

## 5.3 Ranking Theory

Gabriele Kern-Isberner, Niels Skovgaard-Olsen, and Wolfgang Spohn

### Summary

Ranking theory is one of the salient formal representations of doxastic states. It differs from others in being able to represent both belief in a proposition (i.e., taking it to be true) and degrees of belief (i.e., beliefs as more or less firm) and thus to generally account for the dynamics of these beliefs. It does so on the basis of fundamental and compelling rationality postulates and is hence one way of explicating the rational structure of doxastic states. Thereby, it provides foundations for accounts of defeasible or nonmonotonic reasoning. It has widespread applications in philosophy, it proves to be highly useful in artificial intelligence, and it has started to find applications as a model of reasoning in psychology.

#### 1. Belief and Degrees of Belief

Epistemic or doxastic attitudes<sup>1</sup> representing how the world is like come in degrees, whether you call them degrees of belief, of uncertainty, or of plausibility (there are more terms around). There are various accounts of those degrees, amply presented in this handbook.<sup>2</sup> The interests in those accounts are manifold: philosophers are concerned with the rational nature of those degrees, artificial intelligence (AI) researchers are interested in their computational feasibility, psychologists deal with their actual manifestations, and all sides argue about how well they are suited to model human reasoning.

However, we also have the notion of belief *simpliciter*. Related notions are those of acceptance or judgment. These are indeed the more basic notions when it comes to truth, to truly representing the world. Beliefs can be true, but degrees of belief cannot. The latter rather relate to action (see chapter 8.2 by Peterson, this handbook). Accounts of degrees of belief invariably have great difficulties in doing justice to this fundamental point. There is a questionable tendency to take degrees of belief as basic and to belittle those difficulties.

So we need to theoretically account for belief *simpliciter*. The first attempt was doxastic logic (see chapter 5.1 by van Ditmarsch, this handbook). However, it is static and lacks a dynamic perspective. This perspective has been unfolded in belief revision theory (see chapter 5.2 by Rott, this handbook). However, this theory has problems with iterated belief revision, which is required for a complete dynamic account.

Ranking theory promises both to represent belief and degrees of belief *and* to provide a complete dynamics for both. These features give it a prominent place in the spectrum of possible theories. It was first presented in English in Spohn (1988) and fully developed in Spohn (2012). Easy access is provided in Spohn (2009). Its far-reaching applications in philosophy of science, epistemology, and even normative reasoning may be found, for example, in Spohn (2012, 2015, 2020). There is no place here to go into any of them.

Below, we present the basics of the theory in section 2 and its dynamic aspects in section 3. Section 4 is comparative. Section 5 gives a short introduction to the theory's relevance for artificial intelligence, and section 6 explains how it can be put to use in psychology.

#### 2. The Basics of Ranking Theory

Grammatically, “believe” is a transitive verb. In the phrase “*a* believes that *p*,” “*a*” refers to a (human) subject and “that *p*” seems to be the object. What does “that *p*” stand for? That is, what are the objects of belief? This is a difficult and most confusing issue extensively discussed in philosophy (under the rubric “propositions”; see, e.g., McGrath, 2012). Here, we cut short the issue, as is usual in formal epistemology, by saying in a non-committal way that “that *p*” stands for the proposition expressed by “*p*,” where that proposition is its truth condition, the set of possibilities or possible worlds in which *p* obtains or “*p*” is true.

Hence, we simply assume a set *W* of (mutually exclusive and jointly exhaustive) possibilities. These may be

coarse-grained and refer only to a few things of interest; they need not consist in entire possible worlds. Each subset of  $W$ , that is, each element of the power set  $\mathcal{P}(W)$  of  $W$ , is a *proposition*.

Now, the basic representation of a belief state is simply as the set of propositions believed or taken to be true in that state, its *belief set*. Traditionally, a belief set  $\mathcal{B} \subseteq \mathcal{P}(W)$  has to satisfy two rationality requirements:  $\mathcal{B}$  must be *consistent*, that is,  $\bigcap \mathcal{B} \neq \emptyset$ , and  $\mathcal{B}$  must be *deductively closed*, that is, if  $\bigcap \mathcal{B} \subseteq A$ , then  $A \in \mathcal{B}$ .

These two rationality requirements may seem entirely obvious. The rationale of deductive logic is to check what we must not believe and what we are committed to believe. Note, however, that deductive closure is lost when we identify belief with subjective probability above a certain threshold: it easily happens that the probabilities of two propositions are above the threshold, while that of their conjunction is below. Thus, the lottery and the preface paradox and the general desire to stick to a probabilistic representation of belief have led to a contestation of these requirements (see, e.g., Christensen, 2004). Here we stick to them as absolutely basic (see chapter 3.1 by Steinberger, this handbook). Of course, these requirements can be maintained only under a dispositional understanding of belief; occurrent thought cannot be deductively closed (see also section 6 of the introduction, this handbook).

The notion of a belief set is static. However, belief sets continuously change, and we must account for how they change (or should rationally change). We cannot do so on a qualitative level. In those changes, we often give up old beliefs and replace them by new ones, and then we give up less well-entrenched beliefs and keep better-entrenched ones (see chapter 5.2 by Rott, this handbook). Roughly, this calls for some entrenchment order or, indeed, for some kind of degrees of belief measuring the strength of entrenchment. Here, ranking theory commences. Let us start with some brief formal explanations.

*Definition 1:*  $\kappa$  is a *negative ranking function* for  $W$  iff  $\kappa$  is a function from  $\mathcal{P}(W)$  into the set of natural numbers plus infinity  $\infty$  such that for all  $A, B \subseteq W$ ,

- (1)  $\kappa(W) = 0$  and  $\kappa(\emptyset) = \infty$ ,
- (2)  $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$  (the *law of disjunction*).<sup>3</sup>

The basic interpretation is that  $\kappa$  expresses degrees of *disbelief* (whence the qualification “negative”). If  $\kappa(A) = 0$ ,  $A$  is not disbelieved at all. This allows that  $\kappa(\neg A) = 0$  as well (where  $\neg A$  is the negation of  $A$ ); in that case, we have indifference or suspension of judgment regarding  $A$ . If  $\kappa(A) > 0$ ,  $A$  is disbelieved or taken to be false, and

the more so the larger  $\kappa(A)$ . So, positive *belief* in  $A$  is expressed by disbelief in  $\neg A$ , that is, by  $\kappa(\neg A) > 0$  (which implies that  $\kappa(A) = 0$ ).

This interpretation explains axioms (1) and (2): axiom (1) says that the tautology  $W$  is not disbelieved, and hence the contradiction  $\emptyset$  is not believed. This entails that beliefs are consistent according to  $\kappa$ . Axiom (1) moreover says that the contradiction is indeed maximally disbelieved. And axiom (2) states that you cannot disbelieve a disjunction less strongly than its disjuncts. This entails in particular that if you believe two conjuncts, you also believe their conjunction. Hence, beliefs are deductively closed according to  $\kappa$ . In other words, the belief set  $\mathcal{B} = \{A \mid \kappa(\neg A) > 0\}$  associated with  $\kappa$  satisfies the two basic rationality requirements. Note, moreover, that (1) and (2) entail

- (3) either  $\kappa(A) = 0$  or  $\kappa(\neg A) = 0$  (or both) (the *law of negation*),

that is, you cannot (dis)believe  $A$  and  $\neg A$  at once.

For an illustration, consider Tweety. Tweety has, or fails to have, each of three properties: being a bird ( $B$ ), being a penguin ( $P$ ), and being able to fly ( $F$ ). This opens up eight possibilities. Suppose you have no idea who or what Tweety is, but somehow you do not think that it might be a penguin. Then your negative ranks for the eight possibilities (which determine the ranks for all other propositions) may be as given in table 5.3.1 (chosen in some plausible way—but see below how the numbers may be justified).

In this case, the strongest proposition you believe is that Tweety is *either* no penguin and no bird ( $\neg B \wedge \neg P$ ) or a flying bird and no penguin ( $F \wedge B \wedge \neg P$ ); all other possibilities are disbelieved. Hence, you neither believe that Tweety is a bird nor that it is not a bird. You are also indifferent concerning its ability to fly. But you believe, for example, that if Tweety is a bird, it is not a penguin and can fly ( $\neg B \vee (\neg P \wedge F)$ ), and that if Tweety is a penguin, it can fly ( $\neg P \vee F$ )—each if-then taken as material implication. Surely you believe the latter only because you believe that Tweety is not a penguin in the first place. The large ranks in the last column indicate your strong disbelief in penguins *not* being birds. This may suffice as a first illustration.

**Table 5.3.1**  
Prior beliefs about Tweety

$\kappa$	$B \wedge \neg P$	$B \wedge P$	$\neg B \wedge \neg P$	$\neg B \wedge P$
$F$	0	4	0	11
$\neg F$	2	1	0	8

We will see the reasons for starting with negative ranks. But of course, we can also introduce the positive counterpart to  $\kappa$  by defining  $\beta$  to be a *positive ranking function* iff there is a negative ranking function  $\kappa$  such that  $\beta(A) = \kappa(\neg A)$  for all propositions  $A$ . The function  $\beta$  represents *degrees of belief*. Of course, (1) and (2) translate directly into axioms for  $\beta$ .

We may as well represent degrees of belief and degrees of disbelief in a single function by defining  $\tau$  to be a *two-sided ranking function* iff there is a negative ranking function  $\kappa$  and the corresponding positive ranking function  $\beta$  such that  $\tau(A) = \beta(A) - \kappa(A) = \kappa(\neg A) - \kappa(A)$  for all propositions  $A$ . Thus, we have  $\tau(A) > 0$ ,  $< 0$ , or  $= 0$  according to whether  $A$  is believed, disbelieved, or neither in  $\tau$ . Therefore, this is perhaps the most intuitive notion. However, the mathematics is best done in terms of negative ranking functions. It is clear, though, that the three functions are interdefinable.

There is an important interpretational degree of freedom that we have not yet noticed. So far, we said that belief in  $A$  is represented by  $\kappa(\neg A) = \beta(A) = \tau(A) > 0$ . However, we may often find it useful to raise the threshold for belief, as we do informally in asking, “Do you *really* believe  $A$ ?” That is, we may as well say that belief in  $A$  is only represented by  $\kappa(\neg A) = \beta(A) = \tau(A) > z$  for some  $z \geq 0$ . This seems to be a natural move: belief is vague. Where does it commence, and when does it cease? And this vagueness seems well represented by that parameter  $z$ . This move at the same time enlarges the range of suspension of judgment to the interval from  $-z$  to  $z$ . The remarkable point about axioms (1) and (2) is that they guarantee belief sets to be consistent and deductively closed, however we choose the threshold  $z$ . They are indeed equivalent to this general guarantee.

### 3. Conditional Ranks, Reasons, and the Dynamics of Ranks

So far, we have sketched only the static part of ranking theory. However, we mentioned that the numeric ranks are essentially used to account for the dynamics of belief; they are not just to represent greater and lesser firmness of (dis)belief. To achieve this, the crucial notion is that of conditional ranks:

*Definition 2:* Let  $\kappa$  be a negative ranking function for  $W$  and  $\kappa(A) < \infty$ . Then the *conditional rank* of  $B$  given  $A$  is defined as  $\kappa(B|A) = \kappa(A \wedge B) - \kappa(A)$ .

We might rewrite this definition as

$$(4) \quad \kappa(A \wedge B) = \kappa(A) + \kappa(B|A) \quad (\text{the law of conjunction}).$$

This is highly intuitive. For, what is your degree of disbelief in  $A \wedge B$ ? One way for  $A \wedge B$  to be false is that  $A$  is false; this contributes  $\kappa(A)$  to that degree. However, if  $A$  is true,  $B$  must be false, and this adds  $\kappa(B|A)$ .

It immediately follows for all propositions  $A$  and  $B$  with  $\kappa(A) < \infty$  that

$$(5) \quad \kappa(B|A) = 0 \text{ or } \kappa(\neg B|A) = 0 \quad (\text{the conditional law of negation}).$$

This law says that even conditional belief must be consistent. If both  $\kappa(B|A)$  and  $\kappa(\neg B|A)$  were  $> 0$ , both  $B$  and  $\neg B$  would be (dis)believed given  $A$ , and this must be excluded, as long as the condition  $A$  itself is considered possible. Indeed, given definition 2 and axiom (1), we could axiomatize ranking theory also by (5) instead of (2). Hence, the only substantial assumption written into ranking functions is conditional consistency.

Axioms (1) and (2) did not refer to any *cardinal* properties of ranking functions. However, the definition of conditional ranks involves arithmetical operations and thus presupposes a cardinal understanding of ranks. We will see below how this may be justified. We hasten to add that one could as well define positive conditional ranks by  $\beta(B|A) = \kappa(\neg B|A)$  and two-sided conditional ranks by  $\tau(B|A) = \kappa(\neg B|A) - \kappa(B|A)$ .

As an illustration, consider again table 5.3.1 and the conditional beliefs contained therein. We can see that precisely the (material) if-then propositions non-vacuously held true correspond to conditional beliefs. According to the  $\kappa$  specified, you believe, for example, that Tweety can fly given it is a bird (since  $\kappa(\neg F|B) = 1$ ), and also given it is a bird but not a penguin (since  $\kappa(\neg F|B \wedge \neg P) = 2$ ) and that Tweety cannot fly given it is a penguin (since  $\kappa(F|P) = 3$ ). Hence, your vacuous belief in the material implication “If Tweety is a penguin, it can fly” does not amount to a corresponding conditional belief. In other words, “if . . . then . . .” expresses conditional belief rather than material implication, as suggested by the Ramsey test for conditionals (see also chapter 6.1 by Starr, this handbook).

A first fundamental application of conditional ranks lies in the notion of an *epistemic reason*, which is at the center of the entire handbook. It is very natural to say that  $A$  is a reason for  $B$  iff  $A$  speaks in favor of  $B$  or confirms  $B$ , if  $A$  makes  $B$  more plausible or less implausible, or if  $B$  is more credible or less incredible given  $A$  than given  $\neg A$ . This explanation works for any conception of conditional degrees of belief. In a probabilistic interpretation, it amounts to Carnap’s notion of incremental confirmation or positive relevance (Carnap, 1950/1962), which is basic for confirmation theory.<sup>4</sup> Ranking-theoretically, it leads to

*Definition 3:*  $A$  is a reason for  $B$  relative to the negative ranking function  $\kappa$  or the associated two-sided ranking function  $\tau$  iff  $\tau(B|A) > \tau(B|\neg A)$ .

We can show that if  $A$  is a deductive reason for  $B$ , that is, if  $A \subseteq B$ , then  $A$  is also a reason for  $B$  according to definition 3 (given  $\kappa(A), \kappa(\neg B) < \infty$ ). Clearly, this definition, depending entirely on the subject’s doxastic state, provides only a subjectively relativized notion of a reason. Philosophers strive for a more objective notion of a reason,<sup>5</sup> perhaps because they take objective deductive reasons as a paradigm. In our view, the extent to which a more objective notion may be reached is a philosophically fundamental, alas very open, issue (see Spohn, 2018).

On this account, reasons can take four significant forms, depending on whether  $\tau(B|A)$  and  $\tau(B|\neg A)$  are positive or negative. For example,  $A$  is a *sufficient* reason for  $B$  iff  $B$  is believed given  $A$  and not believed given  $\neg A$ . We suggest that this is indeed the core meaning of the term “sufficient reason,” although it is often used differently.

Moreover, we may define that  $B$  is (doxastically) *irrelevant* to or *independent* of  $A$  if neither  $A$  nor  $\neg A$  is a reason for  $B$ . On this basis, a theory of (conditional) independence can be developed in ranking terms in wide-ranging analogy to the probabilistic theory. For instance, the theory of Bayes nets as developed by Pearl (1988) (see also chapter 4.2 by Hartmann, this handbook) could equally well be stated in terms of ranking theory (see Spohn, 1988; Goldszmidt & Pearl, 1996).

A second fundamental application of conditional ranks lies in the *dynamics* of beliefs and ranks. As in probability theory, we may say that, upon learning evidence  $E$ , we should simply move to the degrees of belief conditional on  $E$ . Thereby, though, the evidence  $E$  acquires maximal certainty, that is, probability 1 or positive rank  $\infty$ . This seems too restrictive. In general, evidence may be (slightly) uncertain, and our rules for doxastic change through evidence or learning—we do not attend to changes caused in other ways, like forgetting—should take account of this. In ranking theory, this is achieved by two principles: first, conditional ranks given the evidence  $E$  and given its negation  $\neg E$  are not changed by the evidence itself—how could it change them?—and second, the evidence  $E$  does not become maximally certain but improves its position by  $n$  ranks, where  $n$  is a free parameter characterizing the specific information process.<sup>6</sup> These two assumptions suffice to uniquely determine the kinematics of ranking functions (i.e., ranking-theoretic conditionalization).

In order to see how this works, look again at our Tweety example: suppose you learn in some way, and

accept with firmness 2, that Tweety is a bird. Thus, you shift up  $\neg B$ -possibilities by 2 and keep constant the rank differences within  $B$  and within  $\neg B$ . This results in the posterior ranking function  $\kappa'$  given in table 5.3.2.

In  $\kappa'$ , you believe that Tweety is a bird able to fly but not a penguin; you still neglect this possibility. So, in  $\kappa'$ , you believe more than in  $\kappa$ ; in belief revision theory (cf. chapter 5.2 by Rott, this handbook), this would be called a “belief expansion.”

Next, to your surprise, you tentatively learn and accept, say, with firmness 1 that Tweety is indeed a penguin. This results in another ranking function  $\kappa''$ , which shifts all  $P$ -possibilities down by 1 and all  $\neg P$ -possibilities up by 1, so that  $P$  is indeed believed with firmness 1 (i.e.,  $\kappa''(\neg P) = 1$ ; see table 5.3.3).

So, you have changed your mind and believe in  $\kappa''$  that Tweety is a penguin bird that cannot fly. In belief revision theory, this would be called a “belief revision.” Obviously, belief contraction (cf. chapter 5.2 by Rott, this handbook), where you simply give up a belief previously held without replacing it by a new one, can also be modeled by ranking-theoretic conditionalization. The example already demonstrates that this rule of belief change can be iteratively applied ad libitum.

An important application of ranking-theoretic conditionalization is that it delivers a measurement procedure for ranks that justifies the cardinality of ranks. This procedure refers to iterated belief contraction. Its point is this: if your iterated contractions behave as prescribed by ranking theory,<sup>7</sup> then this behavior uniquely determines your ranking function up to a multiplicative constant. That is, your ranks can thereby be measured on a ratio scale (see Hild & Spohn, 2008). The consequences of the fact that ranks are measured only on a ratio scale await investigation. They imply, for example, a problem analogous to the problem of the interpersonal comparison of utilities.

**Table 5.3.2**  
Learning that Tweety is a bird

$\kappa'$	$B \wedge \neg P$	$B \wedge P$	$\neg B \wedge \neg P$	$\neg B \wedge P$
$F$	0	4	2	13
$\neg F$	2	1	2	10

**Table 5.3.3**  
Learning that Tweety is a penguin

$\kappa''$	$B \wedge \neg P$	$B \wedge P$	$\neg B \wedge \neg P$	$\neg B \wedge P$
$F$	1	3	3	12
$\neg F$	3	0	3	9

#### 4. Comparisons

The formal structure defined by axioms (1) and (2) has been called *Baconian probability* by Cohen (1980). Its first clear appearance is in the functions of potential surprise developed by Shackle (1949). The structure is also hidden in Rescher (1964) and is clearly found in Cohen's own work (1970, 1977). The crucial formal advance of ranking theory lies in the definition of conditional ranks, which cannot be found in these works and makes the theory a properly cardinal one.

Belief revision theory was precisely about the dynamics of belief. However, it only conceived of entrenchment *orders*. And within their ordinal framework, there was no clear solution of the problem of iterated belief revision (see chapter 5.2 by Rott, this handbook).

Possibility theory, building on early work by Zadeh (1978) and developed by Dubois and Prade (see chapter 4.7 by Dubois & Prade, this handbook), is in fact equivalent to ranking theory; axiom (2) is the characteristic property of possibility measures. However, the interpretation of those measures was intentionally left open, leaving considerable formal uncertainty as to how to conceive of conditional degrees of possibility.

The theory of Dempster–Shafer (DS) belief functions, as developed in Shafer (1976), seems to be a far more general theory (see again chapter 4.7 by Dubois & Prade, this handbook), which comprises probability theory and also ranking theory as a special case. Shafer (1976, chapter 10) defines so-called consonant belief functions, which appear to be equivalent to negative ranking functions. However, their respective dynamic behaviors diverge, a fact that prevents reduction of ranking theory to DS theory (see Spohn, 2012, section 11.9).

Of course, the largest comparative issue is how ranking theory relates to probability theory. A comparison of their axioms and their form of conditionalization suggests translating the sum of probabilities into the minimum of ranks and the product and the quotient of probabilities into the addition and the subtraction of ranks, respectively. This translation works only for *negative* ranks; that's why they provide the formally preferred version of ranking theory. And it explains why very many things that can be done with probability theory also work for ranking theory in a meaningful way.

However, the translation does not justify conceiving ranks in probabilistic terms. As mentioned in section 2, belief in  $A$  cannot be probabilistically represented by  $P(A) = 1 - \varepsilon$  if one sticks to the consistency and deductive closure of belief sets. The relation of probability and belief is hotly debated in philosophy, without a clear

solution emerging (see, e.g., the proposals of Leitgeb, 2017, and Raidl & Skovgaard-Olsen, 2017). Therefore, our attitude has always been to independently develop ranking theory as a theory of belief.

#### 5. Ranking Functions in Artificial Intelligence

Besides probability theory and logic, ranking functions are among the most popular formalisms used for knowledge representation<sup>8</sup> and reasoning (KRR), and their popularity is still increasing because they provide a very versatile framework for many central operations in KRR, as already sections 2 to 4 pointed out. Most importantly, ranking functions are a convenient common basic tool for nonmonotonic reasoning and belief revision. Belief revision already has been explained in more detail in section 3, and nonmonotonic reasoning also deals with belief dynamics insofar as conclusions may be given up when new information arrives (so, the consequence relation is not monotonic, as in classical logic). Both fields emerged in the 1980s (partly) as a reaction to the incapability of classical logic to handle problems in everyday life that intelligent systems like robots were expected to tackle. Knowledge or belief about the world is usually uncertain, and the world is always changing. Therefore, AI systems built upon classical logics failed. So-called preferential models (Makinson, 1989) provide an important semantics for nonmonotonic logics. Their basic idea is to order worlds according to normality and to focus on the minimal ones (i.e., the most plausible ones) for reasoning. Likewise, AGM belief revision theory (see chapter 5.2 by Rott, this handbook) needs orderings of worlds to become effective. For both fields, ranking functions offer a quite perfect technical tool that also complies nicely with the intuitions behind the techniques. Moreover, they can also evaluate conditionals and are an attractive qualitative counterpart to probabilities (see section 3).

Judea Pearl was probably the first renowned AI scientist to make use of ranking functions; his famous *system Z* (Pearl, 1990) is based on them. He has continuously emphasized the commonsense structural qualities of probabilities and developed ranking functions as an interesting qualitative counterpart to probabilities. He set up his system  $Z$  as an “ultimate system of nonmonotonic reasoning” in terms of ranking functions. To date, it is one of the best and most convenient approaches to implement high-quality nonmonotonic reasoning.

Consequently, ranking functions are deeply connected with nonmonotonic and uncertain reasoning and with belief change, which are core topics in KRR. Many researchers make use of them in one way or

another even if they rely on more general frameworks. Darwiche and Pearl (1997) presented general postulates for the iterated revision of general epistemic states but illustrated their account with ranking functions. So did Jin and Thielscher (2007) and Delgrande and Jin (2012) when they devised novel postulates for iterated and multiple revision. Interestingly, the independence properties for advanced belief revision that were proposed in those studies can be related to independence with respect to ranking functions (see Spohn, 2012, chapter 7) in analogy with probabilistic independence (see Kern-Isberner & Huvermann, 2017).

Indeed, as suggested in section 3, ranking functions are particularly well suited for iterated belief change because they can easily be modified in accordance with AGM theory, returning new ranking functions that are readily available for a subsequent change operation. The main AGM operations are revision (adopting a belief) and contraction (giving up a belief), related by Levi and Harper identities (see chapter 5.2 by Rott, this handbook). In ranking theory, the connections between these operations are even deeper, since (iterated) contraction is just a special kind of (iterated) ranking conditionalization. Indeed, the results of Kern-Isberner, Bock, Sauerwald, and Beierle (2017) show that iterated revision and contraction can be performed by a common methodology.

Continuing on that, and beyond the practicality and diversity of ranking functions, it is crucial to understand that they are not just a pragmatically good choice but indeed allow for deep theoretical foundations of approaches to reasoning. It is the ease and naturalness with which they can handle conditionals—very similar to probabilities—that make them an excellent formal tool for modeling reasoning. Given that conditionals are, on the one hand, crucial entities for nonmonotonic and commonsense reasoning and belief change, and, on the other hand, formal entities fully accessible to conditional logics, this capability provides a key feature for logic-based approaches connecting nonmonotonic logics and belief change theories with commonsense and general human reasoning. More precisely, conditional ranks give meaning to differences between degrees in belief when observing  $A$  versus  $A \wedge B$  (see the law of conjunction in section 3), and the examples of belief change given in section 3 illustrate nicely how it is easily possible to preserve these differences under change when using ranking functions. This property has been elaborated as a *principle of conditional preservation* in Kern-Isberner (2004), giving rise to defining c-representations and c-revisions (all belief changes shown in section 3 are c-revisions). C-representations are c-revisions starting

from a uniform ranking function and allowing for reasoning from conditional belief bases. Ranking theory is one of the few formal frameworks (probability theory is another) that is rich and expressive enough to allow such a precise formalization of conditional preservation that supports both belief change and inductive reasoning as a common methodology.

## 6. Ranking Theory in Psychology

Potentially, ranking theory has applications in many areas of psychology. To illustrate, psychological research on belief revision has been carried out under the inspiration of AGM theory and probabilistic updating (Baratgin & Politzer, 2010; Oaksford & Chater, 2013; Wolf, Rieger, & Knauff, 2012; see also chapter 5.4 by Gazzo Castañeda & Knauff, this handbook). Such work could be extended by ranking theory, given that one of its central motivations was to represent a notion of full belief, in contrast to probabilistic update mechanisms, while improving upon AGM theory to allow for iterative revisions. However, as Colombo, Elkin, and Hartmann (2018) note, probabilistic Bayesian approaches are currently enjoying a boom in cognitive science to the neglect of alternative formal frameworks, like ranking theory.

As said, possibility theory is mathematically equivalent to ranking theory, yet differs fundamentally in its intended interpretation. In Da Silva Neves, Bonnefon, and Raufaste (2002) and Benferhat, Bonnefon, and Da Silva Neves (2005), possibility theory was subjected to empirical testing. In Da Silva Neves et al. (2002), a direct route was chosen by testing whether participants' possibility judgments satisfy the rationality postulates codified in System P (see chapter 4.4 by Pfeifer, this handbook) augmented by *rational monotonicity* ( $A \vdash C$  and not  $A \vdash \neg B$ ; therefore  $A \wedge B \vdash C$ ),<sup>9</sup> in case their responses violated *monotonicity* ( $A \vdash C$ ; therefore  $A \wedge B \vdash C$ ). Interestingly, no such direct test of ranking theory based on the participants' judgments of disbelief or implausibility has yet been made.

What exists is the following: Isberner and Kern-Isberner (2017) investigated whether belief revision with ranking functions could retrodict findings of temporal delay when processing implausible information in tasks where plausibility judgments would interfere with the task constraints (a finding known as the “epistemic Stroop effect”). A guiding assumption underlying this work is that ranking theory can be used to represent the situational model that participants construct during language comprehension. Moreover, Eichhorn, Kern-Isberner, and Ragni (2018) proposed a conditional-logical model based

on ranking functions that allows for the elaboration of plausible background knowledge.

In Ragni, Eichhorn, Bock, Kern-Isberner, and Tse (2017), it was investigated whether ranking theory could retrodict the suppression effect (Byrne, 1989), where endorsement rates of classically valid modus ponens (MP) ( $A \rightarrow B, A$ ; therefore  $B$ ) and modus tollens (MT) ( $A \rightarrow B, \neg B$ ; therefore  $\neg A$ ) are decreased when further premises indicating possible defeaters are presented. To illustrate: Normally, inferring “Lisa will study late in the library” from the premises “Lisa has an essay to write” and “If Lisa has an essay to write, then Lisa will study late in the library” would be seen as unproblematic. But if the additional premise “If the library is open, then Lisa will study late in the library” becomes available, then participants are much more reluctant to draw this inference. However, as Ragni et al. (2017) show, the inference mechanism exploiting ranking functions does not in itself retrodict the suppression effect. To do so, further assumptions about the underlying knowledge base instantiated in long-term memory need to be made. In addition, Ragni et al. (2017) present experiments that test ranking theory’s ability to predict the participants’ reasoning with MP and MT once keywords indicating nonmonotonicity such as “normally” are inserted. Characteristic of this line of research is that c-representations are used to inductively infer ranking functions that satisfy the constraints set by an assumed knowledge base.

Moreover, the account of conditionals in Spohn (2013, 2015) has inspired a series of experiments. Spohn (2013, 2015) outlines a number of expressive roles of conditionals that go beyond the Ramsey test, which merely takes conditionals to express conditional beliefs. For instance, conditionals may express reason relations as specified in definition 3. In Skovgaard-Olsen (2014), a logistic regression model was presented both to formulate predictions for the participants’ evaluations of the conclusions of MP, MT, AC (affirming the consequent:  $A \rightarrow B, B$ ; therefore  $A$ ), and DA (denying the antecedent:  $A \rightarrow B, \neg A$ ; therefore  $\neg B$ ) inferences and to suggest a solution to the problem of updating based on conditional information. In Skovgaard-Olsen, Singmann, and Klauer (2016), a reason-relation reading of the conditional was contrasted experimentally with the Ramsey test in participants’ probability and acceptability evaluations of indicative conditionals, and support was obtained for the reason-relation reading. In Skovgaard-Olsen, Kellen, Hahn, and Klauer (2019), it was found that there are patterns of individual variation in these results. In Skovgaard-Olsen, Singmann, and Klauer (2017), participants’ perceived relevance and reason-relation evaluations were investigated, and some

first evidence for the explications of reason relations and perceived relevance introduced above was obtained. However, characteristic of this latter line of research is that it adopts an indirect route to testing ideas from ranking theory by exploiting its extension to conditionals in Spohn (2013, 2015) and its parallels to probability theory (see also Henrion, Provan, Del Favero, & Sanders, 1994).

Presently, paradigms operationalizing degrees of beliefs as probabilities are much better established than tasks using ranking functions. This is so despite the fact that the arithmetical operations of ranking theory require much less computational effort than those of probability theory. For instance, it is well known that participants exhibit difficulties in properly integrating information about base rates of the rarity of a given disease in evaluating how likely it is that a person has the disease given a positive test result. Yet, interestingly, Juslin, Nilsson, Winman, and Lindskog (2011) find that the notorious base-rate neglect could be reduced when participants are given the tasks in a logarithmic format. Juslin et al. (2011) therefore conjecture that a linear, additive integration of information becomes more intuitive in the absence of access to overriding analytic rules that use a multiplicative format, like probabilities.

Nevertheless, there is a lack of direct experimental investigations of ranking functions. This is perhaps due to the following challenges:

- (i) Negative ranking functions are useful for conducting proofs. But it initially presents a conceptual challenge to think in terms of disbelief in negations of propositions as a way of representing full beliefs. Two-sided ranking functions solve this problem. But they come at the cost of having different rules applying to the negative and positive ranges of the scale.
- (ii) An operationalization of ranking functions in terms of iterated contractions exists (see above), but it has not received the same kind of experimental implementation as the operationalization of probabilities in terms of betting quotients.
- (iii) Since negative ranks take natural numbers from 0 to infinity, there is no natural way of nonarbitrarily dividing the scale into regions of ascending degrees of disbelief other than a region of zero disbelief, a region of above-zero disbelief, and a region of maximal disbelief. In contrast, the crude division of the probability scale of real numbers between 0 and 1 into decimal regions enables participants and experimenters to make qualitative differentiations between low degrees of belief (e.g., [0.0, 0.3]), medium degrees of belief (e.g., [0.3, 0.7]), and strong degrees of belief (e.g., (0.7, 1.0]).

If challenges such as these can be overcome in future work, ranking theory potentially has a lot to offer to psychology and to cognitive science more generally.

### Notes

1. Strictly speaking, “epistemic” only refers to knowledge, although it is often used more widely. Because we will talk only about belief, we prefer to use “doxastic” throughout. See also chapter 5.1 by van Ditmarsch (this handbook).
2. See chapter 4.1 by Hájek and Staffel, chapter 4.5 by Chater and Oaksford, chapter 4.7 by Dubois and Prade, chapter 8.3 by Glöckner, and chapter 8.4 by Hill (all in this handbook).
3. We use the notation of propositional logic, not the set-theoretic one. No confusion can arise thereby.
4. See chapter 4.3 by Merin (this handbook). Sections 1–4 of that chapter work just as well with ranking theory.
5. See also chapter 2.1 by Broome, chapter 2.2 by Wedgwood, and chapter 12.2 by Smith (all in this handbook).
6. This is completely analogous to Jeffrey conditionalization in probability theory (see Jeffrey, 1983, chapter 11; chapter 4.1 by Hájek & Staffel, this handbook). Shenoy (1991) has contributed an important variant.
7. In fact, we need no more than twofold nonvacuous contractions.
8. In AI, the distinction between knowledge and belief is usually quite vague.
9. Here,  $\vdash$  represents nonmonotonic or default entailment.

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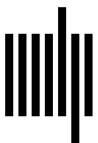
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