Economics 22 (1), 27-59.

[24] Stein, Elias M. and Stein, Jeremy C.(1991), Stock Price Distributions with stochastic Volatility: An Analytic Approach, Review of Financial Studies, Vol. 4, No. 4, 727-752.