First publ. in: COMPSTAT 74. Proceedings in Computational Statistics.
Wien: Physica-Verl. 1974, pp. 193-201

A Dynamic Model for Measuring Individual and Situative Influences on Social Behavior

Wilhelm F. Kempf, Kiel

1. INTRODUCTION

A basic concept in stochastic test theory is the *item characteristic curve* $f_i(\xi_v)$. Let ξ_v be an individual parameter representing e.g. the ability of an individual v in an achievement test or the aggressiveness of the individual in an aggressivity test. Then we introduce a random variable a_{vi} such that $a_{vi} = 1$ if the individual v gives a positive ("correct", "aggressive") response to item number i and $a_{vi} = 0$ if the individual's response is a negative ("incorrect", "non-aggressive") one. $f_i(\xi_v)$ is the probability that the individual v will respond positively to item i, and under the assumption of *local stochastic independency*

$$(1.1) \quad p\{a_{v1}, a_{v2},\ldots,a_{vg}\} = \prod_{i=1}^{g} p\{a_{vi}\} \quad \text{for all } g = 1,\ldots,k$$

the probability distribution of an individual's responses to k items is defined uniquely by

$$(1.2) \quad p\{(a_{vi})\} = \prod_{i=1}^{k} f_i(\xi_v)^{a_{vi}} \cdot (1 - f_i(\xi_v))^{1-a_{vi}}$$

in which $(a_{vi}) = (a_{v1}, a_{v2},\ldots,a_{vk})$ denotes the *response vector* of the individual.

As regards the structural form of $f_i(\xi_v)$ as a function of the latent variable ξ, the literature contains several suggestions. For a number of reasons which are clearly emphasized in the papers by RASCH (1960, 1961) and ANDERSEN (1973a, 1973b), for instance, the logistic test model

$$(1.3)\qquad f_i(\xi_v) = \frac{\xi_v}{\xi_v + \sigma_i} \quad ,$$

in which σ_i is the difficulty of the item, is the most attractive statistical model. According to a well known theorem by NEYMAN & SCOTT (1948) the traditional methods of parameter estimation fail if the number of parameters to be estimated does not tend towards a fixed numeral while the number of observations grows to infinity. From eq. (1.2), however, it follows, that each S that is added to the sample will cause the introduction of an additional parameter. The only models in which consistent estimators exist in such a situation are those in which the individual parameters can be separated from the structural parameters pertaining to the items by use of *conditional inference* methods (cf. ANDERSEN, 1973a) and as ANDERSEN (1972b) has shown, within the framework of stochastic test theory, these are the models suggested by RASCH (1960, 1961) only.

In many psychological applications of stochastic test theory, however, this framework turnes out to be too narrow. Many psychological concepts such as *learning* or *catharsis* conflict with the assumption of local stochastic independency and KEMPF (1974), therefore, has suggested an extension of stochastic test theory to what may be called *dynamic test theory*.

2. THE MODEL

The basic conception of dynamic test theory is to replace the assumption of local stochastic independency by the concept of *local serial dependency*

$$(2.1)\qquad p\{(a_{vi})\} = \prod_{i=1}^{k} p\{a_{vi} | s_{vi}\}$$

in which $(a_{vi}) = (a_{v1}, a_{v2}, \ldots, a_{vk})$ denotes the vector of the individual's responses to k items and in which s_{vi} stands for the partial response vector $(a_{v1}, a_{v2}, \ldots, a_{vi-1})$. The item characteristic curves $f_i(\xi_v)$ are replaced by *conditional item characteristic curves*

$$(2.2)\quad f_{i.s_{vi}}(\xi_v) = p\{a_{vi} = 1 | (a_{v1}, a_{v2}, \ldots, a_{vi-1}) = s_{vi}\}$$

so that

$$(2.3)\quad p\{a_{vi} | s_{vi}\} = f_{i.s_{vi}}(\xi_v)^{a_{vi}} \cdot (1 - f_{i \cdot s_{vi}}(\xi_v))^{1-a_{vi}}$$

On the basis of this formalism, KEMPF (1974) has proposed a dynamic test model in which the conditional item characteristic curves are assumed to depend on the *number of positive responses* to the preceeding items

$$(2.4)\quad r_{vi} = \begin{cases} 0 & \text{for } i = 1 \\ \sum_{j=1}^{i-1} a_{vj} & \text{for } i = 2,3,\ldots,k \end{cases} ,$$

but *not* to depend on *which* of the preceding items were answered positively:

$$(2.5)\quad f_{i \cdot s_{vi}}(\xi_v) = f_{i \cdot r_{vi}}(\xi_v)$$

for all partial response vectors s_{vi} which are compatible with the partial score r_{vi} .

The functions $f_{i \cdot r_{vi}}(\xi_v)$ are defined by

$$(2.6)\qquad f_{i \cdot r_{vi}}(\xi_v) = \frac{\xi_v + \psi_{r_{vi}}}{\xi_v + \sigma_i}$$

in which $\psi_{r_{vi}} \leq \sigma_i$ for all $r_{vi} = 0,\ldots,i-1$ and $i = 1,\ldots,k$.

Before we proceed to a discussion of the model, we note that the *Rasch*-model is a special case of the dynamic model: (2.6) reduces to (1.3) if $\psi_0 = \psi_1 = \ldots = \psi_{k-1} = 0$. This suggests that ξ_v ist a generalized ability parameter and that σ_i is an item difficulty parameter. Fig. 2.1 shows that the probability (2.6) is a monotone *increasing* function of the individual parameter ξ_v and a monotone *decreasing* of the item parameter σ_i. (2.6) tends to 0 for $\sigma_i \to \infty$ and it tends to 1 for $\xi_v \to \infty$ as well as for $\sigma_i \to \psi_r$. If $\xi_v \to 0$, finally, (2.6) tends to ψ_r/σ_i.

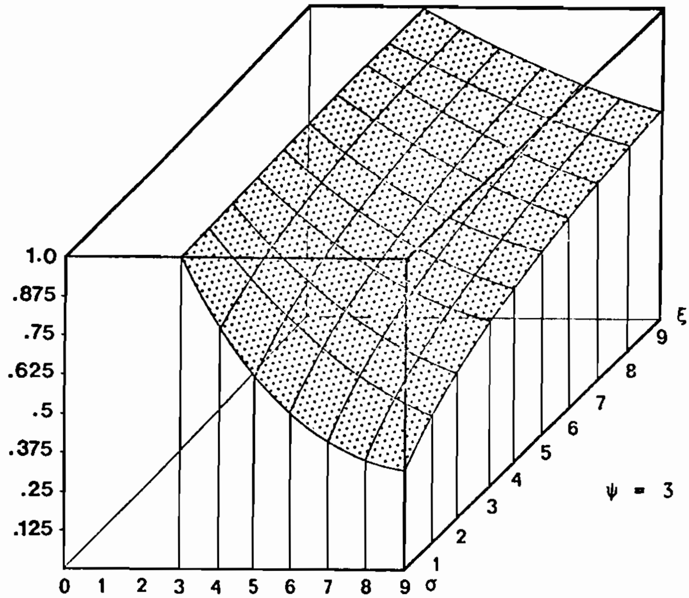
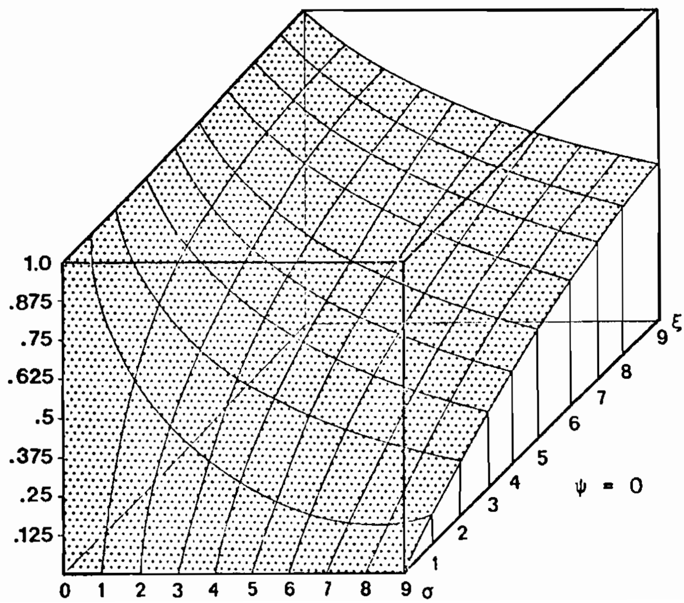


Fig. 2.1.: The conditional item characteristic curve (2.6) as a function of the latent trait variable ξ and of the item difficulty σ; two different values of ψ.



Regardless of an individual's initial ability ξ_v, after r positive responses to the preceeding items, the individual's probability of sucess on item i will not be less than ψ_r/σ_i.

ψ_r will be referred to as a transfer parameter. It describes how an individual's probability of success is effected by his prior responses. (2.6) is a linear increasing function of ψ_r. It tends to 1 if $\psi_r \to \sigma_i$ and to $\xi_v/(\xi_v + \sigma_i)$ if $\psi_r \to 0$. The transfer ψ_r is a *learning effect*, if it is an *increasing* function of r (*positive transfer)*; it is a *reactive inhibition* (such as catharsis, for instance), if it is a *decreasing* function of r (*negative transfer)*, and it is a *fluctuation* that can be explained by concurricung positive and negative transfer, if it is a *non-monotone* function of r.

As for the simple model with $\psi_0 = \psi_1 = \ldots = \psi_{k-1} = 0$, the model is slightly overparametrized by $\xi_v, \ldots, \xi_n$, $\sigma_1, \ldots, \sigma_k$ and $\psi_0, \ldots, \psi_{k-1}$ The parameters of the model are not determined uniquely by eq. (2.6) and the model will still hold when the parameters are multiplied by a positive constant c_1 and when an arbitrary constant c_2 is added to the individual parameters ξ_v, and, at the same time, is subtracted from the item- and transfer-parameters σ_i and ψ_r. The parameters, therefore, are measured on *interval scales* only and we may look for a proper standardization such as

(2.7) $\quad \mathrm{MIN}(\psi_r) = 0 \qquad$ for $r = 0, \ldots, k-1$

and

(2.8) $\quad \prod_{i=1}^{k} \sigma_i = 1 \quad ,$

by which we introduce the same parametrization conditions that are usual in the *Rasch*-model.

As KEMPF (1974) has shown, the dynamic test model (2.6) also has the same mathematical properties as the *Rasch*-model. The test scores $a_{vo} = \Sigma_{i=1}^{k} a_{vi}$ are minimal sufficient statistics for the latent trait parameters ξ_v and the item difficulties σ_i can be estimated by use of the CML-method.

3. SUFFICIENT STATISTICS

The test score a_{vo} is a sufficient statistic for estimating the individual parameter ξ_v *iff* the conditional likelihood of the individual's response vector (a_{vi})

$$(3.1)\quad p\{(a_{vi}) \mid a_{vo}\} = \frac{p\{(a_{vi})\}}{p\{a_{vo}\}}$$

does not depend on the individual parameter ξ_v.
Reformulating (2.6) as

$$(3.2)\quad p\{a_{vi} \mid r_{vi}\} = \frac{(\xi_v + \psi_{r_{vi}})^{a_{vi}} (\sigma_i - \psi_{r_{vi}})^{1-a_{vi}}}{\xi_v + \sigma_i}$$

and inserting (3.2) into (2.1) yields the unconditional likelihood of the response vector

$$(3.3)\quad p\{(a_{vi})\} = \frac{\prod_{i=1}^{k} (\xi_v + \psi_{r_{vi}})^{a_{vi}} (\sigma_i - \psi_{r_{vi}})^{1-a_{vi}}}{\prod_{i=1}^{k} (\xi_v + \sigma_i)} .$$

$$= \frac{\prod_{r=0}^{a_{vo}-1} (\xi_v + \psi_r) \cdot \prod_{i=1}^{k} (\sigma_i - \psi_{r_{vi}})^{1-a_{vi}}}{\prod_{i=1}^{k} (\xi_v + \sigma_i)}$$

The likelihood of the individual's test score a_{vo} is obtained from (3.3) by summation of the probabilities $p\{(a^*_{vi})\}$ of all possible response vectors (a^*_{vi}) which are compatible with the score so that $\Sigma_{i=1}^{k} a^*_{vi} = a_{vo}$

$$(3.4)\quad p\{a_{vo}\} = \sum_{(a^*_{vi}) \mid a_{vo}} p\{(a^*_{vi})\}$$

$$= \frac{\prod_{r=0}^{a_{vo}-1} (\xi_v + \psi_r) \cdot \sum_{(a^*_{vi}) \mid a_{vo}} \prod_{i=1}^{k} (\sigma_i - \psi_{r^*_{vi}})^{1-a^*_{vi}}}{\prod_{i=1}^{k} (\xi_v + \sigma_i)}$$

r^*_{vi} is defined by $r^*_{vi} = \Sigma_{j=1}^{i-1} a^*_{vi}$ for $i = 2,3,\dots,k$ and $r^*_{vi} = 0$ for $i=1$.

The conditional likelihood of the response vector, finally, is obtained from inserting the equations (3.3) and (3.4) into formula (3.1):

$$(3.5) \quad p\{(a_{vi})|a_{vo}\} = \frac{\prod_{i=1}^{k} (\sigma_i - \psi_{r_{vi}})^{1-a_{vi}}}{\sum_{(a^*_{vi})|a_{vo}} \prod_{i=1}^{k} (\sigma_i - \psi_{r^*_{vi}})^{1-a^*_{vi}}}$$

(3.5) is dependent on the item- and transfer-parameters σ_i and ψ_r only and does not depend on the individual parameter ξ_v. Consequently, a_{vo} is a sufficient estimator for ξ_v and any extra information about *which* of the items were answered positively is useless as a source of inference about ξ_v. However, it can be used for inferring the item- and transfer-parameters independently from the individual parameters if $0 < a_{vo} < k$. [1)]

4. CML-ESTIMATORS

Let us consider the responses of n individuals with $0 < a_{vo} < k$. Then, the individual's responses can be arranged in a $n \times k$ response matrix $((a_{vi}))$ and the individual's scores can be arranged in a n-dimensional score vector (a_{vo}). The conditional likelihood of the response matrix follows from inserting (3.5) into

$$(4.1) \quad p\{((a_{vi}))|(a_{vo})\} = \prod_{v=1}^{n} p\{(a_{vi})|a_{vo}\} \quad ,$$

which yields

$$(4.2) \quad p\{((a_{vi}))|(a_{vo})\} = \frac{\prod_{i=1}^{k} \prod_{r=0}^{i=1} (\sigma_i - \psi_r)^{n_{ri}}}{\prod_{v=1}^{n} \sum_{(a^*_{vi})|a_{vo}} \prod_{i=1}^{k} (\sigma_i - \psi_{r^*_{vi}})^{1-a^*_{vi}}} = L$$

---

1) If $a_{vo} = 0$ or $a_{vo} = k$, then $p\{(a_{vi})|a_{vo}\} = 1$ and does not provide any information about the parameters to be estimated.

in which n_{ri} is the number of individuals who responded negatively to item i after $r_{vi} = r$ positive responses to the preceding items $j = 1,2,\dots,i-1$.

Now let N_{k-s} be the number of individuals who gave a total of s negative responses to the k items so that $a_{vo} = k-s$, and let

$$\delta_m(k-s) = \begin{cases} 1 & \text{for } m = 0 \\ \sum_{j_1=0}^{k-s} \sum_{j_2=j_1}^{k-s} \cdots \sum_{j_m=j_{m-1}}^{k-s} \prod_{t=1}^{m} \psi_{j_t} & \text{for } m = 1,2,\dots,s\ . \end{cases}$$

Then, according to KEMPF & HAMPAPA (1974), the denominator in (4.2) can be written as

$$(4.3)\qquad \prod_{v=1}^{n} \sum_{(a^*_{vi})|a_{vo}} \prod_{i=1}^{k} (\sigma_i - \psi_{r^*_{vi}})^{1-a^*_{vi}} =$$

$$= \prod_{s=1}^{k-1} \left\{ \sum_{m=0}^{s} \delta_m(k-s)\cdot\gamma_{s-m}(k)\cdot(-1)^m \right\}^{N_{k-s}}$$

in which $\gamma_{s-m}(k)$ denotes the elementary symmetric function of order s-m of the parameters $\sigma_1,\dots,\sigma_k$. Inserting (4.3) into (4.2) and differentiating

$$(4.4)\qquad \ln(L) =$$

$$= \sum_{i=1}^{k} \sum_{r=0}^{i-1} n_{ri}\cdot\ln(\sigma_i - \psi_r) - \sum_{s=1}^{k-1} N_{k-s}\cdot\ln\left(\sum_{m=0}^{s} \delta_m(k-s)\cdot\gamma_{s-m}(k)\cdot(-1)^m\right)$$

with respect to the item-parameters $\sigma_\alpha, \alpha = 1,\dots,k$ and to the transfer-parameters ψ_β, $\beta = 0,\dots,k-1$, finally, leads to the necessary estimation equations $\partial\ln(L)/\partial\sigma_\alpha = 0$ for $\alpha = 1,\dots,k$ and $\partial\ln(L)/\partial\psi_\beta = 0$ for $\beta = 0,\dots,k-1$. The resulting CML-estimators are

$$(4.5)\qquad \sum_{r=0}^{\alpha-1} \frac{n_{r\alpha}}{\sigma_\alpha - \psi_r} =$$

$$= \sum_{s=1}^{k-1} N_{k-s} \cdot \frac{\sum_{m=0}^{s-1} \delta_m(k-s)\cdot\gamma^{(\alpha)}_{s-m-1}(k)\cdot(-1)^m}{\sum_{m=0}^{s} \delta_m(k-s)\cdot\gamma_{s-m}(k)\cdot(-1)^m}$$

for estimating the item-difficulties σ_α and

$$(4.6)\qquad \sum_{i=\beta+1}^{k} \frac{n_{\beta i}}{\sigma_i-\psi_\beta} =$$

$$= \sum_{\substack{s=1\\ \beta\leq k-s}}^{k-1} N_{k-s} \cdot \frac{\sum_{m=1}^{s} \gamma_{s-m}(k)\cdot\left\{\sum_{j=0}^{m-1} \psi_\beta^j \cdot\delta_{m-1-j}(k-s)\right\}\cdot(-1)^m}{\sum_{m=0}^{s} \delta_m(k-s)\cdot \gamma_{s-m}(k)\cdot(-1)^m}$$

for estimating the transfer parameters ψ_β .

$\gamma_{s-m-1}^{(\alpha)}(k)$ denotes the elementary symmetric function of order $s-m-1$ of the parameters $\sigma_1,\ldots,\sigma_{\alpha-1},\sigma_{\alpha+1},\ldots,\sigma_k$.

A *FORTRAN*-program for the numerical solution of the estimation equations has been written by KEMPF & MACH (1974).

REFERENCES

ANDERSEN, E.B.: Conditional inference and models for measuring. Copenhagen: Mentalhygiejnisk Forlag, 1973a.

ANDERSEN, E.B.: Conditional inference for multiple-choice questionnaires. Br.J.math.statist.Psychol. 1973b, 26, 31-44.

KEMPF, W.F.: Dynamische Modelle zur Messung sozialer Verhaltensdispositionen. In: KEMPF, W.F. (Hrsg.): Probabilistische Modelle in der Sozialpsychologie. Bern: Huber, 1974.

KEMPF, W.F. & P. HAMPAPA : The numerical solution of a set of conditional estimation equations arising in a dynamic test model. Br.J.math.statist.Psychol. 1974 (forthcoming).

KEMPF, W.F. & G. MACH : Unveröffentliches Computerprogramm zur CML-Parameterschätzung in einem dynamischen Testmodell. Kiel:IPN (Inst.f.d.Päd.d.Naturwiss.), 1974.

NEYMAN, J. & E.L. SCOTT : Consistent estimates based on partially consistent observations. Econometrica 1948, 16, 1-32.

RASCH, G.: Probabilistic models for some intelligence and attainment tests. Copenhagen: Nielsen & Lydiche, 1960.

RASCH, G.: On general laws and the meaning of measurement in psychology. In: Proceedings of the fourth Berkeley symposium on mathematical statistics and probability. Vol IV. Berkeley: Univ.of Calif.Press, 1961, 321-334.