Well-posedness and Asymptotics for Coupled Systems of Plate Equations

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Contents

Chapter 1

Introduction

The deflection of a plate, that is, an elastic material which has a comparatively small thickness, is governed by a *scalar* valued partial differential equation, namely by the *plate equation*

$$u_{tt}(t,x) + \Delta^2 u(t,x) = f(t,x). \tag{1.1}$$

However, in order to model more complex physical or chemical phenomena, *systems* of partial differential equations are needed.
In general, on one hand we are not only interested in describing a scalar value, as in this case the deflection of the plate, but we are interested in the behaviour of a whole collection of different quantities, e.g. three dimensional coordinates in space or the concentration of a chemical substance. On the other hand, for the correct description of these quantities, the modeling of other mutually influencing factors is necessary.

For instance, flows in gases and liquids, i.e. three dimensional vector fields representing the movement of matter at any place and any time, are described by the Navier-Stokes equations. These streams are dependent on the prevailing pressure which itself is influenced by the streams. In a car battery, chemical reactions take place, that is, one needs to describe concentrations of substances while at the same time the resulting electric currents are of interest. All these quantities influence each other and are each modeled by different equations which are coupled to form a whole system of partial differential equations.

In the example of the plate equation, the temperature can influence the physical properties of the plate, while at the same time the deflection of the plate and the resulting stresses and frictions can change this very influence of temperature. When modeling a plate, such as a bridges or parts of a car body, the equation (1.1) is no longer sufficient. The additional description of the temperature leads to so-called *thermoelastic plate equations.* In its simplest form, the thermoelastic plate is given by (see [Lag89])

$$\begin{aligned} u_{tt}(t,x) + \Delta^2 u(t,x) + \Delta\theta(t,x) &= f(t,x), \\ \theta_t(t,x) - \Delta\theta(t,x) - \Delta u_t(t,x) &= g(t,x), \end{aligned} \tag{1.2}$$

a system of two partial differential equations, namely a coupled system of the plate equation (1.1) and the heat equation. Here, $u(t,x)$ describes the deflection of the plate at time t at location x and $\theta(t,x)$ is the current temperature (or the difference to a reference temperature θ_0).

Another way to obtain a system of partial differential equations is to couple two scalar valued equations (or even systems) at a transmission interface, also called transmission layer. This leads to so-called *transmission problems.* Here, different equations (or systems) govern the behaviour

in different subregions of the geometry Ω. At the transmission layers, i.e. at the transitions from one section to another one, these equations are coupled via certain *transmission conditions.*
For example, a plate may have different physical properties in different areas of its geometry due to the nature of the material. We ignore the small thickness of the plate at this point and consider a two-dimensional geometry. While in one subdomain $\Omega_2 \subset \Omega \subset \mathbb{R}^2$ the plate is represented by the equation (1.1), in the rest Ω_1 of the area the *structurally damped plate equation*

$$u_{tt}(t,x) + \Delta^2 u(t,x) - \rho\Delta u_t(t,x) = f(t,x) \tag{1.3}$$

with damping factor $\rho > 0$ might be a more appropriate description of the plate. Using transmission conditions at the transmission interface Γ as well as additional boundary conditions imposed on Γ_1 (cf. Fig. 1.1 below), such a plate with different properties also points to a system of partial differential equations. This system simultaneously describes the deflections of two different plates which influence each other. These examples illustrate the relevance of studying

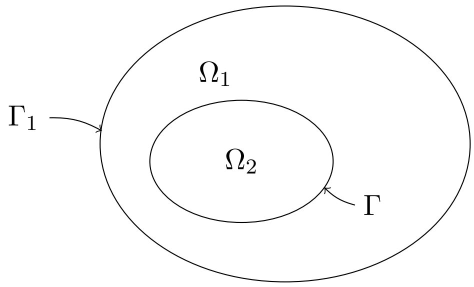


Figure 1.1: *Geometry Ω of the plate, divided in two subdomains Ω_1 and Ω_2, in which the plate possesses different properties.*

systems for applications in physics, chemistry or other sciences. But in fact, even scalar valued equations such as the plate equations (1.1) and (1.3), lead to a system of partial differential equations rather often when treated mathematically: a standard approach for the treatment of *evolution equations*, i.e. time-dependent partial differential equations, is to read the equation as an ordinary differential equation in time. This ordinary differential equation now takes values in a function space X of functions in the space variable. For example, in this case, one would read (1.3) in the form

$$u_{tt}(t) + \Delta^2 u(t) - \rho\Delta u_t(t) = f(t). \tag{1.4}$$

Now, u, u_t, u_{tt} and f are no longer scalar functions. Instead, for all $t \geq 0$, $u(t), u_t(t), u_{tt}(t)$ and $f(t)$ are elements of suitable function spaces.
In applications, these spaces are often L_p Sobolev spaces of the corresponding differentiability order. So, in this case we have $u(t) \in H_p^4(\mathbb{R}^n), u_t(t) \in H_p^2(\mathbb{R}^n)$ as well as $u_{tt}(t), f(t) \in L_p(\mathbb{R}^n)$ for the full-space case $\Omega = \mathbb{R}^n$. Writing (1.4) as a first order system in time

$$U_t - \mathcal{A}U = U_t - \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta \end{pmatrix} U = \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{1.5}$$

with $U = (u, u_t)^\top$, we are able to use powerful tools, as for example the *semigroup theory* (e.g. [Paz83], [EN00]), in order to treat the system (1.5) in the space $X = H^2_p(\mathbb{R}^n) \times L_p(\mathbb{R}^n)$. This will in turn allow us to solve the original equation (1.3) and examine the properties of the solution. Thus, the study of the plate equation can be traced back to the study of the properties of the matrix differential operator of mixed order

$$\mathcal{A}\colon X \supset D(\mathcal{A}) \to X, \quad U \mapsto \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \Delta \end{pmatrix} U$$

with an appropriate domain $D(\mathcal{A}) \subset X$. Hence, for the thermoelastic plate equation (1.2), this approach even yields a 3×3 system of mixed order.
From a mathematical point of view, due to the theories available, systems (in general of mixed order) often present a natural view even for scalar valued equations. Consequently, systems of partial differential equations, usually complemented with boundary values and, in our case, transmission conditions, play an important role in the analysis of partial differential equations and are an elementary object of research.

The main part of this thesis deals with two fundamental questions regarding the above mentioned transmission problems which we will investigate on the basis of plate equations. Apart from their relevance in applications, such systems also serve, as indicated above, as a useful mathematical prototype for mixed-order systems.

The first question has its origin in the theory of *parameter-elliptic boundary value problems*, see for example [ADN59], [AV64], [Ama90], [Ama93], [Ama95], [ADF97] and [DHP03]. This theory can be used to make statements about the solvability of non-linear problems by means of properties of the linear equation, more precisely by properties of the differential operators and the boundary operators. Parameter-elliptic boundary value problems have the so-called property of *maximal L_p-regularity*. This makes it possible to solve non-linear equations by iterative methods on the basis of the linearized equation. Usually, one has to impose smallness conditions on the time horizon or on the initial data due to the use of fixed-point arguments.
In addition to the ability to handle non-linear problems, the maximal L_p-regularity implies, in particular, the analyticity of the corresponding semigroup, which is why parameter-elliptic boundary value problems also have a smoothing property of the solution. Moreover, in bounded domains, such semigroups are exponentially stable. The smoothing property also leads to solutions for initial values with lower regularity than the equation actually requires. As time starts to evolve, these solutions immediately have the required regularity.
Since the theory of parameter-elliptic boundary value problems can not be applied to the systems occurring in this work, we will use the approach via *$\mathcal{R}$-sectorial* solution operators, being equivalent to the property of maximal L_p-regularity (see eg. [Wei01]).
Specifically, we consider the model problem for the coupling of two parabolic plate equations, in this case the structurally damped plate equation

$$\begin{aligned} \partial_t^2 u_1 + \beta_1 \Delta^2 u_1 - \rho_1 \Delta \partial_t u_1 &= f_1 \quad \text{in } (0,\infty) \times \mathbb{R}^n_+, \\ \partial_t^2 u_2 + \beta_2 \Delta^2 u_2 - \rho_2 \Delta \partial_t u_2 &= f_2 \quad \text{in } (0,\infty) \times \mathbb{R}^n_- \end{aligned} \tag{1.6}$$

with canonical transmission conditions

$$\partial_\nu^{j-1} u_1 - \partial_\nu^{j-1} u_2 = 0 \tag{1.7}$$

on $(0,\infty)\times\Gamma=(0,\infty)\times\mathbb{R}^{n-1}$ for $j=1,\dots,4$. Here, $\beta_i,\rho_i>0$ and $f_i\in L_p(\mathbb{R}^n_\pm)$ for $i=1,2$ are given.
The canonical transmission conditions (1.7) state that the composite function $u=\chi_{\mathbb{R}^n_-}u_2+\chi_{\mathbb{R}^n_+}u_1$ belongs to the Sobolev space $H^4_p(\mathbb{R}^n)$, i.e. the solution shall be smooth in the sense of the order of the equation. If one reflects the equation in $\mathbb{R}^n_-$ on the transmission layer Γ, one obtains a boundary value problem in the half-space $\mathbb{R}^n_+$ with four boundary conditions satisfying the so-called *Shapiro-Lopatinskii condition* (cf. [Wlo82] and [DHP03]), which in turn allows us to construct explicit solution operators using the partial Fourier transform. Moreover, we are able to prove appropriate estimates for these solution operators.
As already mentioned, the well-known parameter-elliptic theory can not be applied directly to such a system. This is due to both the mixed order structure of the system and the fact that the functional analytic basis space is not the space L_p but the product space $H^2_p\times L_p$. In fact, even more conditions have to be imposed within the basis space in order to obtain $\mathcal{R}$-sectoriality. We refer to [DD11] for such results.
Although the procedure of using partial Fourier transform and the Shapiro-Lopatinskii condition is analogous to the known theory, the more complex structure of the basis space in combination with the mixed-order structure does not allow a direct transfer of the results of the classical parameter-elliptic theory to such systems.
Generally, the question of $\mathcal{R}$-boundedness of solution operators for systems of boundary and transmission value problems is a central research interest. In the case of parametric-elliptic theory, using this machinery one can show maximal L_p-regularity for a whole class of equations, which, as mentioned before, forms a strong foundation for the study of non-linear equations. Here, the parameter-ellipticity and the Shapiro-Lopatinskii condition together are equivalent to the maximal regularity of the evolution equation.
Correspondingly large are the efforts to set up a similarly meaningful theory for systems. Works in this direction are, for instance, [DF12], [DD11], [DK13] or [KS12], in which necessary and sufficient conditions on the operators and underlying spaces in order to obtain ($\mathcal{R}$-)sectoriality are proven. Specific problems of transmission have also been investigated in the literature in this context. For example, parabolic-elliptic systems appear in the analysis of lithium-ion batteries (see [Seg13], [DS14]).

It would be desirable to have an accurate characterization of systems with matching boundary and transmission conditions that possess the property of maximal L_p-regularity. However, due to the large variability of issues, this seems to be a potentially impossible undertaking. For that reason, it is unavoidable at the present time to have prototypes, as in our case the structurally damped plate equation with canonical transmission conditions, to check for $\mathcal{R}$-sectoriality, and eventually to identify various classes of problems that can be handled similarly. Among other things, this thesis aims to contribute to this topic.

The second central question of this work deals with the properties of solutions of transmission problems, in which equations with different characteristics are coupled.
Being the product of two Schrödinger equations, the plate equation in its simplest form (1.1) does not show either a decay of energy or a smoothing property in the sense of analytic semigroups. In contrast, the thermoelastic plate (1.2) and the structurally damped plate (1.3) possess both. For the thermoelastic plate with so-called *free boundary conditions* this has been shown

in [DS17]. For the structurally damped plate with Dirichlet-Neumann boundary conditions (*clamped boundary conditions*) these results can be found in [CT89] or [DS15]. In fact, these equations even have maximal L_p-regularity, meaning that they have almost all the desirable features.
If we now combine both equations, the question arises which of the positive properties of the thermoelastic or structurally damped plate are transferred to the undamped plate and which properties are lost. If the coupled problem is stable, then the question concerning the rate of decay of energy for the whole system arises.
In this thesis, we investigate the properties of the coupling of the plate equation with the structurally damped plate equation

$$\partial_t^2 u_1 + \Delta^2 u_1 - \rho\Delta\partial_t u_1 = 0 \quad \text{in } (0,\infty)\times\Omega_1, \tag{1.8}$$
$$\partial_t^2 u_2 + \Delta^2 u_2 = 0 \quad \text{in } (0,\infty)\times\Omega_2 \tag{1.9}$$

with *clamped* boundary conditions

$$u_1 = \partial_\nu u_1 = 0 \quad \text{on } \Gamma_1 \tag{1.10}$$

and transmission conditions

$$\begin{aligned} u_1 &= u_2, \\ \partial_\nu u_1 &= \partial_\nu u_2, \\ \Delta u_1 &= \Delta u_2, \\ -\rho\partial_\nu\partial_t u_1 + \partial_\nu\Delta u_1 &= \partial_\nu\Delta u_2 \end{aligned} \tag{1.11}$$

on the transmission interface Γ. In some sense, these transmission conditions are the *natural transmission conditions*, as they yield the dissipativity of the whole system (1.8)-(1.11). All energy decay is caused by the damping term $\rho\Delta\partial_t u_1$.

Similar questions have already been extensively studied in the literature, since the question of stability and of information transfer in the case of control problems for coupled materials is of immense importance from an application point of view.
Not only for plate equations ([Has17], [RO04]), but also in thermo-elasticity theory (such as [MnRN07], [MnRR17]), in the coupling of wave and plate equations ([Her05], [Has16], [LL99], [AG16], [AN10], [CL10]) or transmission problems of heat conduction and wave equations ([RT74], [ZZ07]) the transfer of properties of equations with parabolic character (or equations with dissipative effects) to equations with hyperbolic (or energy-conserving) character was investigated.

However, the question of the transfer of analyticity of one semigroup to the whole system is a surprisingly rare topic in the literature. Apart from indirect results, as in [MnRR17], where the solution of a transmission problem in thermo-elasticity has only polynomial and no exponential decay, implying that the corresponding semigroup can not be analytic according to the theorem of Gearhart-Prüss, one of the few positive results can be found in [SR11]. Here, two thermoelastic plates have been coupled and the authors were able to show analyticity of the semigroup. However, as both equations already have the smoothing property, this result seems

less surprising.
As a part of control theory, transmission problems have also been studied extensively, see for example [LW99], [LW00] and [AV09]. Here questions of stabilization or the exact control of the transmission problem using a control at the outer boundary arise, i.e. one deals with the transmission of information from the outside boundary over various equations that are imposed on different subdomains and are coupled via transmission conditions.

Below we present the structure of this thesis in detail, delimit the topics covered and give an outlook on the main results:

In **Chapter 2**, we cover general properties of systems of partial differential equations.
In the first section, we investigate the necessity of a so-called *parameter-ellipticity in the principal symbol* $A_0(x,\xi)$ of the matrix operator $\mathcal{A} = \mathcal{A}(x,D)$ for generating analytic C_0-semigroups in a very general setting. This easy-to-use tool allows us to show the lack of analyticity, which we use in Chapter 3 when we ask ourselves the question of transferring various properties of parabolic-hyperbolic coupled transmission problems.
The main result Theorem 2.2 shows the necessity of this parameter-ellipticity for the validity of the resolvent estimate

$$\|\lambda(\lambda-\mathcal{A})^{-1}\|_{L(\mathcal{X})} \le C$$

for some $\lambda \in \mathbb{C}$. The two Corollaries 2.4 and 2.5 substantiate the result for generators of semigroups and analytic semigroups. The proof is based on similar arguments presented in [DF12], [ADN59] and [Ama90].

In the second section, we present various perturbation results for sectorial and $\mathcal{R}$-sectorial boundary and transmission value problems, allowing for the study of equations with variable coefficients. Moreover, they pave the way for localization procedures. Especially in the parameter-elliptic theory, one often deals with model problems with constant coefficients in the half-space $\mathbb{R}^n_+$ or full-space $\mathbb{R}^n$, providing important tools as (partial) Fourier transform. Here, such Theorems are unavoidable in order to solve the practically relevant problems with variable coefficients in domains.
Again, the situation is very general: we consider a model transmission problem of the form

$$\begin{aligned}(\lambda-\mathcal{A})u &= f \quad \text{in } \mathbb{R}^n,\\ \mathcal{B}_\gamma u &= g \quad \text{on } \mathbb{R}^{n-1}\end{aligned} \tag{1.12}$$

in certain product spaces of Sobolev space. Here, the operator $\mathcal{A}$ is of the form

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^- & 0 \\ 0 & \mathcal{A}^+ \end{pmatrix},$$

where $\mathcal{A}^-$ and $\mathcal{A}^+$ are matrix differential operators of mixed-order structure and different sizes. The transmission conditions are given by

$$\mathcal{B}_\gamma := \begin{pmatrix} \gamma\mathcal{B}^- & \gamma\mathcal{B}^+ \end{pmatrix} := \begin{pmatrix} \gamma\mathcal{B}_1^- & \gamma\mathcal{B}_1^+ \\ \vdots & \vdots \\ \gamma\mathcal{B}_M^- & \gamma\mathcal{B}_M^+ \end{pmatrix}$$

with γ denoting the trace operator.
For such systems, we examine the transferability of sectoriality, see Proposition 2.16, and the transferability of $\mathcal{R}$-sectoriality, see Proposition 2.18, of the model problem to a modified problem with small perturbations. With that, we are able to consider the problem (1.12) with a curved, sufficiently smooth transmission layer Γ, as the perturbations that occur during localization are covered by the shown results.
If we set, for example, $\mathcal{A}^- = 0$, with (1.12) one can also handle boundary value problems. Thus, we are able to deal with systems of transmission-boundary value problems in smooth, bounded domains.
The approach in this section is similar to [DHP03], Section 7.3, but differs noticeably by the choice of a product space of L_p Sobolev spaces as the base space in contrast to a Banach space valued L_p space. Thus, our setting is much more versatile. This part is based on joint work with Felix Hummel.

Chapter 3 deals with the transmission problem (1.6), (1.7) of two structurally damped plates in the underlying space $H_p^2 \times L_p$ for $1 < p < \infty$ and is based on a corresponding work on the Dirichlet-Neumann boundary value problem in [DS15]. The goal is to show $\mathcal{R}$-sectoriality for the transmission problem, paving the way for non-linear versions of the equation. However, we will not explicitly deal with non-linear equations in this thesis. For this, we proceed analogously to [DS15] and show that the methods used for the boundary value problems are also applicable in order to treat the transmission problem. However, due to the fact that we now obtain four conditions on the transmission interface instead of only two conditions on the boundary, the procedure is even more technical.
Standardly, after reflecting one part of the equation to obtain a boundary value problem, we use partial Fourier transform in order to explicitly construct solution operators. Furthermore, we find suitable $\mathcal{R}$-bounds for certain families of operators in the first selected basis space $H_p^2(\mathbb{R}_+^n) \times L_p(\mathbb{R}_+^n)$, see Theorem 3.11. In this space, the problem is uniquely solvable and we obtain some *a-priori estimates* (Corollary 3.12). Unfortunately, the solution operators are not ($\mathcal{R}$)-sectorial in this space. In particular, we do not get the desired property of maximal L_p-regularity. In order to fix this, further conditions need to be written in the underlying space. To be precise, we need to impose all transmission conditions that are meaningful in the underlying space to the basis space, see Theorem 3.16. This is analogous to the Dirichlet-Neumann boundary value problem, for which the authors in [DS15] have shown $\mathcal{R}$-sectoriality, if these boundary conditions are already realized in the underlying space.
The $\mathcal{R}$-sectoriality obtained in Theorem 3.16 finally leads to a solution of the evolution equation with inhomogeneous data, see Theorem 3.18. With the help of the results from Chapter 2, we can also solve transmission-boundary value problems in sufficiently smooth, bounded domains (cf. Figure 1.1), see Theorem 3.19.

While the parabolic-parabolic coupled transmission problem allows an L_p theoretical access, for coupled systems such as (1.8) - (1.11), which we examine in **Chapter 4**, this is, in general, no longer possible (see [DH18]). Here, the questions of stability, more precisely exponential stability and the analyticity of the corresponding semigroup for the whole system, are at the center of our attention. In this chapter, we will work in a suitable Hilbert space setting.
Using energy methods, the parameter-ellipticity of the structurally damped plate equation and

the theory of *interpolation extrapolation scales* (see [Ama95]), we can prove an exponential decay of energy under low geometric conditions (Theorem 4.20) for the system (1.8) - (1.11). In particular, we do not need a geometric condition on the transmission layer Γ, which is often assumed in the literature.
By contrast, due to the results in Chapter 2, the semigroup associated to the whole system now lacks the property of analyticity. This is shown in Proposition 4.31. In addition, we illustrate the lack of smoothing by using a numerical simulation for the one-dimensional transmission problem, which suggests that the associated spectrum does not lie in a sector with opening angle $\vartheta > \frac{\pi}{2}$ of the complex plane, which is a necessary condition.
Both the statements about exponential stability and the absence of the analyticity of the semigroup can be transferred to plate equations in which the damping is distributed via a C^4 function $\rho\colon \overline{\Omega} \to [0,\infty)$ over the domain, see Theorem 4.30 as well as Proposition 4.32.
In our case, we always consider a bounded domain Ω (see also Figure 1.1). In this situation, the resolvent of the operators are compact which leads to statements about the regularity of the solutions (as in Lemma 4.5) which we use frequently within the proofs. Nevertheless, results like in [MN01] and [MN07] for the wave equation suggest that a certain decay behaviour of the solution may also apply to unbounded domains, provided that the damping is active at infinity. However, we will not deal with such problems at this point.
Parts of this chapter are based on a joint work with Prof. Dr. Robert Denk and were published in [DK18].

In the **Appendix**, we briefly summarize some results on the topics Sobolev spaces, Fourier multipliers, $\mathcal{R}$-sectoriality, maximal L_p-regularity as well as boundary value problems for ordinary differential equations. This serves mainly to increase the readability of the thesis.
Relevant literature is referenced in the appropriate places.

Chapter 2

Ellipticity and localization for systems of transmission and boundary value problems

This chapter deals with quite general questions concerning systems of transmission and boundary value problems in the context of analytic semigroups and ($\mathcal{R}$)-sectorial solution operators. First, we give a necessary condition for the generation of an analytic semigroup, which will turn out to be an easy tool in order to proof lack of analyticity for a semigroup, see Chapter 4.
In the second section we deal with perturbation results for ($\mathcal{R}$)-sectorial transmission and boundary value problems, allowing localization procedures in order to reduce systems in domains with variable coefficients to model problems in the full- or half-space with constant coefficients, respectively. For such model problems, tools like (partial) Fourier transform are available. With such tools, partial differential equations can be handled by solving ordinary differential equations, see Chapter 3.

2.1. On the necessity of an ellipticity condition for a resolvent estimate

Let $\Omega \subset \mathbb{R}^n$ be a domain with boundary $\partial\Omega$. Note that Ω might be unbounded and that the boundary $\partial\Omega$ may consist of several connected components.

Let $N \in \mathbb{N}$ and $s_1, \ldots, s_N \geq 0, t_1, \ldots, t_N > 0$. For $1 < p < \infty$, let X_p, Y_p be the Banach spaces defined by

$$X_p := \prod_{j=1}^{N} H_p^{s_j}(\Omega), \quad Y_p := \prod_{j=1}^{N} H_p^{s_j+t_j}(\Omega)$$

where $H_p^s(\Omega) = W_p^s(\Omega)$ denotes the classical Sobolev space of order $s \in \mathbb{N}_0$. In a canonical way, we equip the spaces X_p and Y_p with the norms $\|U\|_{X_p} := \sum_{j=1}^{N} \|u_j\|_{H_p^{s_j}(\Omega)}$ for $U = (u_1, \ldots, u_N)^\top \in X_p$ and $\|\cdot\|_{Y_p}$ defined analogously.
Using standard multi-index notation

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad \xi^\alpha = \prod_{i=1}^{n} \xi_i^{\alpha_i}, \quad D^\alpha = (-\mathrm{i}\partial_1)^{\alpha_1} \cdot \ldots \cdot (-\mathrm{i}\partial_n)^{\alpha_n} \quad (\alpha \in \mathbb{N}_0^n, \xi \in \mathbb{R}^n)$$

and $\beta \le \alpha \Leftrightarrow (\forall i \in \{1,\dots,n\} : \beta_i \le \alpha_i)$ for $\alpha, \beta \in \mathbb{N}_0^n$, we consider a quadratic matrix differential operator $\mathcal{A}$ of the form

$$\mathcal{A} = \mathcal{A}(x,D) = \Big(A(x,D)_{jk}\Big)_{j,k=1}^{N} \qquad (x \in \Omega) \tag{2.1}$$

where the entries are scalar valued differential operators of the form

$$A(x,D)_{jk} = \sum_{|\alpha| \le m_{jk}} a_\alpha^{jk}(x) D^\alpha \qquad (x \in \Omega).$$

Here, $a_\alpha^{jk} : \Omega \to \mathbb{C}$ are bounded, sufficiently smooth functions on $\overline{\Omega}$ and $m_{jk} \in \mathbb{N}_0 \cup \{-\infty\}$ denotes the order $\operatorname{ord}(A(x,D)_{jk})$ of the scalar valued differential operator $A(x,D)_{jk}$ for $1 \le j,k \le N$, where $\operatorname{ord}(0) := -\infty$.
We assume that the orders m_{jk} satisfy the estimate

$$m_{jk} \le s_k + t_k - s_j \qquad (1 \le j,k \le N),$$

which implies that $\mathcal{A}\colon Y_p \to X_p$ is a bounded linear operator, i.e. $\mathcal{A} \in L(Y_p, X_p)$.

As usual, if E, F are Banach spaces, we denote by $L(E,F)$ the Banach space of all bounded linear operators $T\colon E \to F$ equipped with the operator norm and we set $L(E) := L(E,E)$. The set $L_{\text{is}}(E,F)$ defined by

$$L_{\text{is}}(E,F) := \{T \in L(E,F) : T \text{ is bijective and } T^{-1} \in L(F,E)\} \subset L(E,F)$$

is an open subset of $L(E,F)$. We write $E \hookrightarrow F$ provided that E is continuously embedded in F.

Let $\mathcal{X}$ be a Banach space and $D(\mathcal{A}) \subset \mathcal{X}$ be a sub space such that $\mathcal{X} \hookrightarrow X_p$ and $D(\mathcal{A}) \hookrightarrow Y_p$. In the following, we will assume that the norm on $\mathcal{X}$ is given by the norm of X_p and that $\mathcal{A}\colon \mathcal{X} \supset D(\mathcal{A}) \to \mathcal{X}$ is a well-defined, closed operator. In this case, $D(\mathcal{A})$ equipped with the graph norm $\|U\|_{D(\mathcal{A})} := \|\mathcal{A}U\|_{X_p} + \|U\|_{X_p}$ is a Banach space.
We are interested in necessary conditions on $\mathcal{A}$ for resolvent estimates, connected with analytic semigroups, to hold. In order to formulate such a result, we define the symbol $A(x,\xi)$ of $\mathcal{A}(x,D)$ by setting

$$A(x,\xi) := \Big(A(x,\xi)_{jk}\Big)_{j,k=1}^{N} \qquad (x \in \overline{\Omega}, \xi \in \mathbb{R}^n)$$

where

$$A(x,\xi)_{jk} := \sum_{|\alpha| \le m_{jk}} a_\alpha^{jk}(x) \xi^\alpha \qquad (x \in \overline{\Omega}, \xi \in \mathbb{R}^n)$$

is the symbol of the differential operator $A(x,D)_{jk}$. We call $\mathcal{A}_0 = \mathcal{A}_0(x,D)$ the principal part of $\mathcal{A}$, given by

$$\mathcal{A}_0 := \Big(A_0(x,D)_{jk}\Big)_{j,k=1}^{N} \qquad (x \in \overline{\Omega})$$

with

$$A_0(x,D)_{jk} := \begin{cases} \sum_{|\alpha|=m_{jk}} a_\alpha^{jk}(x)D^\alpha, & m_{jk} = s_k + t_k - s_j, \\ 0 & \text{else.} \end{cases} \qquad (x \in \overline{\Omega})$$

Hence, $\mathcal{A}_0$ is the differential operator matrix where every entry (j,k) consists of those terms of $A(x,D)_{jk}$ that are exactly of order $s_k+t_k-s_j$. The principal symbol $A_0(x,\xi)$ of $\mathcal{A}(x,D)$ is defined as the symbol of $\mathcal{A}_0(x,D)$ and we denote the principal symbol of the scalar valued differential operator $A(x,D)_{jk}$ by $A_0(x,\xi)_{jk}$, i.e. $A_0(x,\xi)_{jk}$ is defined as the symbol of $A_0(x,D)_{jk}$.

2.1 Example. Consider the operator matrix

$$\mathcal{A}\colon H^2(\Omega)\times L_2(\Omega) \supset H^4(\Omega)\times H^2(\Omega) \to H^2(\Omega)\times L_2(\Omega)$$

given by

$$\mathcal{A}(D) := \begin{pmatrix} \partial_1 & 1 \\ \Delta^2+\Delta & -\Delta \end{pmatrix}.$$

Here, we have $s_1 = 2, s_2 = 0$ and $t_1 = t_2 = 2$. Therefore, the principal part $\mathcal{A}_0$ of $\mathcal{A}$ is given by the operator matrix

$$\mathcal{A}_0(D) = \begin{pmatrix} 0 & 1 \\ \Delta^2 & -\Delta \end{pmatrix}$$

with principal symbol

$$A_0(\xi) = \begin{pmatrix} 0 & 1 \\ |\xi|^4 & |\xi|^2 \end{pmatrix}.$$

For $\theta \in [0,2\pi)$, let

$$\mathcal{L}_\theta := \{z \in \mathbb{C} : \arg z = \theta\}$$

be the ray of angle θ in the complex plane. For $\kappa \geq 0$ we define

$$\mathcal{L}_{\theta,\kappa} := \{z \in \mathcal{L}_\theta : |z| \geq \kappa\}$$

to be the ray of angle θ in the complex plane, starting at $\kappa e^{\mathrm{i}\theta}$.
By $\mathscr{D}(U,\mathbb{C}^N)$ we denote the $\mathbb{C}^N$-valued test functions on a set $U \subset \mathbb{R}^n$, that is we let $\mathscr{D}(U,\mathbb{C}^N) = C_0^\infty(U,\mathbb{C}^N)$ be the locally convex space of all infinitely often differentiable functions $\varphi\colon U \to \mathbb{C}^N$ with compact support in U.

Let $\partial\Omega$ be of class C^μ with $\mu := \max\{s_j + t_j : j = 1,\dots,N\}$ and assume that the coefficients $a_\alpha^{jk} \in L_\infty(\overline{\Omega})$ satisfy $a_\alpha^{jk} \in C_b^{s_j}(\overline{\Omega})$ for all $1 \leq j,k \leq N$ and $|\alpha| \leq m_{jk}$, where, as usual,

$$C_b^k(U) := \{f \in C^k(U) : D^\alpha f \in L_\infty(U) \text{ for all } |\alpha| \leq k\}$$

is the space of bounded, $k \in \mathbb{N}_0$ times differentiable functions $f\colon U \to \mathbb{C}$ with bounded derivatives. Then we have the following theorem:

2.2 Theorem. *Let $\mathcal{X}$ be a Banach space and $D(\mathcal{A}) \subset \mathcal{X}$ be a sub space with $\mathcal{X} \hookrightarrow X_p$ and $D(\mathcal{A}) \hookrightarrow Y_p$. Assume that $\mathcal{A}\colon \mathcal{X} \supset D(\mathcal{A}) \to \mathcal{X}$ is a well-defined, closed operator of the form (2.1), where $\mathcal{X}$ is equipped with the canonical norm on X_p. Moreover, the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ satisfies $\mathcal{L}_{\theta,\kappa} \subset \rho(\mathcal{A})$ for some $\kappa \geq 0$ and $\theta \in [0, 2\pi)$. Finally, assume that there exists some $\omega \geq 0$ such that $(\omega - \mathcal{A}) \in L_{\text{is}}\big((D(\mathcal{A}), \|\cdot\|_{Y_p}), \mathcal{X}\big)$.*
If the resolvent estimate

$$\|\lambda(\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{X})} \leq C \quad (\lambda \in \mathcal{L}_{\theta,\kappa}) \tag{2.2}$$

holds true for some constant $C > 0$, then we have

$$\det(\lambda - A_0(x, \xi)) \neq 0 \tag{2.3}$$

for all $\lambda \in \mathcal{L}_{\theta,\kappa}$, $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof. Assume that the resolvent estimate (2.2) holds true.
Since $(\omega - \mathcal{A}) \in L_{\text{is}}\big((D(\mathcal{A}), \|\cdot\|_{Y_p}), \mathcal{X}\big)$ and recalling that $\mathcal{X}$ is equipped with the norm on X_p, we can then estimate

$$\begin{aligned}
\|U\|_{Y_p} &\leq C\|(\omega - \mathcal{A})(\lambda - \mathcal{A})^{-1}(\lambda - \mathcal{A})U\|_{X_p} \\
&= C\|(\omega - \lambda + \lambda - \mathcal{A})(\lambda - \mathcal{A})^{-1}(\lambda - \mathcal{A})U\|_{X_p} \\
&= C\|((\omega - \lambda)(\lambda - \mathcal{A})^{-1} + I)(\lambda - \mathcal{A})U\|_{X_p} \\
&\leq C\left(\frac{|\omega - \lambda|}{|\lambda|} + 1\right)\|(\lambda - \mathcal{A})U\|_{X_p} \\
&\leq C\left(\frac{|\omega|}{|\lambda|} + 2\right)\|(\lambda - \mathcal{A})U\|_{X_p} \\
&\leq C\|(\lambda - \mathcal{A})U\|_{X_p}
\end{aligned}$$

for all $\lambda \in \mathcal{L}_{\theta,\kappa}$ with $|\lambda| \geq |\omega|$ and all $U \in Y_p$. Using this estimate as well as

$$|\lambda|\|U\|_{X_p} \leq |\lambda|\|(\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{X})}\|(\lambda - \mathcal{A})U\|_{X_p} \leq C\|(\lambda - \mathcal{A})U\|_{X_p}$$

for $\lambda \in \mathcal{L}_{\theta,\kappa}$ and $U \in Y_p$, we obtain the estimate

$$|\lambda|\|U\|_{X_p} + \|U\|_{Y_p} \leq C\|(\lambda - \mathcal{A})U\|_{X_p} \quad (\lambda \in \mathcal{L}_{\theta,\kappa},\, U \in Y_p). \tag{2.4}$$

In what follows, we will show that if there are λ_0, x_0, ξ^0 such that $\det(\lambda_0 - A_0(x_0, \xi^0)) = 0$, then (2.4) cannot hold true and in turn, the resolvent estimate (2.2) cannot hold true.
In order to do so, let us assume that there are $\lambda_0 \in \mathcal{L}_{\theta,\kappa}, x_0 \in \overline{\Omega}$ and $\xi^0 \in \mathbb{R}^n \setminus \{0\}$ such that $\det(\lambda_0 - A_0(x_0, \xi^0)) = 0$. Choose a vector $h = (h_1, \dots, h_N)^\top \in \mathbb{C}^N \setminus \{0\}$ with

$$(\lambda_0 - A_0(x_0, \xi^0))h = 0.$$

We set $a_\alpha^{jk} = 0$ if $m_{jk} < s_k + t_k - s_j$. For given $\varepsilon > 0$ we fix a neighbourhood $U = U(x_0)$ of x_0 such that

$$|a_\alpha^{jk}(x) - a_\alpha^{jk}(x_0)| < \frac{\varepsilon}{2} \quad (x \in U(x_0) \cap \overline{\Omega})$$

holds for all $1 \le j,k \le N$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = s_k + t_k - s_j$. Here, we used the continuity of the top order coefficients a_α^{jk} in every entry (j,k) of the operator matrix $\mathcal{A}$.
Now we choose $y_0 \in U(x_0) \cap \Omega$ and $\delta > 0$ such that the open ball

$$B(y_0,\delta) := \{y \in \mathbb{R}^n : |y - y_0| < \delta\}$$

with center y_0 and radius δ satisfies

$$B(y_0,\delta) \subset \Omega \cap U(x_0).$$

Note that we can choose $y_0 = x_0$ if $x_0 \in \Omega$.
For $U = (u_1,\dots,u_N)^\top \in \mathscr{D}(B(y_0,\delta),\mathbb{C}^N) \subset Y_p$ and $j = 1,\dots,N$ we can estimate

$$\begin{aligned}
\|[(\mathcal{A} - \mathcal{A}_0(x_0))U]_j\|_{H_p^{s_j}(\Omega)} &\le \left\| \sum_{k=1}^N \sum_{|\alpha|=s_k+t_k-s_j} \left(a_\alpha^{jk}(\cdot) - a_\alpha^{jk}(x_0)\right) D^\alpha u_k \right\|_{H_p^{s_j}(\Omega)} \\
&\quad + \left\| \sum_{k=1}^N \sum_{|\alpha|<s_k+t_k-s_j} a_\alpha^{jk}(\cdot) D^\alpha u_k \right\|_{H_p^{s_j}(\Omega)} \\
&\le \frac{\varepsilon}{2} \sum_{k=1}^N \|u_k\|_{H_p^{s_k+t_k}(\Omega)} + C \sum_{k=1}^N \|u_k\|_{H_p^{s_k+t_k-1}(\Omega)} \\
&\le \frac{\varepsilon}{2}\|U\|_{Y_p} + \frac{\varepsilon}{2}\|U\|_{Y_p} + C(\tfrac{\varepsilon}{2})\|U\|_{X_p}.
\end{aligned}$$

Here, we used interpolation theory in the last step. Hence, for all $U \in \mathscr{D}(B(y_0,\delta),\mathbb{C}^N)$, the estimate (2.4) reads

$$\begin{aligned}
|\lambda|\|U\|_{X_p} + \|U\|_{Y_p} &\le C\|(\lambda - \mathcal{A})U\|_{X_p} \\
&\le C\|(\mathcal{A} - \mathcal{A}_0(x_0))U\|_{X_p} + C\|(\lambda - \mathcal{A}_0(x_0))U\|_{X_p} \\
&\le \varepsilon\|U\|_{Y_p} + C\|U\|_{X_p} + C\|(\lambda - \mathcal{A}_0(x_0))U\|_{X_p}
\end{aligned}$$

for all $\lambda \in \mathcal{L}_{\theta,\kappa}$. Choosing $\varepsilon = \frac{1}{2}$, this implies

$$|\lambda|\|U\|_{X_p} + \|U\|_{Y_p} \le C'\|(\lambda - \mathcal{A}_0(x_0))U\|_{X_p} \tag{2.5}$$

for all $\lambda \in \mathcal{L}_{\theta,\kappa}$ with $|\lambda| > 2C$ and $U \in \mathscr{D}(B(y_0,\delta),\mathbb{C}^N)$.
For simplicity, set $B := B(y_0,\delta)$ and let $\varphi \in \mathscr{D}(B,\mathbb{C}^N)$ be a test function on B with $\varphi \ge 0$ and $\varphi \ne 0$. For all positive parameters $\tau > 0$ we define $U_\tau = (u_\tau^1,\dots,u_\tau^N)^\top \in \mathscr{D}(B,\mathbb{C}^N)$ by setting

$$u_\tau^j(x) := \varphi(x) e^{\mathrm{i}\tau\xi^0 x} h_j \tau^{-(s_j+t_j)} \qquad (x \in \Omega)$$

for $j \in \{1,\dots,N\}$. We show estimates for the left- and right-hand side of (2.5) which will yield a contradiction to this very estimate.
For all $j \in \{1,\dots,N\}$ we have, using equivalence of norms on finite dimensional spaces,

$$\begin{aligned}
\|u_\tau^j\|_{H_p^{s_j+t_j}(\Omega)} &\ge C \sum_{|\alpha|\le s_j+t_j} \|D^\alpha u_\tau^j\|_{L_p(\Omega)} \\
&= C \sum_{|\alpha|\le s_j+t_j} \left\| \sum_{\beta\le\alpha} \binom{\alpha}{\beta} (D^{\alpha-\beta}\varphi)(D^\beta e^{\mathrm{i}\tau\xi^0\cdot}) h_j \tau^{-(s_j+t_j)} \right\|_{L_p(\Omega)} \\
&\ge C\|\varphi\|_{L_p(\Omega)} |h_j| |\xi_l^0|^{s_j+t_j} - C\tau^{-1},
\end{aligned}$$

where $l \in \{1, \dots N\}$ satisfies $\xi_l^0 \neq 0$. Hence, it follows that

$$\|U_\tau\|_{Y_p} = \gamma + \mathcal{O}(\tau^{-1})$$

for some constant $\gamma > 0$ only depending on $\varphi \in \mathscr{D}(B, \mathbb{C}^N), h \in \mathbb{C}^N \setminus \{0\}$ and $\xi^0 \in \mathbb{R}^n \setminus \{0\}$. Moreover, we clearly have that $\|u_\tau^j\|_{H_p^{s_j}(\Omega)} = \mathcal{O}(\tau^{-t_j})$ for $j \in \{1, \dots, \mathrm{N}\}$ and hence, using $t_j > 0$ it follows that

$$\|U_\tau\|_{X_p} = \mathcal{O}\Big(\tau^{-\min\{t_1,\dots,t_N\}}\Big) = \mathcal{O}(\tau^{-1}).$$

On the other hand, for $j \in \{1, \dots, N\}$ we calculate

$$\begin{aligned}
[\mathcal{A}_0(x_0)U_\tau]_j &= \sum_{k=1}^N A_0(x_0, D)_{jk} u_\tau^k \\
&= \sum_{k=1}^N \sum_{|\alpha| = s_k + t_k - s_j} a_\alpha^{jk}(x_0) D^\alpha \Big(\varphi e^{\mathrm{i}\tau\xi^0 \cdot} h_k \tau^{-(s_k+t_k)}\Big) \\
&= \sum_{k=1}^N \sum_{|\alpha| = s_k + t_k - s_j} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} a_\alpha^{jk}(x_0)(D^{\alpha-\beta}\varphi)(D^\beta e^{\mathrm{i}\tau\xi^0\cdot}) h_k \tau^{-(s_k+t_k)} \\
&= \sum_{k=1}^N \sum_{|\alpha| = s_k + t_k - s_j} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} a_\alpha^{jk}(x_0)(D^{\alpha-\beta}\varphi)\tau^{|\beta|}(\xi^0)^\beta e^{\mathrm{i}\tau\xi^0\cdot} h_k \tau^{-(s_k+t_k)} \\
&= \sum_{k=1}^N \sum_{|\alpha| = s_k + t_k - s_j} \Big(a_\alpha^{jk}(x_0)\tau^{|\alpha|}(\xi^0)^\alpha \varphi e^{\mathrm{i}\tau\xi^0\cdot} h_k \tau^{-(s_k+t_k)} \\
&\qquad + \sum_{\beta < \alpha} \binom{\alpha}{\beta} a_\alpha^{jk}(x_0)(D^{\alpha-\beta}\varphi)\tau^{|\beta|}(\xi^0)^\beta e^{\mathrm{i}\tau\xi^0\cdot} h_k \tau^{-(s_k+t_k)} \Big) \\
&= \tau^{-s_j} \varphi e^{\mathrm{i}\tau\xi^0\cdot} [A_0(x_0, \xi^0)h]_j + \mathcal{O}(\tau^{-1}).
\end{aligned}$$

Let us first assume that $\lambda_0 = 0$, i.e. $A_0(x_0, \xi^0)h = 0$. From what we have just seen, we obtain that $\mathcal{A}_0(x_0)U_\tau$ behaves like $\mathcal{O}(\tau^{-1})$ in this case. Consequently, using $\|U_\tau\|_{X_p} = \mathcal{O}(\tau^{-1})$, the estimate

$$\|(\lambda - \mathcal{A}_0(x_0))U_\tau\|_{X_p} \leq |\lambda| \|U_\tau\|_{X_p} + \mathcal{O}(\tau^{-1}) = \mathcal{O}(\tau^{-1})$$

for some fixed $\lambda \in \mathcal{L}_{\theta,\kappa}$ follows. Together with the fact that $\|U_\tau\|_{Y_p} = \gamma + \mathcal{O}(\tau^{-1})$, this yields a contradiction to (2.5) for $\tau \to \infty$.

In the case $\lambda_0 \neq 0$, we set $t := \max\{t_1, \dots, t_N\} + 1 \geq 2$ and define $\lambda_\tau := \tau^t \lambda_0$ for $\tau > 0$. Then, similar as above, we have

$$\begin{aligned}
\|[(\lambda_\tau - \mathcal{A}_0(x_0))U_\tau]_j\|_{H_p^{s_j}(\Omega)} &\leq \|\lambda_\tau u_\tau^j - \lambda_0 \tau^{-s_j} \varphi e^{\mathrm{i}\tau\xi^0\cdot} h_j\|_{H_p^{s_j}(\Omega)} + \mathcal{O}(\tau^{-1}) \\
&= \|\lambda_\tau u_\tau^j - \tau^{t+t_j-t} \lambda_0 \tau^{-(s_j+t_j)} \varphi e^{\mathrm{i}\tau\xi^0\cdot} h_j\|_{H_p^{s_j}(\Omega)} + \mathcal{O}(\tau^{-1}) \\
&= |\lambda_\tau| \|u_\tau^j - \tau^{t_j - t} u_\tau^j\|_{H_p^{s_j}(\Omega)} + \mathcal{O}(\tau^{-1}) \\
&= |\lambda_\tau| |1 - \tau^{t_j - t}| \|u_\tau^j\|_{H_p^{s_j}(\Omega)} + \mathcal{O}(\tau^{-1})
\end{aligned}$$

for $j \in \{1, \dots, N\}$, where now we have used $A_0(x_0, \xi^0)h = \lambda_0 h$. For $\tau \to \infty$, this implies

$$\|(\lambda_\tau - \mathcal{A}_0(x_0))U_\tau\|_{X_p} \le |\lambda_\tau| \|U_\tau\|_{X_p} + \mathcal{O}(\tau^{-1}),$$

again leading to a contradiction to (2.5). □

2.3 Remark. Similar results concerning the necessity of the so called *parameter-ellipticity* for *a-priori* estimates for solutions of boundary value problems have been studied in the literature. Probably the best known result for scalar boundary value problems can be found in [ADN59]. Another similar result can be found in [Ama90], dealing with systems of reaction-diffusion equations. For systems of mixed-order boundary value problems, we refer to [DF12] where the necessity of parameter ellipticity for a-priori estimates in Sobolev spaces with negative order to hold was proven.
Following the naming in the literature, we might refer to (2.3) as *parameter-elliptic in the principal part.*
Both the authors in [ADN59] and [DF12] also prove the necessity of the so called *complementing* or *Shapiro-Lopatinskii condition*, which is a condition on the boundary operators. In Chapter 3, we will use this condition in order to show $\mathcal{R}$-sectoriality for a certain transmission value problem, i.e. we will use the sufficiency of this condition in a concrete context. For now, we are mostly interested in a criterion concerning non-analyticity of semigroups, which we will use in Chapter 4. Hence, we will not deal with the question of necessity of the Shapiro-Lopatinskii condition for (2.2) to hold.

The following two corollaries specify Theorem 2.2 to the setting of generators of C_0-semigroups. In particular, we obtain a condition for the lack of analyticity of a semigroup.

2.4 Corollary. *Let $\mathcal{X}$ be a Banach space and $D(\mathcal{A}) \subset \mathcal{X}$ be a sub space with $\mathcal{X} \hookrightarrow X_p$ and $D(\mathcal{A}) \hookrightarrow Y_p$. Assume that $\mathcal{A}$ is of the form* (2.1) *and that $\mathcal{A}\colon \mathcal{X} \supset D(\mathcal{A}) \to \mathcal{X}$ is the generator of a C_0-semigroup on $\mathcal{X}$ equipped with the canonical norm on X_p. Furthermore, we assume that the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ satisfies $\mathcal{L}_{\theta,\kappa} \subset \rho(\mathcal{A})$ for some $\kappa \ge 0$ and $\theta \in [0, 2\pi)$.*
If the resolvent estimate (2.2) *holds true for some constant $C > 0$, then* (2.3) *holds for all $\lambda \in \mathcal{L}_{\theta,\kappa}$, $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.*

Proof. By the theorem of Hille-Yosida in the general form (e.g. [Paz83], Corollary 3.8), $\mathcal{A}$ is a closed operator and there exists some $\omega > 0$ with $[\omega, \infty) \subset \rho(\mathcal{A})$. Hence, $(\omega - \mathcal{A})^{-1} \in L(\mathcal{X})$. Using $D(\mathcal{A}) \hookrightarrow Y_p$, we obtain

$$\begin{aligned}
\|U\|_{Y_p} &= \|(\omega - \mathcal{A})^{-1}(\omega - \mathcal{A})U\|_{Y_p} \\
&\le C\|(\omega - \mathcal{A})^{-1}(\omega - \mathcal{A})U\|_{D(\mathcal{A})} \\
&= C\left(\|\mathcal{A}(\omega - \mathcal{A})^{-1}(\omega - \mathcal{A})U\|_{X_p} + \|U\|_{X_p}\right) \\
&= C\left(\|(\omega - \omega + \mathcal{A})(\omega - \mathcal{A})^{-1}(\omega - \mathcal{A})U\|_{X_p} + \|U\|_{X_p}\right) \\
&\le C(\omega)\left(\|(\omega - \mathcal{A})U\|_{X_p} + \|U\|_{X_p}\right)
\end{aligned}$$

for all $U \in D(\mathcal{A})$ and hence $(\omega - \mathcal{A})^{-1} \in L(\mathcal{X}, (D(\mathcal{A}), \|\cdot\|_{Y_p}))$.
Since $(\omega - \mathcal{A}) \in L((D(\mathcal{A}), \|\cdot\|_{Y_p}), \mathcal{X})$, it follows that $(\omega - \mathcal{A}) \in L_{\text{is}}\left((D(\mathcal{A}), \|\cdot\|_{Y_p}), \mathcal{X}\right)$. Now, the assertion follows from Theorem 2.2. □

In particular, taking $\theta \in [0, \frac{\pi}{2}]$, we obtain the following result for generators of analytic semigroups:

2.5 Corollary. *In the setting of Corollary 2.4, assume that $\mathcal{A}$ is the generator of an analytic C_0-semigroup on $\mathcal{X}$ equipped with the canonical norm on X_p.*
Then, there exists some $\kappa \geq 0$ such that (2.3) *holds for all $\lambda \in \{z \in \mathbb{C} \setminus \{0\} : \operatorname{Re} z \geq 0, |z| \geq \kappa\}$, $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$.*

Proof. Using the well-known fact that an operator $A \colon X \supset D(A) \to X$ in a Banach space is the generator of an analytic semigroup on X if and only if $\rho(A)$ contains a shifted open sector $\kappa + \Sigma_\vartheta$ (where $\Sigma_\vartheta = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \vartheta\}$) with $\vartheta > \frac{\pi}{2}$ and

$$\|\lambda(\lambda - A)^{-1}\|_{L(X)} \leq C \quad (\lambda \in \kappa + \Sigma_\vartheta)$$

(e.g. [Lun95]), this follows directly from Corollary 2.4. □

In the case of transmission problems, the domain is separated in two parts $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ where Γ is the common interface of Ω_1 and Ω_2. As we see in the proof, we can apply Theorem 2.2 as well as Corollary 2.4 and Corollary 2.5 also in this situation: let $\Omega_1, \dots \Omega_N$ be domains such that for $k, l \in \{1, \dots N\}$ either $\Omega_k = \Omega_l$ or Ω_k and Ω_l are disjoint.
Now consider the spaces X_p and Y_p defined by

$$X_p := \prod_{j=1}^{N} H_p^{s_j}(\Omega_j), \quad Y_p := \prod_{j=1}^{N} H_p^{s_j+t_j}(\Omega_j).$$

Instead of coefficients $a_\alpha^{jk} \colon \Omega \to \mathbb{C}$ we now consider coefficients

$$a_\alpha^{jk} \colon \Omega_k \to \mathbb{C} \quad (1 \leq j, k \leq N).$$

Then, we can formulate the same results, namely Theorem 2.2, Corollary 2.4 and Corollary 2.5 with the spaces X_p and Y_p we have just introduced. In the proof we extend the test function $\varphi \in \mathscr{D}(B, \mathbb{C}^N)$ to the set $\Omega := \bigcup_{j=1}^{N} \Omega_j$, i.e. we choose $\varphi \in \mathscr{D}(B, \mathbb{C}^N) \cap \mathscr{D}(\Omega, \mathbb{C}^N)$.

2.2. Perturbation theorems for sectorial and $\mathcal{R}$-sectorial systems

The situation in this section is quite similar to the situation above. However, this time we include transmission conditions and ask for permissible perturbations both for the operator as well as for the transmission conditions in order to preserve properties like sectoriality and $\mathcal{R}$-sectoriality. We refer to the appendix, in particular to Section A.2 and Section A.3 and especially to the references mentioned there, for a short review on this topic. However, for convenience reasons, we will state definitions of sectoriality and $\mathcal{R}$-sectoriality below.
Note that we only consider transmission conditions as boundary conditions can be constructed by writing a boundary value problem as a transmission problem where the equation on one side is a trivial equation.
As mentioned before, such perturbations theorems are the basis for localization procedures, allowing us to transfer sectoriality and $\mathcal{R}$-sectoriality of model problems to problems with variable coefficients on domains, which yields generation results for semigroups, analyticity of the semigroup and maximal L_p-regularity, respectively. For the application of such localization methods we also refer to [DS15], [DS17], [DHP03] and Chapter 3.

As promised, we remind on the definition of sectorial and $\mathcal{R}$-sectorial operators:

2.6 Definition. Let X, Y be a Banach space.

a) A family $\mathcal{T} \subset L(X,Y)$ of operators is called $\mathcal{R}$-bounded if there exists a constant $C > 0$ such that for all $m \in \mathbb{N}$, $T_1, \dots T_m \in \mathcal{T}$ and $x_1, \dots x_m \in X$ we have

$$\Big\| \sum_{k=1}^{m} r_k T_k x_k \Big\|_{L_p([0,1];Y)} \leq C \Big\| \sum_{k=1}^{m} r_k x_k \Big\|_{L_p([0,1];X)},$$

where the *Rademacher functions* r_k for $k \in \mathbb{N}$ are given by

$$r_k \colon [0,1] \to \{-1,1\},\ t \mapsto \operatorname{sign}(\sin(2^k \pi t)).$$

b) Let $A \colon X \supset D(A) \to X$ be a closed operator. A is called $(\mathcal{R})$-sectorial if A has dense domain and dense range, and if there exists an angle $\vartheta \in (0,\pi)$ such that

$$\rho(A) \supset \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \pi - \vartheta\} =: \Sigma_{\pi-\vartheta}$$

and the set $\{\lambda(\lambda - A)^{-1} : \lambda \in \Sigma_{\pi-\vartheta}\} \subset L(X)$ is $(\mathcal{R})$-bounded.
The *angle of ($\mathcal{R}$-)boundedness* is defined as the infimum of all ϑ for which this holds.

Both sectoriality and $\mathcal{R}$-sectoriality are desirable properties for operators as they imply analyticity of the corresponding semigroup or even maximal L_p-regularity. The aim of this section is to prove conditions on perturbations such that $(\mathcal{R})$-sectorial operators keep this property under those perturbations.

Let E be a Banach space of class $\mathcal{HT}$ and $N^+, N^- \in \mathbb{N}$. We consider quadratic matrices of differential operators $\mathcal{A}^\pm$ of the form

$$\mathcal{A}^\pm = \mathcal{A}^\pm(x,D) = \Big(A^\pm_{jk}(x,D) \Big)_{j,k=1}^{N^\pm} \qquad (x \in \mathbb{R}^n_\pm) \tag{2.6}$$

where the entries are differential operators of the form

$$A^{\pm}_{jk}(x,D) = \sum_{|\alpha|\le m^{\pm}_{jk}} a^{\pm}_{jk,\alpha}(x)D^{\alpha} \qquad (x\in\mathbb{R}^n_{\pm})$$

and $\mathbb{R}^n_{\pm} := \{x \in \mathbb{R}^n : x_n \gtrless 0\}$ denotes the positive and negative half space, respectively. Here, $a^{\pm}_{jk,\alpha}\colon \mathbb{R}^n_{\pm} \to L(E)$ are bounded, sufficiently smooth functions on the closure $\overline{\mathbb{R}^n_{\pm}}$ of $\mathbb{R}^n_{\pm}$ and $m^{\pm}_{jk} \in \mathbb{N}_0 \cup \{-\infty\}$ denotes the order $\operatorname{ord}(A^{\pm}_{jk}(x,D))$ of the E-valued differential operator $A^{\pm}_{jk}(x,D)$ for $1 \le j,k \le N^{\pm}$, where $\operatorname{ord}(0) := -\infty$.

Now, let $s^{\pm}_1,\dots,s^{\pm}_{N^{\pm}} \ge 0, t^{\pm}_1,\dots,t^{\pm}_{N^{\pm}} > 0$. For $1 < p < \infty$, let $X^{\pm}_p, Y^{\pm}_p$ be the Banach spaces defined by

$$X^{\pm}_p := \prod_{j=1}^{N^{\pm}} H^{s^{\pm}_j}_p(\mathbb{R}^n_{\pm},E), \quad Y^{\pm}_p := \prod_{j=1}^{N^{\pm}} H^{s^{\pm}_j+t^{\pm}_j}_p(\mathbb{R}^n_{\pm},E). \tag{2.7}$$

Here, $H^s_p(U,E)$ is the E-valued Bessel potential space, see Section A.1.
In a canonical way, we equip the spaces $X^{\pm}_p$ and $Y^{\pm}_p$ with the norms $\|U\|_{X^{\pm}_p} := \sum_{j=1}^{N^{\pm}} \|u_j\|_{H^{s^{\pm}_j}_p}$ for $U = (u_1,\dots,u_{N^{\pm}})^{\top} \in X^{\pm}_p$ and $\|\cdot\|_{Y^{\pm}_p}$, the latter being defined analogously.

We require the $s^{\pm}_j$ to be zero if there exists a $k \in \{1,\dots,N^{\pm}\}$ such that

$$\operatorname{ord}(A^{\pm}_{jk}(x,D)) > 0,$$

i.e. in each row containing a differential operator (of order ≥ 1) the corresponding component in the space $X^{\pm}_p$ should be L_p.
Moreover, we assume that the orders $m^{\pm}_{jk}$ satisfy the estimate

$$m^{\pm}_{jk} \le s^{\pm}_k + t^{\pm}_k - s^{\pm}_j \qquad (1 \le j,k \le N^{\pm}),$$

which implies that $\mathcal{A}^{\pm}\colon Y^{\pm}_p \to X^{\pm}_p$ is a bounded linear operator, i.e. $\mathcal{A}^{\pm} \in L(Y^{\pm}_p, X^{\pm}_p)$.

Let now $N := N^- + N^+$ and the operator matrix $\mathcal{A}$ of size $N \times N$ be defined as

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^- & 0 \\ 0 & \mathcal{A}^+ \end{pmatrix}.$$

We want to consider the operator $\mathcal{A}$ in the basis space

$$X_p := \big\{u = (u^-,u^+) = (u^-_1,\dots,u^-_{N^-},u^+_1,\dots,u^+_{N^+}) \in X^-_p \times X^+_p : \\ \chi_{\mathbb{R}^n_-}u^-_j + \chi_{\mathbb{R}^n_+}u^+_j \in H^{r_j}(\mathbb{R}^n,E) \text{ for } j = 1,\dots,\max\{N^-,N^+\}\big\}$$

for some $r_j \in \mathbb{N}_0$ with $r_j \le \min\{s^-_j, s^+_j\}$ where we have set $u^{\pm}_j := 0$ if $j > N^{\pm}$. As usual, χ_A denotes the characteristic function on a set $A \subset \mathbb{R}^n$, i.e. $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.

Therefore, the additional conditions within the definition of X_p are equivalent to the *canonical transmission conditions up to order* $r_j - 1$ given by

$$(\partial_\nu)^k u_j^- = (\partial_\nu)^k u_j^+ \text{ on } \mathbb{R}^{n-1}$$

for $j = 1, \dots, \max\{N^-, N^+\}$ and $k = 0, \dots, r_j - 1$. Note that these canonical transmission conditions can be formulated in the same way on another geometry, for example in a bent half-space. Changing the geometry does not affect the description of these conditions. This is due to the ascending nature of the canonical transmission conditions. In contrast, a single transmission condition as, for example, $\partial_\nu^2 u_j^- = \partial_\nu^2 u_j^+$ on $\mathbb{R}^{n-1}$ would change under change of the geometry from a half-space to a bent half-space.

We impose $M \in \mathbb{N}$ additional transmission conditions (in general of higher order) given by $1 \times N^\pm$ operator matrices

$$\mathcal{B}_i^\pm = \mathcal{B}_i^\pm(x, D),\ i = 1, \dots, M$$

on $\mathbb{R}^n_\pm$ of the form

$$\begin{aligned} \mathcal{B}_i^- &= (B_{i1}^-(x,D), \dots, B_{iN^-}^-(x,D)) \\ &= \left(\sum_{|\beta| \le k_{i1}^-} b_{i1,\beta}^-(x) D^\beta, \dots, \sum_{|\beta| \le k_{iN^-}^-} b_{iN^-,\beta}^-(x) D^\beta \right), \end{aligned} \tag{2.8}$$

$$\begin{aligned} \mathcal{B}_i^+ &= (B_{i1}^+(x,D), \dots, B_{iN^+}^+(x,D)) \\ &= \left(\sum_{|\beta| \le k_{i1}^+} b_{i1,\beta}^+(x) D^\beta, \dots, \sum_{|\beta| \le k_{iN^+}^+} b_{iN^+,\beta}^+(x) D^\beta \right) \end{aligned} \tag{2.9}$$

with sufficiently smooth functions $b_{ij,\beta}^\pm \colon \mathbb{R}^n_\pm \to L(E)$. Without loss of generality, we assume that for all $i \in \{1, \dots M\}$ we have $\mathcal{B}_i^- \neq 0$ or $\mathcal{B}_i^+ \neq 0$. Moreover, let the $M \times N$ operator matrix $\mathcal{B}_\gamma$ be defined by

$$\mathcal{B}_\gamma := \begin{pmatrix} \gamma \mathcal{B}^- & \gamma \mathcal{B}^+ \end{pmatrix} := \begin{pmatrix} \gamma \mathcal{B}_1^- & \gamma \mathcal{B}_1^+ \\ \vdots & \vdots \\ \gamma \mathcal{B}_M^- & \gamma \mathcal{B}_M^+ \end{pmatrix}, \tag{2.10}$$

where γ denotes the trace onto $\mathbb{R}^{n-1}$ and let the $MN \times N$ operator matrix $\mathcal{B}$ be defined by

$$\mathcal{B} := \begin{pmatrix} \mathscr{B}_1^- & 0 \\ \vdots & \vdots \\ \mathscr{B}_M^- & 0 \\ 0 & \mathscr{B}_1^+ \\ \vdots & \vdots \\ 0 & \mathscr{B}_M^+ \end{pmatrix}. \tag{2.11}$$

Here, $\mathscr{B}_i^{\pm} := \operatorname{diag}(B_{i1}^{\pm}, \dots, B_{iN^{\pm}}^{\pm})$ is a $N^{\pm} \times N^{\pm}$ operator matrix.
We assume that the orders $k_{ij}^{\pm} \in \mathbb{N}_0 \cup \{-\infty\}$ of the operators $B_{ij}^{\pm}(x, D)$ satisfy

$$k_{ij}^{\pm} \leq s_j^{\pm} + t_j^{\pm} - 1, \quad i = 1, \dots, M, \, j = 1, \dots, N^{\pm},$$

where again the order of an operator equals $-\infty$ if and only if the operator equals zero.
The operator $\mathcal{A}$ is considered as an operator

$$\mathcal{A}_{Y_p} \colon X_p \supset Y_p \to X_p, u \mapsto \mathcal{A}u$$

with domain

$$Y_p := \big\{u = (u^-, u^+) \in (Y_p^- \times Y_p^+) \cap X_p : \mathcal{A}u \in X_p\big\},$$

i.e. without the additional transmission conditions (of possible higher order) as well as an operator

$$\mathcal{A}_{\mathcal{B}_\gamma} \colon X_p \supset D(\mathcal{A}_{\mathcal{B}_\gamma}) \to X_p, u \mapsto \mathcal{A}u$$

with full transmission conditions, i.e. with the smaller domain

$$D(\mathcal{A}_{\mathcal{B}_\gamma}) := \big\{u \in Y_p : \mathcal{B}_\gamma u = 0\big\}.$$

We endow both, Y_p and $D(\mathcal{A}_{\mathcal{B}_\gamma})$, with the norm of $Y_p^- \times Y_p^+$ which we denote by $\|\cdot\|_{Y_p}$.
It will also be convenient to consider the space

$$Z_p := \prod_{i=1}^{M} \prod_{j=1}^{N^-} H_p^{s_j^- + t_j^- - \max\{k_{ij}^-, 0\}}(\mathbb{R}^n_-, E) \times \prod_{i=1}^{M} \prod_{j=1}^{N^+} H_p^{s_j^+ + t_j^+ - \max\{k_{ij}^+, 0\}}(\mathbb{R}^n_+, E).$$

Note that $\mathcal{B} \colon Y_p \to Z_p$ from (2.11) is well-defined and continuous provided the coefficients $b_{ij,\beta}^{\pm}$ are sufficiently smooth.

The graph norm of $\mathcal{A}$ is given by

$$\|u\|_{D(\mathcal{A})} := \|\mathcal{A}u\|_{X_p} + \|u\|_{X_p} \tag{2.12}$$

for u being an element of the maximal domain $D(\mathcal{A}_{\max}) := \{u \in X_p : \mathcal{A}u \in X_p\}$.
Now we turn our attention to the boundary spaces from which the data on the transmission interface $\mathbb{R}^{n-1}$ will be chosen from. Define

$$\partial Y_p^{\pm} := \partial Y_p^{\pm}(\mathbb{R}^{n-1}) := \prod_{i=1}^{M} B_{pp}^{\mu_i^{\pm} - 1/p}(\mathbb{R}^{n-1}, E)$$

where $\mu_i^{\pm} = \min_{j=1,\dots,N^{\pm}}\{s_j^{\pm} + t_j^{\pm} - k_{ij}^{\pm}\}$ and

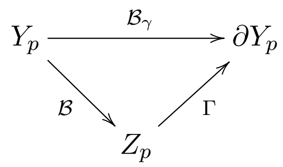
$$\partial Y_p := \partial Y_p(\mathbb{R}^{n-1}) := \partial Y_p^- + \partial Y_p^+ = \prod_{i=1}^{M} B_{pp}^{\min\{\mu_i^-, \mu_i^+\} - 1/p}(\mathbb{R}^{n-1}, E),$$

where $B_{pq}^s(U, E)$ denotes the E-valued Besov space, see Section A.1. The space ∂Y_p will be the space for the data in the transmission problem, whereas $\partial Y_p^{\pm}$ are the spaces of data for the

corresponding boundary value problems in $\mathbb{R}^n_\pm$. In a natural way, we equip ∂Y_p with the product norm of the Besov spaces $B^{\mu_i - 1/p}_{pp}(\mathbb{R}^{n-1}, E)$ where $\mu_i := \min\{\mu_i^-, \mu_i^+\}$.
Introducing the $M \times MN$ operator matrix

$$\Gamma := \begin{pmatrix} \Gamma_- & & & \Gamma_+ & & \\ & \ddots & & & \ddots & \\ & & \Gamma_- & & & \Gamma_+ \end{pmatrix},$$

where $\Gamma_\pm = (\gamma_\pm \ \cdots \ \gamma_\pm)$ is a $1 \times N^\pm$ matrix and $\gamma_\pm$ denotes the trace operator from $\mathbb{R}^n_\pm$ to $\mathbb{R}^{n-1}$, the diagram



commutes.

2.7 Lemma. *The space Y_p is closed in $Y_p^- \times Y_p^+$. In particular, it is a Banach space. Moreover, $D(\mathcal{A}_{\mathcal{B}_\gamma})$ is closed in Y_p, provided that the coefficients of $\mathcal{B}_\gamma$ are pointwise multipliers in the respective spaces (see Definition 2.9 and Remark 2.10 below).*

Proof. Let $y_n = (y^{(n),-}, y^{(n),+})_n \subset Y_p$ be a sequence with $y_n \to y \in Y_p^- \times Y_p^+$ as $n \to \infty$. We have to show that $y \in Y_p$. As $\mathcal{A} \in L(Y_p^- \times Y_p^+, X_p^- \times X_p^+)$, we have that $\mathcal{A}y_n \to \mathcal{A}y \in X_p^- \times X_p^+$. Moreover the mappings $\Gamma_j \colon X_p \to \prod_{l=1}^{r_j} L_p(\mathbb{R}^{n-1}, E)$ given by

$$u = (u^-, u^+) \mapsto \begin{pmatrix} u_j^-|_{\mathbb{R}^{n-1}} - u_j^+|_{\mathbb{R}^{n-1}} \\ \vdots \\ \partial_n^{r_j - 1} u_j^-|_{\mathbb{R}^{n-1}} - \partial_n^{r_j - 1} u_j^+|_{\mathbb{R}^{n-1}} \end{pmatrix}$$

are linear and bounded for $j = 1, \ldots, \max\{N^-, N^+\}$. Hence,

$$0 = \lim_{n\to\infty} \Gamma_j \mathcal{A} y_n = \Gamma_j \lim_{n\to\infty} (\mathcal{A} y_n) = \Gamma_j \mathcal{A} y$$

for all $j = 1, \ldots, \max\{N^-, N^+\}$, i.e. $\mathcal{A}y \in X_p$.
In the same way, considering the mappings $\Gamma_j \colon Y_p \to \prod_{l=1}^{r_j} L_p(\mathbb{R}^{n-1}, E)$, we see that $y \in X_p$ and consequently we have that $y \in Y_p$.
As $D(\mathcal{A}_{\mathcal{B}_\gamma}) = \ker \mathcal{B}_\gamma$ and $\mathcal{B}_\gamma \colon Y_p \to \partial Y_p$ is continuous, the closedness of $D(\mathcal{A}_{\mathcal{B}_\gamma})$ in Y_p follows. □

2.8 Remark. Note that if $\mathcal{A}_{\mathcal{B}_\gamma}$ is closed (e.g. if the resolvent set $\rho(\mathcal{A}_{\mathcal{B}_\gamma})$ of $\mathcal{A}_{\mathcal{B}_\gamma}$ is not empty), then there exists a constant $C > 0$ such that

$$\|u\|_{Y_p} \le C \left(\|u\|_{X_p} + \|\mathcal{A}_{\mathcal{B}_\gamma} u\|_{X_p} \right).$$

In particular, the norms $\|\cdot\|_{Y_p}$ and the graph norm $\|\cdot\|_{\mathcal{A}_{\mathcal{B}_\gamma}} = \|\cdot\|_{X_p} + \|\mathcal{A}_{\mathcal{B}_\gamma} \cdot\|_{X_p}$ are equivalent. Indeed, since $\mathcal{A}_{\mathcal{B}_\gamma}$ is closed, the space $(D(\mathcal{A}_{\mathcal{B}_\gamma}), \|\cdot\|_{\mathcal{A}_{\mathcal{B}_\gamma}})$ is a Banach space. Moreover, by the previous lemma, also $(D(\mathcal{A}_{\mathcal{B}_\gamma}), \|\cdot\|_{Y_p})$ is a Banach space (if the coefficients of $\mathcal{B}_\gamma$ are sufficiently 'nice'). As we have the elementary embedding

$$\mathrm{id}\colon \big(D(\mathcal{A}_{\mathcal{B}_\gamma}), \|\cdot\|_{Y_p}\big) \to \big(D(\mathcal{A}_{\mathcal{B}_\gamma}), \|\cdot\|_{\mathcal{A}_{\mathcal{B}_\gamma}}\big),$$

the equivalence follows from the closed graph theorem.

For the correct smoothness assumptions on the coefficients of $\mathcal{A}$ and $\mathcal{B}_i^\pm$ we will need the notion of pointwise multipliers:

2.9 Definition. Let $\Omega \subset \mathbb{R}^n$ be a domain. Given a normed space X of functions (or of equivalence classes of functions) $g\colon \Omega \to E$, we say that a locally integrable function $f\colon \Omega \to L(E)$ is a pointwise multiplier on X, if

$$X \to X,\ g \mapsto fg$$

is well-defined and continuous. The space of pointwise multipliers on X will be denoted by $M(X)$. We endow this space with the operator (semi-)norm

$$\|f\|_{M(X)} := \sup_{\|g\|_X = 1} \|fg\|_X.$$

2.10 Remark. Let $\Omega \subset \mathbb{R}^n$ be a domain.

(a) One can easily verify that $M(L_p(\Omega, E)) = L_\infty(\Omega, L(E))$ for $1 \leq p \leq \infty$.

(b) Clearly, $W_\infty^k(\Omega, L(E)) \hookrightarrow M(W_p^k(\Omega, E))$ for $k \in \mathbb{N}$ and all $1 \leq p \leq \infty$.

(c) It holds that $B_{pq}^s(\Omega) \hookrightarrow M(B_{pq}^s(\Omega))$ if $s > np$ and $1 \leq p, q \leq \infty$. (see for example [RS96, p. 221/222, Section 4.6.4, Theorem 1])

(d) It holds that $F_{pq}^s(\Omega) \hookrightarrow M(F_{pq}^s(\Omega))$ if $s > np$, $1 \leq p \leq \infty$ and $1 \leq q < \infty$. (see for example [RS96, p. 221/222, Section 4.6.4, Theorem 1])

In the following, we consider the parameter dependent transmission problem with inhomogeneous transmission data

$$\begin{aligned} (\lambda - \mathcal{A})u &= f \quad \text{in } \mathbb{R}^n, \\ \mathcal{B}_\gamma u &= g \ \text{ on } \mathbb{R}^{n-1} \end{aligned} \tag{2.13}$$

for $\lambda \in \mathbb{C}$. Here, $f \in X_p$ and $g = (g_1, \ldots, g_M) \in \partial Y_p$ are given.

Given $\varphi \in (0, \pi]$ we denote the open sector

$$\Sigma_\varphi := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \varphi\}.$$

We also denote by $\mathrm{pr}_j^\pm z$ for $j = 1, \ldots, N^\pm$ the projection on the j-th component of $z \in X_p$ on the positive and the negative half space, respectively.

Often, we will work with the *parameter dependent* norms. On Y_p we introduce the parameter dependent norm component wise

$$\left|\left|\left| u_j^\pm \right|\right|\right|_{H_p^{s_j^\pm + t_j^\pm}} := \|u_j^\pm\|_{H_p^{s_j^\pm + t_j^\pm}} + |\lambda|^{(s_j^\pm + t_j^\pm)/t_j^\pm} \|u_j^\pm\|_{L_p},$$

which, for fixed λ, is equivalent to $\|\cdot\|_{H_p^{s_j^\pm + t_j^\pm}}$. In the same way we define

$$\left|\left|\left| f_j^\pm \right|\right|\right|_{H_p^{s_j^\pm}} := \|f_j^\pm\|_{H_p^{s_j^\pm}} + |\lambda|^{s_j^\pm / t_j^\pm} \|f_j^\pm\|_{L_p}$$

and $|||f|||_{X_p}$ for $f \in X_p$. Finally, on Z_p the parameter dependent norm $|||h|||_{Z_p}$ is component wise given by

$$\left|\left|\left| h_{ij}^\pm \right|\right|\right|_{H_p^{s_j^\pm + t_j^\pm - k_{ij}^\pm}} := \|h_{ij}^\pm\|_{H^{s_j^\pm + t_j^\pm - k_{ij}^\pm}} + |\lambda|^{(s_j^\pm + t_j^\pm - k_{ij}^\pm)/t_j^\pm} \|h_{ij}^\pm\|_{L_p}.$$

Another choice for the norm on ∂Y_p is given by

$$|||g|||_{\partial Y_p} := \inf_{\substack{h \in Z_p \\ \Gamma h = g}} |||h|||_{Z_p} \quad (g \in \partial Y_p)$$

which, in fact, is equivalent to the natural norm on ∂Y_p for fixed λ:

2.11 Lemma. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ be fixed. Then, there exist constants $c_1, c_2 > 0$ such that*

$$c_1 |||g|||_{\partial Y_p} \leq \|g\|_{\partial Y_p} \leq c_2 |||g|||_{\partial Y_p} \quad (g \in \partial Y_p).$$

In particular, $(\partial Y_p, |||\cdot|||_{\partial Y_p})$ is a Banach space.

Proof. Let $g \in \partial Y_p$. Since $\Gamma \colon Z_p \to \partial Y_p$ is continuous, it holds that

$$\|g\|_{\partial Y_p} = \|\Gamma h\|_{\partial Y_p} \leq c\|h\|_{Z_p} \leq c_2 |||h|||_{Z_p}$$

for all $h \in Z_p$ satisfying $\Gamma h = g$. Now, taking the infimum over all such h yields the inequality

$$\|g\|_{\partial Y_p} \leq c_2 |||g|||_{\partial Y_p}.$$

For the other estimate, we choose an extension operator

$$E \in L(B_{pp}^{\mu_i - 1/p}(\mathbb{R}^{n-1}), H_p^{\mu_i}(\mathbb{R}_\pm^n))$$

in order to construct a certain $h \in Z_p$ satisfying $\Gamma h = g$. This h consists only of components of the form Eg_i in the appropriate entry and zero for all the other entries. Then we obtain

$$|||g|||_{\partial Y_p} \leq |||h|||_{Z_p} \leq \|g\|_{\partial Y_p}.$$

To be more precise, for $i \in \{1, \dots M\}$ we first determine whether $\mu_i^- \leq \mu_i^+$ or $\mu_i^- > \mu_i^+$. In the first case we define $\circ := -$ and in the latter case we let $\circ := +$. Now, let

$$j(i) := \operatorname{argmin}_{j=1,\dots,N^\circ} s_j^\circ + t_j^\circ - k_{ij}^\circ.$$

Let $h = (h_{ij}^\pm)$ be defined by $h_{ij(i)}^\circ = Eg_i$ and all the other entries are set to zero. Then $h \in Z_p$ and the assertion follows. □

The following theorem will be the basis for the treatment of boundary and transmission problems in domains via localization. It basically states that if the solution operators for the equation (2.13) satisfy certain a-priori estimates, the unique solvability is preserved under small perturbations in the operators $\mathcal{A}$ and $\mathcal{B}_\gamma$. Later we will see that also the a-priori estimates are stable under those perturbations.

Let $\widetilde{\mathcal{A}}^\pm(x,D)$ be operator matrices be defined similarly as in (2.6) with coefficients $\widetilde{a}^\pm_{jk,\alpha} \in M(H_p^{s_j^\pm}(\mathbb{R}^n_\pm, E))$ and the order of the differential operator $\widetilde{A}^\pm_{jk}(x,D)$ shall be less or equal the order $m^\pm_{jk}$ of the differential operator $A^\pm_{jk}(x,D)$.
For $j \in \{1,\dots,N^\pm\}$ we denote by $\widetilde{\mathcal{A}}^\pm_j$ the j-th row

$$\widetilde{\mathcal{A}}^\pm_j = \Big(\widetilde{A}^\pm_{j1}(x,D),\dots,\widetilde{A}^\pm_{jN^\pm}(x,D)\Big)$$

of $\widetilde{\mathcal{A}}^\pm$.
Also, we require the j-th row of $\widetilde{\mathcal{A}}^\pm$ to equal zero, i.e.

$$\widetilde{\mathcal{A}}^\pm_j = \Big(\widetilde{A}^\pm_{j1}(x,D),\dots,\widetilde{A}^\pm_{jN^\pm}(x,D)\Big) = (0,\dots,0),$$

provided that $s_j^\pm \neq 0$ for all $j = 1,\dots,N^\pm$. This means that we will only consider perturbations of the matrix differential operator $\mathcal{A}$ in the rows where the corresponding basic space is L_p.
Furthermore, let $\widetilde{\mathcal{B}}^\pm_i$ be $1 \times N^\pm$ operator matrices be defined similarly as in (2.8) and (2.9) with coefficients $\widetilde{b}^\pm_{ij,\beta} \in M(H_p^{s_j^\pm+t_j^\pm-k_{ij}^\pm}(\mathbb{R}^n_\pm, E))$ for all $i = 1,\dots,M$, $j = 1,\dots,N^\pm$ and the order of the differential operators $\widetilde{B}^\pm_{ij}(x,D)$ shall be less or equal the order $k^\pm_{ij}$ of the differential operator $B^\pm_{ij}(x,D)$. The operator matrices $\widetilde{\mathcal{B}}_\gamma$ and $\widetilde{\mathcal{B}}$ are defined similarly to (2.10) and (2.11).

2.12 Theorem. *Let $\varphi \in (0,\pi]$ and $\lambda_0 \in \mathbb{R}$. Assume that for any $\lambda \in \lambda_0 + \Sigma_\varphi$, $f \in X_p$ and $g \in \partial Y_p$ there exists a unique solution $u \in Y_p$ of (2.13). Suppose that the solution u can be written as*

$$u = R(\lambda)f + S(\lambda)g$$

with continuous operator $R(\lambda)\colon X_p \to D(\mathcal{A}_{\mathcal{B}_\gamma})$ and a linear operator $S(\lambda)\colon \partial Y_p \to Y_p$ such that $S(\lambda)\colon \partial Y_p^\pm \to Y_p^\pm$ is continuous. Here, $R(\lambda)f$ satisfies (2.13) with $g = 0$ and $S(\lambda)g$ satisfies (2.13) with $f = 0$. We also assume that:

(i) There exists a constant $C > 0$ independent of λ such that the solution operator $R(\lambda_0 + \lambda)$ satisfies the estimates

$$\|\lambda^{(s_j^\pm+t_j^\pm-|\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm R(\lambda_0+\lambda)f\|_{L_p} \le C|||f|||_{X_p} \tag{2.14}$$

for all $\lambda \in \Sigma_\varphi$, $|\alpha| \le s_j^\pm + t_j^\pm$ and $f \in X_p$.

(ii) There exists a constant $C > 0$ independent of λ such that the following estimate for $S(\lambda_0+\lambda)$ holds true: for any $\lambda \in \Sigma_\varphi$ with $|\lambda|$ large enough and for all $j \in \{1,\dots,N^\pm\}$ we have

$$\|\lambda^{(s_j^\pm+t_j^\pm-|\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm S(\lambda_0+\lambda)g\|_{L_p} \le C \inf_{\substack{h\in Z_p,\\ \Gamma h = g}} |||h|||_{Z_p} \qquad (g \in \partial Y_p) \tag{2.15}$$

where $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le s_j^\pm + t_j^\pm$.

Then there exists $\varepsilon > 0$ such that the following holds: If the coefficients of the perturbations $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}_\gamma$ satisfy

$$\|\widetilde{a}^{\pm}_{jk,\alpha}\|_{M(H_p^{s_j^{\pm}}(\mathbb{R}^n_{\pm},E))} \leq \varepsilon, \quad |\alpha| = m^{\pm}_{jk} \tag{2.16}$$

and

$$\|\widetilde{b}^{\pm}_{ij,\beta}\|_{M(H_p^{s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm}}(\mathbb{R}^n_{\pm},E))} \leq \varepsilon, \quad |\beta| = k^{\pm}_{ij},\, i = 1,\dots,M, j = 1,\dots,N^{\pm}, \tag{2.17}$$

then there exists a real number $\nu \geq 0$ such that for all $\lambda \in \nu + \Sigma_\varphi$, $f \in X_p$ and $g = (g_1,\dots,g_M) \in \partial Y_p$ the problem

$$\begin{aligned} (\lambda_0 + \lambda - (\mathcal{A} + \widetilde{\mathcal{A}}))u &= f \quad \text{in } \mathbb{R}^n, \\ (\mathcal{B}_\gamma + \widetilde{\mathcal{B}}_\gamma)u &= g \quad \text{on } \mathbb{R}^{n-1} \end{aligned} \tag{2.18}$$

possesses a unique solution $u = \widetilde{R}(\lambda_0 + \lambda)f + \widetilde{S}(\lambda_0 + \lambda)g \in Y_p$ with the solution operators

$$\begin{aligned} \widetilde{R}(\lambda) &:= \left(1 - R(\lambda)\widetilde{\mathcal{A}} + S(\lambda)\widetilde{\mathcal{B}}_\gamma\right)^{-1} R(\lambda)\colon X_p \to Y_p, \\ \widetilde{S}(\lambda) &:= \left(1 - R(\lambda)\widetilde{\mathcal{A}} + S(\lambda)\widetilde{\mathcal{B}}_\gamma\right)^{-1} S(\lambda)\colon \partial Y_p \to Y_p. \end{aligned}$$

Here, $\widetilde{R}(\lambda)$ solves (2.18) with $g = 0$ and $\widetilde{S}(\lambda)$ solves (2.18) with $f = 0$.

Proof. Let $f \in X_p$ and $g = (g_1,\dots,g_M) \in \partial Y_p$. We write (2.18) in the form

$$\begin{aligned} (\lambda_0 + \lambda - \mathcal{A})u &= f + \widetilde{\mathcal{A}}u, \\ \mathcal{B}_\gamma u &= g - \widetilde{\mathcal{B}}_\gamma u. \end{aligned}$$

Using the solution operators $R(\lambda_0 + \lambda)$ and $S(\lambda_0 + \lambda)$ we obtain

$$u = R(\lambda_0 + \lambda)(f + \widetilde{\mathcal{A}}u) + S(\lambda_0 + \lambda)(g - \widetilde{\mathcal{B}}_\gamma u),$$

i.e.

$$\left(1 - R(\lambda_0 + \lambda)\widetilde{\mathcal{A}} + S(\lambda_0 + \lambda)\widetilde{\mathcal{B}}_\gamma\right) u = R(\lambda_0 + \lambda)f + S(\lambda_0 + \lambda)g.$$

We will show that $(1 - R(\lambda_0 + \lambda)\widetilde{\mathcal{A}} + S(\lambda_0 + \lambda)\widetilde{\mathcal{B}}_\gamma)\colon Y_p \to Y_p$ is continuously invertible for sufficiently large λ using the Neumann series. To do so, we will consider the norm $|\!|\!|\cdot|\!|\!|_{Y_p}$ on Y_p, which is equivalent to $\|\cdot\|_{Y_p}$ for any fixed λ.
First, we need to estimate $\|\widetilde{\mathcal{A}}u\|_{X_p}$. As before, we denote by $\widetilde{\mathcal{A}}^{\pm}_j$ the j-th row of $\widetilde{\mathcal{A}}^{\pm}$ for $j \in \{1,\dots,N^{\pm}\}$. Recall that $\widetilde{\mathcal{A}}^{\pm}_j \neq 0$ only if $s^{\pm}_j = 0$. For such j we have

$$\begin{aligned} \|\widetilde{\mathcal{A}}^{\pm}_j u^{\pm}\|_{L_p} \leq{}& \left\| \sum_{k=1}^{N^{\pm}} \sum_{|\alpha| = m^{\pm}_{jk}} \widetilde{a}^{\pm}_{jk,\alpha} D^\alpha u^{\pm}_k \right\|_{L_p} \\ &+ \left\| \sum_{k=1}^{N^{\pm}} \sum_{|\alpha| < m^{\pm}_{jk}} \widetilde{a}^{\pm}_{jk,\alpha} D^\alpha u^{\pm}_k \right\|_{L_p} \leq \varepsilon |\!|\!|u|\!|\!|_{Y_p} \end{aligned} \tag{2.19}$$

for $|\lambda|$ large enough, where we have used the interpolation inequalities for the second summand. Therefore, we obtain the estimate $\left|\!\left|\!\left|\widetilde{\mathcal{A}}u\right|\!\right|\!\right|_{X_p} \leq \varepsilon |\!|\!|u|\!|\!|_{Y_p}$ and by (2.14) we have

$$\left|\!\left|\!\left|R(\lambda_0+\lambda)\widetilde{\mathcal{A}}u\right|\!\right|\!\right|_{Y_p} \leq C\left|\!\left|\!\left|\widetilde{\mathcal{A}}u\right|\!\right|\!\right|_{X_p} \leq \varepsilon C|\!|\!|u|\!|\!|_{Y_p} \tag{2.20}$$

for any $\varepsilon > 0$ and $|\lambda|$ large enough.
Considering the transmission operators, in the same way we get

$$\begin{aligned} &\|\widetilde{B}_{ij}^{\pm}u_j^{\pm}\|_{H_p^{s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm}}} \\ &\leq C\Bigg(\Big\|\sum_{|\beta|=k_{ij}^{\pm}}\widetilde{b}_{ij,\beta}^{\pm}D^{\beta}u_j^{\pm}\Big\|_{H_p^{s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm}}} + \Big\|\sum_{|\beta|<k_{ij}^{\pm}}\widetilde{b}_{ij,\beta}^{\pm}D^{\beta}u_j^{\pm}\Big\|_{H_p^{s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm}}}\Bigg) \\ &\leq C\big(\varepsilon\|u_j^{\pm}\|_{H_p^{s_j^{\pm}+t_j^{\pm}}} + C(\varepsilon)\|u_j^{\pm}\|_{L_p}\big). \end{aligned} \tag{2.21}$$

Further, for $k_{ij}^{\pm} > 0$ the estimate

$$\begin{aligned} &|\lambda|^{(s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm})/t_j^{\pm}}\|\widetilde{B}_{ij}^{\pm}u_j^{\pm}\|_{L_p} \\ &\leq |\lambda|^{(s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm})/t_j^{\pm}}\left(\varepsilon\|u_j^{\pm}\|_{H_p^{k_{ij}^{\pm}}} + C\|u_j^{\pm}\|_{H_p^{k_{ij}^{\pm}-1}}\right) \\ &\leq |\lambda|^{(s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm})/t_j^{\pm}}\left(\varepsilon\|u_j^{\pm}\|_{H_p^{k_{ij}^{\pm}}} + C(\varepsilon)\|u_j^{\pm}\|_{L_p}\right) \\ &\leq \varepsilon\|u_j^{\pm}\|_{H_p^{s_j^{\pm}+t_j^{\pm}}} + \varepsilon|\lambda|^{(s_j^{\pm}+t_j^{\pm})/t_j^{\pm}}\|u_j^{\pm}\|_{L_p} + C(\varepsilon)|\lambda|^{(s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm})/t_j^{\pm}}\|u_j^{\pm}\|_{L_p} \end{aligned} \tag{2.22}$$

holds true. In the first step we used the multiplier property of the coefficients $\widetilde{b}_{ij,\beta}^{\pm}$. In the second step, we used the interpolation inequality again and the last estimate follows from [Gri85], Theorem 1.4.3.3. In the case $k_{ij}^{\pm} = 0$ we directly obtain

$$|\lambda|^{(s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm})/t_j^{\pm}}\|\widetilde{B}_{ij}^{\pm}u_j^{\pm}\|_{L_p} \leq \varepsilon|\lambda|^{(s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm})/t_j^{\pm}}\|u_j^{\pm}\|_{L_p} \tag{2.23}$$

due to the smallness of the coefficients. Combining (2.21) and (2.22) (or (2.23)) we get

$$\left|\!\left|\!\left|\widetilde{B}_{ij}^{\pm}u_j^{\pm}\right|\!\right|\!\right|_{H_p^{s_j^{\pm}+t_j^{\pm}-k_{ij}^{\pm}}} \leq \varepsilon\left(\|u_j^{\pm}\|_{H_p^{s_j^{\pm}+t_j^{\pm}}} + |\lambda|^{(s_j^{\pm}+t_j^{\pm})/t_j^{\pm}}\|u_j^{\pm}\|_{L_p}\right)$$

for $|\lambda|$ large enough. Consequently, from (2.15) with $|\alpha| \leq s_j^{\pm} + t_j^{\pm}$ we obtain

$$\begin{aligned} &\|\operatorname{pr}_j^{\pm} S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_{\gamma}u\|_{H_p^{s_j^{\pm}+t_j^{\pm}}} + |\lambda|^{(s_j^{\pm}+t_j^{\pm})/t_j^{\pm}}\|\operatorname{pr}_j^{\pm} S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_{\gamma}u\|_{L_p} \\ &\leq C\inf_{\substack{h\in Z_p\\ \Gamma h=\widetilde{\mathcal{B}}_{\gamma}u}}|\!|\!|h|\!|\!|_{Z_p} \leq C\left|\!\left|\!\left|\widetilde{\mathcal{B}}u\right|\!\right|\!\right|_{Z_p} \leq \varepsilon C|\!|\!|u|\!|\!|_{Y_p}, \end{aligned}$$

i.e.

$$\left|\left|\left| S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma u \right|\right|\right|_{Y_p} \leq \varepsilon C |||u|||_{Y_p} \tag{2.24}$$

for $|\lambda|$ large enough. Altogether, (2.20) and (2.24) imply the estimate

$$\left|\left|\left| R(\lambda_0+\lambda)\widetilde{\mathcal{A}}u + S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma u \right|\right|\right|_{Y_p} \leq 2\varepsilon C |||u|||_{Y_p}.$$

For $\varepsilon > 0$ small enough, this shows the unique solvability of (2.18) for all $\lambda \in \nu + \lambda_0 + \Sigma_\varphi$ for some $\nu \geq 0$ large enough with solution operators

$$\widetilde{R}(\lambda) := \left(1 - R(\lambda)\widetilde{\mathcal{A}} + S(\lambda)\widetilde{\mathcal{B}}_\gamma\right)^{-1} R(\lambda) \colon X_p \to Y_p,$$
$$\widetilde{S}(\lambda) := \left(1 - R(\lambda)\widetilde{\mathcal{A}} + S(\lambda)\widetilde{\mathcal{B}}_\gamma\right)^{-1} S(\lambda) \colon \partial Y_p \to Y_p.$$

□

The next theorem shows that the assumed estimates for the solution operators $R(\lambda)$ and $S(\lambda)$ (see (2.14) and (2.15)) are stable under perturbation, i.e. $\widetilde{R}(\lambda)$ and $\widetilde{S}(\lambda)$ satisfy similar estimates as well.

2.13 Theorem. *Consider again the situation of Theorem 2.12.*
Then $\widetilde{R}(\lambda_0+\lambda)$ and $\widetilde{S}(\lambda_0+\lambda)$ satisfy the estimates (2.14) *and* (2.15) *as well, i.e. there exists a constant $C > 0$ independent of $\lambda \in \nu + \Sigma_\varphi$ such that*

$$\|\lambda^{(s_j^\pm + t_j^\pm - |\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda) f\|_{L_p} \leq C |||f|||_{X_p} \quad (f \in X_p)$$

and

$$\|\lambda^{(s_j^\pm + t_j^\pm - |\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm \widetilde{S}(\lambda_0+\lambda) g\|_{L_p} \leq C \inf_{\substack{h \in Z_p,\\ \Gamma h = g}} |||h|||_{Z_p} \quad (g \in \partial Y_p)$$

for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq s_j^\pm + t_j^\pm$.

Proof. We observe that

$$\begin{aligned}
\widetilde{R}(\lambda) &= R(\lambda) + \widetilde{R}(\lambda) - R(\lambda) \\
&= R(\lambda) - (1 - 1 - R(\lambda)\widetilde{\mathcal{A}} + S(\lambda)\widetilde{\mathcal{B}}_\gamma)(1 - R(\lambda)\widetilde{\mathcal{A}} + S(\lambda)\widetilde{\mathcal{B}}_\gamma)^{-1} R(\lambda) \\
&= R(\lambda) + (R(\lambda)\widetilde{\mathcal{A}} - S(\lambda)\widetilde{\mathcal{B}}_\gamma)\widetilde{R}(\lambda)
\end{aligned}$$

and hence

$$\begin{aligned}
\widetilde{R}(\lambda) f &= R(\lambda) f + R(\lambda)\widetilde{\mathcal{A}}\widetilde{R}(\lambda) f - S(\lambda)\widetilde{\mathcal{B}}_\gamma \widetilde{R}(\lambda) f, \\
\widetilde{S}(\lambda) g &= S(\lambda) g + R(\lambda)\widetilde{\mathcal{A}}\widetilde{S}(\lambda) g - S(\lambda)\widetilde{\mathcal{B}}_\gamma \widetilde{S}(\lambda) g,
\end{aligned}$$

where the second equality can be seen as before.
For fixed $\lambda \in \nu + \Sigma_\varphi$ we define the finite set of operators

$$\mathcal{T}(\widetilde{R}, \lambda)^\pm := \left\{ |\lambda|^{(s_j^\pm + t_j^\pm - |\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda) : |\alpha| \leq s_j^\pm + t_j^\pm \right\} \subset L\left((X_p, |||\cdot|||_{X_p}), L_p\right)$$

and

$$\mathcal{T}(\widetilde{R},\lambda) := \mathcal{T}(\widetilde{R},\lambda)^- \ \cup \ \mathcal{T}(\widetilde{R},\lambda)^+.$$

The uniform bounds of these sets are denoted by

$$\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda))^\pm := \max_{\substack{j=1,\dots,N^\pm \\ |\alpha|\le s_j^\pm + t_j^\pm}} \|\lambda^{(s_j^\pm+t_j^\pm-|\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda)\|_{L\left((X_p,|||\cdot|||_{X_p}),L_p\right)}$$

and

$$\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda)) := \max\{\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda))^-,\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda))^+\}.$$

Similarly, we define $\mathcal{T}(R,\lambda)$ and $\mathcal{U}(\mathcal{T}(R,\lambda))$. We will show that there is a constant $C > 0$ independent of λ such that

$$\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda)) \le C$$

for all $\lambda \in \nu + \Sigma_\varphi$ with $|\lambda|$ large enough. For this, we use the representation of $\widetilde{R}(\lambda)$ from above.

$\boldsymbol{R(\lambda_0+\lambda)f}$**:** The estimate for this term is one of the assumptions (see (2.14)) of the theorem.

$\boldsymbol{R(\lambda_0+\lambda)\widetilde{\mathcal{A}}\widetilde{R}(\lambda_0+\lambda)f}$**:** From (2.19) in the proof of Theorem 2.12 and (2.14) we know that

$$\begin{aligned}\left|\left|\left|R(\lambda_0+\lambda)\widetilde{\mathcal{A}}\widetilde{R}(\lambda_0+\lambda)f\right|\right|\right|_{Y_p} &\le \mathcal{U}(\mathcal{T}(R,\lambda))\left|\left|\left|\widetilde{\mathcal{A}}\widetilde{R}(\lambda_0+\lambda)f\right|\right|\right|_{X_p} \\ &\le \varepsilon C\left|\left|\left|\widetilde{R}(\lambda_0+\lambda)f\right|\right|\right|_{Y_p} \\ &\le \varepsilon C\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda))|||f|||_{X_p}\end{aligned}$$

for any $\varepsilon > 0$.

$\boldsymbol{S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{R}(\lambda_0+\lambda)f}$**:** Using (2.24), we get

$$\left|\left|\left|S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{R}(\lambda_0+\lambda)f\right|\right|\right|_{Y_p} \le \varepsilon\left|\left|\left|\widetilde{R}(\lambda_0+\lambda)f\right|\right|\right|_{Y_p} \le \varepsilon\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda))|||f|||_{X_p}.$$

Altogether we have shown that

$$\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda)) \le \mathcal{U}(\mathcal{T}(R,\lambda)) + \varepsilon\,(\mathcal{U}(\mathcal{T}(R,\lambda))+1)\,\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda)).$$

Choosing $\varepsilon > 0$ small enough we obtain

$$\mathcal{U}(\mathcal{T}(\widetilde{R},\lambda)) \le C$$

with a constant independent of λ.
For $\widetilde{S}$ we proceed in the same way: let

$$\mathcal{T}(\widetilde{S},\lambda)^\pm := \left\{|\lambda|^{(s_j^\pm+t_j^\pm-|\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm \widetilde{S}(\lambda_0+\lambda) : |\alpha| \le s_j^\pm + t_j^\pm\right\} \subset L\left((\partial Y_p, |||\cdot|||_{\partial Y_p}), L_p\right)$$

and

$$\mathcal{T}(\widetilde{S},\lambda) := \mathcal{T}(\widetilde{S},\lambda)^- \;\cup\; \mathcal{T}(\widetilde{R},\lambda)^+.$$

The uniform bounds of these sets are denoted by

$$\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda))^\pm := \max_{\substack{j=1,\dots,N^\pm\\ |\alpha|\le s_j^\pm+t_j^\pm}} \|\lambda^{(s_j^\pm+t_j^\pm-|\alpha|)/t_j^\pm} D^\alpha \operatorname{pr}_j^\pm \widetilde{S}(\lambda_0+\lambda)\|_{L\left((\partial Y_p, |\!|\!|\cdot|\!|\!|_{\partial Y_p}), L_p\right)}$$

and

$$\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda)) := \max\{\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda))^-, \mathcal{U}(\mathcal{T}(\widetilde{S},\lambda))^+\}.$$

Similarly, we define $\mathcal{T}(S,\lambda)$ and $\mathcal{U}(\mathcal{T}(S,\lambda))$. We will show that there is a constant $C>0$ independent of λ such that

$$\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda)) \le C$$

for all $\lambda \in \nu+\Sigma_\varphi$ with $|\lambda|$ large enough. For this, we use the representation of $\widetilde{S}(\lambda)$ from above. Note that $(\partial Y_p, |\!|\!|\cdot|\!|\!|_{\partial Y_p})$ is a Banach space according to Lemma 2.11.

$\boldsymbol{S(\lambda_0+\lambda)f}$**:** The estimate for this term is one of the assumptions (see (2.15)) of the theorem.

$\boldsymbol{R(\lambda_0+\lambda)\widetilde{\mathcal{A}}\widetilde{S}(\lambda_0+\lambda)f}$**:** From (2.19) in the proof of Theorem 2.12 and (2.15) we know that

$$\begin{aligned}\left|\!\left|\!\left|R(\lambda_0+\lambda)\widetilde{\mathcal{A}}\widetilde{S}(\lambda_0+\lambda)f\right|\!\right|\!\right|_{Y_p} &\le \mathcal{U}(\mathcal{T}(R,\lambda))\left|\!\left|\!\left|\widetilde{\mathcal{A}}\widetilde{S}(\lambda_0+\lambda)f\right|\!\right|\!\right|_{X_p}\\ &\le \varepsilon C\left|\!\left|\!\left|\widetilde{S}(\lambda_0+\lambda)f\right|\!\right|\!\right|_{Y_p}\\ &\le \varepsilon C\,\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda))|\!|\!|f|\!|\!|_{X_p}\end{aligned}$$

for any $\varepsilon > 0$.

$\boldsymbol{S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{S}(\lambda_0+\lambda)f}$**:** Using (2.24), we get

$$\left|\!\left|\!\left|S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{S}(\lambda_0+\lambda)f\right|\!\right|\!\right|_{Y_p} \le \varepsilon\left|\!\left|\!\left|\widetilde{S}(\lambda_0+\lambda)f\right|\!\right|\!\right|_{Y_p} \le \varepsilon\,\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda))|\!|\!|f|\!|\!|_{X_p}.$$

Altogether we have shown that

$$\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda)) \le \mathcal{U}(\mathcal{T}(S,\lambda)) + \varepsilon\,(\mathcal{U}(\mathcal{T}(S,\lambda))+1)\,\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda)).$$

Choosing $\varepsilon > 0$ small enough we obtain

$$\mathcal{U}(\mathcal{T}(\widetilde{S},\lambda)) \le C$$

with a constant independent of λ. □

Similarly, we are able to preserve $\mathcal{R}$-boundedness for families of solution operators:

2.14 Proposition. *We assume that the assumptions of Theorem 2.13 are satisfied. If the sets of operators* $\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)$ *and* $\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)$ *are $\mathcal{R}$-bounded, then* $\bigcup_{\lambda\in\nu+\Sigma_\varphi}\mathcal{T}(\widetilde{R},\lambda)$ *and* $\bigcup_{\lambda\in\nu+\Sigma_\varphi}\mathcal{T}(\widetilde{S},\lambda)$ *are $\mathcal{R}$-bounded as well.*

Proof. Again we use the representations

$$\widetilde{R}(\lambda)f = R(\lambda)f + R(\lambda)\widetilde{\mathcal{A}}\widetilde{R}(\lambda)f - S(\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{R}(\lambda)f,$$
$$\widetilde{S}(\lambda)g = S(\lambda)g + R(\lambda)\widetilde{\mathcal{A}}\widetilde{S}(\lambda)g - S(\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{S}(\lambda)g.$$

Instead of considering the uniform bounds, we now take the $\mathcal{R}$-bounds of these sets of operators. Hence, we will show that there is a constant $C > 0$ such that

$$\mathcal{R}\big(\bigcup_{\lambda\in\nu+\Sigma_\varphi}\mathcal{T}(\widetilde{R},\lambda)\big) \le C.$$

We will do this by showing that

$$\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big) \le C$$

for any finite collection $\lambda_1,\dots,\lambda_K \in \nu+\Sigma_\varphi$, where C does not depend on $K\in\mathbb{N}$ and the choice of $\lambda_1,\dots,\lambda_K$.

$\boldsymbol{R(\lambda_0+\lambda)}$**:** The $\mathcal{R}$-boundedness for the family of these operators is an assumption.

$\boldsymbol{R(\lambda_0+\lambda)\widetilde{\mathcal{A}}\widetilde{R}(\lambda_0+\lambda)}$**:** From (2.19) in the proof of Theorem 2.12 and (2.14) we know that

$$\begin{aligned}
\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(R\widetilde{\mathcal{A}}\widetilde{R},\lambda_i)\big) &\le \mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)\mathcal{R}_{L(X_p,\|\!|\cdot|\!\|_{X_p})}(\{\widetilde{\mathcal{A}}\widetilde{R}(\lambda_0+\lambda_i): i=1,\dots,K\})\\
&\le \varepsilon C\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)\mathcal{R}_{L((X_p,\|\!|\cdot|\!\|_{X_p}),(Y_p,\|\!|\cdot|\!\|_{Y_p}))}(\{\widetilde{R}(\lambda_0+\lambda_i): i=1,\dots,K\})\\
&\le \varepsilon C\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big)
\end{aligned}$$

for any $\varepsilon > 0$.

$\boldsymbol{S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{R}(\lambda_0+\lambda)}$**:** Using (2.24), we get

$$\begin{aligned}
\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(S\widetilde{B}_\gamma\widetilde{R},\lambda_i)\big) &\le \mathcal{R}_{L(Y_p,\|\!|\cdot|\!\|_{Y_p})}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S\widetilde{B}_\gamma,\lambda)\big)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big)\\
&\le \mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)\big)\|\widetilde{B}_\gamma\|_{L((Y_p,\|\!|\cdot|\!\|_{Y_p}),(\partial Y_p,\|\!|\cdot|\!\|_{\partial Y_p}))}\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big)\\
&\le \varepsilon C\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)\big)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big).
\end{aligned}$$

Altogether we have shown that

$$\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big) \le \mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big) + \varepsilon\left(\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)+\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)\big)\right)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big).$$

Choosing $\varepsilon > 0$ small enough we obtain

$$\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{R},\lambda_i)\big) \le C$$

with a constant independent of λ_i and $K \in \mathbb{N}$. Hence, it holds that

$$\mathcal{R}\big(\bigcup_{\lambda\in\nu+\Sigma_\varphi}\mathcal{T}(\widetilde{R},\lambda)\big) \le C.$$

For $\widetilde{S}$ we proceed in the same way: We will show that there is a constant $C > 0$ independent of λ such that

$$\mathcal{R}\big(\bigcup_{\lambda\in\nu+\Sigma_\varphi}\mathcal{T}(\widetilde{S},\lambda)\big) \le C.$$

We will do this by showing that

$$\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big) \le C$$

for any finite collection $\lambda_1,\dots,\lambda_K \in \nu+\Sigma_\varphi$, where C does not depend on $K \in \mathbb{N}$ and the choice of $\lambda_1,\dots,\lambda_K$. For this, we use the representation of $\widetilde{S}(\lambda)$ from above.

$S(\lambda_0+\lambda)$: The $\mathcal{R}$-boundedness for the family of these operators is an assumption.

$R(\lambda_0+\lambda)\widetilde{\mathcal{A}}\widetilde{S}(\lambda_0+\lambda)$: From (2.19) in the proof of Theorem 2.12 and (2.15) we know that

$$\begin{aligned}\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(R\widetilde{\mathcal{A}}\widetilde{S},\lambda_i)\big) &\le \mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)\mathcal{R}_{L(X_p,|||\cdot|||_{X_p})}(\{\widetilde{\mathcal{A}}\widetilde{S}(\lambda_0+\lambda_i) : i=1,\dots,K\})\\ &\le \varepsilon C\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)\mathcal{R}_{L((X_p,|||\cdot|||_{X_p}),(Y_p,|||\cdot|||_{Y_p}))}(\{\widetilde{S}(\lambda_0+\lambda_i) : i=1,\dots,K\})\\ &\le \varepsilon C\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big)\end{aligned}$$

for any $\varepsilon > 0$.

$\boldsymbol{S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{S}(\lambda_0+\lambda)}$**:** Using (2.24), we get

$$\begin{aligned}\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(S\widetilde{B}_\gamma\widetilde{S},\lambda_i)\big) &\le \mathcal{R}_{L(Y_p,|\!|\!|\cdot|\!|\!|_{Y_p})}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S\widetilde{B}_\gamma,\lambda)\big)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big)\\ &\le \mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)\big)\|\widetilde{B}_\gamma\|_{L((Y_p,|\!|\!|\cdot|\!|\!|_{Y_p}),(\partial Y_p,|\!|\!|\cdot|\!|\!|_{\partial Y_p}))}\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big)\\ &\le \varepsilon C\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)\big)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big).\end{aligned}$$

Altogether we have shown that

$$\begin{aligned}\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big) \le \mathcal{R}\big(&\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)\big)\\ &+\varepsilon\left(\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(R,\lambda)\big)+\mathcal{R}\big(\bigcup_{\lambda\in\Sigma_\varphi}\mathcal{T}(S,\lambda)\big)\right)\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big).\end{aligned}$$

Choosing $\varepsilon>0$ small enough we obtain

$$\mathcal{R}\big(\bigcup_{i=1}^{K}\mathcal{T}(\widetilde{S},\lambda_i)\big)\le C$$

with a constant independent of λ_i and $K\in\mathbb{N}$. Hence, it holds that

$$\mathcal{R}\big(\bigcup_{\lambda\in\nu+\Sigma_\varphi}\mathcal{T}(\widetilde{S},\lambda)\big)\le C.$$

□

2.15 Remark. Consider the situation of Theorem 2.12. Note that, since $\|\cdot\|_{Y_p}$ and $\|\cdot\|_{\mathcal{A}_{\mathcal{B}_\gamma}}$ are equivalent by Remark 2.8, we also have constants $C_1,C_2>0$ such that

$$C_1\|u\|_{Y_p}\le\|(\mathcal{A}+\widetilde{\mathcal{A}})u\|_{X_p}+\|u\|_{X_p}\le C_2\|u\|_{Y_p}$$

for all $u\in Y_p$. Indeed, we have that

$$\begin{aligned}\|\mathcal{A}u\|_{X_p}&\le\|(\mathcal{A}+\widetilde{\mathcal{A}})u\|_{X_p}+\|\widetilde{\mathcal{A}}u\|_{X_p}\le\|(\mathcal{A}+\widetilde{\mathcal{A}})u\|_{X_p}+\varepsilon\|u\|_{Y_p}+C(\varepsilon)\|u\|_{X_p}\\ &\le\|(\mathcal{A}+\widetilde{\mathcal{A}})u\|_{X_p}+\varepsilon C\|\mathcal{A}u\|_{X_p}+C(\varepsilon)\|u\|_{X_p}\end{aligned}$$

where we used (2.19). Hence, for ε small enough we obtain

$$\|u\|_{Y_p}\le\|\mathcal{A}u\|_{X_p}+\|u\|_{X_p}\le\tfrac{1}{1-C\varepsilon}\|(\mathcal{A}+\widetilde{\mathcal{A}})u\|_{X_p}+(1+\tfrac{C(\varepsilon)}{1-C\varepsilon})\|u\|_{X_p}.$$

Applying the same argument to $(\mathcal{A}+\widetilde{\mathcal{A}})-\widetilde{\mathcal{A}}$ yields the continuity of the inverse imbedding.

Now, we are able to show the stability of both sectoriality and $\mathcal{R}$-sectoriality under suitable perturbations, provided we impose an additional condition concerning the orders of the transmission operators: we will assume that these orders are not less than the orders in the underlying space $X_p^\pm$. This is in accordance with results as in [DD11], saying that boundary (or in this case transmission) conditions have to be imposed in the underlying space provided these conditions are meaningful in this space. Hence, our additional condition means that the transmission conditions $\mathcal{B}_\gamma^\pm$ are made up of conditions that cannot be imposed in the space X_p already. Once more, we refer to Chapter 3 where we will impose some of the transmission conditions in the basis space.

2.16 Proposition. *Assume that the assumptions of Theorem 2.13 are satisfied. Furthermore, assume that for all $i = 1, \dots, M$ and all $j = 1, \dots, N^\pm$ the inequality $k_{ij}^\pm \geq s_j^\pm$ holds true if $k_{ij}^\pm \neq -\infty$.*
If the operator $\mathcal{A}_{\mathcal{B}_\gamma}$ is sectorial in the sector with opening angle $\varphi \in (0, \pi]$, then the operator $\nu + (\mathcal{A} + \widetilde{\mathcal{A}})_{(\mathcal{B}+\widetilde{\mathcal{B}})_\gamma}$, defined analogously to $\mathcal{A}_{\mathcal{B}_\gamma}$, is sectorial in the same sector.

2.17 Remark. Note that if $k_{ij}^\pm = -\infty$ for some $i \in \{1, \dots, M\}$ and $j \in \{1, \dots, N^\pm\}$, we have $B_{ij}^\pm = 0$ and hence we have no perturbation $\widetilde{B}_{ij}^\pm$. Hence, all corresponding terms are equal to zero and we do not have to impose $k_{ij}^\pm \geq s_j^\pm$ in this case, as can be seen in the proof.

Proof of Proposition 2.16. Assume that $\mathcal{A}_{\mathcal{B}_\gamma}$ is sectorial in the sector with opening angle $\varphi \in (0, \pi]$. In particular, $\mathcal{A}_{\mathcal{B}_\gamma}$ is a closed operator. By Remark 2.8 the norms $\|\cdot\|_{Y_p}$ and $\|\cdot\|_{\mathcal{A}_{\mathcal{B}_\gamma}}$ are equivalent on $D(\mathcal{A}_{\mathcal{B}_\gamma})$.
First, we note that

$$
\begin{aligned}
\|\widetilde{R}(\lambda)f\|_{Y_p} &\leq C\left(\|(\mathcal{A}+\widetilde{\mathcal{A}})\widetilde{R}(\lambda)f\|_{X_p} + \|\widetilde{R}(\lambda)f\|_{X_p}\right) \\
&\leq C\left(\|(\lambda-\mathcal{A}-\widetilde{\mathcal{A}})\widetilde{R}(\lambda)f\|_{X_p} + \|\lambda\widetilde{R}(\lambda)f\|_{X_p}\right) \\
&\leq C\left(\|f\|_{X_p} + \|\lambda\widetilde{R}(\lambda)f\|_{X_p}\right).
\end{aligned} \tag{2.25}
$$

Again, we use the representation of $\widetilde{R}(\lambda)$ given by

$$
\widetilde{R}(\lambda) = R(\lambda) + R(\lambda)\widetilde{\mathcal{A}}\widetilde{R}(\lambda) - S(\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{R}(\lambda), \tag{2.26}
$$

see Theorem 2.13.
As in (2.19), it holds that $\|\widetilde{\mathcal{A}}u\|_{X_p} \leq \varepsilon C\|u\|_{Y_p} + C(\varepsilon)\|u\|_{X_p}$ for any $u \in Y_p$. Therefore, we get

$$
\|\widetilde{\mathcal{A}}\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} \leq \varepsilon C\|\widetilde{R}(\lambda_0+\lambda)f\|_{Y_p} + C(\varepsilon)\|\widetilde{R}(\lambda_0+\lambda)f\|_{X_p}
$$

Now, choosing $|\lambda|$ large enough (it might depend on ε), using the sectoriality of $R(\lambda)$ and (2.25), we obtain that

$$
\begin{aligned}
\|\lambda R(\lambda_0+\lambda)\widetilde{\mathcal{A}}\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} &\leq \varepsilon C\left(\|\widetilde{R}(\lambda_0+\lambda)f\|_{Y_p} + \|\lambda\widetilde{R}(\lambda_0+\lambda)f\|_{X_p}\right) \\
&\leq \varepsilon C\left(\|\lambda\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} + \|f\|_{X_p}\right)
\end{aligned} \tag{2.27}
$$

with a constant $C > 0$ independent of λ.
By (2.15) we have

$$\|\lambda S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma \widetilde{R}(\lambda_0+\lambda)f\|_{X_p} \le C\Big|\Big|\Big|\widetilde{\mathcal{B}}\widetilde{R}(\lambda_0+\lambda)f\Big|\Big|\Big|_{Z_p}.$$

In order to estimate the right hand side, we consider the summands separately. For large $|\lambda|$, it holds that

$$\begin{aligned}
&\Big|\Big|\Big|\widetilde{B}_{ij}^{\pm}\widetilde{R}(\lambda_0+\lambda)f\Big|\Big|\Big|_{H_p^{s_j^\pm+t_j^\pm-k_{ij}^\pm}} \\
&= \|\widetilde{B}_{ij}^{\pm}\widetilde{R}(\lambda_0+\lambda)f\|_{H_p^{s_j^\pm+t_j^\pm-k_{ij}^\pm}} + |\lambda|^{(s_j^\pm+t_j^\pm-k_{ij}^\pm)/t_j^\pm}\|\widetilde{B}_{ij}^{\pm}\widetilde{R}(\lambda_0+\lambda)f\|_{L_p} \\
&\le \varepsilon\left(\|\operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda)f\|_{H_p^{s_j^\pm+t_j^\pm}} + |\lambda|^{(s_j^\pm+t_j^\pm-k_{ij}^\pm)/t_j^\pm}\|\operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda)f\|_{H_p^{k_{ij}^\pm}}\right).
\end{aligned} \tag{2.28}$$

For the latter summand we have the estimate

$$\begin{aligned}
|\lambda|^{(s_j^\pm+t_j^\pm-k_{ij}^\pm)/t_j^\pm}\|\operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda)f\|_{H_p^{k_{ij}^\pm}} &\le \|\operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda)f\|_{H_p^{s_j^\pm+t_j^\pm}} \\
&\quad + \|\lambda\operatorname{pr}_j^\pm \widetilde{R}(\lambda_0+\lambda)f\|_{H_p^{s_j^\pm}}
\end{aligned} \tag{2.29}$$

due to [Gri85] and $k_{ij}^\pm \ge s_j^\pm$. Together with (2.25) this yields

$$\|\lambda S(\lambda_0+\lambda)\widetilde{\mathcal{B}}_\gamma\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} \le \varepsilon C\|\lambda\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} + \varepsilon C\|f\|_{X_p} \tag{2.30}$$

for $|\lambda|$ large enough.
Altogether, from the representation (2.26) we deduce

$$\|\lambda\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} \le \varepsilon C\|\lambda\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} + C\|f\|_{X_p}$$

due to the sectoriality of $R(\lambda)$ and the estimates (2.27) and (2.30). Hence,

$$\|\lambda\widetilde{R}(\lambda_0+\lambda)f\|_{X_p} \le \tfrac{C}{1-\varepsilon C}\|f\|_{X_p}$$

with a constant $C > 0$ independent of λ. Now, choosing $\varepsilon > 0$ small enough, the assertion follows. □

2.18 Proposition. *Assume that the assumptions of Proposition 2.14 are satisfied. Furthermore, assume that for all $i = 1,\dots,M$ and all $j = 1,\dots,N^\pm$ the inequality $k_{ij}^\pm \ge s_j^\pm$ holds true if $k_{ij}^\pm \ne -\infty$.*
If the operator $\mathcal{A}_{\mathcal{B}_\gamma}$ is $\mathcal{R}$-sectorial in the sector with opening angle $\varphi \in (0,\pi]$, the operator $\nu + (\mathcal{A}+\widetilde{\mathcal{A}})_{(\mathcal{B}+\widetilde{\mathcal{B}})_\gamma}$, defined analogously to $\mathcal{A}_{\mathcal{B}_\gamma}$, is $\mathcal{R}$-sectorial in the same sector.

Proof. Let $\mathcal{A}_{\mathcal{B}_\gamma}$ be $\mathcal{R}$-sectorial in the sector Σ_φ. We show that

$$\mathcal{R}\big(\{\lambda\widetilde{R}(\lambda_0+\lambda) : \lambda \in \nu+\Sigma_\varphi\}\big) < \infty$$

by showing

$$\mathcal{R}\big(\{\lambda_i \widetilde{R}(\lambda_0+\lambda_i) : \lambda_i \in \nu + \Sigma_\varphi, i = 1,\dots,K\}\big) \leq C$$

with a constant $C > 0$ independent of $K \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_K \in \nu + \Sigma_\varphi$. Once more, we use the representation (2.26) of $\widetilde{R}(\lambda)$.
Since $\mathcal{A}_{\mathcal{B}_\gamma}$ is $\mathcal{R}$-sectorial, the corresponding assertion holds for $R(\lambda_0 + \lambda)$.
First, analogously to (2.25) we have for any $K \in \mathbb{N}$ and fixed $\lambda_1, \dots, \lambda_K \in \nu + \Sigma_\varphi$ with $|\lambda_i| \geq 1$

$$\begin{aligned}
\Big\| \sum_{i=1}^K r_i \widetilde{R}(\lambda_i) f_i \Big\|_{Y_p} &\leq C \left(\Big\| (\mathcal{A} + \widetilde{\mathcal{A}}) \sum_{i=1}^K r_i \widetilde{R}(\lambda_i) f_i \Big\|_{X_p} + \Big\| \sum_{i=1}^K r_i \widetilde{R}(\lambda_i) f_i \Big\|_{X_p} \right) \\
&\leq C \Bigg(\Big\| \sum_{i=1}^K r_i (-\lambda_i + \mathcal{A} + \widetilde{\mathcal{A}}) \widetilde{R}(\lambda_i) f_i \Big\|_{X_p} + \Big\| \sum_{i=1}^K r_i \lambda_i \widetilde{R}(\lambda_i) f_i \Big\|_{X_p} \\
&\qquad + \Big\| \sum_{i=1}^K r_i \widetilde{R}(\lambda_i) f_i \Big\|_{X_p} \Bigg) \\
&\leq C \left(1 + \mathcal{R}\big(\{\lambda_i \widetilde{R}(\lambda_i) : i = 1,\dots,K\}\big) \right) \Big\| \sum_{i=1}^K r_i f_i \Big\|_{X_p} \quad (f_i \in X_p),
\end{aligned} \tag{2.31}$$

where we used the contraction principle of Kahane (see [DHP03], Lemma 3.5) in the last step.
Using this estimate, we get for any $K \in \mathbb{N}$ and fixed $\lambda_1, \dots, \lambda_K \in \nu + \Sigma_\varphi$ with $|\lambda_i|$ large enough

$$\begin{aligned}
&\mathcal{R}\big(\{\lambda_i R(\lambda_0+\lambda_i) \widetilde{\mathcal{A}} \widetilde{R}(\lambda_0+\lambda_i) : i = 1,\dots,K\}\big) \\
&\leq \mathcal{R}\big(\{\lambda R(\lambda_0+\lambda) : \lambda \in \Sigma_\varphi\}\big) \mathcal{R}_{L(X_p)}\big(\{\widetilde{\mathcal{A}} \widetilde{R}(\lambda_0+\lambda_i) : i = 1,\dots,K\}\big) \\
&\leq \varepsilon C \mathcal{R}_{L(X_p,Y_p)}\big(\{\widetilde{R}(\lambda_0+\lambda_i) : i = 1,\dots,K\}\big) \\
&\quad + C(\varepsilon) \mathcal{R}_{L(X_p)}\big(\{\widetilde{R}(\lambda_0+\lambda_i) : i = 1,\dots,K\}\big) \\
&\leq \varepsilon C \left(1 + \mathcal{R}\big(\{\lambda_i \widetilde{R}(\lambda_i) : i = 1,\dots,K\}\big) \right),
\end{aligned}$$

where we used the interpolation inequality in the third step and the contraction principle in the last step.
In order to treat the last term in (2.26), we observe that for any $K \in \mathbb{N}$ and any fixed $\lambda_1, \dots, \lambda_K \in \nu + \Sigma_\varphi$ with $|\lambda_i|$ large enough for all $i \in \{1, \dots, K\}$ we have

$$\begin{aligned}
&\Big\| \sum_{i=1}^K S(\lambda_0+\lambda_i) \widetilde{\mathcal{B}}_\gamma \widetilde{R}(\lambda_i) f_i \Big\|_{X_p} \\
&\leq \mathcal{R}_{L((\partial Y_p, \|\!|\cdot|\!\|_{\partial Y_p}), X_p)}\big(\{\lambda S(\lambda_0+\lambda) : \lambda \in \Sigma_\varphi\}\big) \Big|\!\Big|\!\Big| \sum_{i=1}^K r_i \widetilde{\mathcal{B}}_\gamma \widetilde{R}(\lambda_0+\lambda_i) f_i \Big|\!\Big|\!\Big|_{Z_p} \\
&\leq \varepsilon C \left(1 + \Big\| \sum_{i=1}^K r_i \lambda_i \widetilde{R}(\lambda_0+\lambda_i) f_i \Big\|_{X_p} + \Big\| \sum_{i=1}^K r_i \widetilde{R}(\lambda_0+\lambda_i) f_i \Big\|_{X_p} \right) \\
&\leq \varepsilon C \left(\Big\| \sum_{i=1}^K r_i f_i \Big\|_{X_p} + \Big\| \sum_{i=1}^K r_i \lambda_i \widetilde{R}(\lambda_0+\lambda_i) f_i \Big\|_{X_p} \right) \quad (f_i \in X_p)
\end{aligned}$$

thanks to (2.28), (2.29), (2.31) and the contraction principle. Hence, for any $K \in \mathbb{N}$ and any fixed $\lambda_1, \ldots, \lambda_K \in \nu + \Sigma_\varphi$ with $|\lambda_i|$ large enough for all $i \in \{1, \ldots, K\}$ we deduce

$$\begin{aligned}&\mathcal{R}\big(\{\lambda_i \widetilde{R}(\lambda_0 + \lambda_i) : i = 1, \ldots, K\}\big)\\ &\leq C + 2\varepsilon C \left(1 + \mathcal{R}\big(\{\lambda_i \widetilde{R}(\lambda_0 + \lambda_i) : i = 1, \ldots, K\}\big)\right),\end{aligned}$$

i.e.

$$\mathcal{R}\big(\{\lambda_i \widetilde{R}(\lambda_0 + \lambda_i) : i = 1, \ldots, K\}\big) \leq \frac{C + 2\varepsilon C}{1 - 2\varepsilon C},$$

with a constant $C > 0$ independent of $K \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_K \in \nu + \Sigma_\varphi$, wherefore the assertion follows for $\varepsilon > 0$ small enough. □

2.19 Remark. The proofs above show that if we are only interested in perturbations of the operator $\mathcal{A}$ or the transmission operator $\mathcal{B}_\gamma$, it suffices that the corresponding assumptions on $R(\lambda)$ and $S(\lambda)$ are satisfied, respectively.

Before ending this chapter with some remarks on localization procedures, we give an example of a mixed-order boundary value system that fits in our setting.

2.20 Example. We consider the *thermoelastic plate equation*

$$\begin{aligned} u_{tt} + \Delta^2 u + \Delta\theta &= f_1,\\ \theta_t - \Delta\theta - \Delta u_t &= f_2\end{aligned}$$

in $\mathbb{R}^n_+$ with *free boundary conditions*

$$\begin{aligned} \Delta u - (1-\beta)\Delta' u + \theta &= g_1,\\ \partial_\nu(\Delta u + (1-\beta)\Delta' u + \theta) &= g_2,\\ \partial_\nu \theta &= g_3\end{aligned}$$

on $\mathbb{R}^{n-1}$. Here, $\beta \in [0,1)$ is fixed and $\Delta' = \sum_{i=1}^{n-1} \partial_i^2$ stands for the Laplace-Beltrami operator on $\Gamma = \mathbb{R}^{n-1}$.
As indicated above, in order to treat boundary value problems, we let $X_p^- = \{0\}^3$ and $\mathcal{A}^- = 0$. Now, we let $s_1^+ = 2, s_2^+ = s_3^+ = 0$ and $t_1^+ = t_2^+ = t_3^+ = 2$ as well as

$$\mathcal{A}^+ \colon Y_p^+ \to X_p^+, \quad (u, v, \theta)^\top \mapsto A^+(x, D)(u, v, \theta)^\top = \begin{pmatrix} 0 & 1 & 0\\ -\Delta^2 & 0 & -\Delta\\ 0 & \Delta & \Delta\end{pmatrix}\begin{pmatrix} u\\ v\\ \theta\end{pmatrix}.$$

Here, we have defined X_p^+ and Y_p^+ as before, see (2.7). Then, we have that $s_j^\pm = 0$ if there exists a $k \in \{1,2,3\}$ such that $\operatorname{ord}(A_{jk}^+(x,D)) > 0$ and moreover, the orders m_{jk}^+ of $A_{jk}^+(x,D)$ satisfy $m_{jk}^+ \leq s_k^+ + t_k^+ - s_j^+$ for all $1 \leq j,k \leq 3$.
The space X_p is simply given by $X_p := X_p^- \times X_p^+ = \{0\}^3 \times X_p^+$. For the boundary operators, we write $\mathcal{B}_i^- = 0$ for $i = 1,2,3$ and

$$\begin{pmatrix}\mathcal{B}_1^+\\ \mathcal{B}_2^+\\ \mathcal{B}_3^+\end{pmatrix} = \begin{pmatrix} B_{11}^+(x,D) & B_{12}^+(x,D) & B_{13}^+(x,D)\\ B_{21}^+(x,D) & B_{22}^+(x,D) & B_{23}^+(x,D)\\ B_{31}^+(x,D) & B_{32}^+(x,D) & B_{33}^+(x,D)\end{pmatrix} = \begin{pmatrix} \Delta - (1-\beta)\Delta' & 0 & 1\\ \partial_\nu(\Delta + (1-\beta)\Delta') & 0 & 0\\ 0 & 0 & \partial_\nu\end{pmatrix}.$$

Note that the orders k_{ij}^{+} of the operators $B_{ij}^{+}(x, D)$ satisfy $k_{ij}^{+} \leq s_j^{+} + t_j^{+} - 1$ for all $i, j = 1, 2, 3$. Finally, the spaces Z_p and ∂Y_p are given by

$$\begin{aligned} Z_p &= \{0\}^6 \times \prod_{i=1}^{3} \prod_{j=1}^{3} H_p^{s_j^{+}+t_j^{+}-k_{ij}^{+}}(\mathbb{R}_+^n) \\ &= \{0\}^6 \times \left(H_p^2(\mathbb{R}_+^n)^3 \times H_p^1(\mathbb{R}_+^n) \times H_p^2(\mathbb{R}_+^n)^2 \times H_p^4(\mathbb{R}_+^n) \times H_p^2(\mathbb{R}_+^n) \times H_p^1(\mathbb{R}_+^n)\right) \end{aligned}$$

and

$$\partial Y_p = \partial Y_p^{-} + \partial Y_p^{+} = \{0\}^3 + \left(B_{pp}^{2-1/p}(\mathbb{R}^{n-1}) \times B_{pp}^{1-1/p}(\mathbb{R}^{n-1}) \times B_{pp}^{1-1/p}(\mathbb{R}^{n-1})\right).$$

Theorem 1.4 in [DS17] states, amongst other things, that (2.14) and (2.15) are satisfied. In fact, the authors already have shown their corresponding results in domains. However, they are using the model problem and perform the localization procedure (via a bent half-space) by hand. In particular, they show $\mathcal{R}$-boundedness of the solution operators by hand. In contrast, our abstract results directly allow the transfer of ($\mathcal{R}$-)sectoriality from the model problem to the problem in domain.

We finish this chapter with some remarks on how to use the results above in order to deal with variable coefficients, bent half-spaces, domains and transmission-boundary value problems. As all these things are pretty standard, we just give a short introduction and sketch the procedures roughly. Among other literature, we refer to [DHP03], Section 8, [DS17], Section 5 and [Fro16], Chapter 4 for more details on these topics.

The general strategy is as follows:
Assume that the operator $\mathcal{A}_{\mathcal{B}_\gamma}$ is ($\mathcal{R}$)-sectorial for constant coefficients $a_{jk,\alpha}^{\pm}$ and $b_{ij,\beta}^{\pm}$. We are interested in solving

$$(\lambda - \mathcal{A}(x, D))u = f \text{ in } \mathbb{R}^n, \tag{2.32}$$

$$\mathcal{B}_\gamma(x, D)u = 0 \text{ on } \Gamma = \mathbb{R}^{n-1} \tag{2.33}$$

with variable coefficients $a_{jk,\alpha}^{\pm}$ and $b_{ij,\beta}^{\pm}$.
For this, we assume that the limits at infinity $\lim_{|x|\to\infty} \widetilde{a}_{jk,\alpha}^{\pm}(x)$ and $\lim_{|x|\to\infty} \widetilde{b}_{ij,\beta}^{\pm}(x)$ exist. Then, there exists $R > 0$ and an open covering $(U_l)_{l=1}^{L}$ of the compact set $\overline{B(0,R)} := \{x \in \mathbb{R}^n : |x| \leq R\}$ such that the smallness conditions (2.16) and (2.17) are satisfied within $U_l, l = 1, \dots L$ and in $U_0 := \mathbb{R}^n \setminus \overline{B(0,R)}$.
Now, choose a partition of unity $(\varphi_l)_{l=0}^{L}$ for $(U_l)_{0=1}^{L}$. Furthermore, let $(\psi_l)_{l=0}^{L} \subset C^\infty(\mathbb{R}^n)$ with $0 \leq \psi_l \leq 1$, $\operatorname{supp}(\psi_l) \subset U_l$ and $\psi_l = 1$ on $\operatorname{supp}(\varphi_l)$. Here, $\operatorname{supp}(f)$ denotes the support

$$\operatorname{supp}(f) := \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$$

of a function $f \colon \mathbb{R}^n \to \mathbb{C}$.
For $x_l \in U_l$ let $R(\lambda, x_l)f$ denote the solution to the problem

$$\begin{aligned} (\lambda - \mathcal{A}(x_l, D))u &= f \text{ in } \mathbb{R}^n, \\ \mathcal{B}_\gamma(x_l, D)u &= 0 \text{ on } \Gamma = \mathbb{R}^{n-1} \end{aligned}$$

where now $\mathcal{A}(x_l, D)_{\mathcal{B}_\gamma(x_l,D)}$ is $(\mathcal{R})$-sectorial. In order to solve (2.32), we make the ansatz

$$R(\lambda)f := \sum_{l=0}^{L} \varphi_l R(\lambda, x_l)\psi_l f.$$

For $|\lambda|$ large enough, one can show that this is almost the resolvent for $\mathcal{A}(x, D)_{\mathcal{B}_\gamma(x,D)}$. The wrong terms usually have lower order and can often be handled by using interpolation theory, as we have also seen in the proofs above. Moreover, the $(\mathcal{R})$-sectoriality carries over.

Hence, we are able to treat variable coefficients. If we are interested in solving

$$\begin{aligned}(\lambda - \mathcal{A})u &= f,\\ \mathcal{B}_\gamma u &= 0\end{aligned}$$

with a bent transmission interface Γ or in a bounded domain, one uses coordinate transform and shows that the transformed operators fall in the same setting as the variable coefficients. Note that in the case of a bounded domain we can drop the assumption on the existing limits at infinity for the coefficients as the closure of a bounded domain is already compact. Note that the space

$$\begin{aligned}X_p := \big\{u = (u^-, u^+) = (u_1^-, \dots, u_{N^-}^-, u_1^+, \dots, u_{N^+}^+) \in X_p^- \times X_p^+ :\\ \chi_{\mathbb{R}^n_-} u_j^- + \chi_{\mathbb{R}^n_+} u_j^+ \in H^{r_j}(\mathbb{R}^n, E) \text{ for } j \in \max\{N^-, N^+\}\big\}\end{aligned}$$

is invariant under change of coordinates. For example, let $u\colon \mathbb{R}^n_+ \to \mathbb{C}$ and $v\colon \mathbb{R}^n_- \to \mathbb{C}$ be two, sufficiently smooth functions that satisfy $u = v$ on $\Gamma = \mathbb{R}^{n-1}$. Then, $\partial_\tau u = \partial_\tau v$ on Γ where ∂_τ denotes tangential derivatives. If now further $\partial_\nu u = \partial_\nu v$ holds true on Γ, we obtain $\nabla u = \nabla v$ on Γ. Iteratively, ascending transmission conditions of the form $\partial_\nu^k u = \partial_\nu^k v$, $k = 0, \dots r$ for $r \in \mathbb{N}_0$ on Γ imply $D^\alpha u = D^\alpha v$ on Γ for any $|\alpha| \le r$. Hence, the space X_p does not change for transformed functions $\widetilde{u} = u \circ \Phi$ for some coordinate transform Φ. This means, that if we transform the problem from the full-space case to a bent half-space or a domain, the underlying space does not change in the sense that we can use the same description for the underlying space.
In contrast, for example the space

$$\begin{aligned}\widetilde{X}_p := \big\{u = (u^-, u^+) = (u_1^-, \dots, u_{N^-}^-, u_1^+, \dots, u_{N^+}^+) \in X_p^- \times X_p^+ :\\ u_1^- = u_1^+,\ \partial_\nu^2 u_1^- = \partial_\nu^2 u_1^+ \text{ on } \Gamma = \mathbb{R}^{n-1}\big\}\end{aligned}$$

is not invariant under coordinate transform as the additional conditions change under the change of coordinates, i.e. under the change of the unit normal vector. With our choice of the underlying space X_p, we omit the problem of determining the correct model problem we have to investigate in order to solve the originally given problem in a bent half-space or a domain.

Finally, we also want to sketch the procedure for the treatment of transmission-boundary value problems or boundary value problems with different boundary operators on different connected components of the boundary. In fact, we will illustrate the procedure using boundary value problems in order to save time and notation.
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\Gamma = \partial\Omega = \Gamma_1 \dot{\cup} \Gamma_2$ such

that Γ_1 and Γ_2 are both open in the trace topology on Γ. Let $\mathcal{B}_\gamma^1$ and $\mathcal{B}_\gamma^2$ be boundary operators such that

$$\begin{aligned} (\lambda - \mathcal{A})u &= f \quad \text{in } \Omega, \\ \mathcal{B}_\gamma^i u &= 0 \quad \text{on } \Gamma \end{aligned}$$

are ($\mathcal{R}$)-sectorial for $i = 1, 2$. Now, we are interested in the problem

$$\begin{aligned} (\lambda - \mathcal{A})u &= f \quad \text{in } \Omega, \\ \mathcal{B}_\gamma^1 u &= 0 \quad \text{on } \Gamma_1, \\ \mathcal{B}_\gamma^2 u &= 0 \quad \text{on } \Gamma_2. \end{aligned}$$

Let $\varphi_1 \in C^\infty(\overline{\Omega})$ with $0 \le \varphi_1 \le 1$, $\varphi_1 = 1$ in a neighbourhood of Γ_1 and $\operatorname{supp}(\varphi_1) \cap \Gamma_2 = \emptyset$. We set $\varphi_2 := 1 - \varphi_1$ and let $\psi_1, \psi_2 \in C^\infty(\overline{\Omega})$ with $\psi_i = 1$ on $\operatorname{supp}(\varphi_i)$, $\operatorname{supp}(\psi_1) \cap \Gamma_2 = \emptyset$ and $\operatorname{supp} \psi_2 \cap \Gamma_1 = \emptyset$.
Let $u^{(i)} = R_i(\lambda)\psi_i f$ be the unique solution of

$$\begin{aligned} (\lambda - \mathcal{A})u^{(i)} &= \psi_i f, \quad \text{in } \Omega, \\ \mathcal{B}_\gamma^1 u^{(i)} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

Once more, we make the ansatz

$$R(\lambda)f := \varphi_1 R_1(\lambda)\psi_1 f + \varphi_2 R_2(\lambda)\psi_2 f$$

and obtain that $R(\lambda)$ is almost the desired resolvent: it holds that

$$\mathcal{A}R(\lambda)f = (1 + T(\lambda))f$$

for some bounded operator $T(\lambda)$ with norm less than 1 for $|\lambda|$ large enough, wherefore $1 + T(\lambda)$ is invertible. Again, one can show that the ($\mathcal{R}$)-sectoriality carries over. For a concrete example for the biharmonic operator Δ^2 with *clamped* and *free* boundary conditions, we refer to the upcoming paper [BDH+18].

Chapter 3

A transmission problem for structurally damped plate equations in L_p

In this chapter, we consider a transmission problem for a *structurally damped plate equation*. We will solve the *model problem* and show that the solution operators for the stationary problem are $\mathcal{R}$-sectorial, which implies maximal L_p-regularity for the evolution equation. We refer to the Appendix for the notion of maximal L_p-regularity and $\mathcal{R}$-sectorial operators. The localization procedure, introduced in the previous chapter, yields corresponding results for domains with sufficiently smooth boundary and transmission interface.

In [CR82] and [CT89], structural damping was studied in an abstract Hilbert space setting. Here, the main assumption is that the damping operator B (such that the damping is given by $\partial_t B$) is *comparable* to the positive definite, self-adjoint 'main operator' $\mathscr{A}$, i.e., in the case of a plate equation, comparable to the biharmonic operator Δ^2. The authors proved that the structural damping provides a smoothing effect in the sense of analytic semigroups.
The authors of [FL14] were able to show similar results in a Banach space setting, where now the main operator $\mathscr{A} = B^2$ is the square of a sectorial operator B and the structural damping is given by $\partial_t B$.
For the plate equation, even in a non-linear case, where no semigroup is generated by the corresponding operator, the structural damping leads to exponential decay of the energy of the system.
Finally, in [DS15] the authors showed (see also [Fro16]) that structural damping for the clamped plate equation provides both nice properties, the exponential stability and the analyticity of the associated semigroup. In fact, the operator is $\mathcal{R}$-sectorial, i.e. the evolution equation possesses the property of maximal L_p-regularity. In this chapter, we will focus on this last result, generalizing it to the situation of a transmission problem with *canonical* transmission conditions.
However, the method used here is standard and might work for other systems as well, e.g. a coupled system of thermoelastic plates (as the thermoelastic part also yields $\mathcal{R}$-bounded solution operators, see [DR06], [DRS09] and [DS17]).

The model problem in the whole space with canonical transmission conditions reads as follows:

$$\partial_t^2 u_i + \beta_i \Delta^2 u_i - \rho_i \Delta \partial_t u_i = f_i \qquad \text{in } (0,\infty) \times \Omega_i, i = 1, 2, \tag{3.1}$$

$$\partial_\nu^{j-1} u_1 - \partial_\nu^{j-1} u_2 = 0 \qquad \text{on } (0,\infty) \times \Gamma, j = 1, \ldots, 4, \tag{3.2}$$

where β_i and ρ_i are strictly positive, real constants for $i = 1, 2$, Ω_1 is the positive half-space

$$\mathbb{R}^n_+ := \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

and Ω_2 is the negative half-space $\mathbb{R}^n_- = -\mathbb{R}^n_+$. The interface between Ω_1 and Ω_2 is denoted by $\Gamma := \partial\Omega_1$ and ∂_ν stands for the normal derivative with respect to the outer unit normal of Ω_1. We are interested in scalar valued solutions $u_i \colon (0, \infty) \times \Omega_i \to \mathbb{C}$. for $i = 1, 2$.
The transmission conditions (3.2) state that the two plates are connected smoothly, i.e. if we expect the solutions $u_i(t, \cdot)$ to be in the Sobolev space $H^4_p(\Omega_i)$ for all $t > 0$, the solution

$$u(t, \cdot) = \begin{cases} u_1(t, \cdot) & \text{in } \mathbb{R}^n_+, \\ u_2(t, \cdot) & \text{in } \mathbb{R}^n_- \end{cases}$$

in the whole space is an element of $H^4_p(\mathbb{R}^n)$ for all $t > 0$ as well (cf. Lemma A.5).
After studying the model problem where the equations act on the half-spaces and are connected on the interface Γ, we are able to solve the transmission problem in a sufficiently smooth bounded domain $\Omega = \Omega_1 \,\dot\cup\, \Omega_2 \cup \Gamma$ with transmission interface Γ, where Ω_2 lies in the interior of Ω_1. This is done using localization methods. Denoting the inner boundary Γ_2, the outer boundary Γ_1 and the interface between Ω_1 and Ω_2 as Γ, we consider the following transmission problem complemented with *generalized Dirichlet boundary conditions:*

$$\begin{aligned} \partial_t^2 u_i + \beta_i \Delta^2 u_i - \rho_i \Delta \partial_t u_i &= f_i && \text{in } (0, \infty) \times \Omega_i, i = 1, 2, \\ \partial_\nu^{j-1} u_1 - \partial_\nu^{j-1} u_2 &= 0, && \text{on } (0, \infty) \times \Gamma,\ j = 1, \ldots, 4, \\ u_i &= g_i, && \text{on } (0, \infty) \times \Gamma_i, i = 1, 2, \\ \partial_{\nu_i} u_i &= h_i, && \text{on } (0, \infty) \times \Gamma_i, i = 1, 2. \end{aligned}$$

In general, we are mainly interested in *clamped boundary conditions*, i.e. $g_i = h_i = 0$ for $i = 1, 2$. Both the model problem and the transmission problem in a bounded domain are complemented with initial conditions

$$\begin{aligned} u_i|_{t=0} &= \phi_i && \text{in } \Omega_i, i = 1, 2, \\ \partial_t u_i|_{t=0} &= \psi_i && \text{in } \Omega_i, i = 1, 2. \end{aligned}$$

The structurally damped plate equation (without transmission) was studied by R. Denk and R. Schnaubelt in [DS15] in a slightly simpler form. They proved maximal L_p-regularity on bounded time intervals in the whole space $\mathbb{R}^n$ as well as in the half-space $\mathbb{R}^n_+$ and in bounded domains of class C^4, where Dirichlet-Neumann boundary conditions were added. Here, zero boundary conditions had to be taken into the basis space in order to achieve the $\mathcal{R}$-boundedness of the solution operators involved. We will follow their approach to show maximal L_p-regularity for the transmission problems.

For elements $x \in \mathbb{R}^n$ we write $x = (x_1, \ldots, x_n)$ and $x' = (x_1, \ldots, x_{n-1})$. Furthermore, Σ_ϕ denotes the open sector

$$\Sigma_\phi := \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \phi\}$$

in the complex plane for $0 \le \phi \le \pi$. The Fourier transform $\mathscr{F}$, defined for Schwartz functions $f \in \mathscr{S}(\mathbb{R}^n)$, is given by

$$(\mathscr{F} f)(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\mathrm{i}x\xi} f(x)\, dx$$

and its inverse is denoted by $\mathscr{F}^{-1}$.

3.1. The model problem in the half-space

Throughout the whole chapter, let $1 < p < \infty$.
Let $\beta_1, \beta_2, \rho_1, \rho_2 > 0$ be real constants. We deal with the model problem for the transmission problem for the structurally damped plate equation

$$
\begin{aligned}
\partial_t^2 u_1 + \beta_1 \Delta^2 u_1 - \rho_1 \Delta \partial_t u_1 &= f_1, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n_+,\\
\partial_t^2 u_2 + \beta_2 \Delta^2 u_2 - \rho_2 \Delta \partial_t u_2 &= f_2, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n_-,\\
\partial_\nu^{j-1} u_1 - \partial_\nu^{j-1} u_2 &= 0, \quad (t,x) \in (0,\infty) \times \Gamma, j = 1, \ldots, 4,\\
u_i|_{t=0} &= \phi_i, \quad x \in \Omega_i, i = 1,2,\\
\partial_t u_i|_{t=0} &= \psi_i, \quad x \in \Omega_i, i = 1,2,
\end{aligned}
\tag{3.3}
$$

where $\Omega_1 := \mathbb{R}^n_+$, $\Omega_2 := \mathbb{R}^n_-$ and $\Gamma := \partial\mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$. To treat the system (3.3), we will write the equations as first order systems and use reflection in order to handle the transmission conditions. By doing so, the transmission problem can be read as a boundary value problem. First, let $v_1 := (u_1, \partial_t u_1)^\top$ and $v_2 := (u_2, \partial_t u_2)^\top$ where $a^\top$ is the transposed for some matrix a. Then we obtain the first order systems

$$
\begin{aligned}
\partial_t v_1 + A_1(D) v_1 &= \begin{pmatrix} 0 \\ f_1 \end{pmatrix}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n_+,\\
\partial_t v_2 + A_2(D) v_2 &= \begin{pmatrix} 0 \\ f_2 \end{pmatrix}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n_-
\end{aligned}
\tag{3.4}
$$

with initial conditions $v_1|_{t=0} = (\phi_1, \psi_1)^\top$ and $v_2|_{t=0} = (\phi_2, \psi_2)^\top$. Here we have set

$$A_i(D) := \begin{pmatrix} 0 & -1 \\ \beta_i \Delta^2 & -\rho_i \Delta \end{pmatrix}$$

for $i = 1,2$. We put $D := -\mathrm{i}(\partial_1, \ldots, \partial_n)^\top$. The symbols of the differential operators $A_i(D)$ (considered as operators in the whole space) are given by

$$A_i(\xi) := \begin{pmatrix} 0 & -1 \\ \beta_i |\xi|^4 & \rho_i |\xi|^2 \end{pmatrix}, \quad i = 1,2.$$

Since the outer unit normal of $\mathbb{R}^n_+$ is given by $\nu = -e_n = -(0, \ldots, 1)$, the transmission conditions can be written as

$$(-\partial_n)^{j-1}(v_1 - v_2) = 0, \quad j = 1, \ldots, 4. \tag{3.5}$$

For $\lambda \in \mathbb{C}$ we define $A_i(D,\lambda) := \lambda I + A_i(D)$ and similarly $A_i(\xi,\lambda) := \lambda I + A_i(\xi)$ for $i = 1,2$. The corresponding stationary problem to (3.4) with inhomogeneous data reads

$$
\begin{aligned}
A_1(D,\lambda) y_1 &= h_1 \quad \text{in } \mathbb{R}^n_+,\\
A_2(D,\lambda) \widetilde{y}_2 &= \widetilde{h}_2 \quad \text{in } \mathbb{R}^n_-,\\
\partial_\nu^{j-1}(y_1 - \widetilde{y}_2) &= g_j \quad \text{on } \mathbb{R}^{n-1}, j = 1, \ldots, 4.
\end{aligned}
\tag{3.6}
$$

Using the reflection $\tau_n \colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto (x', -x_n)$, we can write (3.6) as a system in $\mathbb{R}^n_+$ where the transmission conditions (3.5) become boundary conditions for the so-obtained system. Hence, let $y_2 := \widetilde{y}_2 \circ \tau_n, h_2 := \widetilde{h}_2 \circ \tau_n, y := (y_1, y_2)^\top, h = (h_1, h_2)^\top$ and $g = (g_1, \ldots, g_4)^\top$. Since $A_2(\xi', \xi_n) = A_2(\xi', -\xi_n)$, we obtain the system

$$\begin{aligned} A(D,\lambda)y &= h \quad \text{in } \mathbb{R}^n_+, \\ \gamma_0 B(D)y &= g \quad \text{on } \mathbb{R}^{n-1} \end{aligned} \tag{3.7}$$

in $\mathbb{R}^n_+$. Here we have set

$$A(D,\lambda) := \begin{pmatrix} \lambda & -1 & 0 & 0 \\ \beta_1\Delta^2 & \lambda - \rho_1\Delta & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & \beta_2\Delta^2 & \lambda - \rho_2\Delta \end{pmatrix}, \quad B(D) := \begin{pmatrix} 1 & 0 & -1 & 0 \\ -\partial_n & 0 & -\partial_n & 0 \\ \partial_n^2 & 0 & -\partial_n^2 & 0 \\ -\partial_n^3 & 0 & -\partial_n^3 & 0 \end{pmatrix} \tag{3.8}$$

and γ_0 denotes the trace on $\mathbb{R}^{n-1}$. The symbol $A(\xi,\lambda)$ of $A(D,\lambda)$ is given by

$$A(\xi,\lambda) := \begin{pmatrix} \lambda & -1 & 0 & 0 \\ \beta_1|\xi|^4 & \lambda + \rho_1|\xi|^2 & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & \beta_2|\xi|^4 & \lambda + \rho_2|\xi|^2 \end{pmatrix},$$

for $\xi \in \mathbb{R}^n$. For $\lambda = 0$ we will write

$$A(D) := A(D,0) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ \beta_1\Delta^2 & -\rho_1\Delta & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \beta_2\Delta^2 & -\rho_2\Delta \end{pmatrix}.$$

The problem (3.7) can be solved as follows: we show that there is a solution $\tilde{y}'$ of

$$A(D,\lambda)\tilde{y}' = e_+ h \text{ in } \mathbb{R}^n,$$

in the whole space $\mathbb{R}^n$, where $e_+ \in L(H_p^s(\mathbb{R}^n_+), H_p^s(\mathbb{R}^n))$ is a global coretraction of the restriction $r_+ \colon L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n_+)$ (or $r_+ \colon H_p^s(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n_+)$, respectively) onto $\mathbb{R}^n_+$ for all $s \in \mathbb{N}_0$ (see e.g. [Ama09], Section 4.4). We set $y' := r_+\tilde{y}'$ and write $y = y' + y''$, wherefore y'' has to solve the boundary value problem

$$\begin{aligned} A(D,\lambda)y'' &= 0 & &\text{in } \mathbb{R}^n_+, \\ \gamma B(D)y'' &= g - \gamma B(D)y' & &\text{on } \mathbb{R}^{n-1}. \end{aligned}$$

Hence, we can split the task of solving (3.7) into two parts: the analysis of the operator $A(D,\lambda)$, considered as an operator in the whole space, and the solution of a boundary value problem with homogeneous right-hand side and inhomogeneous boundary conditions.

3.1.1. The full-space problem

We start with some results for the full space problem

$$A(D,\lambda)y = h \text{ in } \mathbb{R}^n. \tag{3.9}$$

The following results are direct consequences of the results in [DS15], Section 2, where the authors considered the structurally damped plate equation in the full space case. We define

$$\alpha_\pm^{(i)} := \frac{1}{2\beta_i}\left(\rho_i \pm \sqrt{\rho_i^2 - 4\beta_i}\right) = \begin{cases} \frac{1}{2\beta_i}\left(\rho_i \pm \sqrt{\rho_i^2 - 4\beta_i}\right), & \rho_i^2 \geq 4\beta_i, \\ \frac{1}{2\beta_i}\left(\rho_i \pm \mathrm{i}\sqrt{4\beta_i - \rho_i^2}\right), & \rho_i^2 < 4\beta_i, \end{cases}$$

for $i = 1, 2$. Hence, the angle $\pm\vartheta_i = \pm\vartheta_i(\beta_i, \rho_i)$ of $\alpha_\pm^{(i)}$ is

$$\pm\vartheta_i = \begin{cases} 0, & \rho_i^2 \geq 4\beta_i, \\ \pm\arctan\left(\frac{1}{\rho_i}\sqrt{4\beta_i - \rho_i^2}\right), & \rho_i^2 < 4\beta_i. \end{cases}$$

Note that $\alpha_\pm^{(i)} = \frac{1}{\beta_i} e^{\pm \mathrm{i}\vartheta_i}$ for $\rho_i^2 < 4\beta_i$ and $\alpha_\pm^{(i)} > 0$ for $\rho_i^2 \geq 4\beta_i$. Moreover, we see that $\vartheta_i \nearrow \frac{\pi}{2}$ as $\rho_i \searrow 0$. Finally, we define $\vartheta := \max\{\vartheta_1, \vartheta_2\}$.

We are interested in invertibility, resolvent estimates and semigroup generation results for the L_p-realization of the operator $A(D)$ or suitable shifts of the operator, respectively. We employ the spaces

$$\begin{aligned} \mathbb{E} = \mathbb{E}_p &:= H_p^2(\mathbb{R}^n) \times L_p(\mathbb{R}^n), \\ \mathbb{F} = \mathbb{F}_p &:= H_p^4(\mathbb{R}^n) \times H_p^2(\mathbb{R}^n) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}^2 = \mathbb{E}_p^2 &:= \left(H_p^2(\mathbb{R}^n) \times L_p(\mathbb{R}^n)\right)^2 \cong H_p^2(\mathbb{R}^n) \times L_p(\mathbb{R}^n) \times H_p^2(\mathbb{R}^n) \times L_p(\mathbb{R}^n), \\ \mathbb{F}^2 = \mathbb{F}_p^2 &:= \left(H_p^4(\mathbb{R}^n) \times H_p^2(\mathbb{R}^n)\right)^2 \cong H_p^4(\mathbb{R}^n) \times H_p^2(\mathbb{R}^n) \times H_p^4(\mathbb{R}^n) \times H_p^2(\mathbb{R}^n). \end{aligned}$$

As before, norms of the Lebesgue space L_p and the Bessel potential spaces H_p^s for $s \in \mathbb{R}$ are denoted by $\|\cdot\|_{L_p}$ and $\|\cdot\|_{H_p^s}$, respectively. We consider the unbounded operators

$$A_{i,p} \colon \mathbb{E} \supset D(A_{i,p}) \to \mathbb{E}, \quad u \mapsto A_{i,p}u := A_i(D)u$$

and

$$A_p \colon \mathbb{E}^2 \supset D(A_p) \to \mathbb{E}^2, \quad u \mapsto A_p u := A(D)u,$$

where $D(A_{i,p}) := \mathbb{F}$ for $i = 1, 2$ and $D(A_p) := \mathbb{F}^2$. As usual, for $\lambda \in \mathbb{C}$ we denote with λ the scaled identity operator λI. Then, the following holds:

3.1 Proposition. *a) Let $i \in \{1, 2\}$. Then for all $\lambda \in \Sigma_{\pi-\vartheta_i}$, the operator $\lambda + A_{i,p} \colon \mathbb{F} \to \mathbb{E}$ is invertible.*

b) For all $\lambda \in \Sigma_{\pi-\vartheta}$, the operator $\lambda + A_p \colon \mathbb{F}^2 \to \mathbb{E}^2$ is invertible.

c) The operators $A_{1,p}, A_{2,p}$ and A_p are not sectorial in $\mathbb{E}$ and $\mathbb{E}^2$, respectively.

Proof. Writing the block matrix $A(\xi,\lambda)$ in the form

$$A(\xi,\lambda) = \begin{pmatrix} \lambda & -1 & 0 & 0 \\ \beta_1|\xi|^4 & \lambda+\rho_1|\xi|^2 & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & \beta_2|\xi|^4 & \lambda+\rho_2|\xi|^2 \end{pmatrix} = \begin{pmatrix} A_1(\xi,\lambda) & 0 \\ 0 & A_2(\xi,\lambda) \end{pmatrix},$$

we have $\det A(\xi,\lambda) = \det A_1(\xi,\lambda) \det A_2(\xi,\lambda)$. The determinant of the matrix $A_i(\xi,\lambda)$ is given by

$$\det A_i(\xi,\lambda) = \lambda^2 + \lambda\rho_i|\xi|^2 + \beta_i|\xi|^4 = \beta_i \left(\alpha_+^{(i)}\lambda + |\xi|^2\right)\left(\alpha_-^{(i)}\lambda + |\xi|^2\right)$$

for $i = 1, 2$. Therefore, the inverse matrix $A(\xi,\lambda)^{-1}$ of $A(\xi,\lambda)$ exists for all $\lambda \in \Sigma_{\pi-\vartheta}$ and is given by

$$A(\xi,\lambda)^{-1} = \begin{pmatrix} A_1(\xi,\lambda)^{-1} & 0 \\ 0 & A_2(\xi,\lambda)^{-1} \end{pmatrix}$$

where of course

$$A_i(\xi,\lambda)^{-1} = \frac{1}{\det A_i(\xi,\lambda)} \begin{pmatrix} \lambda+\rho_i|\xi|^2 & 1 \\ -\beta_i|\xi|^4 & \lambda \end{pmatrix}$$

for $i = 1, 2$. Now we can proceed as in [DS15], Proposition 2.2. For the reader's convenience, we sketch the approach roughly. We define the matrix operator $S_1(D) := \mathscr{F}^{-1}S_1(\xi)\mathscr{F}$ by setting

$$S_1(\xi) := \begin{pmatrix} 1+|\xi|^2 & 0 \\ 0 & 1 \end{pmatrix}$$

and let $S_2(D) := \mathscr{F}^{-1}(1+|\xi|^2)S_1(\xi)\mathscr{F}$ in order to obtain the isomorphisms

$$\begin{aligned} S_1(D)\colon H_p^2(\mathbb{R}^n) \times L_p(\mathbb{R}^n) &\to L_p(\mathbb{R}^n) \times L_p(\mathbb{R}^n), \\ S_2(D)\colon H_p^4(\mathbb{R}^n) \times H_p^2(\mathbb{R}^n) &\to L_p(\mathbb{R}^n) \times L_p(\mathbb{R}^n). \end{aligned}$$

Since for all fixed $\lambda \in \Sigma_{\pi-\vartheta}$, each of the terms

$$\frac{1+|\xi|^2}{\alpha_\pm^{(i)} + |\xi|^2}, \quad \frac{\lambda}{\alpha_\pm^{(i)} + |\xi|^2} \text{ and } \frac{|\xi|^2}{\alpha_\pm^{(i)} + |\xi|^2}$$

is bounded in ξ, using Mikhlin's theorem (e.g. [Gra08], Theorem 5.2.7), one shows that $M(D,\lambda)$, given by the matrix-valued multiplier symbol

$$\begin{aligned} M(\xi,\lambda) &:= \begin{pmatrix} S_2(\xi) & 0 \\ 0 & S_2(\xi) \end{pmatrix} A(\xi,\lambda)^{-1} \begin{pmatrix} S_1(\xi)^{-1} & 0 \\ 0 & S_1(\xi)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{(1+|\xi|^2)(\lambda+\rho_1|\xi|^2)}{\det A_1(\xi,\lambda)} & \frac{(1+|\xi|^2)^2}{\det A_1(\xi,\lambda)} & 0 & 0 \\ \frac{-\beta_1|\xi|^4}{\det A_1(\xi,\lambda)} & \frac{\lambda(1+|\xi|^2)}{\det A_1(\xi,\lambda)} & 0 & 0 \\ 0 & 0 & \frac{(1+|\xi|^2)(\lambda+\rho_2|\xi|^2)}{\det A_2(\xi,\lambda)} & \frac{(1+|\xi|^2)^2}{\det A_2(\xi,\lambda)} \\ 0 & 0 & \frac{-\beta_2|\xi|^4}{\det A_2(\xi,\lambda)} & \frac{\lambda(1+|\xi|^2)}{\det A_2(\xi,\lambda)} \end{pmatrix}, \end{aligned}$$

is a bounded linear operator on $L_p(\mathbb{R}^n)^4$. Part *c*) follows directly from [DS15], Proposition 2.2 b), where the assertion was shown for $A_{1,p}$ and $A_{2,p}$, using the fact that every L_p-Fourier multiplier is an L_∞-function. $\square$

In the same way, we can state results similar to Theorem 2.3 and Proposition 2.4 in [DS15]:

3.2 Theorem. *Let $i \in \{1,2\}, \varepsilon \in (0, \pi - \vartheta_i), \lambda \in \Sigma_{\pi-\vartheta_i-\varepsilon}$ and $h = (h_1, h_2)^\top \in \mathbb{E}$. Set*

$$y := (y_1, y_2)^\top := (\lambda + A_{i,p})^{-1} h.$$

Further, let $k \in \{0,1,2\}, \alpha \in \mathbb{N}_0^n$ with $|\alpha| = k, \gamma \in \mathbb{N}_0^2$, and $\delta \in \mathbb{N}_0^n$ with $|\delta| = 2$. Then there is a constant $C_\varepsilon > 0$ such that

$$\left\| \lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta y_1 \\ D^\alpha y_2 \end{pmatrix} \right\|_{L_p(\mathbb{R}^n)} \leq C_\varepsilon \left(\|\Delta h_1\|_{L_p(\mathbb{R}^n)} + \|h_2\|_{L_p(\mathbb{R}^n)} \right), \tag{3.10}$$

$$\|\lambda^{2-\frac{k}{2}} D^\alpha y_1\|_{L_p(\mathbb{R}^n)} \leq C_\varepsilon \left(\|\lambda h_1\|_{L_p(\mathbb{R}^n)} + \|h_2\|_{L_p(\mathbb{R}^n)} \right). \tag{3.11}$$

Moreover, the operator families

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta & 0 \\ 0 & D^\alpha \end{pmatrix} A_i(D,\lambda)^{-1} \right] : \lambda \in \Sigma_{\pi-\vartheta_i-\varepsilon} \right\} \tag{3.12}$$

in $L(\mathbb{E}, L_p(\mathbb{R}^n; \mathbb{C}^2))$,

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta & 0 \\ 0 & D^\alpha \end{pmatrix} A_i(D,\lambda)^{-1} \begin{pmatrix} (\lambda - \Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right] : \lambda \in \Sigma_{\pi-\vartheta_i-\varepsilon} \right\}, \tag{3.13}$$

in $L(L_p(\mathbb{R}^n; \mathbb{C}^2))$ and

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[(\lambda - \Delta)^{2-\frac{k}{2}} \begin{pmatrix} D^\alpha & 0 \end{pmatrix} A_i(D,\lambda)^{-1} \begin{pmatrix} (\lambda - \Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right] : \lambda \in \Sigma_{\pi-\vartheta_i-\varepsilon} \right\} \tag{3.14}$$

in $L(L_p(\mathbb{R}^n; \mathbb{C}^2), L_p(\mathbb{R}^n))$ are $\mathcal{R}$-bounded.

3.3 Remark. Note that Theorem 3.2 yields similar estimates and $\mathcal{R}$-bounded families for the operator $A(D, \lambda)$. Let $h = (h_1, \ldots, h_4)^T \in \mathbb{E}^2$ and set $y := (y_1, \ldots, y_4)^T := (\lambda + A_p)^{-1} h$. Then, for example, (3.10) implies

$$\left\| \lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta y_1 \\ D^\alpha y_2 \\ D^\alpha D^\delta y_3 \\ D^\alpha y_4 \end{pmatrix} \right\|_{L_p(\mathbb{R}^n)} \leq C_\varepsilon \left(\|\Delta h_1\|_{L_p(\mathbb{R}^n)} + \|h_2\|_{L_p(\mathbb{R}^n)} + \|\Delta h_3\|_{L_p(\mathbb{R}^n)} + \|h_4\|_{L_p(\mathbb{R}^n)} \right),$$

and by (3.12) the operator family

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{1-\frac{k}{2}} \begin{pmatrix} D^\alpha D^\delta & 0 & 0 & 0 \\ 0 & D^\alpha & 0 & 0 \\ 0 & 0 & D^\alpha D^\delta & 0 \\ 0 & 0 & 0 & D^\alpha \end{pmatrix} A(D,\lambda)^{-1} \right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}$$

is $\mathcal{R}$-bounded in $L(\mathbb{E}^2, L_p(\mathbb{R}^n; \mathbb{C}^4))$.

3.4 Proposition. *For every $\lambda_0 > 0$, the operators $\lambda_0 + A_{i,p}$ are $\mathcal{R}$-sectorial with $\mathcal{R}$-angle ϑ_i. Hence, $\lambda_0 + A_p$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-angle ϑ.*

Proof. The first assertion can be proved as in [DS15], Proposition 2.4. The second assertion is now obvious. □

Proposition 3.4 allows us to solve the full-space evolution equations without transmission conditions. For a sufficiently smooth function $u\colon J \times U \to \mathbb{C}$, $(t,x) \mapsto u(t,x)$ with $U \subset \mathbb{R}^n$ open and $J \subset \mathbb{R}$ an interval, we denote by $\nabla^k u$, $k \in \mathbb{N}$ all (mixed) k-th partial derivatives with respect to x.

3.5 Theorem (cf. [DS15], Theorem 2.5)**.**

(*a*) *The operator $-A_p$ generates an analytic C_0-semigroup on $\mathbb{E}^2$. Moreover, $-A_p$ even has maximal L_q-regularity for any $q \in (1,\infty)$.*

(*b*) *Let $T > 0$ be fixed and let $f_1, f_2 \in L_p((0,T); L_p(\mathbb{R}^n) =: \mathcal{E}$. Furthermore, let $\phi_1, \phi_2 \in W_p^{4-2/p}(\mathbb{R}^n)$ and $\psi_1, \psi_2 \in W_p^{2-2/p}(\mathbb{R}^n)$, where W_p^s denotes the Sobolev-Slobodeckii space, see Section A.1. Then, there exists a unique solution $u = (u_1, u_2)^\top$ with*

$$u_i \in H_p^2((0,T); L_p(\mathbb{R}^n)) \cap L_p((0,T); H_p^4(\mathbb{R}^n)) =: \mathcal{F}_{\mathbb{R}^n}$$

of the system

$$\begin{aligned} \partial_t^2 u_1 + \beta_1 \Delta^2 u_1 - \rho_1 \Delta \partial_t u_1 &= f_1, \quad (t,x) \in (0,T) \times \mathbb{R}^n, \\ \partial_t^2 u_2 + \beta_1 \Delta^2 u_2 - \rho_2 \Delta \partial_t u_2 &= f_2, \quad (t,x) \in (0,T) \times \mathbb{R}^n, \\ u_i|_{t=0} &= \phi_i, \quad x \in \mathbb{R}^n, i = 1,2, \\ \partial_t u_i|_{t=0} &= \psi_i, \quad x \in \mathbb{R}^n, i = 1,2. \end{aligned} \tag{3.15}$$

Additionally, there exists a constant $C = C(p,T) > 0$ such that

$$\|u\|_{\mathcal{F}^2_{\mathbb{R}^n}} \le C \left(\|(f_1,f_2)\|_{\mathcal{E}\times\mathcal{E}} + \|(\phi_1,\phi_2)\|_{W_p^{4-2/p}(\mathbb{R}^n)^2} + \|(\psi_1,\psi_2)\|_{W_p^{2-2/p}(\mathbb{R}^n)^2} \right).$$

(*c*) *Let $f_1, f_2 = 0$, $\phi_1, \phi_2 \in H_p^2(\mathbb{R}^n)$ and $\psi_1, \psi_2 \in L_p(\mathbb{R}^n)$. Then, there exists a unique solution $u = (u_1, u_2)$ of (3.15) with $T = \infty$ and*

$$\partial_t^2 u,\ \partial_t \nabla^2 u,\ \nabla^4 u \in C([\varepsilon,\infty); L_p(\mathbb{R}^n; \mathbb{C}^2))$$

for all $\varepsilon > 0$ and

$$\partial_t u,\ \nabla^2 u \in C([0,\infty); L_p(\mathbb{R}^n; \mathbb{C}^2)).$$

If $\phi_1, \phi_2 \in H_p^4(\mathbb{R}^n)$ and $\psi_1, \psi_2 \in H_p^2(\mathbb{R}^n)$, we may choose $\varepsilon = 0$.

Proof. This can be shown exactly as Theorem 2.5 in [DS15]. □

3.1.2. Solution operators for the stationary boundary value problem

In order to study the parameter-dependent boundary value problem (3.7), we proceed as mentioned before section 3.1.1 and consider the problem with homogeneous right-hand side and inhomogeneous boundary conditions, i.e. for λ lying in a sector Σ_ϕ of the complex plane for some $\phi \in [0, \pi)$, we consider the system

$$\begin{aligned} A(D,\lambda)y &= 0 \quad \text{in } \mathbb{R}^n_+, \\ \partial_\nu^{j-1}(y_1 + (-1)^j y_3) &= g_j \quad \text{on } \mathbb{R}^{n-1}, j = 1, \dots, 4. \end{aligned} \tag{3.16}$$

For now, we assume that the functions g_j are Schwartz functions defined on $\mathbb{R}^{n-1}$. It is a standard method to solve this problem by applying the partial Fourier transform $\mathscr{F}'$ in x'. This approach leads to a system of ordinary differential equations, for which we will try to find stable solutions, see [DHP03], section 6. Defining $w(x_n) := w(\xi', x_n, \lambda) := (\mathscr{F}'y)(\xi', x_n, \lambda)$ and

$$A(\xi', D_n, \lambda) := \begin{pmatrix} \lambda & -1 & 0 & 0 \\ \beta_1(|\xi'|^2 - \partial_n^2)^2 & \lambda + \rho_1(|\xi'|^2 - \partial_n^2) & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & \beta_2(|\xi'|^2 - \partial_n^2)^2 & \lambda + \rho_2(|\xi'|^2 - \partial_n^2) \end{pmatrix},$$

the partial Fourier transform applied to (3.16) leads to the initial value problem

$$A(\xi', D_n, \lambda)w(x_n) = 0, \qquad x_n > 0, \tag{3.17}$$

$$((-\partial_n)^{j-1}(w_1 + (-1)^j w_3))(0) = (\mathscr{F}'g_j)(\xi'), \qquad j = 1, \dots, 4. \tag{3.18}$$

By the definition of $A(\xi', D_n, \lambda)$, (3.17) yields $w_2 = \lambda w_1$ and hence

$$\lambda^2 w_1 + \lambda\rho_1(|\xi'|^2 - \partial_n^2)w_1 + \beta_1(|\xi'|^2 - \partial_n^2)^2 w_1 = 0. \tag{3.19}$$

Similarly, we obtain $w_4 = \lambda w_3$ and

$$\lambda^2 w_3 + \lambda\rho_2(|\xi'|^2 - \partial_n^2)w_3 + \beta_2(|\xi'|^2 - \partial_n^2)^2 w_3 = 0. \tag{3.20}$$

Solutions of these ordinary differential equations are obtained by calculating the roots of the corresponding characteristic polynomials

$$P_i(\tau) := \lambda^2 + \lambda\rho_i(|\xi'|^2 - \tau^2) + \beta_i(|\xi'|^2 - \tau^2)^2 \quad (i = 1, 2).$$

We substitute $\sigma := (|\xi'|^2 - \tau^2)$ and consider the equation $\lambda^2 + \lambda\rho_i\sigma + \beta_i\sigma^2 = 0$. The roots of the latter equation are given by $\sigma_\pm = -\frac{\lambda}{2\beta_i}\left(\rho_i \pm \sqrt{\rho_i^2 - 4\beta_i}\right)$. Recall that $\alpha_\pm^{(i)} = \frac{1}{2\beta_i}\left(\rho_i \pm \sqrt{\rho_i^2 - 4\beta_i}\right)$ for $i = 1, 2$. Then, back substitution yields that the roots of the polynomials P_i are given by

$$\tau^{(i)} = \pm\sqrt{|\xi'|^2 + \alpha_\pm^{(i)}\lambda}.$$

As mentioned before, we are interested in stable solutions of the initial value problem (3.17), (3.18). Therefore, we have to ensure that $|\arg(\alpha_\pm^{(i)}\lambda)| < \pi$ holds. Since $\vartheta = \max\{\vartheta_1, \vartheta_2\}$, we obtain $|\xi'|^2 + \alpha_\pm^{(i)}\lambda \notin (-\infty, 0)$ for all $\xi' \in \mathbb{R}^{n-1}$ and all $\lambda \in \Sigma_{\pi-\vartheta}$, wherefore roots of P_i with positive real part exist for such λ. These roots are given by

$$\begin{aligned} \tau_1^{(i)} &= \tau_1^{(i)}(\xi', \lambda) = \sqrt{|\xi'|^2 + \alpha_+^{(i)}\lambda} \quad \text{and} \\ \tau_2^{(i)} &= \tau_2^{(i)}(\xi', \lambda) = \sqrt{|\xi'|^2 + \alpha_-^{(i)}\lambda} \end{aligned} \tag{3.21}$$

for $i = 1, 2$. In the case $\rho_i^2 = 4\beta_i$, the root $\tau_1^{(i)} = \tau_2^{(i)}$ of P_i has multiplicity two. To simplify notation, we will also write

$$\tau_1 := \tau_1^{(1)},\ \tau_2 := \tau_2^{(1)},\ \sigma_1 := \tau_1^{(2)},\ \sigma_2 := \tau_2^{(2)}$$

if $\rho_i^2 \neq 4\beta_i$ and we will write $\tau := \tau_1^{(1)}$ and $\sigma := \tau_1^{(2)}$ if $\rho_1^2 = 4\beta_1$ and $\rho_2^2 = 4\beta_2$, respectively.

In the following lemma, we will determine a basis of the space of stable solutions of (3.17). Note that there are three different cases $c \in \{1, 2, 3\}$ depending on the choices of ρ_1, ρ_2, β_1 and β_2. In the first case $c = 1$, we have $\rho_i^2 \neq 4\beta_i$ for $i = 1, 2$. In the second case $c = 2$, we assume that $\rho_i^2 = 4\beta_i$ for $i = 1, 2$. By symmetry, the last possibilities, where there are two different roots with positive real part on the one side and one root with positive real part with multiplicity two on the other side, can be reduced two the case $c = 3$ where $\rho_1^2 \neq 4\beta_1$ and $\rho_2^2 = 4\beta_2$.
Now, τ_1, τ_2, τ and $\sigma_1, \sigma_2, \sigma$ represent those roots with positive real part, which will be used to define w_1 and w_3, respectively.

3.6 Lemma. *Let $j \in \{1, \dots, 4\}$.*

(i) *Assume that $\rho_i^2 \neq 4\beta_i$ for $i = 1, 2$. Then the matrix*

$$S_1(\xi', \lambda) := \begin{pmatrix} 1 & 1 & -1 & -1 \\ -\tau_1 & -\tau_2 & -\sigma_1 & -\sigma_2 \\ \tau_1^2 & \tau_2^2 & -\sigma_1^2 & -\sigma_2^2 \\ -\tau_1^3 & -\tau_2^3 & -\sigma_1^3 & -\sigma_2^3 \end{pmatrix}$$

is invertible for all $\xi' \in \mathbb{R}^{n-1}$ and all $\lambda \in \Sigma_{\pi-\vartheta}$. In this case we define

$$\begin{aligned} w_1^{(j)}(x_n) &:= s_{1j}e^{-\tau_1 x_n} + s_{2j}e^{-\tau_2 x_n}, \quad x_n > 0, \\ w_3^{(j)}(x_n) &:= s_{3j}e^{-\sigma_1 x_n} + s_{4j}e^{-\sigma_2 x_n}, \quad x_n > 0. \end{aligned}$$

(ii) *Assume that $\rho_i^2 = 4\beta_i$ for $i = 1, 2$. Then the matrix*

$$S_2(\xi', \lambda) := \begin{pmatrix} 1 & 0 & -1 & 0 \\ -\tau & 1 & -\sigma & 1 \\ \tau^2 & -2\tau & -\sigma^2 & 2\sigma \\ -\tau^3 & 3\tau^2 & -\sigma^3 & 3\sigma^2 \end{pmatrix}$$

is invertible for all $\xi' \in \mathbb{R}^{n-1}$ and all $\lambda \in \Sigma_{\pi-\vartheta}$. In this case we define

$$\begin{aligned} w_1^{(j)}(x_n) &:= s_{1j}e^{-\tau x_n} + s_{2j}x_n e^{-\tau x_n}, \quad x_n > 0, \\ w_3^{(j)}(x_n) &:= s_{3j}e^{-\sigma x_n} + s_{4j}x_n e^{-\sigma x_n}, \quad x_n > 0. \end{aligned}$$

(iii) Assume that $\rho_1^2 \neq 4\beta_1$ and $\rho_2^2 = 4\beta_2$. Then the matrix

$$S_3(\xi',\lambda) := \begin{pmatrix} 1 & 1 & -1 & 0 \\ -\tau_1 & -\tau_2 & -\sigma & 1 \\ \tau_1^2 & \tau_2^2 & -\sigma^2 & 2\sigma \\ -\tau_1^3 & -\tau_2^3 & -\sigma^3 & 3\sigma^2 \end{pmatrix}$$

is invertible for all $\xi' \in \mathbb{R}^{n-1}$ and all $\lambda \in \Sigma_{\pi-\vartheta}$. In this case we define

$$w_1^{(j)}(x_n) := s_{1j}e^{-\tau_1 x_n} + s_{2j}e^{-\tau_2 x_n}, \quad x_n > 0,$$
$$w_3^{(j)}(x_n) := s_{3j}e^{-\sigma x_n} + s_{4j}x_n e^{-\sigma x_n}, \quad x_n > 0.$$

Here, s_{kl} is the (k,l)-th entry of the inverse matrix $(S_c(\xi',\lambda))^{-1}$ of $S_c(\xi',\lambda)$ for $c \in \{1,2,3\}$.

In all three cases we define $w_2^{(j)} := \lambda w_1^{(j)}$ and $w_4^{(j)} := \lambda w_3^{(j)}$. Then, $w^{(j)} = (w_1^{(j)}, w_2^{(j)}, w_3^{(j)}, w_4^{(j)})^\top$ is a solution of (3.17) with initial values

$$\partial_n^{l-1}(w_1^{(j)} + (-1)^l w_3^{(j)})(0) = \delta_{lj}, \quad l = 1,\ldots,4, \tag{3.22}$$

where δ_{lj} denotes the Kronecker delta. Hence, $\{w^{(1)}, w^{(2)}, w^{(3)}, w^{(4)}\}$ is a basis of all stable solutions of (3.17).

Proof. Stable solutions of (3.19) are given by

$$w_1(x_n) = a_1 e^{-\tau_1 x_n} + a_2 e^{-\tau_2 x_n}, \quad x_n > 0$$

if $\tau_1 \neq \tau_2$ (i.e. $\rho_1^2 \neq 4\beta_1$) or

$$w_1(x_n) = a_1 e^{-\tau x_n} + a_2 x_n e^{-\tau x_n}, \quad x_n > 0$$

if $\tau_1 = \tau_2$ (i.e. $\rho_1^2 = 4\beta_1$), where $a_1, a_2 \in \mathbb{C}$ are arbitrary. In the same way, we can represent the stable solutions of (3.20) as

$$w_3(x_n) = b_1 e^{-\sigma_1 x_n} + b_2 e^{-\sigma_2 x_n}, \quad x_n > 0 \quad \text{and}$$
$$w_3(x_n) = b_1 e^{-\sigma x_n} + b_2 x_n e^{-\sigma x_n}, \quad x_n > 0$$

for $b_1, b_2 \in \mathbb{C}$, respectively. Calculating $\partial_n^{l-1}(w_1^{(j)} + (-1)^l w_3^{(j)})(0)$ for $l = 1,\ldots,4$, we obtain the matrix $S_c(\xi',\lambda)$ for the corresponding $c \in \{1,2,3\}$. Then, $w^{(j)} = (w_1^{(j)}, w_2^{(j)}, w_3^{(j)}, w_4^{(j)})^\top$ defined as in the proposition, where the coefficients are given by

$$(a_1, a_2, b_1, b_2)^T = (s_{1j}, s_{2j}, s_{3j}, s_{4j})^T = (S_c(\xi',\lambda))^{-1} e_j,$$

is the unique stable solution of (3.17) satisfying (3.22) provided $(S_c(\xi',\lambda))^{-1}$ exists. Therefore, the proposition is proved if we show the invertibility of $S_c(\xi',\lambda)$. For this purpose, let $\xi' \in \mathbb{R}^{n-1}$ and $\lambda \in \Sigma_{\pi-\vartheta}$.
First, assume that $\rho_i^2 \neq 4\beta_i$ for $i = 1,2$. Let the diagonal matrix $\widetilde{I} \in \mathbb{C}^{4\times 4}$ be defined as

$\widetilde{I} := \operatorname{diag}(1,1,-1,-1)$ where $\operatorname{diag}(v)$ is the $\mathbb{C}^{n\times n}$ diagonal matrix with $v \in \mathbb{C}^n$ on the diagonal. Then the matrix

$$\widetilde{S}_1(\xi',\lambda) := S_1(\xi',\lambda)\widetilde{I} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -\tau_1 & -\tau_2 & \sigma_1 & \sigma_2 \\ (-\tau_1)^2 & (-\tau_2)^2 & \sigma_1^2 & \sigma_2^2 \\ (-\tau_1)^3 & (-\tau_2)^3 & \sigma_1^3 & \sigma_2^3 \end{pmatrix}$$

is a Vandermonde matrix, which is invertible if and only if $-\tau_1, -\tau_2, \sigma_1, \sigma_2$ are pairwise different. Since $\tau_1 \neq \tau_2, \sigma_1 \neq \sigma_2$ by assumption and moreover $-\tau_i \neq \sigma_k$ for $i,k \in \{1,2\}$ as $\operatorname{Re}\tau_i > 0$ and $\operatorname{Re}\sigma_i > 0$, this is the case. Hence, $S_1(\xi',\lambda)$ is invertible.
Now assume that $\rho_i^2 = 4\beta_i$ holds for $i = 1,2$. By using Laplace's formula, one sees that the determinant of $S_2(\xi',\lambda)$ is given by

$$\begin{aligned}\det S_2(\xi',\lambda) &= -\tau^4 - 4\tau^3\sigma - 6\tau^2\sigma^2 - 4\tau\sigma^3 - \sigma^4 \\ &= -(\tau+\sigma)^4.\end{aligned}$$

Consequently, $\det S_2(\xi',\lambda) = 0$ if and only if $\tau = -\sigma$, which is never the case since $\operatorname{Re}\tau, \operatorname{Re}\sigma > 0$. It follows that $S_2(\xi',\lambda)$ is invertible in this case as well.
Finally, let $\rho_1^2 \neq 4\beta_1$ and $\rho_2^2 = 4\beta_2$. Then we obtain

$$\det S_3(\xi',\lambda) = -(\tau_1+\sigma)^2(\tau_1-\tau_2)(\tau_2+\sigma)^2$$

and as before we see that $S_3(\xi',\lambda)$ is invertible. □

3.7 Remark. Lemma 3.6 shows that the canonical transmission conditions satisfy the so-called Shapiro-Lopatinskii condition (see [DHP03], [Wlo82]) for any pair of elliptic differential operators $\mathcal{A}^{(i)}(D) = \sum_{|\alpha|\leq 4} a_\alpha^{(i)} D^\alpha$ with scalar coefficients $a_\alpha^{(i)} \in \mathbb{C}$ for $i = 1,2$ of order 4. One says that the canonical transmission conditions are *completely elliptic*.
In fact, it can be shown that for any elliptic differential operators $A^{(i)}(D) = \sum_{|\alpha|\leq 2m} a_\alpha^{(i)} D^\alpha$ with scalar coefficients $a_\alpha^{(i)} \in \mathbb{C}$ for $i = 1,2$ of order $2m \in 2\mathbb{N}$, the canonical transmission conditions, given by

$$(\partial_\nu)^k u\big|_{\mathbb{R}^n_-} = (\partial_\nu)^k u\big|_{\mathbb{R}^n_+} \quad (k = 0,\dots 2m-1),$$

satisfy the Shapiro-Lopatinskii condition. A proof for this assertion can be found in [Seg13], Theorem 1.9.

We have shown that in all cases $c \in \{1,2,3\}$ the matrix $S_c(\xi',\lambda)$ is invertible and hence there is a unique stable solution for the initial value problem (3.17), (3.22). The solutions are given by Lemma 3.6, where the coefficients are the entries of the inverse matrix $(S_c(\xi',\lambda))^{-1}$. By [RC70], the inverse matrix $(S_1(\xi',\lambda))^{-1}$ of the matrix $S_1(\xi',\lambda)$, which is close to a Vandermonde matrix, is given by

$$\begin{pmatrix} \frac{-\tau_2\sigma_1\sigma_2}{(\tau_1-\tau_2)(\tau_1+\sigma_1)(\tau_1+\sigma_2)} & \frac{\tau_2\sigma_1+\tau_2\sigma_2-\sigma_1\sigma_2}{(\tau_1-\tau_2)(\tau_1+\sigma_1)(\tau_1+\sigma_2)} & \frac{\sigma_1+\sigma_2-\tau_2}{(\tau_1-\tau_2)(\tau_1+\sigma_1)(\tau_1+\sigma_2)} & \frac{-1}{(\tau_1-\tau_2)(\tau_1+\sigma_1)(\tau_1+\sigma_2)} \\ \frac{\tau_1\sigma_1\sigma_2}{(\tau_1-\tau_2)(\tau_2+\sigma_1)(\tau_2+\sigma_2)} & \frac{\sigma_1\sigma_2-\tau_1\sigma_2-\tau_1\sigma_1}{(\tau_1-\tau_2)(\tau_2+\sigma_1)(\tau_2+\sigma_2)} & \frac{\tau_1-\sigma_1-\sigma_2}{(\tau_1-\tau_2)(\tau_2+\sigma_1)(\tau_2+\sigma_2)} & \frac{1}{(\tau_1-\tau_2)(\tau_2+\sigma_1)(\tau_2+\sigma_2)} \\ \frac{-\tau_1\tau_2\sigma_2}{(\sigma_2-\sigma_1)(\sigma_1+\tau_1)(\sigma_1+\tau_2)} & \frac{\tau_1\tau_2-\tau_1\sigma_2-\tau_2\sigma_2}{(\sigma_2-\sigma_1)(\sigma_1+\tau_1)(\sigma_1+\tau_2)} & \frac{-\sigma_2+\tau_1+\tau_2}{(\sigma_2-\sigma_1)(\sigma_1+\tau_1)(\sigma_1+\tau_2)} & \frac{1}{(\sigma_2-\sigma_1)(\sigma_1+\tau_1)(\sigma_1+\tau_2)} \\ \frac{\tau_1\tau_2\sigma_1}{(\sigma_2-\sigma_1)(\sigma_2+\tau_1)(\sigma_2+\tau_2)} & \frac{\tau_1\sigma_1-\tau_1\tau_2+\sigma_1\tau_2}{(\sigma_2-\sigma_1)(\sigma_2+\tau_1)(\sigma_2+\tau_2)} & \frac{-\tau_1-\tau_2+\sigma_1}{(\sigma_2-\sigma_1)(\sigma_2+\tau_1)(\sigma_2+\tau_2)} & \frac{-1}{(\sigma_2-\sigma_1)(\sigma_2+\tau_1)(\sigma_2+\tau_2)} \end{pmatrix}.$$

In the second case $c = 2$ where $\tau = \tau_1 = \tau_2$ and $\sigma = \sigma_1 = \sigma_2$ we have that $(S_2(\xi', \lambda))^{-1}$ equals

$$\frac{1}{(\tau+\sigma)^2}\begin{pmatrix} \frac{\sigma^3+3\tau\sigma^2}{\tau+\sigma} & -\frac{6\tau\sigma}{\tau+\sigma} & \frac{3(\tau-\sigma)}{\tau+\sigma} & \frac{2}{\tau+\sigma} \\ \tau\sigma^2 & \sigma^2 - 2\tau\sigma & \tau - 2\sigma & 1 \\ -\frac{\tau^3+3\sigma\tau^2}{\tau+\sigma} & -\frac{6\tau\sigma}{\tau+\sigma} & \frac{3(\tau-\sigma)}{\tau+\sigma} & \frac{2}{\tau+\sigma} \\ -\tau^2\sigma & \tau^2 - 2\tau\sigma & 2\tau - \sigma & 1 \end{pmatrix}.$$

In the case $c = 3$ we finally obtain that $(S_3(\xi', \lambda))^{-1}$ is given by

$$\begin{pmatrix} -\frac{\tau_2\sigma^2}{(\tau_1-\tau_2)(\tau_1+\sigma)^2} & \frac{(2\tau_2-\sigma)\sigma}{(\tau_1-\tau_2)(\tau_1+\sigma)^2} & \frac{2\sigma-\tau_2}{(\tau_1-\tau_2)(\tau_1+\sigma)^2} & -\frac{1}{(\tau_1-\tau_2)(\tau_1+\sigma)^2} \\ \frac{\tau_1\sigma^2}{(\tau_1-\tau_2)(\tau_2+\sigma)^2} & \frac{\sigma(\sigma-2\tau_1)}{(\tau_1-\tau_2)(\tau_2+\sigma)^2} & \frac{\tau_1-2\sigma}{(\tau_1-\tau_2)(\tau_2+\sigma)^2} & \frac{1}{(\tau_1-\tau_2)(\tau_2+\sigma)^2} \\ -\frac{\tau_1\tau_2(\tau_1(\tau_2+2\sigma)+\sigma(2\tau_2+3\sigma))}{(\tau_1+\sigma)^2(\tau_2+\sigma)^2} & -\frac{\sigma(2\tau_1^2+2\tau_2\tau_1+3\sigma\tau_1+2\tau_2^2+3\tau_2\sigma)}{(\tau_1+\sigma)^2(\tau_2+\sigma)^2} & \frac{\tau_1^2+\tau_2\tau_1+\tau_2^2-3\sigma^2}{(\tau_1+\sigma)^2(\tau_2+\sigma)^2} & \frac{\tau_1+\tau_2+2\sigma}{(\tau_1+\sigma)^2(\tau_2+\sigma)^2} \\ -\frac{\tau_1\tau_2\sigma}{(\tau_1+\sigma)(\tau_2+\sigma)} & \frac{\tau_1(\tau_2-\sigma)-\tau_2\sigma}{(\tau_1+\sigma)(\tau_2+\sigma)} & \frac{\tau_1+\tau_2-\sigma}{(\tau_1+\sigma)(\tau_2+\sigma)} & \frac{1}{(\tau_1+\sigma)(\tau_2+\sigma)} \end{pmatrix}.$$

Later, we will write the solutions $w^{(j)}$ in a more concise form (see Lemma 3.9).
The basis $\{w^{(j)} : j = 1, \dots, 4\}$ now allows us to find solutions of the following boundary value problems: let $\phi\colon \mathbb{R}^n_+ \to \mathbb{C}$ be a restriction of a Schwartz function on $\mathbb{R}^n$, i.e. $\phi = \Phi|_{\mathbb{R}^n_+}$ for some $\Phi \in \mathscr{S}(\mathbb{R}^n)$. For $j \in \{1, \dots, 4\}$ we consider the boundary value problem

$$\begin{aligned} A(D,\lambda)y^{(j)} &= 0 && \text{in } \mathbb{R}^n_+, \\ \partial_n^{l-1}(y_1^{(j)} + (-1)^l y_3^{(j)})(\cdot, 0) &= \delta_{lj}\phi(\cdot, 0) && \text{on } \mathbb{R}^{n-1}, l = 1, \dots, 4. \end{aligned} \tag{3.23}$$

Similar to Proposition 3.3 *a)* in [DS15] we have the following solution result for the boundary value problem (3.23):

3.8 Proposition. *For $i, j \in \{1, \dots, 4\}$ and $\lambda \in \Sigma_{\pi-\vartheta}$ we define the operator $L_i^{(j)}(\lambda)$ by*

$$(L_i^{(j)}(\lambda)\phi)(\cdot, x_n) := -\int_0^\infty (\mathscr{F}')^{-1}\partial_n^{i-1} w_1^{(j)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n)\, dy_n, \quad x_n > 0,$$

if $i \in \{1, 2\}$ and

$$(L_i^{(j)}(\lambda)\phi)(\cdot, x_n) := -\int_0^\infty (\mathscr{F}')^{-1}\partial_n^{i-3} w_3^{(j)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n)\, dy_n, \quad x_n > 0,$$

if $i \in \{3, 4\}$, respectively. Here, $\phi\colon \mathbb{R}^n_+ \to \mathbb{C}$ is the restriction of a Schwartz function on $\mathbb{R}^n$. We define $y_1^{(j)} := L_1^{(j)}(\lambda)\partial_n\phi + L_2^{(j)}(\lambda)\phi$ and $y_3^{(j)} := L_3^{(j)}(\lambda)\partial_n\phi + L_4^{(j)}(\lambda)\phi$. Then,

$$y_i^{(j)}(\cdot, x_n) = (\mathscr{F}')^{-1} w_i^{(j)}(\cdot, x_n, \lambda)(\mathscr{F}'\phi)(\cdot, 0), \quad x_n > 0 \tag{3.24}$$

holds for $i \in \{1, 3\}$ and $y^{(j)} := (y_1^{(j)}, \lambda y_1^{(j)}, y_3^{(j)}, \lambda y_3^{(j)})^\top$ is a solution of (3.23).

Proof. Let $j \in \{1, \dots, 4\}$ and $\lambda \in \Sigma_{\pi-\vartheta}$. First we show (3.24). Integration by parts yields

$$\begin{aligned} y_1^{(j)}(\cdot, x_n) &= (L_1^{(j)}(\lambda)\partial_n\phi)(\cdot, x_n) + (L_2^{(j)}(\lambda)\phi)(\cdot, x_n) \\ &= -\int_0^\infty (\mathscr{F}')^{-1} w_1^{(j)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\partial_n\phi)(\cdot, y_n)\, dy_n \\ &\quad - \int_0^\infty (\mathscr{F}')^{-1} \partial_n w_1^{(j)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n)\, dy_n \\ &= -\Big[(\mathscr{F}')^{-1} w_1^{(j)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n)\Big]_0^\infty \\ &\quad + \int_0^\infty (\mathscr{F}')^{-1} \partial_n w_1^{(j)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n)\, dy_n \\ &\quad - \int_0^\infty (\mathscr{F}')^{-1} \partial_n w_1^{(j)}(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n)\, dy_n \\ &= (\mathscr{F}')^{-1} w_1^{(j)}(\cdot, x_n, \lambda)(\mathscr{F}'\phi)(\cdot, 0) \end{aligned}$$

and hence (3.24) holds for $i = 1$. The case $i = 3$ is shown in the same way. To show that $A(D, \lambda)y^{(j)} = 0$, we use the obtained representation and calculate

$$\begin{aligned} &\beta_1 \Delta^2 \Big[(\mathscr{F}')^{-1} w_1^{(j)}(\cdot, x_n, \lambda)(\mathscr{F}'\phi)(\cdot, 0)\Big] + (\lambda - \rho_1\Delta)\Big[(\mathscr{F}')^{-1} \lambda w_1^{(j)}(\cdot, x_n, \lambda)(\mathscr{F}'\phi)(\cdot, 0)\Big] \\ = \;& (\mathscr{F}')^{-1}(\lambda^2 + \lambda\rho_1(|\xi'|^2 - \partial_n^2) + \beta_1(|\xi'|^2 - \partial_n^2)^2) w_1^{(j)}(\cdot, x_n, \lambda)(\mathscr{F}'\phi)(\cdot, 0) \\ = \;& 0 \end{aligned}$$

by Lemma 3.6. The same calculation for $y_3^{(j)}$ yields $A(D, \lambda)y^{(j)} = 0$. The initial conditions can be verified by plugging in the representation (3.24) and using Lemma 3.6 again. □

The aim of the subsequent considerations is to find technical estimates for the solutions $w^{(j)}$ of (3.17), (3.22) which will lead to $\mathcal{R}$-boundedness of certain operator families related to the solution operators $L_i^{(j)}$.

We collect some estimates for the roots $\tau_j^{(i)}$ of the characteristic polynomials P_i for $i, j = 1, 2$ which we will use frequently in the following. Since we want to apply Lemma A.18 together with Remark A.19, we consider holomorphic extensions $\widetilde{\tau}_j^{(i)} = \widetilde{\tau}_j^{(i)}(z', \lambda)$ of $\tau_j^{(i)} = \tau_j^{(i)}(\xi', \lambda)$ where ξ_k is replaced by z_k in a small sector of the complex plane. Note that this is possible since $\tau_j^{(i)}$ only depends on $|\xi'|$ (and on λ of course) whence $\tau_j^{(i)}$ is symmetric in ξ_k for $k = 1, \dots, n-1$.
Let $\varepsilon > 0$ such that $\pi - \vartheta - 2\varepsilon > \frac{\pi}{2}$. For $z' \in \mathbb{C}^{n-1}$ we set $Z := \sqrt{z_1^2 + \ldots + z_{n-1}^2}$. Now we define

$$\widetilde{\tau}_1^{(i)}(z', \lambda) := \sqrt{Z^2 + \alpha_+^{(i)}\lambda}, \quad \widetilde{\tau}_2^{(i)}(z', \lambda) := \sqrt{Z^2 + \alpha_-^{(i)}\lambda}$$

for $(z', \lambda) \in (\Sigma_{\varepsilon/2})^{n-1} \times \Sigma_{\pi-\vartheta-2\varepsilon}$. Then, as $\alpha_\pm^{(i)}\lambda \in \Sigma_{\pi-2\varepsilon}$, the root $\widetilde{\tau}_j^{(i)}$ is well-defined and holomorphic. By compactness, there exist some constants $C_1, C_2 > 0$ such that

$$0 < C_1 \le |\widetilde{\tau}_j^{(i)}(z', \lambda)| \le C_2 < \infty, \quad (z', \lambda) \in (\overline{\Sigma}_{\varepsilon/2})^{n-1} \times \overline{\Sigma}_{\pi-\vartheta-2\varepsilon} \text{ s.t. } |z'|^2 + |\lambda| = 1.$$

As $r\widetilde{\tau}_j^{(i)}(z',\lambda) = \widetilde{\tau}_j^{(i)}(rz', r^2\lambda)$ for $r > 0$ we obtain

$$C_1(|z'|^2 + |\lambda|)^{1/2} \le |\widetilde{\tau}_j^{(i)}(z',\lambda)| \le C_2(|z'|^2 + |\lambda|)^{1/2}, \quad (z',\lambda) \in (\overline{\Sigma}_{\varepsilon/2})^{n-1} \times \overline{\Sigma}_{\pi-\vartheta-2\varepsilon}. \tag{3.25}$$

Furthermore, since $\theta := \arg(\widetilde{\tau}_j^{(i)}) < \frac{\pi-\varepsilon}{2}$ there exists a constant $C > 0$ such that

$$\operatorname{Re}\widetilde{\tau}_j^{(i)} = |\widetilde{\tau}_j^{(i)}| \operatorname{Re} e^{\mathrm{i}\theta} = |\widetilde{\tau}_j^{(i)}| \cos(\theta) \ge C|\widetilde{\tau}_j^{(i)}|$$

holds for all $(z',\lambda) \in (\overline{\Sigma}_{\varepsilon/2})^{n-1} \times \overline{\Sigma}_{\pi-\vartheta-2\varepsilon}$. As $|\widetilde{\tau}_j^{(i)}| \ge \operatorname{Re}\widetilde{\tau}_j^{(i)}$, these estimates also imply

$$\widetilde{C}_1(|z'|^2 + |\lambda|)^{-1/2} \le |(\widetilde{\tau}_j^{(i)} + \widetilde{\tau}_l^{(k)})(z',\lambda)|^{-1} \le \widetilde{C}_2(|z'|^2 + |\lambda|)^{-1/2}, \ (z',\lambda) \in (\Sigma_{\varepsilon/2})^{n-1} \times \Sigma_{\pi-\vartheta-2\varepsilon} \tag{3.26}$$

for some constants $\widetilde{C}_1, \widetilde{C}_2 > 0$. Concerning points lying on the straight line between $\widetilde{\tau}_j^{(i)}$ and $\widetilde{\tau}_l^{(k)}$, we also obtain

$$C_1(|z'|^2 + |\lambda|)^{1/2} \le |\widetilde{\tau}(r,z',\lambda)| \le C_2(|z'|^2 + |\lambda|)^{1/2}, \quad (z',\lambda) \in (\Sigma_{\varepsilon/2})^{n-1} \times \Sigma_{\pi-\vartheta-2\varepsilon} \tag{3.27}$$

where we have set $\widetilde{\tau}(r,z',\lambda) := \widetilde{\tau}_j^{(i)} + r(\widetilde{\tau}_l^{(k)} - \widetilde{\tau}_j^{(i)})$ for $r \in [0,1]$ and $k,l \in \{1,2\}$ are arbitrary. For convenience, we also write τ_j, σ_j for the complex extensions $\widetilde{\tau}_j^{(1)}$ and $\widetilde{\tau}_j^{(2)}$ for $j = 1,2$, respectively.

3.9 Lemma (cf. [DS15], Lemma 3.2)**.**

a) Assume that $\rho_1^2 \ne 4\beta_1$, *i.e.* $\tau_1 \ne \tau_2$. *For* $\varepsilon > 0, k \in \mathbb{N}$ *and* $l \in \mathbb{Z}$, *we define the function* $f_{k,l}\colon (\Sigma_{\varepsilon/2})^{n-1} \times (0,\infty) \times \Sigma_{\pi-\vartheta-2\varepsilon} \to \mathbb{C}$ *by*

$$f_{k,l}(z',x_n,\lambda) := \frac{x_n^k}{\tau_1 - \tau_2}\left(-\frac{\tau_1^l}{(\tau_1+\sigma_1)(\tau_1+\sigma_2)}e^{-\tau_1 x_n} + \frac{\tau_2^l}{(\tau_2+\sigma_1)(\tau_2+\sigma_2)}e^{-\tau_2 x_n}\right).$$

Then, the estimate

$$|f_{k,l}(z',x_n,\lambda)| \le C(|z'|^2 + |\lambda|)^{(l-k-3)/2}$$

holds for all $(z',x_n,\lambda) \in (\Sigma_{\varepsilon/2})^{n-1} \times (0,\infty) \times \Sigma_{\pi-\vartheta-2\varepsilon}$.

b) Let $w^{(j)}, j = 1,\dots,4$ *be the fundamental solutions from Lemma 3.6. Let* $\varepsilon > 0, l \in \{0,\dots,4\}$ *and* $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| = l$. *Then, for all* $\gamma \in \mathbb{N}_0^2, \beta' \in \mathbb{N}_0^{n-1}, x_n > 0, \lambda \in \Sigma_{\pi-\vartheta-2\varepsilon}, m \in \mathbb{N}_0$ *and* $\xi' \in \mathbb{R}^{n-1}$ *the inequality*

$$\left|\lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \left[\lambda^{2-\frac{l}{2}} (\xi')^{\alpha'} x_n^{m+1} \partial_n^{\alpha_n + k} w_i^{(j)}(\xi',x_n,\lambda)(\lambda + |\xi'|^2)^{(j-k+m-4)/2}\right]\right| \le C, \quad i = 1,3$$

holds for $k \in \{0,1\}$.

Proof. a) For $z \in \Sigma_{(\pi-\varepsilon)/2}$ we define

$$\varphi(z) := -\frac{x_n^k z^l}{(z+\sigma_1)(z+\sigma_2)} e^{-z x_n}$$

for any fixed $x_n > 0$. Note that $\rho_2^2 = 4\beta_2$, i.e. $\sigma = \sigma_1 = \sigma_2$ is possible. For the derivative φ' of φ we obtain

$$\varphi'(z) = \frac{l x_n^k z^{l-1} e^{-zx_n}}{(z+\sigma_1)(z+\sigma_2)} - \frac{x_n^{k+1} z^l e^{-zx_n}}{(z+\sigma_1)(z+\sigma_2)} + \frac{x_n^k z^l e^{-zx_n}}{(z+\sigma_1)^2(z+\sigma_2)} + \frac{x_n^k z^l e^{-zx_n}}{(z+\sigma_1)(z+\sigma_2)^2}.$$

As before we set $\tau(r) := \tau(r, z', \lambda) := \tau_1(z', \lambda) + r(\tau_2(z', \lambda) - \tau_1(z', \lambda))$ for $r \in [0,1]$. By the estimates (3.26), (3.27) and $|(\tau(r)x_n)^k e^{-\tau(r)x_n}| \leq C$, we have

$$\left| \frac{x_n^k \tau(r)^{l-1} e^{-\tau(r)x_n}}{(\tau(r)+\sigma_1)(\tau(r)+\sigma_2)} \right| = \left| \frac{\tau(r)^{l-k-1} (\tau(r)x_n)^k e^{-\tau(r)x_n}}{(\tau(r)+\sigma_1)(\tau(r)+\sigma_2)} \right| \leq C(|z'|^2 + |\lambda|)^{(l-k-1-2)/2}$$

and in the same way we can estimate the remaining terms of $\varphi'(\tau(r))$ and obtain

$$|\varphi'(\tau(r))| \leq C(|z'|^2 + |\lambda|)^{(l-k-3)/2}.$$

Now the assertion follows from

$$|f_{k,l}(z', x_n, \lambda)| = \left| \frac{\varphi(\tau_1) - \varphi(\tau_2)}{\tau_1 - \tau_2} \right| = \left| \int_0^1 \varphi'(\tau(r))\, dr \right|.$$

b) First, assume that $\rho_r^2 \neq 4\beta_r$ for $r = 1, 2$. Inverting the matrix $S_1(\xi', \lambda)$ (see Lemma 3.6 and the observations thereafter), we see that the solutions $w_1^{(j)}$ can be written as

$$\begin{aligned}
w_1^{(1)} &= (\tau_1\tau_2\sigma_1\sigma_2) f_{0,-1}, \\
w_1^{(2)} &= -(\tau_1\tau_2\sigma_1 + \tau_1\tau_2\sigma_2) f_{0,-1} + (\sigma_1\sigma_2) f_{0,0}, \\
w_1^{(3)} &= (\sigma_1 + \sigma_2) f_{0,0} + (\tau_1\tau_2) f_{0,-1}, \\
w_1^{(4)} &= f_{0,0}.
\end{aligned}$$

By part *a)* and $\partial_n f_{0,r} = -f_{0,r+1}$ for all $r \in \mathbb{Z}$, we can estimate

$$\begin{aligned}
\left| x_n^{m+1} \partial_n^{\alpha_n + k} w_1^{(1)} \right| &= \left| x_n^{m+1} \partial_n^{\alpha_n + k} (\tau_1\tau_2\sigma_1\sigma_2) f_{0,-1} \right| \\
&= |(\tau_1\tau_2\sigma_1\sigma_2) f_{m+1, -1+\alpha_n+k}| \\
&\leq C(|z'|^2 + |\lambda|)^{(\alpha_n + k - m - 1)/2}
\end{aligned}$$

for $z' \in \Sigma_{\varepsilon/2}$, where we have considered $w_1^{(1)} = w_1^{(1)}(z', x_n, \lambda)$ as a function of z' instead of ξ', just as we did before. Similarly, we obtain

$$\left| x_n^{m+1} \partial_n^{\alpha_n + k} w_1^{(j)} \right| \leq C(|z'|^2 + |\lambda|)^{(\alpha_n + k - m - j)/2}. \tag{3.28}$$

Slightly changing the definition of $f_{k,l}$, we show this estimate for $w_3^{(j)}$ in this case in the same way.

Now we assume that $\rho_1^2 \neq 4\beta_1$ and $\rho_2^2 = 4\beta_2$. In this case, the solutions $w_1^{(j)}$ can be written as

$$\begin{aligned} w_1^{(1)} &= (\tau_1\tau_2\sigma^2) f_{0,-1}, \\ w_1^{(2)} &= \sigma^2 f_{0,0} - (2\tau_1\tau_2\sigma) f_{0,-1}, \\ w_1^{(3)} &= -2\sigma f_{0,0} + (\tau_1\tau_2) f_{0,-1}, \\ w_1^{(4)} &= f_{0,0}. \end{aligned}$$

The solutions $w_3^{(j)}$ are of the general form

$$w_3^{(j)}(x_n) = a_j e^{-\sigma x_n} + b_j x_n e^{-\sigma x_n}, \quad x_n > 0$$

with coefficients a_j, b_j of the following form: let $p_d = p_d(\tau_1, \tau_2, \sigma)$ denote a polynomial in three variables of total degree d, i.e. the polynomial $\widetilde{p}_d(x) := p_d(x, x, x)$ is of degree d. Then one can write

$$a_j = \frac{p_{5-j}^{(a)}(\tau_1, \tau_2, \sigma)}{(\tau_1 + \sigma)^2(\tau_2 + \sigma)^2}, \quad b_j = \frac{p_{4-j}^{(b)}(\tau_1, \tau_2, \sigma)}{(\tau_1 + \sigma)(\tau_2 + \sigma)}$$

for some polynomials $p_d^{(a)}, p_d^{(b)}$ of total degree d. Note that the difference of the order in a_j and b_j is 'compensated' by the additional x_n in the second term of the solution $w_3^{(j)}$, as

$$|x_n e^{-\sigma x_n}| = |\sigma|^{-1} |\sigma x_n e^{-\sigma x_n}| \leq C|\sigma|^{-1}.$$

Therefore, we obtain the estimate (3.28) also in this case, where $\tau_1 \neq \tau_2$ and $\sigma = \sigma_1 = \sigma_2$. Finally, the case $\rho_r^2 = 4\beta_r$ for $r = 1, 2$ can be treated in the same way as before. The solutions $w_i^{(j)}$ have the form

$$\begin{aligned} w_1^{(j)}(x_n) &= a_j e^{-\tau x_n} + b_j x_n e^{-\tau x_n}, \quad x_n > 0, \\ w_3^{(j)}(x_n) &= c_j e^{-\sigma x_n} + d_j x_n e^{-\sigma x_n}, \quad x_n > 0, \end{aligned}$$

with coefficients

$$a_j = \frac{p_{4-j}^{(a)}(\tau, \sigma)}{(\tau + \sigma)^3}, \ b_j = \frac{p_{4-j}^{(b)}(\tau, \sigma)}{(\tau + \sigma)^2}, \ c_j = \frac{p_{4-j}^{(c)}(\tau, \sigma)}{(\tau + \sigma)^3}, \ d_j = \frac{p_{4-j}^{(d)}(\tau, \sigma)}{(\tau + \sigma)^2}.$$

Hence, in all three cases the estimate (3.28) holds true.
We have

$$\left| \lambda^{2-\frac{l}{2}} (z')^{\alpha'} x_n^{m+1} \partial_n^{\alpha_n + k} w_i^{(j)}(z', x_n, \lambda) \right| \leq C(|z'|^2 + |\lambda|)^{(\alpha_n + k - m - j + 4 - l + \alpha')/2}$$

and together with $|\alpha| = l$ this implies

$$\left| \lambda^{2-\frac{l}{2}} (z')^{\alpha'} x_n^{m+1} \partial_n^{\alpha_n + k} w_i^{(j)}(z', x_n, \lambda)(\lambda + |z'|^2)^{(j-k+m-4)/2} \right| \leq C.$$

By Lemma A.18 and Remark A.19, we get

$$\left| \lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \left[\lambda^{2-\frac{l}{2}} (\xi')^{\alpha'} x_n^{m+1} \partial_n^{\alpha_n + k} w_i^{(j)}(\xi', x_n, \lambda)(\lambda + |\xi'|^2)^{(j-k+m-4)/2} \right] \right| \leq C$$

for $i = 1, 3$ as desired.

□

As a consequence, we obtain an $\mathcal{R}$-boundedness result (cf. [DS15], Proposition 3.3 *b*)), where we will use the following notation: For $i = 1, \dots, 4$, we set $I(i) := (i+1) \bmod 2$, that is $I(1) = 0, I(2) = 1, I(3) = 0, I(4) = 1$.

3.10 Corollary. *Let $\varepsilon > 0, \gamma \in \mathbb{N}_0^2, l \in \{0, \dots, 4\}$ and $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = l$. Then, for all $i, j \in \{1, \dots, 4\}$ the family of operators*

$$\left\{\lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{2-\frac{l}{2}} D^\alpha L_i^{(j)}(\lambda)(\lambda - \Delta')^{(j-I(i)-4)/2}\right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\right\}$$

in $L(L_p(\mathbb{R}_+^n))$ is well-defined and $\mathcal{R}$-bounded. Here, $\Delta' = \partial_1^2 + \ldots + \partial_{n-1}^2$ denotes the Laplacian in the first $n-1$ space variables.

Proof. For $x_n, y_n > 0, \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}, \xi' \in \mathbb{R}^{n-1}$ and $\beta' \in \mathbb{N}_0^{n-1}$, Lemma 3.9 *b*) with $m = 0$ yields

$$\left|\lambda^\gamma \partial_\lambda^\gamma (\xi')^{\beta'} \partial_{\xi'}^{\beta'} \left[\lambda^{2-\frac{l}{2}} (\xi')^{\alpha'} \partial_n^{\alpha_n + I(i)} w_r^{(j)}(\xi', x_n + y_n, \lambda)(\lambda + |\xi'|^2)^{(j-I(i)-4)/2}\right]\right| \leq \frac{C}{x_n + y_n}$$

with a constant $C > 0$ that does not depend on x_n, y_n, λ or ξ'. By the Mikhlin multiplier theorem, the operator families

$$\left\{(\mathscr{F}')^{-1} \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{2-\frac{l}{2}} (\xi')^{\alpha'} \partial_n^{\alpha_n + I(i)} w_r^{(j)}(\xi', x_n + y_n, \lambda)(\lambda + |\xi'|^2)^{(j-I(i)-4)/2}\right] \mathscr{F}' : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\right\}$$

for $r = 1, 3$ are $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^{n-1}))$ with $\mathcal{R}$-bound less or equal $C/(x_n + y_n)$ for all $x_n, y_n > 0$. Since the integral operator

$$(Tg)(x_n) := \int_0^\infty \frac{g(y_n)}{x_n + y_n}\, dy_n \quad (g \in L_p(\mathbb{R}_+))$$

is bounded in $L_p(\mathbb{R}_+)$, the assertion follows from [DHP03], Proposition 4.12. □

This finally allows us to solve the inhomogeneous transmission problem (cf. (3.7))

$$\begin{aligned} A(D,\lambda)y &= h && \text{in } \mathbb{R}_+^n, \\ y_1 - y_3 &= g_1 && \text{on } \mathbb{R}^{n-1}, \\ -\partial_n y_1 - \partial_n y_3 &= g_2 && \text{on } \mathbb{R}^{n-1}, \\ \partial_n^2 y_1 - \partial_n^2 y_3 &= g_3 && \text{on } \mathbb{R}^{n-1}, \\ -\partial_n^3 y_1 - \partial_n^3 y_3 &= g_4 && \text{on } \mathbb{R}^{n-1} \end{aligned} \tag{3.29}$$

in $\mathbb{R}_+^n$ with data

$$h = (h_1, \dots, h_4)^\top \in \mathbb{E}_+^2 = \mathbb{E}_{p,+}^2 := \left(H_p^2(\mathbb{R}_+^n) \times L_p(\mathbb{R}_+^n)\right)^2$$

and

$$g = (g_1, \dots, g_4)^\top \in \mathbb{G} := \prod_{l=0}^{3} B_{pp}^{4-l-1/p}(\mathbb{R}^{n-1})$$

such that the unique solution y belongs to the space

$$y = (y_1, \ldots, y_4)^\top \in \mathbb{F}_+^2 = \mathbb{F}_{p,+}^2 := \left(H_p^4(\mathbb{R}_+^n) \times H_p^2(\mathbb{R}_+^n)\right)^2 .$$

The corresponding L_p-realization $A_{p,+}$ is given as the unbounded operator

$$A_{p,+} \colon \mathbb{E}_+^2 \supset D(A_{p,+}) \to \mathbb{E}_+^2, \quad u \mapsto A_{p,+}u := A(D)u,$$

with domain

$$D(A_{p,+}) := \left\{v \in \mathbb{F}_+^2 : \gamma_0 B(D)v = 0\right\}.$$

Recall that $r_+ \colon L_p(\mathbb{R}^n) \to L_p(\mathbb{R}_+^n)$ denotes the restriction of functions to the half-space and $e_+ \in L(H_p^s(\mathbb{R}_+^n), H_p^s(\mathbb{R}^n))$ is the global coretraction of r_+. Moreover, let $e_0 \colon L_p(\mathbb{R}_+^n) \to L_p(\mathbb{R}^n)$ denote the trivial extension by zero to the whole space. As in [DS15], we will use the parameter-dependent extension operator $E_\lambda \in L(B_{pp}^{k-1/p}(\mathbb{R}^{n-1}), H_p^k(\mathbb{R}_+^n))$ given by

$$(E_\lambda \phi)(\cdot, x_n) := (\mathscr{F}'^{-1}) \exp\left(-(\lambda + |\xi'|^2)^{1/2} x_n\right) \mathscr{F}'\phi \quad (x_n > 0).$$

Then, $\gamma_0 E_\lambda = \mathrm{id}_{B_{pp}^{k-1/p}(\mathbb{R}^{n-1})}$ for all $k \in \mathbb{N}$ and $\lambda \in \Sigma_{\pi-\vartheta}$. Furthermore, it holds that (see [DS15], (3.11) and [ADF97], Proposition 2.3)

$$\partial_n E_\lambda \phi = (-\lambda - \Delta')^{1/2} E_\lambda \phi \qquad (\phi \in B_{pp}^{1-1/p}(\mathbb{R}^{n-1})). \tag{3.30}$$

To shorten notation in the next theorem and thereafter, we introduce the symbols

$$\Lambda_s = \Lambda_s(\lambda) := (\lambda - \Delta)^s, \quad \Lambda_s' = \Lambda_s'(\lambda) := (\lambda - \Delta')^s$$

for $\lambda \in \mathbb{C}$ and $s \in \mathbb{R}$.

3.11 Theorem (cf. [DS15], Theorem 3.5)**.**
For all $\lambda \in \Sigma_{\pi-\vartheta}, h \in \mathbb{E}_+^2$ and $g \in \mathbb{G}$, there exists a unique solution $y \in \mathbb{F}_+^2$ of (3.29). *The solution can be written as*

$$y = R(\lambda)e_+ h + T(\lambda)E_\lambda g$$

with solution operators

$$\begin{aligned}
R(\lambda) &= r_+(\lambda + A_p)^{-1} - T(\lambda)B(D)r_+(\lambda + A_p)^{-1}, \\
T(\lambda) &= T^{(1)}(\lambda)\partial_n + T^{(2)}(\lambda), \\
T^{(i)}(\lambda) &= \begin{pmatrix} L_i^{(1)}(\lambda) & -L_i^{(2)}(\lambda) & L_i^{(3)}(\lambda) & -L_i^{(4)}(\lambda) \\ \lambda L_i^{(1)}(\lambda) & -\lambda L_i^{(2)}(\lambda) & \lambda L_i^{(3)}(\lambda) & -\lambda L_i^{(4)}(\lambda) \\ L_{i+2}^{(1)}(\lambda) & -L_{i+2}^{(2)}(\lambda) & L_{i+2}^{(3)}(\lambda) & -L_{i+2}^{(4)}(\lambda) \\ \lambda L_{i+2}^{(1)}(\lambda) & -\lambda L_{i+2}^{(2)}(\lambda) & \lambda L_{i+2}^{(3)}(\lambda) & -\lambda L_{i+2}^{(4)}(\lambda) \end{pmatrix} \quad (i = 1, 2).
\end{aligned}$$

Moreover, the solution operators have the following $\mathcal{R}$-boundedness properties:
Let $\varepsilon > 0$. Then, for all $k \in \{0, 1, 2\}, |\alpha| = k, |\delta| = 2$ and $\gamma \in \mathbb{N}_0^2$ the families

$$\left\{\lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{1-\frac{k}{2}} D^\alpha \operatorname{diag}\left(D^\delta, 1, D^\delta, 1\right) R(\lambda) \operatorname{diag}\left(\Lambda_{-1}, 1, \Lambda_{-1}, 1\right)\right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\right\} \tag{3.31}$$

in $L(L^p(\mathbb{R}^n;\mathbb{C}^4), L^p(\mathbb{R}^n_+;\mathbb{C}^4))$ *and*

$$\Big\{\lambda^\gamma\partial_\lambda^\gamma\Big[\lambda^{1-\frac{k}{2}}D^\alpha \operatorname{diag}\Big(D^\delta,1,D^\delta,1\Big)T^{(i)}(\lambda) \\ \times \operatorname{diag}\Big(\Lambda'_{(-I(i)-3)/2},\Lambda'_{(-I(i)-2)/2},\Lambda'_{(-I(i)-1)/2},\Lambda'_{-I(i)/2}\Big)\Big] : \lambda\in\Sigma_{\pi-\vartheta-\varepsilon}\Big\} \tag{3.32}$$

in $L(L_p(\mathbb{R}^n_+;\mathbb{C}^4))$ *are* $\mathcal{R}$*-bounded.*

Proof. Let $\lambda\in\Sigma_{\pi-\vartheta}$, $h\in\mathbb{E}^2_+$ and $g\in\mathbb{G}$. As described in the beginning of Section 3.1, we set $y' := r_+(\lambda+A_p)^{-1}e_+h\in\mathbb{F}^2_+$ and write $y=y'+y''$. Consequently, y'' has to solve the problem

$$\begin{aligned} A(D,\lambda)y'' &= 0 && \text{in } \mathbb{R}^n_+, \\ \gamma_0 B(D)y'' &= g-\gamma_0 B(D)y' && \text{on } \mathbb{R}^{n-1}. \end{aligned} \tag{3.33}$$

We define an extension $\widetilde{g}=(\widetilde{g}_1,\dots,\widetilde{g}_4)^\top$ of $g-\gamma_0B(D)y'$ by $\widetilde{g}:=E_\lambda g-B(D)y'$. Due to the remarks on the extension operator E_λ above, it holds that $\widetilde{g}\in\prod_{l=0}^3 H_p^{4-l}(\mathbb{R}^n_+)$.
Corollary 3.10 states that we can extend the operators $L_i^{(j)}(\lambda)$, which were defined for restrictions of Schwartz functions, to bounded operators $L_i^{(j)}(\lambda)\in L(H_p^{4+I(i)-j}(\mathbb{R}^n_+),H_p^4(\mathbb{R}^n_+))$. Indeed, it holds that $\Lambda'_{(4+I(i)-j)/2}\in L(H_p^{4+I(i)-j}(\mathbb{R}^n_+),L_p(\mathbb{R}^n_+))$ and hence we obtain

$$\begin{aligned} \|D^\alpha L_i^{(j)}(\lambda)\phi\|_{L_p(\mathbb{R}^n_+)} &\le |\lambda|^{-2+|\alpha|/2}\|\lambda^{2-|\alpha|/2}D^\alpha L_i^{(j)}(\lambda)\Lambda'_{(j-I(i)-4)/2}\|_{L(L_p(\mathbb{R}^n_+))} \\ &\qquad\cdot\|\Lambda'_{(4+I(i)-j)/2}\|_{L(H_p^{4+I(i)-j}(\mathbb{R}^n_+),L_p(\mathbb{R}^n_+))}\|\phi\|_{H_p^{4+I(i)-j}(\mathbb{R}^n_+)} \\ &\le C\|\phi\|_{H_p^{4+I(i)-j}(\mathbb{R}^n_+)} \end{aligned}$$

for all $|\alpha|\le 4$ with a constant only depending on λ (and, of course, on ρ_1,ρ_2,β_1 and β_2).
Now, by Proposition 3.8, a solution y'' of (3.33) is given by

$$y''=T(\lambda)\widetilde{g}=T^{(1)}(\lambda)\partial_n\widetilde{g}+T^{(2)}(\lambda)\widetilde{g}$$

and

$$\begin{aligned} y := y'+y'' &= r_+(A_p+\lambda)^{-1}e_+h+T(\lambda)(E_\lambda g-B(D)y') \\ &= r_+(A_p+\lambda)^{-1}e_+h+T(\lambda)(E_\lambda g-B(D)r_+(A_p+\lambda)^{-1}e_+h) \\ &= R(\lambda)e_+h+T(\lambda)E_\lambda g\in\mathbb{F}^2_+ \end{aligned}$$

is a solution of (3.29). Together with Lemma 3.6 and, again, Proposition 3.8, it follows that the solution is unique in the space of tempered distributions $\mathscr{S}'(\mathbb{R}^n)$ and therefore in $\mathbb{F}^2_+$.

Let $\varepsilon>0$. In order to show the $\mathcal{R}$-boundedness result for the operator family

$$\Big\{\lambda^\gamma\partial_\lambda^\gamma\Big[\lambda^{1-\frac{k}{2}}D^\alpha\operatorname{diag}\Big(D^\delta,1,D^\delta,1\Big)R(\lambda)\operatorname{diag}(\Lambda_{-1},1,\Lambda_{-1},1)\Big] : \lambda\in\Sigma_{\pi-\vartheta-\varepsilon}\Big\},$$

we use the representation

$$R(\lambda)=r_+(\lambda+A_p)^{-1}-T(\lambda)B(D)r_+(\lambda+A_p)^{-1} \tag{3.34}$$

obtained in the first part of this theorem. Theorem 3.2 and Remark 3.3 state that the family

$$\left\{\lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{1-\frac{k}{2}} D^\alpha \operatorname{diag}\left(D^\delta, 1, D^\delta, 1\right)(\lambda + A_p)^{-1} \operatorname{diag}(\Lambda_{-1}, 1, \Lambda_{-1}, 1)\right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\right\} \tag{3.35}$$

is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n; \mathbb{C}^4))$ and therefore the statement is true for the first term in the representation (3.34) of $R(\lambda)$.
For the second term in (3.34), we write

$$\begin{aligned}
&T(\lambda)B(D)r_+(\lambda + A_p)^{-1} \\
&= \sum_{i=1}^{2} T^{(i)}(\lambda)\partial_n^{2-i} B(D) r_+(\lambda + A_p)^{-1} \\
&= \sum_{i=1}^{2} T^{(i)}(\lambda) \operatorname{diag}\left(\Lambda'_{(-I(i)-3)/2}, \Lambda'_{(-I(i)-2)/2}, \Lambda'_{(-I(i)-1)/2}, \Lambda'_{-I(i)/2}\right) \\
&\quad \times \begin{pmatrix} \Lambda'_{(I(i)+3)/2}\partial_n^{2-i} & 0 & -\Lambda'_{(I(i)+3)/2}\partial_n^{2-i} & 0 \\ -\Lambda'_{(I(i)+2)/2}\partial_n^{3-i} & 0 & -\Lambda'_{(I(i)+2)/2}\partial_n^{3-i} & 0 \\ \Lambda'_{(I(i)+1)/2}\partial_n^{4-i} & 0 & -\Lambda'_{(I(i)+1)/2}\partial_n^{4-i} & 0 \\ -\Lambda'_{I(i)/2}\partial_n^{5-i} & 0 & -\Lambda'_{I(i)/2}\partial_n^{5-i} & 0 \end{pmatrix} r_+(\lambda + A_p)^{-1}.
\end{aligned} \tag{3.36}$$

Note that

$$C_i(\lambda) \begin{pmatrix} \Lambda_{-2} & 0 \\ 0 & 0 \\ 0 & \Lambda_{-2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_2 & 0 \end{pmatrix} = C_i(\lambda)$$

holds true for the matrix

$$C_i(\lambda) = \begin{pmatrix} \Lambda'_{(I(i)+3)/2}\partial_n^{2-i} & 0 & -\Lambda'_{(I(i)+3)/2}\partial_n^{2-i} & 0 \\ -\Lambda'_{(I(i)+2)/2}\partial_n^{3-i} & 0 & -\Lambda'_{(I(i)+2)/2}\partial_n^{3-i} & 0 \\ \Lambda'_{(I(i)+1)/2}\partial_n^{4-i} & 0 & -\Lambda'_{(I(i)+1)/2}\partial_n^{4-i} & 0 \\ -\Lambda'_{I(i)/2}\partial_n^{5-i} & 0 & -\Lambda'_{I(i)/2}\partial_n^{5-i} & 0 \end{pmatrix}$$

and consequently, we have

$$\begin{aligned}
&\lambda^{1-k/2} D^\alpha \operatorname{diag}\left(D^\delta, 1, D^\delta, 1\right) T(\lambda)B(D)r_+(\lambda + A_p)^{-1} \operatorname{diag}(\Lambda_{-1}, 1, \Lambda_{-1}, 1) \\
&= \lambda^{1-k/2} \sum_{i=1}^{2} D^\alpha \operatorname{diag}\left(D^\delta, 1, D^\delta, 1\right) T^{(i)}(\lambda) \operatorname{diag}\left(\Lambda'_{\frac{-I(i)-3}{2}}, \Lambda'_{\frac{-I(i)-2}{2}}, \Lambda'_{\frac{-I(i)-1}{2}}, \Lambda'_{\frac{-I(i}{2}}\right) \\
&\quad \times \begin{pmatrix} \Lambda'_{(I(i)+3)/2}\partial_n^{2-i} & 0 & -\Lambda'_{(I(i)+3)/2}\partial_n^{2-i} & 0 \\ -\Lambda'_{(I(i)+2)/2}\partial_n^{3-i} & 0 & -\Lambda'_{(I(i)+2)/2}\partial_n^{3-i} & 0 \\ \Lambda'_{(I(i)+1)/2}\partial_n^{4-i} & 0 & -\Lambda'_{(I(i)+1)/2}\partial_n^{4-i} & 0 \\ -\Lambda'_{I(i)/2}\partial_n^{5-i} & 0 & -\Lambda'_{I(i)/2}\partial_n^{5-i} & 0 \end{pmatrix} r_+ \begin{pmatrix} \Lambda_{-2} & 0 \\ 0 & 0 \\ 0 & \Lambda_{-2} \\ 0 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \Lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_2 & 0 \end{pmatrix} (\lambda + A_p)^{-1} \operatorname{diag}(\Lambda_{-1}, 1, \Lambda_{-1}, 1)
\end{aligned}$$

$$=: \sum_{i=1}^{2} \lambda^{1-k/2} M_i^{(1)}(\lambda) \times M_i^{(2)}(\lambda) \times M_i^{(3)}(\lambda). \tag{3.37}$$

From Theorem 3.2 we know that

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[(\lambda - \Delta)^{2-\frac{k}{2}} \begin{pmatrix} D^\alpha & 0 \end{pmatrix} A_l(D,\lambda)^{-1} \begin{pmatrix} (\lambda-\Delta)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right] : \lambda \in \Sigma_{\pi-\vartheta_l-\varepsilon} \right\} \quad (l = 1, 2)$$

is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n;\mathbb{C}^2), L_p(\mathbb{R}^n))$. For the full operator $A(D,\lambda)$, this implies that

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[(\lambda - \Delta)^{2-\frac{k}{2}} \begin{pmatrix} D^\alpha & 0 & 0 & 0 \\ 0 & 0 & D^\alpha & 0 \end{pmatrix} A(D,\lambda)^{-1} \operatorname{diag}(\Lambda_{-1}, 1 \Lambda_{-1}, 1) \right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}$$

is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n;\mathbb{C}^4), L_p(\mathbb{R}^n;\mathbb{C}^2))$. Hence, the operator family

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma M_i(\lambda)^{(3)} : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}, \tag{3.38}$$

i.e. the third term in (3.37), is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n;\mathbb{C}^4), L_p(\mathbb{R}^n;\mathbb{C}^2))$.
The (r,s)-th entry of the first term in (3.37) reads

$$\left[\lambda^{1-k/2} M_i^{(1)}(\lambda) \right]_{rs} = \begin{cases} \lambda^{1-k/2} D^{\alpha+\delta} L_i^{(s)}(\lambda) \Lambda'_{(-I(i)+s-4)/2}, & r = 1, \\ \lambda^{2-k/2} D^{\alpha} L_i^{(s)}(\lambda) \Lambda'_{(-I(i)+s-4)/2}, & r = 2, \\ \lambda^{1-k/2} D^{\alpha+\delta} L_{i+2}^{(s)}(\lambda) \Lambda'_{(-I(i)+s-4)/2}, & r = 3, \\ \lambda^{2-k/2} D^{\alpha} L_{i+2}^{(s)}(\lambda) \Lambda'_{(-I(i)+s-4)/2}, & r = 4 \end{cases}$$

for $s = 1, \ldots, 4$. By Corollary 3.10, it holds that

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{2-\frac{l}{2}} D^\alpha L_i^{(j)}(\lambda) (\lambda - \Delta')^{(j-I(i)-4)/2} \right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}$$

is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n_+))$, wherefore also the first term in (3.37) possesses the appropriate $\mathcal{R}$-boundedness, or, to be precise, it holds that the family of operators

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma M_i^{(1)}(\lambda) : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\} \tag{3.39}$$

is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n_+;\mathbb{C}^4))$.
The second term in (3.37) is given by

$$M_i^{(2)}(\lambda) = r_+ \begin{pmatrix} \Lambda'_{(I(i)+3)/2} \partial_n^{2-i} \Lambda_{-2} & -\Lambda'_{(I(i)+3)/2} \partial_n^{2-i} \Lambda_{-2} \\ -\Lambda'_{(I(i)+2)/2} \partial_n^{3-i} \Lambda_{-2} & -\Lambda'_{(I(i)+2)/2} \partial_n^{3-i} \Lambda_{-2} \\ \Lambda'_{(I(i)+1)/2} \partial_n^{4-i} \Lambda_{-2} & -\Lambda'_{(I(i)+1)/2} \partial_n^{4-i} \Lambda_{-2} \\ -\Lambda'_{I(i)/2} \partial_n^{5-i} \Lambda_{-2} & -\Lambda'_{I(i)/2} \partial_n^{5-i} \Lambda_{-2} \end{pmatrix}.$$

Thus, we investigate the symbol of the operators

$$\Lambda'_{(4+I(i)-j)/2} \partial_n^{1+j-i} \Lambda_{-2}$$

for $i = 1, 2$ and $j = 1, \dots, 4$. Those symbols are given by

$$\frac{(\lambda + |\xi'|^2)^{(4+I(i)-j)/2}}{(\lambda + |\xi'|^2 + |\xi_n|^2)^{2-(1+j-i)/2}} \cdot \frac{(i\xi_n)^{1+j-i}}{(\lambda + |\xi'|^2 + |\xi_n|^2)^{(1+j-i)/2}}.$$

As the second factor is smooth and positive homogeneous of degree 0, Lemma A.18 c) yields the $\mathcal{R}$-boundedness of

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \mathscr{F}^{-1} \frac{(i\xi_n)^{1+j-i}}{(\lambda + |\xi'|^2 + |\xi_n|^2)^{(1+j-i)/2}} \mathscr{F} : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\} \subset L(L_p(\mathbb{R}^n)).$$

For the first factor, note that

$$\frac{4 + I(i) - j}{2} = \begin{cases} 2 - \frac{j}{2}, & i = 1, \\ 2 - \frac{j-1}{2}, & i = 2 \end{cases}$$

equals

$$2 - \frac{1 + j - i}{2} = \begin{cases} 2 - \frac{j}{2}, & i = 1, \\ 2 - \frac{j-1}{2}, & i = 2. \end{cases}$$

Thus, by Lemma A.20 the family of operators

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \mathscr{F}^{-1} \frac{(\lambda + |\xi'|^2)^{(4+I(i)-j)/2}}{(\lambda + |\xi'|^2 + |\xi_n|^2)^{2-(1+j-i)/2}} \mathscr{F} : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\} \subset L(L_p(\mathbb{R}^n))$$

is also $\mathcal{R}$-bounded and consequently, by using the Leibniz product rule and Lemma A.16, one sees that the family

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma M_i^{(2)}(\lambda) : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}, \tag{3.40}$$

i.e. the second term in (3.37), is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}^n; \mathbb{C}^2), L_p(\mathbb{R}^n; \mathbb{C}^4))$.
Finally, from (3.38), (3.39), (3.40), the Leibniz rule and Lemma A.16 it follows that (3.31) is $\mathcal{R}$-bounded in $L(L^p(\mathbb{R}^n; \mathbb{C}^4), L^p(\mathbb{R}^n_+; \mathbb{C}^4))$.
The corresponding result for the operator family (3.32) follows directly from Corollary 3.10. □

The $\mathcal{R}$-boundedness of (3.31) and (3.32) implies some a-priori estimates for the unique solution y of (3.29), which we state in the following corollary.

3.12 Corollary (cf. [DS15], Corollary 3.6)**.** *For any $\varepsilon > 0$ and $\lambda_0 > 0$, there exists a constant $C > 0$ depending on ε and λ_0 such that for all $|\alpha| = k \in \{0, 1, 2\}$, and $\lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}$ the estimate*

$$\|\lambda^{1-k/2} D^\alpha y\|_{\mathbb{E}^2_+} \le C \Bigg(\|h\|_{\mathbb{E}^2_+} + \|g\|_{\mathbb{G}} + |\lambda| \|h_1\|_{L_p(\mathbb{R}^n_+)} + |\lambda| \|h_3\|_{L_p(\mathbb{R}^n_+)} + \sum_{l=1}^{4} |\lambda|^{\frac{2-l+1-1/p}{2}} \|g_l\|_{L_p(\mathbb{R}^{n-1})} \Bigg)$$

holds true for all $h = (h_1, \dots, h_4)^\top \in \mathbb{E}^2_+$ and $g = (g_1, \dots, g_4)^\top \in \mathbb{G}$, where y is the unique solution of (3.29).

Proof. In the same way as Corollary 3.6 follows from Theorem 3.5 in [DS15], this statement follows directly from Theorem 3.11 using

$$|||E_\lambda v|||_{k,p,\mathbb{R}^n_+} \le C|||v|||_{k-1/p,p,\mathbb{R}^{n-1}} \quad (v \in W_p^{k-1/p}(\mathbb{R}^{n-1}))$$

with a constant $C > 0$ independent of λ, where the $|||\cdot|||$ norms are given by

$$|||\phi|||_{s,p,U} := \|\phi\|_{W_p^s(U)} + |\lambda|^{s/2}\|\phi\|_{L_p(U)}$$

for $U \subset \mathbb{R}^n$, see Proposition 2.3 in [ADF97]. Note that "$B^s_{pp} = W^s_p$", see Theorem A.2 *d*). □

3.1.3. $\mathcal{R}$-sectoriality for the model problem

Similar as for the boundary value problem of a structurally damped plate equation, additional conditions have to be imposed in the basis space in order to obtain sectoriality, as the following result indicates:

3.13 Proposition (see [DS15], Proposition 3.4)**.** *For any $\lambda_0 \ge 0$, the operator $\lambda_0 + A_{p,+}$ is not sectorial in $\mathbb{E}^2_+$ and therefore, the operator is not the generator of a C_0-semigroup.*

Proof. For any $\lambda_0 \ge 0$, we can apply the results in [DD11], Section 3.2 to the boundary value problem $(\lambda_0 + A(D), B(D))$. In the notion of [DD11], Section 3.2, we let

$$\begin{aligned}(s_1, s_2, s_3, s_4) &= (0, 2, 0, 2),\\ (m_1, m_2, m_3, m_4) &= (2, 0, 2, 0),\\ (r_1, r_2, r_3, r_4) &= (-2, -1, 0, 1).\end{aligned}$$

For $\lambda \in (0, \infty)$ and $h \in \mathbb{E}^2_+$, let $y_\lambda \in D(A_{p,+})$ be the unique solution of $A(D, \lambda)y_\lambda = h$ (see Theorem 3.11). Theorem 3.8 in [DD11] states that if there exists a constant $C > 0$ independent of λ such that $\|\lambda y_\lambda\|_{\mathbb{E}^2_+} \le C$, then we already have $y_1 - y_3 = 0$ and $\partial_n y_1 + \partial_n y_3 = 0$ on $\mathbb{R}^{n-1}$. Hence, the resolvent estimate cannot hold for $h \in \mathbb{E}^2_+$ with $h_1 - h_3 \ne 0$ or $\partial_n h_1 + \partial_n h_3 \ne 0$ on $\mathbb{R}^{n-1}$.
The second statement follows directly from the theorem of Hille-Yosida (e.g. [Paz83], Theorem 3.1 or [EN00], Theorem 3.8). Indeed, assume that $-(\lambda_0 + A_{p,+})$ is the generator of a C_0-semigroup. Then, for $\omega \ge 0$ large enough, $-(\omega + \lambda_0 + A_{p,+})$ generates a bounded C_0-semigroup. By the theorem of Hille-Yosida, this implies that $\omega + \lambda_0 + A_{p,+}$ is sectorial which yields a contradiction to the first part of this proposition. □

Having Proposition 3.13 in mind, it seems tempting to define

$$\begin{aligned}\widetilde{\mathbb{E}}^2_0 = \widetilde{\mathbb{E}}^2_{p,0} &:= \left\{h = (h_1, h_2, h_3, h_4)^\top \in \mathbb{E}^2_+ : h_1 - h_3 = \partial_\nu(h_1 + h_3) = 0 \text{ on } \mathbb{R}^{n-1}\right\},\\ \widetilde{\mathbb{F}}^2_0 = \widetilde{\mathbb{F}}^2_{p,0} &:= \left\{y = (y_1, y_2, y_3, y_4)^\top \in \mathbb{F}^2_+ \cap \mathbb{E}^2_0 : y_2 - y_4 = \partial_\nu(y_2 + y_4) = 0 \text{ on } \mathbb{R}^{n-1}\right\}\end{aligned}$$

and consider the unbounded operator

$$\widetilde{A}_{p,0} \colon \widetilde{\mathbb{E}}^2_0 \supset D(\widetilde{A}_{p,0}) \to \widetilde{\mathbb{E}}^2_0$$

given by $\widetilde{A}_{p,0}U = A(D)U$ for $U \in D(\widetilde{A}_{p,0}) = \{Y \in \widetilde{\mathbb{F}}_0^2 : \gamma_0 B(D)Y = 0\}$.
However, unfortunately we are not able to show $\mathcal{R}$-sectoriality of the operator $\widetilde{A}_{p,0}$ in the space $\widetilde{\mathbb{E}}_0^2$. Hence, instead of using $\widetilde{\mathbb{E}}_0^2$ and $\widetilde{\mathbb{F}}_0^2$, we modify the spaces $\mathbb{E}_+^2$ and $\mathbb{F}_+^2$ as follows: let

$$\begin{aligned}\mathbb{E}_0^2 = \mathbb{E}_{p,0}^2 &:= \big(H_{p,0}^2(\mathbb{R}_+^n) \times L_p(\mathbb{R}_+^n)\big)^2,\\ \mathbb{F}_0^2 = \mathbb{F}_{p,0}^2 &:= \big(\big(H_p^4(\mathbb{R}_+^n) \cap H_{p,0}^2(\mathbb{R}_+^n)\big) \times H_{p,0}^2(\mathbb{R}_+^n)\big)^2\end{aligned}$$

and

$$A_{p,0} \colon \mathbb{E}_0^2 \supset D(A_{p,0}) \to \mathbb{E}_0^2$$

be defined by $A_{p,0}U = A(D)U$ for $U \in D(A_{p,0}) = \{Y \in \mathbb{F}_0^2 : \gamma_0 B(D)Y = 0\}$. Obviously, it holds that $\mathbb{E}_0^2$ is a closed subspace of the Banach space $\widetilde{\mathbb{E}}_0^2$.
Of course, proving $\mathcal{R}$-sectoriality for the operator $A_{p,0}$ in $\mathbb{E}_0^2$ is a weaker result and not optimal, at least in the sense of Proposition 3.13. In order to obtain a precise statement, one would have to show that the operator $\widetilde{A}_{p,0}$ is $\mathcal{R}$-sectorial in $\widetilde{\mathbb{E}}_0^2$ or that this is not the case. But coming from the system of evolution equations (3.1), we are mainly interested in solutions of the resolvent problem (3.29) with right-hand side $h = (0, h_2, 0, h_4)^\top \in \mathbb{E}_0^2$ for $h_2, h_4 \in L_p(\mathbb{R}_+^n)$. Therefore, for maximal L_p-regularity of the evolution equation, proving $\mathcal{R}$-sectoriality of $A_{p,0}$ in $\mathbb{E}_0^2$ is sufficient.

We introduce the multiplication operator M_n given by

$$(M_n f)(x', x_n) := x_n f(x', x_n) \qquad ((x', x_n) \in \mathbb{R}_+^n)$$

for a function $f \colon \mathbb{R}_+^n \to \mathbb{C}$. Moreover, it will be convenient to consider the odd extension operator $e_s \in L(\mathbb{E}_0^2, \mathbb{E})$ defined by

$$(e_s f)(x) := \begin{cases} f(x), & \text{if } x_n \geq 0,\\ -f(x', -x_n), & \text{if } x_n < 0.\end{cases}$$

Using similar methods as in [DS15], we are able to show that this statement in fact holds true. In particular, we will use the following technical results.

3.14 Lemma (Lemma 4.1 and Remark 4.2 in [DS15]).
For every $f \in H_{p,0}^2(\mathbb{R}_+^n)$, we have that $M_n^{-2} f \in L_p(\mathbb{R}_+^n)$. Moreover, it holds that

$$\partial_n^2 \mathscr{F} e_s M_n^{-2} f = -\mathscr{F} e_s f$$

for all $f \in H_{p,0}^2(\mathbb{R}_+^n)$.

3.15 Lemma (Lemma 4.3 in [DS15]). *Let $\varepsilon \in (0, \pi - \vartheta)$ and $b \colon (\mathbb{R}^n \times \overline{\Sigma}_{\pi-\vartheta-\varepsilon}) \setminus \{0\} \to \mathbb{C}$ be infinitely smooth and homogeneous of degree 0 in $(\xi, \lambda^{1/2})$. We set*

$$\begin{aligned} b_0(\xi,\lambda) &:= -(\lambda + |\xi'|^2)\partial_n^2 b(\xi,\lambda),\\ b_1(\xi,\lambda) &:= -2\mathrm{i}(\lambda + |\xi'|^2)^{1/2}\partial_n b(\xi,\lambda),\\ b_2(\xi,\lambda) &:= b(\xi,\lambda)\end{aligned}$$

for $(\xi,\lambda)\in(\mathbb{R}^n\times\overline{\Sigma}_{\pi-\vartheta-\varepsilon})\setminus\{0\}$. *We then obtain*

$$r_+\mathscr{F}^{-1}b(\cdot,\lambda)\mathscr{F}e_sf=\sum_{l=0}^{2}M_n^l(\lambda-\Delta')^{-1+l/2}r_+\mathscr{F}^{-1}b_l(\cdot,\lambda)\mathscr{F}e_sM_n^{-2}f$$

for all $f\in H^2_{p,0}(\mathbb{R}^n_+)$ *and*

$$\|r_+\mathscr{F}^{-1}b_l(\cdot,\lambda)\mathscr{F}e_s\|_{L(L_p(\mathbb{R}^n_+))}\le C\quad(l=0,1,2).$$

Moreover, the operator families

$$\left\{\lambda^\gamma\partial_\lambda^\gamma r_+\mathscr{F}^{-1}b_l(\cdot,\lambda)\mathscr{F}e_s:\lambda\in\overline{\Sigma}_{\pi-\vartheta-\varepsilon}\right\}\subset L(L_p(\mathbb{R}^n_+))$$

are $\mathcal{R}$*-bounded for every* $\gamma\in\mathbb{N}_0^2$ *and* $l=0,1,2$.

Now, we finally obtain the desired $\mathcal{R}$-sectoriality (recall Definition 2.6):

3.16 Theorem (cf [DS15], Theorem 4.4)**.** *For every* $\lambda_0>0$*, the operator* $\lambda_0+A_{p,0}$ *is* $\mathcal{R}$*-sectorial in* $\mathbb{E}_0^2$ *with* $\mathcal{R}$*-angle* $\vartheta=\max\{\vartheta_1,\vartheta_2\}$.

Proof. Note that $\mathscr{D}(\mathbb{R}^n_+)^4=\prod_{k=1}^4\mathscr{D}(\mathbb{R}^n_+)\subset\mathbb{E}_0^2$ is dense and that it is sufficient to prove the $\mathcal{R}$-boundedness for $h\in\mathscr{D}(\mathbb{R}^n_+)^4$.
Let $h=(h_1,h_2,h_3,h_4)^\top\in\mathscr{D}(\mathbb{R}^n_+)^4$, $\varepsilon\in(0,\pi-\vartheta)$ and $\lambda\in\lambda_0+\Sigma_{\pi-\vartheta-\varepsilon}$. By Theorem 3.11, the unique solution $y=(y_1,y_2,y_3,y_4)^\top\in\mathbb{F}_0^2$ of the resolvent problem is given by

$$y=R(\lambda)e_sh=r_+(\lambda+A_p)^{-1}e_sh-T(\lambda)B(D)r_+(\lambda+A_p)^{-1}e_sh.$$

As h is a test function, so is e_sh and we calculate

$$\begin{aligned}(\mathscr{F}e_sh(x',-x_n))(\xi',\xi_n)&=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathrm{e}^{-ix'\xi'}\mathrm{e}^{ix_n\xi_n}(e_sh)(\xi',\xi_n)\,d(\xi',\xi_n)\\&=\frac{1}{(2\pi)^{n/2}}\int_{\xi_n>0}\mathrm{e}^{-\mathrm{i}x'\xi'}\mathrm{e}^{\mathrm{i}x_n\xi_n}h(\xi',\xi_n)\,d(\xi',\xi_n)\\&\quad-\frac{1}{(2\pi)^{n/2}}\int_{\xi_n<0}\mathrm{e}^{-\mathrm{i}x'\xi'}\mathrm{e}^{\mathrm{i}x_n\xi_n}h(\xi',-\xi_n)\,d(\xi',\xi_n)\\&=\frac{1}{(2\pi)^{n/2}}\int_{y_n<0}\mathrm{e}^{-\mathrm{i}x'\xi'}\mathrm{e}^{-\mathrm{i}x_ny_n}h(\xi',-y_n)\,d(\xi',y_n)\\&\quad-\frac{1}{(2\pi)^{n/2}}\int_{y_n>0}\mathrm{e}^{-\mathrm{i}x'\xi'}\mathrm{e}^{-\mathrm{i}x_ny_n}h(\xi',y_n)\,d(\xi',y_n)\\&=-(\mathscr{F}e_sh(x',x_n))(\xi',\xi_n).\end{aligned}$$

In the same way, we have

$$\widetilde{y}(x',-x_n)=\mathscr{F}^{-1}A(\cdot,\lambda)^{-1}\mathscr{F}e_sh(x',-x_n)=-\widetilde{y}(x',x_n)$$

for $\widetilde{y}=(\lambda+A_p)^{-1}e_sh$. Here, we used the fact that $A(\xi',\xi_n,\lambda)=A(\xi',-\xi_n,\lambda)$. This shows that $\widetilde{y}\in\mathbb{E}$ is odd and it holds that $\widetilde{y}_i(x',0)=\partial_n^2\widetilde{y}_i(x',0)=0$ for all $x'\in\mathbb{R}^{n-1}$ and $i=1,3$. Hence, we may assume that $\widetilde{g}_1 = \widetilde{g}_3 = 0$ in the proof of Theorem 3.11. Consequently, we obtain

$$\begin{aligned} y &= r_+(\lambda + A_p)^{-1} e_s h - \sum_{i=1}^{2} T^{(i)}(\lambda)\partial_n^{2-i} B(D) r_+(\lambda + A_p)^{-1} e_s h \\ &= r_+(\lambda + A_p)^{-1} e_s h \\ &\quad - \sum_{i=1}^{2} \begin{pmatrix} 0 & -L_i^{(2)}(\lambda) & 0 & -L_i^{(4)}(\lambda) \\ 0 & -\lambda L_i^{(2)}(\lambda) & 0 & -\lambda L_i^{(4)}(\lambda) \\ 0 & -L_{i+2}^{(2)}(\lambda) & 0 & -L_{i+2}^{(4)}(\lambda) \\ 0 & -\lambda L_{i+2}^{(2)}(\lambda) & 0 & -\lambda L_{i+2}^{(4)}(\lambda) \end{pmatrix} \partial_n^{2-i} \begin{pmatrix} 1 & 0 & -1 & 0 \\ -\partial_n & 0 & -\partial_n & 0 \\ \partial_n^2 & 0 & -\partial_n^2 & 0 \\ -\partial_n^3 & 0 & -\partial_n^3 & 0 \end{pmatrix} r_+(\lambda + A_p)^{-1} e_s h. \end{aligned} \tag{3.41}$$

We can write the second term in (3.41) as

$$\begin{aligned} &- \begin{pmatrix} 1 \\ \lambda \\ 0 \\ 0 \end{pmatrix} \sum_{k=1}^{2}\sum_{i=1}^{2} \left[L_i^{(2)}(\lambda)(-\partial_n)^{3-i} + L_i^{(4)}(\lambda)(-\partial_n)^{5-i} \right] \\ &\qquad\qquad \times r_+ \mathscr{F}^{-1} \left(\widetilde{a}_{1,k}(\cdot,\lambda) \mathscr{F} e_s h_k + \widetilde{a}_{3,k+2}(\cdot,\lambda) \mathscr{F} e_s h_{k+2} \right) \\ &- \begin{pmatrix} 0 \\ 0 \\ 1 \\ \lambda \end{pmatrix} \sum_{k=1}^{2}\sum_{i=1}^{2} \left[L_{i+2}^{(2)}(\lambda)(-\partial_n)^{3-i} + L_{i+2}^{(4)}(\lambda)(-\partial_n)^{5-i} \right] \\ &\qquad\qquad \times r_+ \mathscr{F}^{-1} \left(\widetilde{a}_{1,k}(\cdot,\lambda) \mathscr{F} e_s h_k + \widetilde{a}_{3,k+2}(\cdot,\lambda) \mathscr{F} e_s h_{k+2} \right), \end{aligned} \tag{3.42}$$

where $\widetilde{a}_{i,j}(\xi,\lambda)$ denotes the (i,j)-th entry of the matrix $A(\xi,\lambda)^{-1}$, see Proposition 3.1.
In order to show the assertion of the theorem, we have to find a $\mathcal{R}$-bound for

$$\{\lambda R(\lambda) e_s h : \lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}\}$$

in the space $L(\mathbb{E}_0^2)$. From Proposition 3.4, we already know that

$$\{r_+ \lambda(\lambda + A_p)^{-1} e_s : \lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}\},$$

i.e. the first term in (3.41), is $\mathcal{R}$-bounded in $L(\mathbb{E}_0^2)$.
We can now restrict ourselves to the first term in (3.42), as the second term (with i replaced by $i+2$) can be treated in the same way.
We write

$$\begin{aligned} &L_i^{(j)}(\lambda)(-\partial_n)^{1+j-i} r_+ \mathscr{F}^{-1} \left(\widetilde{a}_{1,k}(\cdot,\lambda) \mathscr{F} e_s h_k + \widetilde{a}_{3,k+2}(\cdot,\lambda) \mathscr{F} e_s h_{k+2} \right) \\ &= L_i^{(j)}(\lambda) \Lambda'_{j/2-I(i)/2-k} r_+ \mathscr{F}^{-1} \Big((\mathrm{i}\xi_n)^{1+j-i} (\lambda + |\xi'|^2)^{k-j/2+I(i)/2} \widetilde{a}_{1,k}(\cdot,\lambda) \mathscr{F} e_s h_k \\ &\qquad\qquad + (\mathrm{i}\xi_n)^{1+j-i} (\lambda + |\xi'|^2)^{k-j/2+I(i)/2} \widetilde{a}_{3,k+2}(\cdot,\lambda) \mathscr{F} e_s h_{k+2} \Big) \\ &=: L_i^{(j)}(\lambda) \Lambda'_{j/2-I(i)/2-k} r_+ \mathscr{F}^{-1} \left(g_{i,j,k}^{(1)}(\cdot,\lambda) \mathscr{F} e_s h_k + g_{i,j,k}^{(3)}(\cdot,\lambda) \mathscr{F} e_s h_{k+2} \right) \end{aligned} \tag{3.43}$$

for $i = 1, 2$, $k = 1, 2$ and $j = 2, 4$.

Let us first consider the case $k = 2$. Since

$$(\widetilde{a}_{1,2}, \widetilde{a}_{3,4})(\xi, \lambda) = \left(\frac{1}{\beta_1(\alpha_+^{(1)}\lambda + |\xi|^2)(\alpha_-^{(1)}\lambda + |\xi|^2)}, \frac{1}{\beta_2(\alpha_+^{(2)}\lambda + |\xi|^2)(\alpha_-^{(2)}\lambda + |\xi|^2)} \right)$$

and

$$4 - j + I(i) + 1 + j - i = 5 + I(i) - i = 4$$

for $j = 2, 4$ and $i = 1, 2$, we have that

$$g_{i,j,2}^{(1)}(\xi, \lambda) = (\mathrm{i}\xi_n)^{1+j-i}(\lambda + |\xi'|^2)^{2-j/2+I(i)/2}\widetilde{a}_{1,2}(\xi, \lambda)$$

and

$$g_{i,j,2}^{(3)}(\xi, \lambda) = (\mathrm{i}\xi_n)^{1+j-i}(\lambda + |\xi'|^2)^{2-j/2+I(i)/2}\widetilde{a}_{3,4}(\xi, \lambda)$$

are homogeneous of degree 0 in $(\xi, \lambda^{1/2})$ for $i = 1, 2$ and $j = 2, 4$, wherefore these functions satisfy the conditions for the Mikhlin multiplier theorem. Lemma A.18 thus implies that

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma r_+ \mathscr{F}^{-1} g_{i,j,2}^{(m)}(\cdot, \lambda) \mathscr{F} e_s : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\} \subset L(L_p(\mathbb{R}_+^n)) \tag{3.44}$$

is $\mathcal{R}$-bounded for $i = 1, 2$, $j = 2, 4$, $m = 1, 3$ and $\gamma \in \mathbb{N}_0^2$.
From Corollary 3.10, we know that

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \left[\lambda^{2-\frac{l}{2}} D^\alpha L_i^{(j)}(\lambda)(\lambda - \Delta')^{(j-I(i)-4)/2} \right] : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon} \right\}$$

is $\mathcal{R}$-bounded in $L(L_p(\mathbb{R}_+^n))$ and altogether, from (3.43) and (3.44) we obtain that the terms in (3.42) for $k = 2$ are $\mathcal{R}$-bounded in $L(\mathbb{E}_0^2)$.

Finally, let $k = 1$. In this case, the power $\frac{j}{2} - \frac{I(i)}{2} - k = \frac{j}{2} - \frac{I(i)}{2} - 1$ of $\Lambda'_{j/2-I(i)/2-1}$ does not fit to Corollary 3.10. Instead, we will use the fact that $h_1, h_3 \in H_{p,0}^2(\mathbb{R}_+^n)$. To be precise, we will use the representation from Lemma 3.15 in (3.43). Note that

$$\begin{aligned} g_{i,j,1}^{(1)}(\xi, \lambda) &= (i\xi_n)^{1+j-i}(\lambda + |\xi'|^2)^{1-j/2+I(i)/2}\widetilde{a}_{1,1}(\xi, \lambda) \\ &= \frac{(i\xi_n)^{1+j-i}(\lambda + |\xi'|^2)^{1-j/2+I(i)/2}(\lambda + \rho_1|\xi|^2)}{\beta_1(\alpha_+^{(1)}\lambda + |\xi|^2)(\alpha_-^{(1)}\lambda + |\xi|^2)} \end{aligned}$$

and similarly $g_{i,j,1}^{(3)}$ are both infinitely smooth homogeneous of degree 0 in $(\xi, \lambda^{1/2})$. Therefore, with Lemma 3.15 we write

$$\begin{aligned} &L_i^{(j)}(\lambda)\Lambda'_{j/2-I(i)/2-1} r_+ \mathscr{F}^{-1} \left(g_{i,j,1}^{(1)}(\cdot, \lambda) \mathscr{F} e_s h_1 + g_{i,j,1}^{(3)}(\cdot, \lambda) \mathscr{F} e_s h_3 \right) \\ &= \sum_{l=0}^{2} L_i^{(j)}(\lambda)\Lambda'_{l/2+j/2-I(i)/2-2} M_n^l r_+ \mathscr{F}^{-1} b_l^{(1)}(\cdot, \lambda) \mathscr{F} e_s M_n^{-2} h_1 \\ &\quad + \sum_{l=0}^{2} L_i^{(j)}(\lambda)\Lambda'_{l/2+j/2-I(i)/2-2} M_n^l r_+ \mathscr{F}^{-1} b_l^{(3)}(\cdot, \lambda) \mathscr{F} e_s M_n^{-2} h_3, \end{aligned} \tag{3.45}$$

where we have set, suppressing the dependence on $i = 1, 2$ and $j = 2, 4$,

$$\begin{aligned} b_0^{(m)}(\xi,\lambda) &:= -(\lambda + |\xi'|^2)\partial_n^2 g_{i,j,1}^{(m)}(\xi,\lambda), \\ b_1^{(m)}(\xi,\lambda) &:= -2\mathrm{i}(\lambda + |\xi'|^2)^{1/2}\partial_n g_{i,j,1}^{(m)}(\xi,\lambda), \\ b_2^{(m)}(\xi,\lambda) &:= g_{i,j,1}^{(m)}(\xi,\lambda) \end{aligned}$$

for $m = 1, 3$. Furthermore, Lemma 3.15 also states that the families of operators

$$\left\{\lambda^\gamma \partial_\lambda^\gamma r_+ \mathscr{F}^{-1} b_l^{(m)}(\cdot,\lambda)\mathscr{F} e_s : \lambda \in \Sigma_{\pi-\vartheta-\varepsilon}\right\} \subset L(L_p(\mathbb{R}^n_+)) \quad (m = 1,3) \tag{3.46}$$

are $\mathcal{R}$-bounded for every $\gamma \in \mathbb{N}_0^2$ and $l = 0, 1, 2$.
Combining (3.45) and (3.46), it is left to show that

$$\left\{\lambda^{2-|\alpha|} D^\alpha L_i^{(j)}(\lambda)\Lambda'_{l/2+j/2-I(i)/2-2} M_n^l : \lambda \in \lambda_0 + \Sigma_{\pi-\vartheta-\varepsilon}\right\} \subset L(L_p(\mathbb{R}^n_+)) \tag{3.47}$$

is $\mathcal{R}$-bounded for $|\alpha| \le 2$, $l \in \{0, 1, 2\}$, $i \in \{1, 2\}$ and $j \in \{2, 4\}$. By Corollary 3.10, this is true for $l = 0$. For $l > 0$, we proceed as in the proof of Proposition 3.8 and Corollary 3.10 and write

$$\begin{aligned} &\lambda^{2-|\alpha|/2} D^\alpha L_i^{(j)}(\lambda)\Lambda'_{l/2+j/2-I(i)/2-2} M_n^l \phi(\cdot, x_n) \\ &= -\int_0^\infty (\mathscr{F}')^{-1} m(\cdot, x_n + y_n, \lambda)(\mathscr{F}'\phi)(\cdot, y_n)\, dy_n \quad (\phi \in \mathscr{S}(\mathbb{R}^n)|_{\mathbb{R}^n_+}) \end{aligned}$$

with the symbol

$$m(\xi', x_n + y_n, \lambda) := \lambda^{2-|\alpha|/2}(\xi')^{\alpha'}(\lambda + |\xi'|^2)^{l/2+j/2-I(i)/2-2} y_n^l \partial_n^{\alpha_n + I(i)} w_1^{(j)}(\xi', x_n + y_n, \lambda)$$

for $x_n, y_n > 0$, $\alpha = (\alpha', \alpha_n)$ and $\xi' \in \mathbb{R}^{n-1}$. Since $y_n^l < (x_n + y_n)^l$ for $x_n > 0$, by Lemma 3.9 *b)* we have that $(x_n + y_n)m(\cdot, x_n + y_n, \lambda)$ satisfies the Mikhlin condition. As in the proof of Corollary 3.10, we conclude the $\mathcal{R}$-boundedness of (3.47). This finishes the proof. □

3.2. The evolution equation in $\mathbb{R}^n_+$ and in bounded domains

In this section, we are finally able to solve the system of evolution equations (3.1)-(3.2) with inhomogeneous transmission conditions, i.e.

$$\begin{aligned} \partial_t^2 u_i + \beta_i \Delta^2 u_i - \rho_i \Delta \partial_t u_i &= f_i \quad \text{in } J \times \Omega_i, i = 1, 2, \\ \partial_\nu^{j-1} u_1 - \partial_\nu^{j-1} u_2 &= g_j \quad \text{on } J \times \Gamma, j = 1, \ldots, 4 \end{aligned} \tag{3.48}$$

with initial conditions

$$\begin{aligned} u_i|_{t=0} &= \phi_i \quad \text{in } \Omega_i, i = 1, 2, \\ \partial_t u_i|_{t=0} &= \psi_i \quad \text{in } \Omega_i, i = 1, 2. \end{aligned} \tag{3.49}$$

Here, $J = (0, T)$ is a finite time interval for some $T \in (0, \infty)$. First, we will use the results from Section 3.1 to solve the problem in the full-space case $\Omega_1 = \mathbb{R}^n_+$, $\Omega_2 = \mathbb{R}^n_-$ and $\Gamma = \mathbb{R}^{n-1}$ for

$n \in \mathbb{N}$. Naturally, the data is taken from the spaces

$$\begin{aligned} f_i \in \mathcal{E}_\pm &:= L_p(J; L_p(\mathbb{R}^n_\pm)),\\ g_j \in \mathcal{G}_j &:= W_p^{2-(j-1)/2-1/(2p)}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; W_p^{4-(j-1)-1/p}(\mathbb{R}^{n-1})),\\ \phi_i \in \mathcal{Y}_\pm &:= W_p^{4-2/p}(\mathbb{R}^n_\pm),\\ \psi_i \in \mathcal{Z}_\pm &:= W_p^{2-2/p}(\mathbb{R}^n_\pm) \end{aligned}$$

for $i = 1, 2$ and $j = 1, 2, 3, 4$. As usual, W_p^s denotes the Sobolev-Slobodeckii space, see Section A.1. Furthermore, we assume that the *compatibility conditions*

$$\begin{aligned} g_j|_{t=0} &= \partial_\nu^{j-1}\phi_1 - \partial_\nu^{j-1}\phi_2 \text{ for } j = 1, \dots, 4 \quad \text{if } p > \tfrac{3}{2}j - 3,\\ \partial_t g_j|_{t=0} &= \partial_\nu^{j-1}\psi_1 - \partial_\nu^{j-1}\psi_2 \text{ for } j = 1, 2 \qquad \text{if } p > \tfrac{3}{2}j \end{aligned} \tag{3.50}$$

are satisfied on $\Gamma = \mathbb{R}^{n-1}$.

We are looking for a unique solution $u = (u_1, u_2)^\top$ in the space $\mathcal{F} := \mathcal{F}_+ \times \mathcal{F}_-$ where

$$\mathcal{F}_\pm := H_p^2(J; L_p(\mathbb{R}^n_\pm)) \cap L_p(J; H_p^4(\mathbb{R}^n_\pm)).$$

3.17 Remark. In order to formulate the compatibility conditions (3.50), we have used Theorem 4.2 in [MS12] which states that

$$\mathcal{G}_j \hookrightarrow BUC(\overline{J}; W_p^{4-(j-1)-3/p}(\mathbb{R}^{n-1}))$$

for $j = 1, \dots, 4$ and

$$\mathcal{G}_j \hookrightarrow BUC^1(\overline{J}; W_p^{2-(j-1)-3/p}(\mathbb{R}^{n-1}))$$

for $j = 1, 2$, provided that $p > \frac{3}{2}j$.

In the following theorem, we solve the system (3.48), (3.49). We will not go into all of the details of the proof, as our main interest was to show maximal regularity for the operator $A_{p,0}$ in $\mathbb{E}_0^2$. This is the assertion in part *a*), which follows from Theorem 3.16. Part *b*) and *c*) are then obtained in the same way as the analogous results in Theorem 3.5. The rest can be shown similarly as in the proof of Theorem 4.5 and Theorem 4.6 in [DS15] (see also Theorem 3.24 in [Fro16]) using Theorem 3.5. We will give a short outline of the proof of this last part of the theorem.

3.18 Theorem (see also [DS15], Theorem 4.5, Theorem 4.6)**.**

a) *The operator $-A_{p,0}$ generates an analytic C_0-semigroup on $\mathbb{E}_0^2$ with maximal L_p-regularity on bounded time intervals for every $q \in (1, \infty)$.*

b) *Let $p \notin \{3/2, 3\}$ and $T \in (0, \infty)$. Then, for all $(f_1, f_2) \in \mathcal{E}_+ \times \mathcal{E}_-$, $(\phi_1, \phi_2) \in \mathcal{Y}_+ \times \mathcal{Y}_-$ and $(\psi_1, \psi_2) \in \mathcal{Z}_+ \times \mathcal{Z}_-$ satisfying the compatibility conditions* (3.50)*, there exists a unique solution $u \in \mathcal{F}$ of* (3.48)*,* (3.49) *with $g = 0$. Moreover, there exists a constant $C = C(p, T) > 0$ such that*

$$\|u\|_{\mathcal{F}} \le C \left(\|(f_1, f_2)\|_{\mathcal{E}_+ \times \mathcal{E}_-} + \|(\phi_1, \phi_2)\|_{\mathcal{Y}_+ \times \mathcal{Y}_-}\| + \|(\psi_1, \psi_2)\|_{\mathcal{Z}_+ \times \mathcal{Z}_-} \right).$$

c) *Let $T = \infty$, $f = 0$, $g = 0$, $(\phi_1, \phi_2) \in H^2_{p,0}(\mathbb{R}^n_+) \times H^2_{p,0}(\mathbb{R}^n_-)$ and $(\psi_1, \psi_2) \in L_p(\mathbb{R}^n_+) \times L_p(\mathbb{R}^n_-)$. Then there exists a unique solution u of* (3.48), (3.49) *with*

$$\partial_t^2 u,\, \partial_t \nabla^2 u,\, \nabla^4 u \in C([\varepsilon, \infty), L_p(\mathbb{R}^n_+) \times L_p(\mathbb{R}^n_-))$$

for each $\varepsilon > 0$ and

$$\partial_t u,\, \nabla^2 u \in C([0, \infty), L_p(\mathbb{R}^n_+) \times L_p(\mathbb{R}^n_-)).$$

If $(\phi_1, \phi_2) \in (H^4_p(\mathbb{R}^n_+) \cap H^2_{p,0}(\mathbb{R}^n_+)) \times (H^4_p(\mathbb{R}^n_-) \cap H^2_{p,0}(\mathbb{R}^n_-))$ and $(\psi_1, \psi_2) \in H^2_{p,0}(\mathbb{R}^n_+) \times H^2_{p,0}(\mathbb{R}^n_-)$, we may choose $\varepsilon = 0$.

d) *Let $p \notin \{3/2, 3\}$ and $T \in (0, \infty)$. Then, for all $(f_1, f_2) \in \mathcal{E}_+ \times \mathcal{E}_-$, $(g_1, \dots g_4) \in \mathcal{G}_1 \times \dots \times \mathcal{G}_4$, $(\phi_1, \phi_2) \in \mathcal{Y}_+ \times \mathcal{Y}_-$ and $(\psi_1, \psi_2) \in \mathcal{Z}_+ \times \mathcal{Z}_-$ satisfying the compatibility conditions* (3.50), *there exists a unique solution $u \in \mathcal{F}$ of* (3.48), (3.49). *Moreover, there exists a constant $C = C(p, T) > 0$ such that*

$$\|u\|_{\mathcal{F}} \le C\bigg(\|(f_1, f_2)\|_{\mathcal{E}_+ \times \mathcal{E}_-} + \sum_{j=1}^{4} \|g_j\|_{\mathcal{G}_j} + \|(\phi_1, \phi_2)\|_{\mathcal{Y}_+ \times \mathcal{Y}_-}\| + \|(\psi_1, \psi_2)\|_{\mathcal{Z}_+ \times \mathcal{Z}_-} \bigg).$$

Sketch of proof. As mentioned above, *a*) follows from 3.16 and *b*) and *c*) are consequences of *a*), cf. Theorem 3.5.
Let $(f_1, f_2) \in \mathcal{E}_+ \times \mathcal{E}_-$, $(g_1, \dots g_4) \in \mathcal{G}_1 \times \dots \times \mathcal{G}_4$, $(\phi_1, \phi_2) \in \mathcal{Y}_+ \times \mathcal{Y}_-$ and $(\psi_1, \psi_2) \in \mathcal{Z}_+ \times \mathcal{Z}_-$ satisfy the compatibility conditions (3.50). We extend f, ϕ_i and ψ_i to $\mathbb{R}^n$ for $i = 1, 2$ using the extension operators $e_\pm \colon H^s_p(\mathbb{R}^n_\pm) \to H^s_p(\mathbb{R}^n)$. Then, by Theorem 3.5, we obtain a solution

$$u' = (u'_1, u'_2) \in H^2_p((0, T); L_p(\mathbb{R}^n)) \cap L_p((0, T); H^4_p(\mathbb{R}^n))^2 = \mathcal{F}^2_{\mathbb{R}^n}$$

of the uncoupled problem

$$\begin{aligned} \partial_t^2 u'_1 + \beta_1 \Delta^2 u'_1 - \rho_1 \Delta \partial_t u'_1 &= f_1, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\ \partial_t^2 u'_2 + \beta_1 \Delta^2 u'_2 - \rho_2 \Delta \partial_t u'_2 &= f_2, \quad (t, x) \in (0, T) \times \mathbb{R}^n, \\ u'_i|_{t=0} &= \phi_i, \quad x \in \mathbb{R}^n, i = 1, 2, \\ \partial_t u'_i|_{t=0} &= \psi_i, \quad x \in \mathbb{R}^n, i = 1, 2. \end{aligned} \tag{3.51}$$

Now, we define $\widetilde{g} = (\widetilde{g}_1, \dots, \widetilde{g}_4)^\top$ by $\widetilde{g}_j = g_j - \gamma_0((-1)^{j-1} \partial_n^{j-1}(u'_1 - u'_2))$ for $j = 1, \dots, 4$. Note that the $\widetilde{g}_j$ satisfy the compatibility conditions (3.50) with $\phi_i = 0$ and $\psi_i = 0$ for $i = 1, 2$. Furthermore, it holds that $\|\widetilde{g}_j\|_{\mathcal{G}_j} \le C(\|g_j\|_{\mathcal{G}_j} + \|u'\|_{\mathcal{F}^2_{\mathbb{R}^n}})$ for $j = 1, \dots, 4$.
Hence, it is sufficient to consider the problem (3.48), (3.49) with $f_i = 0, \phi_i = 0, \psi_i = 0$ for $i = 1, 2$ and $g_j \in \mathcal{G}_j$ for $j = 1, \dots, 4$ satisfying the compatibility conditions (3.50) with $\phi_i = 0$ and $\psi_i = 0$ for $i = 1, 2$. We may assume that the functions $g_j \in \mathcal{G}_j$ are test functions for $j = 1, \dots, 4$.
In the following, we implicitly use reflection on the transmission interface $\Gamma = \mathbb{R}^{n-1}$, in order to use the representation of the solution for the stationary problem given by Theorem 3.11.
We denote the temporal Fourier transform by $\mathscr{F}_t$ and set $\hat{g} := \mathscr{F}_t g$. Having Theorem 3.11 in mind, we define $\hat{y}(\mathrm{i}\tau) := T(\mathrm{i}\tau) E_{\mathrm{i}\tau} \hat{g}$ for $\lambda = \mathrm{i}\tau$ with $\tau \in \mathbb{R}$. Defining $y := \mathscr{F}_t^{-1} \hat{y}$ and $u_1 := y_1$ as

well as $u_2 := y_3$, we have that $\partial_t u_1 = y_2$ and $\partial_t u_2 = y_4$.
Now, as in the proof of Theorem 4.6 of [DS15] using the definition of $T(\lambda)$, one calculates

$$\begin{aligned}\hat{y}(\mathrm{i}\tau) = \sum_{i=1}^{2} \Big[T^{(i)}(\mathrm{i}\tau) \operatorname{diag}\left(\Lambda'_{(-I(i)-3)/2}, \Lambda'_{(-I(i)-2)/2}, \Lambda'_{(-I(i)-1)/2}, \Lambda'_{-I(i)/2}\right) \mathscr{F}_t \\ \times (-1)^i \mathscr{F}_t^{-1} \Big(\operatorname{diag}\left(\Lambda'_{(I(i)-i+5)/2}, \Lambda'_{(I(i)-i+4)/2}, \Lambda'_{(I(i)-i+3)/2}, \Lambda'_{-(I(i)-i+2)/2}\right) E_{\mathrm{i}\cdot}\hat{g} \Big)(\tau) \Big].\end{aligned} \tag{3.52}$$

By [MS12], Lemma 3.1, the operator $L = \mu + \partial_t - \Delta'$ in $L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1}))$ with domain $H^1_{p,0}(\mathbb{R}_+; L_p(\mathbb{R}^{n-1})) \cap L_p(\mathbb{R}_+; H^2_p(\mathbb{R}^{n-1}))$ admits bounded imaginary powers with power angle not larger than $\pi/2$ for any $\mu > 0$. Thus, formally we have to shift the equation, i.e. in fact we are solving the system

$$\begin{aligned}(\mu + \partial_t)U(t) + A(D)U(t) &= 0,\\ \gamma_0 B(D)U(t) &= g(t)\end{aligned}$$

for $U = (u_1, \partial_t u_1, u_2, \partial_t u_2)^\top$. Rescaling the so obtained solution with appropriate estimates, yields the assertion for the original evolution equation without shift.
Let $F_0^{\pm}$ be defined by $(F_0 g_j)(t, x', x_n) := e^{-x_n L^{1/2}} g_j(t, x')$ for $t \in \mathbb{R}$, $x' \in \mathbb{R}^n, x_n > 0$ and $j = 1, \ldots, 4$. If we now write

$$\mathscr{F}_t^{-1} \operatorname{diag}\left(\Lambda'_{(I(i)-i+5)/2}, \Lambda'_{(I(i)-i+4)/2}, \Lambda'_{(I(i)-i+3)/2}, \Lambda'_{-(I(i)-i+2)/2}\right) E_{\mu+\mathrm{i}\cdot}\hat{g} = \begin{pmatrix} LF_0 L g_1 \\ L^{1/2} F_0 L g_2 \\ L F_0 g_3 \\ L^{1/2} F_0 g_4 \end{pmatrix}$$

for $\lambda = \mu + \mathrm{i}\tau$, we see that the norm in $\mathcal{E}_+$ of these functions is bounded by $C \sum_{j=1}^4 \|g_j\|_{\mathcal{G}_j}$. For this estimate, we use the continuity assertions of Lemma 3.5 in [DHP07] for the operator F_0 as well as the fact that

$$Lg_2, g_4 \in (L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1})), D(L))_{\frac{1}{2}-\frac{1}{2p},p} = (L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1})), D(L^{1/2}))_{1-\frac{1}{p},p},$$

see Lemma 3.1 in [MS12]. Here, $(\cdot,\cdot)_{\theta,p}$ denotes the real interpolation functor with parameter $\theta \in (0,1)$. From (3.32) in Theorem 3.11 and the Fourier multiplier theorem (Theorem 3.4 in [Wei01]), we obtain the estimate

$$\|\partial_t^2 u\|_{\mathcal{E}_+(\mathbb{R})\times\mathcal{E}_-(\mathbb{R})} + \|\nabla^4 u\|_{\mathcal{E}_+(\mathbb{R})\times\mathcal{E}_-(\mathbb{R})} \le C \sum_{j=1}^{4} \|g_j\|_{\mathcal{G}_j}.$$

As in the proof of Theorem 4.6 in [DS15], we conclude that u vanishes on $(-\infty, 0)$ and the assertion follows. □

Eventually, we will adapt Theorem 3.18 to the case of a bounded domain together with *clamped boundary conditions*: let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\Gamma_1 := \partial\Omega$ and let $\Omega_2 \subset \Omega$ be open such that $\overline{\Omega_2} \subset \Omega$ holds true. We set $\Gamma := \partial\Omega_2$ and $\Omega_1 := \Omega \setminus \overline{\Omega_2}$. Then, Γ is the common interface between Ω_1 and Ω_2 and $\partial\Omega_1 = \partial\Omega \cup \Gamma$. All domains are assumed to be of class C^4. We refer to Figure 4.1 in Chapter 4 below.

We consider (3.48), (3.49) and add clamped boundary conditions

$$u_1 = \partial_\nu u_1 = 0 \text{ on } \Gamma_1.$$

Similar to the definition of $\mathbb{E}_0^2$ and $\mathbb{F}_0^2$ we introduce the spaces

$$\mathbb{E}_0^{(i)} := H^2_{p,0}(\Omega_i) \times L_p(\Omega_i),$$
$$\mathbb{F}_0^{(i)} := \big(H^4_p(\Omega_i) \cap H^2_{p,0}(\Omega_i)\big) \times H^2_{p,0}(\Omega_i)$$

for $i = 1, 2$.
Then, the following result allows us formulate Theorem 3.18 for the bounded domain case:

3.19 Theorem (cf. [DS15], Theorem 5.1)**.** *There exists a $\lambda_0 > 0$ such that the operator $\lambda_0 + A_{p,0,\Omega}$, given by*

$$A_{p,0,\Omega} \colon \mathbb{E}_0^{(1)} \times \mathbb{E}_0^{(2)} \supset D(A_{p,0,\Omega}) \to \mathbb{E}_0^{(1)} \times \mathbb{E}_0^{(2)},\, U \mapsto A(D)U$$

with domain

$$D(A_{p,0,\Omega}) = \Big\{Y = (y_1, y_2, y_3, y_4)^\top \in \mathbb{F}_0^{(1)} \times \mathbb{F}_0^{(2)} : \gamma_0\partial_\nu^2(y_1 - y_3) = \gamma_0\partial_\nu^3(y_1 - y_3) = 0\Big\}$$

is $\mathcal{R}$-sectorial of angle $\vartheta = \max\{\vartheta_1, \vartheta_2\}$ in $\mathbb{E}_0^{(1)} \times \mathbb{E}_0^{(2)}$. Moreover, $-A_{p,0,\Omega}$ generates an exponentially stable, analytic C_0-semigroup on $\mathbb{E}_0^{(1)} \times \mathbb{E}_0^{(2)}$ with maximal L_q-regularity on $(0,\infty)$ for every $q \in (1,\infty)$.

Sketch of proof. Using the notation from Section 2.2, we introduce the spaces

$$X_p^\pm := H^2_{p,0}(\mathbb{R}^n_\pm) \times L_p(\mathbb{R}^n_\pm),$$
$$X_p := \{(u_1^-, u_2^-, u_1^+, u_2^+) \in X_p^- \times X_p^+ : \chi_{\mathbb{R}^n_-} u_1^- + \chi_{\mathbb{R}^n_+} u_1^+ \in H^2(\mathbb{R}^n)\} = X_p^- \times X_p^+,$$
$$Y_p^\pm := \big(H^4_p(\mathbb{R}^n_\pm) \cap H^2_{p,0}(\mathbb{R}^n_\pm)\big) \times H^2_p(\mathbb{R}^n_\pm)$$

as well as the operator matrix

$$\mathcal{A} = \begin{pmatrix} A_1(D) & 0 \\ 0 & A_2(D) \end{pmatrix}.$$

The additional transmission conditions of order 2 and 3 are given by the transmission operators $\mathcal{B}_i^\pm = \begin{pmatrix} \partial_n^{1+i} & 0 \end{pmatrix}$ for $i = 1, 2$. Finally, we set

$$\mathcal{B}_\gamma := \begin{pmatrix} \gamma\mathcal{B}_1^- & \gamma\mathcal{B}_1^+ \\ \gamma\mathcal{B}_2^- & \gamma\mathcal{B}_2^+ \end{pmatrix}$$

and define the operator

$$\mathcal{A}_{\mathcal{B}_\gamma} \colon X_p \supset D(\mathcal{A}_{\mathcal{B}_\gamma}) \to X_p,\, u \mapsto \mathcal{A}u$$

with domain $D(\mathcal{A}_{\mathcal{B}_\gamma}) = \{u \in Y_p : \mathcal{B}_\gamma u = 0\}$.
From Theorem 3.16, we conclude that for $\lambda_1 > 0$ the operator $\lambda_1 + A_{\mathcal{B}_\gamma}$ is $\mathcal{R}$-sectorial of angle ϑ in X_p.
As we are now in the situation of Section 2.2, we may apply Proposition 2.18 in order to show $\mathcal{R}$-sectoriality for small perturbations of $\lambda_1 + \mathcal{A}_{\mathcal{B}_\gamma}$. Note that in Section 2.2, we originally required $X_p^{\pm}$ to be product spaces of Sobolev spaces. However, as one can see in the proofs, all the results in Section 2.2 are still valid if we use closed subspaces of Sobolev spaces.
Hence, the assertion follows from Proposition 2.18, the remarks in the end of Section 2.2 on localization and solving transmission-boundary value problems together with Theorem 4.4 in [DS15] which states that the structurally damped plate equation with clamped boundary conditions is $\mathcal{R}$-sectorial as well. The fact that the associated semigroup is exponentially stable is obtained in the same way as the corresponding assertion in Theorem 5.1 in [DS15]. That is, we need to show that the spectrum of $-A_{p,0,\Omega}$ is contained in the open right half-plane of the complex plane. We refer to Corollary 4.6 and Proposition 4.8 in the next chapter, where we show a similar result for another transmission problem and which can be adapted easily. □

Chapter 4

Transmission problems for coupled systems of damped-undamped plate equations in L_2

Having studied the properties of a certain parabolic-parabolic transmission problem in the previous chapter, we now turn our interest towards transmission problems of non-parabolic-parabolic type. Here, we are not able to work in a L_p space setting as we did in Chapter 3. It is known that so-called quasi-hyperbolic equations of order larger than 1 may not be well-posed in $L_p(\mathbb{R}^n)$ for $n > 1$ (see e.g. [DH18]). In fact, the standard plate equation

$$\partial_t^2 u + \Delta^2 u = 0$$

is not well-posed in the underlying space $W_p^2(\mathbb{R}^n) \times L_p(\mathbb{R}^n)$ for $p \neq 2$, even if $n = 1$, see Example 3.12 in [DH18].
Therefore, in contrast to parabolic-parabolic systems, maximal L_p-regularity cannot be expected when dealing with a transmission problem that contains a (quasi-)hyperbolic part. Hence, we will work in an L_2 Hilbert space setting.

The main question when studying non-parabolic-parabolic transmission problems is the question on how much influence the parabolic and the non-parabolic part have on the system. While parabolic equations usually provide a damping effect and possess a smoothing character, hyperbolic or other non-parabolic equations may have wave-like solutions which often tend to preserve energy as well as the initial regularity. In applications, for example in engineering of bridges or cars, it is desirable that the *parabolic properties* carry through in order to counteract vibrations which may endanger the stability of the construction.
But also from a pure mathematical point of view, these problems lead to important questions about properties of solutions in dependence of the transmission-, boundary- or boundary-control conditions.

Therefore, plate equations with different types of dissipation, e.g. Kelvin-Voigt damping or damping on the boundary, and transmission problems of such plates have attracted great attention in the theory of partial differential equations. But of course, similar problems for plate equations or in thermo-elasticity have been studied.
In the beginning of the previous chapter, we have already mentioned results on structural and thermoelastic damping for plate equations. However, in Hilbert space theory, a lot more problems concerning partial damping of different structures and the accompanied decay of energy have been studied and different results have been obtained. For example, not all the problems lead to an exponential decay of energy but only to polynomial decay (e.g. [MnRR17]) or to

logarithmic decay (e.g. [Has17] or [Has16]).

In this chapter, we will deal with a transmission problem coupling the standard plate equation

$$\partial_t^2 u + \Delta^2 u = 0$$

with the structurally damped plate equation

$$\partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u = 0$$

with some damping coefficient $\rho > 0$. While the standard plate equation

$$\partial_t^2 u + \Delta^2 u = (\partial_t^2 + \Delta^2)u = (\partial_t + \mathrm{i}\Delta)(\partial_t - \mathrm{i}\Delta)u = 0$$

is energy conservative and lacks a smoothing property in the sense of holomorphic semigroups, the structurally damped plate equation is exponentially stable in bounded domains and the associated operator of the first-order in time system is the generator of an analytic semigroup. We will show that the transmission problem is exponentially stable, i.e. the damping of the structurally damped plate equation is strong enough in order to stabilize the whole system. We will need only few geometric conditions, in particular, we do not need a geometric condition on the transmission interface.
In contrast, we show that the whole system is not the generator of an analytic semigroup. For this result, we will use the general necessity condition for analyticity obtained in Chapter 2.

4.1. A transmission problem for plate equations with discontinuous structural damping

Let $n \le 4$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\Gamma_1 := \partial\Omega$ and let $\Omega_2 \subset \Omega$ be open such that $\overline{\Omega_2} \subset \Omega$ holds true. We set $\Gamma := \partial\Omega_2$ and $\Omega_1 := \Omega \setminus \overline{\Omega_2}$. Then, Γ is the common interface between Ω_1 and Ω_2 and $\partial\Omega_1 = \partial\Omega \cup \Gamma$ (cf. Figures 4.1, 4.2, 4.3). All domains are assumed to be of class C^4. The assumption $n \le 4$ is due to technical reasons. However, the physically most relevant cases $n = 1$ and $n = 2$ are included. Basically, there are three different possible geometric situations, see Figure 4.1, 4.2, 4.3 below:

a) Ω_1 is connected.

b) Ω_1 is not connected but for each connected component C of Ω_1 we have that $\overline{C} \cap \Gamma_1 \neq \emptyset$, in particular $\overline{C} \cap \Gamma_1$ has positive Lebesgue measure.

c) Ω_1 is not connected and there exists a connected component C of Ω_1 such that $\overline{C} \cap \Gamma_1 = \emptyset$.

We will also consider the situation, where Ω_2 (or a connected component of Ω_2, to be precise) has a boundary Γ_2 itself. In this case, we will impose clamped boundary conditions also on Γ_2. For the sake of readability, we postpone this situation to the end of this section.

Let ν denote the outer unit normal on Γ_1. On Γ, ν is the outer normal with respect to Ω_2. As before, ∂_ν is the derivative in normal direction, i.e. $\partial_\nu u(x) = \langle \nu(x), \nabla u(x)\rangle_{\mathbb{R}^n}$ for a differentiable function $u\colon \Omega \to \mathbb{C}$ and some point $x \in \partial\Omega$.

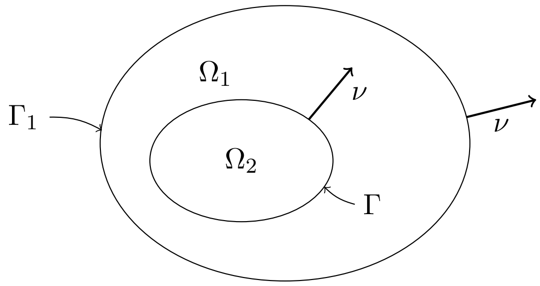


Figure 4.1.: The set $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ where Ω_1 and Ω_2 are connected (geometrical situation *a*)).

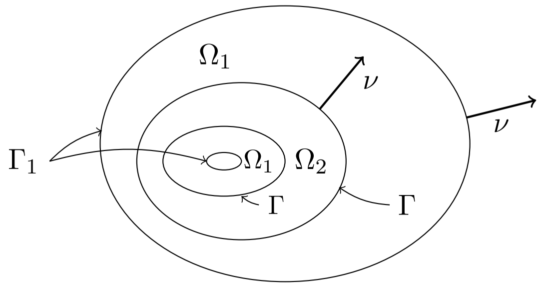


Figure 4.2.: The set $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ where Ω_1 is not connected but each connected component possesses a part of the boundary Γ_1 (geometrical situation *b*)).

We consider a transmission problem for thin plates where the plate in Ω_2 is undamped and the material in Ω_1 is structurally damped. We are looking for solutions $u_i \colon \Omega_i \to \mathbb{C}$ of the system

$$\partial_t^2 u_1 + \Delta^2 u_1 - \rho\Delta\partial_t u_1 = 0 \quad \text{in } (0,\infty)\times\Omega_1, \tag{4.1}$$

$$\partial_t^2 u_2 + \Delta^2 u_2 = 0 \quad \text{in } (0,\infty)\times\Omega_2 \tag{4.2}$$

with *clamped boundary conditions*

$$u_1 = \partial_\nu u_1 = 0 \quad \text{on } \Gamma_1. \tag{4.3}$$

Here, $\rho \in \mathbb{R}_+$ is the damping factor. The transmission conditions on Γ are given by

$$u_1 = u_2, \tag{4.4}$$

$$\partial_\nu u_1 = \partial_\nu u_2, \tag{4.5}$$

$$\Delta u_1 = \Delta u_2, \tag{4.6}$$

$$-\rho\partial_\nu\partial_t u_1 + \partial_\nu\Delta u_1 = \partial_\nu\Delta u_2. \tag{4.7}$$

The problem is completed by the initial conditions

$$u_1(0,\cdot) = u_1^0, \quad \partial_t u_1(0,\cdot) = u_1^1 \quad \text{in } \Omega_1, \tag{4.8}$$

$$u_2(0,\cdot) = u_2^0, \quad \partial_t u_2(0,\cdot) = u_2^1 \quad \text{in } \Omega_2. \tag{4.9}$$

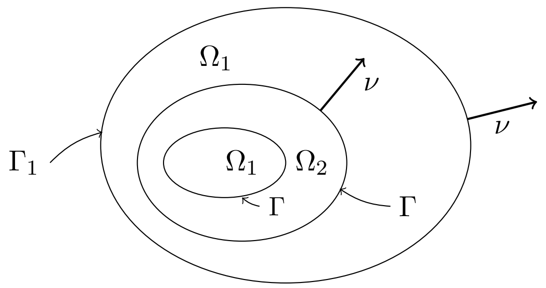


Figure 4.3.: The set $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ where Ω_1 is not connected and at least one component of Ω_1 has no boundary but the transmission interface Γ (geometrical situation c)).

Note that the system (4.1) - (4.2) can be written as as the single equation

$$\partial_t^2 u + \Delta^2 u - \widetilde{\rho}\Delta\partial_t u = 0$$

with a discontinuous coefficient $\widetilde{\rho} \in L_2(\Omega)$ given by

$$\widetilde{\rho}(x) := \begin{cases} \rho, & x \in \Omega_1, \\ 0, & x \in \Omega_2. \end{cases}$$

The energy of the system (4.1)-(4.9) is defined as

$$E(t) := \frac{1}{2}\int_{\Omega_1} |\partial_t u_1|^2 + |\Delta u_1|^2 \, dx + \frac{1}{2}\int_{\Omega_2} |\partial_t u_2|^2 + |\Delta u_2|^2 \, dx. \tag{4.10}$$

If (u_1, u_2) is a solution, integration by parts yields the estimate

$$\frac{d}{dt}E(t) = -\rho\|\nabla\partial_t u_1\|^2_{L_2(\Omega_1)} \leq 0. \tag{4.11}$$

Note that $u_1 = \partial_\nu u_1 = 0$ on Γ_1 implies $\partial_t u_1 = \partial_\nu\partial_t u_1 = 0$ on Γ_1. The estimate shows that the energy of the transmission problem is decreasing in time and the dissipation is caused by the damped part u_1.

Our main goal is to show that the damping in Ω_1 is strong enough to achieve exponential decrease of the energy, i.e. there exist constants $C, \kappa > 0$ such that

$$E(t) \leq CE(0)e^{-\kappa t}$$

holds for all $t \geq 0$.

4.1.1. Well-posedness

As indicated in the beginning of this chapter, we want to work in an L_2 Hilbert space setting. Therefore we will formulate the problem (4.1)-(4.9) as a first order system in time, that is in the form

$$U_t - \mathcal{A}U = 0,$$

in an appropriate complex Hilbert space $\mathcal{H}$. The operator $\mathcal{A}$ acts in form of the matrix

$$A(D) := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Delta^2 & 0 & \rho\Delta & 0 \\ 0 & -\Delta^2 & 0 & 0 \end{pmatrix}.$$

To show well-posedness in the sense of Hadamard, we will use semigroup theory.

In the following, $H^k(\Omega_i) = W_2^k(\Omega_i)$ denotes the standard Sobolev space of order $k \in \mathbb{N}_0$. The space $H_0^k(\Omega_i)$ is the closure of $C_0^\infty(\Omega_i)$ with respect to the Sobolev space norm $\|\cdot\|_{H^k(\Omega_i)}$, which is given by

$$\|u\|_{H^k(\Omega_i)} = \left(\sum_{|\alpha|\le k} \|\partial^\alpha u\|^2_{L_2(\Omega_i)} \right)^{1/2}.$$

Furthermore, $H^s(\Omega_i) = H_2^s(\Omega_i)$ denotes the Bessel potential space of order $s \in \mathbb{R}$ endowed with the norm $\|\cdot\|_{H^s(\Omega_i)}$.

We define

$$X := \left\{ (u_1, u_2) \in H^2(\Omega_1) \times H^2(\Omega_2) : u_1 = \partial_\nu u_1 = 0 \text{ on } \Gamma_1,\ u_1 = u_2,\ \partial_\nu u_1 = \partial_\nu u_2 \text{ on } \Gamma \right\}.$$

4.1 Lemma. *The space X endowed with the norm*

$$\|(u_1, u_2)\|_X := \left(\|\Delta u_1\|^2_{L_2(\Omega_1)} + \|\Delta u_2\|^2_{L_2(\Omega_2)} \right)^{1/2}, \quad (u_1, u_2) \in X$$

is a Hilbert space. Moreover, the norm $\|\cdot\|_X$ and the Sobolev space norm $\|\cdot\|_{H^2(\Omega)}$ are equivalent on X.

Proof. This follows as in [Has17], Proposition 3.1, in which the assertion was shown for the space

$$\left\{ (u_1, u_2) \in H^2(\Omega_1) \times H^2(\Omega_2) : u_1|_{\Gamma_1} = 0, u_1|_\Gamma = u_2|_\Gamma, \partial_\nu u_1|_\Gamma = \partial_\nu u_2|_\Gamma \right\}.$$

It is easily seen that X is a closed subspace thereof and hence the assertion follows. □

Let $\mathcal{H} := X \times H := X \times L_2(\Omega_1) \times L_2(\Omega_2)$. Then we define the operator $\mathcal{A}$ by

$$\mathcal{A}U := \begin{pmatrix} v_1 \\ v_2 \\ -\Delta^2 u_1 + \rho\Delta v_1 \\ -\Delta^2 u_2 \end{pmatrix}$$

for $U = (u_1, u_2, v_1, v_2)^\top$. The operator acts on the domain $D(\mathcal{A})$ defined by

$$D(\mathcal{A}) := \big\{ (u_1, u_2, v_1, v_2) \in \mathcal{H} : (v_1, v_2) \in X, (-\Delta^2 u_1 + \rho\Delta v_1, -\Delta^2 u_2) \in H, \\ \text{(4.6) and (4.7) are weakly satisfied} \big\},$$

where *weakly satisfied* means that

$$\begin{aligned} &\langle -\Delta^2 u_1 + \rho\Delta v_1, \phi_1 \rangle_{L_2(\Omega_1)} - \langle \Delta^2 u_2, \phi_2 \rangle_{L_2(\Omega_2)} \\ &= -\langle \Delta u_1, \Delta\phi_1 \rangle_{L_2(\Omega_1)} - \langle \Delta u_2, \Delta\phi_2 \rangle_{L_2(\Omega_2)} - \rho \langle \nabla v_1, \nabla\phi_1 \rangle_{L_2(\Omega_1)} \end{aligned} \tag{4.12}$$

holds for all $\phi = (\phi_1, \phi_2) \in X$.

4.2 Remark. Later, in Lemma 4.5, we will see that the functions in $D(\mathcal{A})$ are sufficiently smooth and the transmission conditions even hold true in the sense of traces.

The following theorem states that the system (4.1)-(4.9) is well-posed:

4.3 Theorem. *The operator $\mathcal{A}$ is the generator of a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$. Therefore, for all $U^0 \in D(\mathcal{A})$ the Cauchy problem*

$$\begin{aligned} U_t(t) - \mathcal{A}U(t) &= 0, \\ U(0) &= U^0 \end{aligned} \tag{4.13}$$

possesses a unique classical solution $U \in C^1([0,\infty), \mathcal{H})$ with $U(t) \in D(\mathcal{A})$ for all $t \geq 0$.

Proof. We want to apply the theorem of Lumer-Phillips (e.g. [Paz83], Theorem 4.3). Therefore, we have to show that $\mathcal{A}$ is dissipative, densely defined and there exists $\lambda_0 > 0$ such that the range $R(\lambda_0 I - \mathcal{A})$ equals $\mathcal{H}$. For this purpose, let $U = (u_1, u_2, v_1, v_2)^\top \in D(\mathcal{A})$. The definition of the domain $D(\mathcal{A})$ and (4.12) yield

$$\begin{aligned} &\operatorname{Re} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} \\ &\quad= \operatorname{Re} \big(\langle \Delta v_1, \Delta u_1\rangle_{L_2(\Omega_1)} + \langle \Delta v_2, \Delta u_2\rangle_{L_2(\Omega_2)} \\ &\qquad\quad + \langle -\Delta^2 u_1 + \rho\Delta v_1, v_1\rangle_{L_2(\Omega_1)} + \langle -\Delta^2 u_2, v_2\rangle_{L_2(\Omega_2)} \big) \\ &\quad= \operatorname{Re} \big(\langle \Delta v_1, \Delta u_1\rangle_{L_2(\Omega_1)} + \langle \Delta v_2, \Delta u_2\rangle_{L_2(\Omega_2)} \\ &\qquad\quad - \langle \Delta u_1, \Delta v_1\rangle_{L_2(\Omega_1)} - \langle \Delta u_2, \Delta v_2\rangle_{L_2(\Omega_2)} - \rho\langle \nabla v_1, \nabla v_1\rangle_{L_2(\Omega_1)} \big) \\ &\quad= -\rho\|\nabla v_1\|^2_{L_2(\Omega_1)} \leq 0. \end{aligned}$$

Hence, $\mathcal{A}$ is dissipative.
In the second step we show that $R(I - \mathcal{A}) = \mathcal{H}$. For this purpose, let $F = (f_1, f_2, g_1, g_2)^\top \in \mathcal{H}$. We have to find $U = (u_1, u_2, v_1, v_2)^\top \in D(\mathcal{A})$ satisfying

$$\begin{aligned} u_1 - v_1 &= f_1, \\ u_2 - v_2 &= f_2, \\ v_1 + \Delta^2 u_1 - \rho\Delta v_1 &= g_1, \\ v_2 + \Delta^2 u_2 &= g_2. \end{aligned}$$

Plugging in $v_i = u_i - f_i$ for $i = 1, 2$ in the third and fourth equation yields

$$\begin{aligned} u_1 + \Delta^2 u_1 - \rho\Delta u_1 &= g_1 + f_1 - \rho\Delta f_1, \\ u_2 + \Delta^2 u_2 &= g_2 + f_2 \end{aligned} \tag{4.14}$$

as equalities in $L_2(\Omega_1)$ and $L_2(\Omega_2)$, respectively. We define $B\colon X \times X \to \mathbb{C}$ by

$$\begin{aligned} B(u, \phi) = {}& \langle u_1, \phi_1\rangle_{L_2(\Omega_1)} + \langle \Delta u_1, \Delta\phi_1\rangle_{L_2(\Omega_2)} + \rho\langle \nabla u_1, \nabla\phi_1\rangle_{L_2(\Omega_1)} \\ &+ \langle u_2, \phi_2\rangle_{L_2(\Omega_2)} + \langle \Delta u_2, \Delta\phi_2\rangle_{L_2(\Omega_2)} \end{aligned}$$

for $u = (u_1, u_2), \phi = (\phi_1, \phi_2) \in X$. Obviously, B is continuous and sesquilinear. Since

$$\operatorname{Re} B(u, u) \geq \|(u_1, u_2)\|^2_X \qquad (u \in X),$$

B is coercive on X. Obviously, the mapping $\Lambda\colon X \to \mathbb{C}$ defined by

$$\Lambda(\phi) := \langle g_1 + f_1, \phi_1\rangle_{L_2(\Omega_1)} + \rho\langle \nabla f_1, \nabla\phi_1\rangle_{L_2(\Omega_1)} + \langle g_2 + f_2, \phi_2\rangle_{L_2(\Omega_2)}$$

for $\phi = (\phi_1, \phi_2) \in X$ is linear and continuous. By the theorem of Lax-Milgram there exists a unique $u = (u_1, u_2) \in X$ such that $B(u, \phi) = \Lambda(\phi)$ holds for all $\phi \in X$. In particular, for $(\phi_1, \phi_2) \in C_0^\infty(\Omega_1) \times C_0^\infty(\Omega_2) \subset X \times X$ we obtain

$$\begin{aligned} u_1 + \Delta^2 u_1 - \rho\Delta u_1 &= g_1 + f_1 - \rho\Delta f_1 \qquad &&\text{in } \mathscr{D}'(\Omega_1),\\ u_2 + \Delta^2 u_2 &= g_2 + f_2 &&\text{in } \mathscr{D}'(\Omega_2). \end{aligned}$$

Since $g_1, f_1, \Delta f_1 \in L_2(\Omega_1)$ and $g_2, f_2 \in L_2(\Omega_2)$, we obtain (4.14). We define

$$U := (u_1, u_2, v_1, v_2) := (u_1, u_2, u_1 - f_1, u_2 - f_2) \in X \times X.$$

Multiplying (4.14) with $\phi = (\phi_1, \phi_2) \in X$ and summing up leads to

$$0 = \langle u_1 + \Delta^2 u_1 - \rho\Delta v_1 - g_1 - f_1, \phi_1\rangle_{L_2(\Omega_1)} + \langle u_2 + \Delta^2 u_2 - g_2 - f_2, \phi_2\rangle_{L_2(\Omega_2)}$$

and hence, we have

$$\begin{aligned} &\langle -\Delta^2 u_1 + \rho\Delta v_1, \phi_1\rangle_{L_2(\Omega_2)} - \langle \Delta^2 u_2, \phi_2\rangle_{L_2(\Omega_2)}\\ &= B(u,\phi) - \Lambda(\phi) - \langle \Delta u_1, \Delta\phi_1\rangle_{L_2(\Omega_1)} - \langle \Delta u_2, \Delta\phi_2\rangle_{L_2(\Omega_2)} - \rho\langle \nabla v_1, \nabla\phi_1\rangle_{L_2(\Omega_1)}. \end{aligned}$$

Since $B(u, \phi) - \Lambda(\phi) = 0$, equation (4.12) follows and the transmission conditions are weakly satisfied, due to the arbitrariness of $\phi \in X$. Altogether, we have shown that $U \in D(\mathcal{A})$ is the unique solution of $(I - \mathcal{A})U = F$.
It remains to show that $\mathcal{A}$ is densely defined, which follows from $R(I - \mathcal{A}) = \mathcal{H}$, the reflexivity of $\mathcal{H}$ and [Paz83], Theorem 4.6. □

4.4 Remark. The operator $\mathcal{A}$ is continuously invertible, i.e. $0 \in \rho(\mathcal{A})$. This can be seen as in the proof of the surjectivity of $I - \mathcal{A}$ in Theorem 4.3. In this case, we have to solve the problem

$$\begin{aligned} \Delta^2 u_1 &= g_1 - \rho\Delta f_1,\\ \Delta^2 u_2 &= g_2, \end{aligned} \tag{4.15}$$

wherefore we now set $B(u, \phi) = \langle u, \phi\rangle_X$ and

$$\Lambda(\phi) := \langle g_1, \phi_1\rangle_{L_2(\Omega_1)} + \rho\langle \nabla f_1, \nabla\phi_1\rangle_{L_2(\Omega_1)} + \langle g_2, \phi_2\rangle_{L_2(\Omega_2)}$$

for $u = (u_1, u_2), \phi = (\phi_1, \phi_2) \in X$. Due to the Riesz Representation Theorem, there exists a unique solution $u = (u_1, u_2) \in X$ satisfying $B(u, \phi) = \Lambda(\phi)$ for all $\phi \in X$. As $g_1 - \rho\Delta f_1 \in L_2(\Omega_1)$ and $g_2 \in L_2(\Omega_2)$, we obtain (4.15) using test functions $\phi \in C_0^\infty(\Omega_1) \times C_0^\infty(\Omega_2)$. Multiplying this equation by $\phi \in X$, we get

$$\begin{aligned} &\langle -\Delta^2 u_1 + \rho\Delta v_1, \phi_1\rangle_{L_2(\Omega_1)} - \langle \Delta^2 u_2, \phi_2\rangle_{L_2(\Omega_2)}\\ &= \langle -g_1 + \rho\Delta f_1 - \rho\Delta f_1, \phi_1\rangle_{L_2(\Omega_1)} - \langle g_2, \phi_2\rangle_{L_2(\Omega_2)}\\ &= \langle -g_1, \phi_1\rangle_{L_2(\Omega_1)} - \langle g_2, \phi_2\rangle_{L_2(\Omega_2)}\\ &= B(u,\phi) - \Lambda(\phi) - \rho\langle \nabla v_1, \nabla\phi_1\rangle_{L_2(\Omega_1)} - \langle \Delta u_1, \Delta\phi_1\rangle_{L_2(\Omega_1)} - \langle \Delta u_2, \Delta\phi_2\rangle_{L_2(\Omega_2)}, \end{aligned}$$

where we used $v_1 = -f_1$. Therefore, the transmission conditions are weakly satisfied and we deduce that $\mathcal{A}\colon D(\mathcal{A}) \to \mathcal{H}$ (precisely $-\mathcal{A}$) is bijective. Since $\mathcal{A}$ is the generator of a C_0-semigroup, $\mathcal{A}$ is closed and the continuity of $\mathcal{A}^{-1}\colon \mathcal{H} \to \mathcal{H}$ follows.

The following lemma states a higher regularity result for elements in the domain $D(\mathcal{A})$. A consequence is that the transmission conditions on the interface Γ can be understood in the sense of traces. Moreover, we obtain the compactness of the resolvent of $\mathcal{A}$.

4.5 Lemma. *It holds that*

$$D(\mathcal{A}) \subset H^4(\Omega_1) \times H^4(\Omega_2) \times X$$

and consequently

$$\begin{aligned} D(\mathcal{A}) = \big\{(u_1, u_2, v_1, v_2) \in \mathcal{H} : (v_1, v_2) \in X, (u_1, u_2) \in H^4(\Omega_1) \times H^4(\Omega_2), \\ \Delta u_1 = \Delta u_2, -\rho\partial_\nu v_1 + \partial_\nu \Delta u_1 = \partial_\nu \Delta u_2 \text{ on } \Gamma\big\}. \end{aligned}$$

Here, the equalities on Γ can be understood as equalities in the trace spaces $H^{3/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, respectively.

Proof. Let $U \in D(\mathcal{A})$ and $\mathcal{A}U =: F = (f_1, f_2, g_1, g_2)^\top \in \mathcal{H}$. It suffices to show $u_i \in H^4(\Omega_i)$ for $i = 1, 2$. Then, the traces up to order three exist in the corresponding Sobolev spaces of fractional order and, using integration by parts, one sees that the transmission conditions are satisfied in the sense of traces. By Remark 4.4, $u = (u_1, u_2) \in X$ is the unique solution of

$$\langle u, \phi\rangle_X = \Lambda(\phi) = \langle g_1, \phi_1\rangle_{L_2(\Omega_1)} + \rho\langle \nabla f_1, \nabla\phi_1\rangle_{L_2(\Omega_1)} + \langle g_2, \phi_2\rangle_{L_2(\Omega_2)} \quad (\phi = (\phi_1, \phi_2) \in X)\,.$$

In order to show the assertion, we consider the transmission problem strongly, i.e. we consider

$$\Delta^2 w_1 = g_1 - \rho\Delta f_1, \tag{4.16}$$

$$\Delta^2 w_2 = g_2 \tag{4.17}$$

in $L_2(\Omega_1) \times L_2(\Omega_2)$ with boundary conditions

$$w_1 = \partial_\nu w_1 = 0 \text{ on } \Gamma_1 \tag{4.18}$$

and transmission conditions

$$w_1 - w_2 = 0, \tag{4.19}$$

$$\partial_\nu w_1 - \partial_\nu w_2 = 0, \tag{4.20}$$

$$\partial_\nu^2 w_1 - \partial_\nu^2 w_2 = 0, \tag{4.21}$$

$$\partial_\nu^3 w_1 - \partial_\nu^3 w_2 = -\rho\partial_\nu f_1 \tag{4.22}$$

on Γ. Note that thanks to the transmission conditions of order zero and one, the transmission conditions of order three and four can be written again as

$$\begin{aligned} \Delta w_1 - \Delta w_2 = 0, \\ \partial_\nu\Delta w_1 - \partial_\nu\Delta w_2 = -\rho\partial_\nu f_1. \end{aligned}$$

Let us define the operator $\mathcal{B}$ in the basis space $L_2(\Omega)$ associated to the strong transmission problem with homogeneous transmission conditions: we set $\mathcal{B}W := \Delta^2 W$ and

$$D(\mathcal{B}) := H^4(\Omega) \cap H_0^2(\Omega).$$

Elliptic regularity (cf. [Tri78], Section 5.7.1, Theorem 1) states that 0 belongs to the resolvent set of $\mathcal{B}$.
We are now able to construct the strong solution of the transmission problem (4.16) - (4.22) with inhomogeneous transmission conditions. By [Tri78], Section 4.7.1, p. 330, the mapping

$$\mathscr{R}h := \left(h|_{\partial\Omega_1}, \partial_\nu h|_{\partial\Omega_1}, \partial_\nu^2 h|_{\partial\Omega_1}, \partial_\nu^3 h|_{\partial\Omega_1}\right)$$

is a retraction from $H^4(\Omega_1)$ onto $\prod_{j=0}^3 H^{4-1/2-j}(\partial\Omega_1)$. Ergo, there exists a function $h \in H^4(\Omega_1)$ such that

$$\mathscr{R}h = (0, 0, 0, -\chi_\Gamma \rho \partial_\nu f_1),$$

where χ_Γ is the characteristic function on Γ, i.e. $\chi_\Gamma(x) = 1$ provided that $x \in \Gamma$ and $\chi_\Gamma(x) = 0$ otherwise. Moreover, h depends continuously on the right-hand side. In particular, using trace theorems we have

$$\|h\|_{H^4(\Omega_1)} \le C\|f_1\|_{H^2(\Omega_1)} \le C\|F\|_{\mathcal{H}}.$$

We define $\widetilde{W} := \mathcal{B}^{-1}G \in H^4(\Omega) \cap H_0^2(\Omega)$, where

$$G = \chi_1(g_1 - \rho\Delta f_1 - \Delta^2 h) + \chi_2 g_2 \in L_2(\Omega).$$

Of course, χ_i denotes the characteristic function on Ω_i for $i = 1, 2$.
We set $w_1 := \widetilde{W}|_{\Omega_1} + h$ and $w_2 := \widetilde{W}|_{\Omega_2})$. Then, $w = (w_1, w_2) \in H^4(\Omega_1) \times H^4(\Omega_2)$ satisfies the strong transmission problem (4.16) - (4.22).
On the other hand, for $\phi = (\phi_1, \phi_2) \in X$ we calculate

$$\begin{aligned}\langle w, \phi\rangle_X - \Lambda(\phi) = {} & \langle \Delta w_1, \Delta\phi_1\rangle_{L_2(\Omega_1)} + \langle \Delta w_2, \Delta\phi_2\rangle_{L_2(\Omega_2)} \\ & - \langle g_1, \phi_1\rangle_{L_2(\Omega_1)} - \rho\langle \nabla f_1, \nabla\phi_1\rangle_{L_2(\Omega_1)} - \langle g_2, \phi_2\rangle_{L_2(\Omega_2)} = 0,\end{aligned}$$

using integration by parts and the fact, that w solves the transmission problem. Therefore, $(u_1, u_2) = (w_1, w_2) \in H^4(\Omega_1) \times H^4(\Omega_2)$ and we have seen that the embedding

$$D(\mathcal{A}) \to H^4(\Omega_1) \times H^4(\Omega_2) \times X$$

is continuous. □

4.6 Corollary. *The inverse operator $\mathcal{A}^{-1}$ is a compact operator in the Hilbert space $\mathcal{H}$.*

Proof. This follows from Remark 4.4, Lemma 4.5, the closedness of X and the Rellich-Kondrachov Theorem A.4. Indeed, it holds that $\mathcal{A}^{-1} \in L(\mathcal{H}, H^4(\Omega_1) \times H^4(\Omega_2) \times X)$ and we have

$$\begin{aligned}\mathcal{H} \xrightarrow{\mathcal{A}^{-1}} D(\mathcal{A}) & \longrightarrow \left[\left(H^4(\Omega_1) \times H^4(\Omega_2)\right) \cap X\right] \times X \\ & \xrightarrow{c} \left[\left(H^2(\Omega_1) \times H^2(\Omega_1)\right) \cap X\right] \times H = X \times H = \mathcal{H},\end{aligned}$$

where $\xrightarrow{c}$ means that the embedding is compact. □

4.1.2. The spectrum of the first-order system

In this section we will show that the imaginary axis is a subset of the resolvent set of $\mathcal{A}$. Together with the Arendt-Batty-Lyubich-Phong Theorem (e.g. [AB88], Stability Theorem) this will yield the strong stability of the semigroup generated by the operator $\mathcal{A}$. Afterwards, we will continue with a more detailed analysis of the resolvent in order to obtain exponential stability.

We begin with the following fact on eigenfunctions of the Laplace operator in bounded domains with Dirichlet and Neumann boundary conditions.

4.7 Lemma. *Let $U \subset \mathbb{R}^n$ be a domain with boundary of class C^2 and let $\lambda \in \mathbb{C}$. If $w \in H_0^2(U)$ is a solution of*

$$\Delta w + \lambda w = 0 \quad \text{in } U,$$

then $w = 0$ already.

Proof. Let $w \in H_0^2(U)$ be a solution of $\Delta w + \lambda w = 0$. Since $w = \partial_\nu w = 0$ on ∂U, by Lemma A.5 we can extend w by zero to a function $W \in H^2(\mathbb{R}^n)$ satisfying $\Delta W + \lambda W = 0$. Applying Fourier transform yields $(-|\xi|^2 + \lambda)\widehat{W}(\xi) = 0$ for almost all $\xi \in \mathbb{R}^n$. Hence, $\widehat{W} = 0$ which in turn implies $W = 0$ and therefore $w = 0$. □

4.8 Proposition. *The imaginary axis is a subset of the resolvent set of $\mathcal{A}$, i.e. $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. We already know that $0 \in \rho(\mathcal{A})$. Moreover, since $\mathcal{A}^{-1}$ is compact it is sufficient to show that there is no point spectrum on the imaginary line. Let $\lambda \in \mathbb{R} \setminus \{0\}$ such that there exists $U = (u_1, u_2, v_1, v_2)^\top \in D(\mathcal{A})$ satisfying $(-\mathrm{i}\lambda + \mathcal{A})U = 0$, i.e.

$$\begin{aligned} -\mathrm{i}\lambda u_1 + v_1 &= 0, \\ -\mathrm{i}\lambda u_2 + v_2 &= 0, \\ -\mathrm{i}\lambda v_1 - \Delta^2 u_1 + \rho \Delta v_1 &= 0, \\ -\mathrm{i}\lambda v_2 - \Delta^2 u_2 &= 0. \end{aligned}$$

Hence, (u_1, u_2) satisfies the system

$$-\Delta^2 u_1 + \mathrm{i}\lambda\rho\Delta u_1 + \lambda^2 u_1 = 0 \quad \text{in } \Omega_1, \tag{4.23}$$

$$-\Delta^2 u_2 + \lambda^2 u_2 = 0 \quad \text{in } \Omega_2 \tag{4.24}$$

with boundary conditions $u_1 = \partial_\nu u_1 = 0$ on Γ_1 and transmission conditions

$$\begin{aligned} u_1 &= u_2, \\ \partial_\nu u_1 &= \partial_\nu u_2, \\ \Delta u_1 &= \Delta u_2, \\ -\mathrm{i}\lambda\rho\partial_\nu u_1 + \partial_\nu \Delta u_1 &= \partial_\nu \Delta u_2 \end{aligned}$$

on the common interface Γ.
We will show that $(u_1, u_2) = 0$. We multiply (4.23) and (4.24) with $\overline{u_1}$ and $\overline{u_2}$, respectively. Summing up and performing an integration by parts yields

$$-\|\Delta u_1\|^2_{L_2(\Omega_1)} - \mathrm{i}\lambda\rho\|\nabla u_1\|^2_{L_2(\Omega_1)} + \lambda^2\|u_1\|^2_{L_2(\Omega_1)} - \|\Delta u_2\|^2_{L_2(\Omega_2)} + \lambda^2\|u_2\|^2_{L_2(\Omega_2)} = 0.$$

Here we have used the boundary conditions as well as the transmission conditions on Γ. Considering the imaginary part only, we get $\|\nabla u_1\|_{L_2(\Omega_1)} = 0$. Hence, u_1 is constant on each connected component of Ω_1. Let C be a connected component of Ω_1. Then, $u_1|_C$ satisfies

$$-\Delta^2 u_1|_C + \mathrm{i}\lambda\rho\Delta u_1|_C + \lambda^2 u_1|_C = 0,$$

i.e. $\lambda^2 u_1|_C = 0$ on the component C. Since $\lambda \neq 0$ we obtain $u_1|_C = 0$ on each component C and therefore $u_1 = 0$ in Ω_1. Therefore, in this situations u_2 satisfies the boundary value problem

$$\begin{aligned} -\Delta^2 u_2 + \lambda^2 u_2 = 0 \quad & \text{in } \Omega_2, \\ u_2 = \partial_\nu u_2 = \Delta u_2 = \partial_\nu \Delta u_2 = 0 \quad & \text{on } \Gamma = \partial\Omega_2. \end{aligned} \tag{4.25}$$

We define $w := -\Delta u_2 + \lambda u_2$. From the regularity of u_2 and the boundary values of u_2 it follows that $w \in H_0^2(\Omega_2)$ and w clearly satisfies $\Delta w + \lambda w = 0$. Lemma 4.7 implies $w = 0$ which reads $\Delta u_2 - \lambda u_2 = 0$. Applying Lemma 4.7 again, we obtain $u_2 = 0$. This shows $(u_1, u_2) = 0$ and therefore $\mathrm{i}\mathbb{R} \cap \sigma_p(\mathcal{A}) = \emptyset$.

□

As mentioned in the introduction of this subsection we obtain the following corollary.

4.9 Corollary. *The semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ in $\mathcal{H}$ is strongly stable, that is for any initial $U^0 \in \mathcal{H}$ we have*

$$\|\mathcal{T}(t)U^0\|_{\mathcal{H}} \to 0 \quad (t \to \infty).$$

4.1.3. A-priori estimates for the damped plate equation

For the proof of exponential stability of the coupled damped-undamped plate equation, we need some a-priori estimates for the damped part. For this, we will apply the theory of interpolation-extrapolation scales due to Amann (see [Ama95], Chapter V).
Throughout this section, let $U \subset \mathbb{R}^n$ be a bounded domain with boundary ∂U of class C^4 and let the operator A in $H_0^2(U) \times L_2(U)$ be defined by

$$\begin{aligned} A\colon H_0^2(U) \times L_2(U) \supset D(A) := \big(H^4(U) \cap H_0^2(U)\big) \times H_0^2(U) \to H_0^2(U) \times L_2(U), \\ A := \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta \end{pmatrix}. \end{aligned} \tag{4.26}$$

It was shown in [CT89] that A generates an analytic C_0-semigroup of contractions in $H_0^2(U) \times L_2(U)$. To extrapolate this result to spaces of negative regularity, we need to determine the adjoint operator A' considered in the dual spaces. In the following, $\langle \cdot, \cdot \rangle_{E' \times E}$ denotes the dual pairing in a Banach space E.
We begin with a small observation on the biharmonic operator.

4.10 Remark. Under the above assumptions on U, the operator $\Delta^2\colon H_0^2(U) \to H^{-2}(U)$ is an isomorphism. In fact, we have the coercive estimate

$$\langle \Delta^2 u, u \rangle_{H^{-2}(U) \times H_0^2(U)} = \|\Delta u\|_{L_2(U)}^2 \geq C\|u\|_{H^2(U)}^2 \quad (u \in H_0^2(U)).$$

Here the last inequality holds by elliptic regularity and invertibility of the Dirichlet Laplacian $\Delta_D\colon H^2(U) \cap H_0^1(U) \to L_2(U)$. Now an application of the Lax-Milgram theorem yields the invertibility of $\Delta^2\colon H_0^2(U) \to H^{-2}(U)$.

4.11 Lemma. *The adjoint operator A' of A is given by*

$$A' \colon H^{-2}(U) \times L_2(U) \supset D(A') := L_2(U) \times H_0^2(U) \to H^{-2}(U) \times L_2(U),$$
$$A' := \begin{pmatrix} 0 & -\Delta^2 \\ 1 & \rho\Delta \end{pmatrix}.$$

Proof. Let $E := H_0^2(U) \times L_2(U)$ and $\widetilde{D} := L_2(U) \times H_0^2(U) \subset E'$. Then, $E' = H^{-2}(U) \times L_2(U)$ and for all $v = (v_1, v_2) \in \widetilde{D}$ and $u = (u_1, u_2) \in D(A) = \big(H^4(U) \cap H_0^2(U)\big) \times H_0^2(U)$, integration by parts and the definition of distributional derivatives yield

$$\begin{aligned} v(Au) &= \langle v_1, u_2\rangle_{L_2(U)} + \langle v_2, -\Delta^2 u_1\rangle_{L_2(U)} + \langle v_2, \rho\Delta u_2\rangle_{L_2(U)} \\ &= \langle v_1, u_2\rangle_{L_2(U)} + \langle -\Delta v_2, \Delta u_1\rangle_{L_2(U)} + \langle \rho\Delta v_2, u_2\rangle_{L_2(U)} \\ &= \langle -\Delta^2 v_2, u_1\rangle_{H^{-2}(U)\times H_0^2(U)} + \langle v_1, u_2\rangle_{L_2(U)} + \langle \rho\Delta v_2, u_2\rangle_{L_2(U)} \\ &= w_1(u_1) + w_2(u_2), \end{aligned}$$

with $w_1 := -\Delta^2 v_2 \in H^{-2}(U)$ and $w_2 := v_1 + \rho\Delta v_2 \in L_2(U)$. Recall that $\langle\cdot,\cdot\rangle_{Y'\times Y}$ denotes the dual pairing in a Banach space Y. Therefore, we set

$$\widetilde{A} := \begin{pmatrix} 0 & -\Delta^2 \\ 1 & \rho\Delta \end{pmatrix}$$

with $D(\widetilde{A}) := \widetilde{D}$. With this definition, we have $v(Au) = (\widetilde{A}v)(u)$ for all $u \in D(A)$ and all $v \in \widetilde{D}$. Moreover, for all $v \in \widetilde{D}$ the mapping $[u \mapsto v(Au)] \colon D(A) \to \mathbb{C}$ is continuous with respect to $\|\cdot\|_E$. Hence, we have $\widetilde{A} \subset A'$.

Let $u \in D(A)$ and $v \in E'$. Then

$$v(Au) = \langle v_1, u_2\rangle_{H^{-2}(U)\times H_0^2(U)} + \langle v_2, -\Delta^2 u_1\rangle_{L_2(U)} + \langle v_2, \rho\Delta u_2\rangle_{L_2(U)}. \tag{4.27}$$

Now, let $v \in D(A')$. Then, the mapping $[u \mapsto v(Au)] \colon D(A) \to \mathbb{C}$ can be extended to a linear, continuous mapping from E to $\mathbb{C}$. In particular, considering

$$|\langle v_2, \Delta^2 u_1\rangle_{L_2(U)}| = |v(A(u_1, 0))| \le C\|(u_1,0)\|_E = C\|u_1\|_{H^2(U)} \quad (u_1 \in H^4(U) \cap H_0^2(U)),$$

it holds that

$$\varphi \colon H^4(U) \cap H_0^2(U) \to \mathbb{C}, \quad u_1 \mapsto \varphi(u_1) := \langle v_2, \Delta^2 u_1\rangle_{L_2(U)} \tag{4.28}$$

is continuous with respect to $\|\cdot\|_{H^2(U)}$. By Remark 4.10,

$$\Delta^2 \colon H_0^2(U) \to H^{-2}(U) \tag{4.29}$$

is an isomorphism. Therefore, (4.28) and (4.29) imply that

$$\left[\widetilde{u}_1 \mapsto \varphi\left((\Delta^2)^{-1}\widetilde{u}_1\right) = \langle v_2, \widetilde{u}_1\rangle_{L_2(U)}\right] : L_2(U) \to \mathbb{C}$$

is continuous considered as a mapping from $(L_2(U), \|\cdot\|_{H^{-2}(U)})$ to $\mathbb{C}$. Since $L_2(U) \subset H^{-2}(U)$ is dense, there exists a unique continuous extension $\widetilde{\varphi} \in \big(H^{-2}(U)\big)' = H_0^2(U)$ of this mapping. Together with

$$\langle \widetilde{\varphi}, \widetilde{u}_1 \rangle_{H_0^2(U) \times H^{-2}(U)} = \langle v_2, \widetilde{u}_1 \rangle_{H_0^2(U) \times H^{-2}(U)}$$

for $\widetilde{u}_1 \in L_2(U)$, we deduce $v_2 = \widetilde{\varphi} \in H_0^2(U)$.
The fact that $v_2 \in H_0^2(U)$ implies that the last term in (4.27),

$$\Big[u_2 \mapsto \langle v_2, \rho\Delta u_2 \rangle_{L_2(U)} = \langle v_2, \rho\Delta u_2 \rangle_{H_0^2(U) \times H^{-2}(U)}\Big] : H_0^2(U) \to \mathbb{C},$$

is continuous on $L_2(U)$. Since (4.27) needs to be continuous, by setting $u_1 = 0$ it follows that also the first term

$$\Big[u_2 \mapsto \langle v_1, u_2 \rangle_{H^{-2}(U) \times H_0^2(U)}\Big] : H_0^2(U) \to \mathbb{C}$$

can be extended continuously to $L_2(U)$, which means $v_1 \in L_2(U)$.
We have shown that $v \in D(A')$ implies $v_2 \in H_0^2(U)$ and $v_1 \in L_2(U)$, i.e. $v \in \widetilde{D}$. Hence, we obtain $\widetilde{D} = D(A')$ and therefore $\widetilde{A} = A'$. □

Recall that $\Sigma_\phi = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \phi\}$ (see chapter 3).

4.12 Theorem. *The operator A defined in* (4.26) *is the generator of an analytic, exponentially stable C_0-semigroup on $H_0^2(U) \times L_2(U)$ and we have the following a-priori estimates: there exist constants $C_1, C_2 > 0$ such that for any $\lambda \in \rho(A) \supset \overline{\Sigma_{\pi/2}} \setminus \{0\}$, the solution $u = (u_1, v_1) \in D(A)$ of*

$$(\lambda - A)u = F \in H_0^2(U) \times L_2(U)$$

satisfies the estimates

$$\|u\|_{H^4(U)\times H^2(U)} \leq C_1 \|F\|_{H^2(U)\times L_2(U)}, \tag{4.30}$$

$$\|u\|_{H^2(U)\times L_2(U)} \leq C_2 \|F\|_{L_2(U)\times H^{-2}(U)}. \tag{4.31}$$

Consequently, by interpolation we obtain the estimate

$$\|u\|_{H^{2+\theta}(U)\times H^\theta(U)} \leq C \|F\|_{H^\theta(U)\times H^{-2+\theta}(U)}$$

for $\theta \in [0, 2]$. In particular, for $u = (u_1, v_1) \in D(A)$ solving

$$(\lambda - A)u = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

with $f \in L_2(U)$ the estimates

$$\|u_1\|_{H^4(U)} \leq C_1 \|f\|_{L_2(U)}, \tag{4.32}$$

$$\|u_1\|_{H^2(U)} \leq C_2 \|f\|_{H^{-2}(U)} \tag{4.33}$$

and consequently

$$\|u_1\|_{H^{2+\theta}(U)} \leq C' \|f\|_{H^{-2+\theta}(U)} \quad (\theta \in [0, 2])$$

hold true.

Proof. By [CT89], A is the generator of an analytic, strongly continuous semigroup of contractions on $H_0^2(U) \times L_2(U)$. Therefore, we only need to show the estimates (4.32) and (4.33). The first one follows for example from the analysis of the corresponding boundary value problem in the half-space, see [DS15], Theorem 3.5 and Corollary 3.6. Nevertheless, since our approach for the second estimate will work for the first one as well, we will state a proof for both estimates. As mentioned above, we will use the theory of interpolation-extrapolation scales.

We define the following spaces: let $A^\sharp := A'$ be the adjoint operator of A and set

$$\begin{aligned}
E_0 &:= H_0^2(U) \times L_2(U), \\
E_1 &:= D(A) = \big(H^4(U) \cap H_0^2(U)\big) \times H_0^2(U), \\
E_0^\sharp &:= E_0' = H^{-2}(U) \times L_2(U), \\
E_1^\sharp &:= D(A^\sharp).
\end{aligned}$$

Obviously, E_0 is reflexive and E_1 is dense in E_0. Since A is the generator of an analytic C_0-semigroup on E_0 with domain E_1, in symbols $A \in \mathcal{H}(E_1, E_0)$, by [Ama95], p. 13, Proposition 1.2.3, the same holds true for $A^\sharp$ on $E_0^\sharp$ with domain $E_1^\sharp$, i.e. $A^\sharp \in \mathcal{H}(E_1^\sharp, E_0^\sharp)$.
Hence, we can define the interpolation-extrapolation scales $\{(A_\alpha, E_\alpha) : \alpha \in \mathbb{R}\}$ and its dual scale $\{(A_\alpha^\sharp, E_\alpha^\sharp) : \alpha \in \mathbb{R}\}$. Then, Theorem 1.5.12 on page 282 of [Ama95] states that E_α is reflexive and we have

$$(E_\alpha)' = E_{-\alpha}^\sharp \text{ and } (A_\alpha)' = A_{-\alpha}^\sharp$$

for all $\alpha \in \mathbb{R}$. Moreover, by [Ama88], Theorem 6.1 and [Ama95], Theorem 2.1.3 it holds that A_α and $A_\alpha^\sharp$ are generators of analytic C_0-semigroups in E_α with domain $E_{\alpha+1}$ and $E_\alpha^\sharp$ with domain $E_{\alpha+1}^\sharp$ for all $\alpha \in \mathbb{R}$, respectively. Again, we write $A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha)$ and $A_\alpha^\sharp \in \mathcal{H}(E_{\alpha+1}^\sharp, E_\alpha^\sharp)$.
In particular, A_{-1} is the generator of an analytic C_0-semigroup on E_{-1} with domain E_0. By [Ama95], Theorem 2.1.3 on page 289, $\lambda - A_{-1}$ is an isomorphism from E_0 to E_{-1} and we have

$$\|(\mu - A_{-1})^{-1}\|_{L(E_{-1},E_0)} \leq C\|(\mu - A)^{-1}\|_{L(E_0,E_1)} \leq C'$$

for all $\mu \in \rho(A)$ with a constant C' independent of μ. By Lemma 4.11, the space E_{-1} equals

$$E_{-1} = (E_{-1})'' = \left(E_1^\sharp\right)' = \big(D(A')\big)' = \big(L_2(U) \times H_0^2(U)\big)' = L_2(U) \times H^{-2}(U).$$

Finally, we obtain the estimate

$$\|u_1\|_{H^2(U)} \leq \|u\|_{E_0} \leq C \left\| \begin{pmatrix} 0 \\ f \end{pmatrix} \right\|_{E_{-1}} = C\|f\|_{H^{-2}(U)}.$$

The estimate (4.32) is obtained in the same way, using A instead of the extrapolated operator A_{-1}. Indeed, we have

$$\|u_1\|_{H^4(U)} \leq \|u\|_{E_1} \leq C \left\| \begin{pmatrix} 0 \\ f \end{pmatrix} \right\|_{E_0} = C\|f\|_{L_2(U)}.$$

□

4.13 Corollary. *There exists a constant $C > 0$ such that for any $\lambda \in \rho(A)$, the solution $u = (u_1, v_1) \in D(A)$ of $(\lambda - A)u = F$ with $F \in H_0^2(U) \times L_2(U)$ satisfies the estimate*

$$\|D^\alpha u\|_{H^2(U)\times L_2(U)} \le C\|F\|_{H^2(U)\times L_2(U)} \quad (\alpha \in \mathbb{N}_0^n, |\alpha| = 2)$$

holds true.

Proof. This follows immediately from the estimate (4.30) and the fact that

$$D^\alpha\colon H^4(U) \times H^2(U) \to H^2(U) \times L_2(U)$$

is linear and continuous. □

4.1.4. Exponential stability

In what follows, we continue the analysis of the coupled system (4.1) - (4.2). We will estimate the resolvent $(\mathcal{A} - \mathrm{i}\lambda)^{-1}$ on the imaginary axis for large $|\lambda|$. We aim for uniform boundedness of the resolvent for $\lambda \in \mathbb{R}, |\lambda| > \lambda_0$ for some $\lambda_0 > 0$. The Theorem of Prüss (see [Prü84], Corollary 4) implies exponential stability of the semigroup in this case.

We start with some useful identities. In order to reduce notation, we will use summation convention, i.e. we write $x_k y_k := \sum_{k=1}^N x_k y_k$ for $x, y \in \mathbb{C}^N$.

4.14 Proposition (Rellich's identity)**.** *Let $U \subset \mathbb{R}^n$ be an open and bounded set with boundary of class C^4 and let $w \in H^4(U)$. Then the identity*

$$\begin{aligned} 2\,\mathrm{Re} \int_U \Delta^2 w (q\nabla\overline{w})\,dx = &- \int_U \operatorname{div}(q)|\Delta w|^2\,dx + 2\,\mathrm{Re}\int_U \Delta q_k \partial_k w \Delta\overline{w}\,dx \\ &+ 4\,\mathrm{Re}\int_U \nabla q_k \nabla(\partial_k w)\Delta\overline{w}\,dx + \int_{\partial U}(q\nu)|\Delta w|^2\,dS \\ &+ 2\,\mathrm{Re}\int_{\partial U}(q\nabla w)\partial_\nu(\Delta\overline{w}) - \Delta w \partial_\nu(q\nabla\overline{w})\,dS \end{aligned}$$

holds true, where $q \in C^2(U, \mathbb{R}^n)$ is a vector field and ν denotes the outer unit normal on ∂U.

Proof. Define the vector field $V \in C^1(U)$ by

$$V(x) := q(x)|\Delta w(x)|^2 + 2(q(x)\nabla\overline{w}(x))\nabla\Delta w(x) - 2\Delta w(x)\nabla(q(x)\nabla\overline{w}(x)) \quad (x \in U).$$

Then

$$\begin{aligned} \operatorname{div} V &= \operatorname{div}(q|\Delta w|^2) + 2\operatorname{div}\left((q\nabla\overline{w})\nabla\Delta w\right) - 2\operatorname{div}(\Delta w\nabla(q\nabla\overline{w})) \\ &= \operatorname{div}(q)|\Delta w|^2 + q_i\left(\partial_i\Delta w\Delta\overline{w} + \Delta w\partial_i\Delta\overline{w}\right) \\ &\quad + 2\left(\Delta^2 w(q\nabla\overline{w}) + \nabla\Delta w\nabla(q\nabla\overline{w})\right) \\ &\quad - 2\left(\Delta(q\nabla\overline{w})\Delta w - \nabla\Delta w\nabla(q\nabla\overline{w})\right) \\ &= \operatorname{div}(q)|\Delta w|^2 + q_i\left((\partial_i\Delta w\Delta\overline{w}) + \Delta w\partial_i\Delta\overline{w}\right) \\ &\quad + 2\Delta^2 w(q\nabla\overline{w}) - 2\left(\Delta q_i\partial_i\overline{w} + 2\nabla q_i\nabla\partial_i\overline{w} + q_i\partial_i\Delta\overline{w}\right)\Delta w. \end{aligned}$$

Taking real parts, integrating over U and using the Divergence Theorem, the assertion follows. □

4.15 Remark. The identity above can be found in [Man13], p. 238 in a more general way. To be precise, it is claimed that the identity

$$\begin{aligned} 2\operatorname{Re}\int_U \Delta^2 w(q\nabla\overline{w})\,dx = &-\int_U \operatorname{div}(q)|\Delta w|^2\,dx + 2\int_U \Delta q_k\partial_k w\Delta\overline{w}\,dx \\ &+ 4\int_U \nabla q_k\nabla(\partial_k w)\Delta\overline{w}\,dx + \int_{\partial U}(q\nu)|\Delta w|^2\,dS \\ &+ 2\int_{\partial U}(q\nabla w)\partial_\nu(\Delta\overline{w}) - \Delta w\partial_\nu(q\nabla\overline{w})\,dS \end{aligned}$$

holds. However, there is no proof in this article and we are not able to show the latter version. Despite, our version of Rellich's identity is good enough for our purposes. We also refer to [Mit93], Proposition 2.2 for a version for real valued functions.

We introduce the following vector field: for fixed $x_0 \in \mathbb{R}^n$ we define $q\colon \Omega \to \mathbb{R}^n$ by

$$q(x) := x - x_0. \tag{4.34}$$

Recall that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ was the whole domain in which the transmission problem (4.1)-(4.9) is considered.
As an immediate consequence we obtain the following identity.

4.16 Corollary. *Let $U \subset \mathbb{R}^n$ be open and bounded with boundary of class C^4 and let $w \in H^4(U)$. Define $q\colon \mathbb{R}^n \to \mathbb{R}^n$ as in (4.34). Then*

$$\begin{aligned} 2\operatorname{Re}\int_U \Delta^2 w(q\nabla\overline{w})\,dx = &(4-n)\|\Delta w\|^2_{L_2(U)} + \int_{\partial U}(q\nu)|\Delta w|^2\,dS \\ &+ 2\operatorname{Re}\int_{\partial U}(q\nabla w)\partial_\nu\Delta\overline{w} - \Delta w\partial_\nu(q\nabla\overline{w})\,dS. \end{aligned}$$

The following lemma provides an identity which will be useful for the resolvent estimate of both the damped and the undamped part of our transmission problem.

4.17 Lemma. *Let $U \subset \mathbb{R}^n$ be a bounded open set with boundary of class C^4. Further, let $\lambda \in \mathbb{R}\setminus\{0\}$, $z \in L_2(U)$ and $w \in H^4(U)$ be a solution of*

$$-\Delta^2 w + \lambda^2 w = z.$$

Then the identity

$$\begin{aligned} n\lambda^2\|w\|^2_{L_2(U)} + (4-n)\|\Delta w\|^2_{L_2(U)} + \int_{\partial U}(q\nu)|\Delta w|^2\,dS \\ = -2\operatorname{Re}\int_U z(q\nabla\overline{w})\,dx + \lambda^2\int_{\partial U}(q\nu)|w|^2\,dS \\ - 2\operatorname{Re}\int_{\partial U}(q\nabla w)\partial_\nu\Delta\overline{w} - \Delta w\partial_\nu(q\nabla\overline{w})\,dS \end{aligned}$$

holds true, where q is defined as in (4.34) for some $x_0 \in \mathbb{R}^n$ and again, ν denotes the outer unit normal on ∂U.

Proof. Multiplying the equation with $q\nabla\overline{w}$, integrating over U and taking real parts yields

$$-\operatorname{Re}\int_U \Delta^2 w(q\nabla\overline{w})\,dx + \lambda^2 \operatorname{Re}\int_U w(q\nabla\overline{w})\,dx = \operatorname{Re}\int_U z(q\nabla\overline{w})\,dx.$$

Using integration by parts and $\operatorname{div}(q) = n$, we obtain

$$\begin{aligned}\operatorname{Re}\int_U w(q\nabla\overline{w})\,dx &= \operatorname{Re}\int_U w\left(\operatorname{div}(q\overline{w}) - n\overline{w}\right)\,dx \\ &= -n\|w\|^2_{L_2(U)} + \operatorname{Re}\int_U w\operatorname{div}(q\overline{w})\,dx \\ &= -n\|w\|^2_{L_2(U)} + \int_{\partial U}(q\nu)|w|^2\,dS - \operatorname{Re}\int_U w(q\nabla\overline{w})\,dx,\end{aligned}$$

and therefore

$$\operatorname{Re}\int_U w(q\nabla\overline{w})\,dx = -\frac{n}{2}\|w\|^2_{L_2(U)} + \frac{1}{2}\int_{\partial U}(q\nu)|w|^2\,dS.$$

Together with Corollary 4.16, the assertion follows. □

A proof of the next lemma can be found in the proof of Theorem 2.2 in [Man13], p. 239.

4.18 Lemma. *Let $U \subset \mathbb{R}^n$ be open with boundary of class C^3. Furthermore, let $S \subset \partial U$ be a nontrivial part of the boundary. Then, for $w \in H^3(U)$ satisfying $w = \partial_\nu w = 0$ on S, we have*

$$\partial_\nu(q\nabla w) = (q\nu)\Delta w$$

on S, where q is defined as above and ν denotes the outer unit normal on ∂U.

The following a-priori estimate will be crucial for the proof of the exponential stability for the transmission problem (4.1) - (4.9). Here, we will consider a resolvent problem with a specific right hand side $F \in \mathcal{H}$. This resolvent problem will appear in the proof of the exponential stability later.

4.19 Proposition. *Let $w_1 \in H^4(\Omega_1)$ and $g_2 \in L_2(\Omega_2)$. Then, there exists $\lambda_0 > 0$ and a constant $C > 0$ (only depending on n, ρ and λ_0) such that for any solution $U = (u_1, u_2, v_1, v_2)^\top \in X \times X$ with $u_i \in H^4(\Omega_i)$ for $i = 1, 2$ of*

$$(-\mathrm{i}\lambda + \mathcal{A})U = (0,0,0,g_2)^\top =: F, \tag{4.35}$$

satisfying

$$\left.\begin{aligned}\Delta u_1 &= \Delta u_2, \\ -\mathrm{i}\lambda\rho\partial_\nu u_1 + \partial_\nu\Delta u_1 &= \partial_\nu\Delta u_2 + \mathrm{i}\lambda\rho\partial_\nu w_1\end{aligned}\right\}\ \text{on } \Gamma, \tag{4.36}$$

the estimate

$$\|U\|_{\mathcal{H}} \le C\left(\|g_2\|_{L_2(\Omega_2)} + |\lambda|\|\partial_\nu w_1\|_{L_2(\Gamma)}\right) \quad (\lambda \in \mathbb{R}, |\lambda| > \lambda_0)$$

holds true.

Proof. Let $\lambda \in \mathbb{R}$ with $|\lambda| \gg 1$ and $U = (u_1, u_2, v_1, v_2)$ be a solution of (4.35) satisfying the transmission conditions (4.36), i.e. U satisfies

$$-\mathrm{i}\lambda u_1 + v_1 = 0 \quad \text{in } \Omega_1, \tag{4.37}$$
$$-\Delta^2 u_1 + \rho\Delta v_1 - \mathrm{i}\lambda v_1 = 0 \quad \text{in } \Omega_1, \tag{4.38}$$
$$-\mathrm{i}\lambda u_2 + v_2 = 0 \quad \text{in } \Omega_2, \tag{4.39}$$
$$-\Delta^2 u_2 - \mathrm{i}\lambda v_2 = g_2 \text{ in } \Omega_2 \tag{4.40}$$

with boundary conditions $u_1 = \partial_\nu u_1 = 0$ on Γ_1. Hence, (u_1, u_2) is a solution of

$$-\Delta^2 u_1 + \mathrm{i}\lambda\rho\Delta u_1 + \lambda^2 u_1 = 0, \tag{4.41}$$
$$-\Delta^2 u_2 + \lambda^2 u_2 = g_2 \tag{4.42}$$

in Ω_1 and Ω_2, respectively, satisfying the transmission conditions (4.36).
In order to show the assertion of the theorem, we need to establish an estimate of the form

$$\|U\|_{\mathcal{H}}^2 \leq \varepsilon\|U\|_{\mathcal{H}}^2 + C_\varepsilon\left(\|g_2\|_{L_2(\Omega_2)}^2 + |\lambda|^2\|\partial_\nu w_1\|_{L_2(\Gamma)}^2\right)$$

for some $\varepsilon = \varepsilon(\rho) \in (0,1)$ independent of λ.
With C we denote generic constants which may change from time to time but which do not depend on λ, U or F. By C_ε we denote generic constants only depending on some fixed $\varepsilon > 0$. We also use generic constants $C(\rho)$ to emphasize the dependence on the damping factor ρ. In the same way we emphasize the dependence on ρ for ε and C_ε.
Similar to the proof of the dissipativity of $\mathcal{A}$ in Theorem 4.3, we obtain

$$\mathrm{Re}\langle F, U\rangle_{\mathcal{H}} = \mathrm{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\rho\|\nabla v_1\|_{L_2(\Omega_1)}^2 - \mathrm{Re}\int_\Gamma \mathrm{i}\lambda\rho\overline{v_1}\partial_\nu w_1\, dS,$$

wherefore the trace theorem and Young's inequality yield

$$\begin{aligned}\|\nabla v_1\|_{L_2(\Omega_1)}^2 &\leq C(\rho)\left(\|g_2\|_{L_2(\Omega_2)}\|U\|_{\mathcal{H}} + |\lambda|\|\partial_\nu w_1\|_{L_2(\Gamma)}\|v_1\|_{H^1(\Omega_1)}\right)\\ &\leq \varepsilon(\rho)\left(\|U\|_{\mathcal{H}}^2 + \|v_1\|_{H^1(\Omega_1)}^2\right) + C_{\varepsilon(\rho)}\left(\|g_2\|_{L_2(\Omega_2)}^2 + |\lambda|^2\|\partial_\nu w_1\|_{L_2(\Gamma)}^2\right)\end{aligned}$$

that is

$$\|\nabla u_1\|_{L_2(\Omega_1)}^2 \leq \varepsilon(\rho)|\lambda|^{-2}\left(\|U\|_{\mathcal{H}}^2 + \|v_1\|_{H^1(\Omega_1)}^2\right) + C_{\varepsilon(\rho)}|\lambda|^{-2}\left(\|g_2\|_{L_2(\Omega_2)}^2 + |\lambda|^2\|\partial_\nu w_1\|_{L_2(\Gamma)}^2\right).$$

Furthermore, it holds that

$$\|u_1\|_{H^1(\Omega_1)} \leq C\left(\|\nabla u_1\|_{L_2(\Omega_1)} + \|\nabla u_2\|_{L_2(\Omega_2)}\right)$$

since $z := \chi_{\Omega_1}u_1 + \chi_{\Omega_2}u_2 \in H_0^2(\Omega)$ and therefore $\|z\|_{L_2(\Omega)} \leq C\|\nabla z\|_{L_2(\Omega)}$ due to the Poincaré inequality. Moreover, interpolation theory states

$$\begin{aligned}\|\nabla u_2\|_{L_2(\Omega_2)}^2 &\leq \varepsilon\|u_2\|_{H^2(\Omega_2)}^2 + C_\varepsilon\|u_2\|_{L_2(\Omega_2)}^2\\ &\leq \varepsilon\|U\|_{\mathcal{H}}^2 + C_\varepsilon\|u_2\|_{L_2(\Omega_2)}^2\end{aligned}$$

which leads us to

$$
\begin{aligned}
\|u_1\|^2_{H^1(\Omega_1)} &\le C\|\nabla u_1\|^2_{L_2(\Omega_1)} + \varepsilon\|U\|^2_{\mathcal{H}} + C_\varepsilon\|u_2\|^2_{L_2(\Omega_2)} \\
&\le \varepsilon(\rho)|\lambda|^{-2}\left(\|U\|^2_{\mathcal{H}} + \|v_1\|^2_{H^1(\Omega_1)}\right) \\
&\quad + C_{\varepsilon(\rho)}|\lambda|^{-2}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)}\right) + C_{\varepsilon(\rho)}\|u_2\|^2_{L_2(\Omega_2)} \\
&= \varepsilon(\rho)\left(|\lambda|^{-2}\|U\|^2_{\mathcal{H}} + \|u_1\|^2_{H^1(\Omega_1)}\right) \\
&\quad + C_{\varepsilon(\rho)}|\lambda|^{-2}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)}\right) + C_{\varepsilon(\rho)}\|u_2\|^2_{L_2(\Omega_2)}
\end{aligned}
$$

and hence, choosing $\varepsilon(\rho)$ small enough,

$$
\|u_1\|^2_{H^1(\Omega_1)} \le \varepsilon(\rho)|\lambda|^{-2}\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(|\lambda|^{-2}\|g_2\|^2_{L_2(\Omega_2)} + \|\partial_\nu w_1\|^2_{L_2(\Gamma)} + \|u_2\|^2_{L_2(\Omega_2)}\right), \tag{4.43}
$$

as well as

$$
\|v_1\|^2_{H^1(\Omega_1)} \le \varepsilon(\rho)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)} + \|v_2\|^2_{L_2(\Omega_2)}\right) \tag{4.44}
$$

where we used $v_j = \mathrm{i}\lambda u_j$ for $j = 1, 2$.
We need to estimate $\|\Delta u_1\|^2_{L_2(\Omega_1)}, \|\Delta u_2\|^2_{L_2(\Omega_2)}$ and $\|v_2\|^2_{L_2(\Omega_2)}$. To do so, we multiply (4.41) by $-\overline{u_1}$ and (4.42) with $-\overline{u_2}$. Integration by parts and summing up yields

$$
\begin{aligned}
&\|\Delta u_1\|^2_{L_2(\Omega_1)} + \|\Delta u_2\|^2_{L_2(\Omega_2)} + \mathrm{i}\lambda\rho\|\nabla u_1\|^2_{L_2(\Omega_2)} \\
&\qquad = \lambda^2\left(\|u_1\|^2_{L_2(\Omega_1)} + \|u_2\|^2_{L_2(\Omega_2)}\right) - \langle g_2, u_2\rangle_{L_2(\Omega_2)} + \mathrm{i}\lambda\rho\int_\Gamma \overline{u_1}\partial_\nu w_1\, dS,
\end{aligned}
$$

where we have used the transmission conditions (4.36) and $u_1 = \partial_\nu u_1 = 0$ on Γ_1. By the trace theorem we get, after taking real parts

$$
\begin{aligned}
\|\Delta u_1\|^2_{L_2(\Omega_1)} + \|\Delta u_2\|^2_{L_2(\Omega_2)} &\le |\lambda|^2\left(\|u_1\|^2_{L_2(\Omega_1)} + \|u_2\|^2_{L_2(\Omega_2)}\right) \\
&\quad + \|g_2\|_{L_2(\Omega_2)}\|u_2\|_{L_2(\Omega_2)} + |\lambda|\rho\|u_1\|_{L_2(\Gamma)}\|\partial_\nu w_1\|_{L_2(\Gamma)} \\
&\le C(\rho)\Bigg(|\lambda|^2\left(\|u_1\|^2_{L_2(\Omega_1)} + \|u_2\|^2_{L_2(\Omega_2)}\right) \\
&\quad + \|g_2\|_{L_2(\Omega_2)}\|u_2\|_{L_2(\Omega_2)} + |\lambda|\|u_1\|_{H^1(\Omega_1)}\|\partial_\nu w_1\|_{L_2(\Gamma)}\Bigg).
\end{aligned}
$$

Plugging in (4.43) and (4.44), respectively, we thus obtain

$$
\begin{aligned}
&\|\Delta u_1\|^2_{L_2(\Omega_1)} + \|\Delta u_2\|^2_{L_2(\Omega_2)} \\
&\qquad \le \varepsilon(\rho)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)} + \|v_2\|^2_{L_2(\Omega_2)}\right) \\
&\qquad\quad + |\lambda|^2\|u_2\|^2_{L_2(\Omega_2)} + \varepsilon\|U\|^2_{\mathcal{H}} + C_\varepsilon\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)} \\
&\qquad \le \varepsilon(\rho)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)} + \|v_2\|^2_{L_2(\Omega_2)}\right)
\end{aligned} \tag{4.45}
$$

by Young's inequality and $|\lambda| \gg 1$.
It remains to show an appropriate estimate for $|\lambda|^2 \|u_2\|^2_{L_2(\Omega_2)} = \|v_2\|^2_{L_2(\Omega_2)}$.
By (4.42) and Lemma 4.17 it holds that

$$\begin{aligned} n\|v_2\|^2_{L_2(\Omega_2)} &= n|\lambda|^2\|u_2\|^2_{L_2(\Omega_2)} \\ &= -(4-n)\|\Delta u_2\|^2_{L_2(\Omega_2)} - \int_\Gamma (q\nu)|\Delta u_2|^2\, dS - 2\operatorname{Re}\int_{\Omega_2} g_2(q\nabla\overline{u_2})\, dx \\ &\quad + \lambda^2 \int_\Gamma (q\nu)|u_2|^2\, dS - 2\operatorname{Re}\int_\Gamma (q\nabla u_2)\partial_\nu \Delta\overline{u_2} - \Delta u_2 \partial_\nu(q\nabla\overline{u_2})\, dS. \end{aligned} \tag{4.46}$$

In the same way, we apply Lemma 4.17 in Ω_1 with $w = u_1$ and $z = -\rho\Delta v_1$. Here, u_1 fulfils $-\Delta^2 u_1 + \lambda^2 u_1 = -\rho\Delta v_1$ due to (4.37) and (4.38). Moreover, note that the orientation of the normal vector ν on Γ is different; on the part Γ of $\partial\Omega_1$ it points inside Ω_2 and on Γ it points inside Ω_1. We obtain

$$\begin{aligned} n\|v_1\|^2_{L_2(\Omega_1)} &= -(4-n)\|\Delta u_1\|^2_{L_2(\Omega_1)} - \int_{\Gamma_1} (q\nu)|\Delta u_1|^2\, dS \\ &\quad + \int_\Gamma (q\nu)|\Delta u_1|^2\, dS + 2\operatorname{Re}\int_{\Omega_1} (q\nabla\overline{u_1})\rho\Delta v_1\, dx \\ &\quad + \lambda^2 \int_{\Gamma_1} (q\nu)|u_1|^2\, dS - \lambda^2 \int_\Gamma (q\nu)|u_1|^2\, dS \\ &\quad - 2\operatorname{Re}\int_{\Gamma_1} \left[(q\nabla u_1)\partial_\nu\Delta\overline{u_1} - \Delta u_1\partial_\nu(q\nabla\overline{u_1})\right] dS \\ &\quad + 2\operatorname{Re}\int_{\Gamma} \left[(q\nabla u_1)\partial_\nu\Delta\overline{u_1} - \Delta u_1\partial_\nu(q\nabla\overline{u_1})\right] dS. \end{aligned} \tag{4.47}$$

Recall the transmission conditions (4.36), i.e.

$$u_1 = u_2, \quad \nabla u_1 = \nabla u_2, \quad \Delta u_1 = \Delta u_2, \quad \partial_\nu\Delta u_2 = \partial_\nu \Delta u_1 - \mathrm{i}\lambda\rho\partial_\nu(u_1 + w_1)$$

on Γ. Hence we have

$$\partial_\nu(q\nabla u_1) = \partial_\nu(q\nabla u_2)$$

on Γ. Indeed, in order to see this, we apply Lemma 4.18 to the function $u_1 - \widetilde{u_2}|_{\Omega_1}$, where $\widetilde{u_2} \in H^3(\Omega)$ denotes a regular extension of u_2 to Ω. Then we obtain

$$\partial_\nu(q\nabla(u_1 - \widetilde{u_2}|_{\Omega_1})) = (q\nu)\Delta(u_1 - \widetilde{u_2}|_{\Omega_1}) = 0$$

on Γ, which implies the assertion.
Using Lemma 4.18 again yields

$$u_1 = 0, \nabla u_1 = 0, \partial_\nu(q\nabla u_1) = (q\nu)\Delta u_1$$

on Γ_1. Therefore, adding (4.46) and (4.47) we arrive at

$$\begin{aligned} n\left(\|v_1\|^2_{L_2(\Omega_1)} + \|v_2\|^2_{L_2(\Omega_2)}\right) &= -(4-n)\left(\|\Delta u_1\|^2_{L_2(\Omega_2)} + \|\Delta u_2\|^2_{L_2(\Omega_2)}\right) \\ &\quad + 2\operatorname{Re}\left[\mathrm{i}\lambda\rho\int_{\Omega_1}(q\nabla\overline{u_1})\Delta u_1\, dx\right] - 2\operatorname{Re}\int_{\Omega_2}(q\nabla\overline{u_2})g_2\, dx \\ &\quad + 2\operatorname{Re}\left[\mathrm{i}\lambda\rho\int_\Gamma (q\nabla u_1)\partial_\nu(\overline{u_1 + w_1})\, dS\right] + \int_{\Gamma_1}(q\nu)|\Delta u_1|^2\, dS, \end{aligned} \tag{4.48}$$

which implies

$$\|v_2\|^2_{L_2(\Omega_2)} \le C\Big[|\lambda|\|u_1\|_{H^1(\Omega_1)}\|\Delta u_1\|_{L_2(\Omega_1)} + \|\nabla u_2\|_{L_2(\Omega_2)}\|g_2\|_{L_2(\Omega_2)}$$
$$+ |\lambda|\|u_1\|^2_{H^1(\Gamma)} + |\lambda|\|u_1\|_{H^1(\Gamma)}\|\partial_\nu w_1\|_{L_2(\Gamma)} + \|u_1\|^2_{H^2(\Gamma_1)}\Big]. \tag{4.49}$$

The first term can be estimated by (4.43), $\|\Delta u_1\|_{L_2(\Omega_1)} \le \|U\|_{\mathcal{H}}$ and Young's inequality. We obtain

$$\begin{aligned}|\lambda|\|u_1\|_{H^1(\Omega_1)}\|\Delta u_1\|_{L_2(\Omega_1)} &\le \Big(\varepsilon(\rho)\|U\|_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|_{L_2(\Omega_2)} + |\lambda|\|\partial_\nu w_1\|_{L_2(\Gamma)}\right)\Big)\|U\|_{\mathcal{H}}\\ &\le \varepsilon(\rho)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|_{L_2(\Omega_2)} + |\lambda|\|\partial_\nu w_1\|_{L_2(\Gamma)}\right)^2.\end{aligned} \tag{4.50}$$

For the second term in (4.49), observe that

$$\begin{aligned}\|\nabla u_2\|^2_{L_2(\Omega_2)} &\le \|\Delta u_2\|_{L_2(\Omega_2)}\|u_2\|_{L_2(\Omega_2)} + \|u_1\|^2_{H^2(\Omega_1)}\\ &\le C_\varepsilon\|\Delta u_2\|^2_{L_2(\Omega_2)} + \varepsilon\|u_2\|^2_{L_2(\Omega_2)} + \|u_1\|^2_{H^2(\Omega_1)} \le C\|U\|^2_{\mathcal{H}},\end{aligned}$$

which is obtained by using Greens formula, Cauchy-Schwarz, Young's inequality and the trace theorem. Therefore, we conclude for the second term in (4.49)

$$\|\nabla u_2\|_{L_2(\Omega_2)}\|g_2\|_{L_2(\Omega_2)} \le \varepsilon\|U\|^2_{\mathcal{H}} + C_\varepsilon\|g_2\|^2_{L_2(\Omega_2)}.$$

The trace theorem, interpolation and (4.43) yields

$$\begin{aligned}|\lambda|\|u_1\|^2_{H^1(\Gamma)} &\le C|\lambda|\|u_1\|^2_{H^{3/2}(\Omega_1)}\\ &\le C|\lambda|\|u_1\|_{H^1(\Omega_1)}\|u_1\|_{H^2(\Omega_1)}\\ &\le \varepsilon(\rho)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|\|\partial_\nu w_1\|^2_{L_2(\Gamma)}\right),\end{aligned} \tag{4.51}$$

wherefore the third term in (4.49) is estimated. For the fourth term in (4.49) we write

$$|\lambda|\|u_1\|_{H^1(\Gamma)}\|\partial_\nu w_1\|_{L_2(\Gamma)} \le \varepsilon\|u_1\|^2_{H^1(\Gamma)} + C_\varepsilon|\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)}.$$

As $|\lambda| \ge 1$, this can be estimated by the right-hand side of (4.51). Altogether, we obtain

$$\|v_2\|^2_{L_2(\Omega_2)} \le \varepsilon(\rho)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)}\right) + C\|u_1\|^2_{H^2(\Gamma_1)}. \tag{4.52}$$

What is left is to show an appropriate estimate for the boundary term $\|u_1\|_{H^2(\Gamma)}$. We introduce a cut-off function $\chi \in C^\infty(\overline{\Omega_1}), 0 \le \chi \le 1$, satisfying $\chi = 1$ in a neighbourhood of Γ_1 and $\chi = 0$ in a neighbourhood of the transmission interface Γ. Now, set

$$z_1 := \chi u_1, \quad z_2 := \mathrm{i}\lambda z_1, \quad z := (z_1, z_2)^\top.$$

Then, since u_1 is a solution of (4.41), z satisfies

$$(-\mathrm{i}\lambda + A)\, z = \begin{pmatrix} 0 \\ \widetilde{f} \end{pmatrix},$$

where A is defined as in (4.26) and

$$\begin{aligned}\widetilde{f} &= (-\Delta^2\chi)u_1 - 2(\nabla\Delta\chi)\cdot\nabla u_1 - \Delta\chi\Delta u_1 - 2\Delta(\nabla\chi\cdot\nabla u_1)\\ &\quad - \Delta\chi\Delta u_1 - 2\nabla\chi\cdot\nabla\Delta u_1 + \mathrm{i}\lambda\rho\left((\Delta\chi)u_1 + 2\nabla\chi\cdot\nabla u_1\right) \in L_2(\Omega_1).\end{aligned}$$

We also write

$$\widetilde{f} = B_3(D,\chi)u_1 + \mathrm{i}\lambda\rho\left((\Delta\chi)u_1 + 2\nabla\chi\cdot\nabla u_1\right)$$

with a λ-independent differential operator $B_3(D,\chi)$ of order 3 with coefficients only consisting of derivatives of the C^∞-function χ. Hence, $B_3(D,\chi) \in L(H^{3/2}(\Omega_1), H^{-3/2}(\Omega_1))$. From Theorem 4.12 and interpolation theory, we obtain

$$\begin{aligned}\|z_1\|_{H^{5/2}(\Omega_1)} &\le C\|\widetilde{f}\|_{H^{-3/2}(\Omega_1)}\\ &\le C(\rho)\left(\|B_3(D,\chi)u_1\|_{H^{-3/2}(\Omega_1)} + |\lambda|\|u_1\|_{H^1(\Omega_1)}\right)\\ &\le C(\rho)\left(\|u_1\|_{H^{3/2}(\Omega_1)} + |\lambda|\|u_1\|_{H^1(\Omega_1)}\right).\end{aligned}$$

Finally, from (4.43) and (4.51) we deduce

$$\begin{aligned}\int_{\Gamma_1}|\Delta u_1|^2\,dS = \int_{\Gamma_1}|\Delta z_1|^2\,dS &\le C\|z_1\|^2_{H^{5/2}(\Omega_1)}\\ &\le \varepsilon(\rho)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho)}\left(\|g_2\|^2_{L_2(\Omega_2)} + |\lambda|^2\|\partial_\nu w_1\|^2_{L_2(\Gamma)}\right). \quad (4.53)\end{aligned}$$

The assertion now follows from (4.44), (4.45), (4.52) and (4.53). □

Having established this a-priori estimate, we are now able to prove the exponential stability of the semigroup generated by $\mathcal{A}$. This is done via subtraction of the solution of the resolvent problem associated to the damped plate equation in the whole domain Ω. The difference of the solution of the resolvent problem for $\mathcal{A}$ on the imaginary line and the solution of the resolvent problem to the damped equation satisfies the problem we dealt with in the last proposition.

4.20 Theorem. *There exists a constant $C = C(\rho) > 0$ such that*

$$\|(-\mathrm{i}\lambda + \mathcal{A})^{-1}\|_{L(\mathcal{H})} \le C \quad (\lambda \in \mathbb{R}\setminus\{0\}, |\lambda| > \lambda_0)$$

for some $\lambda_0 > 0$. Consequently, the C_0-semigroup $(\mathcal{T}(t))_{t\ge0}$ generated by $\mathcal{A}$ is exponentially stable, i.e. there exist constants $M > 0$ and $\kappa > 0$ such that

$$\|\mathcal{T}(t)U^0\|_{\mathcal{H}} \le Me^{-\kappa t}\|U^0\|_{\mathcal{H}} \quad (t \ge 0)$$

holds for all $U^0 \in \mathcal{H}$.

Proof. Let $\lambda \in \mathbb{R}$ with $|\lambda| \gg 1$ and let $F = (f_1, f_2, g_1, g_2) \in \mathcal{H}$. Let $U = (u_1, u_2, v_1, v_2) \in D(\mathcal{A})$ be the unique solution of

$$(-\mathrm{i}\lambda + \mathcal{A})U = F,$$

i.e. U satisfies

$$\begin{aligned} -\mathrm{i}\lambda u_1 + v_1 &= f_1 \quad \text{in } \Omega_1, \\ -\Delta^2 u_1 + \rho\Delta v_1 - \mathrm{i}\lambda v_1 &= g_1 \quad \text{in } \Omega_1, \\ -\mathrm{i}\lambda u_2 + v_2 &= f_2 \quad \text{in } \Omega_2, \\ -\Delta^2 u_2 - \mathrm{i}\lambda v_2 &= g_2 \quad \text{in } \Omega_2. \end{aligned}$$

In order to show the assertion, we will subtract the solution W of a structurally damped plate equation with clamped boundary conditions on the whole domain Ω from U. For this difference we will be able to use the a-priori estimate from Proposition 4.19, whereas for W an appropriate estimate is known.
Recall the definition of the operator A from (4.26) and define

$$\widetilde{W} = (w, z) \in D(A) = \left(H^4(\Omega) \cap H_0^2(\Omega)\right) \times H_0^2(\Omega)$$

by

$$\widetilde{W} := (-\mathrm{i}\lambda + A)^{-1} \begin{pmatrix} \chi_1 f_1 + \chi_2 f_2 \\ \chi_1 g_1 + \chi_2 g_2 \end{pmatrix},$$

where χ_i is the characteristic function on Ω_i for $i = 1, 2$. Since $\chi_1 f_1 + \chi_2 f_2 \in H_0^2(\Omega)$ and $\chi_1 g_1 + \chi_2 g_2 \in L_2(\Omega)$ due to the definition of $\mathcal{H}$, by Theorem 4.12, $\widetilde{W}$ is well-defined. In the following, we denote the restrictions of the components of $\widetilde{W}$ by $w_i := w|_{\Omega_i}$ and $z_i := z|_{\Omega_i}$ for $i = 1, 2$. Finally, we set

$$W := (w_1, w_2, z_1, z_2) \in X \times X.$$

Note that $u_i \in H^4(\Omega_i)$ for $i = 1, 2$. With this definitions, we obtain that the difference $U - W$ satisfies

$$(-\mathrm{i}\lambda + \mathcal{A})(U - W) = F - (-\mathrm{i}\lambda + \mathcal{A})W = (0, 0, 0, \widetilde{g}_2)^\top \tag{4.54}$$

with $\widetilde{g}_2 = \rho\Delta z_2 \in L_2(\Omega_2)$, subject to the transmission conditions

$$\begin{cases} \Delta(u_1 - w_1) = \Delta(u_2 - w_2), \\ -\mathrm{i}\lambda\rho\partial_\nu(u_1 - w_1) + \partial_\nu\Delta(u_1 - w_1) = \partial_\nu\Delta(u_2 - w_2) + \mathrm{i}\lambda\rho\partial_\nu w_1. \end{cases}$$

Thanks to Proposition 4.19, we have

$$\|U - W\|_{\mathcal{H}} \le C \left(\|\widetilde{g}_2\|_{L_2(\Omega_2)} + |\lambda| \|\partial_\nu w_1\|_{L_2(\Gamma)} \right).$$

Corollary 4.13 states

$$\|\widetilde{g}_2\|_{L_2(\Omega_2)} = \rho\|\Delta z_2\|_{L_2(\Omega_2)} \le C\|F\|_{\mathcal{H}},$$

wherefore it follows that

$$\|U - W\|_{\mathcal{H}} \le C\|F\|_{\mathcal{H}}.$$

Here, we have used the trace theorem and the fact that

$$|\lambda| \|w_1\|_{H^2(\Omega_1)} \leq C\|F\|_{\mathcal{H}},$$

since A is the generator of a bounded, analytic C_0-semigroup on $H_0^2(\Omega) \times L_2(\Omega)$ by Theorem 4.12. Invoking Theorem 4.12 again, we deduce

$$\|U\|_{\mathcal{H}} \leq \|U - W\|_{\mathcal{H}} + \|W\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}$$

with a constant $C = C(\rho) > 0$. This proves the theorem. □

Finally, we want to comment on a geometry, where Ω_2 (or a connected component of Ω_2, to be precise) also has a boundary, called Γ_2. This generalizes the situation considered before as the first situation corresponds to the new geometry with $\Gamma_2 = \emptyset$. In this case, the situation can be described as follows: Let $n \leq 4$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega = \Gamma_1 \cap \Gamma_2$ such that Γ_1 and Γ_2 are of class C^4. Moreover, we assume that $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. Let Ω be divided in two open sets Ω_1 and Ω_2, i.e. $\Omega = \Omega_1 \dot\cup \Gamma \dot\cup \Omega_2$ such that the common interface Γ is of class C^4 again. We assume that $\overline{\Omega_i} \cap \Gamma_i = \Gamma_i$, i.e. $\Gamma_i \subset \partial\Omega_i$ for $i = 1, 2$.

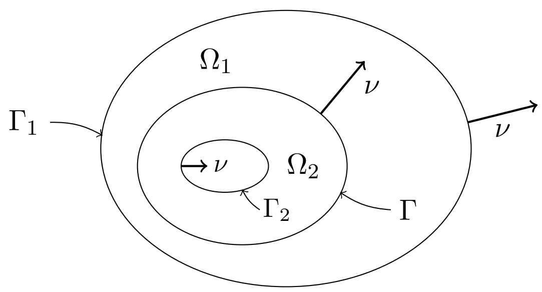


Figure 4.4.: The set $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ with additional boundary Γ_2 for Ω_2.

Now, we consider the same problem as above, where we have to add boundary conditions on Γ_2. As we did on Γ_1, we assume that the plate is clamped on Γ_2, i.e. we assume the Dirichlet-Neumann boundary conditions

$$u_2 = \partial_\nu u_2 = 0 \text{ on } \Gamma_2,$$

where ν denotes the outer unit normal on Γ_2 with respect to Ω_2.

Obviously, defining the operator $\mathcal{A}$ and the spaces $\mathcal{H}, D(\mathcal{A})$ as before, but now setting

$$X := \big\{(u_1, u_2) \in H^2(\Omega_1) \times H^2(\Omega_2) : u_j = \partial_\nu u_j = 0 \text{ on } \Gamma_j, j = 1, 2, \\ u_1 = u_2, \partial_\nu u_1 = \partial_\nu u_2 \text{ on } \Gamma\big\}$$

leads to an appropriate system to describe the problem with additional boundary Γ_2. The fact that $\mathcal{A}\colon \mathcal{H} \supset D(\mathcal{A}) \to \mathcal{H}$ is the generator of a C_0-semigroup is obtained exactly as in Theorem 4.3. In this case, exponential stability is obtained, provided a geometric condition is satisfied.

4.21 Theorem. *Assume that there exists some $x_0 \in \mathbb{R}^n$ such that the region enclosed by Γ_2 is star-shaped with respect to x_0, i.e.*

$$q\nu \leq 0$$

holds on Γ_2 where $q = q_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ is given by $q_{x_0}(x) := x - x_0$.
Then, there exists a constant $C = C(\rho) > 0$ such that

$$\|(-\mathrm{i}\lambda + \mathcal{A})^{-1}\|_{L(\mathcal{H})} \leq C \quad (\lambda \in \mathbb{R} \setminus \{0\}, |\lambda| > \lambda_0)$$

for some $\lambda_0 > 0$. Consequently, the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is exponentially stable, i.e. there exist constants $M > 0$ and $\kappa > 0$ such that

$$\|\mathcal{T}(t)U^0\|_{\mathcal{H}} \leq Me^{-\kappa t}\|U^0\|_{\mathcal{H}} \quad (t \geq 0)$$

holds for all $U^0 \in \mathcal{H}$.

The proof can be adapted step by step from the proof above. There are only two things, we need to take care of. The first step, where we have to be a bit more careful, is when we adapt Proposition 4.8 in order to show $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$ and therefore, to show the strong stability of the semigroup generated by $\mathcal{A}$. In the proof of Proposition 4.8 we arrive at (cf. (4.25))

$$\begin{aligned} -\Delta^2 u_2 + \lambda^2 u_2 &= 0 \quad \text{in } \Omega_2, \\ u_2 = \partial_\nu u_2 = \Delta u_2 = \partial_\nu \Delta u_2 &= 0 \quad \text{on } \Gamma, \\ u_2 = \partial_\nu u_2 &= 0 \quad \text{on } \Gamma_2 \end{aligned}$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. In order to show $u_2 = 0$ we substitute Lemma 4.7 by the following:

4.22 Lemma. *Let $U \subset \mathbb{R}^n$ be a domain with boundary $\partial U = \gamma_1 \cup \gamma_2$ of class C^4 such that $\overline{\gamma_1} \cap \overline{\gamma_2} = \emptyset$. Moreover, assume that the region enclosed by γ_2 is star-shaped with respect to some $x_0 \in \mathbb{R}^n$, i.e. $q\nu \leq 0$ on γ_2 where $q = q_{x_0}$ is defined as above and ν denotes the outer unit normal.*
Let $\lambda \in \mathbb{R} \setminus \{0\}$ be arbitrary. If $w \in H^4(U) \cap H_0^2(U)$ satisfies

$$\begin{aligned} -\Delta^2 w + \lambda^2 w &= 0 \quad \text{in } U, \\ \Delta w = \partial_\nu \Delta w &= 0 \quad \text{on } \gamma_1, \end{aligned} \tag{4.55}$$

then $w = 0$ already.

Proof. We define the closed set $V \subset \mathbb{R}^n$ to be the region enclosed by γ_2, i.e. we set

$$V := \{x \in \mathbb{R}^n : \exists y \in \gamma_2 \text{ s.t. } x = \mu x_0 + (1-\mu)y \text{ for some } \mu \in [0,1]\}.$$

Let $w \in H^4(U) \cap H_0^2(U)$ be a solution of (4.55). Extend w to a function $W \in H^4(\mathcal{O})$ by letting $W(x) := 0$ for $x \in \mathbb{R}^n \setminus (V \cup U)$. Here we have set $\mathcal{O} := \mathbb{R}^n \setminus V$. Then W satisfies

$$\begin{aligned} -\Delta^2 W + \lambda^2 W &= 0 \quad \text{in } \mathcal{O}, \\ W = \partial_\nu W &= 0 \quad \text{on } \gamma_2 = \partial\mathcal{O}. \end{aligned}$$

Applying Lemma 4.17, Lemma 4.18 and the fact that $W = \partial_\nu W = 0$ on γ_2 yields

$$\begin{aligned}
n\lambda^2\|W\|^2_{L_2(\mathcal{O})} + (4-n)\|\Delta W\|^2_{L_2(\mathcal{O})} + \int_{\gamma_2} (q\nu)|\Delta W|^2\,dS \\
&= \lambda^2 \int_{\gamma_2} (q\nu)|W|^2\,dS - 2\,\mathrm{Re}\int_{\gamma_2} (q\nabla W)\partial_\nu \Delta\overline{W} - \Delta W\partial_\nu(q\nabla\overline{W})\,dS \\
&= 2\,\mathrm{Re}\int_{\gamma_2} (q\nu)|\Delta W|^2\,dS \\
&\le 2\int_{\gamma_2} (q\nu)|\Delta W|^2\,dS.
\end{aligned}$$

Since $q\nu \le 0$ on γ_2 we obtain $\|W\|^2_{L_2(\mathcal{O})} + \|\Delta W\|^2_{L_2(\mathcal{O})} = 0$ and therefore $W = 0$. In particular, it holds that $w = W|_U = 0$. □

The second thing is that due to the boundary conditions on Γ_2 we now obtain

$$\begin{aligned}
n\|v_2\|^2_{L_2(\Omega_2)} &= n|\lambda|^2\|u_2\|^2_{L_2(\Omega_2)} \\
&= -(4-n)\|\Delta u_2\|^2_{L_2(\Omega_2)} - \int_{\Gamma\cup\Gamma_2} (q\nu)|\Delta u_2|^2\,dS - 2\,\mathrm{Re}\int_{\Omega_2} g_2(q\nabla\overline{u_2})\,dx \\
&\quad + \lambda^2\int_{\Gamma\cup\Gamma_2} (q\nu)|u_2|^2\,dS - 2\,\mathrm{Re}\int_{\Gamma\cup\Gamma_2} (q\nabla u_2)\partial_\nu\Delta\overline{u_2} - \Delta u_2\partial_\nu(q\nabla\overline{u_2})\,dS \\
&= -(4-n)\|\Delta u_2\|^2_{L_2(\Omega_2)} - \int_{\Gamma} (q\nu)|\Delta u_2|^2\,dS - 2\,\mathrm{Re}\int_{\Omega_2} g_2(q\nabla\overline{u_2})\,dx \\
&\quad + \lambda^2\int_{\Gamma\cup\Gamma_2} (q\nu)|u_2|^2\,dS + \int_{\Gamma_2} (q\nu)|\Delta u_2|^2\,dS
\end{aligned}$$

instead of (4.46) in the proof of Proposition 4.19, where now

$$\lambda^2\int_{\Gamma_2} (q\nu)|u_2|^2\,dS + \int_{\Gamma_2} (q\nu)|\Delta u_2|^2\,dS \le 0.$$

Using this remarks we obtain exponential stability also in the case with an additional boundary Γ_2, provided it satisfies the geometric condition.

4.23 Remark. Note that in the one-dimensional case, i.e. where $n = 1$, we obtain exponential stability for the transmission problem where the damping is effective only on the right-hand part (or, by symmetry, in the left-hand part) of the domain, which in this case is an interval.

4.2. A plate equation with continuous partial damping

In this section we want to consider a plate equation where damping effects are continuously distributed over the domain $\Omega \subset \mathbb{R}^n$. Different to the situation in Section 4.1 we have no discontinuity at some interface Γ, wherefore we now consider canonical transmission conditions everywhere, that means we are looking for a solution $u \in H^4(\Omega)$ in contrast to the solution of the above problem, which only had H^4-regularity on the subdomains Ω_1 and Ω_2 but not on the whole domain Ω. Provided that the damping is active in a neighbourhood of the outer boundary Γ_1 we can apply the same methods as above to show exponential stability of a similar problem.

The geometrical situation is similar as in the last part of Section 4.1 (cf. Fig. 4.4): Let $n \leq 4$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Here, Γ_1 and Γ_2 are of class C^4 and satisfy $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. In the whole section we assume that there exists some $x_0 \in \mathbb{R}^n$ such that

$$(x - x_0)\nu(x) \leq 0 \quad (x \in \Gamma_2), \tag{4.56}$$

where ν denotes the outer unit normal on Γ_2. Hence, Γ_2 can be considered as the inner boundary of the domain Ω and the region enclosed by Γ_2 is star-shaped.

In what follows, we will often work in small neighbourhoods of surfaces within a domain in $\mathbb{R}^n$. Therefore, we introduce the following definition.

4.24 Definition. An open set $U \subset \mathbb{R}^n$ is said to be a *neighbourhood in a domain $G \subset \mathbb{R}^n$ of some part $\gamma \subset \partial G$* if $U = G \cap V$ for some open connected $V \subset \mathbb{R}^n$ containing γ.

Let $\rho \in C^4(\overline{\Omega}; \mathbb{R})$ be a non-negative function. Further, let $\rho_0 > 0$ be a constant and assume that $\rho \geq 2\rho_0$ in a neighbourhood Ω_1 of Γ_1 in Ω. Then, the boundary of Ω_1 is of class C^4.

We consider the problem

$$\partial_t^2 u + \Delta^2 u - \operatorname{div}(\rho \nabla \partial_t u) = 0 \quad \text{in } (0, \infty) \times \Omega \tag{4.57}$$

with clamped boundary conditions

$$u = \partial_\nu u = 0 \quad \text{on } \partial\Omega. \tag{4.58}$$

The problem is completed by the initial conditions

$$u(0, \cdot) = u^0, \quad \partial_t u(0, \cdot) = u^1 \quad \text{in } \Omega. \tag{4.59}$$

Since we are looking for H^4 solutions in Ω, on the interface Γ separating Ω_1 where $\rho > \rho_0$ and Ω_2 where we only have $\rho \geq 0$, canonical transmission conditions

$$\partial_\nu^j u_1 = \partial_\nu^j u_2, \quad j = 0, \dots, 3 \tag{4.60}$$

shall hold. This can also be written as

$$\begin{aligned} u_1 &= u_2, \\ \partial_\nu u_1 &= \partial_\nu u_2, \\ \Delta u_1 &= \Delta u_2, \\ \partial_\nu \Delta u_1 &= \partial_\nu \Delta u_2 \end{aligned}$$

on Γ. In fact, $u \in H^4(\Omega)$ is equivalent to the assumption that (4.60) holds on each embedded C^4-curve being contained in Ω.
Using the definitions of Ω_1, Ω_2 and Γ as well as (4.60) allows us to read the boundary value problem (4.57) - (4.59) as an transmission problem with the artificial transmission interface Γ. In this setting we are able to adapt the steps in Section 4.1 to our new problem with the now on the whole domain Ω variable damping coefficient ρ.

Let $\mathcal{H} := H_0^2(\Omega) \times L_2(\Omega)$ and consider the operator $\mathcal{A}\colon \mathcal{H} \supset D(\mathcal{A}) \to \mathcal{H}$ given by

$$\mathcal{A}U := \begin{pmatrix} v \\ -\Delta^2 u + \operatorname{div}(\rho \nabla v) \end{pmatrix}$$

for $U = (u, v)^\top \in D(\mathcal{A})$ where the domain $D(\mathcal{A})$ is defined by

$$D(\mathcal{A}) := \left\{(u, v) \in \mathcal{H} : v \in H_0^2(\Omega), -\Delta^2 u \in L_2(\Omega)\right\}.$$

By the invertibility of $\Delta^2\colon L_2(\Omega) \cap \left(H^4(\Omega) \cap H_0^2(\Omega)\right) \to L_2(\Omega)$ (e.g. [CL10], p. 29) we even have

$$D(\mathcal{A}) = \left(H^4(\Omega) \cap H_0^2(\Omega)\right) \times H_0^2(\Omega).$$

The space $\mathcal{H}$ equipped with the scalar product

$$\langle (u_1, v_1), (u_2, v_2)\rangle_{\mathcal{H}} := \langle \Delta u_1, \Delta u_2\rangle_{L_2(\Omega)} + \langle v_1, v_2\rangle_{L_2(\Omega)} \quad ((u_i, v_i) \in \mathcal{H},\ i = 1, 2)$$

is a Hilbert space.

4.25 Theorem. *The operator $\mathcal{A}$ is the generator of a C_0-semigroup of contractions on the Hilbert space $\mathcal{H}$. Therefore, for all $U^0 \in D(\mathcal{A})$ the Cauchy problem*

$$\begin{aligned} U_t(t) - \mathcal{A}U(t) &= 0, \\ U(0) &= U^0 \end{aligned}$$

possesses a unique classical solution $U \in C^1([0, \infty), \mathcal{H})$ with $U(t) \in D(\mathcal{A})$ for all $t \geq 0$. Moreover, $0 \in \rho(\mathcal{A})$ and $\mathcal{A}$ has compact resolvent. In particular, the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is discrete.

Proof. The proof is similar to the proof of Theorem 4.3. For the dissipativity we now obtain

$$\operatorname{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\int_\Omega \rho(x) |\nabla u(x)|^2 \, dx \leq 0.$$

The surjectivity of $1 - \mathcal{A}$ can be shown by using the form $B\colon H_0^2(\Omega) \times H_0^2(\Omega) \to \mathbb{C}$ given by

$$B(u, \phi) := \langle u, \phi\rangle_{L_2(\Omega)} + \langle \Delta u, \Delta \phi\rangle_{L_2(\Omega)} + \langle \rho \nabla u, \nabla \phi\rangle_{L_2(\Omega)}.$$

In order to solve

$$(1 - \mathcal{A})U = F = (f, g)^\top \in \mathcal{H},$$

that is

$$u - v = f,$$
$$v + \Delta^2 u - \operatorname{div}(\rho \nabla v) = g,$$

which is equivalent to the problem

$$u + \Delta^2 u - \operatorname{div}(\rho \nabla u) = g + f - \operatorname{div}(\rho \nabla f) \tag{4.61}$$

we define the linear functional $\Lambda \in H_0^2(\Omega)' = H^{-2}(\Omega)$ given by

$$\Lambda(\phi) := \langle g + f, \phi \rangle_{L_2(\Omega)} + \langle \rho \nabla f, \nabla \phi \rangle_{L_2(\Omega)} \quad (\phi \in H_0^2(\Omega))$$

and apply the theorem of Lax-Milgram to the coercive form B. This yields a unique $u \in H_0^2(\Omega)$ satisfying $B(u, \phi) = \Lambda(\phi)$ for all $\phi \in H_0^2(\Omega)$, in particular (4.61) holds in $\mathscr{D}'(\Omega)$ and since the right-hand side of (4.61) is in $L_2(\Omega)$, we obtain $\Delta^2 u \in L_2(\Omega)$, which, as mentioned above, means $u \in H^4(\Omega)$. Now, let $v := u - f \in H_0^2(\Omega)$.
In order to show $0 \in \rho(\mathcal{A})$ we proceed similarly. Let $F = (f, g)^\top \in \mathcal{H}$. Then $-\mathcal{A}U = F$ is equivalent to

$$-v = f,$$
$$\Delta^2 u - \operatorname{div}(\rho \nabla v) = g.$$

Plugging in yields the equation

$$\Delta^2 u = g - \operatorname{div}(\rho \nabla f) \in L_2(\Omega)$$

which possesses a unique solution $u \in H^4(\Omega) \cap H_0^2(\Omega)$ since the biharmonic operator with Dirichlet-Neumann conditions is invertible as mentioned above.
The compactness of the resolvent follows from the Rellich-Kondrachov embedding theorem (see Theorem A.4). □

Similar to Proposition 4.8 we want to state that the imaginary axis belongs to the resolvent set of $\mathcal{A}$. In order to use integration by parts and the trace theorem without problems, we need further assumptions on the damping coefficient ρ in the case $n \in \{2, 3, 4\}$: we will assume that the parts of the domain with positive n-dimensional Lebesgue measure where no damping is active, i.e. where $\rho = 0$, are of class C^4. Due to the inner boundary Γ_2, this is not automatically fulfilled. Without such an assumption the situation could be as in the following figure 4.5. The non-smooth wedges generally cause problems when taking the trace of Sobolev functions. In the one-dimensional case there are no non-smooth boundaries and hence there is no need for additional assumptions on ρ for $n = 1$.

4.26 Proposition. *Assume that the function ρ additionally satisfies the following assumption: if $Z \subset \{x \in \Omega : \rho(x) = 0\}$ is a connected component of $\{x \in \Omega : \rho(x) = 0\}$ and has positive n-dimensional Lebesgue measure $\lambda(Z) > 0$, then ∂Z is of class C^4.*
Then the imaginary axis is a subset of the resolvent set of $\mathcal{A}$, i.e. $\mathrm{i}\mathbb{R} \subset \rho(\mathcal{A})$.

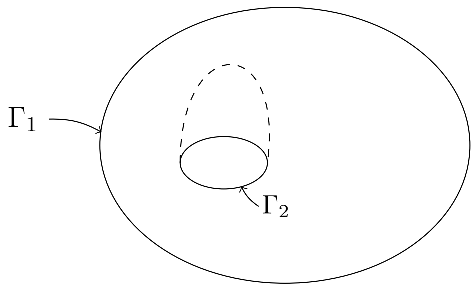


Figure 4.5.: In the area enclosed by the dashed line it holds that $\rho = 0$. This area is non-smooth and integration by parts or trace theorems cannot be applied in general.

Proof. This can be proved in the same way as Proposition 4.8 was proved combined with the remarks we added in the end of Section 4.1 using Lemma 4.22:
Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $U = (u, v)^\top \in D(\mathcal{A})$ solve $(-\mathrm{i}\lambda + \mathcal{A})U = 0$. Then $u \in H^4(\Omega) \cap H_0^2(\Omega)$ satisfies

$$\lambda^2 u - \Delta^2 u + \mathrm{i}\lambda \operatorname{div}(\rho \nabla u) = 0. \tag{4.62}$$

Multiplying with $\overline{u}$ and performing an integration by parts we obtain, after taking the imaginary part,

$$-\int_\Omega \rho(x)|\nabla u(x)|^2 \, dx = 0$$

and therefore $\nabla u = 0$ on the support $S := \operatorname{supp}(\rho)$ of ρ. Note that $\{x \in S : \rho(x) = 0\}$ has Lebesgue measure zero since ρ is continuous. Hence, u is constant on S and from (4.62) we deduce $u = 0$ on S. Lemma 4.22 implies $u = 0$ on $\Omega \setminus S$ and therefore $u = 0$. Since $v = \mathrm{i}\lambda u$ it follows that $U = 0$. □

4.27 Remark. Note that the assumption on ρ in Proposition 4.26 is particularly fulfilled, provided that $\rho = 0$ or $\rho \geq \eta > 0$ for some positive η in a neighbourhood of Γ_2 within Ω. In fact, due to the C^4-regularity of the damping coefficient ρ, the connected components $Z \subset \{x \in \Omega : \rho(x) = 0\}$ with positive Lebesgue measure are of class C^4, given that $\partial Z \cap \Gamma_2 = \emptyset$. However, of course, this is not necessary.

4.28 Remark. Assume that the additional assumption on ρ in Proposition 4.26 does not hold and consider the situation given in Figure 4.5. Let γ denote the dashed line and Z be the area enclosed by γ and Γ_2. In order to prove Proposition 4.26 in this case, we would have to show the following: if $u \in H^4(Z)$ satisfies $\lambda^2 u - \Delta^2 u = 0$ in Z with boundary conditions $u = \partial_\nu u = 0$ on ∂Z as well as $\Delta u = \partial_\nu \Delta u = 0$ on γ, then $u = 0$ already.

The following proposition shows the boundedness of the resolvent on the imaginary axis applied to a right-hand side $F \in \mathcal{H}$ of the form $F = (0, g)^\top$ for some $g \in L_2(\Omega)$. This statement plays the same role as Proposition 4.19 did before, where we had to consider an non-homogeneous right hand side in the transmission condition of order three at some interface.

4.29 Proposition. *Assume that the function ρ additionally satisfies the following assumption: if $Z \subset \{x \in \Omega : \rho(x) = 0\}$ is a connected component of $\{x \in \Omega : \rho(x) = 0\}$ and has positive n-dimensional Lebesgue measure $\lambda(Z) > 0$, then ∂Z is of class C^4.*
Let $g \in L_2(\Omega)$. Then, there exists $\lambda_0 > 0$ and a constant $C > 0$ (only depending on n, ρ and λ_0) such that for the solution $U = (u, v)^\top \in D(\mathcal{A})$ of

$$(-\mathrm{i}\lambda + \mathcal{A})U = (0, g)^\top =: F, \tag{4.63}$$

the estimate

$$\|U\|_{\mathcal{H}} \leq C\|g\|_{L_2(\Omega)} \quad (\lambda \in \mathbb{R}, |\lambda| > \lambda_0)$$

holds true.

Proof. Let $\lambda \in \mathbb{R}$ with $|\lambda| \gg 1$ and $U = (u, v)^\top \in D(\mathcal{A})$ be the solution of (4.63). Then $\mathrm{Re}\langle F, U\rangle_{\mathcal{H}} = -\int_\Omega \rho(x)|\nabla v(x)|^2\,dx$ and hence

$$\int_\Omega \rho(x)|\nabla v(x)|^2\,dx \leq \|U\|_{\mathcal{H}}\|g\|_{L_2(\Omega)} \leq \varepsilon\|U\|_{\mathcal{H}}^2 + C_\varepsilon\|g\|_{L_2(\Omega)}^2, \tag{4.64}$$

or, since $v = \mathrm{i}\lambda u$,

$$\int_\Omega \rho(x)|\nabla u(x)|^2\,dx \leq |\lambda|^{-2}\left(\varepsilon\|U\|_{\mathcal{H}}^2 + C_\varepsilon\|g\|_{L_2(\Omega)}^2\right). \tag{4.65}$$

In particular, in Ω_1 where $\rho \geq 2\rho_0 > 0$ we obtain

$$\|\nabla v|_{\Omega_1}\|_{L_2(\Omega_1)}^2 \leq \varepsilon(\rho_0)\|U\|_{\mathcal{H}}^2 + C_{\varepsilon(\rho_0)}\|g\|_{L_2(\Omega)}^2$$

and

$$\|\nabla u|_{\Omega_1}\|_{L_2(\Omega_1)}^2 \leq |\lambda|^{-2}\left(\varepsilon(\rho_0)\|U\|_{\mathcal{H}}^2 + C_{\varepsilon(\rho_0)}\|g\|_{L_2(\Omega)}^2\right),$$

respectively. Recall that Ω_1 is a neighbourhood within Ω of the outer boundary Γ_1. Hence, we get

$$\|u|_{\Omega_1}\|_{H^1(\Omega_1)}^2 \leq |\lambda|^{-2}\left(\varepsilon(\rho_0)\|U\|_{\mathcal{H}}^2 + C_{\varepsilon(\rho_0)}\|g\|_{L_2(\Omega)}^2\right) \tag{4.66}$$

using Poincaré's inequality.
Since U satisfies (4.63), $u \in H^4(\Omega)$ satisfies the equation

$$\lambda^2 u - \Delta^2 u = g - \mathrm{i}\lambda\,\mathrm{div}(\rho\nabla u). \tag{4.67}$$

Multiplying (4.67) with $-\overline{u}$, integration by parts and taking real parts yields

$$\begin{aligned}\|\Delta u\|_{L_2(\Omega)}^2 &\leq \|g\|_{L_2(\Omega)}\|u\|_{L_2(\Omega)} + \lambda^2\|u\|_{L_2(\Omega)}^2 \\ &\leq \varepsilon\|U\|_{\mathcal{H}}^2 + C_\varepsilon\|g\|_{L_2(\Omega)}^2 + \|v\|_{L_2(\Omega)}^2\end{aligned} \tag{4.68}$$

due to Young's inequality. In order to estimate $\|v\|^2_{L_2(\Omega)}$ we use Rellich's identity 4.14. Since u satisfies (4.67) and on Γ_2 we have $q\nu \le 0$ for $q(x) = x - x_0$ we get

$$\begin{aligned} n\|v\|^2_{L_2(\Omega)} &= n\lambda^2\|u\|^2_{L_2(\Omega)} \\ &\le \int_{\Gamma_1} (q\nu)|\Delta u|^2\,dS - 2\operatorname{Re}\int_\Omega (g - \mathrm{i}\lambda\operatorname{div}(\rho\nabla u))(q\nabla\overline{u})\,dx \\ &\le \int_{\Gamma_1} (q\nu)|\Delta u|^2\,dS + \varepsilon\|U\|^2_{\mathcal{H}} + C_\varepsilon\|g\|^2_{L_2(\Omega)} \\ &\quad + C|\lambda|\|\nabla\rho\|_{L_\infty(\Omega)}\|\nabla u\|^2_{L_2(\Omega)} + C_\varepsilon|\lambda|^2\|\rho^{1/2}\|_{L_\infty(\Omega)}\|\rho^{1/2}\nabla u\|^2_{L_2(\Omega)} + \varepsilon\|\Delta u\|^2_{L_2(\Omega)} \end{aligned}$$

as $u = \partial_\nu u = 0$ on both Γ_1 and Γ_2. Interpolation theory and (4.65) now imply

$$\begin{aligned} n\|v\|^2_{L_2(\Omega)} &= n\lambda^2\|u\|^2_{L_2(\Omega)} \\ &\le \int_{\Gamma_1} (q\nu)|\Delta u|^2\,dS + \varepsilon\|U\|^2_{\mathcal{H}} + C_\varepsilon\|g\|^2_{L_2(\Omega)} \\ &\quad + \frac{1}{2}|\lambda|^2\|u\|^2_{L_2(\Omega)} + \frac{1}{2}\|\Delta u\|^2_{L_2(\Omega)} + \varepsilon\|U\|^2_{\mathcal{H}} + C_\varepsilon\|g\|^2_{L_2(\Omega)} + \varepsilon\|\Delta u\|^2_{L_2(\Omega)} \end{aligned}$$

and hence

$$\left(n - \tfrac{1}{2}\right)\|v\|^2_{L_2(\Omega)} \le \int_{\Gamma_1} (q\nu)|\Delta u|^2\,dS + \left(\varepsilon + \tfrac{1}{2}\right)\|U\|^2_{\mathcal{H}} + C_\varepsilon\|g\|^2_{L_2(\Omega)},$$

that is

$$\|v\|^2_{L_2(\Omega)} \le \int_{\Gamma_1} (q\nu)|\Delta u|^2\,dS + \left(\varepsilon + \tfrac{1}{2}\right)\|U\|^2_{\mathcal{H}} + C_\varepsilon\|g\|^2_{L_2(\Omega)}. \tag{4.69}$$

Eventually, we need an estimate for the term $\int_{\Gamma_1}(q\nu)|\Delta u|^2\,dS$. This will be achieved similarly to the proof of Proposition 4.19: First, for $r > 0$ we define

$$U_r := \{x \in \overline{\Omega} : \operatorname{dist}(x, \Gamma_1) < r\},$$

where $\operatorname{dist}(y, A) := \inf_{a\in A}|y - a|$ denotes the distance between a point $y \in \mathbb{R}^n$ and a set $A \subset \mathbb{R}^n$. Consider the operator A given by

$$A\colon \mathcal{H} \supset D(\mathcal{A}) \to \mathcal{H}, \quad U \mapsto \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho_0\Delta \end{pmatrix} U.$$

We introduce a cut-off function $\chi \in C^\infty(\overline{\Omega})$, $0 \le \chi \le 1$ satisfying $\chi = 1$ in $U_{r/2}$ and $\chi = 0$ in $\overline{\Omega}\backslash U_r$, where $r > 0$ is chosen in such a way that $\overline{U_r} \subset \Omega_1$ holds true. We define $z_1 := \chi u$, $z_2 = \mathrm{i}\lambda z_1$ and $z := (z_1, z_2)^\top$. Then $z_1 \in H^4(\Omega) \cap H^2_0(\Omega)$ satisfies

$$\begin{aligned} \lambda^2 z_1 - \Delta^2 z_1 + \mathrm{i}\lambda\rho_0\Delta z_1 &= \chi\left(g - \mathrm{i}\lambda\operatorname{div}(\rho\nabla u)\right) + B_3(D,\chi)u \\ &\quad + \mathrm{i}\lambda\rho_0\left((\Delta\chi)u + 2\nabla\chi\nabla u + \chi\Delta u\right) \\ &= \chi\left(g + \mathrm{i}\lambda\operatorname{div}((\rho - \rho_0)\nabla u)\right) + B_3(D,\chi)u \\ &\quad + \mathrm{i}\lambda\rho_0\left((\Delta\chi)u + 2\nabla\chi\nabla u\right) =: \widetilde{f} \end{aligned}$$

for some third order, λ-independent differential operator $B_3(D,\chi)$ having C^∞-coefficients dependent on χ, i.e. $B_3(D,\chi)\colon H^{3/2}(\Omega) \to H^{-3/2}(\Omega)$ is bounded. Moreover, it holds that $\operatorname{supp}(B_3(D,\chi)u \subset \overline{U_r}$. In terms of the operator A we have $(-i\lambda + A)z = (0, \widetilde{f})^\top \in H_0^2(\Omega)\times L_2(\Omega)$ and, using Theorem 4.12, we obtain the estimate

$$\|z_1\|^2_{H^{5/2}(\Omega)} \le C\Big(\|\chi g\|^2_{H^{-3/2}(\Omega)} + |\lambda|^2 \left[\|\chi\nabla\rho\nabla u\|^2_{H^{-3/2}(\Omega)} + \|(\rho-\rho_0)\chi\Delta u\|^2_{H^{-3/2}(\Omega)}\right] + \|B_3(D,\chi)u\|^2_{H^{-3/2}(\Omega)} + \rho_0^2|\lambda|^2\|(\Delta\chi)u\|^2_{H^{-3/2}(\Omega)} + \rho_0^2\|\nabla\chi\nabla u\|^2_{H^{-3/2}(\Omega)}\Big). \tag{4.70}$$

We estimate the terms separately. Recall that ρ is continuously differentiable, i.e. $|\rho|$ and $|\nabla\rho|$ both attain their maximum in $\overline{\Omega}$. Since $\overline{U_r} \subset \Omega_1$, from (4.66) it follows that

$$|\lambda|^2 \left(\|\chi\nabla\rho\nabla u\|^2_{L_2(\Omega)} + \rho_0^2\|(\Delta\chi)u\|^2_{L_2(\Omega)} + \rho_0^2\|\nabla\chi\nabla u\|^2_{L_2(\Omega)}\right) \le \varepsilon(\rho_0)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho_0)}\|g\|^2_{L_2(\Omega)}. \tag{4.71}$$

Furthermore,using the product rule for the Laplace operator, we see that

$$\begin{aligned} |\lambda|^2\|(\rho-\rho_0)\chi\Delta u\|^2_{H^{-3/2}(\Omega)} &\le C|\lambda|^2\|\chi\Delta u\|^2_{H^{-3/2}(\Omega)} \\ &= C|\lambda|^2\|\Delta(\chi u) - (\Delta\chi)u - 2\nabla\chi\nabla u\|^2_{H^{-3/2}(\Omega)} \\ &\le C|\lambda|^2\left(\|\chi u\|^2_{H^{1/2}(\Omega)} + \|(\Delta\chi)u\|^2_{L_2(\Omega)} + \|\nabla\chi\nabla u\|^2_{L_2(\Omega)}\right) \\ &\le \varepsilon(\rho_0)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho_0)}\|g\|^2_{L_2(\Omega)}, \end{aligned} \tag{4.72}$$

where we used $H^{1/2}(\Omega) \hookrightarrow H^1(\Omega)$ and (4.71) in the last step. For the term $B_3(D,\chi)u$, by interpolation theory and (4.66) we have

$$\begin{aligned} \|B_3(D,\chi)u\|^2_{H^{-3/2}(\Omega)} &\le C\|u|_{\Omega_1}\|^2_{H^{3/2}(\Omega_1)} \\ &\le \varepsilon\|u|_{\Omega_1}\|^2_{H^2(\Omega)} + C_\varepsilon\|u|_{\Omega_1}\|^2_{H^1(\Omega_1)} \\ &\le \varepsilon(\rho_0)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho_0)}\|g\|^2_{L_2(\Omega)}. \end{aligned} \tag{4.73}$$

From the trace theorem and (4.70)-(4.73) we get

$$\int_{\Gamma_1} (q\nu)|\Delta|^2\, dS \le C\|z_1\|^2_{H^{5/2}(\Omega)} \le \varepsilon(\rho_0)\|U\|^2_{\mathcal{H}} + C_{\varepsilon(\rho_0)}\|g\|^2_{L_2(\Omega)}. \tag{4.74}$$

Now the assertion of follows from combining (4.68), (4.69) and (4.74). □

With the same method as in Theorem 4.20 we can now prove the exponential stability for our problem with variable coefficient $\rho \in C^4(\overline{\Omega})$ satisfying $\rho \ge 2\rho_0 > 0$ in a neighbourhood Ω_1 of Γ_1 within Ω.

4.30 Theorem. *Assume that the function ρ additionally satisfies the following assumption: if $Z \subset \{x\in\Omega : \rho(x) = 0\}$ is a connected component of $\{x\in\Omega : \rho(x) = 0\}$ and has positive*

n-dimensional Lebesgue measure $\lambda(Z) > 0$, then ∂Z is of class C^4. Then there exists a constant $C = C(\rho) > 0$ such that

$$\|(-\mathrm{i}\lambda + \mathcal{A})^{-1}\|_{L(\mathcal{H})} \leq C \quad (\lambda \in \mathbb{R} \setminus \{0\}, |\lambda| > \lambda_0)$$

for some $\lambda_0 > 0$. Consequently, the C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ generated by $\mathcal{A}$ is exponentially stable, i.e. there exist constants $M > 0$ and $\kappa > 0$ such that

$$\|\mathcal{T}(t)U^0\|_{\mathcal{H}} \leq Me^{-\kappa t}\|U^0\|_{\mathcal{H}} \quad (t \geq 0)$$

holds for all $U^0 \in \mathcal{H}$.

Proof. Let $\lambda \in \mathbb{R}$ with $|\lambda| \gg 1$ and $F = (f, g)^\top \in \mathcal{H}$. Let $U = (u, v)^\top \in D(\mathcal{A})$ be the unique solution to $(-\mathrm{i}\lambda + \mathcal{A})U = F$, i.e.

$$\begin{aligned} -\mathrm{i}\lambda u + v &= f, \\ -\mathrm{i}\lambda v - \Delta^2 u + \operatorname{div}(\rho\nabla v) &= g \end{aligned}$$

in Ω. Plugging in $v = f + \mathrm{i}\lambda u$ into the second equation, we see that $u \in H^4(\Omega) \cap H_0^2(\Omega)$ satisfies

$$\lambda^2 u - \Delta^2 u = g + \mathrm{i}\lambda f + \operatorname{div}(\rho\nabla f).$$

As before, consider the operator

$$A \colon \mathcal{H} \supset D(\mathcal{A}) \to \mathcal{H}, \quad U \mapsto \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho_0\Delta \end{pmatrix} U$$

and let $W = (w, z)^\top \in D(\mathcal{A})$ be the unique solution to $(-\mathrm{i}\lambda + A)W = F$. Then, the difference $U - W$ satisfies

$$\begin{aligned} (-\mathrm{i}\lambda + \mathcal{A})(U - W) &= F - \begin{pmatrix} -\mathrm{i}\lambda w + z \\ -\mathrm{i}\lambda z - \Delta^2 w + \operatorname{div}(\rho\nabla z) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ g + \mathrm{i}\lambda z + \Delta^2 w - \rho_0\Delta z + \operatorname{div}((\rho_0 - \rho)\nabla z) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \operatorname{div}((\rho_0 - \rho)\nabla z) \end{pmatrix}. \end{aligned}$$

With Proposition 4.29 it follows that

$$\|U - W\|_{\mathcal{H}} \leq C\|\operatorname{div}((\rho_0 - \rho)\nabla z)\|_{L_2(\Omega)} \leq C\|\rho\|_{C^1(\Omega)}\|z\|_{H^2(\Omega)}.$$

Together with Corollary 4.13 as well as Theorem 4.12 we obtain

$$\|U\|_{\mathcal{H}} \leq \|U - W\|_{\mathcal{H}} + \|W\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}}$$

which shows the assertion. □

4.3. Lack of analyticity

Having seen that both problems in Section 4.1 and 4.2 possess exponentially stable solutions, the question arises, whether the semigroups associated to these problems are analytic which would immediately lead to a strong smoothing effect for the solutions. We will see that this is not the case. First, we will show that the resolvents of the associated operators cannot decay as $1/|\lambda|$ for $\lambda \in \mathrm{i}\mathbb{R}$, already proving the lack of analyticity of the semigroups. After that, using a numerical approach, we will try to locate the spectrum of the transmission problem discussed in Section 4.1 in the complex plane. The numerical analysis of the 1-dimensional transmission problem will suggest that not only the decay of the resolvent is not strong enough in order to lead to an analytic semigroup but that there is no angle $\vartheta > \frac{\pi}{2}$ such that $\Sigma_\vartheta \subset \rho(\mathcal{A})$ is satisfied.

Let us first consider the transmission problem from Section 4.1, i.e. we consider the operator $\mathcal{A}\colon \mathcal{H} \supset D(\mathcal{A}) \to \mathcal{H}$, $U \mapsto A(D)U$, where

$$A(D) := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\Delta^2 & 0 & \rho\Delta & 0 \\ 0 & -\Delta^2 & 0 & 0 \end{pmatrix}$$

with some constant damping factor $\rho > 0$. The damping is effective only in Ω_1. The underlying Hilbert space $\mathcal{H}$ was defined by $\mathcal{H} = X \times H$ with

$$X := \big\{(u_1, u_2) \in H^2(\Omega_1) \times H^2(\Omega_2) : u_1 = \partial_\nu u_1 = 0 \text{ on } \Gamma_1,\ u_1 = u_2,\ \partial_\nu u_1 = \partial_\nu u_2 \text{ on } \Gamma\big\}$$

and $H = L_2(\Omega_1) \times L_2(\Omega_2)$ and the domain $D(\mathcal{A})$ of $\mathcal{A}$ turned out to be equal to the space

$$\begin{aligned} D(\mathcal{A}) = \big\{(u_1, u_2, v_1, v_2) \in \mathcal{H} : (v_1, v_2) \in X, (u_1, u_2) \in H^4(\Omega_1) \times H^4(\Omega_2), \\ \Delta u_1 = \Delta u_2, -\rho\partial_\nu v_1 + \partial_\nu \Delta u_1 = \partial_\nu \Delta u_2 \text{ on } \Gamma\big\}, \end{aligned}$$

see Lemma 4.5.

4.31 Proposition. *Let the assumptions of Theorem 4.21 be satisfied. Then, the C_0-semigroup generated by the operator $\mathcal{A}$ from Section 4.1 is not analytic.*

Proof. Using notation from Chapter 2, Section 2.1, we have

$$X_2 = \prod_{j=1}^{4} H^{s_j}(U_j), \quad Y_2 = \prod_{j=1}^{4} H^{s_j+t_j}(U_j)$$

with

$$\begin{aligned} t_1 = t_2 = t_3 = t_4 = 2, \\ s_1 = s_2 = 2, \\ s_3 = s_4 = 0, \\ U_1 = U_3 = \Omega_1, \\ U_2 = U_4 = \Omega_2. \end{aligned}$$

Hence, the principle symbol A_0 of $\mathcal{A}$ is given by

$$A_0(\xi) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -|\xi|^4 & 0 & -\rho|\xi|^2 & 0 \\ 0 & -|\xi|^4 & 0 & 0 \end{pmatrix} \quad (\xi \in \mathbb{R}^n \setminus \{0\}).$$

Also in terms of the notation from Section 2.1, we denote $\mathcal{X} = \mathcal{H}$.
From Theorem 4.3 and Proposition 4.8, it follows that all assumptions of Corollary 2.4 are satisfied. Therefore, using Corollary 2.5 and the remarks thereafter, it follows that if $\mathcal{A}$ generates an analytic C_0-semigroup on $\mathcal{X} = \mathcal{H}$, then $\det(\lambda - A_0(\xi)) \neq 0$ must hold true for all $\lambda \in \mathrm{i}\mathbb{R} \setminus \{0\}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$. However, since

$$\det(\lambda - A_0(\xi)) = (\lambda^2 + |\xi|^4)(\lambda^2 + \rho\lambda|\xi|^2 + |\xi|^4)$$

this assumption is violated (for example, let $\lambda_0 = \mathrm{i}\kappa^2$ for any $\kappa > 0$ and $\xi^0 \in \mathbb{R}^n \setminus \{0\}$ such that $|\xi^0| = \kappa$) and the assertion follows. □

A similar result holds for the problem studied in Section 4.2. There, we considered the operator $\mathcal{A}\colon \mathcal{H} \supset D(\mathcal{A}) \to \mathcal{H}$ given by

$$\mathcal{A}U := \begin{pmatrix} v \\ -\Delta^2 u + \operatorname{div}(\rho\nabla v) \end{pmatrix}$$

for $U = (u, v)^\top \in D(\mathcal{A})$, where $\mathcal{H} = H_0^2(\Omega) \times L_2(\Omega)$ and the domain $D(\mathcal{A})$ was defined by

$$D(\mathcal{A}) := \left\{(u, v) \in \mathcal{H} : v \in H_0^2(\Omega), -\Delta^2 u \in L_2(\Omega)\right\}.$$

Similar to the situation above, higher regularity results showed that in fact

$$D(\mathcal{A}) = \left(H^4(\Omega) \cap H_0^2(\Omega)\right) \times H_0^2(\Omega).$$

The function $\rho \in C^4(\overline{\Omega}; \mathbb{R})$ was assumed to be non-negative, satisfying $\rho \geq \rho_0$ for some $\rho_0 > 0$ in a neighbourhood Ω_1 of Γ_1, the outer boundary of Ω. Using the product rule, we see that the operator $\mathcal{A}$ acts in form of the matrix differential operator

$$A(D) = \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta + \nabla\rho\nabla \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta + \sum_{i=1}^n (\partial_i\rho)\partial_i \end{pmatrix}.$$

For this operator, given the assumptions we made in Section 4.2, we have the following result:

4.32 Proposition. *Let the assumptions of Theorem 4.30 be satisfied. Then, the C_0-semigroup generated by the operator $\mathcal{A}$ from Section 4.2 is analytic if and only if $\rho(x) > 0$ for all $x \in \overline{\Omega}$.*

Proof. First, assume that $\rho(x_0) = 0$ for some $x_0 \in \overline{\Omega}$. Using notation from Chapter 2, Section 2.1, we have

$$X_2 = H^2(\Omega) \times L_2(\Omega), \quad Y_2 = H^4(\Omega) \times H^2(\Omega),$$

i.e. $s_1 = 2, s_2 = 0$ and $t_1 = t_2 = 2$. Hence, the principle symbol A_0 of $\mathcal{A}$ is given by

$$A_0(x, \xi) = \begin{pmatrix} 0 & 1 \\ -|\xi|^4 & -\rho(x)|\xi|^2 \end{pmatrix} \quad (x \in \overline{\Omega},\, \xi \in \mathbb{R}^n \setminus \{0\}).$$

Calculating

$$\det(\lambda - A_0(x_0, \xi)) = \lambda^2 + \lambda\rho(x_0)|\xi|^2 + |\xi|^4 = \lambda^2 + |\xi|^4,$$

it follows from Corollary 2.5 that the semigroup generated by $\mathcal{A}$ is not analytic.

Now, assume that $\rho > 0$ on $\overline{\Omega}$. Since ρ is, in particular, continuous and $\overline{\Omega}$ is compact, we can define $\rho_1 := \min_{x\in\overline{\Omega}} \rho(x) > 0$, i.e. $\rho(x) \geq \rho_1$ holds for all $x \in \overline{\Omega}$.
In order to show that $\mathcal{A}$ is the generator of an analytic C_0-semigroup, we write $\mathcal{A}$ as

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sum_{i=1}^n (\partial_i\rho)\partial_i \end{pmatrix} =: \mathcal{B} + \mathcal{L}.$$

In the following, we will show that $\mathcal{B}\colon \mathcal{H} \supset D(\mathcal{B}) := D(\mathcal{A}) \to \mathcal{H}$ is the generator of an analytic C_0-semigroup on the Hilbert space $\mathcal{H} = H_0^2(\Omega) \times L_2(\Omega)$ and that $\mathcal{L}$ is a admissible perturbation (in terms of analytic semigroups) for the operator $\mathcal{B}$. The latter is easy to see: consider $\mathcal{L}$ as an operator, for example let

$$\mathcal{L}\colon \mathcal{H} \supset D(\mathcal{L}) := L_2(\Omega) \times H_0^1(\Omega) \supset D(\mathcal{B}) \to \mathcal{H}.$$

Then, using interpolation theory, for any $\varepsilon > 0$ we can estimate

$$\|\mathcal{L}U\|_{\mathcal{H}} = \left\| \sum_{i=1}^n (\partial_i\rho)\partial_i u_2 \right\|_{L_2(\Omega)} \leq C\|\nabla u_2\|_{L_2(\Omega)} \leq \varepsilon\|\mathcal{B}U\|_{\mathcal{H}} + C\|U\|_{\mathcal{H}}$$

for all $U = (u_1, u_2)^\top \in D(\mathcal{B}) = D(\mathcal{A})$ with some $C > 0$ independent of U. Theorem 2.1 in Section 3.2 of [Paz83] states that if $\mathcal{B}$ is the generator of an analytic C_0-semigroup on $\mathcal{H}$, so is $\mathcal{A} = \mathcal{B} + \mathcal{L}$.
We already know from Theorem 4.12 that the operator

$$\mathcal{B}_\mu\colon \mathcal{H} \supset D(\mathcal{B}_1) := D(\mathcal{A}) \to \mathcal{H}, \quad U = (u_1, u_2)^\top \mapsto B_\mu(D)U := \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \mu\Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

is the generator of an analytic C_0-semigroup on $\mathcal{H}$ for any real number $\mu > 0$. Now, the assertion that $\mathcal{B}$ is the generator of an analytic C_0-semigroup follows in the usual way using a partition of unity and the perturbation result Proposition 2.16 together with the remarks at the end of Chapter 2, Section 2.2. □

We have seen in Proposition 4.31 that the operator $\mathcal{A}$ from Section 4.1 is not analytic since an appropriate resolvent estimate is not satisfied. However, the following numerical simulation suggests that also the condition $\rho(\mathcal{A}) \supset \Sigma_{\pi/2+\varepsilon}$ is violated for any $\varepsilon > 0$.
We consider the one-dimensional resolvent problem for $\mathcal{A}$, where the system of partial differential equations in fact becomes a system of ordinary differential equations. Let $\Omega = (-1, 1) \subset \mathbb{R}$,

$\Omega_1 = (0,1)$ and $\Omega_2 = (-1,0)$. The transmission interface is then given by $\Gamma = \{0\}$. Note that Ω_2 has an additional boundary $\Gamma_2 = \{-1\}$ which clearly satisfies the geometrical condition given in (4.56). Now, after using reflection on $\Gamma = \{0\}$, we obtain a boundary value problem for a system of ordinary differential equations. The unique solvability of such a problem can be investigated using the *characteristic determinant*, see Theorem A.32.
In terms of Theorem A.32, we obtain

$$F = F(\lambda) = \left(\begin{array}{cccc|cccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\lambda^2 & 0 & \lambda\rho & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\lambda^2 & 0 & 0 & 0 \end{array}\right), \; B = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

and

$$A = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\lambda\rho & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

We intend to plot the spectrum of the transmission problem for the resolvent of $\mathcal{A}$. By Theorem A.32, the boundary value problem

$$y'(t) = F(\lambda)y(t), \quad Ay(0) + By(1) = c \qquad (t \in \mathbb{R})$$

is uniquely solvable for $c \in \mathbb{C}^8$, if and only if the characteristic matrix

$$C_Y := AY(0) + BY(1)$$

is invertible. Using MATLAB, we are looking for $\lambda \in \mathbb{C}$ such that this is not the case. Hence, we plot the characteristic determinant (or, to be precise, the logarithm $\ln \det C_Y(\lambda)$ of $\det C_Y(\lambda)$ due to the large values of the latter expression) in dependence of $\lambda \in \mathbb{C}$, see Figure 4.6.
Plotting the roots of the characteristic determinant (i.e. the small holes one can see in Figure 4.6), we obtain Figure 4.7. As the roots seem to lie on a curve which increases superlinearly, this picture indeed suggests that the condition $\rho(\mathcal{A}) \supset \Sigma_{\pi/2+\varepsilon}$ is violated for any $\varepsilon > 0$.

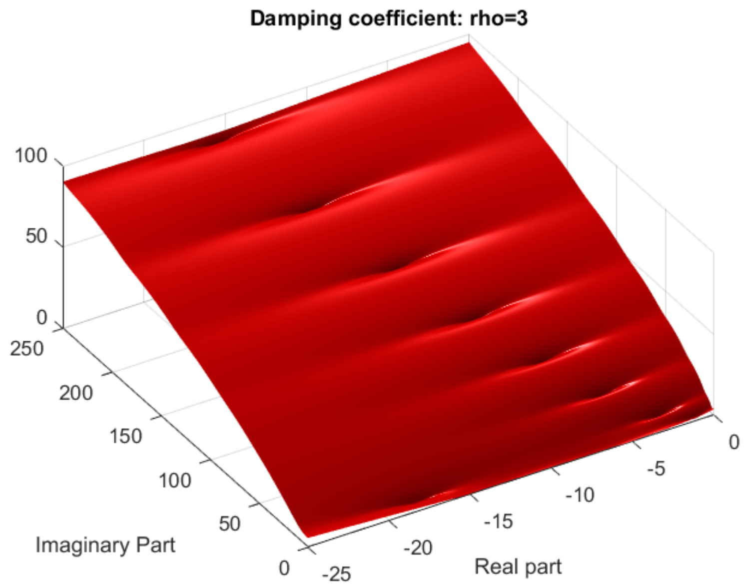


Figure 4.6.: Plot of the characteristic determinant for the 1-dimensional transmission problem.

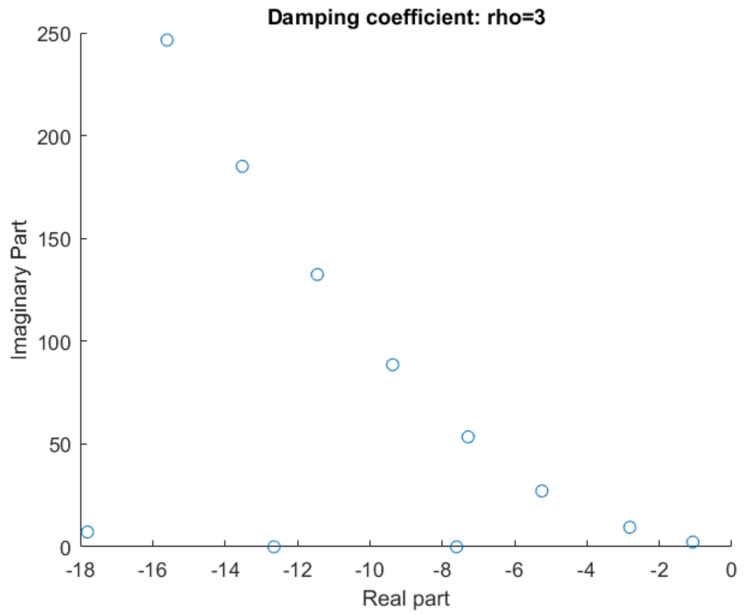


Figure 4.7.: Plot of the roots of the characteristic determinant for the 1-dimensional transmission problem.

Appendix

For the readers convenience, in this appendix, we will state some definitions and results on Sobolev spaces, Fourier multipliers, $\mathcal{R}$-boundedness and maximal L_p-regularity. Moreover, we state a result on boundary value problems for ordinary differential equations, which was used to visualize spectrum of the damped-undamped plate equation in Chapter 4, Section 4.3.

A.1. Sobolev spaces

In this section we introduce Sobolev, Besov and Bessel potential spaces and cite embedding, interpolation and trace theorems for these spaces. For details and proof in the scalar valued case, see [AF03], [BL76], [Tri78] and [Tri83]. Many well known theorems can be generalized to the vector valued case. We mention [Ama00] and [SSS12] for such generalizations. For some results in the vector valued case, it is necessary that E is of class $\mathcal{HT}$ which is equivalent to E being an UMD space (see [Ama95]).
Many theorems in the following are stated for the full and half space case. But for sufficiently smooth domains $\Omega \subset \mathbb{R}^n$ one obtains corresponding assertions with the help of retractions and coretractions. Again, we mention [Ama00].

Let $\Omega \subset \mathbb{R}^n$ be open, E be a $\mathbb{C}$-Banach space and $1 \leq p \leq \infty$. As usually, for real Banach spaces we use the complexification (cf. [Ama95]).
As usual, we use multi-index notation, i.e. we define

$$\begin{aligned} \partial^\alpha &:= \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \\ x^\alpha &:= x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \\ |\alpha| &:= \alpha_1 + \ldots + \alpha_n, \end{aligned}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

$\mathscr{D}(\Omega, E) = C_0^\infty(\Omega, E)$ denotes the space of all E valued *test functions.*

$$\mathscr{D}'(\Omega, E) := L(\mathscr{D}(\Omega, \mathbb{C}), E)$$

denotes the space of E valued *distributions* on Ω. We define the space of E valued *Schwartz functions* $\mathscr{S}(\mathbb{R}^n, E)$ as the space of all smooth functions $f \in C^\infty(\mathbb{R}^n, E)$ such that

$$\sup_{x \in \mathbb{R}^n} |(1 + |x|^2)^{k/2} \partial^\alpha f(x)| < \infty$$

for all $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$. Then, similarly as above, we let $\mathscr{S}'(\Omega, E)$ denote the space of E valued *tempered distributions.*

For $m \in \mathbb{N}_0$ we define the *Sobolev space* $W_p^m(\Omega, E)$ by

$$W_p^m(\Omega, E) := \{u \in L_p(\Omega, E) : \partial^\alpha u \in L^p(\Omega, E),\ |\alpha| \le m\}.$$

Here, $\partial^\alpha u = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} u$ is the distributional derivative of $u \in L_p(\Omega, E)$. Endowed with the norm

$$\|u\|_{W_p^m} := \left(\sum_{|\alpha| \le m} \|\partial^\alpha u\|_p^p \right)^{1/p},$$

$W_p^m(\Omega, E)$ is a Banach space.

$BUC^m(\Omega, E)$ is the closed subspace of $W_\infty^m(\Omega, E)$ of all functions u for which $\partial^\alpha u$ for all $|\alpha| \le m$ is bounded and uniformly continuous.

For $0 < \theta < 1$ let

$$I_{\theta,p}(u) := \begin{cases} \left(\int_{\Omega\times\Omega} \frac{\|u(x)-u(y)\|_E^p}{|x-y|^{n+\theta p}}\, d(x,y) \right)^{1/p}, & p < \infty \\ \sup_{\substack{x,y\in\Omega,\\ x\neq y}} \frac{\|u(x)-u(y)\|_E}{|x-y|^\theta}, & p = \infty. \end{cases}$$

Then

$$W_p^{m+\theta}(\Omega, E) := \left\{ u \in W_p^m(\Omega, E) : \|u\|_{W_p^{m+\theta}} < \infty \right\}$$

is called *Sobolev-Slobodeckii space* with norm given by

$$\|u\|_{W_p^{m+\theta}} := \|u\|_{W_p^m} + \max_{|\alpha|=m} I_{\theta,p}(\partial^\alpha u).$$

With $BUC^{m+\theta}(\Omega, E) \subset BUC^m(\Omega, E)$ we denote the space of all functions $u \in BUC^m(\Omega, E)$ such that $\partial^\alpha u$ with $|\alpha| = m$ is Hölder continuous with exponent θ. Then

$$W_\infty^{m+\theta}(\Omega, E) = BUC^{m+\theta}(\Omega, E) \tag{A.1}$$

holds. For $m \in \mathbb{N}$ and $0 \le \theta < 1$ let $W_p^{-m+\theta}(\Omega, E)$ be the space of all E valued distributions on Ω that possess a representation of the form

$$u = \sum_{|\alpha| \le m} \partial^\alpha u_\alpha \tag{A.2}$$

with $u_\alpha \in W_p^\theta(\Omega, E)$. $W_p^{-m+\theta}(\Omega, E)$ is endowed with the norm

$$\|u\|_{W_p^{-m+\theta}} := \inf_{|\alpha| \le m} \|u_\alpha\|_{W_p^\theta},$$

where we take the infimum over all representations of u of the form (A.2). Using (A.1), one defines $BUC^{-m+\theta}(\Omega, E)$ in the same way. All these spaces are Banach spaces as well.

Now we introduce vector valued Bessel potential and Besov spaces. We fix $\psi \in \mathscr{D}(\mathbb{R}^n) := \mathscr{D}(\mathbb{R}^n, \mathbb{C})$ with $\psi(\xi) = 1$ for $|\xi| < 1$, and $\psi(\xi) = 0$ for $|\xi| \ge 2$. Define

$$\psi_k(\xi) := \psi(2^{-k}\xi) - \psi(2^{-k+1}\xi), \quad \xi \in \mathbb{R}^n$$

and $\psi_k(D) := \mathscr{F}^{-1}\psi_k\mathscr{F}$ for $k \in \mathbb{N}$, where $\mathscr{F}$ denotes the *Fourier transform* on $\mathscr{S}'(\mathbb{R}^n, E)$. Then the *Besov space* $B^s_{p,q}(\mathbb{R}^n, E)$ for $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, defined by

$$B^s_{p,q}(\mathbb{R}^n, E) := \{u \in \mathscr{S}'(\mathbb{R}^n, E) : \|u\|_{s,p,q} < \infty\}$$

with

$$\|u\|_{s,p,q} := \left\| \left(2^{sk} \|\psi_k(D)u\|_{L_p(\mathbb{R}^n,E)} \right)_{k\in\mathbb{N}} \right\|_{\ell_q},$$

is a Banach space. The spaces are independent of the choice of ψ; different choices of ψ lead to the same spaces with equivalent norms.
For $1 < p < \infty$ and $s \in \mathbb{R}$ the *Bessel potential space* $H^s_p(\mathbb{R}^n, E)$, defined by

$$H^s_p(\mathbb{R}^n, E) := \{u \in \mathscr{S}'(\mathbb{R}^n, E) : \|u\|_{H^s_p} < \infty\}$$

with

$$\|u\|_{H^s_p} := \left\| \mathscr{F}^{-1}(1 + |\xi|^2)^{s/2}\mathscr{F}u \right\|_{L_p(\mathbb{R}^n,E)},$$

is a Banach space.

A.1 Theorem (H = W, [SSS12], Remark 2.11)**.** *Let $1 < p < \infty$. Then we have*

$$H^m_p(\mathbb{R}^n, E) = W^m_p(\mathbb{R}^n, E), \qquad m \in \mathbb{N},$$

if and only if E is of class $\mathcal{HT}$.

In the following we write $W^s_p := W^s_p(\mathbb{R}^n, E), B^s_{p,q} := B^s_{p,q}(\mathbb{R}^n, E)$ and $BUC^s := BUC^s(\mathbb{R}^n, E)$. Furthermore, write $\mathscr{S} := \mathscr{S}(\mathbb{R}^n, E)$ and $\mathscr{S}' := \mathscr{S}'(\mathbb{R}^n, E)$.

A.2 Theorem. *Let $0 < \theta < 1$. For $s_0, s_1 \in \mathbb{R}$ satisfying $s_0 \neq s_1$ we define $s_\theta := (1-\theta)s_0 + \theta s_1$. Then the following assertions hold.*

a)

$$\mathscr{S} \hookrightarrow B^{s_1}_{p,q_1} \hookrightarrow B^{s_0}_{p,q_2} \hookrightarrow \mathscr{S}', \qquad s_1 > s_0,$$

and

$$B^s_{p,q_0} \hookrightarrow B^s_{p,q_1}, \qquad q_0 < q_1.$$

b)

$$B^{s_1}_{p_1,q} \hookrightarrow B^{s_0}_{p_0,q}, \qquad s_1 > s_0,\ s_1 - n/p_1 = s_0 - n/p_0.$$

c) Besov spaces are compatible with real interpolation:

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} = B^{s_\theta}_{p,q}, \qquad s_0 \neq s_1.$$

d) For $s \in \mathbb{R}\backslash\mathbb{Z}$ we have

$$W^s_p = B^s_{p,p}$$

and for $m \in \mathbb{Z}$ we have

$$B^m_{p,1} \hookrightarrow W^m_p \hookrightarrow B^m_{p,\infty}, \qquad p < \infty.$$

Furthermore, $B^m_{p,p} \neq W^m_p$, unless $p = 2$ and E is a Hilbert space.

e) *Real interpolation spaces of Sobolev-Slobodeckii spaces are Besov spaces:*

$$(W_p^{s_0}, W_p^{s_1})_{\theta,q} = B_{p,q}^{s_\theta}, \quad s_0 \neq s_1,\, p < \infty.$$

f) *For $m \in \mathbb{Z}$*

$$B_{\infty,1}^m \hookrightarrow BUC^m \hookrightarrow B_{\infty,\infty}^m$$

holds.

g) *Real interpolation spaces of BUC^s are Besov spaces as well:*

$$(BUC^{s_0}, BUC^{s_1})_{\theta,q} = B_{\infty,q}^{s_\theta}.$$

A.3 Theorem (Trace theorem)**.** *Let $1 < p < \infty$. Then there exists a continuous operator*

$$\operatorname{tr} \in L(W_p^m(\mathbb{R}^n, E), B_{p,p}^{m-1/p}(\mathbb{R}^{n-1}, E))$$

satisfying $\operatorname{tr} u = u|_{\mathbb{R}^{n-1}}$ *for $u \in C(\mathbb{R}^n, E)$. For $s > 1/p$*

$$\operatorname{tr} \in L(H_p^s(\mathbb{R}^n, E), B_{p,p}^{s-1/p}(\mathbb{R}^{n-1}, E))$$

holds true. The operator tr *is called trace operator.*

Now we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary. Then $B_{p,q}^s(\Omega, E)$ is defined by means of the restriction $\mathbb{R}^n \to \overline{\Omega}$. Let

$$r_{\overline{\Omega}} \in L(C(\mathbb{R}^n, E), C(\overline{\Omega}, E))$$

denote the point wise restriction of continuous functions given by $u \mapsto u|_{\overline{\Omega}}$. The distributional restriction $r_\Omega \in L(\mathscr{D}'(\mathbb{R}^n, E), \mathscr{D}'(\Omega, E))$ is given by

$$r_\Omega u(\phi) := u(\phi), \quad u \in \mathscr{D}'(\mathbb{R}^n, E),\, \phi \in \mathscr{D}(\Omega).$$

Set

$$B_{p,q}^s(\Omega, E) := r_\Omega B_{p,q}^s(\mathbb{R}^n, E).$$

Similarly one defines $H_p^s(\Omega, E)$. Then the assertions from above still hold true when we replace $\mathbb{R}^n$ by Ω and $\mathbb{R}^{n-1}$ by $\partial\Omega$, respectively.

A.4 Theorem (Rellich-Kondrachov)**.** *Let E_0 and E_1 be Banach spaces such that E_1 is compactly embedded in E_0. For $s_1 > s_0$ and $s_1 - n/p_1 > s_0 - n/p_0$ we have that the embedding*

$$W_{p_1}^{s_1}(\Omega, E_1) \hookrightarrow W_{p_0}^{s_0}(\Omega, E_0)$$

is well-defined and compact. If $m \in \mathbb{N}_0$ and $s \in \mathbb{R}$ such that $s - n/p > m$ then the embedding

$$W_p^s(\Omega, E_1) \hookrightarrow C^m(\overline{\Omega}, E_0)$$

is well-defined and compact.

In the following, $\mathbb{R}^n_+$ denotes the positive half space in $\mathbb{R}^n$, i.e.

$$\mathbb{R}^n_+ := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

Similarly, $\mathbb{R}^n_- := -\mathbb{R}^n_+$.

A.5 Lemma. *Let $1 < p < \infty, k \in \mathbb{N}, u_1 \in W^k_p(\mathbb{R}^n_+)$ and $u_2 \in W^k_p(\mathbb{R}^n_-)$. Then the connected function*

$$u(x) := \begin{cases} u_1(x), & x \in \mathbb{R}^n_+, \\ u_2(x), & x \in \mathbb{R}^n_- \end{cases}$$

satisfies $u \in W^k_p(\mathbb{R}^n)$ if and only if

$$\partial^l_n u_1 = \partial^l_n u_2, \quad l = 0, \dots, k-1,$$

holds on $\partial\mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$. The same is true for Besov spaces $B^{k+s}_{pq}(\mathbb{R}^n)$ with $1 \le p, q < \infty$ and for Bessel potential spaces $H^{k+s}_p(\mathbb{R}^n)$ with $0 \le s < 1/p$.

A proof can be found in [Ama09], Theorem 4.7.3.

A.2. Fourier multipliers and $\mathcal{R}$-bounded operator families

For a detailed discussion of these topics we refer, for example, to [Wei01], [KW04] and [DHP03].

A.6 Definition. Let $1 \le p \le \infty$. A function $m \in L_\infty(\mathbb{R}^n)$ is called *Fourier multiplier* in L_p, if $f \mapsto \mathscr{F}^{-1} m \mathscr{F} f$ defines a bounded linear operator $M \in L(L_p(\mathbb{R}^n))$. Precisely: m is a Fourier multiplier, if the mapping

$$\widetilde{M} \colon \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n), \, \varphi \mapsto \mathscr{F}^{-1} m \mathscr{F} \varphi$$

satisfies $R(\widetilde{M}) \subset L_p(\mathbb{R}^n)$ and

$$\|\widetilde{M}\varphi\|_{L_p(\mathbb{R}^n)} \le C \|\varphi\|_{L_p(\mathbb{R}^n)} \quad (\varphi \in \mathscr{S}(\mathbb{R}^n)),$$

i.e. $\widetilde{M}$ can be uniquely extended to $M \in L(L_p(\mathbb{R}^n))$. In this case, we call m the *symbol* of the operator M. We write $\mathrm{op}(m) := \mathscr{F}^{-1} m \mathscr{F} := M$ and $\mathrm{symb}(M) := m$.

A.7 Remark. By Plancherel's theorem, we have $\mathrm{op}(m) \in L(L_2(\mathbb{R}^n))$ if and only if $g \mapsto mg$ is continuous in $L_2(\mathbb{R}^n)$. This is equivalent to $m \in L_\infty(\mathbb{R}^n)$. Indeed: Suppose that $m \in L_\infty(\mathbb{R}^n)$. Then we have $\|mg\|_{L_2(\mathbb{R}^n)} \le \|m\|_{L_\infty(\mathbb{R}^n)} \|g\|_{L_2(\mathbb{R}^n)}$. On the other hand, if $m \notin L_\infty(\mathbb{R}^n)$, then there exists a sequence $(A_k)_{k\in\mathbb{N}}$ of measurable sets with $0 < \lambda(A_k) < \infty$ and $(c_k)_{k\in\mathbb{N}} \subset [0, \infty)$ with $c_k \to \infty$ for $k \to \infty$ such that $|m| \ge c_k$ on A_k. For $g_k := \chi_{A_k} \in L_2(\mathbb{R}^n)$ we obtain

$$\|mg_k\|_{L_2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |m(\xi) g_k(\xi)|^2 \, d\xi \ge c_k^2 \lambda(A_k) = c_k^2 \|g_k\|^2_{L_2(\mathbb{R}^n)},$$

i.e. $\mathrm{op}(m)$ is not bounded in $L_2(\mathbb{R}^n)$. Here, χ_{A_k} is the indicator function on A_k.

For $n \in \mathbb{N}$ we denote with $\lfloor \frac{n}{2} \rfloor$ the largest integer less or equal $\frac{n}{2}$.

A.8 Theorem (Mikhlin). *Let $1 < p < \infty$ and $m\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be a function. If one of the following conditions*

(*i*) $m \in C^{\lfloor \frac{n}{2} \rfloor + 1}(\mathbb{R}^n \setminus \{0\})$ *and*

$$|\xi|^{|\beta|}|D^\beta m(\xi)| \le C_M \quad \left(\xi \in \mathbb{R}^n \setminus \{0\}, |\beta| \le \left\lfloor \frac{n}{2} \right\rfloor + 1\right),$$

(*ii*) $m \in C^n(\mathbb{R}^n \setminus \{0\})$ *and*

$$|\xi^\beta D^\beta m(\xi)| \le C_M \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0,1\}^n)$$

holds with a constant $C_M > 0$, then m is a L_p Fourier multiplier satisfying

$$\| \operatorname{op}(m)\|_{L(L_p(\mathbb{R}^n))} \le c(n,p)C_M$$

and the constant $c(n,p)$ does only depend on n and p.

A.9 Remark. a) Let $m\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ be homogeneous in ξ of degree $d \in \mathbb{R}$, i.e.

$$m(\rho\xi) = \rho^d m(\xi) \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \rho > 0).$$

If $m \in C^k(\mathbb{R}^n \setminus \{0\})$ for some $k \in \mathbb{N}$ then $\mathscr{D}^\beta m$ is homogeneous in ξ of degree $d - |\beta|$ for all $|\beta| \le k$. To see this, let $\beta = (1, 0, \dots, 0) \in \mathbb{N}_0^n$. Then we have

$$\begin{aligned}(\partial_{\xi_1} m)(\rho\xi) &= \lim_{h\to 0} \frac{m(\rho\xi + he_1) - m(\rho\xi)}{h} = \lim_{h\to 0} \rho^d \frac{m(\xi + \frac{h}{\rho}e_1) - m(\xi)}{\rho\frac{h}{\rho}} \\ &= \rho^{d-1}\partial_{\xi_1} m(\xi).\end{aligned}$$

Now the assertion for arbitrary β follows iteratively.

b) Suppose that $m \in C^{\lfloor \frac{n}{2} \rfloor + 1}(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree 0. Then m satisfies the Mikhlin condtion. Part a) of this remark states that $m_\beta(\xi) := |\xi|^{|\beta|} D^\beta m(\xi)$ is homogeneous of degree 0 for $|\beta| \le \lfloor \frac{n}{2} \rfloor + 1$. This implies

$$|m_\beta(\xi)| = \left| m_\beta\left(\frac{\xi}{|\xi|}\right)\right| \le \max_{|\eta|=1} |m_\beta(\eta)| < \infty \quad (\xi \in \mathbb{R}^n \setminus \{0\}).$$

We now introduce the definition of $\mathcal{R}$-boundedness and state some properties of $\mathcal{R}$-bounded operator families.

A.10 Definition. For $n \in \mathbb{N}$ we define $r_n\colon [0,1] \to \{-1, 1\}$ by

$$r_n(t) := \operatorname{sign}\sin(2^n \pi t) \quad (t \in [0,1]).$$

The functions r_n $(n \in \mathbb{N})$ are called *Rademacher functions.*

A.11 Remark. It is easy to verify that

$$\int_0^1 r_n(t) r_m(t)\, dt = \delta_{nm} \quad (n, m \in \mathbb{N}),$$

where δ denotes the Kronecker delta. Moreover, we have that

$$\lambda\left(\{t \in [0,1] : r_{n_1}(t) = z_1, \dots r_{n_M}(t) = z_M\}\right) = \frac{1}{2^M} = \prod_{j=1}^{M} \lambda\left(\{t \in [0,1] : r_{n_j}(t) = z_j\}\right),$$

where λ denotes the (here one-dimensional) *Lebesgue measure*. Hence, $(r_n)_{n\in\mathbb{N}}$ is a sequence of independent, identically distributed, symmetric random variables on the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$.

A.12 Definition. A familiy of bounded linear operators $\mathcal{T} \subset L(X,Y)$ is called $\mathcal{R}$-*bounded*, if there exists a constant $C > 0$ and $p \in [1, \infty)$ such that for all $N \in \mathbb{N}, T_j \in \mathcal{T}$ and $x_j \in X$ the inequality

$$\Big\| \sum_{j=1}^{N} r_j T_j x_j \Big\|_{L_p([0,1],Y)} \le C \Big\| \sum_{j=1}^{N} r_j x_j \Big\|_{L_p([0,1],X)} \tag{A.3}$$

holds true. The smallest possible C is called the $\mathcal{R}$-*bound* of $\mathcal{T}$ and we write $\mathcal{R}(\mathcal{T}) = C$.

In fact, the definition of $\mathcal{R}$-boundedness is independent of $p \in [1, \infty)$:

A.13 Theorem (Kahane's inequality)**.** *Let $1 \le p < \infty$. Then there exists a constant $C_p > 0$ sucht that*

$$\frac{1}{C_p} \Big\| \sum_{j=1}^{N} r_j x_j \Big\|_{L_2([0,1],X)} \le \Big\| \sum_{j=1}^{N} r_j x_j \Big\|_{L_p([0,1],X)} \le C_p \Big\| \sum_{j=1}^{N} r_j x_j \Big\|_{L_2([0,1],X)}$$

is valid for each $N \in \mathbb{N}$ and $x_j \in X$.

A.14 Corollary. *If the condition* (A.3) *in Definition A.12 is satisfied for one $p \in [1, \infty)$, then it is satisfied for all $p \in [1, \infty)$. The corresponding $\mathcal{R}$-bounds satisfy*

$$\frac{1}{C_p^2} \mathcal{R}_2(\mathcal{T}) \le \mathcal{R}_p(\mathcal{T}) \le C_p^2 \mathcal{R}_2(\mathcal{T}).$$

A.15 Remark. a) If $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded, then $\mathcal{T}$ is uniformly bounded with

$$\sup_{T\in\mathcal{T}} \|T\|_{L(X,Y)} \le \mathcal{R}(\mathcal{T}).$$

This follows directly from Definition A.12 with $N = 1$.

b) If X and Y are Hilbert spaces, then $\mathcal{T} \subset L(X,Y)$ is $\mathcal{R}$-bounded if and only if $\mathcal{T}$ is uniformly bounded. Indeed: In this case, $L_2([0,1], X)$ and $L_2([0,1], Y)$ are Hilbert spaces as well. Moreover, $(r_n x_n)_{n\in\mathbb{N}} \subset L_2([0,1], X)$ and $(r_n T_n x_n)_{n\in\mathbb{N}} \subset L_2([0,1], Y)$ are orthogonal sequences. Now we assume that $\mathcal{T}$ is uniformly bounded, i.e. there exists a constant $C_{\mathcal{T}}$ such that $\|T\|_{L(X,Y)} \leq C_{\mathcal{T}}$ for all $T \in \mathcal{T}$. We obtain

$$\Big\| \sum_{n=1}^{N} r_n T_n x_n \Big\|^2_{L_2([0,1],Y)} = \sum_{n=1}^{N} \|r_n T_n x_n\|^2_{L_2([0,1],Y)} = \sum_{n=1}^{N} \|T_n x_n\|^2_Y$$
$$\leq C_{\mathcal{T}}^2 \sum_{n=1}^{N} \|x_n\|_X^2 = C_{\mathcal{T}}^2 \Big\| \sum_{n=1}^{N} r_n x_n \Big\|^2_{L_2([0,1],X)}.$$

A.16 Lemma. *Let X, Y, Z be Banach spaces and $\mathcal{T}, \mathcal{S} \subset L(X,Y)$ and $\mathcal{U} \subset L(Y,Z)$ be $\mathcal{R}$-bounded. Then*

$$\mathcal{T} + \mathcal{S} := \{T + S : T \in \mathcal{T}, S \in \mathcal{S}\}$$

and

$$\mathcal{U}\mathcal{T} := \{UT : U \in \mathcal{U}, T \in \mathcal{T}\}$$

are also $\mathcal{R}$-bounded with

$$\mathcal{R}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}(\mathcal{T}) + \mathcal{R}(\mathcal{S}),\ \mathcal{R}(\mathcal{U}\mathcal{T}) \leq \mathcal{R}(\mathcal{U})\mathcal{R}(\mathcal{T}).$$

A.17 Lemma (Contraction principle of Kahane)**.** *Let $1 \leq p < \infty, N \in \mathbb{N}$ and $x_j \in X$ for $j = 1, \dots, N$. Furthermore, let $a_j, b_j \in \mathbb{C}$ with $|a_j| \leq |b_j|$ for $j = 1, \dots, N$. Then*

$$\Big\| \sum_{j=1}^{N} a_j r_j x_j \Big\|_{L_p([0,1],X)} \leq 2 \Big\| \sum_{j=1}^{N} b_j r_j x_j \Big\|_{L_p([0,1],X)}.$$

The following lemma gives a sufficient condition when a family of homogeneous symbols does not only lead Fourier multipliers but also to an $\mathcal{R}$-bounded family of operators.

A.18 Lemma (see [Fro16], Lemma 2.7)**.** *Let $\omega \in (0, \pi]$ and $\gamma \in \mathbb{N}_0^2$. Further, let $b\colon (\mathbb{R}^n \times \overline{\Sigma}_\omega) \setminus \{0\} \to \mathbb{C}$ be infinitely often differentiable and positive homogeneous in $(\xi, \lambda^{1/2})$ of degree 0, i.e. $b(r\xi, r^2\lambda) = b(\xi, \lambda)$ for any $r > 0$, $\xi \in \mathbb{R}^n$ and $\lambda \in \Sigma_\omega$. Then, the following assertions hold true:*

(a) For all $\lambda \in \Sigma_\omega$ we have that $\partial_\lambda^\gamma b(\cdot, \lambda)$ is a L_p Fourier multiplier. Moreover, the mapping

$$\Sigma_\omega \ni \lambda \mapsto \mathscr{F}^{-1} b(\cdot, \lambda) \mathscr{F} \in L(L_p(\mathbb{R}^n))$$

is infinitely often partially differentiable with

$$\partial_\lambda^\gamma \mathscr{F}^{-1} b(\cdot, \lambda) \mathscr{F} = \mathscr{F}^{-1} \partial_\lambda^\gamma b(\cdot, \lambda) \mathscr{F}.$$

(b) There exists a constant $C = C(\gamma, \omega, n, p, b) > 0$ such that

$$\|\lambda^\gamma \partial_\lambda^\gamma \mathscr{F}^{-1} b(\cdot, \lambda) \mathscr{F}\|_{L(L_p(\mathbb{R}^n))} \leq C$$

for all $\lambda \in \Sigma_\omega$.

(c) The family

$$\left\{\lambda^\gamma \partial_\lambda^\gamma \mathscr{F}^{-1} b(\cdot, \lambda) \mathscr{F} : \lambda \in \Sigma_\omega\right\} \subset L(L_p(\mathbb{R}^n))$$

is $\mathcal{R}$-bounded.

A.19 Remark. The homogenity in Lemma A.18 can be replaced as follows:
Let $m\colon (\mathbb{R}^n \times \overline{\Sigma}_\omega) \setminus \{0\} \to \mathbb{C}$, $(\xi, \lambda) \mapsto m(\xi, \lambda)$ be a symbol which has a bounded holomorphic extension $\widetilde{m}\colon (\Sigma_\phi)^n \times \Sigma_{\omega'} \to \mathbb{C}$, $(z, \lambda) \mapsto \widetilde{m}(z, \lambda)$ with $\phi > 0$ and $\omega' > \omega$. By Cauchy's integral formula we have that

$$\partial_\lambda^k \widetilde{m}(z, \lambda) = \frac{1}{2\pi \mathrm{i} k!} \int_{\gamma_{\lambda,\varepsilon}} \widetilde{m}(z, \mu)(\lambda - \mu)^{-k-1} \, d\mu$$

for $k \in \mathbb{N}$. Here, $\gamma_{\lambda,\varepsilon}$ denotes the positively oriented circle with center λ and radius $\frac{1}{2}\sin(\varepsilon)|\lambda|$ with $\varepsilon = \omega' - \omega$. By assumtion, $\widetilde{m}(z, \lambda)$ is bounded for $\mu \in \gamma_{\lambda,\varepsilon}$. Thus, we get that $\lambda^k \partial_\lambda^k \widetilde{m}(z, \lambda)$ is bounded for all $\lambda \in \overline{\Sigma}_\omega \setminus \{0\}$ and $k \in \mathbb{N}$. In the same way, we treat derivatives with respect to z and obtain

$$|\lambda^\gamma \partial_\lambda^\gamma \xi^\alpha \partial_\xi^\alpha m(\xi, \lambda)| \leq C_{\gamma,\alpha} \quad (\lambda \in \overline{\Sigma}_\omega, \xi \in \mathbb{R}^n \setminus \{0\}).$$

Hence, $\lambda^\gamma \partial_\lambda^\gamma m(\cdot, \lambda)$ satisfies the Mikhlin condition and Theorem 3.2 in [GW03] shows that

$$\left\{ \lambda^\gamma \partial_\lambda^\gamma \mathscr{F}^{-1} m(\cdot, \lambda) \mathscr{F} : \lambda \in \overline{\Sigma}_\omega \right\} \subset L(L_p(\mathbb{R}^n))$$

is $\mathcal{R}$-bounded.

Finally, we will also be using a similar result for a symbol where Lemma A.18 does not apply:

A.20 Lemma (see [Fro16], Lemma 2.10)**.** *Let $n \geq 2, \varepsilon \in (0, \pi), s > 0$ and*

$$b(\xi, \lambda) := \frac{(\lambda + |\xi'|^2)^{s/2}}{(\lambda + |\xi|^2)^{s/2}}$$

for $\xi = (\xi', \xi_n) \in \mathbb{R}^n$ and $\lambda \in \Sigma_{\pi-\varepsilon}$. Then, the assertions of Lemma A.18 also hold for b. In part (b) of Lemma A.18, the constant now depends on $\gamma, \varepsilon, n, p$ and s.

A.3. Maximal L_p-regularity

The purpose of this section is motivating the definition of maximal L_p regularity and to show the connection between maximal regularity and vector-valued Fourier multipliers. Maximal regularity and vector-valued multipliers are closely connected to $\mathcal{R}$-boundedness of a certain operator family.
Let X be a (separable) Banach space. We consider the abstract Cauchy problem

$$\begin{aligned} \partial_t u - Au &= f, \quad t \in (0, T), \\ u(0) &= u_0 \end{aligned} \tag{A.4}$$

with a linear closed operator $A\colon X \supset D(A) \to X$. If we assume $f \in L_p((0, T), X)$, then we would require v to satisfy

$$\partial_t v \in L_p((0, T), X) \text{ and } v \in L_p((0, T), D(A)).$$

u_0 is assumed to be an element of the *trace space*

$$I_p(A) := \{x = u|_{t=0} : u \in \mathbb{E} := W_p^1((0, T), X) \cap L_p((0, T), D(A))\},$$

which is a Banach space endowed with the norm $\|x\|_{I_p(A)} := \inf\{\|u\|_{\mathbb{E}} : u|_{t=0} = x\}$. This space is an intermediate space of $D(A)$ and X in the sense that

$$D(A) \hookrightarrow I_p(A) \hookrightarrow X.$$

The space $I_p(A)$ coincides with the real interpolation space $(X, D(A))_{1-1/p,p}$. In the case of a differential operator of order $m \in \mathbb{N}$ we would set $X := L_p(\mathbb{R}^n), D(A) := W_p^m(\mathbb{R}^n)$. This leads to the trace space $(X, D(A))_{1-1/p,p} = (L_p(\mathbb{R}^n), W_p^m(\mathbb{R}^n))_{1-1/p,p} = B_{pp}^{m-m/p}(\mathbb{R}^n)$ (provided that $p > 1$), where $B_{pq}^s(\mathbb{R}^n)$ is the Besov space of order $s \in \mathbb{R}$. More details will be given in the lecture on Interpolation of Banach spaces.

A.21 Definition. Let $T \in (0, \infty], J := (0, T), 1 \leq p \leq \infty$ and $A\colon D(A) \to X$ be a closed linear operator. Then A has *maximal L_p regularity,* if for all $(f, u_0) \in L_p(J, X) \times I_p(A) =: \mathbb{F}$ there exists u solving (A.4) almost everywhere and if there exists a constant $C = C(J) > 0$ such that the estimate

$$\|\partial_t u\|_{L_p(J,X)} + \|Au\|_{L_p(J,X)} \leq C(\|f\|_{L_p(J,X)} + \|u_0\|_{I_p(A)}) \tag{A.5}$$

holds for all $(f, u_0) \in \mathbb{F}$. In this case we write $A \in MR_p(J, X)$ or $A \in MR_p(X)$ if $J = (0, \infty)$.

A.22 Remark. The definition A.21 requires u to satisfy $\partial_t u \in L_p(J, X)$. If J is finite or 0 belongs to the resolvent set of A, one can replace $\|\partial_t u\|_{L_p(J,X)}$ with $\|u\|_{W_p^1(J,X)}$ in (A.5). In this case, A has maximal regularity if and only if

$$\begin{pmatrix} \partial_t + A \\ \gamma_0 \end{pmatrix} : \mathbb{E} \to \mathbb{F}$$

is an isomorphism of Banach spaces. Here, γ_0 denotes the trace operator, i.e. $\gamma_0(u) = u(0)$.

In the following we assume u_0 to be zero. This can always be done. The functions u and f will be extended to the whole line $t \in \mathbb{R}$ by zero. We will apply Fourier transform with respect to time

$$(\mathscr{F}_t u)(\tau) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-\mathrm{i}t\tau} u(t)\, dt.$$

Note that

$$[\mathscr{F}_t(\partial_t u)](\tau) = \mathrm{i}\tau(\mathscr{F}_t u)(\tau).$$

Hence, applying temporal Fourier transform $\mathscr{F}_t$, we get

$$(\mathrm{i}\tau - A)(\mathscr{F}_t u)(\tau) = (\mathscr{F}_t f)(\tau).$$

For A having maximal regularity we need

$$\partial_t u = \mathscr{F}_t^{-1} \mathrm{i}\tau(\mathrm{i}\tau - A)^{-1} \mathscr{F}_t f \in L_p((0, \infty), X).$$

In fact, one can show the following theorem:

A.23 Theorem. *The operator A has maximal L_p regularity if and only if*

$$\mathscr{F}_t^{-1} \mathrm{i}\tau(\mathrm{i}\tau - A)^{-1} \mathscr{F}_t$$

defines a continuous operator in $L_p(\mathbb{R}, X)$.

A.4. Vector-valued Fourier multipliers

In order to prove maximal regularity for an operator A, it is crucial to find sufficient conditions for $m(\tau) := \mathrm{i}\tau(\mathrm{i}\tau - A)^{-1}$ being a Fourier multiplier in $L_p(\mathbb{R}, X)$. If $X \neq \mathbb{C}$ we cannot apply the Mikhlin Multiplier Theorem A.8. Therefore, we need will need other criteria if $X \neq \mathbb{C}$ is a more general Banach space.

To get such results, we first have to impose some restrictions (which are not that restrictive in applications) on our Banach space X.

A.24 Definition. a) The *Hilbert transform* $\mathcal{H}\colon \mathscr{S}(\mathbb{R}, X) \to \mathscr{S}'(\mathbb{R}, X)$ is defined by

$$\mathcal{H}f := \operatorname{op}(m)f \text{ with } m(\xi) := \frac{\mathrm{i}\xi}{|\xi|} \quad (\xi \in \mathbb{R} \setminus \{0\}).$$

b) The Banach space X is called *of class* $\mathcal{HT}$ if for some $p \in (1, \infty)$ we have $\mathcal{H} \in L(L_p(\mathbb{R}, X))$.

A.25 Remark. a) If b) in Definition A.24 holds for one $p \in (1, \infty)$, then it holds for all $p \in (1, \infty)$.

b) If X is a Hilbert space, then X is of class $\mathcal{HT}$, because $m \in L_\infty(\mathbb{R})$.

c) If X is of class $\mathcal{HT}$ then X is reflexive.

d) Assume that X is of class $\mathcal{HT}$ and let (Σ, μ) be a σ-finite measurable space, then $L_p(\Sigma, \mu, X)$ is of class $\mathcal{HT}$ for all $p \in (1, \infty)$.

e) If (X_0, X_1) is an interpolation pair such that X_0 and X_1 are both spaces of class $\mathcal{HT}$, then the real and complex interpolation spaces $[X_0, X_1]_\theta$ and $(E_0, E_1)_{\theta,p}$ are of class $\mathcal{HT}$ as well for every $\theta \in (0, 1)$ and all $p \in (1, \infty)$.

A.26 Theorem (Weis 2001)**.** *Let $p \in (1, \infty)$. Assume that X is of class $\mathcal{HT}$ and let $m \in C^n(\mathbb{R}^n \setminus \{0\}, L(X))$ with*

$$\mathcal{R}\left(\left\{\xi^\beta \partial_\xi^\beta m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta \in \{0, 1\}^n\right\}\right) < \infty.$$

Then m is a Fourier multiplier, i.e. $\operatorname{op}(m) \in L(L_p(\mathbb{R}^n, X))$.

Theorem A.26 provides a sufficient condition for m being a Fourier multiplier in $L_p(\mathbb{R}^n, X)$ and the corresponding operator is $\operatorname{op}(m)$ is bounded in $L_p(\mathbb{R}^n, X)$. Roughly speaking we have that

$\mathcal{R}$-bounded symbols lead to bounded operators.

This is not good enough: If we want to solve an evolution equation, it might be useful to use Fourier transform in time and in space separately. This leads to iteration of Fourier transform and hence we need a result of the type

$\mathcal{R}$-bounded symbols lead to $\mathcal{R}$-bounded operators.

For such results we need another geometric property of the Banach space X.

A.27 Definition. A Banach space X has *property* (α) if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}, \alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \leq 1$ and all $x_{ij} \in X$ we have

$$\int_0^1 \int_0^1 \Big\| \sum_{i,j=1}^N r_i(i) r_j(u) \alpha_{ij} x_{ij} \Big\|_X du\, dv \leq C \int_0^1 \int_0^1 \Big\| \sum_{i,j=1}^N r_i(u) r_j(v) x_{ij} \Big\|_X du\, dv.$$

As before, r_i, r_j are the Rademacher functions.

A.28 Remark. a) If $X = L_p(\Sigma, \mu)$ with a σ-finite measure μ, then X has property (α). Closed subspaces of $L_p(G, \mu)$ also have property (α).

b) If X has property (α) and μ is a σ-finite measure on G, then $L_p(G, \mu, X)$ has property (α).

A.29 Theorem. *Let X, Y be Banach spaces of class $\mathcal{HT}$ with property (α). Assume that $\mathcal{T} \subset L(X, Y)$ is $\mathcal{R}$-bounded. Consider the set*

$$M := \{m \in C^n(\mathbb{R}^n \setminus \{0\}, L(X, Y)) : \xi^\alpha D^\alpha m(\xi) \in \mathcal{T} \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \{0,1\}^n\}.$$

Then $\{\mathrm{op}(m) : m \in M\} \subset L(L_p(\mathbb{R}^n, X), L_p(\mathbb{R}^n, Y))$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound less or equal $C\mathcal{R}(\mathcal{T})$ with some constant $C > 0$ only depending on p, m, X and Y. Again we have set $\mathrm{op}(m) := \mathscr{F}^{-1} m \mathscr{F}$.

A proof can be found in [KW04]

A.30 Corollary. *Let $\{m_\lambda : \lambda \in \Lambda\}$ be a family of matrix-valued functions*

$$m_\lambda \in C^n\left(\mathbb{R}^n \setminus \{0\}, \mathbb{C}^{N\times N}\right)$$

such that

$$|\xi^\alpha D^\alpha m_\lambda(\xi)| \leq C_0 \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \{0,1\}^n, \lambda \in \Lambda).$$

Then $\{\mathrm{op}(m_\lambda) : \lambda \in \Lambda\} \subset L(L_p(\mathbb{R}^n, \mathbb{C}^N))$ is $\mathcal{R}$-bounded with $\mathcal{R}$-bound $C \cdot C_0$, where C only depends on p and N.

Proof. $X = \mathbb{C}^N$ is of class $\mathcal{HT}$ and has property (α). By assumtion,

$$\{\xi^\alpha D^\alpha m_\lambda(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \{0,1\}^n, \lambda \in \Lambda\} \subset L(X)$$

is bounded and since X is a Hilbert space, this family of operators is also $\mathcal{R}$-bounded. Now choose $\mathcal{T} := \{A \in \mathbb{C}^{N\times N} : |A| \leq C_0\}$ in Theorem A.29. □

The following summarizes the connection between maximal L_p regularity, vector-valued Fourier multipliers and $\mathcal{R}$-boundedness:

A.31 Theorem (Weis). *Let $1 < p < \infty$, let X be a Banach space of class $\mathcal{HT}$ and let A be a sectorial operator, i.e. A is closed, densely defined and it holds that*

$$\rho(A) \supset \overline{\mathbb{C}}_+, \ \|\lambda(\lambda - A)^{-1}\|_{L(X)} \leq M \quad (\lambda \in \overline{\mathbb{C}}_+)$$

for a constant $M > 0$. Here, $\overline{\mathbb{C}}_+ := \{z \in \mathbb{C} : \mathrm{Re}\, z \geq 0\}$ is the closed right half plane of the complex plane $\mathbb{C}$.
Then the following statements are equivalent:

(i) *A has maximal L_p regularity.*

(ii) *The $L(X)$-valued function $m(\tau) := \mathrm{i}\tau(\mathrm{i}\tau - A)^{-1}$ is a L_p Fourier multiplier.*

(iii) *The set $\{\lambda(\lambda - A)^{-1} : \mathrm{Re}\, z \geq 0\}$ is $\mathcal{R}$-bounded (i.e. A is $\mathcal{R}$-sectorial).*

A.5. Boundary value problems for ordinary differential equations

We consider the boundary value problem

$$\begin{aligned} y'(t) &= F(t)y(t) + h(t) \quad (t \in \mathbb{R}) \\ Ay(a) + By(b) &= c. \end{aligned} \tag{A.6}$$

Here, $a < b$, $F \in C([a,b];\mathbb{C}^{N\times N})$, $h \in C([a,b];\mathbb{C}^N)$, $A, B \in \mathbb{C}^{N\times N}$ and $c \in \mathbb{C}^N$. Let $Y \in C^1([a,b];\mathbb{C}^{N\times N})$ be a fundamental system of the homogeneous system, i.e.

$$Y'(t) = F(t)Y(t) \quad (t \in \mathbb{R}).$$

A.32 Theorem ([DR11])**.** *The boundary value problem* (A.6) *is uniquely solvable for all* $h \in C([a,b];\mathbb{C}^N)$ *and* $c \in \mathbb{C}^N$*, if and only if the characteristic matrix*

$$C_Y := AY(a) + BY(b)$$

is invertible. This holds true if and only if the homogeneous boundary value problem

$$y'(t) = F(t)y(t), \quad Ay(a) + By(b) = 0$$

only possesses the trivial solution $y = 0$.

Deutsche Zusammenfassung

Während die Auslenkung einer Platte, sprich eines elastischen Materials, welches eine vergleichsweise geringe Dicke aufweist, noch durch eine skalarwertige partielle Differentialgleichung, nämlich durch die *Plattengleichung*

$$u_{tt}(t,x) + \Delta^2 u(t,x) = f(t,x) \tag{Z.1}$$

beschrieben werden kann, werden für die Modellierung komplexerer physikalischer oder chemischer Phänomene Systeme von partiellen Differentialgleichungen benötigt.
Im Allgemeinen soll einerseits nicht nur ein skalarer Wert, wie in diesem Beispiel die Auslenkung der Platte, eines Phänomens beschrieben werden, sondern gleich eine ganze Ansammlung verschiedener Größen, wie beispielsweise dreidimensionaler Koordinaten im Raum oder die Konzentration einer chemischen Substanz; andererseits ist zu deren korrekten Beschreibung häufig auch die Modellierung weiterer, sich gegenseitig beeinflussender Größen nötig.

So werden Strömungen in Gasen und Flüssigkeiten, das heißt dreidimensionale Vektorfelder, welche die Bewegung des Stoffes an jedem Ort und zu jeder Zeit darstellen, durch Navier-Stokes-Gleichungen in Abhängigkeit vom herrschenden Druck, welcher sich durch die Strömung selbst ändert und diese im Gegenzug ebenso beeinflusst, beschrieben.
In einer Autobatterie finden chemische Reaktionen statt, d.h. es müssen Konzentrationen von Stoffen beschrieben werden, während man gleichzeitig an den entstehenden elektrischen Strömen interessiert ist. All diese Größen beeinflussen sich gegenseitig und werden jeweils durch verschiedene Gleichungen modelliert, welche gekoppelt ein ganzes System von partiellen Differentialgleichungen formen.

Im Beispiel der Plattengleichung kann etwa die Temperatur Einfluss auf die physikalischen Eigenschaften der Platte haben, während gleichzeitig die Auslenkung der Platte und dadurch entstehende Spannungen und Reibungen wiederum die Temperatur ändern können. Will man diese Effekte bei der Modellierung einer Platte, wie beispielsweise einer Brücke oder Teilen der Karosserie eines Automobils, berücksichtigen, ist die Gleichung ($Z.1$) hierfür nicht mehr ausreichend. Die zusätzliche Beschreibung der Temperatur führt auf sogenannte *thermoelastische Plattengleichungen*. In ihrer einfachsten Form ist die thermoelastische Platte durch (siehe [Lag89])

$$\begin{aligned} u_{tt}(t,x) + \Delta^2 u(t,x) + \Delta\theta(t,x) &= f(t,x), \\ \theta_t(t,x) - \Delta\theta(t,x) - \Delta u_t(t,x) &= g(t,x) \end{aligned} \tag{Z.2}$$

gegeben, einem System zweier partieller Differentialgleichungen, genauer gesagt einem gekoppelten System der Plattengleichung ($Z.1$) sowie der Wärmeleitungsgleichung. Hier beschreibt $u(t,x)$ die Auslenkung der Platte zur Zeit t am Ort x und $\theta(t,x)$ die Temperatur (beziehungsweise die

Differenz zu einer Referenztemperatur θ_0).

Aber auch durch die Kopplung zweier skalarer Gleichungen (oder natürlich auch von Systemen) an einer sogenannten Transmissionsschicht erhält man ein System partieller Differentialgleichungen, sogenannte *Transmissionsprobleme*. Hier treten in verschiedenen Teilbereichen der Geometrie Ω jeweils verschiedene Gleichungen (bzw. Systeme) auf, welche an den Transmissionsschichten, d.h. an den Übergängen von einem Teilbereich zum Nächsten, über gewisse *Transmissionsbedingungen* gekoppelt sind.
Eine Platte kann zum Beispiel materialbedingt an verschiedenen Stellen unterschiedliche physikalische Eigenschaften besitzen. Wir vernachlässigen die geringe Dicke der Platte an dieser Stelle und betrachten eine zweidimensionale Geometrie. Während in einem Teilgebiet $\Omega_2 \subset \Omega \subset \mathbb{R}^2$ die Platte beispielsweise durch die Gleichung $(Z.1)$ beschrieben wird, könnte im Rest Ω_1 des Gebietes die *strukturell gedämpfte Plattengleichung*

$$u_{tt}(t,x) + \Delta^2 u(t,x) - \rho \Delta u_t(t,x) = f(t,x) \qquad (Z.3)$$

mit dem Dämpfungsfaktor $\rho > 0$ eine passendere Beschreibung der Platte sein. Mithilfe von Transmissionsbedingungen an der Transmissionsschicht Γ sowie zusätzlicher Randbedingungen auf Γ_1 (vgl. Abbildung Abb.1 unten) führt eine solche Platte mit verschiedenen Eigenschaften ebenfalls auf ein System partieller Differentialgleichungen, denn hier werden die Auslenkungen zweier verschiedener Platten, welche sich gegenseitig beeinflussen, zeitgleich beschrieben.

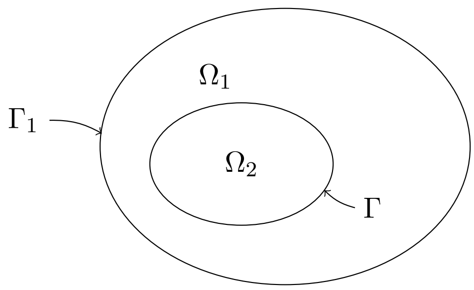


Abbildung Abb.1: *Geometrie Ω der Platte, aufgeteilt in zwei Teilgebiete Ω_1 und Ω_2, in welchen die Platte unterschiedliche Eigenschaften besitzt.*

Diese Beispiele verdeutlichen die Relevanz des Studiums von Systemen für die Anwendung in der Physik, der Chemie oder anderen Wissenschaften. Tatsächlich führen jedoch bereits skalare Gleichungen, wie die Plattengleichungen $(Z.1)$ und $(Z.3)$, mathematisch gesehen häufig auf Systeme partieller Differentialgleichungen: Ein Standardansatz für die Behandlung von *Evolutionsgleichungen*, d.h. zeitabhängiger partieller Differentialgleichungen, ist es, die Gleichung als gewöhnliche Differentialgleichung in der Zeit zu lesen, welche nun Werte in einem Funktionenraum X im Ort annimmt. In diesem Falle würde man also $(Z.3)$ in der Form

$$u_{tt}(t) + \Delta^2 u(t) - \rho \Delta u_t(t) = f(t) \qquad (Z.4)$$

lesen, wobei nun u, u_t, u_{tt} und f nicht mehr skalarwertige Funktionen, sondern für jedes $t \geq 0$ sind $u(t), u_t(t), u_{tt}(t)$ und $f(t)$ Elemente passender Funktionenräume.

In der Anwendung sind diese häufig L_p-Sobolevräume der entsprechenden Differenzierbarkeitsordnung, in diesem Falle also $u(t) \in H_p^4(\Omega), u_t(t) \in H_p^2(\Omega)$ sowie $u_{tt}(t), f(t) \in L_p(\Omega)$. Schreibt man nun $(Z.4)$ als System erster Ordnung in der Zeit

$$U_t(t) - \mathcal{A}U(t) = U_t(t) - \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \rho\Delta \end{pmatrix} U(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \qquad (Z.5)$$

mit $U = (u, u_t)^\top$, so erhält man zum Beispiel mit der *Halbgruppentheorie* (siehe [Paz83], [EN00]) ein mächtiges Werkzeug an die Hand, um das System $(Z.5)$ im Raum $X = H_p^2(\mathbb{R}^n) \times L_p(\mathbb{R}^n)$ und damit wiederum die ursprüngliche Gleichung $(Z.3)$ auf Wohlgestelltheit und Eigenschaften der Lösung zu untersuchen. Damit wurde das Studium der Plattengleichung auf das Studium der Eigenschaften des Matrixdifferentialoperators gemischter Ordnung

$$\mathcal{A}\colon X \supset D(\mathcal{A}) \to X, \quad U \mapsto \begin{pmatrix} 0 & 1 \\ -\Delta^2 & \Delta \end{pmatrix} U$$

mit geeignetem Definitionsbereich $D(\mathcal{A}) \subset X$ zurückgeführt. Dieses Vorgehen liefert für die thermoelastische Platte $(Z.2)$ demnach sogar ein 3×3 System gemischter Ordnung.
Von einem mathematischen Standpunkt aus betrachtet bilden Systeme (i.A. von gemischter Ordnung) aufgrund der zur Verfügung stehenden Theorien also selbst bei skalaren Gleichungen häufig eine natürliche Sichtweise. Folgerichtig spielen Systeme partieller Differentialgleichungen mit ihren Rand- und Transmissionsbedingungen eine wichtige Rolle in der Analysis und sind elementarer Gegenstand der Forschung.

Der Hauptteil dieser Arbeit befasst sich mit zwei wesentlichen Fragestellungen im Bezug auf die schon erwähnten Transmissionsprobleme, welche anhand von Plattengleichungen untersucht werden. Abgesehen von ihrer anwendungstechnischen Relevanz dienen solche Systeme auch, wie oben angedeutet, als nützlicher mathematischer Prototyp für Systeme gemischter Ordnung.

Die erste Frage hat ihren Ursprung in der Theorie *parameter-elliptischer Randwertprobleme*, siehe beispielsweise [ADN59], [AV64], [Ama90], [Ama93], [Ama95], [ADF97] und [DHP03]. Mit dieser lassen sich über Eigenschaften der linearen Gleichung, genauer gesagt über Eigenschaften des Differentialoperators sowie der Randoperatoren, Aussagen zur Lösbarkeit von nichtlinearen Problemen treffen. Parameter-elliptische Randwertprobleme besitzen die Eigenschaft der sogenannten *maximalen L_p-Regularität*. Diese erlaubt es, nichtlineare Gleichungen durch Iterationsverfahren anhand der linearisierten Gleichung zu lösen, wofür aufgrund der Anwendung von Fixpunktargumenten häufig Kleinheitsbedingungen an den Zeithorizont oder die Anfangsdaten gestellt werden müssen.
Neben der Möglichkeit, nichtlineare Probleme zu behandeln, impliziert die maximale L_p-Regularität insbesondere die Holomorphie der zugehörigen Halbgruppe, weshalb parameter-elliptische Randwertprobleme auch eine Glättungseigenschaft der Lösung besitzen und, in beschränkten Gebieten, exponentiell stabil sind. Durch die Glättungseigenschaft existieren auch zu Anfangswerten mit geringerer Regularität, als die Gleichung eigentlich fordert, Lösungen, welche die erforderliche Regularität erhalten.
Da die Theorie parameter-elliptischer Randwertprobleme auf die in dieser Arbeit auftretenden Systeme nicht angewendet werden kann, beschäftigen wir uns hier mit dem zur maximalen L_p-Regularität äquivalenten Zugang über $\mathcal{R}$-*sektorielle* Lösungsoperatoren (siehe z.B. [Wei01]).

Konkret betrachten wir das Modellproblem für die Kopplung zweier parabolischer Plattengleichungen, in diesem Falle der strukturell gedämpften Platte

$$\begin{aligned} \partial_t^2 u_1 + \beta_1 \Delta^2 u_1 - \rho_1 \Delta \partial_t u_1 &= f_1 \quad \text{in } (0,\infty) \times \mathbb{R}^n_+, \\ \partial_t^2 u_2 + \beta_2 \Delta^2 u_2 - \rho_2 \Delta \partial_t u_2 &= f_2 \quad \text{in } (0,\infty) \times \mathbb{R}^n_- \end{aligned} \tag{Z.6}$$

mit kanonischen Transmissionsbedingungen

$$\partial_\nu^{j-1} u_1 - \partial_\nu^{j-1} u_2 = 0 \tag{Z.7}$$

auf $(0,\infty) \times \Gamma = (0,\infty) \times \mathbb{R}^{n-1}$ für $j = 1, \ldots, 4$. Hierbei seien $\beta_i, \rho_i > 0$ und $f_i \in L_p(\mathbb{R}^n_\pm)$ für $i = 1, 2$ gegeben.
Die kanonischen Transmissionsbedingungen ($Z.7$) bedeuten, dass die zusammengesetzte Funktion $u = \chi_{\mathbb{R}^n_-} u_2 + \chi_{\mathbb{R}^n_+} u_1$ im Sobolevraum $H_p^4(\mathbb{R}^n)$ liegen soll, d.h. dass die Lösung insgesamt glatt im Sinne der Ordnung der Gleichung sein soll. Spiegelt man die Gleichung in $\mathbb{R}^n_-$ an der Transmissionsschicht Γ auf die andere Seite, so erhält man ein Randwertproblem im Halbraum $\mathbb{R}^n_+$ mit vier Randbedingungen, welche die sogenannte *Shapiro-Lopatinskii-Bedingung* (siehe bspw. [Wlo82] und [DHP03]) erfüllen. Dies wiederum erlaubt es explizite Lösungsoperatoren mithilfe der partiellen Fouriertransformation zu konstruieren sowie passende Abschätzungen für diese zu beweisen.
Wie bereits erwähnt, lässt sich die bekannte parameter-elliptische Theorie für ein solches System nicht direkt anwenden, was sowohl an der gemischten Ordnung des Systems liegt, als auch an der Tatsache, dass der funktionalanalytische Grundraum nicht der Raum L_p, sondern der Produktraum $H_p^2 \times L_p$ ist. In der Tat müssen sogar weitere Bedingungen in den Grundraum geschrieben werden, um die $\mathcal{R}$-Sektorialität zu erhalten (siehe auch [DD11]).
Das Vorgehen über die partielle Fouriertransformation und die Shapiro-Lopatinskii-Bedingung ist zwar analog zur bekannten Theorie, die komplexere Struktur des Grundraumes im Zusammenspiel mit den gemischten Ordnungen erlauben aber keine direkte Übertragung der Ergebnisse auf solche Systeme.
Allgemein ist die Frage nach der $\mathcal{R}$-Beschränktheit von Lösungsoperatoren zu Systemen von Rand- und Transmissionsproblemen ein zentrales Forschungsinteresse. Im Falle der parameter-elliptischen Theorie kann mit dieser Maschinerie für eine ganze Klasse an Gleichungen maximale L_p-Regularität gezeigt werden, was wie erwähnt eine starke Grundlage zum Studium nichtlinearer Gleichungen bildet. Hier sind die Parameter-Elliptizität und die Shapiro-Lopatinskii-Bedingung gemeinsam äquivalent zur maximalen Regularität der Evolutionsgleichung.
Entsprechend groß sind die Bemühungen, eine ähnlich aussagekräftige Theorie für Systeme aufzustellen. Arbeiten in dieser Richtung sind u.a. [DF12], [DD11], [DK13] oder [KS12], in welchen notwendige und hinreichende Bedingungen für ($\mathcal{R}$-)Sektorialität an die Operatoren und die zugrunde liegenden Räume in verschiedenen Situationen bewiesen werden. Auch konkrete Transmissionsprobleme wurden in der Literatur in diesem Kontext untersucht. So treten beispielsweise parabolisch-elliptische Systeme bei der Analyse von Lithium-Ionen Batterien auf (vgl. [Seg13], [DS14]).
Wünschenswert wäre eine genaue Charakterisierung von Systemen mit passenden Rand- und Transmissionsbedingungen, welche maximale L_p-Regularität besitzen. Dies scheint jedoch aufgrund der großen Variabilität an Fragestellungen ein vielleicht unmögliches Unterfangen zu sein, weshalb es zum jetzigen Zeitpunkt unumgänglich ist, Prototypen, wie in unserem Falle die

strukturell gedämpfte Platte mit kanonischen Transmissionsbedingungen, auf $\mathcal{R}$-Sektorialität zu überprüfen und somit schließlich im Laufe der Zeit verschiedene Klassen von Problemen zu identifizieren, welche gemeinsam behandelt werden können. Hierzu soll diese Arbeit unter anderem einen Beitrag leisten.

Die zweite zentrale Fragestellung dieser Arbeit beschäftigt sich mit den Eigenschaften von Lösungen von Transmissionsproblemen, bei welchen Gleichungen mit unterschiedlicher Charakteristik gekoppelt werden.
Während die Plattengleichung in ihrer einfachsten Form (Z.1) als Produkt zweier Schrödingergleichungen weder einen Energieabfall noch eine Glättungseigenschaft im Sinne analytischer Halbgruppen aufweist, ist für die thermoelastische Platte (Z.2) und die strukturell gedämpfte Platte (Z.3) beides erfüllt. Für die thermoelastische Platte mit sogenannten *freien Randbedingungen* wurde dies in [DS17], für die strukturell gedämpfte Platte mit Dirichlet-Neumann Randbedingungen (*clamped Randbedingungen*) in [CT89] oder auch [DS15] gezeigt. In der Tat besitzen diese Gleichungen sogar maximale L_p-Regularität. Sie verfügen also über nahezu alle wünschenswerten Eigenschaften.
Koppelt man nun beide Gleichungen, so stellt sich sogleich die Frage, welche der positiven Eigenschaften der thermoelastischen bzw. strukturell gedämpften Platte sich auf die ungedämpfte Platte übertragen und welche Eigenschaften hingegen verloren gehen. Falls das gekoppelte Problem stabil ist, so stellt sich auch die Frage nach der Abklingrate der Energie für das Gesamtsystem.
In dieser Arbeit untersuchen wir die Eigenschaften für die Kopplung der Platte mit der strukturell gedämpften Platte

$$\partial_t^2 u_1 + \Delta^2 u_1 - \rho\Delta\partial_t u_1 = 0 \quad \text{in } (0,\infty)\times\Omega_1, \tag{Z.8}$$

$$\partial_t^2 u_2 + \Delta^2 u_2 = 0 \quad \text{in } (0,\infty)\times\Omega_2 \tag{Z.9}$$

mit *clamped* Randbedingungen

$$u_1 = \partial_\nu u_1 = 0 \quad \text{auf } \Gamma_1 \tag{Z.10}$$

und Transmissionsbedingungen

$$\begin{aligned} u_1 &= u_2, \\ \partial_\nu u_1 &= \partial_\nu u_2, \\ \Delta u_1 &= \Delta u_2, \\ -\rho\partial_\nu\partial_t u_1 + \partial_\nu\Delta u_1 &= \partial_\nu\Delta u_2 \end{aligned} \tag{Z.11}$$

auf der Transmissionsschicht Γ. Diese Transmissionsbedingungen sind in gewisser Weise die *natürlichen Transmissionsbedingungen*, denn vermöge dieser Bedingungen wird das Gesamtsystem (Z.8)-(Z.11) *dissipativ.* Sämtlicher Energieverlust wird hierbei durch den Dämpfungsterm $\rho\Delta\partial_t u_1$ verursacht.

Ähnliche Fragestellungen wurden bereits umfangreich in der Literatur untersucht, ist doch auch aus Anwendungssicht gerade die Frage nach der Stabilität sowie der Informationsübertragung bei Kontrollproblemen für gekoppelte Materialien von immenser Bedeutung.

Nicht nur für Plattengleichungen ([Has17], [RO04]), auch in der Thermoelastizitätslehre (etwa [MnRN07], [MnRR17]), bei der Kopplung von Wellen- und Plattengleichungen ([Her05], [Has16], [LL99], [AG16], [AN10], [CL10]) oder anhand von Transmissionsproblemen von Wärmeleitungs- und Wellengleichung ([RT74], [ZZ07]) wurde die Übertragung von Eigenschaften von Gleichungen parabolischen Charakters (beziehungsweise Gleichungen mit dissipativen Effekten) auf Gleichungen hyperbolischen (bzw. energiekonservierenden) Charakters untersucht.
Die Frage nach der Übertragung von Analytizität der einen Halbgruppe auf das Gesamtsystem findet sich in der Literatur hingegen überraschend selten. Abgesehen von indirekten Resultaten, wie beispielsweise in [MnRR17], wo die Lösung eines thermoelastischen Transmissionsproblems nur polynomielles und kein exponentielles Abklingen aufweist und damit die zugehörige Halbgruppe nach dem Satz von Gearhart-Prüss nicht analytisch sein kann, findet sich eines der wenigen positiven Ergebnisse für die Kopplung zweier thermoelastischer Platten in [SR11]. Hier besitzen jedoch beide Gleichungen bereits die Glättungseigenschaft, was dieses Resultat weniger überraschend erscheinen lässt.
Im Rahmen der Kontrolltheorie wurden Transmissionsprobleme ebenfalls ausführlich studiert, siehe etwa [LW99], [LW00] oder auch [AV09]. Hierbei stellen sich beispielsweise Fragen nach der Stabilisierung oder der genauen Steuerung des Transmissionsproblems durch alleinige Kontrolle am äußeren Rand, d.h. man beschäftigt sich mit der Informationsübertragung von außen über verschiedentlich geartete Gleichungen im Gebiet hinweg.

Im Folgenden stellen wir die Gliederung dieser Arbeit detailliert vor, grenzen die behandelten Themen ein und geben einen Ausblick auf die Hauptresultate der Arbeit:

In **Kapitel 2** beschäftigen wir uns mit allgemeinen Eigenschaften von Systemen partieller Differentialgleichungen.
Im ersten Abschnitt untersuchen wir die Notwendigkeit einer sogenannten *Parameter-Elliptizität im Hauptsymbol* $A_0(x, \xi)$ des Matrixoperators $\mathcal{A} = \mathcal{A}(x, D)$ für die Erzeugung holomorpher C_0-Halbgruppen in einer sehr allgemeinen Situation. Dieses einfach handhabbare Werkzeug erlaubt es, das Fehlen von Analytizität zu zeigen, was wir in Kapitel 3 aufgreifen werden, wenn wir uns die Frage nach der Übertragung verschiedener Eigenschaften von parabolisch-hyperbolisch gekoppelten Transmissionsproblemen stellen.
Das Hauptresultat in Theorem 2.2 zeigt die Notwendigkeit dieser Parameter-Elliptizität für die Gültigkeit der Resolventenabschätzung

$$\|\lambda(\lambda - \mathcal{A})^{-1}\|_{L(\mathcal{X})} \leq C$$

für gewisse $\lambda \in \mathbb{C}$. Die beiden Korollare 2.4 und 2.5 konkretisieren das Ergebnis für die Erzeuger von Halbgruppen und analytischen Halbgruppen. Der Beweis orientiert sich an ähnlichen Aussagen in [DF12], [ADN59] oder auch [Ama90].
Im zweiten Abschnitt stellen wir verschiedene Störungsresultate für sektorielle und $\mathcal{R}$-sektorielle Transmissions- und Randwertsysteme vor, welche zum einen das Studium von Gleichungen mit variablen Koeffizienten erlauben und zum anderen den Weg für Lokalisierungsprozeduren eröffnen. Gerade in der parameter-elliptischen Theorie, in welcher man häufig Modellprobleme mit konstanten Koeffizienten im Halb- und Ganzraum betrachtet, womit man wichtige Hilfsmittel wie die (partielle) Fouriertransformation zur Verfügung hat, sind solche Sätze unumgänglich, um die praktisch relevanten Probleme mit variablen Koeffizienten in unterschiedlichen Gebieten

lösen zu können.
Auch hier ist die Situation sehr allgemein gehalten: Wir betrachten ein Modell-Transmissionsproblem der Form

$$\begin{aligned} (\lambda - \mathcal{A})u &= f \quad \text{in } \mathbb{R}^n, \\ \mathcal{B}_\gamma u &= g \quad \text{auf } \mathbb{R}^{n-1} \end{aligned} \tag{Z.12}$$

in gewissen Produkträumen von Sobolevräumen. Hier ist der Operator $\mathcal{A}$ von der Form

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^- & 0 \\ 0 & \mathcal{A}^+ \end{pmatrix},$$

wobei $\mathcal{A}^-$ und $\mathcal{A}^+$ jeweils Matrixdifferentialoperatoren mit gemischten Ordnungen und unterschiedlicher Größe sind. Die Transmissionsbedingungen sind durch

$$\mathcal{B}_\gamma := \begin{pmatrix} \gamma\mathcal{B}^- & \gamma\mathcal{B}^+ \end{pmatrix} := \begin{pmatrix} \gamma\mathcal{B}_1^- & \gamma\mathcal{B}_1^+ \\ \vdots & \vdots \\ \gamma\mathcal{B}_M^- & \gamma\mathcal{B}_M^+ \end{pmatrix}$$

gegeben, wobei γ den Spuroperator bezeichne.
Für solche Systeme untersuchen wir die Übertragbarkeit von Sektorialität, siehe Proposition 2.16, sowie die Übertragbarkeit von $\mathcal{R}$-Sektorialität, siehe Proposition 2.18, des Modellproblems auf ein modifiziertes Problem mit kleinen Störungen. Dies ermöglicht es, das Problem (Z.12) auch mit einer gekrümmten, hinreichend glatten Transmissionsschicht Γ zu betrachten, denn die beim Lokalisieren auftretenden Störungen der Operatoren sind durch die gezeigten Resultate abgedeckt.
Setzt man zum Beispiel $\mathcal{A}^- = 0$, so kann man mit (Z.12) auch Randwertprobleme behandeln und insgesamt lassen sich somit Systeme von Transmissions-Randwertproblemen auf zum Beispiel beschränkten, glatten Gebieten behandeln.
Das Vorgehen in diesem Abschnitt ist ähnlich zu [DHP03], Abschnitt 7.3, unterscheidet sich jedoch durch die Wahl eines Produktraumes von L_p-Sobolevräumen als Grundraum anstatt eines Banachraumwertigen L_p-Raumes deutlich. In Folge sind unsere Ergebnisse deutlich vielseitiger anwendbar. Dieser Teil beruht auf gemeinsamer Arbeit mit Felix Hummel.

Kapitel 3 behandelt das Transmissionsproblem (Z.6), (Z.7) zweier strukturell gedämpfter Platten im Grundraum $H_p^2 \times L_p$ für $1 < p < \infty$ und ist an eine entsprechende Arbeit über das Dirichlet-Neumann-Randwertproblem in [DS15] angelehnt. Ziel ist es, $\mathcal{R}$-Sektorialität für das Transmissionsproblem zu zeigen und damit den Weg für nichtlineare Varianten des Transmissionsproblems zu ebnen, auch wenn wir uns hier nicht explizit mit nichtlinearen Gleichungen beschäftigen. Hierfür gehen wir analog zu [DS15] vor und zeigen, dass die Methoden, mit welchen man das Randwertproblem behandeln konnte, auch für das Transmissionsproblem zielführend sind. Da man nun jedoch vier Bedingungen auf der Transmissionsschicht anstatt zwei Bedingungen auf dem Rand behandeln muss, ist der Vorgang technisch noch einmal aufwändiger.
Standardmäßig führt uns der Weg, nach Spiegelung und Überführung auf ein Randwertproblem, über die partielle Fouriertransformation zur expliziten Konstruktion von Lösungsoperatoren und geeigneten $\mathcal{R}$-Schranken für gewisse Operatorfamilien, siehe Theorem 3.11, im zunächst gewählten Grundraum $H_p^2(\mathbb{R}_+^n) \times L_p(\mathbb{R}_+^n)$. Hier ist das Problem zwar eindeutig lösbar und man

erhält auch einige *a-priori*-Abschätzungen (Korollar 3.12), doch die $\mathcal{R}$-Sektorialität und damit die maximale L_p-Regularität ist in diesem Grundraum nicht gegeben (Proposition 3.13). Um diese zu erhalten, müssen weitere Bedingungen in den Grundraum geschrieben werden; konkret müssen die im Grundraum existierenden Transmissionsbedingungen schon hier integriert werden, siehe Theorem 3.16. Dies ist analog zum Dirichlet-Neumann-Randwertproblem, für welches die Autoren in [DS15] $\mathcal{R}$-Sektorialität zeigen konnten, falls eben diese Randbedingungen bereits im Grundraum realisiert sind.
Theorem 3.16 erlaubt es uns schließlich, das zeitabhängige System mit inhomogenen Daten zu lösen, siehe Theorem 3.18. Mithilfe der Resultate aus Kapitel 2 lassen sich nun auch Transmissions-Randwertprobleme in hinreichend glatten beschränkten Gebieten (vgl. Abbildung Abb.1) lösen, siehe Theorem 3.19.

Während das parabolisch-parabolisch gekoppelte Transmissionsproblem einen L_p-theoretischen Zugang erlaubt, ist dies für gekoppelte Systeme wie (Z.8)-(Z.11), welches wir in **Kapitel 4** untersuchen, im Allgemeinen nicht mehr möglich (vgl. [DH18]). Hier stehen die Fragen nach Stabilität, genauer exponentieller Stabilität, und der Analytizität der zugehörigen Halbgruppe in einem geeigneten Hilbertraum für das Gesamtsystem im Zentrum der Aufmerksamkeit.
Mithilfe von Energiemethoden, der Parameter-Elliptizität der strukturell gedämpften Platte sowie der Theorie von *Interpolations-Extrapolationsskalen* (siehe [Ama95]) können wir für das System (Z.8)-(Z.11) unter geringen geometrischen Bedingungen ein exponentielles Abklingen der Energie beweisen (Theorem 4.20). Insbesondere benötigen wir keine geometrische Bedingung an die Transmissionsschicht Γ, welche sonst häufig in der Literatur zu finden ist.
Die Analytizität ist für das Gesamtsystem nach den Ergebnissen aus Kapitel 2 hingegen nicht mehr gegeben, siehe Proposition 4.31. Darüber hinaus verdeutlichen wir das Fehlen der Glättungseigenschaft mithilfe einer numerischen Simulation für das eindimensionale Transmissionsproblem, welche nahelegt, dass das zugehörige Spektrum nicht in einem Sektor mit Öffnungswinkel $\vartheta > \frac{\pi}{2}$ der komplexen Ebene liegt, was eine notwendige Bedingung darstellt.
Sowohl die Aussagen über exponentielle Stabilität als auch die über das Fehlen der Analytizität übertragen wir weiterhin auf Plattengleichungen, in welchen die Dämpfung mithilfe einer C^4-Funktion $\rho\colon \overline{\Omega} \to [0,\infty)$ über das Gebiet verteilt wird, siehe Theorem 4.30 und Proposition 4.32. In unserem Falle betrachten wir stets ein beschränktes Gebiet Ω (siehe auch Abbildung Abb.1). Damit erhalten wir die Kompaktheit der Resolvente der Operatoren und somit Aussagen über die Regularität der Lösungen (wie in Lemma 4.5), welche wir in den Beweisen immer wieder ausnutzen. Dennoch legen Ergebnisse wie in [MN01] und [MN07] für die Wellengleichung nahe, dass auch für unbeschränkte Gebiete ein gewisses Abklingverhalten der Lösung gelten kann, sofern die Dämpfung im Unendlichen wirkt. Mit solchen Problemen werden wir uns an dieser Stelle jedoch nicht befassen.
Teile dieses Kapitels basieren auf gemeinsamer Arbeit mit Prof. Dr. Robert Denk und wurden in [DK18] veröffentlicht.

Im **Anhang** fassen wir einige Ergebnisse zu den Themen Sobolevräume, Fouriermultiplikatoren, $\mathcal{R}$-Sektorialität, maximale L_p-Regularität sowie Randwertprobleme gewöhnlicher Differentialgleichungen kurz zusammen. Dies dient hauptsächlich der Lesbarkeit dieser Arbeit.
Auf entsprechende Literatur wird an den passenden Stellen verwiesen.

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Symbols

Functions

$\text{dist}(x, A)$	Distance of a point $x \in \mathbb{R}^n$ from a set $A \subset \mathbb{R}^n$, page 106
χ_A	Indicator function on the set A, page 119
τ_n	Reflection in $\mathbb{R}^n$, page 43

Function spaces and spaces of distributions

$B^s_{p,q}(\mathbb{R}^n, E)$	E valued Besov space on $\mathbb{R}^n$, page 117
$B^s_{p,q}(\Omega, E)$	E valued Besov space on Ω, page 118
$BUC^m(\Omega, E)$	E valued space of $m \in \mathbb{N}_0$ times differentiable bounded uniformly continuous functions on Ω, page 116
$BUC^{m+s}(\Omega, E)$	E valued space of m times differentiable bounded uniformly continuous functions on Ω such that m-th derivatives are Hölder continuous with exponent $s \in (0, 1)$, page 116
$C^\infty_0(\Omega, E)$	Space of infinitely often differentiable functions $\varphi\colon \Omega \to E$ having compact support, page 115
$C^k_b(U)$	Space of bounded, $k \in \mathbb{N}_0$ times differentiable functions $f\colon U \to \mathbb{C}$ with bounded derivatives, page 11
$\mathscr{D}(\Omega, E)$	E valued test functions on a domain Ω, page 115
$\mathscr{D}'(\Omega, E)$	E valued distributions on Ω, page 115
$H^k(\Omega)$	$\mathbb{C}$ valued Sobolev space of order $k \in \mathbb{N}$, page 79
$H^k_0(\Omega)$	Closure of $C^\infty_0(\Omega)$ in $H^k(\Omega)$, page 79
$H^s_p(\mathbb{R}^n, E)$	E valued Bessel potential space on $\mathbb{R}^n$, page 117
$H^s_p(\Omega, E)$	E valued Bessel potential space on Ω, page 118
$\mathscr{S}(\mathbb{R}^n, E)$	E valued Schwartz functions, page 115
$\mathscr{S}'(\mathbb{R}^n, E)$	E valued tempered distributions, page 115
$W^m_p(\Omega, E)$	E valued Sobolev space on a domain Ω, page 116

$W_p^s(\Omega, E)$	E valued Sobolev-Slobodeckii space on Ω, page 116

Operators

$\overline{\mathbb{C}}_+$	Closed right half plane $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\} \subset \mathbb{C}$, page 126
e_0	Trivial extension by zero for functions on $\mathbb{R}^n_+$ to $\mathbb{R}^n$, page 59
E_λ	Parameter-dependent extension operator from $\mathbb{R}^{n-1}$ to $\mathbb{R}^n_+$, page 59
e_s	Odd extension operator from $H^2_{p,0}(\mathbb{R}^n_+)$ to $H^2_p(\mathbb{R}^n)$, page 65
e_+	Smooth extension operator from $H^s_p(\mathbb{R}^n_+)$ to $H^s_p(\mathbb{R}^n)$, page 44
$\mathscr{F}$	Fourier transform, page 42
$\mathscr{F}_t$	Temporal Fourier transform, page 71
I	Identity operator, page 45
Λ_s	Shift operator $(\lambda - \Delta)^s$ in $\mathbb{R}^n$, page 59
Λ'_s	Shift operator $(\lambda - \Delta')^s$ in $\mathbb{R}^{n-1}$, page 59
M_n	Multiplication operator in the n-th variable, page 65
∂^α	Derivative with multi-index α, page 115
∇^k	All mixed k-th partial derivatives with respect to the space variable x, page 48
$\operatorname{op}(m)$	Operator defined by the symbol m, page 119
r_+	Restriction operator from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n_+)$, page 44
$r_{\overline{\Omega}}$	Point-wise restriction on $\overline{\Omega}$, page 118
γ_0	Trace operator, page 124
γ	Trace operator, page 19

Others

$\operatorname{diag}(v)$	$\mathbb{C}^{n \times n}$ diagonal matrix with $v \in \mathbb{C}^n$ on the diagonal, page 52
$\langle \cdot, \cdot \rangle_{X' \times X}$	Dual pairing in a Banach space X, page 86
$\lfloor s \rfloor$	Largest integer less or equal $s > 0$, page 119
$\mathcal{H}(E_1, E_0)$	The set of all generators A of an analytic C_0-semigroup on E_0 with domain E_1, page 88
$(\cdot, \cdot)_{\theta,p}$	Real interpolation functor, page 72
λ	Lebesgue measure in $\mathbb{R}^n$, page 121

$L(E,F)$	Banach space of all bounded linear operators from a Banach space E to a Banach space F, page 10
$L_{\mathrm{is}}(E,F)$	The set of all continuously invertible operators $T \in L(E,F)$, page 10
$\mathcal{L}_\theta$	The ray of angle $\theta \in [0,2\pi)$ in the complex plane, page 11
$\mathcal{L}_{\theta,\kappa}$	The ray of angle $\theta \in [0,2\pi)$ in the complex plane, starting at $\kappa e^{\mathrm{i}\theta}$, page 11
x^α	Product with multi-index α, page 115
$\lvert\alpha\rvert$	Order of a multi-index α, page 115
$\mathcal{R}(\mathcal{T})$	$\mathcal{R}$-bound of the operator family $\mathcal{T}$, page 121
$\mathbb{R}^n_+$	The half-space $\{x_n > 0\}$ in $\mathbb{R}^n$, page 42
$\mathbb{R}^n_-$	The negative half-space $\{x_n < 0\}$ in $\mathbb{R}^n$, page 42
Σ_ϕ	Symmetric sector in $\mathbb{C}$ with angle ϕ, page 42
$\mathrm{supp}(f)$	Support of a function f, page 37
$\mathrm{symb}(M)$	Symbol of the operator M, page 119