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Structures in Arbitrary Object Theory

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Abstract: This article critically compares two approaches that seek to explain *ante rem* mathematical structures in terms of arbitrary objects. The first is a theory that was developed by Kit Fine, and the second is a view that was more recently developed by me. I make a plea for a synthesis between the two approaches.

Keywords: Mathematical structuralism, arbitrary object theory, permutation problem.

1. Introduction

Two kinds of mathematical structures can be distinguished: rigid structures, and non-rigid structures. This division is defined in terms of a natural notion of symmetry on a structure. A rigid structure is a structure that admits non-trivial automorphisms, i.e., non-trivial isomorphisms from the structure to itself. A non-rigid structure does not have non-trivial automorphisms.

We will be concerned with the nature of rigid and non-rigid mathematical structures, explicated in an *ante rem* manner[1] in terms of arbitrary objects. The idea of connecting *ante rem* structuralism with the theory of arbitrary objects was first explored by Fine.[2] The present article intends to be a contribution to a discussion between Kit Fine and me about these matters. We have engaged in an exchange of constructive criticism concerning our respective proposals.[3] In this paper I argue that our mutual critiques point in the direction of a *synthesis* of the two approaches that might satisfy both parties.

To some extent, this will be a tale of book reviews. There are book reviews, and then there are *book reviews*. This article is to a significant extent about a couple of examples of the latter: (Burgess, 1999), (Burgess, 2008) and especially (Fine, 2022). We will explore how one might improve on the account of mathematical structures in (Horsten, 2019b) and (Horsten, 2019a), in the light of objections and friendly suggestions by Fine in his book review (Fine, 2022).

[1] The *locus classicus* for the discussion of *ante rem* structuralism is (Shapiro, 1997).
[2] See (Fine, 1998, Section VI).
[3] I have criticised Fine's account in (Horsten, 2019b, Section 8.4.3), and in (Horsten, 2019a, Section 6.1).

I will discuss the problem of describing the nature of rigid and non-rigid mathematical structures on the basis of a particularly simple example of each: the *natural number structure* on the one hand, and the *complete two-element graph* on the other hand. But the philosophical conclusions drawn from these examples are supposed to generalise to all rigid and non-rigid structures.

The structure of this article is as follows. I start with a brief revisit of the main ingredients of Fine's metaphysical theory of arbitrary objects, which I will call **F-theory**, and of my alternative to it, which I will call **H-theory**. I then move on to the way in which Fine and I employ arbitrary object theory to give a philosophical account of *ante rem* mathematical structures. Special attention is given to Fine's claim that my theory cannot do justice to the uniqueness of subject matter of categorical mathematical theories. Also, close attention is given to the metaphysical treatment of non-rigid mathematical structures.

2. Two theories of arbitrary objects

The basic idea of arbitrary object theory is that beside the particular objects that we are all familiar with (chairs, people, individual atoms,...), there are also *arbitrary objects*. An arbitrary object is an object that can take particular objects as values, but is not numerically identical to any particular object. The best case for the existence of such a quaint sort of entities lies, in my view, in its applications in metaphysics. It has been argued, for instance, that random variables can best be seen as arbitrary objects. In this paper, we explore the view that mathematical structures can best be viewed as arbitrary objects.

There are two competing metaphysical theories of arbitrary objects: the one pioneered by Fine in (Fine, 1985) and further developed in later publications (*F-theory*), and the theory proposed in (Horsten, 2019b) and applied in later publications such as (Horsten, 2019a) (*H-theory*).

The bare bones of F-theory are as follows.[4] Arbitrary objects are entities that have *values*. Here 'taking a value' is a primitive metaphysical concept, which is not to be confused with functional application (even though arbitrary objects can be *modeled* as functions.) For any collection of ordinary (*particular*) objects S, there is an *independent arbitrary* S (a_S) such that for each $o \in S$, the arbitrary object a_S takes the value o (and a_S takes no other values). The set S then forms the *value range* of a_S. For instance, if $S = \mathbb{N}$, then a_S is an independent arbitrary natural number. But there are also arbitrary objects that *depend on* other arbitrary objects. For instance, there is a *dependent* a'_S such that, whenever a_S takes the value n, a'_S takes the value $2 \times n$. Thus the dependence relation is an irreducible conceptual component of Fine's theory of arbitrary objects.

In (Horsten, 2019b), an alternative metaphysical account of arbitrary objects is proposed. According to H-theory, arbitrary objects are objects that can be in *states* that make up a *state space*. Thus an arbitrary number can be in the state of being the particular number 29. There are states s such that more than one arbitrary object can simultaneously be in state s. Thus there is a state such that an arbitrary natural number a_1 "is" the number 29 while another arbitrary object a_2 "is" the number 23.

These two metaphysical accounts are closely related to each other.[5] Fine's notion of taking a value corresponds to the H-theory's notion of being in a state, and Fine's related notion of value space corresponds to the H-theory's related notion of state space. But there is no primitive notion of dependence in the H-theory. Rather, in H-theory dependence is a defined notion

[4] See (Fine, 1985, chapter 1) for an introduction to F-theory.

[5] Indeed, Fine conjectures that *mathematically*, F-theory and H-theory are equivalent (Fine, 2022, p. 616–617).

(Horsten, 2019b, section 9.4): it is a matter of correlation of values across states. This then is a fundamental metaphysical difference between the two accounts.

In H-theory, a notion of possibility plays a role. H-theory takes the notion of possibility that is at play to be a peculiar modality (*afthairetic modality*). The reason for this is that it does not seem to make much sense to ask what the *actual* state is that a given arbitrary number is in: so it is a notion of modality without an accompanying notion of actuality. Fine, however, is sceptical about this modality. Moreover, he believes that it is important that his account of arbitrary objects is regarded as a *non-modal* account (Fine, 2022, p. 604):

> "[...] is the underlying theory of the kind of object involved in such statements itself modal? [...] [M]y own view is that it is not. [...] we should dispense with this peculiar modality and talk explicitly about the values that x and y take in different possible situations."

It is not clear that the disagreement between the two accounts on this issue is substantial. It is not an uncommon view that any satisfactory maximally expressive theory of metaphysical possibility should directly quantify over possible worlds—this happens, for instance, when possible worlds semantics for a modal language is explained in a metalanguage. (My metaphysical theory does quantify directly over 'afthairetically possible situations', of course.) Moreover, Fine would presumably agree that it does not make much sense to ask whether there is any situation in the *actual world* where a_S takes the value 21.[6]

At any rate, even if Fine would in the final analysis prefer to avoid mentioning situations at all in his theory of arbitrary objects, modality arguably nonetheless implicitly plays a role in Fine's dependence relation. The conditional that is used to express dependence relations ('if...then...') is clearly not the material conditional. Rather, it is an indicative conditional that is to semantically be understood in modal terms.

H-theory can be seen as an *abstractionist* account of arbitrary objects. It holds that systems of arbitrary objects *supervene* on their behaviour in states. On the one hand, this means that it suffices for the specification of a system $\langle a_1, \ldots, a_n \rangle$ of arbitrary objects to determine what the value of each a_i (for $i \leq n$) is on a set S of states (taking the arbitrary objects a_i to be undefined outside of S). This is in effect a *comprehension principle* for arbitrary objects. On the other hand, it implies a principle of *ontological economy* for arbitrary objects (Horsten, 2019b, p. 45):[7]

> Suppose that a and b are arbitrary objects. Then $a = b$ if and only if for every situation s, the value that a takes in s is identical to the value that b takes in s.

This is in effect a *principle of extensionality* for arbitrary objects.

Even though Fine accepts a principle that is close to the aforementioned extensionality principle, he sees matters differently. For Fine, the dependence relation is primitive and does not supervene on the values that arbitrary objects take. He holds that there is an *initial* arbitrary natural number—let us call it a.[8] Then there also is an arbitrary natural number b that *depends* on a and that takes the value 13 whenever a takes the value 11 and *vice versa*; and moreover there is an arbitrary natural number c that depends on b that takes the value 13 whenever arbitrary

[6] Observe, however, that it seems perfectly reasonable to ask, after reading the above quote by Fine: "But which values to x and y *actually* take?"
[7] Fine accepts a closely related principle: see (Fine, 1985, p. 34).
[8] Cf. infra, p. 39.

natural number b takes the value 11 and *vice versa*. Then a and c always take the same values. However, c (indirectly) depends on a, but not *vice versa*.

At the informal level there is a connection between (mathematical) structures and arbitrariness (Burgess, 2008, p. 403):

> [Both (*in rebus* structuralism and *ante rem* structuralism)] may be taken to start with the idea that 'the real numbers' means 'the arbitrary complete ordered field' and then diverge over the interpretation of 'arbitrary'. On the majority view, 'the arbitrary F' denotes nothing by itself: 'The arbitrary F is a G' just means All Fs are Gs'. On the minority view (as in Fine, 1985) the arbitrary F is a specific F but an extraordinary one in that it has no properties not shared by all Fs (though it is distinguished from ordinary Fs by the 'meta-property'[9] of being arbitrary).

Both F-theory and H-theory take the 'minority' approach: they aim to develop the thought that arbitrary object theories can be fruitfully employed in the construction of theories of *ante rem* mathematical structures.[10]

3. Fine's theory of mathematical structure

Fine's arbitrary objects account of mathematical structures is based on (Fine, 1985). It is sketched in a condensed form in (Fine, 1998, section VI), and was later slightly modified in (Fine, 2022). In this section, we concentrate on the presentation of the account in (Fine, 1998, section VI).

The process of obtaining a structure in terms of arbitrary objects consists of three stages. When we apply this to the case of the natural number structure, the account is as follows (Fine, 1998, p. 630):

Stage I We consider the *independent* arbitrary object N which, for every ω-sequence of particular objects $\underline{N}$, has $\underline{N}$ as one of its values (and has no other values). This arbitrary object N is called the *prototype* of the natural number structure.

Stage II We consider the sequence of *dependent* arbitrary objects

$$a_0^N, a_0^N, \ldots, a_i^N, \ldots$$

such that for every natural number i, a_i^N is the arbitrary object that takes for every value $\underline{N}$, of N, the i-th element of $\underline{N}$ as a value (and takes no other values).

Stage III The natural number structure is then defined as the following structure (in the classical set theoretic sense of the word):

$$\mathbb{N}_F = \langle \{a_0^N, a_1^N, \ldots, a_i^N, \ldots\}, < \rangle,$$

where $<$ is the ordering naturally induced by the orderings on the values of N.

According to Fine's account, arithmetic is then about one unique subject matter (Fine, 2022, p. 606–608). This subject matter is not an arbitrary object, but a *particular* object: it is a particular structure in the classical *set theoretic* sense of the word, i.e., a particular *set*. Thus Fine's account

[9] This relates to Fine's distinction between *generic* and *classical* conditions (Fine, 1985, p. 14), which I leave aside here.

[10] One might be sceptical about *ante rem* structures in general, of course, but in this article we do not enter into the discussion about the relative advantages and disadvantages of *ante rem* structuralism and *in rebus* structuralism.

of the natural number structure is a version of *set theoretic structuralism*. Indeed, we immediately see that:

Proposition 3.1. $\mathbb{N}_F$ *is an* ω*-sequence.*

So there is a possible situation in which $\mathbb{N}_F$ takes itself as its value.

Fine distinguishes between representational and non-representational kinds of objects: "Following Hallett . . . , we shall say that an account of the types of some kind is *representational* if each type of the given type is of that type." (Fine, 1998, p. 623) Thus, for instance, von Neumann's definition of cardinal numbers is representational: the cardinal number of a two-element set contains two elements. On the other hand, Frege's definition of cardinal numbers is not: the cardinal number two is an infinite class. The previous proposition then says that Fine's definition of natural number structure is representational.

I have argued[11] that the F-theory suffers from a *permutation problem* in the sense of (Hellman, 2006, p. 546). The objection goes as follows. Consider the structure $\mathbb{N}'_F$, which is just like $\mathbb{N}_F$, except that the places of a_0^N and a_1^N in the ordering $<$ are reversed. Clearly this also is an ω-sequence. According to Fine's account, $\mathbb{N}'_F$ is not, however, the natural number structure. Nonetheless, for familiar Benacerrafian reasons, it is hard to give convincing reasons for why $\mathbb{N}_F$, rather than $\mathbb{N}'_F$ is the 'real' natural number structure.

Even if this worry is dismissed, a closely related argument raises a complication for F-theory. Fine holds that there are not just one but many *independent* arbitrary real numbers (Fine, 2022, p. 609),[12] so presumably also many independent arbitrary natural numbers, and many natural number prototypes. Each of the latter gives rise to an ω-sequence that is a candidate for being the subject matter of arithmetic: will the privileged ω-sequence please rise? Fine argues that among the independent arbitrary reals, only one is the *initial* independent arbitrary real—the Ur-arbitrary real, we might say (Fine, 2022, p. 609). Likewise, presumably, there is a unique initial natural number prototype, and the subject matter of arithmetic is the ω-sequence that *this* naturally gives rise to. Thus the notion of being 'initial' appears to be an additional primitive notion in Fine's theory of arbitrary objects; it has no counterpart in H-theory.

4. The prototype account

In (Horsten, 2019a), an an alternative explication of *ante rem* structuralism in terms of arbitrary objects is given, which we will call the **prototype account** (for reasons that will become clear). Again, we illustrate his approach on the basis of what it says about the natural number structure.

The prototype account is simpler than Fine's account. It identifies the natural number structure $\mathbb{N}_H$ with an arbitrary object that can, for any ω-sequence $\underline{N}$, be in the state of being $\underline{N}$, and can be in no other states. Such an arbitrary object is called a *generic* ω-sequence.

Also for the prototype account, a Benacerrafian worry rears its head. In (Horsten, 2019b, section 6.4), it is argued that there are in fact *many* arbitrary ω-sequences. It is then not clear what is meant when arithmetic is said to have *the* natural number structure as its subject matter. We will return to this problem in section 6.

[11] See (Horsten, 2019b, section 7.5, and p. 158), (Horsten, 2019a, p. 373).
[12] On this point, Fine's view has evolved since Fine (1985).

To this account of the natural number structure, the prototype account adds an account of the individual natural numbers. The natural number n, in this account, is identified with the arbitrary object that, in the state where $\mathbb{N}_H$ takes the value $\underline{N}$, takes the value of the n-th element in this ω-sequence.

This looks much like Fine's account of the individual natural numbers. Indeed, when we compare it with Fine's account of the natural number structure, then, if we disregard Fine's qualification of 'independence' (which is not a primitive notion in H-theory), we can say that the prototype account simply identifies the natural number structure with what Fine calls the *prototype N* that we have encountered in our description of Stage I of Fine's account of the natural numbers structure. Moreover, the account of the individual natural numbers is no more than a copy of Fine's stage II. But the prototype account does *not* go on to build an ω-sequence out of the individual natural numbers and identify the natural number structure with that ω-sequence. In other words, Fine's stage III is dropped altogether.

We immediately see that:

Proposition 4.1. $\mathbb{N}_H$ *is not an* ω*-sequence.*

So, in contrast to Fine's theory of the natural number structure, the prototype account of natural numbers is *non-representational*, whereby a Hellman-style permutation objection cannot be formulated against it.

Observe, incidentally, that a version of the prototype account can be held within Fine's metaphysical framework also. Indeed, I submit that Fine might be well-advised to do so if he does not want to be open to the permutation problem. Of course, this means abandoning the ambition of giving a *representational* account of the natural numbers.

5. Non-rigid structures

The natural number structure is rigid: it allows no non-trivial automorphisms. But we know since (Burgess, 1999)—the third book review that plays an important role in this article—that the mathematical structuralist had better not forget about non-rigid structures. Indeed, we will now see that it is not completely straightforward how F-theory and H-theory apply to non-rigid structures. We investigate this question on the basis of a simple example: the complete two-element graph. But as with the natural number structure, the lessons drawn are intended to generalise.

Consider Fine's account first. We start, in Stage I, by considering the prototype G_2, which is the independent arbitrary object that takes every complete two-element graph *system* as one of its values (and takes no other values). Ultimately, Fine wants G_2 *itself* to be a complete 2-element graph. So what are its two elements a, b? In analogy with the individual natural numbers, we want a^{G_2} to be a dependent arbitrary object which, for any value $\underline{G}$ of G_2, takes one of its two elements as its value. But *which element* of $\underline{G}$ takes a as its value? Any choice here seems completely arbitrary. Of course, if it is not clear what a, b are, then it is also not clear what the complete two-element graph is, on Fine's account.

On the non-representational prototype account, there is no mystery about what the complete two-element graph is: it is (roughly) what Fine calls the prototype G_2. Just as on the prototype account the natural number structure is not itself an ω-sequence, G_2 is not itself a two-element graph. Nonetheless, the prototype account faces a similar problem to the one that Fine faces.

One would like to say that even though it is not a two-element graph in the set theoretic sense of the word, G_2 somehow contains exactly two elements a, b, which are themselves arbitrary objects. But it is not immediately clear what these elements are. Consider a *system* S_1 that instantiates the complete two-element graph. S_1 then consists of two elements a_1, b_1 that stand in a (total) relation to each other. It seems natural to take a, b to be arbitrary objects that can be in a state such that a is a_1, and b is b_1. But why does a take the value a_1 rather than the value b_1? Again, there seems no good answer to this question.

Fine addresses the problem with non-rigid structures in (Fine, 2022). He proposes to extend his account so that arbitrary objects are also allowed to be *multi-valued*. Given this amendment, concerning the complete two-element graph, Fine describes the nature of its two elements v and w as follows (Fine, 2022, p. 613):

> [...] in the particular complete two-element graph G_0 with vertices v_0 and w_0 as a value for [the prototype] G, v will take v_0 and w take w_0 as a value but also, v will take w_0 and w take v_0 as a value.

Fine rightly observes (Fine, 2022, p. 613) that the prototype account can also make use of this solution.[13] The proponent of the prototype theory then simply says that, in the example under consideration, there are *two* situations in which the relevant prototype is in the state of being G_0. One of them is such that v is in the state of being v_0 and w is in the state of being w_0, and the other is such that v is in the state of being w_0 and w is in the state of being v_0. That seems a perfectly satisfactory solution of the problem. Let us therefore amend the prototype theory by adopting Fine's recommendation.

6. Uniqueness

We saw earlier that according to (Horsten, 2019b) and (Horsten, 2019a), there is not just one, but rather there are many arbitrary ω-sequences; moreover, there is no priviliged 'initial' one. On this account, arbitrary ω-sequences together form a *structure* of entities that share a state space. For any arbitrary ω-sequence A_1 in this structure, there is another ω-sequence A_1 such that, in some state s, A_1 and A_2 'are' different particular ω-sequences.

If that is so, then it is not clear how arithmetic can be said to be about a unique structure. Concerning this, I wrote that it is, in a way, a matter of perspective (Horsten, 2019b, p. 112):

> The apparent conflict results from a difference between regarding **N** as a structure or universe on the one hand, and regarding **N** as an element of a larger structure or universe on the other hand. [...] [I]f you are doing number theory (without making use of 'higher mathematics' to obtain number theoretic results), then you are working within *the* generic natural number structure. But the generic natural number structure is itself an entity belonging to a larger universe. If you are making use of generic ω-sequences or are investigating them as a class, then you are dealing with multiple copies of the generic natural number structure that are modally connected with each other.

[13] In (Horsten, 2019a, section 6.8), an alternative solution is proposed for the problem that non-rigid structures pose for an arbitrary objects-account of mathematical structures. This solution proposal was rightly dismissed by Fine as unsatisfactory (Fine, 2022, p. 612–613), and will not be discussed here.

Fine demurs (Fine, 2022, p. 608):

> But the *ante rem* structuralist (as opposed to his eliminative counterpart) is not someone who thinks that it is only relative to a certain perspective that we might talk of the structure of natural numbers as one, even though in fact it is many; and nor, it seems to me [i.e., Fine], is this the attitude of the mathematician towards the complete two-element graph. Indeed, if this were their view, then it is not at all clear why they should not have adopted the Berkeleyan [i.e., *in rebus*] line right from the start and talked about a particular ω-sequence or a particular complete two-element graph as if it were the exemplar of all ω-sequences or all complete two-element graphs.

Certainly Fine has a point. Observe also that Fine's objection affects the prototype account of non-rigid structures as much as its account of rigid structures. Therefore, the prototype accounts at least needs to be further amended. I will now seek to do so.

There is in H-theory a metaphysical correlate to what Fine in this quote refers to as the "perspective from which we might talk of the structure of the natural numbers as one".[14] Recall that a *state* (in my use of the term) is such that one or more arbitrary entities can be in it simultaneously. Consider the generic ω-sequence Ω that can be in any (and only such) state s such that in s, Ω "is" some particular ω-sequence *and no other arbitrary object* (in particular, no other generic ω-sequence) *is in that state* s. By the comprehension principle discussed in section 2, this generic ω-sequence Ω exists. Moreover, by the extensionality principle discussed in that same section, Ω is *unique*.

The generic ω-sequence Ω is an example of an *incomplete arbitrary object space* (Horsten, 2019b, p. 55), i.e., an arbitrary object space containing fewer arbitrary objects than states. Incomplete arbitrary object spaces hitherto received almost no attention in H-theory. This may be the reason why no use was made of them in the account of mathematical structures in (Horsten, 2019a).

Ω forms a 1-element system of arbitrary objects that is completely isolated from—i.e., not correlated with—any other arbitrary object.[15] It is thus "independent" from all other generic ω-sequences. In this respect, Ω differs from Fine's initial ω-sequence. No ω-sequences depend on Ω,—except Ω itself, of course—nor do any other arbitrary objects, whereas many arbitrary objects depend on Fine's initial ω-sequence.

Arithmetic is a self-standing mathematical discipline, in the sense that it is about a single, independent structure, which is instantiated by each ω-sequence. The generic sequence Ω is the *unique* arbitrary ω-sequence that does not belong to a ≥ 2-element system of correlated arbitrary ω-sequences. So we are not faced with a version of Benacerraf's problem of arbitrarily having to choose from a collection of equally suitable arbitrary ω-sequences. In this way, the Ur-generic ω-sequence Ω fits the bill perfectly. I propose that we take it to be the subject matter of arithmetic.

We have seen how the arbtirary ω-sequence Ω *induces*, in the Finean sense,[16] a particular ω-sequence Ω^*. It will by now not come as a surprise that I object to identifying Ω^* with the natural number structure, for this would give rise to a version of the permutation problem.

[14] As, if Fine is right that his and my account are mathematically equivalent, one would expect that there would be.
[15] For a formal discussion of how arbitrary objects are organised in systems in H-theory, see (Steinkrauss and Horsten, 2025).
[16] Cfr supra, section 3.

There is also a lesson here for Fine's friendly suggestion, discussed in section 5, for a satisfactory prototype account of non-rigid structures. Consider again the case of the complete two-element graph, but now suppose that there are *two* particular systems

$$S_1 = \langle \{a_1, b_1\}, R_1 \rangle$$
$$S_2 = \langle \{a_2, b_2\}, R_2 \rangle$$

to be considered. Following Fine's suggestion for defining the elements a, b of the prototype complete two-element graph S (and using his terminology of arbitrary objects x taking a value $v(x)$ in a state), we get a state space $\{s_1, s_2, s_3, s_4\}$ such that:

- in $s_1 : v(S) = S_1, v(a) = a_1, v(b) = b_1$;
- in $s_2 : v(S) = S_1, v(a) = b_1, v(b) = a_1$;
- in $s_3 : v(S) = S_2, v(a) = a_2, v(b) = b_2$;
- in $s_4 : v(S) = S_2, v(a) = b_2, v(b) = a_2$.

So far, so good. But someone might object that there are also arbitrary objects a^*, b^* that behave like a, b, respectively, in s_1 and s_2, but "switch roles" in s_3 and s_4. We have $a \neq a^*$ and $b \neq b^*$. But a^*, b^* seem equally good candidates for being the elements of the complete two-element graph. The proper response to this is that it should be part of the *specification* of each s_i that *no arbitrary objects other than* a, b take values in it.

7. On balance

It is now time to take stock. We have been concerned with the application of arbitrary object theory to the problem of the nature of mathematical structures. In particular, we have discussed the relative merits and demerits of Fine's account of mathematical structures on the one hand, and those of the prototype theory of mathematical structures on the other hand.

We have seen how the original version of Fine's account as well as the original version of the prototype account fail to give a satisfactory account of mathematical structures that admit non-trivial automorphisms: both of them contain unmotivated choice points. In a slight amendment of his original view, Fine proposes a satisfactory way out of this problem. Moreover, he rightly states that this solution can and should also be adopted by the prototype theory.

As Fine in addition points out, the prototype theory of mathematical structures is in addition beset by a uniqueness problem. According to the prototype theory in its first incarnation, there are always *many* prototypes, all of which are equally serviceable as *ante rem* mathematical structures. Any choice between them would be arbitrary, and any appeal to an implicit quantification over all of them would push us in the direction of *in rebus* structuralism. In short, we need a notion of prototype that ensures the relevant uniqueness.

In the form of incomplete arbitrary object spaces, H-theory contains the resources to identify the right notion of prototype: they are in some sense analogues in H-theory of Finean *initial* arbitrary objects. The resulting (further) **amended prototype theory** is then no longer vulnerable to Fine's non-uniqueness objection. Indeed, the amended prototype theory seems, at least so far, a perfectly satisfactory arbitrary object theoretical account of mathematical structures.

Because of its representationality, Fine's own amended account of mathematical structures is in addition marred by a version of Hellman's permutation problem. The prototype theory is not representational and is therefore not vulnerable to any permutation challenges. Moreover, the prototype theory, even in its amended version, is perfectly compatible with F-theory. I therefore recommend Fine to simply abandon representationality and adopt the amended prototype theory also.

At least on the face of it, none of all this tells against the background F-theory or against the background H-theory of arbitrary objects. This is in itself an interesting finding. True, because it has fewer primitive notions, H-theory is more economical. But there may well be decisive reasons for taking the relation of dependence between arbitrary objects as a primitive element of arbitrary object theory. Maybe the future will tell.

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