Proofs of the Nyquist-Shannon Sampling Theorem

Bachelorarbeit

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Zusammenfassung

Der Abtastsatz (engl. sampling theorem) besagt, dass eine bandbreitenbeschränkte Funktion aus ihren Funktionswerten an diskreten Stellen vollständig rekonstruiert werden kann, wenn der Abstand zwischen diesen Stellen ausreichend klein ist. Damit ist dieser Satz die Grundlage der Digitalisierung von kontinuierlichen Signalen. Das Ziel dieser Arbeit ist der Beweis von sechs verschiedenen Varianten des Abtastsatzes. Diese unterscheiden sich in den Voraussetzungen, aber auch in den Beweismethoden.

In Kapitel 1 wird eine kurze Einführung in die Fragestellung des Abtastsatzes sowie seine Geschichte und einige seiner Anwendungen gegeben. Danach wird die Struktur dieser Arbeit vorgestellt. Sie gliedert sich in Grundlagen (Kapitel 2), Fourieranalysis (Kapitel 3), den Abtastsatz (Kapitel 4) und eine Zusammenfassung (Kapitel 5).

In den Grundlagen werden zunächst die Notationen eingeführt. Danach folgt eine kurze Einführung in die Integrationstheorie von Lebesgue mit den wichtigsten Sätzen. Darauf aufbauend werden die L^p-Räume eingeführt. Ein Spezialfall ist L^2, da dieser die Struktur eines Hilbertraumes trägt. Deren Eigenschaften werden ebenfalls kurz vorgestellt. Schließlich benötigen wir noch lokalkonvexe Räume für die Schwartz-Funktionen. Außerdem werden später ein paar Resultate aus der Funktionentheorie verwendet, die sich ebenfalls in den Grundlagen finden. Zum Schluss werden Distributionen eingeführt.

Kapitel 3 widmet sich der Fourieranalysis. Darunter verstehen wir die Theorie von Fouriertransformation und Fourierreihen im Mehrdimensionalen. Die Fouriertransformation wird zunächst auf den Schwartzfunktionen definiert und dann auf L^1- und L^2-Funktionen übertragen. Anschließend werden die Eigenschaften der Fouriertransformation vorgestellt. Die temperierten Distributionen werden als Elemente des Dualraumes der Schwartzfunktionen definiert. Damit kann man die Fouriertransformation für temperierte Distributionen definieren.

Für Fourierreihen ist es zunächst notwendig, periodische Funktionen einzuführen. Periodische Funktionen kann man als Funktionen auffassen, die auf dem Torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ definiert sind. Aus diesem Grund bezeichnen wir die Fourierreihen auch als Fouriertransformation auf dem Torus. Die Fouriertransformation auf dem Torus bildet unendlich oft differenzierbare Funktionen auf dem Torus auf Schwartzfunktionen auf dem Ganzzahlgitter $\mathbb{Z}^n$ ab. Diese erfüllen ähnliche Abklingeigenschaften wie die normalen Schwartzfunktionen, sind aber nur für ganzzahlige Vektoren definiert. Damit ist deren Konvergenz immer gesichert. Für andere periodische Funktionen ist aber oft nicht klar, ob deren Fourierreihe konvergiert und wenn ja, ob sie gegen die ursprüngliche Funktion konvergiert. Diese Fragen behandeln wir kurz in einem eigenen Abschnitt. Eine Anwendung der Fourierreihen ist die Poisson'sche Summenformel. Anschließend wird die Fourierreihe auf periodische Distributionen übertragen. Dabei werden periodische Distributionen als Elemente des Dualraums der unendlich oft differenzierbaren Funktionen auf dem Torus definiert. Damit lässt sich die Fourierreihe einer periodischen Distribution mit Hilfe der oben definierten Fouriertransformation auf dem Torus erklären. Es wird gezeigt, dass die Fouriertransformation auf dem Torus sowohl für die unendlich oft differenzierbaren periodischen Funktionen als auch für die periodischen Distributionen ein Homöomorphismus in den zugehörigen Schwartz-(Dual)-Raum ist.

In Kapitel 4 werden schließlich verschiedene Versionen des Abtastsatzes bewiesen. Zunächst wird dafür das Konzept der Bandbreitenbeschränktheit eingeführt. Dann werden einige Zusammenhänge zwischen Bandbreitenbeschränkung und exponentiell beschränktem Wachstum für

komplexe Argumente gezeigt. Eine wichtige Rolle spielen hierbei die Sätze von Paley und Wiener.

Der erste Beweis beruht auf der Vertauschbarkeit der Faktoren im halbdiskreten Faltungsprodukt. Die abzutastende Funktion muss dafür W-bandbreitenbeschränkt und stetig sein und in $\mathrm{L}^2(\mathbb{R})$ liegen. Dann kann man die Funktion mit der Abtastreihe aus den Abtastwerten rekonstruieren, wenn die Werte mit einer Frequenz von W/π aufgenommen wurden. Die Abtastreihe is absolut und gleichmäßig konvergent.

Für den Beweis mit Hilfe der verallgemeinerten Parsevalgleichung muss die abzutastende Funktion als Fourierrücktransformation einer $\mathrm{L}^2([-W,W]^n)$-Funktion geschrieben werden können. Dieser Beweis liefert eine Verallgemeinerung auf n Dimensionen. Die anderen Resultate sind die selben wie im vorherigen Beweis.

Als nächstes wird mit Hilfe einer Schauderbasis der Abtastsatz für Funktionen bewiesen, die als Rücktransformation einer $\mathrm{L}^p([-W,W]^n)$-Funktion für $1 < p \leqslant \infty$ geschrieben werden können. Zusätzlich muss gefordert werden, dass die Reihe der Abtastwerte absolut konvergiert. Dann konvergiert die Abtastreihe absolut und gleichmäßig.

Der Satz von Paley und Wiener stellt, wie oben angedeutet, einen Zusammenhang zwischen einer Funktion mit kompakten Träger und Eigenschaften ihrer Fouriertransformierten wie Fortsetzbarkeit zu einer ganzen Funktion und exponentiell beschränktes Wachstum für komplexe Argumente her. Es wird gezeigt, dass die Menge der um ganze Argumente verschobene sinc-Funktionen eine Orthonormalbasis im Hilbertraum aller exponentiell beschränkten ganzen und quadratintegrierbaren Funktionen bildet. Dann ist der Abtastsatz einfach die Entwicklung der Funktion in dieser Basis. Dieser Hilbertraum kann mit dem Hilbertraum aus den ersten beiden Beweisen identifiziert werden, sodass die Aussagen ebenfalls die selben sind.

Der Abtastsatz kann für ganze exponentiell beschränkte Funktionen mittels Pfadintegralen bewiesen werden. Diese Voraussetzung lässt auch nicht integrierbare Funktionen wie die Sinus-Funktion zu. Damit geht allerdings einher, dass dass die Abtastfrequenz erhöht werden muss, keine absolute Konvergenz mehr gegeben ist und gleichmäßige Konvergenz nur noch auf beschränkten Mengen gilt. Eine Verallgemeinerung auf n Dimensionen scheint möglich, erfordert aber tiefergehende Theorien.

Schließlich wird der Abtastsatz noch für Funktionen bewiesen, deren Fouriertransformierte nur als Distribution mit kompaktem Träger gegeben ist. In diesem allgemeinen Fall ist ebenfalls eine erhöhte Abtastfrequenz erforderlich. Außerdem muss die sinc-Funktion durch die Fouriertransformierten einer glatten Variante der charakteristischen Funktion ersetzt werden, um Konvergenz zu erhalten. Die Abtastreihe konvergiert gleichmäßig auf kompakten Teilmengen von $\mathbb{C}^n$ und der Beweis gilt für n Dimensionen.

In Kapitel 5 werden die Ergebnisse des Abtastsatzes noch einmal zusammengefasst. Außerdem wird ein Ausblick auf weitere Fragestellungen gegeben, die in diesem Zusammenhang untersucht werden könnten.

Abstract

The sampling theorem states that a band limited function can be fully reconstructed by its discrete samples if they are close enough. Therefore, it is the basis for digitalization of continuous signals. In this work, we prove six versions of the sampling theorem with different methods and under different assumptions. First, we provide a basic overview over the basic theorems used later on. Then we present the theory of Fourier transforms and Fourier series. We introduce it for functions as well as for distributions. Based on this, we can introduce the concept of band limited functions and distributions, which is crucial for the sampling theorem. Then we prove the sampling theorem using the commutativity of semi-discrete convolution product, the generalized Parseval formula, a Schauder basis, an orthogonal system, contour integration and, finally, the Fourier series for distributions. Whenever possible, the sampling theorem is shown in a multidimensional setting. The prerequisites are relaxed from band-limited L^2-functions to functions, having a distributional Fourier transform with compact support. As a consequence, the absolute and uniform convergence of the sampling series weakens to uniform convergence on compact sets and the sinc interpolation function has to be replaced.

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Contents

1 Introduction

Today, almost everything is digital - music, television, books. The change of our everyday life due to the impact of the digital world can be compared to the industrial revolution [Thi]. The word “digital” refers to the main feature of the digital world: Any kind of data, as sounds, pictures or text, is represented by a sequence of the binary digits 0 and 1. But - apart from quantum mechanics - real world is continuous. Music, for example, is a continuous variation of air pressure over time, from the physical point of view. Thus, how can you represent music by discrete numbers, which can be pressed into a CD or stored on a flash drive? At this point, mathematics comes into play.

Abstracting from the physical details, we are concerned with the following questions: When is it possible to totally represent a function by discrete values? How can one reconstruct the original function if the samples are known? These questions have been answered for communication theory by Shannon [Sha49] for the western hemisphere and independently by Kotel'nikov [Kot33] in soviet literature. For successful sampling and reconstruction, the function has to be band limited and the sampling frequency must be at least twice the band width. The converse was shown some years before by Nyquist. He showed that the number of independent pulse samples that can be sent through a telegraph line is limited by the band with of the transmission system [Nyq28]. However, the topic showed up in mathematical literature even before. Whittaker investigated the interpolation problem of finding a function passing through given points [Whi15]. He found the sampling series and noted that the resulting function is band limited. A rather complete history of the sampling theorem can be found in [Hig77].

Today, the sampling theorem is not only of interest in communication theory, but also for time-varying systems, boundary value problems, splines or in optics and crystallography, analysis of meteorological data or models of cardiac pacemakers, only to mention a few of the examples in [Jer77, Section VII]. It can also be applied in quantum mechanics to reconstruct wave functions [Ray97]. As a more real world application, the sampling theorem is used in computerized tomography [Nat89]. Finally, the sampling frequency of audio files is often 44.1 kHz. This is slightly more than twice the 20 kHz, humans can hear. We will find the reason for this oversampling in the proofs for the more general versions of the sampling theorem. But first, we introduce the structure of this work.

Chapter 2 provides the basic facts needed throughout the thesis. This includes, after introducing the symbols used, Lebesgue's integration theory, L^p-spaces, Hilbert spaces, locally convex spaces, a few results of complex analysis and distributions.

Based on this, Fourier transform and Fourier series are established in Chapter 3. Starting with Schwartz functions, the properties of the Fourier transform are stated and extended to L^1 and L^2. Then the Fourier transform is introduced for tempered distributions. For periodic functions the Fourier transform is defined on the torus, which yields the Fourier series. A big issue is the convergence of the Fourier series, which is only discussed shortly. One consequence of the expansion as Fourier series is Poisson's summation formula, which is proved for Schwartz functions. Finally, the concept of Fourier series is carried over to periodic distributions, which will be needed for the distributional version of the sampling theorem.

In Chapter 4, the concept of band limited functions and distributions is introduced. This is the basis for the six proofs of slightly different versions of the Nyquist-Shannon sampling theorem. The first one makes use of the commutativity of the semi-discrete convolution product and is valid for band-limited functions of $\mathrm{L}^2(\mathbb{R})$. The next proof uses the generalized Parseval formula being valid for any function which can be written as inverse Fourier transform of a function of $\mathrm{L}^2([-W,W]^n)$. This result is extended to functions of $\mathrm{L}^p([-W,W])$ for $1 < p \leqslant \infty$ by means of a Schauder basis in $\mathrm{L}^p([-W,W])$. In the fourth proof it is shown that the set of shifted sinc_W functions forms a complete orthogonal set in the Hilbert space of all W-band limited $\mathrm{L}^2(\mathbb{R}^n)$ functions. Then the sampling theorem is simply the expansion of any function in this basis. Via contour integration in the complex plane it is possible to prove the sampling theorem without using any Fourier transform. It only requires the function to be bounded on the real line and growing at most exponentially for complex arguments. Finally, a sampling expansion is given for functions of polynomial growth on the real line, whose Fourier transform can only be given as a distribution with compact support.

2 Foundations

In this chapter the fundamentals for the Nyquist-Shannon sampling theorem are given. First, the symbols and notations for classical derivation and integration are introduced. Then L^p, Hilbert and locally convex spaces are presented. We also give some results of complex analysis, which are necessary for one of the proofs. Finally, the theory of distributions is presented.

2.1 Notations

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. First, we want to consider continuous and differentiable functions. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\Omega \subseteq \mathbb{K}^n$ be a domain for some fixed $n \in \mathbb{N}$. For $x \in \mathbb{K}^n$ we write $x = (x_1, \dots, x_n)$ when it is necessary to talk about the components. For $z \in \mathbb{C}^n$ we define the real part $\operatorname{Re} z := (\operatorname{Re} z_1, \dots, \operatorname{Re} z_n)$ and the imaginary part $\operatorname{Im} z := (\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)$, hence $z = \operatorname{Re} z + \mathrm{i} \operatorname{Im} z$.

2.1 Definition. Let X be a topological space and $f : X \to \mathbb{K}$. Then the *support* of f is defined as

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}},$$

where the bar denotes the topological closure.

For the case of multidimensional derivation a multi-index notation is introduced.

2.2 Definition. Let $\alpha, \beta \in \mathbb{N}_0^n$ and $x, y \in \mathbb{K}^n$. Then

(a) $|\alpha| := \sum_{i=1}^n \alpha_i$, $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(b) $x^\alpha := \prod_{i=1}^n x_i^{\alpha_i} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

(c) $xy := \sum_{i=1}^n x_i y_i$

(d) $\partial^\alpha := \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}}$

2.3 Definition. (a) $C(\Omega)$ is the vector space of the continuous functions from Ω to $\mathbb{K}$.

(b) $C_c(\Omega)$ is the set of all $f \in C(\Omega)$ with compact support.

(c) $C^k(\Omega)$ is the vector space of k times differentiable functions f from Ω to $\mathbb{K}$, where the kth derivative $f^{(k)}$ is continuous. In symbols

$$C^k(\Omega) = \{f \in C(\Omega) : \forall \alpha \in \mathbb{N}_0^n : |\alpha| \leqslant k \Rightarrow \partial^\alpha f \in C(\Omega)\}.$$

(d) The infinitely continuously differentiable functions are defined as

$$C^\infty(\Omega) := \bigcap_{k \in \mathbb{N}} C^k(\Omega) = \{f \in C(\Omega) : \partial^\alpha f \in C(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^n\}.$$

2.2 Lebesgue's Integration Theory

For integration, we use Lebesgue's theory. Since in this work we focus on the results of this theory, only a short overview over the basics is given. Details can be found, for example, in [Rud66]. While Riemann integration uses intervals of decreasing length to approximate the integral, Lebesgue allows these sets to be more general, so called *measurable*. The crucial properties are that the intersection of a countable family of measurable sets is measurable and that the complement of every measurable set is measurable. The set of all measurable sets is called σ-algebra and often denoted by $\mathcal{A}$. Since the idea of approximating the integral by rectangles remains the same, one has to extend the idea of "length" to these sets. This is called *measure* and it has the property that the measure of the union of pairwise disjoint measurable sets is the sum over the measures of the single sets, called *countable additivity*. Further one has to define *measurable functions* to avoid problems. A function is measurable if the preimage of every measurable set is measurable (with respect to the associated σ-algebras). The triple $(X, \mathcal{A}, \mu)$ with a set X, an associated σ-algebra $\mathcal{A}$ and an associated measure μ is called *measure space*. Given a measure space, we can define integration of measurable functions.

2.4 Definition (Lebesgue integration)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) $f\colon X \to [0, \infty)$ is called a *simple function* if its range consists only of finitely many points, i.e. it has a representation

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i},$$

with c_i being the (distinct) values of f, $A_i := \{x \in X : f(x) = c_i\}$ and χ_A being the characteristic function of A.

(b) Let f be a measurable simple function. Then the integral is defined as

$$\int f \mathrm{d}\mu := \sum_{i=1}^{n} c_i \mu(A_i).$$

(c) Let $f\colon X \to [0, \infty]$ be a measurable function. Then the integral is defined as

$$\int f \mathrm{d}\mu := \sup \left\{ \int s \mathrm{d}\mu : s \text{ is a simple function with } 0 \leqslant s \leqslant f \right\}.$$

(d) Let $f\colon X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be a measurable function. Then f can be separated into $f = f^+ - f^-$ with $f^+ := \max\{0, f\}$ and $f^- := -\min\{0, f\}$, both being non-negative and measurable. The integral is defined by

$$\int f \mathrm{d}\mu = \int f^+ \mathrm{d}\mu - \int f^- \mathrm{d}\mu,$$

if at least one of the integrals is finite.

(e) A measurable function f is called *integrable* if the integral over $|f|$ is finite. The set of all integrable functions (with respect to a measure μ) is called $\mathcal{L}^1(\mu)$.

(f) Let $f\colon X \to \mathbb{C}$ be an integrable function. Then f can be split into its real part u and its imaginary part v, both being measurable and real-valued. The integral is defined as

$$\int f \mathrm{d}\mu = \int u \mathrm{d}\mu + \mathrm{i} \int v \mathrm{d}\mu.$$

Since $u+ \leqslant |u| \leqslant |f|$, etc., all integrals are finite.

(g) Let $A \in \mathcal{A}$ and f be a measurable function. Then we define

$$\int_A f \mathrm{d}\mu := \int f \chi_A \mathrm{d}\mu.$$

2.5 Remark. Note that measurable functions (but not simple functions) are allowed to have the value ∞, which allows to incorporate many special cases into the general theory. Summation and multiplication of infinity is defined by $a + \infty = \infty + a = \infty$ if $0 \leqslant a \leqslant \infty$ and $a \cdot \infty = \infty \cdot a = \infty$ if $0 < a \leqslant \infty$ and $0 \cdot \infty = \infty \cdot 0 = 0$. The implications $ac = bc \Rightarrow a = b$ and $a + c = b + c \Rightarrow a = b$ only hold for $c \neq \infty$ (cf. [Rud73, 1.22, p. 18]).

2.6 Remark. It can be shown that for a measurable function f the integral $\int f \mathrm{d}\mu$ is finite if and only if $\int |f| \mathrm{d}\mu$ is finite.

From the definitions given above, it follows that the integral over any function over a set of measure zero is zero. Therefore, it is possible to change the values of a function on a set of measure zero without changing the value of the integral. This leads to the definition of almost everywhere.

2.7 Definition. Let P be a statement for a point and $A \in \mathcal{A}$. Then P is said to hold *almost everywhere (a.e.)* on A iff there is an $N \in \mathcal{A}$ with $\mu(N) = 0$, $N \subseteq A$ and P holds for all $x \in A \backslash N$.

Note that the concept of a.e. depends on the measure. Usually, it is clear from the context, which measure is meant.

The following important theorems of Lebesgue's theory are also formulated in the general setting. Later on, we will use the Lebesgue-measure λ and the Borel-σ-algebra, which is generated by the open sets of $\mathbb{R}^n$. This implies that all continuous functions are measurable.

2.8 Theorem (monotone convergence)**.** *([Rud66, 1.26, p. 21]) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions from X to $\overline{\mathbb{R}}$ with $0 \leqslant f_1(x) \leqslant f_2(x) \leqslant \ldots \leqslant \infty$ and define $f(x) := \lim_{n \to \infty} f_n(x)$ for almost every $x \in X$. Then f is measurable and*

$$\lim_{n \to \infty} \int_X f_n \mathrm{d}\mu = \int_X f \mathrm{d}\mu = \int_X \lim_{n \to \infty} f_n \mathrm{d}\mu.$$

2.9 Theorem (dominated convergence by Lebesgue)**.** *([Rud66, 1.34, p.26]) Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of complex measurable functions on X for which $f(x) = \lim_{n \to \infty} f_n(x)$ exists for almost every $x \in X$. If there is a function $g \in \mathcal{L}^1(\mu)$ with $|f_n(x)| \leqslant g(x)$ for all $n \in \mathbb{N}$ and almost every $x \in X$, then $f \in \mathcal{L}^1(\mu)$,*

$$\lim_{n \to \infty} \int |f_n - f| \mathrm{d}\mu = 0 \quad \textit{and} \quad \lim_{n \to \infty} \int f_n \mathrm{d}\mu = \int f \mathrm{d}\mu = \int \lim_{n \to \infty} f_n \mathrm{d}\mu.$$

2.10 Theorem. *([Rud66, 1.38, p. 28]) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex measurable functions defined a.e. on X with $\sum_{n=1}^{\infty}\int|f_n|\mathrm{d}\mu < \infty$. Then the series $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for almost all $x \in X$, $f \in \mathcal{L}^1(\mu)$ and*

$$\sum_{n=1}^{\infty}\int f_n\mathrm{d}\mu = \int f\mathrm{d}\mu = \int\sum_{n=1}^{\infty} f_n\mathrm{d}\mu.$$

2.11 Theorem. *([DR11, 13.43, p. 196]) Let $U, V \subset \mathbb{R}^n$ be open and let $\Phi : U \to V$ be a bijective, continuously differentiable function with continuously differentiable inverse Φ^{-1} (a diffeomorphism). A measurable function $f : V \to \mathbb{R}^n$ is integrable on $V = \Phi(U)$ if and only if $(f \circ \Phi)|\det \Phi'| : U \to \mathbb{C}$ is integrable on U. In this case*

$$\int_{\Phi(U)} f(y)\mathrm{d}y = \int_U |\det \Phi'(x)|f(\Phi(x))\mathrm{d}x$$

holds.

2.3 L^p-spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space. This space is rather general and does not provide many structure. Therefore, we want to introduce a norm and choose a subspace in order to obtain a Banach space.

2.12 Definition. Let $f : X \to \mathbb{K}$ be a measurable function. For $1 \leqslant p < \infty$ define

$$\|f\|_p := \left(\int_X |f|^p\,\mathrm{d}\mu\right)^{1/p}$$

$$\text{and } \mathcal{L}^p(\mu) := \{f : X \to \mathbb{K} : f \text{ measurable}, \|f\|_p < \infty\}.$$

For $p = \infty$ define $g := |f|$ and

$$\|f\|_\infty := \operatorname{esssup} f := \inf\left(\{r \geqslant 0 : \mu(g^{-1}((r, \infty])) = 0\} \cup \{\infty\}\right),$$

where $\operatorname{esssup} f$ is called the *essential supremum* of f (cf. [Rud66, 3.7, p. 64]). Similarly, $\mathcal{L}^\infty(\mu)$ contains all measurable functions f with $\|f\|_\infty < \infty$. If μ is the Lebesgue measure and $X =: \Omega \subseteq \mathbb{K}^n$ is measurable, we write $\mathcal{L}^p(\Omega)$.

2.13 Theorem. *([Rud66, 3.5, p. 62]) Let $1 \leqslant p, q \leqslant \infty$ with $1/p + 1/q = 1$, $f, g : X \to [0, \infty]$ measurable. Then*

$$\int fg\mathrm{d}\mu \leqslant \left(\int f^p\mathrm{d}\mu\right)^{1/p}\left(\int g^q\mathrm{d}\mu\right)^{1/q}$$

(Hölder's inequality) and

$$\left(\int (f+g)^p\mathrm{d}\mu\right)^{1/p} \leqslant \left(\int f^p\mathrm{d}\mu\right)^{1/p} + \left(\int g^p\mathrm{d}\mu\right)^{1/p}$$

(Minkowski's inequality).

The importance of these inequalities becomes clear if one considers the function $|f| : X \to [0, \infty]$ for $f : X \to \mathbb{K}^n$. In this case the integrals convert to the norm defined above.

2.14 Remark. Using Minkowski's inequality[1], one can show that $\|\cdot\|_p$ fulfills the triangle

[1] for $1 < p < \infty$ and the triangle inequality in $\mathbb{K}$ for $p \in \{1, \infty\}$

inequality. From the properties of the integral[2] the homogeneity of the mapping $\|\cdot\|_p$ is easily seen. But $\|\cdot\|_p$ defines only a seminorm on $\mathcal{L}^p$, because $\|f\|_p = 0$ precisely if $f(x) = 0$ *almost* everywhere. To overcome this problem, we define an equivalence relation by $f \sim g$ iff $\|f - g\|_p = 0$.

2.15 Definition. Let $\mathcal{L}^p(\mu)$ and the equivalence relation $\sim$ be defined as above. Then we set

$$\mathrm{L}^p(\mu) := \mathcal{L}^p(\mu)/\{f \in \mathcal{L}^p(\mu) : f \sim 0\} = \{[f] : f \in \mathcal{L}^p(\mu)\},$$

i.e. $\mathrm{L}^p(\mu)$ consists of the equivalence classes of $\mathcal{L}^p(\mu)$. $[f]$ denotes the equivalence class of f with respect to $\sim$.

2.16 Example. If we choose $\subset \mathbb{N}$, $\mathcal{A} := \mathcal{P}(\mathbb{N})$ the power set of $\mathbb{N}$ and $\mu := \xi$ to be the counting measure. Then the space $\ell^p(\mathbb{N}) := \mathrm{L}^p(xi)$ is the space of all sequences which converge in the ℓ^p-norm $\|(a_i)_{i\in\mathbb{N}}\|_p = \left(\sum_{i=1}^n |a_i|^p\right)^{1/p}$. In this case there is no set of measure zero, except for the empty set. Therefore, it is not necessary to consider equivalence classes.

Despite the fact that the elements of $\mathrm{L}^p(\mu)$ in general are equivalence classes, we continue calling them functions. But we have to keep in mind that these functions are only defined almost everywhere and cannot be evaluated at a single point. Fortunately, we have the following theorem:

2.17 Theorem. *([Rud66, 3.14, p. 68]) Let $\Omega \subseteq \mathbb{K}^n$ be open. For $1 \leqslant p < \infty$, $C_c(\Omega)$ is dense in $\mathrm{L}^p(\Omega)$.*

Hence every L^p-function ($1 \leqslant p < \infty$) can be approximated by a sequence of continuous functions with compact support.

Additionally, one can show the following:

2.18 Theorem. *([Rud66, 3.11, p. 66]) $\mathrm{L}^p(\Omega)$ is a complete normed vector space, i.e. a Banach space, for $1 \leqslant p \leqslant \infty$.*

Therefore, for for $1 \leqslant p < \infty$ $\mathrm{L}^p(\Omega)$ can be considered as the completion of $C_c(\Omega)$ with respect to the norm $\|\cdot\|_p$. This implies many the definitions introduced for continuous functions can be extended to L^p-functions. From now on, we will only consider the Lebesgue-measure and the Borel-σ-algebra if not stated otherwise. We use the symbol $\mathrm{d}x$ rather than $\mathrm{d}\mu(x)$ to indicate the integration variable.

2.4 Hilbert spaces

L^2 is a special L^p-space, because it is a Hilbert space. In this section, we will present some of the properties of Hilbert spaces.

2.19 Definition. Let H be a vector space over $\mathbb{K}$ and let $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{K}$ be an inner product on H, i.e.

(i) $\langle\cdot, v\rangle : H \to \mathbb{K}$ is linear for all $v \in H$,

[2] and the supremum for $p = \infty$

(ii) $\langle v, w\rangle = \overline{\langle w, v\rangle}$ for all $v, w \in H$, where the bar denotes complex conjugation,

(iii) $\langle v, v\rangle > 0$ for all $v \in H\backslash\{0\}$ and $\langle v, v\rangle = 0$ only if $v = 0$.

If H is complete with respect to the norm $\|v\| := \sqrt{\langle v, v\rangle}$ ($v \in H$), then H is called *Hilbert space.*

For the remaining section let H always denote a Hilbert space.

2.20 Examples. (i) $\mathbb{C}^n$ is a Hilbert space if the inner product is defined as

$$\langle x, y\rangle_{\mathbb{C}^n} := \sum_{i=1}^{n} x_i \overline{y_i}$$

for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{C}^n$.

(ii) $\mathrm{L}^2(\mathbb{R}^n)$ is a Hilbert space if the inner product is defined as

$$\langle u, v\rangle_{\mathrm{L}^2(\mathbb{R}^n)} := \int_{\mathbb{R}^n} u(x)\overline{v(x)}\mathrm{d}x \quad (u, v \in \mathrm{L}^2(\mathbb{R}^n)).$$

(iii) Let $K \subset \mathbb{C}^n$ compact with $\lambda(K) > 0$, where $\lambda(K)$ is the Lebesgue-measure of K. Then $\mathrm{L}^2(K)$ with is a Hilbert space if the inner product is defined as

$$\langle u, v\rangle_{\mathrm{L}^2(K)} = \frac{1}{\lambda(K)} \int_K u(x)\overline{v(x)}\mathrm{d}x \quad (u, v \in \mathrm{L}^2(K)).$$

Note that $K \Subset \mathbb{R}^n$ implies $K \Subset \mathbb{C}^n$.

The inner product enables us to establish the concept of orthogonality.

2.21 Definition. (i) $x \in H$ is *orthogonal* to $y \in H$ if and only if $\langle x, y\rangle = 0$, in symbols $x \perp y$.

(ii) For sets $E, F \subset H$ we say that E is orthogonal to F, in symbols $E \perp F$, if for each $x \in E$ and $y \in F$ holds: $x \perp y$.

(iii) $x^\perp$ is the *orthogonal complement* of x defined by $x^\perp := \{y \in H : \langle x, y\rangle = 0\}$.

(iv) Let $M \subset H$ be a subspace. Then $M^\perp := \{y \in H : \forall x \in M : \langle x, y\rangle = 0\}$.

(v) Let A be an index set, $(u_\alpha)_{\alpha\in A} \subset H$ a sequence of vectors in a Hilbert space H. The set $\{u_\alpha : \alpha \in A\}$ is called *orthonormal* if $\langle u_\alpha, u_\beta\rangle = 0$ for all $\alpha, \beta \in A$ with $\alpha \neq \beta$ and $\|u_\alpha\| = 1$ for every $\alpha \in A$.

2.22 Lemma. *([Rud66, p. 84]) For any $x \in H$ and any orthonormal set $\{u_\alpha : \alpha \in A\} \subset H$, the cardinality of $\{\alpha \in A : \langle x, u_\alpha\rangle \neq 0\}$ is at most countable.*

2.23 Theorem (Bessel's Inequality)**.** *([Rud66, 4.16, p. 84]) If $\{u_\alpha : \alpha \in A\}$ is any orthonormal set in H then*

$$\sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2 \leqslant \|x\|^2$$

for all $x \in H$.

2.24 Theorem. *([Rud66, 4.18, p. 85]) Let $\{u_\alpha : \alpha \in A\}$ be an orthonormal set in H. The following four statements are equivalent:*

(a) $\{u_\alpha : \alpha \in A\}$ is a maximal orthonormal set in H.

(b) The set S of all finite linear combinations of members of $\{u_\alpha : \alpha \in A\}$ is dense in H.

(c) For every $x \in H$, we have $\|x\| = \sum_{\alpha\in A} |\langle x, u_\alpha\rangle|^2$.

(d) If $x, y \in H$, then

$$\langle x, y\rangle = \sum_{\alpha\in A} \langle x, u_\alpha\rangle \overline{\langle y, u_\alpha\rangle}. \tag{2.1}$$

This equation is known as Parseval's identity.

2.25 Remark. Maximal orthogonal sets are also called *complete* orthogonal sets.

2.26 Lemma. *([Rud66, p. 85]) $\{u_\alpha : \alpha \in A\} \subset H$ is a maximal orthonormal set if for every $x \in H$ the property $\langle x, u_\alpha\rangle = 0$ for every $\alpha \in A$ implies $x = 0$.*

The converse result is the Riesz-Fischer theorem.

2.27 Theorem (Riesz-Fischer)**.** *([Hig77, p. 16]) If $\{u_\alpha : \alpha \in A\} \subset H$ is a maximal orthonormal set then $\langle x, u_\alpha\rangle = \langle y, u_\alpha\rangle$ for every $\alpha \in A$ implies $x = y$ for $x, y \in H$.*

2.5 Locally convex spaces

In some cases it is not possible to describe a topological vector space with a single norm or even a single seminorm, but with a family of seminorms. Since all spaces of distributions we will consider in this work are of this type, we will briefly introduce the concept of locally convex spaces.

2.28 Definition. Let X be a vector space over $\mathbb{K}$ and $\mathcal{T} \subset \mathcal{P}(X)$.

(a) $\mathcal{T}$ is called *topology* on X if $\varnothing, X \in \mathcal{T}$ and if arbitrary unions and finite intersections of elements of $\mathcal{T}$ are contained in $\mathcal{T}$. $(X, \mathcal{T})$ is called topological space.

(b) $O \subset X$ is called *open* if $O \in \mathcal{T}$.

(c) $U \subset X$ is called *neighbourhood* of $x \in X$ if there is $O \in \mathcal{T}$ with $x \in O$ and $O \subset U$. The set of all neighbourhoods of $x \in X$ is denoted $\mathcal{U}(x)$.

(d) For $x \in X$ a subset $\mathcal{V}(x) \subset \mathcal{U}(x)$ is called *local basis* or neighbourhood basis of x if for all $U \in \mathcal{U}(x)$ there is $V \in \mathcal{V}(x)$ with $V \subset U$.

2.29 Definition. Let X, Y be topological spaces and $T : X \to Y$. Then f is *continuous at* $x_0 \in X$ if for every neighbourhood V of $T(x_0)$ there is a neighbourhood W of x_0 with $T(W) \subset V$.

2.30 Definition. Let X be a vector space over $\mathbb{K}$ and $P = \{p_\lambda : \lambda \in \Lambda\}$ be a family of seminorms $p_\lambda : X \to [0, \infty)$. Define $B^{(\lambda)}(x, r) := \{y \in X : p_\lambda(x - y) < r\}$ for $x \in X, \lambda \in \Lambda$ and $r > 0$. The topology $\mathcal{T}_P$ generated by

$$U_0 := \left\{B^{(\lambda)}(x, r) : x \in X, r > 0, \lambda \in \Lambda\right\}$$

is called the locally convex topology on X generated by P.

A topological space $(X, \mathcal{T})$ is called locally convex if there exists a family of seminorms P whose generated topology as defined above is the same, i.e. $\mathcal{T} = \mathcal{T}_P$. We denote a locally convex space by (X, P).

2.31 Lemma. *([Den12, 1.2, p.1]) Let (X, P) be a locally convex space. Let Λ be an index set for the family of seminorms P. Then the set*

$$U_x := \left\{ \bigcap_{\lambda \in L} B^{(\lambda)}(x, r) : L \subset \Lambda \text{ is finite}, r > 0 \right\}$$

is a local base for $x \in X$.

2.32 Theorem. *([Den12, 1.11, p. 5]) Let X and Y be locally convex vector spaces and let $\mathcal{X}$ and $\mathcal{Y}$ be sets of seminorms which induce the topology on X and Y, respectively. Let $T : X \to Y$ be linear. Then the following statements are equivalent:*

(i) T is continuous

(ii) T is continuous at 0

(iii) If $q\colon Y \to [0, \infty)$ is a continuous seminorm (not necessarily in $\mathcal{Y}$, then $q \circ T : X \to [0, \infty)$ is a continuous seminorm

(iv) For every $q \in \mathcal{Y}$ there is $\mathcal{Z} \subset \mathcal{X}$ finite and $M > 0$ such that $q(Tx) \leqslant M \max_{p \in \mathcal{Z}} p(x)$ for each $x \in X$.

2.33 Corollary. *Let X be a locally convex vector space and let $\mathcal{X}$ be a set of seminorms which induce the topology on X. Let $T : X \to \mathbb{K}$ be linear. Then T is continuous if and only if there is $\mathcal{Z} \subset \mathcal{X}$ and $M > 0$ such that $|Tx| \leqslant M \max_{p \in \mathcal{Z}} p(x)$ holds for all $x \in X$.*

2.34 Definition. Let $(X, \mathcal{X}), (Y, \mathcal{Y})$ be locally convex vector spaces. Define

$$L(X, Y) := \{ f\colon X \to Y : f \text{ is linear and continuous} \}.$$

Furthermore, we define the *dual space* of X by $X' := L(X, \mathbb{K})$.

For the dual space there are several frequently used topologies.

2.35 Definition. Let X be a topological vector space and X' its dual space.

(a) A set $B \subset X$ is called bounded, if there is $\lambda > 0$ for every neighbourhood U of zero, such that $B \subset \lambda U$.

(b) The weak-$*$-topology on X' is the smallest (coarsest) topology on X', such that the mapping $x'\colon X' \to \mathbb{K}, f \mapsto f(x)$ is continuous for every $x \in X$.

(c) The strong topology on X' is defined as the locally convex topology generated by the seminorms

$$p_S(f) := \sup_{x \in B} |f(x)| \quad f \in X'$$

for every $B \subset X$ such that for every $f \in X'$ there holds $\sup_{x \in B} |f(x)| < \infty$ (cf. [Heu06, p. 372]).

The topologies can also be characterized by their convergence properties.

2.36 Lemma. *([Tre67, p. 197f]) Let X be a topological vector space and X' its dual space. A sequence $(f_n)_{n\in\mathbb{N}} \subset X'$ converges to zero*

(a) in the weak-$*$-topology, if and only if for each $x \in X$ the sequence $(f_n(x))_{n\in\mathbb{N}} \subset \mathbb{K}$ converges to zero,

(b) in the strong topology, if and only if the function sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on every bounded subset of X.

However, both convergence types coincide in some cases. We will only give the result necessary for our proof.

2.37 Theorem. *([Tre67, Corollary 2, p. 358]) Let $\Omega \subset \mathbb{R}^n$ be open. In the strong dual space of $C^\infty(\Omega)$ every weakly converging sequence is strongly converging.*

2.6 Complex Analysis

Since we will be dealing with functions with complex variables, we will briefly introduce the most important concepts of complex analysis.

2.38 Definition. Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. Then f is

(a) *complex differentiable* in $z_0 \in \Omega$ if and only if the limit

$$f'(z) := \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists,

(b) *holomorphic* in Ω if and only if f is complex differentiable for every $z \in \Omega$,

(c) *entire* if $\Omega = \mathbb{C}$ and f is holomorphic on $\mathbb{C}$.

This can be generalized to n dimensions:

2.39 Definition. Let $\Omega \subset \mathbb{C}^n$ be open and $f : \Omega \to \mathbb{C}$. Then f is

(a) *holomorphic* in Ω if and only if it is holomorphic in each variable separately, i.e. for all $(a_1, \ldots, a_n) \in \Omega$ each of the functions

$$g_i : \Omega_i \to \mathbb{C}, \quad \lambda \mapsto f(a_1, \ldots, a_{i-1}, a_i + \lambda, a_{i+1}, \ldots, a_n) \quad (i = 1, \ldots, n)$$

is holomorphic in some neighborhood of 0;

(b) *entire* if and only if it is holomorphic in $\Omega = \mathbb{C}^n$.

2.40 Lemma. *([Rud66, corollary, p.210]) Let $\Omega \subset \mathbb{C}$ be open, let $f, g : \Omega \to \mathbb{C}$ be holomorphic with $f(z) = g(z)$ for all z in some set which has a limit point in Ω. Then $f(z) = g(z)$ for all $z \in \Omega$.*

2.41 Lemma. *([Rud73, 7.21, p. 181]) If $f : \mathbb{C}^n \to \mathbb{C}$ is entire and $f(z) = 0$ for all $z \in \mathbb{R}^n$, then $f(z) = 0$ for all $z \in \mathbb{C}^n$.*

2.42 Definition. Let $\Omega \subset \mathbb{C}$ be open, let $a \in \Omega$ and let $f\colon \Omega\backslash\{a\} \to \mathbb{C}$ be holomorphic. Let a be a pole of f of order k. Then f has the *Laurent series* expansion

$$f(z) = \sum_{i=-k}^{\infty} c_i (z-a)^i \quad (z \in \Omega\backslash\{a\})$$

with coefficients $c_i \in \mathbb{C}$. The coefficient c_{-1} is called the *residue* of f at a, in symbols

$$c_{-1} = \operatorname{res}(f; a).$$

2.43 Lemma. *([DR12, 9.9, p. 114]) Let $a \in \mathbb{C}$, $\Omega \subset \mathbb{C}$ open and let $f\colon \Omega\backslash\{a\} \to \mathbb{C}$ be holomorphic. Let a be a a pole of f of order $(n+1)$. Then*

$$\operatorname{res}(f; a) = \frac{1}{n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} (z-a)^{n+1} f(z)\Big|_{z=a}.$$

2.44 Definition. Let $\Omega \subset \mathbb{C}$ be open, $A \subset \Omega$ without limit point in Ω and $f\colon \Omega\backslash A \to \mathbb{C}$ holomorphic. If each point of A is a pole of f, then f is called *meromorphic.*

2.45 Definition and Theorem. *([Rud66, 13.13, p. 260]) Let $\Omega \subset \mathbb{C}$ be open, $f : \Omega\backslash A \to \mathbb{C}$ be meromorphic.. If γ is a closed path in $\Omega\backslash A$, then*

$$\int_\gamma f(z)\mathrm{d}z = 2\pi\,\mathrm{i} \sum_{a \in A} \operatorname{res}(f; a) \operatorname{ind}_\gamma(z),$$

where

$$\operatorname{ind}_\gamma(z) = \frac{1}{2\pi\,\mathrm{i}} \int_\gamma \frac{\mathrm{d}\zeta}{\zeta - z} \quad (z \in \mathbb{C}\backslash R(\gamma))$$

is the index of z with respect to γ.

2.46 Remark. The index $\operatorname{ind}_\gamma(z)$ is often called *winding number*, since it tells us the number of times that γ winds around z (cf. [Rud66, remark, p. 205]).

2.7 Distributions

The idea of distributions is to generalize functions in such a way that every distribution has partial derivatives, which are also distributions. On the other hand, we want to maintain some properties of the classical (continuous) functions. Therefore we request the following. First, we want all continuous functions to be distributions. Second, all the established rules of calculus may be applied to distributions too. Finally, the derivative of a classically differentiable function should coincide with the derivative of the generalized function.

2.47 Definition. Let $\varnothing \neq \Omega \subset \mathbb{R}^n$ be open. We use the notation $K \Subset \Omega$, to say that K is a compact subset of Ω.

(a) The set $C_0^\infty(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp}(f) \Subset \Omega\}$ is called the set of *test functions.* For $K \Subset \Omega$ we define $\mathcal{D}_K(\Omega) := \{f \in C_0^\infty(\Omega) : \operatorname{supp}(f) \subset K\}$.

(b) A sequence $(\varphi_k)_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ is defined to converge to zero iff

(i) there is $K \Subset \Omega$ with $\operatorname{supp}(\varphi_k) \subset K$ for every $k \in \mathbb{N}$ and

(ii) the sequence $(\partial^\alpha \varphi_k)_{k\in\mathbb{N}}$ converges uniformly, i.e. $\sup_{x\in\Omega} |\partial^\alpha \varphi_i(x)| \to 0$ for $i \to \infty$ for every multi-index $\alpha \in \mathbb{N}_0^n$.

(c) If $\mathbb{C}_0^\infty(\Omega)$ is equipped with the topology belonging to the definition of convergence in the previous statement, one obtains a complete locally convex topological vector space, which is denoted by $\mathcal{D}(\Omega)$.

(d) For $N \in \mathbb{N}_0$ and $K \Subset \Omega$

$$\varphi \mapsto p_{N,K}(\varphi) := \max\{|\partial^\alpha \varphi(x)| : x \in K, \alpha \in \mathbb{N}_0^n, |\alpha| \leqslant N\}$$

defines a seminorm on $\mathcal{D}_K$.

2.48 Definition. Let $L : \mathcal{D}(\Omega) \to \mathbb{K}$ be linear and continuous with respect to the topology defined above. Then L is called a *distribution.* We denote the set of all distributions by the symbol $\mathcal{D}'(\Omega)$.

2.49 Theorem. *([Rud73, p. 141]) A linear functional $L : \mathcal{D}(\Omega) \to \mathbb{K}$ is a distribution iff for every $K \Subset \Omega$ there are an $N \in \mathbb{N}_0$ and a constant $c < \infty$ such that*

$$|L\varphi| \leqslant c \cdot p_{N,K}(\varphi)$$

holds for every $\varphi \in \mathcal{D}_K$.

2.50 Example. Let $x_0 \in \mathbb{R}^n$. The *Dirac distribution* δ_{x_0} is defined via

$$\delta_{x_0} : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}, \quad \varphi \mapsto \varphi(x_0).$$

The linearity is obvious and since $|\varphi(x_0)| \leqslant p_{0,K}(\varphi)$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $K \Subset \mathbb{R}^n$, δ_{x_0} is a distribution according to Theorem 2.49.

2.51 Example (Dirac comb)**.** For $a > 0$ we call the functional

$$Ш_a : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}, \quad \varphi \mapsto \sum_{k\in\mathbb{Z}^n} \delta_{ak}(\varphi)$$

Dirac comb. Since all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ have compact support, the sum is always finite, hence $Ш_a$ is well-defined. Using the linearity of the Dirac delta we obtain that $Ш_a$ is also linear. Again, we need not care about the convergence, because the sum is finite. Finally, we want to show that $Ш_a$ is a distribution. Let $K \Subset \mathbb{R}^n$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then

$$|Ш_a(\varphi)| = \left| \sum_{k\in\mathbb{Z}^n} \delta_{ak}(\varphi) \right| \leqslant \sum_{k\in\mathbb{Z}^n} |\delta_{ak}(\varphi)| \,.$$

Since $|\delta_{x_0}(\varphi)| \leqslant p_{0,K}(\varphi)$ for every $x_0 \in \mathbb{R}^n$ according to the previous example, we obtain

$$|Ш_a(\varphi)| \leqslant p_{0,K}(\varphi) \sum_{k\in\mathbb{Z}^n \cap aK} 1 = C p_{0,K}(\varphi)$$

with C being the finite cardinality of the set $\mathbb{Z}^n \cap aK$, where $aK := \{ax : x \in K\}$. According to Theorem 2.49, $Ш_a$ is a distribution.

2.52 Example. Let $\Omega \subseteq \mathbb{R}^n$ be open, $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega) := \{f\colon \Omega \to \mathbb{C} \text{ measurable } : \forall K \Subset \Omega : f|_K \in \mathrm{L}^1(K)\}$. Then

$$[f]\colon \mathcal{D}(\Omega) \to \mathbb{K}, \quad \varphi \mapsto \int_\Omega f(x)\varphi(x)\mathrm{d}x$$

defines a distribution.

Proof. Obviously $[f]$ is a linear functional. Let $K \Subset \Omega$ and $\varphi \in \mathcal{D}_K$. Then

$$\begin{aligned} |[f](\varphi)| &= \left|\int_\Omega f(x)\varphi(x)\mathrm{d}x\right| \overset{\operatorname{supp}\varphi\subseteq K}{=} \left|\int_K f(x)\varphi(x)\mathrm{d}x\right| \\ &\leqslant \left|\int_K f(x)\mathrm{d}x\right| \max\{|\varphi(x)| : x \in K\} \\ &=: c_K p_{0,K}(\varphi). \end{aligned}$$

Note that $c_K < \infty$ since $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$. By Theorem 2.49, $[f]$ is a distribution. $\square$

2.53 Definition. A distribution $u \in \mathcal{D}'(\Omega)$ is called *regular* if there is a function $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega)$ with $u = [f]$.

2.54 Definition. Let $L \in \mathcal{D}'(\Omega)$ be a distribution.

(a) The number

$$\operatorname{ord}(L) := \inf\{N \in \mathbb{N}_0 : \forall K \Subset \Omega\ \exists c_K > 0\ \forall \varphi \in \mathcal{D}(\Omega) : |L\varphi| \leqslant c_K p_{N,K}(\varphi)\}$$

is called the *order* of L. If there is no such infimum, L is said to be of infinite order and $\operatorname{ord}(L) := \infty$.

(b) Let $U \subset \Omega$ be open. L *vanishes in* U iff $L(\varphi) = 0$ for every $\varphi \in \mathcal{D}(U)$.

(c) The *support* of L is defined as

$$\operatorname{supp}(L) := \Omega \backslash \bigcup_{U \in U_L} U = \bigcap_{C \in C_L} C$$

with $U_L := \{U \subseteq \Omega : U \text{ open}, L \text{ vanishes in } U\}$ and $C_L := \{C \subseteq \Omega : C \text{ closed}, L \text{ vanishes in } \Omega\backslash C\}$.

2.55 Example. (i) As can be seen in examples 2.50 and 2.52, any regular distribution as well as the Dirac distribution has order 0.

(ii) Let $\Omega \subseteq \mathbb{R}^n$ be open, $x_0 \in \Omega$ and $\varphi \in \mathcal{D}(\Omega)$. Let $C \subset \Omega$ be closed with $x_0 \in C$. Then $\delta_{x_0}(\varphi) = \varphi(x_0) = 0$ for every $\varphi \in \mathcal{D}(\Omega\backslash C)$. Hence

$$\operatorname{supp}(\delta_{x_0}) \subset \bigcap_{\substack{C \subset \Omega \text{ closed} \\ x_0 \in C}} C = \{x_0\}.$$

If $x_0 \notin \operatorname{supp}(\delta_{x_0})$ then $\delta(\varphi) = 0$ for every $\varphi \in \mathcal{D}(\Omega)$, hence $\delta_{x_0} = 0$, which is a contradiction.

(iii) Let $f \in C(\mathbb{R}^n)$. Let $C \subset \Omega$ be closed. Then we obtain

$$[f](\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)\mathrm{d}x = \int_{\Omega \backslash C} f(x)\varphi(x)\mathrm{d}x = 0$$

for all $\varphi \in \mathcal{D}(\Omega\backslash C)$ if and only if $f = 0$ a.e. on $\Omega\backslash C$. This is equivalent to $\operatorname{supp}(f) \subseteq C$. Hence

$$\operatorname{supp}([f]) = \bigcap_{\substack{C \subset \Omega \text{ closed} \\ \operatorname{supp}(f) \subseteq C}} C = \operatorname{supp}(f).$$

2.56 Definition. Let $\alpha \in \mathbb{N}_0^n$ be a multi-index and $L \in \mathcal{D}'(\Omega)$. The derivative $\partial^\alpha L$ of L is defined as

$$\partial^\alpha L : \mathcal{D}(\Omega) \to \mathbb{K}, \quad \varphi \mapsto (\partial^\alpha L)(\varphi) := (-1)^{|\alpha|} L(\partial^\alpha \varphi).$$

2.57 Remark. Since locally integrable and differentiable functions can also be considered as distributions, we may ask if the distribution generated by the classical derivative coincides with the distributional derivative, i.e. $\partial^\alpha[f] = [\partial^\alpha f]$ provided that the classical derivative exists. The answer is given by integration by parts[3], as we need to show

$$(-1)^{|\alpha|} \int_\Omega f(x)(\partial^\alpha \varphi)(x)\mathrm{d}x = \int_\Omega (\partial^\alpha f)(x)\varphi(x)\mathrm{d}x$$

for every $\varphi \in \mathcal{D}(\Omega)$. The boundary terms vanish, since $\varphi|_{\partial\Omega} = 0$, because $\operatorname{supp}(\varphi) \Subset \Omega$ is compact and Ω is open.

2.58 Definition. Let $f : \mathbb{K}^n \supset \Omega \to \mathbb{K}^m$ be a function and $x \in \mathbb{K}^n$. We define

$$\tau_x f := f(\cdot - x) \quad \text{and} \quad \check{f} := f(-\cdot).$$

Let $u \in \mathcal{D}'(\mathbb{K}^n)$ be a distribution and $x \in \mathbb{K}^n$. We define

$$\tau_x u := \varphi \mapsto u(\tau_{-x}\varphi) \quad \text{and} \quad \check{u} := \varphi \mapsto u(\check{(\varphi)}).$$

2.59 Definition (Convolution)**.** (a) Let $u, v : \mathbb{R}^n \to \mathbb{C}$ be measurable. The convolution $u * v$ is given as

$$u * v : \mathbb{R}^n \to \mathbb{C},\ x \mapsto \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(y)v(x-y)\mathrm{d}y = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(y)(\tau_x \check{v})(y)\mathrm{d}y,$$

if the integral exists for almost every $x \in \mathbb{R}^n$.

(b) Let $u \in \mathcal{D}'(\mathbb{R}^n), \varphi \in \mathcal{D}(\mathbb{R}^n)$. Then the convolution $u * \varphi$ is a function and defined as

$$u * \varphi : \mathbb{R}^n \to \mathbb{C},\ x \mapsto (2\pi)^{-n/2} u(\tau_x \check{\varphi}).$$

(c) Let $u, v \in \mathcal{D}'(\mathbb{R}^n)$ with at least one of them having compact support. Then $u * v \in \mathcal{D}'(\mathbb{R}^n)$ is defined as the unique distribution $w := (u * v) \in \mathcal{D}'(\mathbb{R}^n)$ (cf. [Hö83, 4.2.2, p. 101]) which fulfills

$$w * \varphi = (u * v) * \varphi = u * (v * \varphi) \quad (\varphi \in \mathcal{D}(\mathbb{R}^n)).$$

[3] or the Divergence Theorem in the multidimensional case

A pleasant property of convolution is that it commutes with translation and differentiation as stated in the next theorem.

2.60 Theorem. *([Rud73, p. 156, p.160]) Let* $u, v, w \in \mathcal{D}'(\mathbb{R}^n)$, $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$. *Then*

(a) $\tau_x(u * \varphi) = (\tau_x u) * \varphi = u * (\tau_x \varphi)$ *for every* $x \in \mathbb{R}^n$,

(b) $u * \varphi \in C^\infty$ *and* $\partial^\alpha(u * \varphi) = (\partial^\alpha u) * \varphi = u * (\partial^\alpha \varphi)$ *for every multi-index* $\alpha \in \mathbb{N}_0^n$,

(c) $u * (\varphi * \psi) = (u * \varphi) * \psi$.

(d) If at least one of u, v *has compact support, then* $u * v = v * u$.

(e) If at least two of u, v, w *have compact support, then* $(u * v) * w = u * (v * w)$.

(f) Let δ_0 *be the Dirac distribution and* $\alpha \in \mathbb{N}_0^n$ *be a multi-index, then* $(2\pi)^{-n/2}\partial^\alpha u = (\partial^\alpha \delta_0) * u$. *In particular,* $(2\pi)^{-n/2} u = \delta_0 * u$.

2.61 Corollary. *Let* $u \in \mathcal{D}'(\mathbb{R}^n)$ *be a distribution,* $x_0 \in \mathbb{R}^n$, δ_{x_0} *be the Dirac distribution and* $\alpha \in \mathbb{N}_0^n$ *a multi-index, then* $(\partial^\alpha \delta_{x_0}) * u = (2\pi)^{-n/2}\partial^\alpha \tau_{x_0} u$. *In particular,* $\delta_{x_0} * u = (2\pi)^{-n/2}\tau_{x_0} u$.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then $(\delta_{x_0} * \varphi)$ is a function by Theorem 2.60(b). For $x \in \mathbb{R}^n$, we have

$$\begin{aligned}((\partial^\alpha \delta_{x_0}) * \varphi)(x) &= (2\pi)^{-n/2}(\partial^\alpha \delta_{x_0})(\tau_x \check{\varphi}) = (2\pi)^{-n/2}\partial^\alpha(\tau_x \check{\varphi})(x_0) = (2\pi)^{-n/2}\partial^\alpha \check{\varphi}(x_0 - x)\\ &= (2\pi)^{-n/2}\partial^\alpha \varphi(x - x_0) = (2\pi)^{-n/2}\tau_{x_0}(\partial^\alpha \varphi)(x).\end{aligned}$$

Therefore, we get using (a), (b), (d) and (e) of Theorem 2.60

$$\begin{aligned}((\partial^\alpha \delta_{x_0}) * u) * \varphi &\overset{(d)}{=} (u * (\partial^\alpha \delta_{x_0})) * \varphi \overset{(e)}{=} u * ((\partial^\alpha)\delta_{x_0} * \varphi) = (2\pi)^{-n/2} u * \tau_{x_0}(\partial^\alpha \varphi)(x)\\ &\overset{(a)}{=} (2\pi)^{-n/2}\tau_{x_0} u * \partial^\alpha \varphi \overset{(b)}{=} (2\pi)^{-n/2}(\tau_{x_0}\partial^\alpha u) * \varphi,\end{aligned}$$

hence

$$(\partial^\alpha \delta_{x_0}) * u = (2\pi)^{-n/2}\tau_{x_0}\partial^\alpha u.$$

□

3 Fourier Analysis

Fourier analysis is a crucial tool for several proofs of the sampling theorem. Therefore we given an introduction into Fourier transform and Fourier series for both functions and distributions. The idea of Fourier analysis is to decompose a function into a sum of periodic functions, in particular sine and cosine. Of course, this is difficult to achieve for arbitrary functions. Therefore, one has to substitute the sum by an integral. However, it is sufficient to consider a series of trigonometric functions if the function is periodic.

3.1 Fourier transform

3.1.1 Schwartz functions and the Fourier transform

3.1 Definition. The set of all rapidly decreasing functions on $\mathbb{R}^n$ is called *Schwartz space* $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$. This can be formally written by introducing a family of seminorms $\|.\|_{\alpha,\beta}$ defined by

$$\|\varphi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left| x^\beta \partial^\alpha \varphi(x) \right|$$

for $\varphi \in C^\infty(\mathbb{R}^n)$ and multi-indices α and β. Then we define

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in C^\infty(\mathbb{R}^n) : \|\varphi\|_{\alpha,\beta} < \infty \text{ for all } \alpha, \beta \in \mathbb{N}_0^n\}.$$

Obviously $C_0^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.

3.2 Example. The function $x \mapsto \exp(-a|x|^2)$ is a Schwartz function if $a > 0$.

Proof. Let $a > 0$.

(a) First consider $x^l \exp(-ax^2)$ for some $l \in \mathbb{N}$ and $x \in \mathbb{R}$. Then

$$\left| x^l \exp(-ax^2) \right| = \left| \frac{\exp(ax^2)}{x^l} \right|^{-1} = \left| \sum_{k=0}^{\infty} \frac{(ax^2)^k}{x^l k!} \right|^{-1} \leqslant \left| \frac{a^l x^l}{l!} \right|^{-1} = \frac{l!/a^l}{|x^l|} \to 0$$

for $|x| \to \infty$. With the laws for summing limits this result can be extended to polynomials instead of the monomial x^l.

(b) Now let $\alpha, \beta \in \mathbb{N}_0^n$. Note that $\partial^\alpha \exp(-a|x|^2) = P_\alpha(x) \exp(-a|x|^2)$ for some polynomial P_α. Since $x^\beta P_\alpha(x) \exp(-a|x|^2)$ is continuous, it is bounded on every compact subset and it remains to show that this function is bounded for $|x| \to \infty$. The estimate [1] $|x^\beta| \leqslant |x|^{|\beta|}$

[1] $|x^\beta| = \left| \prod_{i=1}^n x_i^{\beta_i} \right| = \prod |x_i|^{\beta_i} \leqslant \prod_{i=1}^n |x|^{\beta_i} = |x|^{|\beta|}$

yields $|P_\alpha(x)| \leqslant P_\alpha(|x|) := \sum_\delta |c_\delta||x|^{|\delta|}$ for the polynomial P_α given as $P_\alpha(x) := \sum_\delta c_\delta x^\delta$. Then we get

$$\left|x^\beta P_\alpha(x)\exp(-a|x|^2)\right| \leqslant |x|^{|\beta|}|P_\alpha|(|x|)\exp(-a|x|^2) \to 0,$$

for $|x| \to \infty$ by (a). Hence $\exp(-a|x|^2) \in \mathcal{S}(\mathbb{R}^n)$.

□

3.3 Remark. In Definition 3.1 one could equivalently use the family of seminorms

$$p_{\alpha,m} : \varphi \mapsto \sup_{x\in\mathbb{R}^n}(1+|x|^2)^m|\partial^\alpha\varphi(x)|$$

for $\varphi \in \mathbb{C}^\infty(\mathbb{R}^n)$, any multi-index $\alpha \in \mathbb{N}_0^n$ and $m \in \mathbb{N}$. This can be seen as follows: If we define $N := |\beta| + 1$ if $|\beta|$ is odd and $N := |\beta|$ else, we obtain using again $|x^\beta| \leqslant |x|^{|\beta|}$

$$\left|x^\beta\right| \leqslant |x|^{|\beta|} \leqslant \left(1+|x|^N\right) \leqslant \left(1+|x|^2\right)^{N/2}.$$

On the other hand, we have

$$\left(1+|x|^2\right)^N \leqslant \left(\sum_{n=1}^N \binom{n}{N}|x|^{2n}\right) \leqslant \left(\sum_{n=1}^N \binom{n}{N}(Nx_1^2)^n\right) \leqslant c\max\{|x^{(2N,0,\dots)}|, 1\}.$$

Hence $\beta = 0$ or $\beta = (2N, 0, \dots) \in \mathbb{N}_0^n$. For the estimate we used that without loss of generality $x_1^2 \geqslant x_i^2$ $(i = 1, \dots, n)$ and therefore $Nx_1^2 \geqslant |x|^2$. In the last step we rated against the largest term, which is 1 if $|x_1| \leqslant 1$ or $|x_1|^{2N}$ if $|x_1| > 1$.

3.4 Definition. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then we call

$$\mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\exp(-\mathrm{i}\,\xi x)f(x)\mathrm{d}x \tag{3.1}$$

the *Fourier transform* of f. Recall that $\xi x := \sum_{i=1}^n \xi_i x_i$.

3.5 Example. Let $a > 0$. Then $\varphi : x \mapsto \exp(-a|x|^2) \in \mathcal{S}(\mathbb{R}^n)$ by Example 3.2. Hence the Fourier transform is given by

$$\begin{aligned}
\mathcal{F}[\varphi](\xi) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\exp(-\mathrm{i}\,x\xi)\exp(-a|x|^2)\mathrm{d}x \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\exp\left(-\sum_{i=1}^n\left(ax_i^2 + \mathrm{i}\,x_i\xi_i - \frac{\xi_i^2}{4a}\right) - \sum_{i=1}^n\frac{\xi_i^2}{4a}\right)\mathrm{d}x \\
&= \frac{1}{(2\pi)^{n/2}}\exp\left(-\frac{|\xi|^2}{4a}\right)\int_{\mathbb{R}^n}\exp\left(-\sum_{i=1}^n\left(\sqrt{a}x_i + \mathrm{i}\,\frac{\xi_i}{2\sqrt{a}}\right)^2\right)\mathrm{d}x.
\end{aligned}$$

Using substitution $\zeta_i := \sqrt{a}x_i + \mathrm{i}\,\xi_i/2\sqrt{a}$, the integral becomes

$$\int_{\mathbb{R}^n}\exp\left(-\sum_{i=1}^n\left(\sqrt{a}x_i + \mathrm{i}\,\frac{\xi_i}{2\sqrt{a}}\right)^2\right)\mathrm{d}x = \int_{\mathbb{R}^n}\exp(-|\zeta|^2)\frac{1}{\sqrt{a}^n}\mathrm{d}\zeta = \left(\frac{\pi}{a}\right)^{n/2}.$$

Inserting this result into the equation above, we obtain

$$\mathcal{F}\left[\varphi\right](\xi) = \frac{1}{(2a)^{n/2}} \exp\left(-\frac{|\xi|^2}{4a}\right).$$

This is again a Gaussian function, but the width changes from $1/\sqrt{a}$ to $2\sqrt{a}$. For $a = 1/2$ the function coincides with its Fourier transform.

3.6 Theorem. *([Rud73, 7.4(d), p. 168]) The Fourier transform is a continuous linear mapping of $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$.*

Since $|f(\xi)\exp(-x\xi)| = |f(\xi)|$ the integral in the definition of the Fourier transform (3.1) is well-defined for $f \in \mathrm{L}^1(\mathbb{R}^n)$. Therefore, one can extend Definition 3.4 as follows:

3.7 Definition. For $f \in \mathrm{L}^1(\mathbb{R}^n)$ we define the Fourier transform of f as in Equation (3.1).

But this extension has one drawback. The Fourier transform of an integrable function is not necessarily integrable. This can be seen in the following example.

3.8 Example. The Fourier transform of $\chi_{[-a,a]} \in \mathrm{L}^1(\mathbb{R})$ is given by

$$\begin{aligned}
\mathcal{F}\left[\chi_{[-a,a]}\right](\xi) &= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} \chi_{[-a,a]}(x) \exp(-\mathrm{i}\, x\xi) \mathrm{d}x = \frac{1}{(2\pi)^{1/2}} \int_{-a}^{a} \exp(-\mathrm{i}\, x\xi) \mathrm{d}x \\
&= \frac{1}{-\mathrm{i}\,\xi\sqrt{2\pi}} \left(\exp(-\mathrm{i}\, a\xi) - \exp(\mathrm{i}\, a\xi)\right) = \frac{\sqrt{2\pi}a}{\pi\xi a} \frac{1}{2\,\mathrm{i}} \left(\exp(\mathrm{i}\, a\xi) - \exp(-\mathrm{i}\, a\xi)\right) \\
&= \frac{\sqrt{2\pi}a}{\pi} \frac{\sin(a\xi)}{a\xi}.
\end{aligned}$$

But $\xi \mapsto \sin(a\xi)/a\xi$ is not integrable.

3.1.2 Properties of the Fourier transform

3.9 Theorem (Properties of the Fourier transform)**.** *([Rud73, 7.2, p. 167]) Let $f, g \in \mathrm{L}^1(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\alpha \in \mathbb{C}$, then the following properties hold*

(a) $\mathcal{F}\left[\alpha f + g\right] = \alpha\mathcal{F}\left[f\right] + \mathcal{F}\left[g\right]$,

(b) $\mathcal{F}\left[\tau_x f\right] = \exp(-\mathrm{i}\, x\cdot)\mathcal{F}\left[f\right]$ *and* $\mathcal{F}\left[\exp(\mathrm{i}\, x\cdot) f\right] = \tau_x \mathcal{F}\left[f\right]$.

(c) For $\lambda > 0$ *and* $h(x) := f(x/\lambda)$ *we have* $\mathcal{F}\left[h\right](\xi) = \lambda^n \mathcal{F}\left[f\right](\lambda\xi)$.

(d) $\mathcal{F}\left[\check{f}\right] = \widetilde{\mathcal{F}\left[f\right]}$ *(cf. [Gra08, 2.2.11, p. 100])*

(e) $\mathcal{F}\left[(f * g)\right] = \mathcal{F}\left[f\right]\mathcal{F}\left[g\right]$

(f) If $f, g \in \mathcal{S}(\mathbb{R}^n)$*, there holds* $\mathcal{F}\left[fg\right] = \mathcal{F}\left[f\right] * \mathcal{F}\left[g\right]$ *([Rud73, 7.8, p. 172]),*

3.10 Definition. The mapping

$$\mathcal{F}^{-1} \colon \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad f \mapsto \left(x \mapsto \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi) \exp(\mathrm{i}\, x\xi) \mathrm{d}\xi\right)$$

is called the inverse Fourier transform.

The right-hand side can also be applied to $f \in \mathrm{L}^1(\mathbb{R}^n)$. Therefore, we call the right hand side the inverse Fourier transform of f.

3.11 Theorem (inversion theorem). *([Rud73, 7.7, p. 170]) If $g \in \mathcal{S}(\mathbb{R}^n)$, then*

$$g(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}[g](\xi) \exp(\mathrm{i}\, x\xi) \mathrm{d}\xi \tag{3.2}$$

for every $x \in \mathbb{R}^n$. Further, the Fourier transform is a continuous, linear, one-to-one mapping of $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$, whose inverse is also continuous. Its period is four, that is $\mathcal{F}^4 = \mathrm{id}_{\mathcal{S}(\mathbb{R}^n)}$. If we have $g \in \mathrm{L}^1(\mathbb{R}^n)$ and $\mathcal{F}[g] \in \mathrm{L}^1(\mathbb{R}^n)$ then (3.2) holds for almost every $x \in \mathbb{R}^n$.

3.12 Corollary. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. Then $\mathcal{F}^2[f] = \check{f}$ and $\mathcal{F}^{-1}[f] = \widetilde{\mathcal{F}[f]} = \mathcal{F}\left[\check{f}\right]$. The latter remains true for every measurable function $f\colon \mathbb{R}^n \to \mathbb{C}$, provided its Fourier transform or its inverse Fourier transform is finite everywhere.*

Proof. Recall that $\check{f}(x) = f(-x)$. Since $f \in \mathcal{S}(\mathbb{R}^n)$, we can use the inversion theorem 3.11 to get

$$f(-x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathcal{F}[f](\xi) \exp(-\mathrm{i}\, x\xi) \mathrm{d}\xi = \mathcal{F}[\mathcal{F}[f]](x) = \mathcal{F}^2[f](x)$$

for every $x \in \mathbb{R}^n$ by definition of the Fourier transform. The right-hand side is well-defined, since the Fourier transform of a Schwartz function is again a Schwartz function by Theorem 3.6 and for Schwartz functions the Fourier transform exists.

By definition we have for $x \in \mathbb{R}^n$

$$\mathcal{F}^{-1}[f](x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\xi) \exp(-\mathrm{i}(-x)\xi) \mathrm{d}\xi = \mathcal{F}[f](-x) = \widetilde{\mathcal{F}[f]}(x) \overset{3.9(d)}{=} \mathcal{F}\left[\check{f}\right](x).$$

□

3.13 Theorem (Plancherel's theorem). *([Rud73, 7.9, p. 172]) The Fourier transform can be extended to a linear isometry of $\mathrm{L}^2(\mathbb{R}^n)$ onto $\mathrm{L}^2(\mathbb{R}^n)$.*

We have seen that the Fourier transform can be applied to functions of $\mathrm{L}^1(\mathbb{R}^n)$ and $\mathrm{L}^2(\mathbb{R}^n)$. Moreover, it can also be applied to functions of $\mathrm{L}^p(\mathbb{R}^n)$ with $1 \leqslant p \leqslant 2$ as is shown in the next theorem.

3.14 Theorem (Hausdorff-Young). *([Hig96, 2.17, p. 19]) If $f \in \mathrm{L}^p(\mathbb{R}^n)$ and $1 \leqslant p \leqslant 2$, then there is a constant C_p depending only on p, such that*

$$\|\mathcal{F}[f]\|_q \leqslant C_p \|f\|_p ,$$

where $1/p + 1/q = 1$. In particular, $\mathcal{F}[f] \in \mathrm{L}^q(\mathbb{R}^n)$.

Since many common functions such as polynomials or trigonometric functions are not integrable on $\mathbb{R}$, we want to extend the Fourier transform to distributions.

3.1.3 Tempered distributions and the Fourier transform

3.15 Definition. A distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is called *tempered* if it has a continuous extension to $\mathcal{S}(\mathbb{R}^n)$.

Equivalently, a continuous linear functional $f : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is called *tempered distribution*, i.e. f is an element of the dual space $\mathcal{S}'(\mathbb{R}^n)$ of the Schwartz space. (cf. [Rud73, 7.11, p. 174]) (Continuity has to be checked in the sense of the topology induced by the family of seminorms.)

A useful characterization is given by Corollary 2.33.

3.16 Example. Ш_a is a tempered distribution for $a \in \mathbb{R}$. To show this, let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\text{Ш}_a \varphi = \sum_{k \in \mathbb{Z}^n} \delta_{ak} \varphi = \sum_{k \in \mathbb{Z}^n} \varphi(ak).$$

Since φ is a tempered distribution, we have according to remark 3.3

$$|\varphi(ak)| \leqslant p_{0,n}(\varphi) \frac{1}{(1 + |ak|^2)^n}$$

with n being the dimension of the vector space where φ is defined. Hence

$$\left| \sum_{k \in \mathbb{Z}^n} \varphi(ak) \right| \leqslant \sum_{k \in \mathbb{Z}^n} \frac{p_{0,n}(\varphi)}{(1 + |ak|^2)^n} = \sum_{l \in \mathbb{N}_0} \sum_{\substack{k \in \mathbb{Z}^n \\ l-1 < \max_i |k_i| \leqslant l}} \frac{p_{0,n}(\varphi)}{(1 + |ak|^2)^n}.$$

The reordering in the last step is possible because the series converges absolutely if we show that it is convergent. The inner sum describes the difference of two n-dimensional cubes with side lengths $2l + 1$ and $2l - 1$, hence there are $(2l + 1)^n - (2l - 1)^n \leqslant \tilde{c} l^{n-1}$ elements of $\mathbb{Z}^n$ in this hollow cube and for each of it we have the estimate $|k|^2 \geqslant l^2$. Hence we get

$$\left| \sum_{k \in \mathbb{Z}^n} \varphi(ak) \right| \leqslant p_{0,n}(\varphi) \tilde{c} \left(1 + \sum_{l \in \mathbb{N}} \frac{l^{n-1}}{(1 + |al|^2)^n} \right) \leqslant p_{0,n}(\varphi) \tilde{c} \left(c + a^{-n-1} \sum_{\substack{l \in \mathbb{N} \\ al > 1}} \frac{1}{|al|^{n+1}} \right) < \infty$$

where c is the sum over all terms with $|al| < 1$. Therefore, Ш_a is well-defined. It inherits the linearity from the Dirac distribution. Now it remains to show that Ш_a is continuous. According to Corollary 2.33 we choose $N = 1$ and M as the constant factor in the equation above and get that Ш_a is a tempered distribution.

3.17 Definition. For $u \in \mathcal{S}'(\mathbb{R}^n)$ define the Fourier transform by

$$\mathcal{F}[u](\varphi) = u(\mathcal{F}[\varphi])$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

3.18 Theorem. *([Rud73, 7.15, p. 176]) The Fourier transform is a continuous, linear, one-to-one mapping of $\mathcal{S}'(\mathbb{R}^n)$ onto $\mathcal{S}'(\mathbb{R}^n)$, of period 4, whose inverse is also continuous. Continuity refers to the weak-$*$-topology induced by $\mathcal{S}(\mathbb{R}^n)$ on $\mathcal{S}'(\mathbb{R}^n)$.*

3.19 Remark. If we consider functions as regular distributions then the Fourier transform of the distribution coincides with the regular distribution generated by the Fourier transform of the function whenever the Fourier transform of the function exists. To see this, we choose $f \in \mathrm{L}^1(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned}\mathcal{F}\left[[f]\right](\varphi) \quad &= \quad [f](\mathcal{F}\left[\varphi\right]) = \int_{\mathbb{R}^n} f(x)\mathcal{F}\left[\varphi\right](x)\mathrm{d}x = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\varphi(\xi)\exp(-\mathrm{i}\,x\xi)\mathrm{d}\xi\mathrm{d}x \\ &\overset{\text{Tonelli}}{=} \int_{\mathbb{R}^n} \varphi(\xi)\mathcal{F}\left[f\right](\xi)\mathrm{d}\xi = [\mathcal{F}\left[f\right]](\varphi).\end{aligned}$$

3.20 Examples. Let us consider the Fourier transform of some non-L^1 functions. We obtain:

(i) Let $1 : \mathbb{R}^n \to \mathbb{C}, x \mapsto 1$ be the constant function. Then $\mathcal{F}\left[[1]\right] = (2\pi)^{n/2}\delta_0$.

(ii) Let $x_0 \in \mathbb{R}^n$. Then $\mathcal{F}\left[\delta_{x_0}\right] = \left[\frac{1}{(2\pi)^{n/2}}\exp(-\mathrm{i}\,x_0\cdot)\right]$. In particular for $x_0 = 0$: $\mathcal{F}\left[\delta_0\right] = \left[\frac{1}{(2\pi)^{n/2}}\right]$.

(iii) Let $\sin_\omega : \mathbb{R} \to \mathbb{C}, x \mapsto \sin(\omega x)$ for $\omega \in \mathbb{R}$. Then $\mathcal{F}\left[[\sin_\omega]\right] = -\mathrm{i}\sqrt{\pi/2}(\delta_{-\omega} - \delta_\omega)$.

Proof. (i) For the Fourier transform of $1 : \mathbb{R}^n \to \mathbb{C}, x \mapsto 1$ we have to consider the Fourier transform of its corresponding regular distribution $[1] : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}, \varphi \mapsto \int_{\mathbb{R}^n} 1\varphi(x)\mathrm{d}x$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then we have

$$\begin{aligned}\mathcal{F}\left[[1]\right](\varphi) =&[1](\mathcal{F}\left[\varphi\right]) = \int_{\mathbb{R}^n} 1\mathcal{F}\left[\varphi\right](\xi)\mathrm{d}\xi = (2\pi)^{n/2}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\exp(\mathrm{i}\,0\xi)\mathcal{F}\left[\varphi\right](\xi)\mathrm{d}\xi \\ =&(2\pi)^{n/2}\varphi(0)\end{aligned}$$

due to the inversion theorem 3.11. Using the Dirac distribution, we can write

$$\mathcal{F}\left[[1]\right] = (2\pi)^{n/2}\delta_0.$$

(ii) By definition, we have for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned}\mathcal{F}\left[\delta_{x_0}\right](\varphi) =&\delta_{x_0}(\mathcal{F}\left[\varphi\right]) = \mathcal{F}\left[\varphi\right](x_0) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\exp(-\mathrm{i}\,x_0\xi)\varphi(\xi)\mathrm{d}\xi \\ =&\left[\frac{1}{(2\pi)^{n/2}}\exp(-\mathrm{i}\,x_0\cdot)\right](\varphi).\end{aligned}$$

(iii) Again we have to consider the corresponding regular distribution $[\sin_\omega]$ to obtain the Fourier transform. Let $\varphi \in \mathcal{S}(\mathbb{R})$. Then we have

$$\begin{aligned}\mathcal{F}\left[[\sin_\omega]\right](\varphi) =&[\sin_\omega](\mathcal{F}\left[\varphi\right]) = \int_{\mathbb{R}^n}\sin(\omega\xi)\mathcal{F}\left[\varphi\right](\xi)\mathrm{d}\xi \\ =&\frac{(2\pi)^{1/2}}{(2\pi)^{1/2}}\frac{1}{2i}\left(\int_{\mathbb{R}^n}\exp(\mathrm{i}\,\omega\xi)\mathcal{F}\left[\varphi\right](\xi)\mathrm{d}\xi - \int_{\mathbb{R}^n}\exp(\mathrm{i}(-\omega)\xi)\mathcal{F}\left[\varphi\right](\xi)\mathrm{d}\xi\right) \\ =&\sqrt{\frac{\pi}{2}}\imath\,(\delta_{-\omega} - \delta_\omega)\end{aligned}$$

due to the inversion theorem 3.11.

□

3.21 Lemma. *([Gra08, 2.2.5, p. 106])* $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $\mathrm{L}^p(\mathbb{R}^n)$ *for* $1 \leqslant p < \infty$.

3.22 Remark. The previous lemma is important, since it enables us to transfer many properties of the Fourier transform from Schwartz functions to L^p-functions. Often, the properties can be shown easily for Schwartz functions. Then, one can approximate any function of $\mathrm{L}^p(\mathbb{R}^n)$ by a sequence of Schwartz functions and apply the property to the Schwartz functions. The most difficult step is to show that the resulting sequence converges to the desired result.

3.2 Fourier series

The Fourier series is the discrete version of the Fourier transformation. It has been developed to approximate periodic continuous functions by linear combinations of sine and cosine functions, so called trigonometric polynomials.

3.2.1 Periodic functions

First, we have to define periodic functions in a multidimensional setting.

3.23 Definition. Let $\Lambda := \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{K}^n$ be a set of n linearly independent vectors and $\lambda > 0$. A function $f\colon \mathbb{K}^n \to \mathbb{K}$ is called

(i) Λ-*periodic* if $f(x + \sum_{i=1}^n k_i\lambda_i) = f(x)$ holds for every $x \in \mathbb{K}^n$ and $k := (k_1, \dots, k_n) \in \mathbb{Z}^n$.

(ii) λ-periodic if f is Λ-periodic with $\Lambda = \{\lambda e_i : i = 1, \dots, n\}$, where $\{e_i : i = 1, \dots, n\}$ is an orthogonal basis of $\mathbb{K}^n$, i.e. when the periodicity along all axes is the same.

Although the definition of periodicity via the basis Λ seems to be more general, it is sufficient to consider λ-periodic or even 1-periodic functions, since they can be transformed into each other via a linear variable transform.

3.24 Lemma. *For each set of linearly independent vectors* $\Lambda := \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{K}^n$ *there is an invertible matrix* $A \in \mathbb{K}^{n\times n}$ *such that for every* Λ*-periodic function* $f : \mathbb{K}^n \to \mathbb{K}$ *the function* $g\colon \mathbb{K}^n \to \mathbb{K}$ *defined by* $f \circ A$ *is* 1*-periodic*[2].

Proof. Let $E := \{e_i : i = 1, \dots, n\}$ be an orthogonal basis of $\mathbb{K}^n$. Define $A := (a_{ij})_{ij} \in \mathbb{K}^{n\times n}$ by $a_{ij} = \langle \lambda_i, e_j\rangle_{\mathbb{C}^n}$, where $\langle\cdot,\cdot\rangle_{\mathbb{C}^n}$ is the inner product of $\mathbb{K}^n$ introduced in Example 2.20. Since Λ is a basis, A has full rank and is therefore invertible and by definition we have $Ae_i = \lambda_i$ $(i = 1, \dots, n)$. Let $f : \mathbb{K}^n \to \mathbb{K}$, $x \in \mathbb{K}^n$ and $k \in \mathbb{Z}^n$. Then, we get for $g := f \circ A$

$$g\left(x + \sum_i k_i e_i\right) = f\left(Ax + \sum_i k_i A e_i\right) = f\left(Ax + \sum_i k_i \lambda_i\right) \overset{\Lambda\text{-periodic}}{=} f(Ax) = g(x),$$

hence g is 1-periodic. □

3.25 Remark. Many books, for example [Gra08] or [RT10], restrict themselves to the case of 1-periodic functions. Then, however, it is sometimes difficult to figure out where the period

[2] We do not distinguish between the matrix A and its corresponding linear mapping $A\colon \mathbb{K}^n \to \mathbb{K}^n, x \mapsto Ax$.

enters the formulae. To avoid this problem, the following theorems are formulated for λ-periodic functions. This case is still simple, as we can perform scalar operations. On the other hand it enables us to keep track of the influence of the periodicity on the results. In most cases one can switch from λ-periodic to Λ-periodic by replacing λ by A, $1/\lambda$ by A^{-1} and λ^n by $\det A$, where $A \in \mathbb{R}^{n\times n}$ is defined as above.

If f is λ-periodic, we can identify the points $x, y \in \mathbb{K}^n$ if $x - y \in \lambda\mathbb{Z}^n$, since the values of f at both points are the same. This is an equivalence relation which leads to the following definition.

3.26 Definition. Let $\lambda > 0$. We define $\lambda\mathbb{T}^n := \mathbb{R}^n/\lambda\mathbb{Z}^n$ as the set of all equivalence classes induced by $\lambda\mathbb{Z}^n$ with $\lambda\mathbb{Z}^n := \{\lambda k : k \in \mathbb{Z}^n\}$. $\lambda\mathbb{T}^n$ is called the *n-torus of period λ*.

A function on $\lambda\mathbb{T}^n$ is a function f on $\mathbb{R}^n$ which satisfies $f(x) = f(x + \lambda k)$ for every $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$. Hence, $C^k(\lambda\mathbb{T}^n) := C^k(\mathbb{R}^n) \cap \{f : \mathbb{R}^n \to \mathbb{K} : f \text{ is } \lambda\text{-periodic}\}$ for $k \in \mathbb{N}_0 \cup \{\infty\}$.

The topology on $C^\infty(\lambda\mathbb{T}^n)$ is induced by the seminorms

$$p_\alpha : C^\infty(\lambda\mathbb{T}^n) \to [0,\infty), \quad \varphi \mapsto \sup_{x\in\lambda\mathbb{T}^n} |\partial^\alpha\varphi(x)|$$

for $\alpha \in \mathbb{N}_0^n$ (cf. [Gra08, p. 162f]).

For the periodic analogue of $\mathrm{L}^p(\mathbb{R}^n)$ we need another norm, since periodic functions are not integrable. But because of periodicity all information about the function is contained in the n-torus.

3.27 Definition. Let $\lambda > 0$, $1 \leqslant p < \infty$. We define

$$\|f\|_{\lambda,p} := \frac{1}{\lambda^n}\left(\int_{\lambda\mathbb{T}^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p} \mathrm{d}x = \frac{1}{\lambda^n}\left(\int_{[-\lambda/2,\lambda/2]^n} |f(x)|^p \,\mathrm{d}x\right)^{1/p} \mathrm{d}x.$$

Due to the periodicity one can replace the integral domain $[-\lambda/2, \lambda/2]^n$ by any interval of the form $[a_1, \lambda + a_1] \times \ldots \times [a_n, \lambda + a_n]$ for any $(a_1, \ldots, a_n) \in \mathbb{R}^n$.

Defining $\mathcal{L}^p(\lambda\mathbb{T}^n) := \{f : \mathbb{R}^n \to \mathbb{K} : f \text{ is } \lambda\text{-periodic }, \|f\|_{\lambda,p} < \infty\}$ and switching over to equivalence classes as in section 2.3, one obtains the normed vector space $\mathrm{L}^p(\lambda\mathbb{T}^n)$.

3.28 Remark. Note that in the definition of $\|\cdot\|_{\lambda,p}$ we integrate over a finite interval. Hence $\|1\|_{\lambda,q} = 1$ for every $1 \leqslant q < \infty$. Using Hölder's inequality, we get for $1 \leqslant q \leqslant \infty$ with $1/p + 1/q = 1$ and $f \in \mathrm{L}^p(\lambda\mathbb{T}^n)$

$$\|f\|_{\lambda,1} \leqslant \|f\|_{\lambda,p} \|1\|_{\lambda,q} < \infty.$$

Hence $\mathrm{L}^p(\lambda\mathbb{T}^n) \subseteq \mathrm{L}^1(\lambda\mathbb{T}^n)$ for every $1 \leqslant p \leqslant \infty$. An important choice is $p = 2$, since $\mathrm{L}^2(\lambda\mathbb{T}^n)$ is a Hilbert space.

Now trigonometric polynomials are introduced.

3.29 Definition. Let $\lambda > 0$. A trigonometric polynomial on $\lambda\mathbb{T}^n$ is a function

$$P : \mathbb{R}^n \to \mathbb{K}, \quad x \mapsto \sum_{k\in\mathbb{Z}^n} a_k e^{\mathrm{i}\,\frac{2\pi}{\lambda} kx} \tag{3.3}$$

with $\{a_k\}_{k\in\mathbb{Z}^n} \subset \mathbb{C}$ being a sequence with $\{k \in \mathbb{Z}^n : a_k \neq 0\}$ having finite cardinality. The number $N := \max\{\sum_{i=1}^n |q_i| : q \in \mathbb{Z}^n, a_q \neq 0\}$ is called *degree* of P.

3.30 Example. In the one-dimensional case (3.3) can be rewritten as

$$f : \mathbb{R} \to \mathbb{C}, \quad t \mapsto a_0 + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{2\pi}{\lambda} nt\right) + b_n \sin\left(\frac{2\pi}{\lambda} nt\right) \right)$$

with $N \in \mathbb{N}$ being the degree and $a_0, \ldots, a_N, b_1, \ldots, b_N \in \mathbb{C}$.

If and how one can represent a periodic function by a trigonometric polynomial is subject to the Fourier series theory.

3.2.2 Fourier transform on the torus

The previously introduced Fourier transform can also be called euclidean Fourier transform, since the functions are defined on the whole euclidean space $\mathbb{R}^n$. Now we will introduce the Fourier transform on the torus. Similarly, we start with Schwartz functions and extend it via duality to distributions.

3.31 Definition. The *Schwartz space on the integers* is defined as

$$\mathcal{S}(\mathbb{Z}^n) := \left\{ \varphi \colon \mathbb{Z}^n \to \mathbb{C} : \forall M < \infty \; \exists C_{\varphi,M} > 0 \; \forall k \in \mathbb{Z}^n : |\varphi(k)| \leqslant C_{\varphi,M} (1 + |k|^2)^{-M/2} \right\}.$$

The topology on $\mathcal{S}(\mathbb{Z}^n)$ is induced by the seminorms

$$p_m : \mathcal{S}(\mathbb{Z}^n) \to [0, \infty), \quad \varphi \mapsto \sup_{k \in \mathbb{Z}^n} (1 + |k|^2)^{m/2} |\varphi(k)|$$

for $m \in \mathbb{N}_0$.

3.32 Definition. The Fourier transform on the n-torus is given by

$$\mathcal{F}_{\mathbb{T}} \colon C^\infty(\lambda \mathbb{T}^n) \to \mathcal{S}(\mathbb{Z}^n), \quad f \mapsto \left(k \mapsto \frac{1}{\lambda^n} \int_{\lambda \mathbb{T}^n} \mathrm{e}^{-\mathrm{i} \frac{2\pi}{\lambda} kx} f(x) \mathrm{d}x \right).$$

The number $\mathcal{F}_{\mathbb{T}}[f](k)$ is called kth-Fourier coefficient of f for $f \in C^\infty(\lambda \mathbb{T}^n)$, $k \in \mathbb{Z}^n$. In the following, we will also use the notation $\widehat{f}(k) := \mathcal{F}_{\mathbb{T}}[f](k)$ to keep the expressions handy.

3.33 Theorem (Properties of $\mathcal{F}_{\mathbb{T}}$)**.** *([Gra08, 3.1.2, p. 164]) For $\varphi \in \mathcal{S}(\mathbb{Z}^n)$, $\psi, \psi_1, \psi_2 \in \mathbb{C}^\infty(\lambda \mathbb{T}^n)$, $k \in \mathbb{Z}^n$, $\lambda \in \mathbb{C}$ we have*

(i) $\mathcal{F}_{\mathbb{T}}[\psi_1 + \psi_2](k) = \mathcal{F}_{\mathbb{T}}[\psi_1](k) + \mathcal{F}_{\mathbb{T}}[\psi_2](k)$

(ii) $\mathcal{F}_{\mathbb{T}}[\lambda \psi] = \lambda \mathcal{F}_{\mathbb{T}}[\psi]$

(iii) $\mathcal{F}_{\mathbb{T}}\left[\check{\psi}\right] = \widetilde{\mathcal{F}_{\mathbb{T}}[\psi]}$, $\mathcal{F}_{\mathbb{T}}^{-1}\left[\check{\psi}\right] = \widetilde{\mathcal{F}_{\mathbb{T}}^{-1}[\psi]}$

(iv) $\mathcal{F}_{\mathbb{T}}[\partial^\alpha \psi](k) = (2\pi \mathrm{i}\, k)^\alpha \mathcal{F}_{\mathbb{T}}[\psi](k)$ *for* $k \in \mathbb{Z}^n$, $\alpha \in \mathbb{N}_0^n$ *and* $\psi \in C^\alpha$.

In order to show that the Fourier transform on the torus is a homeomorphism, we need the following uniqueness assertion.

3.34 Theorem. *([Gra08, 3.1.13, p. 169]) Let $f, g \in \mathrm{L}^1(\lambda \mathbb{T}^n)$ satisfy $\widehat{f}(k) = \widehat{g}(k)$ for all $k \in \mathbb{Z}^n$, then $f = g$ a.e.*

3.35 Theorem. *([RT10, 3.1.10, p. 301]) The Fourier transform $\mathcal{F}_{\mathbb{T}}$ is bijective and continuous with the continuous inverse given by*

$$\mathcal{F}_{\mathbb{T}}^{-1} \colon \mathcal{S}(\mathbb{Z}^n) \to C^{\infty}(\lambda\mathbb{T}^n), \quad f \mapsto \left(x \mapsto \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}xk} f(k) \right),$$

so that

$$\mathcal{F}_{\mathbb{T}}^{-1} \circ \mathcal{F}_{\mathbb{T}} = \mathrm{id}_{C^{\infty}(\lambda\mathbb{T}^n))} \qquad \textit{and} \qquad \mathcal{F}_{\mathbb{T}} \circ \mathcal{F}_{\mathbb{T}}^{-1} = \mathrm{id}_{\mathcal{S}(\mathbb{Z}^n)}\,.$$

Proof. $\mathcal{F}_{\mathbb{T}}$ is injective: Let $f, g \in C^{\infty}(\lambda\mathbb{T}^n)$ with $\mathcal{F}_{\mathbb{T}}[f] = \mathcal{F}_{\mathbb{T}}[g]$, i.e. $\mathcal{F}_{\mathbb{T}}[f](k) = \mathcal{F}_{\mathbb{T}}[g](k)$ for every $k \in \mathbb{Z}^n$. Since continuous functions are integrable on compact sets we have $f = g$ almost everywhere by 3.34 and as f and g are continuous we have $f = g$ everywhere.

$\mathcal{F}_{\mathbb{T}}$ is surjective: Let $g \in \mathcal{S}(\mathbb{Z}^n)$. Define

$$f \colon \mathbb{R}^n \to \mathbb{C}, \quad x \mapsto \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}kx} g(k).$$

Then, we have for $\alpha \in \mathbb{N}_0^n$

$$|\partial^{\alpha} f(x)| \leqslant \sum_{k\in\mathbb{Z}^n} |\partial^{\alpha}\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}kx} g(k)| = \sum_{k\in\mathbb{Z}^n} \left(\frac{2\pi}{\lambda}\right)^{|\alpha|} |k^{\alpha}||g(k)| \leqslant \left(\frac{2\pi}{\lambda}\right)^{|\alpha|} \sum_{k\in\mathbb{Z}^n} \frac{|k^{\alpha}|}{(1+|k|^2)^{M/2}}.$$

The sum on the right-hand side converges for M large enough. Therefore, $p_{\alpha}(f) < \infty$ for every $\alpha \in \mathbb{N}_0^n$ with p_{α} being one of the seminorms inducing the topology on $C^{\infty}(\lambda\mathbb{T}^n)$ as introduced in Definition 3.26. Furthermore, we have for $l \in \mathbb{Z}^n$

$$f(x+\lambda l) = \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}k(x+\lambda l)} g(k) = \sum_{k\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}kx} \mathrm{e}^{\mathrm{i}\,2\pi kl} g(k) = f(x),$$

hence $f \in \mathbb{C}^{\infty}(\lambda\mathbb{T}^n)$. Then, we have for $k \in \mathbb{Z}^n$

$$\begin{aligned}\mathcal{F}_{\mathbb{T}}[f](k) &= \frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n} \mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{\lambda}xk} \sum_{l\in\mathbb{Z}^n} \mathrm{e}^{\frac{2\pi}{\lambda}lx} g(l)\mathrm{d}x = \sum_{l\in\mathbb{Z}^n} g(l)\frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}x(l-k)}\mathrm{d}x \\ &= \sum_{l\in\mathbb{Z}^n} g(l)\delta_{kl} = g(k),\end{aligned}$$

where we used Theorem 2.10 to interchange summation and integration. This was possible, since for $f_l(x) := \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}(l-k)x} g(l)$ we have $|f_l(x)| = |g(l)|$. Since $g \in \mathcal{S}(\mathbb{Z}^n)$, the sum $\sum_{l\in\mathbb{Z}^n} |f_l| = \sum_{l\in\mathbb{Z}^n} |g(l)| \leqslant \sum_{l\in\mathbb{Z}^n} C_{g,M}(1+|l|^2)^{-M/2}$ converges as we can choose M large enough. Furthermore, we used the fact that for $k \in \mathbb{Z}^n$ the mappings $x \mapsto \mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{\lambda}kx}$ form an orthonormal set in $\mathrm{L}^2(\lambda\mathbb{T}^n)$ as will be shown in Lemma 4.15. Hence, f is a preimage of g under the Fourier transform.

To show continuity, we want to apply Theorem 2.32. The topology on $C^{\infty}(\lambda\mathbb{T}^n)$ is given by the seminorms

$$p_{\alpha}^{\infty}(\varphi) := \sup_{x\in\lambda\mathbb{T}^n} |\partial^{\alpha}\varphi(x)|$$

for $\varphi \in \mathrm{C}^\infty(\lambda\mathbb{T}^n)$ and $\alpha \in \mathbb{N}_0^n$, whereas the topology on $\mathcal{S}(\mathbb{Z}^n)$ is given by the seminorms

$$p_k^S(\psi) := \sup_{\xi\in\mathbb{Z}^n}(1+|\xi|^2)^{k/2}|\psi(\xi)|$$

for $\psi \in \mathcal{S}(\mathbb{Z}^n)$ and $k \in \mathbb{N}_0$.

Let $k \in \mathbb{N}_0$ and $\varphi \in C^\infty(\lambda\mathbb{T}^n)$. Then we have for $\xi = 0 \in \mathbb{Z}^n$

$$\begin{aligned}(1+|\xi|^2)^{k/2}\left|\mathcal{F}_\mathbb{T}[\varphi](\xi)\right| &= \left|\lambda^{-n}\int_{\lambda\mathbb{T}^n} \mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{\lambda}x0}\varphi(x)\mathrm{d}x\right| \leqslant \lambda^{-n}\int_{\lambda\mathbb{T}^n}|\varphi(x)|\mathrm{d}x \leqslant \sup_{x\in\lambda\mathbb{T}^n}|\varphi(x)|\\ &= p_0^\infty(\varphi).\end{aligned}$$

For $0 \neq \xi \in \mathbb{Z}^n$ we get

$$(1+|\xi|^2)^{k/2}\left|\mathcal{F}_\mathbb{T}[\varphi](\xi)\right| = (1+|\xi|^2)^{k/2}\left|\frac{\mathcal{F}_\mathbb{T}[\partial^\alpha\varphi]}{(\mathrm{i}\,2\pi\xi)^\alpha}\right| = \left(\frac{1+|\xi|^2}{|(2\pi\xi)^{2\alpha/k}|}\right)^{k/2}\sup_{x\in\lambda\mathbb{T}^n}|\partial^\alpha\varphi(x)|$$

using property 3.33(iv). Let $i_0 \in \{1,\dots,n\}$ be the index with $\xi_{i_0} = \max\{|\xi_i| : 1 \leqslant i \leqslant n\}$. Since $\xi \neq 0$, we have $\xi_{i_0} \neq 0$. Choosing $\alpha^{(i_0)} \in \mathbb{N}_0^n$ by

$$\alpha_i^{(i_0)} = \begin{cases} k, & i = i_0 \\ 0, & i \neq i_0 \end{cases} \qquad (1 \leqslant i \leqslant n)$$

yields

$$\begin{aligned}\frac{1+|\xi|^2}{|(2\pi\xi)^{2\alpha^{(i_0)}/k}|} &= \frac{1}{\prod_{i=1}^n(2\pi|\xi_i|)^{2\alpha_i^{(i_0)}/k}} + \sum_{j=1}^n \frac{\xi_j^2}{\prod_{i=1}^n(2\pi|\xi_i|)^{2\alpha_i^{(i_0)}/k}}\\ &\leqslant \frac{1}{(2\pi)^{2n}\xi_{i_0}^2} + \sum_{j=1}^n \frac{1}{(2\pi)^{2n}}\frac{\xi_i^2}{\xi_{i_0}^2} \leqslant \frac{n+1}{(2\pi)^{2n}} \leqslant 1.\end{aligned}$$

For $i \in \{1,\dots,n\}$ define $A_i := \{\xi \in \mathbb{Z}^n\backslash\{0\} : |\xi_i| \geqslant \xi_j, 1 \leqslant j \leqslant n\}$ and $A_0 := \{0\}$. Then $\bigcup_{i=0}^n A_i = \mathbb{Z}^n$. By use of all estimates one obtains

$$\begin{aligned}p_k^S(\mathcal{F}_\mathbb{T}[\varphi]) &= \sup_{\xi\in\mathbb{Z}^n}(1+|\xi|^2)^{k/2}|\varphi(\xi)| = \max\left\{\sup_{\xi\in A_i}(1+|\xi|^2)^{k/2}|\varphi(\xi)| : 0 \leqslant i \leqslant n\right\}\\ &\leqslant \max\left\{p_0^\infty(\varphi)\right\} \cup \left\{\lambda^n p_{\alpha^{(i)}}^\infty(\varphi) : 1 \leqslant i \leqslant n\right\},\end{aligned}$$

using the vectors $\alpha^{(i)} \in \mathbb{N}_0^n$ as defined above. If we define $\mathcal{Z} := \{p_0^\infty\} \cup \{p_{\alpha^{(i)}}^\infty : 1 \leqslant i \leqslant n\}$ and $M := 1$, we can write the estimate above as

$$p_k^S(\mathcal{F}_\mathbb{T}[\varphi]) = \max_{p\in\mathcal{Z}} p(\varphi) \qquad (\varphi \in C^\infty(\lambda\mathbb{T}^n)),$$

with $\mathcal{Z}$ having $n+1$ elements, hence being finite. By Theorem 2.32, $\mathcal{F}_\mathbb{T}$ is continuous.

For the continuity of $\mathcal{F}_{\mathbb{T}}^{-1}$ we can use the same theorem. Again using property 3.33(iv) (applied on the inverse transform) one obtains for $\alpha \in \mathbb{N}_0^n$ and $x \in \lambda\mathbb{T}^n$, $\psi \in S(\mathbb{Z}^n)$

$$\begin{aligned}\left|\partial^\alpha \mathcal{F}_{\mathbb{T}}^{-1}[\psi](x)\right| &= \left|\mathcal{F}_{\mathbb{T}}^{-1}\left[\left(\frac{\mathrm{i}\,2\pi}{\lambda}\cdot\right)^\alpha \psi\right](x)\right| = \left|\sum_{\xi\in\mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}\xi x}\left(\frac{\mathrm{i}\,2\pi\xi}{\lambda}\right)^\alpha \psi(\xi)\right| \\ &\leqslant \sum_{\xi\in\mathbb{Z}^n}\left|\frac{2\pi\xi}{\lambda}\right|^{|\alpha|}|\psi(\xi)|\frac{(1+|\xi|^2)^{k/2}}{(1+|\xi|^2)^{k/2}} \\ &\leqslant \left(\sup_{\xi\in\mathbb{Z}^n}|\psi(\xi)|(1+|\xi|^2)^{k/2}\right)\left(\frac{2\pi}{\lambda}\right)^{|\alpha|}\sum_{\xi\in\mathbb{Z}^n}\frac{|\xi|^{|\alpha|}}{(1+|\xi|^2)^{k/2}}.\end{aligned}$$

The series converges if k is chosen large enough. The argument is similar to the one given in Example 3.16. Hence, we have

$$p_\alpha^\infty(\mathcal{F}_{\mathbb{T}}^{-1}[\psi]) = \sup_{x\in\lambda\mathbb{T}^n}|\partial^\alpha\mathcal{F}_{\mathbb{T}}^{-1}[\psi](x)| \leqslant C p_k^S(\psi)$$

for some constant $C > 0$ and $k \in \mathbb{N}_0$, which shows the continuity of $\mathcal{F}_{\mathbb{T}}^{-1}$. □

It is also possible to define the Fourier series for classes of functions of lower regularity. But in this case, the convergence of the Fourier series is a delicate problem. The definition of the Fourier coefficients is the same as above, but the Fourier series has to be understood a priori as a formal series.

3.36 Definition. Let $\lambda > 0$ and $f \in \mathrm{L}^1(\lambda\mathbb{T}^n)$.

(i) The kth *Fourier coefficient* of f is given by

$$\hat{f}(k) = \frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n} f(x)\exp\left(-\mathrm{i}\,\frac{2\pi}{\lambda}kx\right)\mathrm{d}x \tag{3.4}$$

for every $k \in \mathbb{Z}^n$.

(ii) The *Fourier series* of f at $x \in \lambda\mathbb{T}^n$ is

$$\sum_{k\in\mathbb{Z}^n}\hat{f}(k)\exp\left(\mathrm{i}\,\frac{2\pi}{\lambda}kx\right) \tag{3.5}$$

with the partial sums

$$s_N(x) = \sum_{|k|\leqslant N}\hat{f}(n)\exp\left(\mathrm{i}\,\frac{2\pi}{\lambda}nx\right).$$

(cf. [Rud66, 4.26, p. 91], [Gra08, 3.1.6, p.165]).

3.2.3 Convergence of the Fourier series

The following theorem shows that the Fourier series can be used to approximate any periodic continuous function.

3.37 Theorem. *([Gra08, 3.1.11, p. 168]) For any $f \in C(\lambda\mathbb{T}^n)$ the arithmetic means of the partial sums of the Fourier series*

$$\overline{s}_N(x) := \sum_{\substack{k\in\mathbb{Z}^n \\ |k_i|\leqslant N}} \left(1 - \frac{|k_1|}{N+1}\right)\cdots\left(1 - \frac{|k_n|}{N+1}\right)\widehat{f}(k)\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}kx}$$

converge uniformly to f.

Furthermore, we already know by Theorem 3.34 that the Fourier coefficients are uniquely determined. The completeness of $\mathrm{L}^2(\lambda\mathbb{T}^n)$ gives us the following result:

3.38 Theorem. *([Rud73, p. 92]) Every $f \in \mathrm{L}^2(\lambda\mathbb{T}^n)$ is the L^2-limit of the partial sums of its Fourier series.*

One can also obtain a similar convergence result for $\mathrm{L}^p(\lambda\mathbb{T}^n)$ with $1 < p < \infty$:

3.39 Theorem. *([Gra08, 3.5.7, p. 217]) Let $1 < p < \infty$ and $f \in \mathrm{L}^p(\lambda\mathbb{T}^n)$. Then*

$$\Big\| f - \sum_{\substack{k\in\mathbb{Z}^n \\ |k_i|\leqslant N}} \widehat{f}(k)\,\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}k\cdot}\Big\|_p \to 0, \quad N \to \infty.$$

3.40 Remark. Note, that index sets of the partial sums are "squares", since any component k_i has to be smaller than the partial sum index N. This does not seem to make much difference. However, for circular partial sums, i.e. the condition $|k| < N$, there is $f \in \mathrm{L}^p(\lambda\mathbb{T}^n)$ such that the circular partial sums do not converge in $\mathrm{L}^p(\lambda\mathbb{T}^n)$ $(1 < p \neq 2 < \infty)$. See Grafakos [Gra08, p. 213] for details. One can also find functions in $\mathrm{L}^1(\lambda\mathbb{T}^n)$ whose Fourier series do not converge in $\mathrm{L}^1(\lambda\mathbb{T}^n)$.

This shows that one must be very careful, when considering convergence of Fourier series. Even more delicate is the question of pointwise convergence of the Fourier series. In general, this is not even given for continuous functions. Many attempts have been made to extract conditions on the functions to obtain pointwise convergence, see for example [Car66] and [Mar35]. The conditions often try to control the oscillatory behaviour of the functions, which leads to the concept of absolute continuity in the next section.

First, we want to state a theorem affirming pointwise convergence, if the function is sufficiently neat.

3.41 Theorem. *([Hö83, 7.2.2. p. 179]) Let $k > n$ and let $f \in C^k(\mathbb{R}^n)$ be λ-periodic. Then the Fourier series* (3.5) *of f with the Fourier coefficients* (3.4) *converges uniformly to f.*

3.2.4 Poisson's summation formula

3.42 Definition. A complex function f on $\mathbb{R}$ is called *absolutely continuous* if to every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{i=1}^{N}(\beta_i - \alpha_i) < \delta \quad \text{implies} \quad \sum_{i=1}^{N}|f(\beta_i) - f(\alpha_i)| < \varepsilon,$$

whenever $((\alpha_i, \beta_i))_{i=1}^N$ is a sequence of disjoint intervals.

3.43 Theorem. *([Rud66, 8.17, p. 165]) If $g \in \mathrm{L}^1(\mathbb{R})$ and*

$$f(x) = \int_{-\infty}^{x} g(t)\mathrm{d}t \quad (x \in \mathbb{R})$$

then f is absolutely continuous and $f'(x) = g(x)$ holds almost everywhere (with respect to the Lebesgue measure on $\mathbb{R}$).

3.44 Theorem (Poisson's summation formula)**.** *([BSS88, p. 7]) Let $f \in \mathrm{L}^1(\mathbb{R})$ be absolutely continuous with $f' \in \mathrm{L}^1(\mathbb{R})$, then $f^* := \frac{\lambda}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} f(\cdot + \lambda k) \in \mathrm{L}^1_\lambda$ and*

$$f^*(t) = \sum_{k=-\infty}^{\infty} \mathcal{F}[f]\left(\frac{2k\pi}{\lambda}\right) \exp(\mathrm{i}\, 2k\pi t/\lambda). \quad (t \in \mathbb{R}) \tag{3.6}$$

for any $\lambda \in \mathbb{R}$.

3.45 Theorem (Poisson's summation formula for Schwartz functions)**.** *([Den89, p. 11, Lemma 2]) Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$. Then we have for every $x \in \mathbb{R}^n$*

$$\sum_{k \in \mathbb{Z}^n} \varphi\left(x + \frac{2\pi k}{\lambda}\right) = \left(\frac{\lambda}{\sqrt{2\pi}}\right)^n \sum_{k \in \mathbb{Z}^n} \mathcal{F}[\varphi](\lambda k) \exp(\mathrm{i}\, \lambda k x) \tag{3.7}$$

and equivalently

$$\sum_{k \in \mathbb{Z}^n} \varphi(x + k\lambda) = \left(\frac{\sqrt{2\pi}}{\lambda}\right)^n \sum_{k \in \mathbb{Z}^n} \mathcal{F}[\varphi]\left(\frac{2\pi k}{\lambda}\right) \exp\left(\mathrm{i}\, \frac{2\pi k}{\lambda} x\right). \tag{3.8}$$

Proof. Define

$$\Phi : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \sum_{k \in \mathbb{Z}^n} \varphi(x + k\lambda).$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the convergence of this series can be obtained analogously to Example 3.16.

By definition, Φ is λ-periodic[3]. Since Φ is at least $n+1$ times differentiable, we can expand Φ into its Fourier series according to Theorem 3.41

$$\Phi(x) = \sum_{l \in \mathbb{Z}^n} \widehat{\Phi}(l) \exp\left(\mathrm{i}\, \frac{2\pi}{\lambda} l x\right),$$

with the Fourier coefficients

$$\widehat{\Phi}(l) = \frac{1}{\lambda^n} \int\limits_{[-\frac{\lambda}{2}, \frac{\lambda}{2}]^n} \sum_{k \in \mathbb{Z}^n} \varphi(x + k\lambda) \exp\left(-\mathrm{i}\, \frac{2\pi}{\lambda} l x\right) \mathrm{d}x.$$

To interchange summation and integration we Theorem 2.10, whose requirements will be shown later on. Then

$$\widehat{\Phi}(l) = \frac{1}{\lambda^n} \sum_{k \in \mathbb{Z}^n} \int\limits_{[-\frac{\lambda}{2}, \frac{\lambda}{2}]^n} \varphi(x + k\lambda) \exp\left(-\mathrm{i}\, \frac{2\pi}{\lambda} l x\right) \mathrm{d}x.$$

[3] $\Phi(x + \lambda l) = \sum_{k \in \mathbb{Z}^n} \varphi(x + \lambda(k + l)) \overset{m=k+l}{=} \sum_{m \in \mathbb{Z}^n} \varphi(x + \lambda m) = \Phi(x)$

The substitution $u = x + k\lambda$ leads to

$$\begin{aligned}\widehat{\Phi}\,(l) &= \frac{1}{\lambda^n} \sum_{k\in\mathbb{Z}^n} \int\limits_{k\lambda+[-\frac{\lambda}{2},\frac{\lambda}{2}]^n} \varphi(u) \exp\left(-\,\mathrm{i}\,\frac{2\pi}{\lambda} lu\right) \exp\left(\mathrm{i}\,\frac{2\pi}{\lambda}\lambda lk\right) \mathrm{d}u \\ &= \frac{1}{\lambda^n}(2\pi)^{n/2}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \varphi(x)\exp\left(-\,\mathrm{i}\,x\frac{2\pi}{\lambda}l\right)\mathrm{d}x \\ &= \frac{1}{\lambda^n}(2\pi)^{n/2}\mathcal{F}\left[\varphi\right]\left(\frac{2\pi l}{\lambda}\right),\end{aligned}$$

since $\{k + [-\lambda/2, \lambda/2] : k \in \mathbb{Z}^n\} = \mathbb{R}^n$ and the intersections are sets of measure zero. In total, we get

$$\sum_{k\in\mathbb{Z}^n} \varphi(x + k\lambda) = \left(\frac{\sqrt{2\pi}}{\lambda}\right)^n \sum_{k\in\mathbb{Z}^n} \mathcal{F}\left[\varphi\right]\left(\frac{2\pi l}{\lambda}\right)\exp\left(\mathrm{i}\,\frac{2\pi l}{\lambda}x\right).$$

Since this result holds for every $\lambda \in \mathbb{R}^n$, we can substitute λ by $\tilde{\lambda} := 2\pi/\lambda$ and obtain

$$\sum_{k\in\mathbb{Z}^n} \varphi\left(x + \frac{2\pi}{\lambda}k\right) = \left(\frac{\lambda}{\sqrt{2\pi}}\right)^n \sum_{k\in\mathbb{Z}^n} \mathcal{F}\left[\varphi\right](\lambda k)\exp\left(\mathrm{i}\,\lambda kx\right).$$

For the requirements of Theorem 2.10, we have to show that $f_k := \varphi(\cdot + k\lambda)\exp\left(-\,\mathrm{i}\,\frac{2\pi}{\lambda}l\cdot\right)$ is measurable and $\sum_{k\in\mathbb{Z}^n}\int_{[-\frac{\lambda}{2},\frac{\lambda}{2}]^n}|f_n| < \infty$. Since φ and the exponential function are infinitely differentiable, their product is also infinitely differentiable and in particular continuous. Therefore, f_k is measurable. For the convergence, we see, using the substitution $u = x+k\lambda$ as above

$$\sum_{k\in\mathbb{Z}^n}\int_{[-\frac{\lambda}{2},\frac{\lambda}{2}]^n}|f_k(x)|\,\mathrm{d}x = \int_{\mathbb{R}^n}|\varphi(u)|\left|\exp\left(-\,\mathrm{i}\,\frac{2\pi}{\lambda}lu\right)\right|\mathrm{d}u = \int_{\mathbb{R}^n}|\varphi(u)|\mathrm{d}u < \infty,$$

because $\varphi \in \mathcal{S}(\mathbb{R}^n)$. □

3.46 Lemma. *Let $a > 0$. Then $\mathcal{F}\left[\text{Ш}_a\right] = (\sqrt{2\pi}/a)^n\text{Ш}_{2\pi/a}$.*

Proof. Using Poisson's summation formula (3.7) for $x = 0$, we obtain for $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\mathcal{F}\left[\text{Ш}_a\right](\varphi) = \text{Ш}_a\left(\mathcal{F}\left[\varphi\right]\right) = \sum_{k\in\mathbb{Z}^n}\mathcal{F}\left[\varphi\right](ak) \overset{\text{PSF}}{=} \left(\frac{\sqrt{2\pi}}{a}\right)^n\sum_{k\in\mathbb{Z}^n}\varphi(\tilde{a}k) = \left(\frac{\sqrt{2\pi}}{a}\right)^n\text{Ш}_{\tilde{a}}(\varphi),$$

with $\tilde{a} = 2\pi/a$. Hence, the Fourier transform of the Dirac comb is again a Dirac comb and for $a = \sqrt{2\pi}$ it is a fixed point of the Fourier transform. □

As the Hausdorff-Young inequality 3.14 generalizes Plancherel's theorem 3.13, one can find a generalization for the Poisson's summation formula:

3.47 Theorem. *([Hig96, 2.18, p. 19]) Let $f \in \mathrm{L}^p(\lambda\mathbb{T})$, $1 < p \leqslant 2$. Then there is a constant $D_{\lambda,p}$, depending only on λ and p, such that*

$$\mathcal{F}_{\mathbb{T}}\left[f\right]_q \leqslant D_{\lambda,p}\left\|f\right\|_{\lambda,p} \Leftrightarrow \left(\sum_{k\in\mathbb{Z}}|\mathcal{F}_{\mathbb{T}}\left[f\right](k)|^q\right)^{1/q} \leqslant \frac{D_{\lambda,p}}{\lambda}\left(\int_{\lambda\mathbb{T}}|f(x)|^p\mathrm{d}x\right)^{1/p}.$$

It seems to be unknown, whether there is an analogue for higher dimensions.

3.2.5 Periodic distributions

Now we can ask if there is also a Fourier series representation for periodic distributions. But first we have to define what a periodic distributions is. There are two ways to achieve this, similar to the definition of periodic functions in 3.23. Recall that τ_x is the translation operator with $\tau_x u(\varphi) = u(\tau_{-x}\varphi)$ for a distribution $u \in \mathcal{D}'(\mathbb{R}^n)$.

The first one is the direct translation from functions to distributions.

3.48 Definition. Let $\lambda > 0$. Then $u \in \mathcal{D}'(\mathbb{R}^n)$ is called λ-periodic if $\tau_{\lambda k} u = u$ holds for every $k \in \mathbb{Z}^n$.

Alternatively, one can define periodic distributions as continuous linear functionals of the functions on the torus.

3.49 Definition. An element of the dual space $\mathcal{D}'(\lambda\mathbb{T}^n) = L(C^\infty(\lambda\mathbb{T}^n), \mathbb{C})$ of $C^\infty(\lambda\mathbb{T}^n)$ is called a λ-*periodic distribution*.

3.50 Remark. Note that the previous definitions differ in the spaces used to define periodic distributions. Although we used the same name in the two definitions, it is not clear if it is always the same. For regular distributions it is indeed the same, as one can see in the example below. But for general distributions it is not clear how to replicate the distribution to get from $\mathcal{D}'(\lambda\mathbb{T}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ or how to cut out one period to get back. In the following we will use the latter definition.

3.51 Example. ([RT10, 3.1.25, p. 304]) Let $\varphi \in C^\infty(\lambda\mathbb{T}^n)$. Then

$$[\varphi]_{\lambda\mathbb{T}^n} : C^\infty(\lambda\mathbb{T}^n) \to \mathbb{C}, \quad \psi \mapsto \int_{\lambda\mathbb{T}^n} \varphi(x)\psi(x)\mathrm{d}x$$

is a periodic distribution.

This definition is the same as for regular distributions in $\mathcal{D}'(\mathbb{R}^n)$. Therefore, we also use the rectangular parentheses to denote the distribution induced by the function. If it is clear from the context, we will omit the index $\lambda\mathbb{T}^n$.

Given a distribution with compact support, we can construct a periodic distribution as follows:

3.52 Lemma. *(cf. [Bre65, 3.12, p. 26]) Let $W > 0$ and $u \in \mathcal{D}'(\mathbb{R}^n)$ with $\operatorname{supp} u \subset (-\lambda/2, \lambda/2)^n$. Then there is $\tilde{u} \in \mathcal{D}'(\lambda\mathbb{T}^n)$ with $\tilde{u}(\varphi) = u(\varphi)$ for all $\varphi \in \{u \in \mathcal{D}(\mathbb{R}^n) : \operatorname{supp} u \subset [-\lambda/2, \lambda/2]^n\} = \mathcal{D}(\mathbb{R}^n) \cap C^\infty(\lambda\mathbb{T}^n)$.*

Proof. Let $\theta \in \mathcal{D}(\mathbb{R}^n)$ with $\theta \equiv 1$ on $\operatorname{supp} u$ and $\operatorname{supp} \theta \subset (-\lambda/2, \lambda/2)^n$. Define

$$\tilde{u} \colon C^\infty(\lambda\mathbb{T}^n) \to \mathbb{C}, \quad \varphi \mapsto u(\theta\varphi_p)$$

with φ_p being the natural periodic continuation of φ on $\mathbb{R}^n$. Then, the distribution $\tilde{u}$ is well-defined, since $\theta\varphi \in \mathcal{D}(\mathbb{R}^n)$ and for $(\varphi_k)_{k\in\mathbb{N}} \subset C^\infty(\lambda\mathbb{T}^n)$ with $\varphi_k \to 0$ $(k \to \infty)$ we have $u(\theta\varphi_k) \to 0$ for $k \to \infty$ because $(\theta\varphi_k)_k \subset \mathcal{D}(\mathbb{R}^n)$ and $u \in \mathcal{D}'(\mathbb{R}^n)$.

Let $\varphi \in \{u \in \mathcal{D}(\mathbb{R}^n) : \operatorname{supp} u \subset [-\lambda/2, \lambda/2]^n\}$. Then, we have $\varphi \in C^\infty(\lambda\mathbb{T}^n)$ by the natural periodic continuation and $\varphi(x) = \theta(x)\varphi(x)$ for $x \in \operatorname{supp} u$. Hence,

$$\tilde{u}(\varphi) = u(\theta\varphi) = u(\varphi).$$

□

Since u and $\tilde{u}$ coincide for functions which can be considered to be elements of $\mathcal{D}(\mathbb{R}^n)$ as well as of $C^\infty(\lambda\mathbb{T}^n)$, we can consider u to be in $\mathcal{D}'(\lambda\mathbb{T}^n)$ via $\tilde{u}$.

3.2.6 Fourier series for periodic distributions

The Fourier series for periodic distributions is defined analogously to the Fourier transform for distributions. However, one has to bear in mind that the domain and the image of the Fourier transform on the torus are not the same.

3.53 Definition. *Tempered distributions* on the lattice are elements of the dual space of the Schwartz space on the lattice $\mathcal{S}'(\mathbb{Z}^n) := L(S(\mathbb{Z}^n), \mathbb{C})$.

3.54 Remark. ([RT10, 3.1.7, p. 300]) Any tempered distributions $u \in \mathcal{S}'(\mathbb{Z}^n)$ is of the form

$$u(\psi) = \sum_{\xi \in \mathbb{Z}^n} f(\xi)\psi(\xi) \quad (\psi \in \mathcal{S}(\mathbb{Z}^n)),$$

where $f : \mathbb{Z}^n \to \mathbb{C}$ grows at most polynomially, i.e. there are $C > 0$, $M < \infty$ with

$$|f(\xi)| \leqslant C(1 + |\xi|^2)^{M/2} \quad (\xi \in \mathbb{Z}^n).$$

Hence, u is pointwise well-defined. This can be seen if we use the Schwartz function

$$\delta_k : \mathbb{Z}^n \to \mathbb{C}, \quad \xi \mapsto \begin{cases} 1 & \xi = k \\ 0 & \xi \neq k \end{cases},$$

which yields

$$u(\delta_k) = \sum_{\xi \in \mathbb{Z}^n} f(\xi)\delta_k(\xi) = f(k).$$

Therefore, we write $u(k)$ and mean $u(\delta_k)$ for $k \in \mathbb{Z}^n$.

3.55 Definition. For a periodic distribution $u \in \mathcal{D}'(\lambda\mathbb{T}^n)$ the Fourier transform is given by

$$\mathcal{F}_{\mathbb{T}} : \mathcal{D}'(\lambda\mathbb{T}^n) \to \mathcal{S}'(\mathbb{Z}^n), \quad \mathcal{F}_{\mathbb{T}}[u](\varphi) = u(\mathcal{F}_{\mathbb{T}}^{-1}[\check{\varphi}]) \quad (\varphi \in \mathcal{S}(\mathbb{Z}^n)).$$

3.56 Lemma. *Let $u \in \mathcal{D}'(\lambda\mathbb{T}^n)$ and $k \in \mathbb{Z}^n$. The kth Fourier coefficient can be expressed by*

$$\widehat{u}(k) := \mathcal{F}_{\mathbb{T}}[u](k) = u\left(\exp\left(-\frac{2\pi}{\lambda}k\cdot\right)\right).$$

This yields the representation

$$\mathcal{F}_{\mathbb{T}}[u] : C^\infty(\lambda\mathbb{T}^n) \to \mathbb{C}, \quad \varphi \mapsto \sum_{k \in \mathbb{Z}^n} \widehat{u}(k)\,\varphi(k).$$

Proof. For $u \in \mathcal{D}'(\lambda\mathbb{T}^n)$ and $k \in \mathbb{Z}^n$ we have, considering Remark 3.54,

$$\mathcal{F}_{\mathbb{T}}[u](k) = u\left(\mathcal{F}_{\mathbb{T}}^{-1}\left[\check{\delta_k}\right]\right) = u\left(x \mapsto \sum_{\xi \in \mathbb{Z}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}x\xi}\delta_{-k}(\xi)\right) = u\left(x \mapsto \mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{\lambda}xk}\right),$$

where we have used $\check{\delta_k} = \delta_{-k}$. □

3.57 Theorem. *The Fourier transform $\mathcal{F}_{\mathbb{T}}$ for periodic distributions is a bijection and continuous with the continuous inverse given by*

$$\mathcal{F}_{\mathbb{T}}^{-1}\colon \mathcal{S}'(\mathbb{Z}^n) \to \mathcal{D}'(\lambda\mathbb{T}^n), \quad \mathcal{F}_{\mathbb{T}}^{-1}[u](\varphi) = u(\mathcal{F}_{\mathbb{T}}[\breve{\varphi}]),$$

so that

$$\mathcal{F}_{\mathbb{T}}^{-1} \circ \mathcal{F}_{\mathbb{T}} = \mathrm{id}_{\mathcal{D}'(\lambda\mathbb{T}^n)} \quad \text{and} \quad \mathcal{F}_{\mathbb{T}} \circ \mathcal{F}_{\mathbb{T}}^{-1} = \mathrm{id}_{\mathcal{S}'(\mathbb{Z}^n)}.$$

Proof. $\mathcal{F}_{\mathbb{T}}$ is an injection: Let $u, v \in \mathcal{D}'(\lambda\mathbb{T}^n)$ with $\mathcal{F}_{\mathbb{T}}[u] = \mathcal{F}_{\mathbb{T}}[v]$, i.e. $u(\mathcal{F}_{\mathbb{T}}^{-1}[\breve{\varphi}]) = v(\mathcal{F}_{\mathbb{T}}^{-1}[\breve{\varphi}])$ for all $\varphi \in \mathcal{S}(\mathbb{Z}^n)$. Since $\varphi \mapsto \breve{\varphi}$ is an automorphism and $\mathcal{F}_{\mathbb{T}}^{-1}$ is a bijection from $\mathcal{S}(\mathbb{Z}^n)$ to $C^\infty(\lambda\mathbb{T}^n)$, we can identify each $\breve{\varphi} \in \mathcal{S}(\mathbb{Z}^n)$ with exactly one $\psi \in C^\infty(\lambda\mathbb{T}^n)$. Hence, we have $u(\psi) = v(\psi)$ for all $\psi \in C^\infty(\lambda\mathbb{T}^n)$, i.e. $u = v$ in $\mathcal{D}'(\lambda\mathbb{T}^n)$.

$\mathcal{F}_{\mathbb{T}}$ is a surjection: Let $v \in \mathcal{S}'(\mathbb{Z}^n)$. Define

$$u\colon C^\infty(\mathbb{T}^n) \to \mathbb{C}, \quad \psi \mapsto v\left(\mathcal{F}_{\mathbb{T}}\left[\breve{\psi}\right]\right).$$

As $\mathcal{F}_{\mathbb{T}}$ is a bijection from $C^\infty(\lambda\mathbb{T}^n)$ to $\mathcal{S}(\mathbb{Z}^n)$, there is for each $\psi \in C^\infty(\lambda\mathbb{T}^n)$ exactly one $\varphi := \mathcal{F}_{\mathbb{T}}[\psi] \in \mathcal{S}(\mathbb{Z}^n)$. Therefore, u is well-defined. Furthermore, since v, $\mathcal{F}_{\mathbb{T}}$ and inversion are continuous, hence it is u. Therefore, $u \in \mathcal{D}'(\lambda\mathbb{T}^n)$. Then, we have for $\varphi \in \mathcal{S}(\mathbb{Z}^n)$ with $r(\varphi) := \breve{\varphi}$

$$\mathcal{F}_{\mathbb{T}}[u](\varphi) = u(\mathcal{F}_{\mathbb{T}}^{-1} \circ r \circ \varphi) = v(\mathcal{F}_{\mathbb{T}} \circ r \circ \mathcal{F}_{\mathbb{T}}^{-1} \circ r(\varphi)) = v(\mathcal{F}_{\mathbb{T}} \circ \mathcal{F}_{\mathbb{T}}^{-1} \circ r \circ r(\varphi)) = v(\varphi),$$

since $r \circ \mathcal{F}_{\mathbb{T}}^{-1}(\varphi) = \mathcal{F}_{\mathbb{T}}^{-1} \circ r(\varphi)$ by Theorem 3.33, $r \circ r = \mathrm{id}$ and $\mathcal{F}_{\mathbb{T}} \circ \mathcal{F}_{\mathbb{T}}^{-1} = \mathrm{id}_{\mathcal{S}(\mathbb{Z}^n)}$ by Theorem 3.35.

Therefore, $\mathcal{F}_{\mathbb{T}}$ is a bijection. Next, we show the two identities. Let $u \in \mathcal{D}'(\lambda\mathbb{T}^n)$. Then, we have for $\varphi \in C^\infty(\lambda\mathbb{T}^n)$

$$\mathcal{F}_{\mathbb{T}}^{-1} \circ \mathcal{F}_{\mathbb{T}}[u](\varphi) = u(\mathcal{F}_{\mathbb{T}} \circ r \circ \mathcal{F}_{\mathbb{T}}^{-1} \circ r(\varphi)) = u(\mathcal{F}_{\mathbb{T}} \circ \mathcal{F}_{\mathbb{T}}^{-1} \circ r \circ r(\varphi)) = u(\varphi)$$

as above. Hence,

$$\mathcal{F}_{\mathbb{T}}^{-1} \circ \mathcal{F}_{\mathbb{T}} = \mathrm{id}_{\mathcal{D}'(\lambda\mathbb{T}^n)}$$

and a similar calculation shows the other identity.

To show that $\mathcal{F}_{\mathbb{T}}$ is continuous, it is sufficient to show that $\mathcal{F}_{\mathbb{T}}$ is continuous at 0, since $\mathcal{F}_{\mathbb{T}}$ is linear. Let $V \subset \mathcal{S}'(\mathbb{Z}^n)$ be a neighbourhood of 0. Then, there are $\varphi_1, \dots, \varphi_k \in \mathcal{S}(\mathbb{Z}^n)$ for some $k \in \mathbb{N}$ such that

$$\{u \in \mathcal{S}'(\mathbb{Z}^n) : |u(\varphi_i)| < 1, 1 \leqslant i \leqslant k\} \subset V.$$

Now define

$$W := \{u \in \mathcal{D}'(\lambda\mathbb{T}^n) : |u(\mathcal{F}_{\mathbb{T}}^{-1}[\breve{\varphi}_i])| < 1, 1 \leqslant i \leqslant k\}.$$

Since $\psi_i := \mathcal{F}_{\mathbb{T}}^{-1}[\breve{\varphi}_i] \in C^\infty(\lambda\mathbb{T}^n)$, W is a neighbourhood of 0 in $\mathcal{D}'(\lambda\mathbb{T}^n)$. Furthermore, we have for $w \in W$

$$|\mathcal{F}_{\mathbb{T}}[W](\varphi_i)| = |w(\mathcal{F}_{\mathbb{T}}^{-1}[\breve{\varphi}_i])| < 1,$$

hence $\mathcal{F}_{\mathbb{T}}(W) \subset V$. Hence, $\mathcal{F}_{\mathbb{T}}$ is continuous by Definition 2.29.

Interchanging $\mathcal{F}_{\mathbb{T}}$ and $\mathcal{F}_{\mathbb{T}}^{-1}$ above yields the continuity of $\mathcal{F}_{\mathbb{T}}^{-1}$. $\square$

3.58 Corollary. *For $u \in \mathcal{D}'(\lambda\mathbb{T}^n)$, we have the representation*

$$u = \sum_{k\in\mathbb{Z}^n} \widehat{u}\,(k) \left[\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}k\cdot}\right],$$

where $\left[\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}k\cdot}\right]$ *denotes the regular distribution induced by the function* $x \mapsto \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}kx}$ *in* $\mathcal{D}'(\lambda\mathbb{T}^n)$.

Proof. Since $\mathcal{F}_{\mathbb{T}}^{-1} \circ \mathcal{F}_{\mathbb{T}} = \mathrm{id}_{\mathcal{D}'(\lambda\mathbb{T}^n)}$, we have for $\varphi \in \mathbb{C}^{\infty}(\lambda\mathbb{T}^n)$

$$u(\varphi) = \mathcal{F}_{\mathbb{T}}^{-1} \circ \mathcal{F}_{\mathbb{T}} u\varphi = \mathcal{F}_{\mathbb{T}}\,[u]\,(\mathcal{F}_{\mathbb{T}}\,[\check{\varphi}]) = \sum_{k\in\mathbb{Z}^n} \widehat{u}\,(k)\,\mathcal{F}_{\mathbb{T}}\left[\check{\psi}\right](k),$$

where we have used Lemma 3.56 for the representation of $\mathcal{F}_{\mathbb{T}}\,[u]$. Furthermore, we have for $k \in \mathbb{Z}^n$

$$\mathcal{F}_{\mathbb{T}}\,[\check{\varphi}](k) = \int_{\lambda\mathbb{T}^n} \mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{\lambda}kx}\varphi(-x)\mathrm{d}x = \int_{\lambda\mathbb{T}^n} \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}kx}\varphi(x)\mathrm{d}x = \left[\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}\cdot}\right](\varphi).$$

This yields

$$u(\varphi) = \sum_{k\in\mathbb{Z}^n} \widehat{u}\,(k) \left[\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}k\cdot}\right](\varphi)$$

for every $\varphi \in C^{\infty}(\lambda\mathbb{T}^n)$, which concludes the proof. □

4 The Sampling Theorem

In his original publication [Sha49, p. 448], Shannon makes use of the fact, that a compactly supported function can be expanded as a Fourier series. Although he doesn't take into account the prerequisites for both Fourier transformation and Fourier series expansion, the idea can be turned into a formally valid proof as is done in [Mar01, p. 52ff], [Nat89, p. 56] and [Den89, p. 9, Beweis 1]. In the meantime, many other methods have been developed, of which we will elaborate some. But first, we have to establish the general prerequisites for any kind of sampling theorem - the concept of band limitation.

4.1 Band limited functions and distributions

4.1 Definition. Let $W > 0$.

(i) For $1 \leqslant p \leqslant \infty$ a function $f \in \mathrm{L}^p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ is called *band limited with band-width* W or W-band limited for short if there is $g \in \mathrm{L}^1([-W, W]^n)$ such that $f = \mathcal{F}^{-1}[g]$.

(ii) $u \in \mathcal{S}'(\mathbb{R}^n)$ is called *band limited with band-width* W or W-band limited for short if $\operatorname{supp} \mathcal{F}[u] \subseteq [-W, W]^n$.

(iii) An entire function $f : \mathbb{C}^n \to \mathbb{C}$ is of at most *exponential type* $\sigma \geqslant 0$ if there is $C > 0$ such that $|f(z)| \leqslant C \exp(\sigma |\operatorname{Im} z|)$ for every $z \in \mathbb{C}^n$.

(iv) $B^p_\sigma(\mathbb{C}^n)$ $(1 \leqslant p \leqslant \infty)$ is the class of all entire functions of at most exponential type σ whose restriction to $\mathbb{R}^n$ belongs to $\mathrm{L}^p(\mathbb{R}^n)$.

(v) The *Paley-Wiener space* $\mathrm{PW}_W(\mathbb{C}^n)$ is defined as

$$\mathrm{PW}_W(\mathbb{C}^n) := \{f : \mathbb{C}^n \to \mathbb{C} \text{ entire}, \exists A \geqslant 0, N \in \mathbb{N}_0 : \forall z \in \mathbb{C}^n : |f(z)| \leqslant A(1+|z|)^N \mathrm{e}^{W|\operatorname{Im} z|}\}.$$

Furthermore, for $1 \leqslant p \leqslant \infty$ we define

$$\mathrm{PW}^p_W(\mathbb{C}^n) := \{f \in \mathrm{PW}_W(\mathbb{C}^n) : \exists g \in \mathrm{L}^p([-W, W]^n) : f|_{\mathbb{R}^n} = \mathcal{F}^{-1}[g]\}.$$

4.2 Lemma. *For $1 \leqslant p \leqslant \infty$ and $\sigma \geqslant 0$, $B^p_\sigma(\mathbb{C}^n)$ forms a vector space over $\mathbb{C}$ by pointwise addition and scalar multiplication.*

Proof. Since the space of all entire functions forms a vector space as well as $\mathrm{L}^p(\mathbb{R}^n)$ does, it remains only to show that the sum of two elements of $B^p_\sigma(\mathbb{C}^n)$ again is of at most exponential type σ. This can be verified using the triangular inequality. □

4.3 Lemma. *([BSS88, Eq. 2.2, p. 6]) Let $f \in B^p_\sigma(\mathbb{C})$. Then the following inequalities hold for every $h > 0$:*

$$\|f\|_p \leqslant \sup_{u \in \mathbb{R}} \left(\frac{h}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} |f(u - hk)|^p \right)^{1/p} \leqslant (1 + h\sigma) \|f\|_p \tag{4.1}$$

(Nikol'skiĭ's inequality) and

$$\left\| f^{(r)} \right\|_p \leqslant \sigma^r \left\| f \right\|_p \tag{4.2}$$

(Bernstein's inequality), where $f^{(r)}$ denote the rth (complex) derivative of f ($r \in \mathbb{N}_0$).

4.4 Theorem (Paley-Wiener for functions). *([Rud73, 7.22, p.181])*

(a) If $\varphi \in \mathcal{D}(\mathbb{R}^n)$ has its support in $rB := \{x \in \mathbb{R}^n : |x| \leqslant r\}$, and if

$$f(z) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \varphi(t) \exp(-\mathrm{i}\, zt) \mathrm{d}t \quad (z \in \mathbb{C}^n), \tag{4.3}$$

then f is entire and there are constants $\gamma_N < \infty$ such that

$$|f(z)| \leqslant \gamma_N (1 + |z|)^{-N} e^{r|\operatorname{Im} z|} \quad (z \in \mathbb{C}^n, N \in \mathbb{N}_0). \tag{4.4}$$

(b) Conversely, if an entire function f satisfies the conditions (4.4), then there exists $\varphi \in \mathcal{D}(\mathbb{R}^n)$, with support in rB such that (4.3) holds.

4.5 Theorem (Paley-Wiener for distributions). *([Rud73, 7.23, p. 183])*

(a) If $u \in \mathcal{D}'(\mathbb{R}^n)$ has order $N \in \mathbb{N}_0$, its support is contained in $rB := \{x \in \mathbb{R}^n : |x| \leqslant r\}$ and if

$$f(z) := (2\pi)^{-n/2} u(\exp(-\mathrm{i}\, z\cdot)) \quad (z \in \mathbb{C}^n), \tag{4.5}$$

then f is entire, the restriction of f to $\mathbb{R}^n$ is the Fourier transform of u and there is a constant $\gamma < \infty$ such that

$$|f(z)| \leqslant \gamma (1 + |z|)^{N} e^{r|\operatorname{Im} z|} \quad (z \in \mathbb{C}^n) \tag{4.6}$$

(b) Conversely, if f is an entire function in $\mathbb{C}^n$ which satisfies (4.6) for some $N \in \mathbb{N}_0$ and some γ, then there exists $u \in \mathcal{D}'(\mathbb{R}^n)$ with support in rB, such that (4.5) holds.

4.6 Lemma. *Let $f \in \mathrm{L}^p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be W-band limited. Then $f \in C^\infty(\mathbb{R}^n)$.*

Proof. Let $f \in \mathrm{L}^p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ be W-band limited. By definition, there is $g \in \mathrm{L}^1([-W, W]^n)$ such that $f = \mathcal{F}^{-1}[g]$. $[\check{g}]$ is a regular distribution with support contained in $\{x \in \mathbb{R}^n : |x| \leqslant \sqrt{n}W\}$. $[g]]$ has order 0 as proved in Example 2.55, because it is a regular distribution. Define $\tilde{f} : \mathbb{C}^n \to \mathbb{C}$ by $\tilde{f}(z) := (2\pi)^{-\check{g}/2}(\exp(-z\cdot))$. Then, the Paley-Wiener Theorem 4.5 yields that $\tilde{f}$ is entire and that $\tilde{f}$ restricted to $\mathbb{R}^n$ is the Fourier transform of $[\check{g}]$. Since $\check{g} \in \mathrm{L}^1([-W, W]^n)$, its Fourier transform exists. By Remark 3.19, we have $\mathcal{F}[[\check{g}]] = [\mathcal{F}[\check{g}]] = [\mathcal{F}^{-1}[g]]$. Hence, we infer $\tilde{f}(x) = \mathcal{F}^{-1}[g](x) = f(x)$ for $x \in \mathbb{R}^n$. Since $\tilde{f}$ is entire, it is infinitely differentiable for all $x \in \mathbb{R}^n$. Therefore, $f \in C^\infty(\mathbb{R}^n)$. □

4.7 Lemma. *([Hig96, 6.3, p. 52]) Let $1 < p \leqslant 2$, $1/p + 1/q = 1$ then $\mathrm{PW}^p_\sigma(\mathbb{C}^n) \subset B^q_\sigma(\mathbb{C}^n)$.*

Proof. Let $f \in \mathrm{PW}^p_\sigma(\mathbb{C}^n)$. Then there is $g \in \mathrm{L}^p([-\sigma, \sigma]^n)$ with

$$f(z) = (2\pi)^{-n/2} \int_{[-\sigma,\sigma]^n} g(x) \mathrm{e}^{\mathrm{i}\, zx} \mathrm{d}x.$$

for $z \in \mathbb{R}^n$. Since f is entire, this holds also for $z \in \mathbb{C}^n$ due to the Identity Theorem 2.41. Using $\mathrm{L}^p([-\sigma,\sigma]^n) \subset \mathrm{L}^1([-\sigma,\sigma]^n)$, we get

$$|f(z)| \leqslant (2\pi)^{-n/2} \int_{[-\sigma,\sigma]} |g(x)|\, \mathrm{e}^{-x|\operatorname{Im} z|} \mathrm{d}x \leqslant (2\pi)^{-n/2} \|g\|_1 \, \mathrm{e}^{\sigma|\operatorname{Im} z|},$$

hence f is of exponential type at most σ. By the Hausdorff-Young Theorem 3.14, which is also true for the inverse Fourier transform, we get $f \in \mathrm{L}^q(\mathbb{R}^n)$. Therefore, $f \in B^q_\sigma(\mathbb{C}^n)$. ☐

For $p = 2$, we have equality as stated in the following lemma:

4.8 Lemma. *Let* $\sigma > 0$. *Then* $\mathrm{PW}^2_\sigma(\mathbb{C}^n) = B^2_\sigma(\mathbb{C}^n)$.

Proof. We only have to show $B^2_\sigma(\mathbb{C}^n) \subset \mathrm{PW}^2_\sigma(\mathbb{C}^n)$. Let $f \in B^2_\sigma(\mathbb{C}^n)$. Then $f|_{\mathbb{R}^n} \in \mathrm{L}^2(\mathbb{R}^n)$. The Hausdorff-Young Theorem 3.14 provides $\mathcal{F}[f] \in \mathrm{L}^2(\mathbb{R}^n)$. The Paley-Wiener Theorem 4.5 shows that $\mathcal{F}[f]$ is a distribution with support in $\sigma B \subset [-\sigma,\sigma]^n$. Both Fourier transforms coincide by Remark 3.19. Hence, we conclude $f|_{\mathbb{R}^n} = \mathcal{F}^{-1}[g]$ for some $g \in \mathrm{L}^2([-\sigma,\sigma]^n)$. Furthermore, there holds $|f(z)| \leqslant C(1+|z|)^0 \mathrm{e}^{\sigma|\operatorname{Im} z|}$ by definition of $B^2_\sigma(\mathbb{C}^n)$. Therefore, $f \in \mathrm{PW}^2_\sigma(\mathbb{C}^n)$. ☐

4.9 Definition. For $a > 0$, the sinc_a-function is defined as

$$\mathrm{sinc}_a : \mathbb{R} \to \mathbb{R}, x \mapsto \begin{cases} \frac{\sin(ax)}{ax}, & x \neq 0 \\ 1, & x = 0 \end{cases}. \tag{4.7}$$

This definition can be extended to $\mathbb{R}^n$ by

$$\mathrm{sinc}_a : \mathbb{R}^n \to \mathbb{R}, x = (x_1, \ldots, x_n) \mapsto \prod_{k=1}^n \mathrm{sinc}_a(x_k).$$

4.10 Lemma. *(i)* $\mathrm{sinc}_a \in \mathrm{L}^2(\mathbb{R}^n)$ *and* sinc_a *is a-band limited,*

(ii) $\mathrm{sinc}_a(k\pi/a) = \delta_{k0}$ *for* $k \in \mathbb{Z}^n$ *with* δ_{k0} *being Kronecker's delta (*$\delta_{k0} = 1$ *if* $k = 0$ *and* $\delta_{k0} = 0$ *if* $k \neq 0$*).*

Proof. (i) First we consider $n = 1$. By definition

$$\|\mathrm{sinc}_a\|_2^2 = \int_{\mathbb{R}} \left|\frac{\sin(ax)}{ax}\right|^2 \mathrm{d}x = 2\int_0^{\pi/a} \frac{\sin^2(ax)}{(ax)^2} \mathrm{d}x + 2\int_{\pi/a}^{\infty} \frac{\sin^2(ax)}{(ax)^2} \mathrm{d}x.$$

For the expansion, we exploit the fact that $(\sin(x)/x)^2$ is symmetric. Since $\sin^2(x) \leqslant 1$ for all $x \in \mathbb{R}$, we get the estimate

$$\int_{\pi/a}^{\infty} \frac{\sin^2(ax)}{(ax)^2} \mathrm{d}x \leqslant \int_{\pi/a}^{\infty} \frac{1}{(ax)^2} = \frac{1}{a\pi}. \tag{4.8}$$

As $\sin^2(x)/x^2$ is continuous in $[0, \pi/a]$, it is bounded by some constant c and we get the estimate

$$\int_0^{\pi/a} \frac{\sin^2(ax)}{(ax)^2} \leqslant c\pi/a. \tag{4.9}$$

Using Equations (4.9) and (4.8), we get $\|\mathrm{sinc}_a\|_2^2 < \infty$.

Now we have to consider the Fourier transform of sinc_a. We already know by Example 3.8 that $\mathcal{F}\left[\chi_{[-a,a]}\right] = (\sqrt{2\pi}a/\pi)\,\mathrm{sinc}_a$. Since $\mathrm{sinc}_a, \chi_{[-a,a]} \in \mathrm{L}^2(\mathbb{R})$ and the Fourier transform is a linear isometry on $\mathrm{L}^2(\mathbb{R})$, we can use Corollary 3.12 and get

$$\chi_{[-a,a]}(x) = \chi_{[-a,a]}(-x) = \mathcal{F}^2\left[\chi_{[-a,a]}\right](x) = \mathcal{F}\left[\frac{\sqrt{2\pi}a}{\pi}\,\mathrm{sinc}_a\right](x) = \frac{\sqrt{2\pi}a}{\pi}\mathcal{F}\left[\mathrm{sinc}_a\right](x)$$

for almost every $x \in \mathbb{R}$. Hence

$$\mathcal{F}\left[\mathrm{sinc}_a\right] = \frac{\pi}{\sqrt{2\pi}a}\chi_{[-a,a]}$$

and therefore sinc_a is a-band limited.

For $n > 1$, we get by an iterated application of Tonelli's theorem

$$\|\mathrm{sinc}_a\|_2^2 = \int_{\mathbb{R}^n} |\mathrm{sinc}_a(x)|^2\,\mathrm{d}x = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{i=1}^{n} |\mathrm{sinc}(x_i)|^2\,\mathrm{d}x_1 \cdots \mathrm{d}x_n \leqslant \left(\|\mathrm{sinc}_a\|_{2,1}^n\right)^2 < \infty,$$

where $\|\mathrm{sinc}_a\|_{2,1}$ is the norm of the one dimensional sinc_a as evaluated above. Hence $\mathrm{sinc}_a \in \mathrm{L}^2(\mathbb{R}^n)$. Similarly, the Fourier transform can be evaluated component-wise because the exponential term splits into factors, hence

$$\mathcal{F}\left[\mathrm{sinc}_a\right](x) = \left(\frac{\pi}{\sqrt{2\pi}a}\right)^n \prod_{i=1}^{n} \chi_{[-a,a]}(x_i) = \left(\frac{\pi}{\sqrt{2\pi}a}\right)^n \chi_{[-a,a]^n}(x) \tag{4.10}$$

for almost every $x \in \mathbb{R}^n$. Therefore sinc_a is a-band limited.

(ii) For $n = 1$, the statement $\mathrm{sinc}_a(k\pi/a) = \delta_{k0}$ is clear, since $\mathrm{sinc}_a(0) = 1$ by definition and $\mathrm{sinc}_a(k\pi/a) = \sin(k\pi)/k\pi = 0$ for every $k \in \mathbb{Z}\backslash\{0\}$. For $n > 1$, we have for $k \in \mathbb{Z}^n$

$$\mathrm{sinc}_a\left(\frac{k\pi}{a}\right) = \prod_{i=1}^{n} \mathrm{sinc}_a\left(\frac{k_i\pi}{a}\right) = \prod_{i=1}^{n} \delta_{k_i 0} = \delta_{k0}.$$

□

4.2 Proof based on the commutativity of the semi-discrete convolution product

This proof follows the idea mentioned in Butzer *et al.* [BSS88, p. 9f], which is also used in [Mar01, p. 50] and in more detail in [Den89, p. 11, Beweis 2].

4.11 Theorem. *Let $f \in \mathrm{L}^2(\mathbb{R})$ be band limited with band width W. Then*

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{W}\right)\mathrm{sinc}_W\left(t - \frac{k\pi}{W}\right) \tag{4.11}$$

holds for every $t \in \mathbb{R}$ and the series converges absolutely and uniformly.

Before we can prove this theorem, we first have to prove some auxiliary results.

4.12 Lemma. *Let $g \in \mathrm{L}^1(\mathbb{R}) \cap C(\mathbb{R})$ be $2W$-band limited. Then*

$$\frac{\pi}{\sqrt{2\pi}W} \sum_{k=-\infty}^{\infty} g\left(\frac{k\pi}{W}\right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x)\mathrm{d}x. \tag{4.12}$$

Proof. Let $g \in \mathrm{L}^1(\mathbb{R}) \cap C(\mathbb{R})$ be $2W$-band limited.

From Lemma 4.6, we know $g \in C^\infty(\mathbb{R})$, in particular, there holds $g(x) = \int_{-\infty}^x g'(x)\mathrm{d}x$, where g' is the derivative of g and by Bernstein's inequality (4.2) we also have $g' \in \mathrm{L}^1(\mathbb{R})$. Hence g is absolutely continuous by Theorem 3.43. Then Poisson's summation formula (3.44) with $\lambda = \pi/W$ leads to

$$\frac{\pi}{\sqrt{2\pi}W} \sum_{k=-\infty}^{\infty} g\left(\frac{k\pi}{W}\right) = \sum_{k=-\infty}^{\infty} \left(\mathcal{F}\left[g\right]\right)(2kW) = (\mathcal{F}\left[g\right])(0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x)\mathrm{d}x,$$

because $\mathcal{F}\left[g\right](\xi) = 0$ for $|\xi| \geqslant 2W$ as g is $2W$-band limited and the Inversion Theorem 3.11 holds. ☐

4.13 Lemma. *Let $1 \leqslant p \leqslant \infty$ and $1/p + 1/q = 1$ and $f_1 \in \mathrm{L}^p(\mathbb{R})$, $f_2 \in \mathrm{L}^q(\mathbb{R})$ both be band limited with band width W. Then*

$$(f_1 * f_2)(t) = \frac{\pi}{\sqrt{2\pi}W} \sum_{k=-\infty}^{\infty} f_1\left(\frac{k\pi}{W}\right) f_2\left(t - \frac{k\pi}{W}\right) \quad (t \in \mathbb{R}) \tag{4.13}$$

and the series converges absolutely and uniformly for all real $t \in \mathbb{R}$.

Proof. We want to apply Lemma 4.12 to $\tilde{f}_t := f_1(\cdot)f_2(t-\cdot), t \in \mathbb{R}$. Using Hölder's inequality, we infer

$$\left\|\tilde{f}_t\right\|_1 = \|f_1(\cdot)f_2(t-\cdot)\|_1 \leqslant \|f_1\|_p \|f_2(t-\cdot)\|_q = \|f_1\|_p \|f_2\|_q < \infty, \tag{4.14}$$

hence $\tilde{f}_t \in \mathrm{L}^1(\mathbb{R})$ for every $t \in \mathbb{R}$. Next it is shown that $\tilde{f}_t$ is $2W$-band limited. Therefore, we exploit the fact that multiplication converts to convolution after a Fourier transform as stated in Theorem 3.9, hence

$$\begin{aligned}\mathcal{F}\left[\tilde{f}_t\right] &= \mathcal{F}\left[(f_1(\cdot)f_2(t-\cdot))\right] = \mathcal{F}\left[f_1\right] * \mathcal{F}\left[f_2(t-\cdot)\right] = -\mathcal{F}\left[f_1\right] * \mathcal{F}\left[f_2(\cdot - t)\right] \\ &= \mathcal{F}\left[f_1\right] * \mathcal{F}\left[f_2\right] \mathrm{e}^{-\mathrm{i}\,t\cdot}.\end{aligned}$$

Expanding the convolution into its definition, we obtain

$$\mathcal{F}\left[\tilde{f}_t\right](\xi) = -\int_{\mathbb{R}} \mathcal{F}\left[f_1\right](x)\mathcal{F}\left[f_2\right](\xi - x)\mathrm{e}^{-\mathrm{i}\,t(\xi-x)}\mathrm{d}x.$$

As f_1 and f_2 are both W-band limited, the integrand vanishes for $x \notin (-W, W)$ or $\xi - x \notin (-W, W)$. Putting this together, the right hand side vanishes for $\xi \notin (-2W, 2W)$, hence

$$\operatorname{supp} \mathcal{F}\left[f_1(\cdot)f_2(t-\cdot)\right] \subseteq [-2W, 2W].$$

The Inversion Theorem 3.11 holds, since $\tilde{f}_t \in \mathrm{L}^1(\mathbb{R})$ and $\mathcal{F}\left[\tilde{f}_t\right] \in \mathrm{L}^1$. Therefore, $\tilde{f}_t$ is $2W$-band limited and we can apply Lemma 4.12. We obtain

$$(f_1 * f_2)(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_1(x) f_2(t-x) \mathrm{d}x = \frac{\pi}{\sqrt{2\pi} W} \sum_{k=-\infty}^{\infty} f_1\left(\frac{k\pi}{W}\right) f_2\left(t - \frac{k\pi}{W}\right).$$

Regarding the absolute convergence, we consider inequality (4.1) with $h = \pi/W$ for $p = 1$. This yields

$$\begin{aligned}
&\frac{\pi}{\sqrt{2\pi} W} \sum_{k=-\infty}^{\infty} \left| f_1\left(\frac{k\pi}{W}\right) f_2\left(t - \frac{k\pi}{W}\right) \right| \\
\leqslant\ & \sup_{u \in \mathbb{R}} \frac{\pi}{\sqrt{2\pi} W} \sum_{k=-\infty}^{\infty} \left| f_1\left(-\left(u - \frac{k\pi}{W}\right)\right) f_2\left(t + \left(u - \frac{k\pi}{W}\right)\right) \right| \\
\overset{(4.1)}{\leqslant}\ & (1+\pi) \left\| f_1(-\cdot) f_2(t+\cdot) \right\|_1 \\
\overset{\text{Hölder}}{\leqslant}\ & (1+\pi) \left\| f_1 \right\|_p \left\| f_2 \right\|_q .
\end{aligned}$$

Now it remains to show the uniform convergence of the series. To this end, we estimate

$$\begin{aligned}
&\frac{\pi}{\sqrt{2\pi} W} \sum_{|k| \geqslant N} \left| f_1\left(\frac{k\pi}{W}\right) f_2\left(t - \frac{k\pi}{W}\right) \right| \\
=\ & \frac{\pi}{\sqrt{2\pi} W} \sum_{k=-\infty}^{\infty} \left| f_1\left(\frac{k\pi}{W}\right) \chi_{\{|x| \geqslant N\pi/W\}}\left(\frac{k\pi}{W}\right) f_2\left(t - \frac{k\pi}{W}\right) \right| \\
\overset{(4.1)}{\leqslant}\ & (1+\pi) \left\| x \mapsto f_1(-x) \chi_{\{|x| \geqslant N\pi/W\}}(-x) f_2(t+x) \right\|_1 \\
\overset{\text{Hölder}}{\leqslant}\ & (1+\pi) \left\| f_1 \chi_{\{|x| \geqslant N\pi/W\}} \right\|_p \left\| f_2(t+\cdot) \right\|_q
\end{aligned}$$

as above but using the characteristic function to cut off the beginning of the series. Since the L^p-norm is invariant under translation, there holds $\left\| f_2(t+\cdot) \right\|_q = \left\| f_2 \right\|_q$. Since $\left\| f_1 \chi_{|x| \geqslant N\pi/W\}} \right\|_p \to 0$ for $N \to \infty$, we have

$$\sup_{t \in \mathbb{R}} \frac{\pi}{\sqrt{2\pi} W} \sum_{|k| \geqslant N} \left| f_1\left(\frac{k\pi}{W}\right) f_2\left(t - \frac{k\pi}{W}\right) \right| \leqslant (1+\pi) \left\| f_2 \right\|_q \left\| f_1 \chi_{\{|x| \geqslant N\pi/W\}} \right\|_p \to 0 \quad (N \to \infty),$$

hence uniform convergence with respect to $t \in \mathbb{R}$. □

4.14 Corollary. *In the situation of Lemma 4.13, we have*

$$\sum_{k=-\infty}^{\infty} f_1\left(\frac{k\pi}{W}\right) f_2\left(t - \frac{k\pi}{W}\right) = \sum_{k=-\infty}^{\infty} f_1\left(t - \frac{k\pi}{W}\right) f_2\left(\frac{k\pi}{W}\right) \qquad (t \in \mathbb{R})$$

Proof. As the convolution is commutative, we can change substitute f_1 with f_2 and vice versa without changing the left hand side. Therefore the right hand sides must coincide. □

Now we can proof the Sampling Theorem 4.11 for $\mathrm{L}^2(\mathbb{R})$-functions.

Proof. Since $f \in \mathrm{L}^2(\mathbb{R})$ is W-band limited and $\mathrm{sinc}_W \in \mathrm{L}^2(\mathbb{R})$ is also W-band limited (according to Lemma 4.10), Corollary 4.14 can be used to obtain

$$\sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{W}\right) \mathrm{sinc}_W\left(t - \frac{k\pi}{W}\right) = \sum_{k=-\infty}^{\infty} f\left(t - \frac{k\pi}{W}\right) \mathrm{sinc}_W\left(\frac{k\pi}{W}\right) = f(t).$$

Recall that $\mathrm{sinc}_W(k\pi/W) = \delta_{k0}$. Lemma 4.13 yields the absolute and uniform convergence of the series. $\square$

4.3 Proof based on the generalized Parseval formula

In this section, we want to make use of Parseval's identity (2.1) on $\mathrm{L}^2(\lambda\mathbb{T}^n)$ ($\lambda > 0$) to obtain the sampling theorem. Therefore, we first show that the set $\{x \mapsto \mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}kx} : k \in \mathbb{Z}^n\}$ forms a complete orthogonal set in $\mathrm{L}^2(\lambda\mathbb{T}^n)$. Then we expand any $f \in \mathrm{L}^2(\lambda\mathbb{T}^n)$ with respect to this basis and find the sampling theorem. This idea can be found in [Mar01, p. 55] and [BSS88, p. 11]. The Hilbert space $\mathrm{L}^2(\lambda\mathbb{T}^n)$ is defined in Example 2.20(iii), where we identify $\lambda\mathbb{T}^n := [-\lambda/2, \lambda/2]^n$.

4.15 Lemma. *Let $\lambda > 0$. Then the set $\{x \mapsto \mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}kx} : k \in \mathbb{Z}^n\}$ forms a complete orthonormal set in $\mathrm{L}^2(\lambda\mathbb{T}^n)$.*

Proof. For $k \in \mathbb{Z}^n$ define $\varphi_k : \lambda\mathbb{T}^n \to \mathbb{C}, x \mapsto \mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}kx}$. Then $\varphi_k \in \mathrm{L}^2(\lambda\mathbb{T}^n)$, since $|\varphi_k(x)| = 1$ for every $x \in \lambda\mathbb{T}^n$. Next,

$$\|\varphi_k\|^2_{\mathrm{L}^2(\lambda\mathbb{T}^n)} = \frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n} |\varphi_k(x)|^2 \mathrm{d}x = \frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n} 1\mathrm{d}x = 1,$$

hence φ_k is normalized. Furthermore, for $k, l \in \mathbb{Z}^n$ with $k \neq l$ we have

$$\begin{aligned}\langle \varphi_k, \varphi_l\rangle_{\mathrm{L}^2(\lambda\mathbb{T}^n)} &= \frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n} \mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}(k-l)x}\mathrm{d}x = \prod_{i=1}^{n}\frac{1}{\lambda}\int_{[-\lambda/2,\lambda/2]} \mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}(k_i-l_i)x_i}\mathrm{d}x_i \\ &= \prod_{\substack{i\in\{1,\dots,n\}\\ l_i\neq k_i}} \frac{1}{\mathrm{i}\,2\pi(k_i-l_i)}\left[\mathrm{e}^{\mathrm{i}\,\pi(k_i-l_i)} - \mathrm{e}^{-\mathrm{i}\,\pi(k_i-l_i)}\right] = 0,\end{aligned}$$

where the separation of the integrals is justified by Fubini's theorem. In the case $k_i = l_i$, there holds $\mathrm{e}^{\mathrm{i}\frac{2\pi}{\lambda}(l_i-k_i)x_i} = 1$ ($x \in \lambda\mathbb{T}^n$), hence the integral corresponding to index i is 1. Since we assumed $k \neq l$, there is at least one index $i \in \{1, \dots, n\}$ with $l_i \neq k_i$, hence the whole integral vanishes. Now we have shown that $\{\varphi_k : k \in \mathbb{Z}^n\}$ is an orthonormal set. It remains to show that it is maximal.

Let $f \in \mathrm{L}^2(\lambda\mathbb{T}^n)$. Then

$$\langle f, \varphi_k\rangle = \frac{1}{\lambda^n}\int_{\lambda\mathbb{T}^n} f(x)\mathrm{e}^{-\mathrm{i}\frac{2\pi}{\lambda}kx}\mathrm{d}x = \widehat{f}\,(k)\,,$$

where $\widehat{f}\,(k)$ denotes the kth Fourier coefficient introduced in Equation (3.4) in Definition 3.36. Note that $\mathrm{L}^2(\lambda\mathbb{T}^n) \subseteq \mathrm{L}^1(\lambda\mathbb{T}^n)$, which implies the existence of the integral. Using the uniqueness of the Fourier coefficients as stated in Theorem 3.34, it follows $f = 0$ a.e. if $\langle f, \varphi_k \rangle = 0$ for every $k \in \mathbb{Z}^n$. Hence, the only vector being orthogonal to $\{\varphi_k : k \in \mathbb{Z}^n\}$ is the null vector and therefore the set is maximal. □

Let us state the sampling theorem again.

4.16 Theorem. *Let $W > 0$, $\hat{f} \in \mathrm{L}^2([-W, W]^n)$ and $f \in C^\infty(\mathbb{R}^n)$ with $f = \mathcal{F}^{-1}\left[\hat{f}\right]$. Then*

$$f(x) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right) \operatorname{sinc}_W\left(x + \frac{k\pi}{W}\right) \quad (x \in \mathbb{R}^n),$$

the series converging absolutely and uniformly.

Proof. Let $\varphi_k = \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{2W}k\cdot} \in \mathrm{L}^2(2W\mathbb{T}^n)$ be defined as above for $k \in \mathbb{Z}^n$. Consider $\hat{f}_p \in \mathrm{L}^2(2W\mathbb{T}^n)$ defined as the $2W$-periodic continuation of $\hat{f}$. Then

$$\begin{aligned}\left\langle \hat{f}_p, \varphi_k \right\rangle &= \frac{1}{(2W)^n}\int_{2W\mathbb{T}^n} \hat{f}_p(t)\mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{2W}kt}\mathrm{d}t = \frac{(2\pi)^{n/2}}{(2W)^n}\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \hat{f}(t)\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi k}{2W}\right)t}\mathrm{d}t \\ &= \left(\frac{\sqrt{2\pi}}{2W}\right)^n \mathcal{F}^{-1}\left[\hat{f}\right]\left(-\frac{\pi k}{W}\right) = \left(\frac{\sqrt{2\pi}}{2W}\right)^n f\left(-\frac{\pi k}{W}\right).\end{aligned}$$

Similarly, we obtain for $x \in \mathbb{R}^n$ and $\psi_x : 2W\mathbb{T}^n \to \mathbb{C}, t \mapsto \mathrm{e}^{-\mathrm{i}\,xt} \in \mathrm{L}^2(2W\mathbb{T}^n)$:

$$\begin{aligned}\langle \psi_x, \varphi_k \rangle &= \frac{1}{(2W)^n}\int_{2W\mathbb{T}^n} \mathrm{e}^{-\mathrm{i}\,xt}\mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{2W}kt}\mathrm{d}t \\ &= \left(\frac{\sqrt{2\pi}}{2W}\right)^n \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\,xt}\chi_{[-W,W]^n}(t)\mathrm{e}^{-\mathrm{i}\,\frac{2\pi}{2W}kt}\mathrm{d}t \\ &= \left(\frac{\sqrt{2\pi}}{2W}\right)^n \mathcal{F}\left[e^{-\mathrm{i}\,x\cdot}\chi_{[-W,W]^n}\right]\left(\frac{2\pi}{2W}k\right) \\ &= \operatorname{sinc}_W\left(\frac{\pi}{W}k + x\right),\end{aligned}$$

noting that $\chi_{[-W,W]^n} \in \mathrm{L}^1(\mathbb{R}^n)$ whose Fourier transform is given in Example 3.8 and using the translation property in Theorem 3.9(b) of the Fourier transform. In the previous lemma, we showed that $\{\varphi_k : k \in \mathbb{Z}^n\} \subset \mathrm{L}^2(2W\mathbb{T}^n)$ is a complete orthogonal set, hence we can apply Parseval's identity (Theorem 2.24) to $\hat{f}_p$ and ψ_x:

$$\begin{aligned}\left\langle \hat{f}_p, \psi_x \right\rangle &= \sum_{k\in\mathbb{Z}^n} \left\langle \hat{f}_p, \varphi_k \right\rangle \overline{\langle \psi_x, \varphi_k \rangle} \qquad (4.15)\\ &= \sum_{k\in\mathbb{Z}^n} \left(\frac{\sqrt{2\pi}}{2W}\right)^n f\left(-\frac{\pi k}{W}\right) \operatorname{sinc}_W\left(\frac{\pi}{W}k + x\right).\end{aligned}$$

The left-hand side can be expanded as

$$\begin{aligned}\left\langle \hat{f}_p, \psi_x \right\rangle &= \frac{1}{(2W)^n}\int_{2W\mathbb{T}^n} \hat{f}_p(t)\overline{\psi_x(t)}\mathrm{d}t = \left(\frac{\sqrt{2\pi}}{2W}\right)^n \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \hat{f}(t)\mathrm{e}^{\mathrm{i}\,xt}\mathrm{d}t \\ &= \left(\frac{\sqrt{2\pi}}{2W}\right)^n \mathcal{F}^{-1}\left[\hat{f}\right](x) = \left(\frac{\sqrt{2\pi}}{2W}\right)^n f(x),\end{aligned}$$

since $\operatorname{supp}(\hat{f}) \subset [-W, W]^n$. In total we get after cancelling and the index shift $k \to -k$

$$f(x) = \sum_{k \in \mathbb{Z}^n} f\left(\frac{k\pi}{W}\right) \operatorname{sinc}_W\left(x - \frac{k\pi}{W}\right).$$

To show the convergence properties, it is sufficient to consider the representation (4.15), since the sampling theorem is obtained from it by equalities. By Theorem 2.24(c), we have

$$\sum_{k \in \mathbb{Z}^n} \left|\left\langle \tilde{f}_p, \varphi_k \right\rangle\right|^2 = \left\|\tilde{f}_p\right\| < \infty, \qquad \text{hence} \qquad \left(\left\langle \tilde{f}_p, \varphi_k \right\rangle\right)_{k \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n).$$

By the same argument, we have $(\langle \psi_x, \varphi_k \rangle)_{k \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$ with

$$\|\psi_x\|^2_{\mathrm{L}^2(2W\mathbb{T}^n)} = \sum_{k \in \mathbb{Z}^n} |\langle \psi_x, \varphi_k \rangle|^2 = \frac{1}{(2W)^n} \int_{2W\mathbb{T}^n} \mathrm{e}^{-\mathrm{i}\, xt} \mathrm{e}^{\mathrm{i}\, xt} \mathrm{d}t = 1.$$

This implies $\left(\left\langle \tilde{f}_p, \varphi_k \right\rangle \overline{\langle \psi_x, \varphi_k \rangle}\right)_{k \in \mathbb{Z}^n} \in \ell^1(\mathbb{Z}^n)$ by Hölder's inequality. Therefore, we have absolute convergence. Using again Hölder's inequality we obtain from (4.15)

$$\begin{aligned}
&\left(\frac{2W}{\sqrt{2\pi}}\right)^n \left| f(x) - \sum_{\substack{k \in \mathbb{Z}^n \\ k < N}} f\left(\frac{k\pi}{W}\right) \operatorname{sinc}_W\left(x - \frac{k\pi}{W}\right) \right| = \left| \left\langle \tilde{f}_p, \psi_x \right\rangle - \sum_{|k| \leqslant N} \left\langle \tilde{f}_p, \varphi_k \right\rangle \overline{\langle \psi_x, \varphi_k \rangle)} \right| \\
= &\left| \sum_{|k| \geqslant N} \left\langle \tilde{f}_p, \varphi_k \right\rangle \overline{\langle \psi_x, \varphi_k \rangle)} \right| \leqslant \sum_{|k| \geqslant N} \left| \left\langle \tilde{f}_p, \varphi_k \right\rangle \overline{\langle \psi_x, \varphi_k \rangle)} \right| \\
\overset{\text{Hölder}}{\leqslant} &\left(\sum_{|k| \geqslant N} \left|\left\langle \tilde{f}_p, \varphi_k \right\rangle\right|^2 \right)^{1/2} \left(\sum_{|k| \geqslant N} |\langle \psi_x, \varphi_k \rangle|^2 \right)^{1/2} \leqslant \left(\sum_{|k| \geqslant N} \left|\left\langle \tilde{f}_p, \varphi_k \right\rangle\right|^2 \right)^{1/2} \|\psi_x\|_{\mathrm{L}^2(2W\mathbb{T}^n)} \\
= &\left(\sum_{|k| \geqslant N} \left|\left\langle \tilde{f}_p, \varphi_k \right\rangle\right|^2 \right)^{1/2}
\end{aligned}$$

for $N \in \mathbb{N}$ and every $x \in \mathbb{R}^n$. Since $(\left\langle \tilde{f}_p, \varphi_k \right\rangle)_{k \in \mathbb{Z}^n} \in \ell^2(\mathbb{Z}^n)$, the right hand side converges to zero not depending on x, hence uniformly. □

4.4 Proof based on a Schauder basis

This proof is similar to the previous one, but valid for functions whose Fourier transform is in $\mathrm{L}^p([-W, W]^n)$ for $1 < p < 2 \leqslant \infty$. The case $p \geqslant 2$ is covered by the previous proof, since $\mathrm{L}^p([-W, W]^n) \subset \mathrm{L}^2([-W, W]^n)$ for $p \geqslant 2$. The idea is provided in [GMHM98, Thm 3.3, p. 49].

4.17 Definition. Let $(X, \|\cdot\|_X)$ be a Banach space over $\mathbb{K}$. Then a sequence $(x^{(i)})_{i \in \mathbb{N}_0} \subset X$ is said to be a *Schauder basis* for X if for each $x \in X$ there is a sequence $(c_i)_{i \in \mathbb{N}} \subset \mathbb{K}$ such that

$$\left\| x - \sum_{i=1}^{n} c_i x_i \right\|_X \to 0, \quad n \to \infty.$$

We then write $x = \sum_{i=1}^{n} c_i x_i$ (cf. [You80, p. 1]).

4.18 Lemma. *The system* $(\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}k\cdot})_{k\in\mathbb{Z}^n}$ *forms a Schauder basis in* $\mathrm{L}^p(\lambda\mathbb{T}^n)$ $(1 < p < \infty)$ *if it is ordered properly.*

4.19 Remark. When $n = 2$, a possible ordering are 'rectangular' partial sums where additional terms are added to one of the sides of the rectangle filling out that side before moving on (stated by Morten Nielsen, Aalborg University, Denmark in individual communication).

4.20 Remark. One should note that the convergence above is conditional, i.e. a permutation of the basis elements may not converge as discussed after Theorem 3.41. It is still an open question which orderings yield a Schauder base. However, we have that the exponentials are complete in $\mathrm{L}^p(\lambda\mathbb{T}^n)$, i.e., every $f \in \mathrm{L}^p(\lambda\mathbb{T}^n)$ can be approximated with arbitrary precision by a linear combination of the exponentials (a trigonometric polynomial, cf. [Gra08, 3.1.10, p. 168]). In Hilbert spaces both concepts agree, but in Banach spaces they may differ as can be seen here.

4.21 Theorem. *Let* $1 < p < 2$*,* $\hat{f} \in \mathrm{L}^p([-W, W]^n)$ *and* $f = \mathcal{F}^{-1}\left[\hat{f}\right]$ *with* $\sum_{k\in\mathbb{Z}^n} |f(k\pi/W)| < \infty$*. Then*

$$f(x) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right) \mathrm{sinc}_W \left(x - \frac{k\pi}{W}\right)$$

for $x \in \mathbb{R}^n$*, where the series converges absolutely and uniformly.*

Proof. Since $(\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{2W}k\cdot})_{k\in\mathbb{Z}^n}$ forms a Schauder basis in $\mathrm{L}^p(2W\mathbb{T}^n)$ by Lemma 4.18, there is a sequence $(c_k)_{k\in\mathbb{Z}^n} \subset \mathbb{C}$ and an ordering $o : \mathbb{Z}^n \to \mathbb{N}$ such that

$$\left\| \hat{f} - \sum_{\substack{k\in\mathbb{Z}^n \\ o(k)<N}} c_k \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{2W}k\cdot} \right\|_p \to 0, \quad N \to \infty,$$

when we identify $\mathrm{L}^p([-W, W]^n)$ with $\mathrm{L}^p(2W\mathbb{T}^n)$. Now consider the regular distribution $[\hat{f}] \in \mathcal{D}'(\lambda\mathbb{T}^n)$. Then for $\varphi \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}\left| [\hat{f}](\varphi) - \sum_{o(k)<N} c_k \left[\mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}k\cdot}\right](\varphi) \right| &= \left| \int_{2W\mathbb{T}^n} \left(\hat{f}(x) - \sum_{o(k)<N} c_k \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx} \right) \varphi(x)\mathrm{d}x \right| \\ &\leqslant \left\| \hat{f} - \sum_{o(k)<N} c_k \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}k\cdot} \right\|_p \left\| \varphi\chi_{[-W,W]^n} \right\|_q\end{aligned}$$

by linearity of the integral and Hölder's inequality with $1/p + 1/q = 1$. As the series converges in the p-norm to $\hat{f}$ and $\left\|\varphi\chi_{[-W,W]^n}\right\| < \infty$ due to the compact support, we have convergence in the space of distributions with compact support.

To obtain the coefficients c_k for $k \in \mathbb{Z}^n$, we calculate as follows

$$\begin{aligned} f\left(\frac{k\pi}{W}\right) &= \mathcal{F}^{-1}\left[\hat{f}\right](k) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx}\hat{f}(x)\mathrm{d}x \\ &= (2\pi)^{-n/2}\int_{\lambda\mathbb{T}^n} \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx}\sum_{m\in\mathbb{Z}^n} c_m \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}mx}\mathrm{d}x \\ &= (2\pi)^{-n/2}\sum_{m\in\mathbb{Z}^n} c_m \int_{\lambda\mathbb{T}^n} \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx}\mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}mx}\mathrm{d}x \\ &\overset{(4.3)}{=} (2\pi)^{-n/2}(2W)^{-n}\sum_{m\in\mathbb{Z}^n} c_m \operatorname{sinc}_W\left(-\frac{\pi}{W}k - \frac{\pi}{W}m\right) \\ &= (2\pi)^{-n/2}(2W)^{-n}\sum_{m\in\mathbb{Z}^n} c_m\delta_{km} = (2\pi)^{-n/2}(2W)^{-n}c_k \end{aligned}$$

For the interchange of sum and integral in the third equality, we need $\sum_{k\in\mathbb{Z}^n}|c_k| < \infty$ using 2.10, which we required in the assumptions. Furthermore, we used the interpolation property 4.10(ii) of the sinc_W function.

Now we can apply the inverse Fourier transform to $\hat{f}$, regarded as an element of $\mathrm{L}^p([-W,W]^n)$. By $\sum_{k\in\mathbb{Z}^n}|c_k| < \infty$, we can again interchange summation and integration by Theorem 2.10 and obtain

$$\begin{aligned} f(z) &= (2\pi)^{-n/2}\int_{\mathbb{R}^n}\hat{f}(x)\mathrm{e}^{\mathrm{i}\,xz}\mathrm{d}x = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\frac{(2W)^n}{(2\pi)^{n/2}}(2\pi)^{-n/2}\int_{\mathbb{R}^n}\chi_{[-W,W]^n}(x)\mathrm{e}^{\mathrm{i}\,xz}\mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx}\mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\operatorname{sinc}_W\left(z - \frac{k\pi}{W}\right), \end{aligned}$$

where we identified $\chi_{[-W,W]^n}$ (multiplied with an appropriate constant factor) as the Fourier transform of sinc_W (see (4.10)).

The Hausdorff-Young inequality 3.14 (for the inverse Fourier transform) yields $\left\|\mathcal{F}^{-1}[g]\right\|_\infty \leqslant C\left\|g\right\|_1$ if $g \in \mathrm{L}^1(\mathbb{R}^n)$. Since $\hat{f} \in \mathrm{L}^p([-W,W]^n) \subset \mathrm{L}^1(\mathbb{R}^n)$, we have uniform convergence by

$$\begin{aligned} \left\| f - \sum_{o(k)<N} f(k\pi/W)\tau_{k\pi/W}\operatorname{sinc}_W \right\|_\infty &\leqslant C\left\| \hat{f} - \sum_{o(k)<N} c_k \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}k\cdot} \right\|_1 \\ &\leqslant C\left\|\chi_{[-W,W]^n}\right\|_q \left\| \hat{f} - \sum_{o(k)<N} c_k \mathrm{e}^{\mathrm{i}\,\frac{2\pi}{\lambda}k\cdot} \right\|_p, \end{aligned}$$

where we again used Hölder's inequality.

Absolute convergence is given by the estimate

$$\sum_{k\in\mathbb{Z}^n}\left| f\left(\frac{k\pi}{W}\right)\operatorname{sinc}_W\left(x - \frac{k\pi}{W}\right)\right| \leqslant \sum_{k\in\mathbb{Z}^n}\left| f\left(\frac{k\pi}{W}\right)\right| < \infty,$$

since $|\operatorname{sinc}_W(x)| \leqslant 1$ for all $x \in \mathbb{R}^n$ and the series converges by assumption. □

4.5 Proof based on an orthogonal system

In this proof, we show that $\{\mathrm{sinc}_W(\cdot - k)\}_{k\in\mathbb{Z}^n}$ is a complete orthogonal system in the Hilbert space $B^2_W(\mathbb{C}^n)$. Hence, the sampling theorem is nothing else than the expansion of $f \in B^2_W(\mathbb{C}^n)$ with respect to this basis. The proof follows [BSS88, p. 11] ([Den89, p. 13, Beweis 3]).

4.22 Lemma. *$B^2_\sigma(\mathbb{C}^n)$ is a Hilbert space by virtue of the inner product*

$$\langle \cdot, \cdot \rangle_\sigma : B^2_\sigma(\mathbb{C}^n) \times B^2_\sigma(\mathbb{C}^n) \to \mathbb{C}, \quad (u, v) \mapsto \int_{\mathbb{R}^n} u(x)\overline{v(x)}\mathrm{d}x.$$

Note that integration is only performed in $\mathbb{R}^n$.

Proof. First, we show that $\langle \cdot, \cdot \rangle_\sigma$ is indeed an inner product. Let $u, v \in B^2_\sigma(\mathbb{C}^n)$. By Hölder's inequality the inner product is well-defined: $\langle u, v\rangle_\sigma \leqslant \|u\|_{\mathrm{L}^2(\mathbb{R}^n)} \|v\|_{\mathrm{L}^2(\mathbb{R}^n)} < \infty$.

(a) $\langle \cdot, u\rangle_\sigma$ is linear for every $u \in B^2_\sigma(\mathbb{C}^n)$ because of the linearity of the integral.

(b) $\langle u, v\rangle_\sigma = \overline{\langle v, u\rangle_\sigma}$ holds by definition.

(c) For $u \in B^2_\sigma(\mathbb{C}^n)$, we have $\langle u, u\rangle_\sigma = \langle u, u\rangle_{\mathrm{L}^2(\mathbb{R}^n)} > 0$ if $u \neq 0$ and from $\langle u, u\rangle_\sigma = \langle u, u\rangle_{\mathrm{L}^2(\mathbb{R}^n)} = 0$ it follows $u(x) = 0$ for almost every $x \in \mathbb{R}^n$, since $\langle \cdot, \cdot\rangle_{\mathrm{L}^2(\mathbb{R}^n)}$ is a scalar product on $\mathrm{L}^2(\mathbb{R}^n)$. As u is entire, in particular continuous, we have $u(x) = 0$ for all $x \in \mathbb{R}^n$. By Lemma 2.41, we infer $u(z) = 0$ for all $z \in \mathbb{C}^n$.

Hence $\langle \cdot, \cdot\rangle_\sigma$ is a scalar product on $B^2_\sigma(\mathbb{C}^n)$.

Now we have to show that $B^2_\sigma(\mathbb{C}^n)$ is complete with respect to the norm $\|u\|_\sigma := \sqrt{\langle u, u\rangle_\sigma} = \sqrt{\langle u, u\rangle_{\mathrm{L}^2}} = \|u\|_{\mathrm{L}^2}$, where we write L^2 as abbreviation for $\mathrm{L}^2(\mathbb{R}^n)$.

Let $(u_n)_{n\in\mathbb{N}} \subset B^2_\sigma(\mathbb{C}^n)$ be a Cauchy-Sequence, i.e.

$$\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} : \forall n, m \in \mathbb{N} n, m \geqslant n_0 : \|f_n - f_m\|_\sigma < \varepsilon.$$

We need to show that there is $f \in B^2_\sigma$ with $\|f - f_n\|_\sigma \to 0$, $n \to \infty$. Using Plancherel's theorem 3.13, we have

$$\|f_n - f_m\|_\sigma = \|f_n - f_m\|_{\mathrm{L}^2} = \|\mathcal{F}[f_n - f_m]\|_{\mathrm{L}^2} = \|\mathcal{F}[f_n] - \mathcal{F}[f_m]\|_{\mathrm{L}^2},$$

hence $(\mathcal{F}[f_n])_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^2(\mathbb{R}^n)$. Moreover, by the Lemma 4.8 each $\mathcal{F}[f_n]$ has its support in $\sigma B \subset [-\sigma, \sigma]^n$. Hence, we can also consider the sequence to be in $\mathrm{L}^2(\sigma\mathbb{T}^n)$. Since this is a Hilbert space, it is complete and therefore there is $\tilde{f} \in \mathrm{L}^2(\sigma\mathbb{T}^n)$ with $\left\|\tilde{f} - \mathcal{F}[f_n]\right\|_{\mathrm{L}^2(\sigma\mathbb{T}^n)} \to 0$ for $n \to \infty$. □

4.23 Lemma. *Let $\sigma > 0$. The set $\{s_k\}_{k\in\mathbb{Z}^n}$ defined by $s_k := (\sigma/\pi)^{n/2}\tau_{k\pi/\sigma}\,\mathrm{sinc}_\sigma$ is a complete orthogonal set in $B^2_\sigma(\mathbb{C}^n)$.*

Proof. We can reduce this problem to Lemma 4.15. Let $f, g \in B^2_\sigma(\mathbb{C}^n)$. Then, using Parseval's identity 2.24, we have

$$\langle f, g\rangle_\sigma = \langle f, g\rangle_{\mathrm{L}^2(\mathbb{R}^n)} = \langle \mathcal{F}[f], \mathcal{F}[g]\rangle_{\mathrm{L}^2(\mathbb{R}^n)} = (2\sigma)^n \langle \mathcal{F}[f], \mathcal{F}[g]\rangle_{\mathrm{L}^2(2\sigma\mathbb{T}^n)}$$

because $\mathcal{F}[f]$ is a function with $\operatorname{supp}\mathcal{F}[f] \subset [-\sigma,\sigma]^n$ due to Lemma 4.8. The same applies to $\mathcal{F}[g]$. Hence f, g are orthogonal in $B^2_\sigma(\mathbb{C}^n)$ if and only if $\mathcal{F}[f]$ and $\mathcal{F}[g]$ are orthogonal in $\mathrm{L}^2(2\sigma\mathbb{T}^n)$. Since

$$\mathcal{F}[s_k] = \mathcal{F}\left[\left(\frac{\sigma}{\pi}\right)^{n/2} \tau_{k\pi/\sigma}\operatorname{sinc}_\sigma\right] = \left(\frac{1}{2\sigma}\right)^{n/2} \mathrm{e}^{-\mathrm{i}\,\frac{2k\pi}{2\sigma}\cdot}\chi_{[-\sigma,\sigma]},$$

we have $\|s_k\|_\sigma = 1$ and $\langle s_k, s_l\rangle_\sigma = 0$ for $k, l \in \mathbb{Z}^n$, $k \neq l$, because $\{x \mapsto \mathrm{e}^{-\mathrm{i}\,\frac{2k\pi}{2\sigma}x} : k \in \mathbb{Z}^n\}$ is orthonormal in $\mathrm{L}^2(2\sigma\mathbb{T}^n)$ by Lemma 4.15. Hence, the set $\{s_k\}_{k\in\mathbb{Z}^n}$ is orthonormal in $B^2_\sigma(\mathbb{C}^n)$.

To show completeness, let $f \in B^2_\sigma(\mathbb{C}^n)$ and suppose $\langle f, s_k\rangle = 0$ for every $k \in \mathbb{Z}^n$. Then

$$\langle f, s_k\rangle_\sigma = (2\sigma)^{n/2}\left\langle \mathcal{F}[f], \mathrm{e}^{-\mathrm{i}\,\frac{2\pi k}{2\sigma}\cdot}\right\rangle_{\mathrm{L}^2(2\sigma\mathbb{T}^n)}$$

as above. As the set $\{x \mapsto \mathrm{e}^{-\mathrm{i}\,\frac{2k\pi}{2\sigma}\cdot} : k \in \mathbb{Z}^n\}$ is a complete orthogonal set in $\mathrm{L}^2(2\sigma\mathbb{T}^n)$ by Lemma 4.15, we conclude $\mathcal{F}[f] = 0$ in $\mathrm{L}^2(2\sigma\mathbb{T}^n)$ and also in $\mathrm{L}^2(\mathbb{R}^n)$. Since the Fourier transform is an isometry on $\mathrm{L}^2(\mathbb{R}^n)$ by Plancherel's Theorem 3.13, this implies $f = 0$ in $\mathrm{L}^2(\mathbb{R}^n)$ and therefore in $B^2_\sigma(\mathbb{C}^n)$.

□

4.24 Theorem. *Let $W > 0$ and $f \in B^2_W(\mathbb{C}^n)$. Then*

$$f(x) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\operatorname{sinc}_W\left(x - \frac{k\pi}{W}\right) \qquad (x \in \mathbb{R}^n),$$

the series converging absolutely and uniformly.

Proof. By Lemma 4.23, the set $\{s_k : k \in \mathbb{Z}^n\}$ is a complete orthogonal set in $B^2_W(\mathbb{C}^n)$. Hence, we can expand f in terms of this basis. For $k \in \mathbb{Z}^n$, we deduce

$$\begin{aligned}\langle f, s_k\rangle_W &= (2W)^{n/2}\left\langle \mathcal{F}[f], \mathrm{e}^{-\mathrm{i}\,\frac{k\pi}{W}\cdot}\right\rangle_{\mathrm{L}^2(2W\mathbb{T}^n)}\\ &= \left(\frac{\pi}{W}\right)^{n/2}\frac{1}{(2\pi)^{n/2}}\int_{[-W,W]^n}\mathcal{F}[f](x)\mathrm{e}^{\mathrm{i}\,\frac{k\pi}{W}x}\mathrm{d}x\\ &= \left(\frac{\pi}{W}\right)^{n/2}\mathcal{F}^{-1}[\mathcal{F}[f]]\left(\frac{k\pi}{W}\right) = \left(\frac{\pi}{W}\right)^{n/2} f\left(\frac{k\pi}{W}\right).\end{aligned}$$

Hence

$$f(x) = \sum_{k\in\mathbb{Z}^n}\langle f, s_k\rangle\, s_k(x) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\operatorname{sinc}_W\left(x - \frac{k\pi}{W}\right)$$

for almost every $x \in \mathbb{R}^n$. An application of Hölder's inequality yields for $x \in \mathbb{R}^n$ and $N \in \mathbb{N}$

$$
\begin{aligned}
\left| f(x) - \sum_{|k| \leqslant N} f\left(\frac{k\pi}{W}\right) \operatorname{sinc}_W \left(x - \frac{k\pi}{W}\right)\right| &= \left| \sum_{|k|>N} f\left(\frac{k\pi}{W}\right) \operatorname{sinc}_W \left(x - \frac{k\pi}{W}\right)\right| \\
&\leqslant \left(\sum_{|k|>N} f\left(\frac{k\pi}{W}\right)\right)^{1/2} \left\| \tau_{k\pi/W} \operatorname{sinc}_W \right\|_{\mathrm{L}^2(\mathbb{R}^n)} \\
&= \left(\sum_{|k|>N} f\left(\frac{k\pi}{W}\right)\right)^{1/2} \left(\frac{\sigma}{\pi}\right)^{n/2}.
\end{aligned}
$$

By Theorem 2.24(c), there holds $\|f\|^2_{\mathrm{L}^2(\mathbb{R}^n)} = \sum_{k\in\mathbb{Z}^n} |\langle f, s_k\rangle|^2 = \sum_{k\in\mathbb{Z}^n} |f(k\pi/W)|^2$. Therefore, the right hand side converges to zero independently of $x \in \mathbb{R}^n$, hence uniformly. This uniform convergence also implies convergence for all $x \in \mathbb{R}^n$. Absolute convergence can be obtained again by Hölder's inequality noting that $f, s_k \in \mathrm{L}^2(\mathbb{R}^n)$. □

4.6 Proof based on contour integration

This prove does not rely on the Fourier transform but on the exponential growth condition. This enables us to proof the sampling theorem also functions in L^p for $2 < p < \infty$. We follow the idea of [Wun71, p. 114].

4.25 Lemma. *Let $r := (k + 1/2)\pi$ for $k \in \mathbb{N}_0$. Then the inequality*

$$\left|\sin\left(r\mathrm{e}^{\mathrm{i}\varphi}\right)\right| \geqslant \frac{1}{3}\mathrm{e}^{r|\sin\varphi|}$$

holds for every $\varphi \in [0, 2\pi)$.

Proof. For $\varphi \in [0, 2\pi)$ and $r > 0$, we have

$$
\begin{aligned}
\left|\sin\left(r\mathrm{e}^{\mathrm{i}\varphi}\right)\right|^2 &= \sin\left(r\mathrm{e}^{\mathrm{i}\varphi}\right)\overline{\sin\left(r\mathrm{e}^{\mathrm{i}\varphi}\right)} \\
&= \frac{1}{2\,\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\, r\cos\varphi - r\sin\varphi} - \mathrm{e}^{-\mathrm{i}\, r\cos\varphi + r\sin\varphi}\right)\frac{1}{-2\,\mathrm{i}}\left(\mathrm{e}^{-\mathrm{i}\, r\cos\varphi - r\sin\varphi} - \mathrm{e}^{\mathrm{i}\, r\cos\varphi + r\sin\varphi}\right) \\
&= \frac{1}{4}\left(\mathrm{e}^{-2r\sin\varphi} + \mathrm{e}^{2r\sin\varphi} - \left(\mathrm{e}^{\mathrm{i}\,2r\cos\varphi} + \mathrm{e}^{-\mathrm{i}\,2r\cos\varphi}\right)\right) \\
&= \frac{1}{2}\left(\cosh(2r\sin\varphi) - \cos(2r\cos\varphi)\right).
\end{aligned}
$$

The function $f_r : [0, 2\pi) \to \mathbb{R}, \varphi \mapsto \left|\sin\left(r\exp(\mathrm{i}\,\varphi)\right)\right|^2$ is symmetric, since cos and cosh are symmetric and it is π-periodic, since

$$
\begin{aligned}
f_r(\varphi + \pi) &= \frac{1}{2}\left(\cosh(2r\sin(\varphi + \pi)) - \cos(2r\cos(\varphi + \pi))\right) \\
&= \frac{1}{2}\left(\cosh(-2r\sin\varphi) - \cos(-2r\cos\varphi)\right) = f_r(\varphi),
\end{aligned}
$$

where we again used the symmetry of cosh and cos. Symmetry and π-periodicity also hold true for $\exp(r|\sin(\varphi)|)/3$, because $|\sin(\varphi)|$ is symmetric and π-periodic. Therefore, it is sufficient to show the inequality for $\varphi \in [0, \pi/2]$. On $[0, \pi/2]$ both sin and cos are one-to-one, hence one can apply the variable substitution $\sin\varphi \mapsto x$ and $\cos\varphi \mapsto \sqrt{1-x^2}$. Now define

$$g\colon [0,1] \to \mathbb{R}, \quad x \mapsto \frac{1}{2}\left(\cosh(2rx) - \cos(2r\sqrt{1-x^2})\right) - \frac{e^{2rx}}{9}.$$

After a reordering of the terms, we get

$$g(x) = e^{2rx}\left(\frac{1}{4} - \frac{1}{9}\right) + \frac{1}{4}e^{-2rx} - \frac{\cos(2r\sqrt{1-x^2})}{2}.$$

Now we will check, where the exponential terms are smaller than $1/2$. There is equality

$$\left(\frac{1}{4} - \frac{1}{9}\right)e^{2rx} + \frac{1}{4}e^{-2rx} = \frac{1}{2} \text{ iff } x \in \left\{\frac{1}{2r}\ln\left(\frac{9-2}{9-4}\right), \frac{1}{2r}\ln\left(\frac{9+2}{9-4}\right)\right\},$$

as one obtains by solving the quadratic equation in e^{2rx}. Since for $x = 1$

$$e^{2r}\left(\frac{1}{4} - \frac{1}{9}\right) + e^{-2r}/4 \geqslant e^{\pi}\left(\frac{5}{36}\right) \geqslant \frac{40}{36} \geqslant 1,$$

we know $g(x) \geqslant 0$ for $x \geqslant \ln(11/5)/2r$ due to the continuity of g and the boundedness of cos.

If we can show that $h(x) := \cos(2r\sqrt{1-x^2}) \leqslant 0$ for $x \leqslant (11/5)/2r$ we are done. Therefore, we look for the zeros of h. For $x \in [0,1]$, the argument $2r\sqrt{1-x^2}$ is in the interval $[0, 2r] = [0, (2k+1)\pi]$, considering $r = (k+1/2)\pi$ for $k \in \mathbb{Z}$ as required. Therefore, there are exactly $2k+1$ zeros to be found. The ansatz $2r\sqrt{1-x_l^2} = (l+1/2)\pi$ for $l \in \mathbb{N}_0$ yields

$$x_l = \frac{\pi}{2r}((2k+1)^2 - (l+1/2)^2)^{1/2}$$

as solutions for the zeros of h, where $0 \leqslant l \leqslant 2k$. For $x = 0$, we have $h(0) = \cos((2k+1)\pi) = \cos(\pi) = -1 < 0$. Since h is continuous this is true for all $x \leqslant \min\{x_l : 0 \leqslant l \leqslant 2k\}$. The least x_l is given for $l = 2k$ with $x_l = (2k+3/4)^{1/2}\pi/2r \geqslant \sqrt{3/4}\pi/2r$.

We have $\ln(11/5) \leqslant 1$, since $11/5 < e$ and on the other hand $\pi\sqrt{3/4} \geqslant 2$, hence

$$\frac{1}{2r}\ln\left(\frac{c+2}{c-4}\right) \leqslant \frac{1}{2r}2 \leqslant \frac{1}{2r}\sqrt{\frac{3}{4}}\pi \leqslant \min\{x_l : 0 \leqslant l \leqslant 2k\}.$$

Therefore, $h(x) \leqslant 0$ for $x \leqslant (11/9)/2r$ and thus $g(x) \geqslant 0$ for all $x \in [0,1]$. Hence,

$$\left|\sin\left(re^{i\varphi}\right)\right|^2 \geqslant \frac{e^{2r|\sin\varphi|}}{9}.$$

Since the square root is monotonically increasing, the inequality is conserved and we obtain the claimed result. □

4.26 Lemma. *Let $u : \mathbb{C} \to \mathbb{C}$ be entire, $z \in \mathbb{C}$ and $\omega > 0$. Define*

$$f : \mathbb{C}\backslash\left\{\frac{k\pi}{\omega} : k \in \mathbb{Z}\right\} \cup \{z\} \to \mathbb{C}, \quad \zeta \mapsto \frac{u(\zeta)}{\sin(\omega\zeta)(\zeta - z)}.$$

Then the residues of f are

$$\operatorname{res}(f; z) = \frac{u(z)}{\sin(\omega z)} \quad \text{and} \quad \operatorname{res}\left(f; \frac{k\pi}{\omega}\right), = \frac{u(k\pi/\omega)}{\omega\cos(k\pi)(k\pi/\omega - z)},$$

provided $z \neq k\pi/\omega$ $(k \in \mathbb{Z})$ and

$$\operatorname{res}\left(f; z = \frac{k\pi}{\omega}\right) = \frac{u'(k\pi/\omega)}{\cos(k\pi)\omega}$$

if $z = k\pi/\omega$ for some $k \in \mathbb{Z}$.

Proof. Let $z \neq k\pi/\omega$ for $k \in \mathbb{Z}$. Then $\sin(\omega z) \neq 0$, hence f has a pole of order 1 at $\zeta = z$. With Lemma 2.43, we get

$$\operatorname{res}(f, z) = \frac{u(z)}{\sin(\omega z)}.$$

At $\zeta = k\pi/\omega$, $\sin(\omega z)$ has a single zero but $(\zeta - z) \neq 0$, hence f has a pole of order 1. With Lemma 2.43, we get

$$\operatorname{res}\left(f, \frac{k\pi}{\omega}\right) = \frac{u(\zeta)}{(\zeta - z)} \frac{(\zeta - k\pi/\omega)}{\sin(\omega\zeta)}\bigg|_{\zeta = k\pi/\omega} = \frac{u(k\pi/\omega)}{(k\pi/\omega - z)} \frac{1}{\omega\cos(k\pi)}$$

using the rule of l'Hôpital.

In the case $z = k\pi/\omega$ for some $k \in \mathbb{Z}$, the denominator of f has a root of order 2. Therefore, we get with Lemma 2.43

$$\begin{aligned}
\operatorname{res}\left(f, z = \frac{k\pi}{\omega}\right) &= \frac{\mathrm{d}}{\mathrm{d}\zeta}(\zeta - z)^2 \frac{u(\zeta)}{\sin(\omega\zeta)(\zeta - z)}\bigg|_{\zeta = z} \\
&= u'(\zeta)\frac{z - \zeta}{\sin(\omega\zeta)} + u(\zeta)\frac{\sin(\omega\zeta) - (\zeta - z)\cos(\omega\zeta)\omega}{\sin^2(\omega\zeta)}\bigg|_{\zeta = z} \\
&= \frac{u'(\zeta)}{\omega\cos(\omega\zeta)} + u(\zeta)\frac{\omega\cos(\omega\zeta) - \omega\cos(\omega\zeta) + \omega^2(\zeta - z)\sin(\omega\zeta)}{2\sin(\omega\zeta)\cos(\omega\zeta)\omega}\bigg|_{\zeta = z} \\
&= \frac{u'(z)}{\cos(\omega z)} = \frac{u'(k\pi/\omega)}{\cos(k\pi)}.
\end{aligned}$$

□

4.27 Lemma. *For $\alpha \in \mathbb{R}^n \backslash \mathbb{Z}^n$, the series*

$$\sum_{k \in \mathbb{Z}^n} \operatorname{sinc}(\alpha\pi - k\pi)$$

does not converge absolutely.

Proof. Suppose the series converges absolutely. Then we can interchange the terms

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n}|\mathrm{sinc}(\alpha\pi-k\pi)| &= \sum_{k\in\mathbb{Z}^n}\prod_{i=1}^{n}\frac{|\sin((\alpha_i-k_i)\pi)|}{|(\alpha_i-k_i)\pi|} = \sum_{k\in\mathbb{Z}^n}\prod_{i=1}^{n}\frac{\left|(-1)^{k_i}\right||\sin(\alpha_i\pi))|}{\pi\,|(\alpha_i-k_i)|}\\ &= \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}}\cdots\sum_{k_n\in\mathbb{Z}}\prod_{i=1}^{n}\frac{|\sin(\alpha_i\pi)}{\pi|(\alpha_i-k_i)|} = \prod_{i=1}^{n}\frac{|\sin(\alpha_i\pi)|}{\pi}\left(\sum_{k\in\mathbb{Z}}\frac{1}{|\alpha_i-k|}\right)\\ &= \prod_{i=1}^{n}\frac{|\sin(\alpha_i\pi)|}{\pi}\left(C_i+\sum_{\substack{n\in\mathbb{N}\\ n>|\alpha_i|}}\frac{1}{n-\alpha_i}+\frac{1}{n+\alpha_i}\right),\end{aligned}$$

where the separability of the product was used to interchange summation and multiplication in the second line. In the third line terms with k and $-k$ were summed provided $k>|\alpha_i|$ to resolve the modulus. All terms with $k<\alpha_i$ are summarized in C_i. Since

$$\frac{1}{k-\alpha_i}+\frac{1}{k+\alpha_i}=\frac{2k}{k^2-\alpha_i^2}=\frac{2}{k-\alpha_i^2/k}\geqslant\frac{2}{k},$$

the series diverges by the direct comparison test, because it dominates $2\sum_{k\in\mathbb{N}}1/k$, which diverges. □

Now we can state and prove the sampling theorem.

4.28 Theorem. *Let $W>0$ and $f:\mathbb{C}\to\mathbb{C}$ be of at most exponential type W. Then*

$$f(x)=\sum_{k\in\mathbb{Z}}f\left(\frac{k\pi}{\omega}\right)\mathrm{sinc}_\omega(x-k\pi/\omega) \tag{4.16}$$

for $\omega>W$. The series converges uniformly on bounded sets .

If f fulfills $|f(z)|\leqslant(1+|z|)^{-1}\exp(W|\operatorname{Im}z|)$, then (4.16) remains valid for $\omega=W$.

Proof. Define

$$F_n:\mathbb{C}\to\mathbb{C},\quad z\mapsto\frac{1}{2\pi\,\mathrm{i}}\oint_{C_n}\frac{f(\zeta)}{\sin(\omega\zeta)(\zeta-z)}\mathrm{d}\zeta,\qquad\text{with}\qquad C_n:[0,2\pi]\to\mathbb{C},\quad\varphi\mapsto r_n\mathrm{e}^{\mathrm{i}\varphi}$$

and $r_n:=(n+1/2)\pi/\omega$. Let $z\in\mathbb{C}$ and $z\neq k\pi/\omega$ for all $k\in\mathbb{Z}$. By the Residue Theorem 2.45

$$F_n(z)=\sum_{|k|\leqslant n}\frac{f(k\pi/\omega)}{\omega\cos(k\pi)(k\pi/\omega-z)}+\frac{f(z)}{\sin(\omega z)}$$

holds with the residues calculated in Lemma 4.26, provided n is large enough such that C_n contains z.

On the other hand, we will show $|F_n(z)|\to0$ for $n\to\infty$. Consider

$$\begin{aligned}|F_n(z)| &= \left|\frac{1}{2\pi\,\mathrm{i}}\int_0^{2\pi}\frac{f(r_n\mathrm{e}^{i\varphi}}{\sin(\omega r_n\mathrm{e}^{i\varphi})(r_n\mathrm{e}^{\mathrm{i}\varphi})}\,\mathrm{i}\,r\mathrm{e}^{\mathrm{i}\varphi}\mathrm{d}\varphi\right|\\ &\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(r_n\mathrm{e}^{\mathrm{i}\varphi})|}{|\sin((k+1/2)\pi\mathrm{e}^{\mathrm{i}\varphi})|\,|1-\mathrm{e}^{-\mathrm{i}\varphi}z/r_n|}\mathrm{d}\varphi\end{aligned}$$

The estimate of Lemma 4.25 and the exponential type inequality yield

$$|F_n(z)| \leqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{\gamma e^{W|\sin(\varphi)|}}{\frac{\exp(\omega r_n |\sin\varphi|)}{3} |1 - e^{-i\varphi} z/r_n|} d\varphi.$$

Since $|1 - e^{i\varphi} z/r_n| \to 1$ for $n \to \infty$, there is an index $n_0 \in \mathbb{N}$ such that $|1 - e^{i\varphi} z/r_n| \geqslant 1/2$ for $n \geqslant n_0$. In this case we have

$$|F_n(z)| \leqslant \frac{3}{\pi} \int_0^{2\pi} \gamma e^{(W-\omega) r_n |\sin\varphi|} d\varphi. \tag{4.17}$$

As $|\sin(\varphi)|$ is symmetric and π-periodic, we can replace the integral over $[0, \pi]$ by 4 times the integral over $[0, \pi/2]$. Additionally, using the estimate $\sin(\varphi) \geqslant 2\varphi/\pi$ ($\varphi \in [0, 2\pi]$), and noting that due to $\omega > W$ the exponent is negative, we get

$$|F_n(z)| \leqslant \frac{12\gamma}{\pi} \int_0^{\pi/2} e^{(W-\omega) r_n 2\varphi/\pi} = \frac{6\gamma}{(W-\omega) r_n} \left(e^{(W-\omega) r_n 2\varphi/\pi} - 1 \right),$$

hence $|F_n(z)| \to 0$ for $n \to \infty$. This implies

$$0 = \sum_{k \in \mathbb{Z}} \frac{f(k\pi/\omega)}{\omega \cos(k\pi)(k\pi/\omega - z)} - \frac{f(z)}{\sin(\omega z)}.$$

Solving for $f(z)$ and using $\cos(k\pi) = (-1)^k$ as well as $(-1)^k \sin(x) = \sin(x - k\pi)$ yields

$$f(z) = \sum_{k \in \mathbb{Z}} \frac{f(k\pi/\omega)}{z - \omega(k\pi/\omega)} (-1)^k \sin(\omega z) = \sum_{k \in \mathbb{Z}} \left(f(\frac{k\pi}{\omega} \right) \operatorname{sinc}_\omega \left(z - \frac{k\pi}{\omega} \right). \tag{4.18}$$

This result is true under the restriction $z \neq k\pi/\omega$. But since $\operatorname{sinc}_\omega$ is entire[1] the whole right hand side must be entire. Now we have two entire functions which coincide almost everywhere on $\mathbb{R}$, hence Lemma 2.40 yields (4.16) to hold for every $z \in \mathbb{C}$, especially for $z \in \mathbb{R}$, which was to prove.

Let $K \subset \mathbb{C}^n$ be bounded. Then there is $n \in \mathbb{N}$ such that $|z| < r_n$ for every $z \in K$. Hence $|1 - e^{i\varphi} z/r_{n'}| > 1/2$ for $n' > 2n$. This makes the estimate (4.17) independent of z, providing the estimate

$$\begin{aligned} \sup_{x \in \mathbb{C}^n} \left| f(z) - \sum_{k=-n}^{n} f\left(\frac{k\pi}{\omega}\right) \operatorname{sinc}_\omega \left(z - \frac{k\pi}{\omega} \right) \right| &= \sup_{x \in \mathbb{C}^n} |\sin(\omega z) F_n(z)| \\ &\leqslant \frac{6\gamma}{(W-\omega) r_n} \left(e^{(W-\omega) r_n 2\varphi/\pi} - 1 \right), \end{aligned}$$

which shows uniform convergence.

Absolute convergence cannot be guaranteed, since for example the constant function 1 is of at most exponential type σ for any $\sigma > 0$. For $x \in \mathbb{R}$ and $\omega > 0$ the sampling series is

$$1 = \sum_{k \in \mathbb{Z}} \operatorname{sinc}(\omega x - k\pi),$$

[1]This can be seen using the series expansion $\operatorname{sinc}_\omega(z) = \sum_{i=0}^{\infty} (-1)^k (\omega z)^{2k}/(1+2k)!$.

which was shown not to converge absolutely for $\omega x \in \mathbb{R}\backslash\pi\mathbb{Z}$ in Lemma 4.27.

If we have $|f(z)| \leqslant (1+|z|)^{-1}\exp(W|\operatorname{Im} z|)$, we get an additional factor in (4.17):

$$|F_n(z)| \leqslant \frac{3\gamma}{\pi}\int_0^{2\pi} \frac{\mathrm{e}^{(W-\omega)r_n|\sin\varphi|}}{(1+r_n)}\mathrm{d}\varphi,$$

which leaves the foregoing estimates unchanged ($(1+r_n)\geqslant 1$) but in the case $\omega = W$ we still have

$$|F_n(z)| \leqslant \frac{6\gamma}{1+r_n} \to 0 \quad , n\to\infty.$$

□

To show, that the condition $\omega > W$ is necessary if $f \in \mathrm{L}^\infty(\mathbb{R})$, we consider the following

4.29 Example. Let $f\colon \mathbb{C}\to\mathbb{C}, x\mapsto \sin(\pi x)$. Then $f|_{\mathbb{R}} \in \mathrm{L}^\infty(\mathbb{R})$, since sin is bounded on the real axis. Further we have

$$|\sin(\pi z)| = \frac{1}{2}\left|\mathrm{e}^{\mathrm{i}\,\pi z} - \mathrm{e}^{-\mathrm{i}\,\pi z}\right| \leqslant \frac{1}{2}\left(\mathrm{e}^{-\pi\operatorname{Im} z} + \mathrm{e}^{\pi\operatorname{Im} z}\right) \leqslant \mathrm{e}^{\pi|\operatorname{Im} z|}.$$

Hence, f is of exponential type at most π. But if we sample at the points $k\pi/\pi = k \in \mathbb{Z}$, we have $\sin(k\pi) = 0$ for all sample points. Therefore we would obtain

$$f(z) = \sum_{k\in\mathbb{Z}} f(k)\operatorname{sinc}_\pi(z-k) = 0 \neq \sin(z)$$

if $z \notin \mathbb{Z}$.

4.30 Remark. A generalization of this proof to n dimensions is not trivial, since one needs a residue theorem for n dimensions. Griffiths [GH78, p. 467] provides this residue theory but the required theory is beyond the scope of this work.

4.7 Proof based on Fourier series for distributions

We follow the idea of [GMHM98, p. 50], but here we use periodic distributions defined on the torus instead.

4.31 Lemma. *Let $K \Subset \mathbb{C}^n$ be compact and $M := \{\exp(\mathrm{i}\,z\cdot) \in C^\infty(\mathbb{R}^n) : z \in K\}$. Then M is bounded in $C^\infty(\mathbb{R}^n)$.*

Proof. Let V be a neighbourhood of 0. The seminorms inducing the locally convex topology on $C^\infty(\mathbb{R}^n)$ are defined as $p_{\alpha,N}(\varphi) : \sup_{|x|\leqslant N}|\partial^\alpha\varphi(x)|$ for $\alpha \in \mathbb{N}_0^n$ and $N \in \mathbb{N}$. Then there is a finite index set $I \subset \mathbb{N}$, $r_i > 0$, $N_i \in \mathbb{N}$ and $\alpha_i \in \mathbb{N}_0^n$ for $i \in I$, such that

$$U := \{\varphi \in C^\infty(\mathbb{R}^n) : p_{\alpha_i,N_i}(\varphi) = \sup_{|x|\leqslant N_i}|\partial^{\alpha_i}\varphi(x)| < r_i, i \in I\} \subset V.$$

Since K is compact, there is $N_0 \in \mathbb{N}$, such that $|z| \leqslant N_0$ for every $z \in K$. Define $N := \max\{N_i : i \in I \cup \{0\}\}$, $t' := \max_{|x|,|z|\leqslant N}\max_{i\in I}|\partial^{\alpha_i}\exp(\mathrm{i}\,zx)|$ and $t := \max\{t'/r_i : i \in I\}$. Then, $tr_i \geqslant t'$ and we have for each $z \in K$ and $i \in I$

$$\sup_{|x|\leqslant N_i}|\partial^{\alpha_i}\exp(\mathrm{i}\,zx)| \leqslant t' \leqslant r_i t.$$

Therefore, $M \subset tU \subset tV$. Hence, M is bounded. □

4.32 Theorem. *Let* $W > 0$*,* $f \in \mathrm{PW}_W(\mathbb{C}^n)$*,* $u := \mathcal{F}^{-1}[f]$*,* $\operatorname{supp} u \cap \partial[-W,W]^n = \varnothing$*,* $\theta \in \mathcal{D}(\mathbb{R}^n)$ *with* $\operatorname{supp}(\theta) \subset [-W,W]^n$ *and* $\theta \equiv 1$ *on an open subset containing* $\operatorname{supp} u$*. Then*

$$f(z) = \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\left(\frac{\sqrt{2\pi}}{2W}\right)^n \mathcal{F}[\theta]\left(z - \frac{k\pi}{W}\right).$$

with uniform convergence on compact subsets of $\mathbb{C}^n$*.*

Proof. By Paley-Wiener Thereom 4.5, there is $u \in \mathcal{D}'(\mathbb{R}^n)$ with $\operatorname{supp}(u) \subset [-W,W]^n$ and

$$f(z) = (2\pi)^{-n/2} u(\exp(-\mathrm{i}\, z\cdot)) \quad (z \in \mathbb{C}^n). \tag{4.19}$$

Lemma 3.52 yields the existence of is $\theta \in \mathcal{D}(\mathbb{R}^n)$ with $\theta \equiv 1$ on $\operatorname{supp} u$ and $\operatorname{supp}\theta \subset (-W,W)^n$ such that $\tilde{u} := \theta u \in \mathcal{D}'(2W\mathbb{T}^n)$ and $u(\varphi) = \tilde{u}(\varphi)$ for $\varphi \in \mathcal{D}'(\mathbb{R}^n)$.

Theorem 3.58 allows us to write $\tilde{u}$ as

$$\tilde{u} = \sum_{k\in\mathbb{Z}^n} \widehat{\tilde{u}}(k)\left[\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{2W}k\cdot}\right]_{2W\mathbb{T}^n}$$

with

$$\widehat{\tilde{u}}(k) = (2W)^{-n} u(\theta \mathrm{e}^{-i\frac{2\pi}{2W}k\cdot}) = (2W)^{-n}(2\pi)^{n/2}(2\pi)^{-n/2}u(\mathrm{e}^{-\mathrm{i}\,\frac{\pi}{W}k\cdot}) \overset{(4.19)}{=} \left(\frac{\sqrt{2\pi}}{2W}\right)^n f\left(\frac{k\pi}{W}\right),$$

since $\theta \equiv 1$ on $\operatorname{supp} u$. Hence, for $\varphi \in \mathcal{D}'(\mathbb{R}^n)$ we obtain

$$u(\varphi) = \tilde{u}(\varphi) = \left(\frac{\sqrt{2\pi}}{2W}\right)^n \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\left[\mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}k\cdot}\right]_{2W\mathbb{T}^n}(\varphi).$$

Note that the regular distribution has still to be integrated over $2W\mathbb{T}^n$. To make this distribution an element of $\mathcal{D}'(\mathbb{R}^n)$, we can use again $u(\varphi) = \theta u(\varphi)$, hence

$$\begin{aligned}
(2\pi)^{-n/2}u(\varphi) &= (2\pi)^{-n/2}\theta u(\varphi) = (2\pi)^{-n/2}u(\theta\varphi) = (2W)^{-n}\sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\left[\mathrm{e}^{\mathrm{i}\,\frac{2\pi}{2W}k\cdot}\right](\theta\varphi) \\
&= (2W)^{-n}\sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\int_{2W\mathbb{T}^n} \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx}\theta(x)\varphi(x)\mathrm{d}x \\
&= (2W)^{-n}\sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx}\theta(x)\varphi(x)\mathrm{d}x \\
&= (2W)^{-n}\sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\left[\mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}k\cdot}\theta\right]_{\mathbb{R}^n}(\varphi),
\end{aligned}$$

where the regular distribution is now integrated over $\mathbb{R}^n$ but still has compact support. Using the result of Paley-Wiener (4.19), one obtains for $z \in \mathbb{C}^n$

$$\begin{aligned}
f(z) = (2\pi)^{-n/2}u(\mathrm{e}^{-\mathrm{i}\,z\cdot}) &= (2W)^{-n}\sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}kx}\mathrm{e}^{-\mathrm{i}\,zx}\theta(x)\mathrm{d}x \\
&= \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right)\left(\frac{\sqrt{2\pi}}{2W}\right)^n \mathcal{F}[\theta]\left(z - \frac{k\pi}{W}\right).
\end{aligned}$$

To show uniform convergence on compact subsets, we exploit Theorem 2.37. Since the series

$$(2\pi)^{-n/2}u(\varphi) = (2W)^{-n} \sum_{k\in\mathbb{Z}^n} f\left(\frac{k\pi}{W}\right) \left[\mathrm{e}^{\mathrm{i}\,\frac{\pi}{W}k\cdot}\theta\right]_{\mathbb{R}^n}(\varphi)$$

converges pointwise to u, we have uniform convergence on bounded sets of $C^\infty(\mathbb{R}^n)$ by Theorem 2.37. In Lemma 4.31 we showed that the set $\{\exp(-\mathrm{i}\,z\cdot) \in C^\infty(\mathbb{R}^n) : z \in K\}$ is bounded, provided $K \subset \mathbb{C}^n$ is compact. Hence, the sampling series converges uniformly to f on compact subsets of $\mathbb{C}$. □

4.33 Remark. If we could choose $\theta = \chi_{[-W,W]^n}$, we would retrieve the simple sampling theorem, since by (4.10), we have

$$\left(\frac{\sqrt{2\pi}}{2W}\right)^n \mathcal{F}\left[\chi_{[-W,W]^n}\right] = \mathrm{sinc}_W\,.$$

4.8 Further methods

One can find the sampling theorem as solution of a minimization problem, namely 'given the sampling values of a function f, which functions s_k minimize the difference between f and $\sum_{|k|<N} f(k)s_k$ as sketched in [Mar01, p. 56] and [BSS88, p. 12]. In [Mar01, p. 57] another method is presented, using the Hilbert transform

The distributional version can also be proved with the Fourier series for periodic distributions in $\mathcal{D}'(\mathbb{R}^n)$ as done originally in [GMHM98]. Furthermore, there is the possibility to exploit Poisson's summation formula as in [Cam68, p. 628].

5 Conclusions and Outlook

We have seen several proofs of the Nyquist-Shannon-Sampling theorem. Each of them had slightly different prerequisites concerning either the growth condition for complex arguments or the function space of the Fourier transform. To conclude, we will summarize the requirements and results of the different approaches.

We started with a one dimensional sampling theorem. The proof based on the *semi-discrete convolution product* required the W-band limited sampling function to be continuous and in $\mathrm{L}^2(\mathbb{R})$. The sampling series reconstructs the function if samples are taken with frequency W/π, which is called Nyquist frequency. Furthermore, the series is absolutely and uniformly convergent.

The *generalized Parseval formula* required the sampling function to be the inverse Fourier transform of an $\mathrm{L}^2([-W, W]^n)$-function. Therefore, it generalizes the sampling theorem to n-dimensions. The prerequisites are basically the same, since Plancherel's theorem asserts f to be in L^2 if and only if $\mathcal{F}[f]$ is in L^2. Thus, it is not astonishing, that sampling is possible at the Nyquist frequency and the sampling series converges absolutely and uniformly, too.

The first step loosening the prerequisites builds on the concept of *Schauder bases*. Here, the sampling function has to be the inverse Fourier transform of an $\mathrm{L}^p([-W, W]^n)$-function for $1 < p \leqslant \infty$. Additionally, we required the series of sampling values to converge absolutely. The sampling theorem still provides sampling at Nyquist freqency and uniform convergence. Absolute convergence is given as prerequisite.

As the Paley-Wiener theorems relate compact supports to the growth properties and entire continuations of the corresponding Fourier transforms, it is also possible to characterize the functions for the sampling theorem by their growth properties. We prove that $\tau_k \operatorname{sinc}_W$ $(k \in \mathbb{Z}^n)$ is an *orthogonal system* in the Hilbert space of all entire functions of exponential type at most W, which belong to $\mathrm{L}^2(\mathbb{R}^n)$ when restricted to $\mathbb{R}^n$. Then, the sampling theorem is simply the expansion of the sampling function with respect to this basis. It is shown that this Hilbert space can be identified with the one in the first two proofs. Therefore, we obtain the same results.

Following the idea of the growth properties, we can prove the sampling theorem for entire functions of exponential type at most W. Therefor, we exploit the properties of *contour integration* in the complex plane. This proof generalizes the sampling theorem to functions which are not integrable over $\mathbb{R}$, like the sine function or any constant function. However, this generalization has to be paid with an increased sampling frequency and the loss of convergence properties. The sampling series only reconstructs the function if we sample with $\omega/\pi > W/\pi$ and absolute convergence is lost. Uniform convergence is only given for bounded subsets of $\mathbb{C}$. A generalization to n dimensions may be possible, but requires heavy theory.

Finally, we give a sampling theorem for entire functions, whose Fourier transform may only be given by a *distribution* with compact support. This loss of regularity requires us to sample with

$\omega/\pi > W/\pi$. Furthermore, it forces us to substitute the sinc-function in the series by the Fourier transform of a smooth version of the characteristic function in order to provide convergence of the series. On compact subsets of $\mathbb{C}^n$ the convergence is uniform. This sampling theorem is valid for n dimensions.

For use in applications, one can further investigate how the sampling frequency changes if the support of the Fourier transform does not contain the origin. This is called single-sideband-modulation. As in real world, signals can only be sampled over a finite time, one can also consider these truncation errors. Furthermore, one always finds perturbances both on the signal as well as the sampling device. Therefore, it can be investigated how time jitter influences the reconstructed signal and how much one has to oversample to obtain the real signal back again.

Apart from the applications, it can be shown that the sampling theorem, Cauchy's integral formula and Poisson's summation formula are equivalent in the sense that each of them can be deduced by the other using elementary methods, see [BSS88, section 6.1, p. 45].

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