

ON THE DEFINABILITY OF RADICALS IN SUPERSIMPLE GROUPS

CÉDRIC MILLIET

Abstract. If G is a group with a supersimple theory having a finite SU -rank, then the subgroup of G generated by all of its normal nilpotent subgroups is definable and nilpotent. This answers a question asked by Elwes, Jaligot, Macpherson and Ryten. If H is any group with a supersimple theory, then the subgroup of H generated by all of its normal soluble subgroups is definable and soluble.

§1. Introduction. Among the problems in the model theory of groups, is the one of knowing which subgroups of a group G are definable by a formula. For example, the centraliser of an element a in G is defined by the quantifier free formula $xa = ax$ and the centre of G by $(\forall y) xy = yx$. Similarly, finite sets, centralisers of finite sets and iterated centres of G are always definable. But this is mostly the end of the list: almost every other characteristic subgroup such as the commutator subgroup G' , the FC -centre, the Fitting subgroup or the soluble radical may not be definable, not in first order logic at least: they all are countable union of definable sets. The situation is even more complicated for the iterated FC -centres, the FC -soluble radical or the FC -Fitting subgroup who have a higher complexity in the hierarchy of definable sets.

In an algebraic group over an algebraically closed field, every subgroup cited above is definable. The situation is far less straightforward in a group G which is merely stable. Wagner has shown that the Fitting subgroup of G is always definable [18]. The question is still open for the soluble radical of G , but Baudish [3] has proved that it is definable provided that G be superstable. The starting point of their investigation was a theorem of Poizat [14] that every nilpotent (respectively soluble) subgroup of G is contained in a definable nilpotent (respectively soluble) one of the same nilpotency class (resp. derived length). Recently, many attempts have been made to extend these results to a wider context: let us cite [17, Shelah] and [1, Aldama] for groups with dependent theory, [2, Altinel Baginski] for groups with the descending chain condition on centralisers, [12, Milliet] for groups with a simple theory and [6, Elwes Jaligot Macpherson Ryten], for supersimple groups, where it is shown that the soluble radical of a supersimple group G of finite rank is definable

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and soluble provided that G^{eq} eliminates \exists^∞ . Note that the formulation *supersimple group* is a short-hand for *group with a supersimple theory*, not a variation on *simple groups*. The authors of [6] also asked whether such a group G had a largest nilpotent normal subgroup F and if F would be definable. We give a positive answer here while proving that the Fitting subgroup of a supersimple group of finite SU -rank is definable and nilpotent. We also show that the soluble radical of a supersimple group of arbitrary rank is a definable and soluble subgroup. As a corollary, the FC -soluble radical of a supersimple group is virtually soluble and definable.

§2. Preliminaries on groups. If G is a group and x an element of G , we write x^G for the conjugacy class $\{g^{-1}xg : g \in G\}$ of x , and $C(x)$ for its centraliser $\{g \in G : g^{-1}xg = x\}$ in G . If y is another element of G , we write $[x, y]$ for the commutator $x^{-1}y^{-1}xy$. When A and B are subsets of G , we write $[A, B]$ for the set of commutators $[a, b]$ where a and b range over A and B respectively. We write $G^{(n)}$ for the n th term of the derived series of G defined inductively on n by putting $G^{(0)}$ equal to G and $G^{(n+1)}$ the subgroup generated by the set $[G^{(n)}, G^{(n)}]$. The group G is *soluble of derived length* n if n is the smallest natural number such that $G^{(n)}$ is $\{1\}$.

The FC -centre of a group G is written $FC(G)$ and is defined to be the set of g in G such that g^G is finite. By definition, the group G is an FC -group if $FC(G)$ equals G . Inductively on n , we call $FC_{n+1}(G)$ the preimage in G of $FC(G/FC_n(G))$, with the convention that $FC_0(G)$ is $\{1\}$. This defines an ascending chain of characteristic subgroups of G . The group G is called FC -nilpotent if G equals $FC_n(G)$ for some natural number n , the least such we call the FC -nilpotency class of G . Finite groups and nilpotent ones are both examples of FC -nilpotent groups. If G/N is a quotient group modulo a normal subgroup N of G , we write $FC_G(G/N)$ for the preimage of $FC(G/N)$ in G by the canonical surjection from G onto G/N .

A group G is *virtually- P* if it has a subgroup of finite index with property P .

THEOREM 2.1 (Neumann [13]). *Suppose that G is an FC -group whose conjugacy classes are bounded by a natural number. Then the derived subgroup G' is finite and G is virtually nilpotent of class 2.*

LEMMA 2.2. *If N is a finite normal subgroup of G , then $FC(G)$ equals $FC_G(G/N)$.*

PROOF. The canonical surjection from G onto G/N has a finite kernel. It follows that the conjugacy class x^G is finite if and only if $(xN)^{G/N}$ is finite. \dashv

LEMMA 2.3. *If H and N are two normal subgroups of G with $N \leq H$, then $FC_G(G/H)/N$ equals $FC_{G/N}(G/N/H/N)$.*

PROOF. There is a canonical homomorphism from G/H onto $G/N/H/N$. It follows that $(xH)^{G/H}$ is finite if and only if $((xN)H/N)^{G/N/H/N}$ is finite. This means precisely that $x \in FC_G(G/H)$ if and only if $xN \in FC_{G/N}(G/N/H/N)$. \dashv

LEMMA 2.4. *If for some natural number n the quotient $FC_{n+1}(G)/FC_n(G)$ is finite, then $FC_{n+2}(G)$ equals $FC_{n+1}(G)$.*

PROOF. We have

$$\frac{FC_{n+2}(G)}{FC_n(G)} = \frac{FC_G(G/FC_{n+1}(G))}{FC_n(G)}.$$

By Lemma 2.3

$$\frac{FC_{n+2}(G)}{FC_n(G)} = FC_{G/FC_n G} \left(\frac{G/FC_n(G)}{FC_{n+1}(G)/FC_n(G)} \right).$$

As $FC_{n+1}(G)/FC_n(G)$ is finite, applying Lemma 2.2 we get

$$\frac{FC_{n+2}(G)}{FC_n(G)} = FC \left(\frac{G}{FC_n(G)} \right) = \frac{FC_{n+1}(G)}{FC_n(G)}. \quad \dashv$$

Two subgroups of a given group G are *commensurable* if the index of their intersection is finite in both of them. Commensurability is an equivalence relation on the set of subgroups of G .

THEOREM 2.5 (Schlichting [16]). *Let G be a group and H a subgroup of G such that $H/H \cap H^g$ remains finite and bounded by a natural number for all g in G . Then, there exists a normal subgroup N of G such that $H/H \cap N$ and $N/N \cap H$ are finite. Moreover, N is a finite extension of a finite intersection of G -conjugates of H . In particular, if H is definable then so is N .*

§3. Preliminaries on supersimple groups. A supersimple group G is equipped with a rank function taking values in the ordinals, and ranking every definable subset of G . We write $SU(X)$ for the rank of a definable subset X of G . As there is no other rank considered in the paper, we will simply say *rank* instead of *SU-rank*. We shall not need the precise definition of the rank (we refer to [19] for more details), but only some of its properties that we recall now. The rank is increasing: if $X \subset Y$ are two definable subsets of G , then $SU(X)$ is smaller than or equal to $SU(Y)$. If G is supersimple, then so is each of its elementary extensions, and so is G^{eq} , meaning that every quotient group G/N by a definable normal subgroup N has an ordinal rank. A definable set (in G^{eq}) has rank zero if and only if it is finite. In particular, if N is a definable normal subgroup of G , then $SU(G/N)$ equals zero if and only if N has finite index in G .

The following comes from [7, Remark 3.5] as a particular case of [19, Theorem 5.5.4].

THEOREM 3.1 (Wagner’s version of Zilber’s Indecomposability Theorem). *Let G be a supersimple group of finite rank, $(X_i)_{i \in I}$ a family of definable subsets of G . Then, there exists a definable subgroup H of G such that*

- (1) H is a subgroup of $\langle X_i : i \in I \rangle$.
- (2) Finitely many translates of H cover X_i for every i .

If the sets X_i are normal in G , then H may be chosen normal in G .

COROLLARY 3.2. *If G is a supersimple group with finite rank, then the derived subgroup G' is definable.*

PROOF. We follow exactly the proof of [9, Corollary 7.5]. Let C be the set of commutators of G . By Theorem 3.1, there is a definable subgroup H of G' with H normal in G such that finitely many translates of H cover C . It follows that the set of commutators in G/H is finite, so the derived group $(G/H)'$ is finite by [10, p. 110]. The group G' is a finite union of cosets of H hence definable. \dashv

Any ordinal α decomposes in base ω : there are unique ordinals $\alpha_1 > \dots > \alpha_n$ and non-zero natural numbers k_1, \dots, k_n such that α equals $\omega^{\alpha_1} \cdot k_1 + \dots + \omega^{\alpha_n} \cdot k_n$.

If α and β are two ordinals, we may assume that α equals $\omega^{\alpha_1}.k_1 + \dots + \omega^{\alpha_n}.k_n$ and β equals $\omega^{\alpha_1}.\ell_1 + \dots + \omega^{\alpha_n}.\ell_n$ for the same $\alpha_1, \dots, \alpha_n$, adding some additional possibly zero k_i and ℓ_i if necessary. We write $\alpha \oplus \beta$ for their *Cantor sum* defined by

$$\alpha \oplus \beta = \omega^{\alpha_1}.(k_1 + \ell_1) + \dots + \omega^{\alpha_n}.(k_n + \ell_n).$$

THEOREM 3.3 (Lascar inequalities). *Let G be a supersimple group, and H a definable normal subgroup of G . Then*

$$SU(H) + SU(G/H) \leq SU(G) \leq SU(H) \oplus SU(G/H).$$

As a consequence, note that two definable subgroups of a supersimple group which are commensurable have the same rank.

PROPOSITION 3.4. *Let G be a supersimple group of rank $\omega^{\alpha_1}.k_1 + \dots + \omega^{\alpha_n}.k_n$ with $\alpha_1 > \dots > \alpha_n$. Then for every natural number i such that $1 \leq i \leq n$, there is a definable normal subgroup H of G of rank $\omega^{\alpha_1}.k_1 + \dots + \omega^{\alpha_i}.k_i$. The group H is unique up to commensurability.*

Remark 3.5. Proposition 3.4 is the definable version of [20, Wagner, Corollary 4.2]. It generalises what is known for superstable groups [4, Corollary 2.7 p. 27].

PROOF. We may assume that G is κ -saturated for some infinite cardinal κ and we say that a set is *small* if its cardinal is smaller than κ . We write β_i for $\omega^{\alpha_1}.k_1 + \dots + \omega^{\alpha_i}.k_i$. By [20, Corollary 4.2], there is a type-definable normal subgroup H of G having rank β_i . Recall that β_i is by definition the rank of each of the generic types of H . By [20, Theorem 4.4], the group H is the intersection of definable groups H_i for i in I . We may close this family by finite intersections, remove the members that do not have minimal rank and assume that every H_i has rank β say and that they are all commensurable. It follows that for every i , the group H has small index in H_i so H is a generic type of H_i by [19, Lemma 4.1.15]. Thus β equals β_i . Take any H_i . As H is normal in G , H_i^g and H_i are commensurable for every g in G . Let $FN(H_i)$ stand for the set of g in G such that $H_i/H_i \cap H_i^g$ is finite. On the one hand, the group $FN(H_i)$ is the countable union of the definable sets $FN_m(H_i)$ when m ranges over \mathbb{N} and where $FN_m(H_i)$ stands for $\{g \in G : |H_i : H_i \cap H_i^g| \leq m\}$. On the other hand, by [19, Lemma 4.1.15] and [19, Remark 4.1.5], it is type-definable. It must be definable by compactness and saturation. It follows that $H_i/H_i \cap H_i^g$ remains bounded by some natural number when g ranges over G . By Theorem 2.5, there is a definable normal subgroup N of G commensurable with H hence of rank β_i . If K is another group satisfying the desired requirements, then $K/N \cap K$ and $N/N \cap K$ are small according to [20, Corollary 4.2] hence finite by compactness and saturation. −

LEMMA 3.6. [12, Proposition 4.1] *If G is a group with (super)simple theory, its FC-centre is definable.*

PROOF. It is shown in [12] that $FC(G)$ is definable by a formula ψ provided that G is an \aleph_0 -saturated extension of G . Actually the same formula ψ computed in G defines $FC(G)$. −

§4. The Fitting subgroup. Let G be any group. We call the *Fitting subgroup* of G the subgroup generated by all of its nilpotent normal subgroups. We write it as $Fit(G)$. It is worth mentioning that the Fitting subgroup is definable if it is nilpotent. Namely x belongs to $Fit(G)$ if and only if the subgroup generated by its conjugacy class x^G is nilpotent. It follows that

$$Fit(G) = \bigcup_{n \geq 1} \left\{ x \in G : [x^G \cup x^{-G}, \dots, {}_n x^G \cup x^{-G}] = \{1\} \right\}$$

where for every subset X of G the set $[X, \dots, {}_n X]$ is defined inductively on $n \geq 0$ by

$$[X, {}_0 X] = X \quad \text{and} \quad [X, \dots, {}_{n+1} X] = [[X, \dots, {}_n X], X].$$

The observation that $Fit(G)$ is definable if it is nilpotent was first made by Ould Houcine, and the simple proof above was independently provided by the referee of [2].

PROPOSITION 4.1. *Let G be a group and F a normal subgroup of G . Assume that $F \leq FC_n(G)$ for some natural number n (in particular, F is FC -nilpotent). If G/F is FC -nilpotent, then so is G .*

PROOF. Assume that G/F is FC -nilpotent of class m . There is a surjection from G/F onto $G/FC_n(G)$. As recalled in [12], the image of an FC -nilpotent group by a group homomorphism is FC -nilpotent. It follows that $G/FC_n(G)$ is FC -nilpotent of class at most m , so that $FC_{m+n}(G)$ equals G . ⊣

We recall Hall's criterion for nilpotence.

THEOREM 4.2 (Hall [8]). *Let G be a group and N a normal subgroup of G . If G/N' and N are nilpotent, then G is nilpotent.*

Two other proofs of Theorem 4.2 can be found in [15] and [11], with a bound on the nilpotency class of G depending on the classes of N and G/N' in [11]. Note that since $G/N''/N'/N''$ and G/N' are isomorphic, a straightforward induction on the nilpotency class of N reduces the proof to the case where N is 2-nilpotent.

PROPOSITION 4.3 (adapted from Wagner [12, Proposition 4.3]). *Let G be a group with a (super)simple theory. If G is FC -nilpotent of class n , then G has a definable normal subgroup of finite index which is nilpotent of class at most $2n$.*

THEOREM 4.4 (Milliet [12, Corollary 4.5]). *Let G be a group with a (super)simple theory. If N is a normal nilpotent subgroup of class n , then N is contained in a normal definable nilpotent subgroup of class at most $3n$.*

We can now answer the question asked in [6].

THEOREM 4.5. *Let G be a supersimple group with finite rank. The Fitting subgroup of G is definable and nilpotent.*

PROOF. By Lemma 2.4, for big enough n , the quotient $FC_{n+1}(G)/FC_n(G)$ is either trivial or infinite. By Lascar's equality, there exists a natural number such that $FC_n(G) = FC_{n+1}(G)$. We call G_n the quotient group $G/FC_n(G)$ so that G_n has a trivial FC -centre. Let F_n be its Fitting subgroup.

CLAIM 1. *We may assume that $FC(G)$ is trivial.*

PROOF OF CLAIM 1. We need just assume that F_n is definable and nilpotent and show that $Fit(G)$ is definable and nilpotent too. Note that F_n is FC -nilpotent. Let F be its pull-back in G so that we have $F/FC_n(G) = F_n$. The group F is FC -nilpotent

by Proposition 4.1, and definable. By Proposition 4.3, F has a definable subgroup of finite index which is nilpotent, so it must have a normal one N . It follows that F contains a maximal normal (in G) nilpotent subgroup H of finite index so that H equals $\text{Fit}(G)$. Being a finite extension of N , $\text{Fit}(G)$ is definable. \dashv CLAIM 1

CLAIM 2. *We may assume that G has a definable normal 2-nilpotent subgroup.*

PROOF OF CLAIM 2. On the one hand, if every nilpotent normal subgroup of G is abelian, then $\text{Fit}(G)$ is abelian. In this case, by the remark made at the beginning of this section (or Theorem 4.4), $\text{Fit}(G)$ must be definable. On the other hand, if there is a non abelian nilpotent normal subgroup, then there is a definable one by Theorem 4.4. Call it N . The group $Z_2(N)$ has the required properties. \dashv CLAIM 2

We proceed by induction on $SU(G)$ to prove Theorem 4.5. If $SU(G)$ is zero, then G is finite and so is $\text{Fit}(G)$. If $SU(G)$ equals $n + 1$, by Claim 2, there is a normal nilpotent definable subgroup N of G of nilpotency class 2. By Corollary 3.2, the derived subgroup N' is definable. As N' is normal in G , it is infinite by Claim 1, so $SU(N') \geq 1$. By Lascar's equality, we have $SU(G/N') \leq n$, and we may apply the induction hypothesis to G/N' . It follows that $\text{Fit}(G/N')$ is definable and nilpotent. Let F be its preimage in G so that we have $F/N' = \text{Fit}(G/N')$. By Theorem 4.2, the group F is nilpotent. Thus F equals $\text{Fit}(G)$. \dashv

§5. The soluble radical. We call the *soluble radical* of G the subgroup generated by all soluble normal subgroups and write it as $R(G)$. It is a locally soluble subgroup.

We recall a simple and useful remark by Ould Houcine:

LEMMA 5.1 (Ould Houcine). *Let G be any group and suppose that $R(G)$ is soluble. Then $R(G)$ is definable.*

PROOF. An element x belongs to $R(G)$ if and only if the subgroup generated by its conjugacy class x^G is soluble. Note that the derived subgroup $\langle x^G \rangle^{(1)}$ is generated by all commutators of the form $[a^g, b^h]$ where a and b equal x or x^{-1} and g and h range over G . Thus the following equality holds

$$R(G) = \bigcup_{n \geq 1} \left\{ x \in G : (x^G \cup x^{-G})^{(n)} = \{1\} \right\}$$

where for every subset X of G the set $X^{(n)}$ is defined inductively on $n \geq 0$ by

$$X^{(0)} = X, \quad X^{(1)} = [X, X] \quad \text{and} \quad X^{(n+1)} = [X^{(n)}, X^{(n)}].$$

It follows that $R(G)$ is a countable union of increasing definable sets. As $R(G)$ is soluble, this union is actually a finite one, and $R(G)$ is definable. \dashv

THEOREM 5.2 (Milliet [12, Corollary 4.11]). *Let G be a group with a (super)simple theory and S a normal soluble subgroup of derived length n . Then S is contained in a definable soluble subgroup of derived length at most $3n$.*

THEOREM 5.3. *The soluble radical of a supersimple group is definable and soluble.*

PROOF. We shall proceed by transfinite induction on the rank of G . For that, we first show the following claim:

CLAIM. *If G has a normal subgroup H such that both $R(G/H)$ and $R(H)$ are soluble, then $R(G)$ is definable and soluble.*

PROOF OF THE CLAIM. Let ℓ be a natural number bounding the derived length of both $R(G/H)$ and $R(H)$. Let S be a normal soluble subgroup of G of derived length n . It follows that SH/H and $S \cap H$ are soluble of derived length no greater than ℓ . As SH/H is isomorphic to $S/S \cap H$, we must have $S^{(\ell)} \subset S \cap H$ which in turn yields $S^{(2\ell)} = \{1\}$. It follows that n is less than or equal to 2ℓ , so that the solubility class of S is bounded, independently on S . This means that $R(G)$ is soluble, hence definable by Lemma 5.1. ⊣_{CLAIM}

We can now inductively prove our theorem: if G has rank zero, then G is finite and so is $R(G)$.

If G has a non-monomial rank, there is a natural number $k > 0$ and ordinals α and β such that $SU(G)$ equals $\omega^\alpha.k + \beta$ with $0 < \beta < \omega^\alpha$. By Proposition 3.4, there is a normal subgroup H of G having rank $\omega^\alpha.k$. By the Lascar inequalities, both $SU(G/H)$ and $SU(H)$ are less than $SU(G)$ so we may apply the induction hypothesis to H and G/H and it follows from the Claim that $R(G)$ is soluble and definable.

If G has a monomial rank, it is of the form $\omega^\alpha.k$. Let us first suppose that there is some a in $R(G)$ with a^G having rank at least ω^α . By Theorem 5.2, the conjugacy class a^G is contained in a definable normal soluble group S . As the rank is increasing, S must have rank at least ω^α . In that case, either S and G have the same rank so G is virtually soluble and we are done, or $SU(S) < SU(G)$. Then, by the Lascar inequalities 3.3 we have $SU(G/S) < SU(G)$ and we may again apply the Claim. One last case to deal with: we may have $SU(a^G) < \omega^\alpha$ for all a in $R(G)$. As a^G and $G/C(a)$ are in definable bijection it follows that $SU(G/C(a)) < \omega^\alpha$ for all a in $R(G)$. By the Lascar inequalities, this is equivalent to saying that $SU(G/C(a))$ is zero for all a in $R(G)$. So $R(G)$ is a subgroup of the FC -centre of G which is definable by Lemma 3.6 and virtually nilpotent of class 2 by Theorem 2.1. It follows that $R(G)$ is also virtually nilpotent of class 2 (and locally soluble) hence soluble. In every case, $R(G)$ is definable and soluble. ⊣

§6. The FC -soluble radical.

DEFINITION 6.1 (adapted from Duguid, McLain [5]). A group G is FC -soluble if there exists a finite sequence of subgroups G_0, G_1, \dots, G_n of G such that

$$G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_n = \{1\}$$

and such that G_i/G_{i+1} is an FC -group for all i . We call the least such natural number n the FC -solubility class of G .

If N is a normal subgroup of G , then G is FC -soluble if and only if G/N and N are FC -soluble.

We define the FC -soluble radical of a group to be the subgroup generated by every normal FC -soluble subgroup. This is a locally FC -soluble subgroup:

LEMMA 6.2. *Let H and K be two normal FC -soluble subgroups of a group G of class h and k . The product HK is FC -soluble of FC -solubility class at most $h + k$.*

PROOF. The quotient HK/K is isomorphic to $H/H \cap K$. So K and HK/K both are FC -soluble. ⊣

PROPOSITION 6.3 (Milliet [12, Corollary 4.9]). *A (super)simple FC -soluble group is virtually-soluble.*

COROLLARY 6.4. *The FC-soluble radical of a supersimple group is definable and virtually soluble.*

PROOF. By Theorem 5.3, $R(G)$ is definable so the quotient $G/R(G)$ is supersimple and has no non-trivial normal soluble subgroup. Let us write it as G_R . By Proposition 6.3, an FC-soluble subgroup of G_R is virtually-soluble, hence finite, so every normal FC-soluble subgroup is contained in $FC(G_R)$. By Lemma 3.6 and Theorem 2.1, the group $FC(G_R)$ must be finite. Its preimage in G is definable, virtually soluble, and contains every normal FC-soluble subgroup of G . \dashv

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UNIVERSITÉ GALATASARAY

FACULTÉ DE SCIENCES ET DE LETTRES, DÉPARTEMENT DE MATHÉMATIQUES

CIRAĞAN CADDESİ N. 36, 34357 ORTAKÖY, İSTAMBOUL, TURQUIE

E-mail: milliet@math.univ-lyon1.fr