

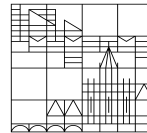
# **Formal Representations of Suspended Judgment**

**Doctoral thesis for obtaining the  
academic degree Doctor of  
Philosophy (Dr.phil.)**

submitted by  
Ali Zolfagharian

at the

Universität  
Konstanz



Faculty of Humanities  
Department of Philosophy

Konstanz, 2019



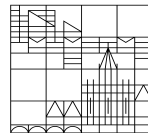
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Date of the oral examination: 23.04.2020

1. Reviewer: Prof. Spohn

2. Reviewer: Prof. Müller

3. Reviewer: Jun.-Prof. Dr. Carolin Antos-Kuby

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To Sahar

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FORMAL REPRESENTATIONS OF  
**SUSPENDED JUDGMENT**

## **Abstract**

This work investigates various theories in Formal Epistemology in order to observe their capacity of distinguishing suspended judgment from ignorance. Besides, it suggests how every theory in formal epistemology could be changed and improved to represent suspended judgment properly.

After an inquiry about the nature of suspended judgment and introducing its characteristics, I observe and suggest some improvement in various theories in formal epistemology namely AGM belief revision, Indeterministic belief revision, Bayesian Epistemology, Dempster Shafer theory of evidence, and Ranking Theory.

The text also suggests a new theory, the Acceptance Revision, which can properly represent all doxastic attitudes namely belief, disbelief, suspended judgment and ignorance.

FORMALE DARSTELLUNGEN VON  
**URTEILSENTHALTUNG**

## **Zusammenfassung**

Diese Arbeit untersucht verschiedene Theorien der formalen Erkenntnistheorie, um ihre Leistungsfähigkeit dahingehend zu beurteilen, ob sie die doxastische Einstellung der Urteilsenthaltung von bloßem Nichtwissen unterscheiden können.

Außerdem werden in der Arbeit Vorschläge gemacht, wie jede Theorie in der formalen Erkenntnistheorie geändert und verbessert werden kann, um die Urteilsenthaltung angemessen abzubilden.

Nach einer Untersuchung des Phänomens der Urteilsenthaltung und einer Einführung in ihre Charakteristika, schlage ich eine Verbesserung verschiedener Theorien in der formalen Erkenntnistheorie vor, nämlich AGM Belief Revision, Indeterministic Belief Revision, Bayesian Epistemology, Dempster Shafer Theory of Evidence und Ranking Theory.

In meiner Arbeit schlage ich auch eine neue Theorie vor, die ich Acceptance Revision nenne, die alle doxastischen Einstellungen richtig darstellen kann, nämlich Glaube, Ablehnung, Urteilsenthaltung und Nichtwissen.



# 0

## **Introduction**

Ideas and perspectives related to the suspension of judgment are scattered like an archipelago. This text gives a detailed map and insight into the meaning of suspension of judgment and its formal representation in various prevalent theories in formal epistemology. I found that reaching a unified account of suspended judgment is not achievable (it was not achievable regarding belief as well). During the inquiry, I tried to keep the plurality of various possible approaches, written and unwritten; to help the reader to form her ideas if she does not agree with the conclusion.

In the first chapter, the nature of suspended judgment is discussed. There are five key questions which shed light on the nature of the suspension of judgment (suspension). This inquiry leads us into ten formulae which hold in the entire

dissertation. This chapter contains the main assumptions, the adopted philosophical approaches, and the definitions of basic and primary concepts.

From the second chapter, we play the game of mathematics-meet-epistemology to find the proper formal representation of suspension. The starting point of the chapter is the preliminary report of the belief revision theory which considers the qualitative notion of belief to represent our doxastic states. The inadequacy of the belief revision in distinguishing ignorance and suspension compels us to turn to indeterministic belief revision. This version can represent formally all four possible doxastic states namely; belief, disbelief, suspension, and ignorance. For a better interpretation of the indeterministic belief revision, the four-valued logic by Belnap is applied. The chapter plays an instructive role in understanding the issues related to the formal representation of suspended judgment.

Chapter three addresses the quantitative notion of belief by discussing Bayesian epistemology. Bayesian probability and the development of probability theory by Kolmogorov helped epistemologists to construct a formal representation of degrees of belief. Bayesian epistemology is the received view in formal epistemology. One of its pillars is the Dutch book argument. I give some counterexample to invalidate the theory. Also, it fails to distinguish suspension and ignorance. The problem arises because of the principle of indifference.

We learn about Dempster-Shafer theory of evidence as a generalization of Bayesian epistemology in chapter four. The principle of indifference would not be applied in the theory of evidence. It creates an opportunity to distinguish the degrees of suspension from ignorance. By adopting the Lockean thesis, the Lottery paradox and the relationship between degrees of belief and four basic doxastic attitudes, namely suspension, belief, disbelief, and ignorance are solved. As this theory is a general version of Bayesian epistemology, we could say that the long-lasting problem in the formal epistemology, which is the lottery paradox, is not a paradox any longer. The idea of the movable or adjustable threshold, which I introduced in this chapter, is an

innovative achievement in the field of formal epistemology and it shifts the research questions from some open theoretical questions to some practical puzzles.

Ranking theory comes in chapter five. Ranking theory is a unique theory because it is able to represent the qualitative and the quantitative belief, disbelief, and suspension at the same time. The notion of degree of neutrality or un-opinionatedness inspired me to introduce the notion of the adjustable threshold. There is a nice relationship between this degree and the degree of suspension. Ranking theory is open for various accounts about the nature of ranks. I tried to give one of the possible accounts and interpretations based on the notion of the degree of contradiction (or surprise). In the last section of this chapter, general version of the ranking theory is introduced. It helps us to represent ignorance as well as suspension.

At the end of the journey, chapter six, we have the map of the paths, bridges, and islands that we passed. For gaining an overall view of the research, we compare the various formal representation of belief and suspension by mentioning their common assumptions, their different approaches, and their unwanted issues that remain open for further investigation. At the end of the chapter, I introduce my theory, Acceptance Revision, which captures the intuitions behind the Belief revision, Bayesian Epistemology, DS theory of Evidence, and Ranking Theory. The acceptance revision could be considered as the formal conclusion of the dissertation.

*We use 'I withhold assent' as short for 'I am unable to say which of the alternatives proposed I ought to believe and which I ought not believe,' indicating that the matters appear equal to us as regards credibility and incredibility...we maintain no firm opinion, ... And withholding assent [epochè] is so called from the intellect's being held back [epechesthai] in such a way as neither to assert nor deny, because of the equipollence of the matters in question.*  
*Outlines of Pyrronism XXII, Sextus Empiricus*

# 1

## Suspension of Judgment

Belief, as a doxastic attitude, is central to epistemology<sup>1</sup>. This central role has been extended to formal epistemology, and consequently, there is a natural tendency to define all epistemic states based on belief (or degrees of belief), e.g., defining disbelieving  $B$  as believing  $\neg B$ .

The central role of *belief* in mainstream epistemology was connected to the tripartite analysis of knowledge (justified true belief) and its inconclusive debate about

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<sup>1</sup> Heil, J. 2010. Belief, in *A Companion to Epistemology*, edited by Dancy J., and Steup, M., Wiley-Blackwell. Page: 254 also Huber, F., 2016, SEP, [plato.stanford.edu/entries/formal-belief](http://plato.stanford.edu/entries/formal-belief)

skepticism and knowledge. The widely accepted view is that when an agent knows *B*, it entails that she believes *B*. After Gettier counterexamples<sup>2</sup>, the debate on knowledge and skepticism started and shed light on the notion of knowledge, justification, and belief; e.g., we learned that our beliefs are fallible, and we should always be prepared to revise our beliefs<sup>3</sup>. We learned that the word *know* and *believe* are context sensitive and vary from one context to another. However, we learned a lot about belief; but the quantitative notion of belief, degrees of belief, and doxastic attitudes like ignorance were neglected in traditional epistemology.

The importance of suspending judgment and its role in the scientific investigation is neglected in epistemology and philosophy of science for the same reason. Unfortunately, the focus on *belief* has distracted us from other primary doxastic attitudes. A column in the New York Times shows that the importance of suspending judgment was acknowledged by a columnist on 2 Oct 1876:

*If there is one quality of mind more than another which can be said to be scientific in its bearing, it is that which is known as the power of suspending judgment. Almost every writer on scientific subjects lays it down as a necessary prerequisite for successful investigation...<sup>4</sup>*

In contrast with many research projects which neglect the role of suspension of judgment in formal epistemology, I agree with this columnist in the 19<sup>th</sup> century. Suspension of judgment plays crucial roles in our epistemic activities, and it demands a detailed investigation to shed light on the nature of suspended judgment as a doxastic attitude. Some questions about belief, could be asked about suspended

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<sup>2</sup> Gettier, E., 1963. Is Justified True Belief Knowledge? *Analysis*, 23(6), 121-123.

<sup>3</sup> Haack, S. 1979. Fallibilism and Necessity. *Synthese*, 41(1), 37-63. P: 37

<sup>4</sup> New York Times, Retrieved from <http://www.nytimes.com/ref/membercenter/nytarchive.html> by search. Keyword: suspension of judgment

judgment as well. Question about its object, its qualitative and quantitative notions and their relationship, and its relationship with other doxastic attitudes are all relevant.

During my research into formal representations of suspended judgment, I came to the conclusion that the answers to the following A-E questions, could bring us enough chalk to write about the possible formal representation of suspended judgment:

- A. How to distinguish suspension and ignorance?
- B. Is not-believing  $B$  (failing to believe<sup>5</sup>) and not-disbelieving  $B$  (not-believing-*not*- $B$  or failing to disbelieve), a necessary condition for suspension?
- C. Could an agent rationally suspend  $B$  and believe (or disbelieve)  $B$  simultaneously?
- D. Is suspension negation-invariant? Or does suspending  $B$  entail suspending  $\neg B$ ?
- E. Do we have degrees of suspension or quantitative notion of suspension?

As this inquiry is restricted to the above questions, it remains neutral about other possible epistemological aspects of the notion of suspension as far as it does not make change in its formal representation. By answering questions mentioned above, I reach some formulae which are valid in the entire text, and they are independently valid in the various theories which I discuss.

## 1 The distinction between ignorance and suspension

What does suspended judgment means? Some assert that suspension of judgment is non-belief attitude, which is *not-believing*  $B$  and *not-believing*  $\neg B$ <sup>6</sup>. If this suggestion

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<sup>5</sup> Salmon applies the same term, *failing to believe*, in his paper but his account about the relation between *suspension* and *failing to belief* differ from me.

Salmon, N. 1995. *Being of two minds: Belief with Doubt*, *Nous*, 29(1), 1-20, p: 1

<sup>6</sup> Chisholm, as Friedman says, asserts the position that withholding  $A$  is not accepting  $A$  and not accepting not  $A$ . She took acceptance and belief, and withholding and suspension of judgment the same.

Friedman, J., 2011, *Suspended Judgment*, *Philosophical studies*, Volume 162, issue 2, pp 165-181

works, then the notion of belief could play its central role flawlessly because suspension will be defined based on the notion of belief. Unfortunately, it is too simplistic to portray suspension as a non-belief attitude, and it could not capture suspension properly. Let  $Bel(A, B)$  be ‘the agent  $A$  believes that  $B$ ’ and  $Sus(A, B)$  be ‘the agent  $A$  suspends whether  $B$ ’.

$$Sus(A, B) \leftrightarrow \{ \neg Bel(A, B) \wedge \neg Bel(A, \neg B) \} \quad (1)$$

The formula (1), proposes that the *non-belief attitude* (*not-believing  $B$  and not-believing  $\neg B$* ), is a sufficient condition for defining suspension. It is flawed because it could not distinguish suspension and ignorance. The main reason is that failing to grasp  $B$ , should not entail suspending  $B$ . Besides, one who is ignorant about  $B$ , is not suspending whether  $B$ . For instance, I do not believe that your desk is oak. I do not believe that your desk is not oak. I simply do not know whether you have a desk or not. I have never thought of that before. I cannot say that I suspend whether you have an oak desk. I am simply ignorant. Another example is the cavemen example. According to (1), cavemen suspends whether Quarks exist because they do not believe or disbelieve that they exist! Obviously, (1) is flawed. Here comes the first question: how to distinguish suspension from ignorance?

For making a distinction between suspension and ignorance, we need to apply the *act* operation.  $B^{act}$  is an action that entails the acceptance of  $B$ . Actions like *asserting  $B$* , or doing something that needs the acceptance of  $B$ . Notice that the term *acceptance* is not necessarily the technical term which was coined by Cohen in epistemology<sup>7</sup>.  $B^{act}$  is doable for a rational agent, if and only if she believes that  $B$  or

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and Chisholm, R., 1976., *Person and Object*, Routledge (2013), P: 27

<sup>7</sup> I agree with many ideas in Cohen’s distinction between belief and acceptance like ‘a person who does not fully believe that  $P$  can nevertheless justifiably accept that  $P$ . The most important difference between Cohen and my account is that he ignores the fact that someone can accept a proposition and its negation at the same time. Besides, he disagrees that an agent who believes  $B$ , also accepts  $B$ . The

suspends whether  $B$ . A rational agent does not disbelieve  $B$  while she is performing  $B^{act}$ . She should at least find  $B$  permissible and she should think that the possibility that  $B$  is true, is not epistemically ignorable. In contrast to the cavemen, a scientist who suspends whether Quarks exist or not, might agree (by considering agreement as an action  $B^{act}$ ) with another scientist, who says that 'quarks exist'. She agrees without believing that 'Quarks exist'. (I think this account illuminates the meaning of the word *agreement*). Let  $Acc(A, B)$  be ' $A$  accepts  $B$ '. Then  $Acc(A, B)$  entails  $Sus(A, B) \vee Bel(A, B)$ . ( $A$  is a rational agent.  $A$  will not do  $B^{act}$  while she is ignorance about it or disbelieves  $B$ ). Now, we are ready to distinguish suspension from ignorance:

$$Sus(A, B) \leftrightarrow ( Acc(A, B) \wedge Acc(A, \neg B) ) \quad (2)$$

$$Ign(A, B) \leftrightarrow ( \neg Acc(A, B) \wedge \neg Acc(A, \neg B) ) \quad (3)$$

Accepting  $B$  means being committed that  $B$  is appropriate to be used for reasoning. The agent is committed because she has evidence in favor of the proposition. Acceptance could be translated as 'having evidence'. Accepting means having enough evidence for  $B$  to use it for reasoning. Notice that someone can accept  $B$  without believing  $B$ . For example, a scientist may accept that light is particle without believing that it is a particle. Other words can be used instead of *accepting* like endorsing or admitting.

Suspension is the acceptance of both sides,  $B$ , and  $\neg B$ , while ignorance is being unable to accept even one side. Ignorance is not-accepting (failing to accept)  $B$  and not-accepting  $\neg B$ . One can also define acceptance by *doubt*. If an agent doubt whether  $B$ , then she accepts  $\neg B$ . Therefore, suspending a proposition entails doubting the

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problem is that he assumes that the notion of commitment is related only to *acceptance* and not to *belief*. He says 'Acceptance implies commitment to policy of premising that  $P$ ... belief... not one goes along with the proposition as a premises ...'

There is not enough space here to discuss these issues. The meaning of acceptance regarding to the distinction between ignorance and suspension, make it enough clear for further application (I hope).  
Cohen, L. J. 1989. *Belief and Acceptance*, Mind, OUP, new series, Vol. 98, No. 391, pp. 367-389, retrieved from <http://www.jstor.org/stable/2254849>

proposition and its negation. So, the acceptance of a proposition necessitates the doubt of its negation. And if a proposition is unknown (ignorance), then the agent does not doubt the proposition as well as its negation.

## 2 The necessity of non-belief for suspension

Is *non-belief* a necessary condition for suspension? Friedman says that the answer is no<sup>8</sup>. I think her argument is not convincing. She showed that some definitions which endorse that non-belief is the necessary condition of the suspension, are flawed. I detail her thought-provoking observation; then we seek the answer to the question B.

The hypothesis that the state of non-belief is a necessary condition for suspension could be written by the following formula:

$$Sus(A, B) \rightarrow (\neg Bel(A, B) \wedge \neg Bel(A, \neg B)) \quad (4)$$

Being in the state of suspension necessitates being in the state of *non-belief*. There are some proposals that they add a condition to non-belief to capture the notion of suspension. A noticeable idea in this framework is that if *B* is suspended, then it should be considered, or it should be entertained by *A*<sup>9</sup>. Let *Con*(*A*, *B*) be '*B* is considered by *A*'. Then the second proposal is the following formula:

$$Sus(A, B) \rightarrow (\neg Bel(A, B) \wedge \neg Bel(A, \neg B)) \wedge Con(A, B) \quad (5)$$

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<sup>8</sup> Friedman, J. 2011. Suspended Judgment, *Philosophical studies*, Volume 162, issue 2, pp 165-181

<sup>9</sup> Wedgwood, R. 2002. *The aim of belief*, *Nous*, Vol. 36, Issue 16, p: 227:

"As I shall use the term, one 'suspends judgment' about *p* when one consciously considers *p*, but neither believes nor disbelieves *p*. (To 'consider' *p* is just to 'entertain' *p*; it is for *p* to 'occur' to one in occurrent thinking.)"

Hajek mentioned the same idea which he received from Daniel Stoljar. He used the term 'entertained' and add it as a condition in order to avoid accepting that cavemen were gnostic whether Quarks exist. Hajek, A. 1998. *Agnosticism meets Bayesianism*, *Analysis*, Vol. 58, No. 3, pp. 199-206, P: 205

(5) says that an agent,  $A$ , suspends whether  $B$ , only if she is considering  $B$  and is in the state of non-belief. Friedman says that *consideration* plus non-belief is no help to capture the concept of suspension because an agent who considers something and then she ignores that issue have no attitude (and consequently she does not suspend)<sup>10</sup>. For instance, I consider whether tomorrow the library is open or not, and I am in the state of non-belief, but I check my calendar, and I see that I should go to the doctor tomorrow. I stop thinking about the library. I considered whether tomorrow the library is open or not, but I do not suspend it, I simply just ignore all things about whether tomorrow the library is open or not. I did not form any doxastic attitude. I cannot say that it is a suspended judgment.

Another idea is that *suspension* is related to *belief resistance*. Let  $Res(A, B)$  be ' $A$  is in the state of belief resistance about  $B$ '. Another non-belief account is the following:

$$Sus(A, B) \rightarrow (\neg Bel(A, B) \wedge \neg Bel(A, \neg B) \wedge Res(A, B)) \quad (6)$$

(6) says that  $B$  is suspended if and only if  $A$  is in the state of non-belief and belief-resistance. Friedman says that an arachnophobic has a reason to stop forming any belief about spiders, but we could not say that she does suspend her judgment. Consider a proposition like a spider is an insect. I do not think that she suspends her judgment toward spider is an insect. Criticism seems compelling. (6) like (5) is not acceptable. Rejecting (5) and (6) leads us to the third idea. Maybe suspension happens because of having an epistemic reason. An agent who suspends whether  $B$ , have epistemic reason to suspends  $B$ . Let  $Epi(A, B)$  be  $A$  has epistemic reason to suspend whether  $B$ . The last non-belief account is:

$$Sus(A, B) \rightarrow (\neg Bel(A, B) \wedge \neg Bel(A, \neg B) \wedge Epi(A, B)) \quad (7)$$

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<sup>10</sup> Friedman, J. 2011. *Suspended Judgment*, Philosophical studies, Volume 162, issue 2, pp 165-181, p: 169

A class of counterexamples is those in which the agent suspends her judgment because of an epistemic reason, and then loses her reason<sup>11</sup>. Imagine an agent suspends whether there is life in Kepler 438b or not because she thinks that she could not know it at least in her lifetime. One day she learns from the radio that the Harvard-Smithsonian Center, which is one of the largest astrophysical institutions on our planet, is doing new research about Kepler 438b and they think that they will find an answer. After hearing this news from one of her reliable sources of knowledge, she loses her reason to suspend whether there is life in Kepler 438b or not, but still, she is in the state of suspension.

I think the above counter-example (and similar examples) cannot play its role properly. The first critical issue is the assumption that an agent who grasps *B* and does not know it is true or false, is in the state of suspension! As it was stated before, she is not in the state of suspension because she could be ignorant. Ignorance does not necessitate 'not-grasping.' Having zero evidence about the truth of a proposition is closer to ignorance and not suspension. If the agent knows that Kepler 438b is the most Earth-like planet (ESI(Earth similarity index)=.88), and also knows that radiation superflares make this planet uninhabitable, and thus suspends her judgment, then it is rational to say she suspends her judgment because she thinks ESI=.88 increases the likelihood and it is not ignorable possibility however she finds that the fact about radiation is also noticeable. But an agent that just knows that Kepler 438b is a planet, without further information is simply ignorant. Grasping and having no information should be called ignorance and not suspended judgment. Another issue is that after receiving the announcement from radio she loses her epistemic reason, but she has a new epistemic reason to suspend her judgment. She knows that in the following days, the new reports will give her more information and still she has all of those contradicting evidence.

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<sup>11</sup> Friedman, J. 2011. *Suspended Judgment*, Philosophical studies, Volume 162, issue 2, pp 165-181, 175

There is another problem that requires a plausible answer. The problem of forgotten evidence is a famous problem in traditional epistemology. Goldman raised the problem of forgotten evidence against evidentialism. He says 'Many justified beliefs are ones for which an agent once had adequate evidence that she subsequently forgot. At the time of epistemic appraisal, she no longer possesses adequate evidence that is retrievable from memory'<sup>12</sup>. In parallel fashion, we could make a new kind of counterexample. An agent suspends her judgment because of the epistemic reason (ER) at the time  $t$ , and then she forgot that epistemic reason at the time  $t + 1$  while she is still in the state of suspension. If we ask the agent, why she is suspending whether  $B$ , then she says that she does not have any specific reason. She just knows that she had a good reason to suspend her judgment. It seems one with zero evidence can suspend her judgment. But there is a question. Imagine an agent who believes that  $B$ . She does not have any evidence, and she knows that she formed this belief via a reliable belief-forming process. She receives new information against  $B$  from a reliable source. She thinks it is compelling for an ignorant agent to disbelieve  $B$  because of the evidence. She tries to remember her evidence, but she cannot. Therefore, she cannot compare her evidence to form the right doxastic attitude. She is wondering whether to suspend, disbelieve or believe  $B$ . I think she should disbelieve because she is committed to what she knows. Now, imagine she receives new evidence for  $B$ , and she does not know whether this is the same evidence that convinced her to believe  $B$  or not. If the answer is yes, she knows that she should suspend her judgment, if no, then she will believe  $B$ . What should she do? She will suspend because she is committed to what she knows. I think the forgotten evidence or forgotten epistemic reason is not a problem for this account. However, I know many might find my explanation inadequate.

I tried to explain Friedman's arguments against non-belief accounts of suspension. Even if we accept that all above non-belief accounts are flawed, it does not follow that another non-belief account cannot capture suspension properly. We can assume that

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<sup>12</sup> Goldman, A. 1999. *Internalism exposed*, Journal of philosophy, Vol 96, No 6, P: 271-293

non-belief is a necessary condition, or we can continue our investigation about the relationship between suspension and non-belief. I think non-belief is a necessary condition for suspension and ignorance. Notice that Friedman was interested to prove that suspended judgment is *sui generis*, and in her paper, I think, she was dealing with the phenomena and not with the rationality of suspending judgment. I think it is not rational to suspend a proposition without being in the state of non-belief. Also, it is not rational to believe and disbelieve a proposition at the same time. Further investigation demands a clear answer to the question C, which is also an answer to the question B.

Before going to the next section, I need to distinguish three ways that people mostly use the expression suspended judgment. First notion is the static suspended judgement, which means the agent has conflicting evidence for and against a proposition, and she cannot believe or disbelieve the proposition. It is what I am interested. The second case is about updating. If someone asks me about whether you will believe tomorrow that 'humans are causing global warming', I have to say that I do not know because it is indeterminate, and it depends on the new information. At any time  $t$  I cannot tell that at  $t + m$  whether I believe that humans are causing global warming or not. I think this does not mean that I should suspend my judgment for ever. I can believe, disbelieve or suspend my judgment based on my evidence that I have, while I know that I might change my mind in future. Third case is when an agent does not have enough evidence. If an agent grasps a proposition but she does not have enough evidence about the proposition, then she is ignorance. Ignorance and suspended judgment are not the same. From 'I understand  $B$  and I have no information about  $B$ ', does not follow that I suspend  $B$ . I call the first and the last case, respectively suspension and ignorance. Regarding to this text, in all theories in formal epistemology that they have updating rule, they are considering the second notion of suspension, and there is no problem there. I found that I should investigate into the first and the third notion of suspension, which with belief and disbelief are four basic doxastic attitudes.

### 3 Believing and suspending at the same time

One of the motivations for rejecting the idea that the state of non-belief is a necessary condition for suspension is that; sometimes, it seems that an agent, simultaneously, suspends and believes (or disbelieves) *B*. If an agent could believe and suspend at the same time, then non-belief is not a necessary condition for suspension, because the agent is not in the state of non-belief. We could split the examples in two groups. The first one relates to the difference between sense and reference and it opens the discussion about the object of doxastic states; the second one relates to what our action says about our doxastic attitudes and it leads us to the definition of rational agent and the notion of acceptance.

#### 3.1 Two-minded<sup>13</sup>

Assume *B* is 'Lake Konstanz is not the largest lake in Europe', and *C* is 'Bodensee is not the largest lake in Europe.' *A* believes *B* and suspends *C*. Unlike us; she does not know that Bodensee is the Konstanz lake. She learned from a reliable source that Konstanz is not the largest Lake. She has contradictory evidence about Bodensee, and she suspends whether *C*. Should we consider this case as a counterexample for a non-belief account? The short answer is no. The long answer is that *B* and *C* may refer to the same lake, but for the agent, *A*, they may be two different lakes. Also, notice that this is not a problem just for suspension. It may happen that an agent believes *B* and disbelieves *C*<sup>14</sup>.

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<sup>13</sup> However, I used the title of Salmon's paper, I think that we need not to discuss about the Frege's puzzle.

Salmon, N. 1995. Being of two minds: Belief with Doubt, *Nous*, 29(1), 1-20

<sup>14</sup> One might argue that the object of belief and suspension are different in nature. The conjunction of belief and suspension are different: 'A believes that *B*' and 'A suspends whether *B*'. We use the word *whether*. The conjunction whether is just a sign that the epistemic modal operator is not truth sensitive. We can say 'A knows whether *B*' and it is exactly like 'A knows whether  $\neg B$ '. 'A knows whether *B*' means *A* believes or disbelieves *B* and is not doubtful.

### 3.2 Suspending while having a high degree of belief

Sometimes it seems that our action contradicts our doxastic state. Imagine *A* has lost her pen while she was climbing a tree next to the river. She thinks that it is highly more probable that it is in the river than it is in the mud. She knows that if it is in the water, then she cannot find it anymore and it is gone. She starts to probe the mud to find the pen. Vasechi<sup>15</sup>, her friend, asks her: do you believe that your pen is in the mud? She answers: no. She asks: why are you probing the mud so?

The above example demands our explanation. It seems the agent is in the states of belief and disbelief, or suspense and disbelief at the same time. For finding the answer, we need to come back to the relation between action and the doxastic states.  $B^{act}$  is an action which entails the acceptance of  $B$ . Let  $Do(A, B^{act})$  be  $A$  is doing  $B^{act}$ . Doing  $B$  requires accepting  $B$ .

$$Do(A, B^{act}) \rightarrow Acc(A, B) \quad (8)$$

It is interesting to find the relationship between  $Acc(A, B)$  and basic doxastic states (belief, disbelief, suspension, and ignorance). In the formulas (2) and (3) its relationship with suspension and ignorance have been illustrated. If you agree that belief entails acceptance, then you find the following formulas convincing. Let  $Dis(A, B)$  be  $A$  disbelieves  $B$ . We have:

$$Bel(A, B) \leftrightarrow (Acc(A, B) \wedge \neg Acc(A, \neg B)) \quad (9)$$

$$Dis(A, B) \leftrightarrow (\neg Acc(A, B) \wedge Acc(A, \neg B)) \quad (10)$$

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<sup>15</sup> Imaginary character. It means literally 'For what'.

From (8) and (9) follows that

$$Bel(A, B) \leftrightarrow Dis(A, \neg B) \quad (11)$$

The above formula does not mean or entail that  $\neg Bel(A, B) \leftrightarrow Dis(A, B)$  or another similar-looking formula  $\neg Bel(A, B) \leftrightarrow Bel(A, \neg B)$ . Not believing B might be disbelieving B. When a proposition is not believed, then the agent might be in any other doxastic states namely disbelief, suspension or ignorance.  $Bel(A, B) \vee Bel(A, \neg B)$  is not a tautology in formal epistemology, but the formula  $Bel(A, B) \vee \neg Bel(A, B)$  is clearly a tautology.

Now, from (2), (3), (8), (9) and (10), we can reach to the formula which illuminates the answer to our question:

$$Do(A, B^{act}) \rightarrow (Bel(A, B) \vee Sus(A, B)) \quad (12)$$

The above formula says that if an agent is doing  $B^{act}$ , then she believes that  $B$  or she suspends whether  $B$ . In our example,  $A$  is doing  $B^{act}$ , and  $B$  is 'Pen is in the mud'. According to the formula (12), she believes that  $B$ , or suspends whether  $B$ . As she said to Vasechi that she does not believe that it is in the mud, we can conclude that she suspends whether it is in the mud or not. If a rational agent does  $B^{act}$  and she does not believe  $B$ , then she is in the state of suspension. Clearly, our action may have conflict. An agent, on the one side, can assert that she disbelieves  $B$  and on the other side, can do  $B^{act}$ . I think we know that it is not an ideal agent that we discuss.

Another candle that sheds light on the issue is the difference between the quantitative and qualitative notion of belief. In many cases that the agent suspends her judgment and it seems that she believes or disbelieves that proposition simultaneously, in fact, she just has very high or very low degree of belief. In the example, the agent  $A$  thinks that it is highly probable that the pen is in the river and it is not in the mud. So

she has a very low degree of belief that it is in the mud. Having a low degree of belief does not guarantee that the agent is in the state of disbelief.

The way that we apply the word *disbelief* is tricky. It should not be confused with 'not believed' or 'low degree of belief.' The answer to the question (C) is 'no'; a rational agent could not be in the state of suspension and belief (or disbelief) at the same time. Consequently, I prefer to agree with people who think non-belief is a necessary condition for suspension of judgment.

#### 4 Is suspension a negation-invariant operator?

Is suspension a negation-invariant<sup>16</sup> epistemic operator?  $Sus(A, B) \leftrightarrow Sus(A, \neg B)$  Does suspending  $B$  entail suspending  $\neg B$ ? An epistemic modal operation,  $Ope(A, B)$  could be called a negation-invariant operator, if and only if  $Ope(A, B) \leftrightarrow Ope(A, \neg B)$ . *Belief* and *disbelief* are not negation-invariant. If an agent believes  $B$ , she could not believe  $\neg B$ . Assume that suspension is not negation-invariant. Then we have three possibilities. i)  $Sus(A, B) \wedge Bel(A, \neg B)$ . The proposition  $B$  is suspended and its negation is believed. This cannot happen. Because if an agent believes  $\neg B$ , then she disbelieves  $B$  and as we showed before a rational agent cannot disbelieve and suspend  $B$  at the same time. ii)  $Sus(A, B) \wedge Dis(A, \neg B)$ . This is also rationally impossible because disbelieving  $\neg B$  entails believing  $B$  and an agent could not rationally believe and suspend whether  $B$  at the same time. Notice that I just applied the formula (11). iii)  $Sus(A, B) \wedge Ign(A, \neg B)$ . Having evidence or information about  $B$ , entails that we have information and evidence about  $\neg B$ . Therefore, we could not be in the state of ignorance about a proposition while we suspend its negation. Consequently,

$$Sus(A, B) \leftrightarrow Sus(A, \neg B) \quad (13)$$

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<sup>16</sup> It was hard to find a proper term for this property. Spohn suggested that negation-invariant might be a suitable option. It was one of many suggestions from him. I thought this term is the best option.

$$Ign(A, B) \leftrightarrow Ign(A, \neg B) \quad (14)$$

The argument for (14) is like (13). The formulas (2), (3), (9) and (10) which says about the relationship between acceptance and four doxastic states, endorse (13) and (14), we could consider it as an alternative proof.

## 5 Degrees of suspended judgment

Do we have degrees of suspension or a quantitative notion of suspension? The simplest case for illustrating the degrees of suspension is suspending our judgment as per the objective chances. Imagine we have two dices. The first one is an eight-sided dice, and the other one is a four-sided dice. An agent suspends whether it will be three in both cases. In the first case the chance is  $.125$  and in the second case is  $.25$ . (if we ask the agent that there is dice that we just know that it is not four-sided. She will not suspend her judgment whether it is seven or not because she thinks seven might not be a possibility. She thinks if it is a six-sided dice then there is no seven). Eight and four-sided dice gives the agent two different degrees of belief. Obviously, the degree of suspense, when the dice is a four-sided dice, is higher than the eight-sided dice because as per the relationship between acceptance and suspension, the degree of acceptance (commitment) is higher. According to (13), suspension is negation-invariant, we could conclude that the agent suspends her judgment whether it comes up three or not as well. The degree of belief for four-sided dice is  $.75$  and for eight-sided dice is  $.875$ . Which one could be the degree of suspension?  $.75$  or  $.25$ ?  $.125$  or  $.875$ ? How could we calculate the degree of suspension here? Let's come back to the definition of the qualitative notion of suspension. The suspension is about having contradictory evidence, and if the degree of belief or acceptance, represents the weight of evidence, then the degree of suspension could be the degree of contradictory evidence, which in this case, is the minimum of the degree of belief and disbelief. For four-sided dice is  $.25$  and for eight-sided dice is  $.125$ . Briefly, the degree of suspension is the degree of contradiction which is the degree of bilateral acceptance. The maximum degree of bilateral acceptance in four-sided dice is  $.25$ , and it is the degree of suspension. I found

some standards for defining the quantitative notion of suspension which helps us to avoid some obvious mistakes:

i) If we make a model to represent suspended judgment, then the agent should be able to turn from one doxastic attitude to another when she updates her epistemic state. This standard looks obvious. Any representation of doxastic attitudes should guarantee this possibility. This standard is important because there is a representation by Van Fraassen<sup>17</sup> that could not satisfy this standard and it is a crucial problem as Hajek already observed<sup>18</sup>. Van Fraassen proposes that for representing suspended judgment, one can use interval or vague probability functions.

ii) In case that the agent is in the state of non-belief (not believing and not disbelieving), the higher degree of suspension entails the lower degree of ignorance and vice versa.

iii) The degree of suspension for  $B$  and its negation is the same. However, their belief functions might be different. One can say the same thing about degrees of ignorance.

As I said in this text, the degree of suspension is the minimum degree of acceptance (or belief) of a proposition and its negation. For example, Dempster Shafer theory of evidence can work with this definition by defining the degree of suspension as the minimum degree of belief and disbelief (I show it in the chapter 5). There might be other suggestions which are like the notion of quantitative suspended judgment. For example, one might say that the degree of suspension is the degree of evidential support that an agent ignores to stay suspended. I explain it in the chapter 6. The threshold that define how much evidential support should be ignored is called the degree of neutrality or neutrality threshold as we have it in ranking theory<sup>19</sup>. I think the

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<sup>17</sup> in Van Fraassen, B. C. 1998. The agnostic subtly probabilified, *Analysis*, 58(3), p: 212-220

<sup>18</sup> Hajek, A. 1998. Agnosticism meets Bayesianism, *Analysis*, 58(3), 199-206

<sup>19</sup> Spohn W. 2012. *The laws of belief*, OUP, P: 76

degree of ignoring is not the degree of suspension. It seems the term ‘degree of unopinionatedness’ is a proper term as Spohn applied it<sup>20</sup>. I try (chapter 6) to show how degree of suspension and unopinionatedness are connected. Degree of unopinionatedness is very helpful way of explaining some epistemic phenomena. For instance, I have noticed that in all cases that the degree of ignoring is high, people use the terms like suspension of belief or suspension of disbelief. Because in their mind, a high degree of belief and disbelief are interchangeable. In this text, as it should be a technical text, a high degree of belief and qualitative belief are not interchangeable. I discuss later in chapter 6 about ranking theory and these standards.

Let’s finish this section with repeating those three standards which should be held and the definition of the degree of suspended judgment: i) agent should be able to turn from one qualitative and quantitative state to another. ii) In the state of non-belief, a higher degree of suspension entails a lower degree of ignorance and vice versa. iii) The degree of suspension of a proposition and its negation is the same. In addition, the quantitative suspended judgment is the minimum degree of acceptance of a proposition and its negation.

$$Sus^o(B) = \min \{Acc^o(B), Acc^o(\neg B)\} \quad (15)$$

## 6 Qualitative and Quantitative epistemic states

As on the one side we have a degree of belief, disbelief, suspension, and ignorance, and on the other side we have belief, disbelief, suspension, and ignorance; it seems reasonable to ask about the relationship between quantitative and qualitative doxastic attitudes. Which one is prior? There are three possible answers, and every answer seems in some aspect reasonable. i) the qualitative notion of belief (and other states) is prior, ii) quantitative notion of belief is prior, iii) neither. It means they are two

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<sup>20</sup> Spohn W. 2008. A survey of ranking theory, P: 23

independent concepts of belief, and we cannot explain one of them with the other one, but we might be able explain their interaction.

From believing  $B$  follows that the agent has a degree of belief. I mean in the presence of belief, always there is a degree of belief. But an agent may have a degree of belief, and no basic qualitative doxastic attitude. This fact may convince many to think that quantitative belief is prior.

The priority of quantitative notion of belief leads us to two key questions. i) How to represent degrees of belief? Which one could represent degrees of belief better: sharp or imprecise probability? Single or multiple probability functions? Additive or non-additive probability function? And ii) What is the relationship between the quantitative and qualitative notion of belief? A narrower question is how an agent takes a doxastic attitude, based on her degrees of belief? In the following chapters, I will discuss these issues.

There are cases that an agent with a low degree of acceptance, prefers to accept, qualitatively, a proposition. Like accepting the logical consequence of scientific findings that looks counterintuitive like Banach-Tarski paradox. In contrast to Banach-Tarski paradox, there are mathematicians who do not accept to change their option in Monty Hall problem<sup>21</sup> however they know that mathematically they should. Another case is when a person knows it is very likely that she loses her money in lottery, but she accepts to buy the ticket. Buying the ticket in lottery sometimes means that she accepts that she is winning. These cases lead us to the notion of the degree of commitment.

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<sup>21</sup> The problem is stated as follows: Assume that a room is equipped with three doors. Behind two are goats, and behind the third is a shiny new car. You are asked to pick a door, and will win whatever is behind it. Let's say you pick door 1. Before the door is opened, however, someone who knows what's behind the doors (Monty Hall) opens *one of the other* two doors, revealing a goat, and asks you if you wish to change your selection to the third door (i.e., the door which neither you picked nor he opened). The Monty Hall problem is deciding whether you do. The correct answer is that you *do* want to switch. Wolfram Math World: Retrieved from <http://mathworld.wolfram.com/MontyHallProblem.html>

For assigning the degree of commitment or acceptance to a proposition based on a qualitative set of updates, I thought I need to define a logic of possibilities. The degree of each possibility will be defined based on the degree of contradiction, or degree of surprise after receiving new information. The degree is computable in different ways like using various sorting algorithms to sort possibilities. I made the theory to give an interpretation about what ranks in ranking theory means, however it was not completely successful.

The third position is *neither*. The lottery paradox is one of the motivations for saying that quantitative and qualitative notion of belief are independent. I think a normative theory in formal epistemology should show their relationship. There are lots of assumption that we could revise to solve problems like lottery paradox. Assumptions like ‘all propositions should have the same threshold’, or ‘the belief threshold should be higher than .5’. My solution is working on degrees of acceptance and the qualitative notion of acceptance instead of belief. As accepting a proposition and its negation at the same time is possible (suspension), an agent can accept qualitatively a proposition  $B$ , however her degree of acceptance of  $B$  is less than the degree of acceptance of  $\neg B$ . This approach allows me to revise the assumption that the acceptance threshold should be higher than .5.

## 7 Doxastic attitudes: definitions and relationships

I, almost, like many researchers in formal epistemology, agree to take that the object of belief is a proposition. Besides, I assumed that *believing*  $B$  is equivalent with *disbelieving*  $\neg B$ . By assuming those assumptions, I tried to answer some questions about the nature of suspended judgment. During my research I reached to eleven formulae, from (A1) to (A11), which are valid in the entire dissertation:

$$\text{Sus}(A, B) \leftrightarrow (\text{Acc}(A, B) \wedge \text{Acc}(A, \neg B)) \quad (\text{A.1})$$

$$\text{Ign}(A, B) \leftrightarrow (\neg \text{Acc}(A, B) \wedge \neg \text{Acc}(A, \neg B)) \quad (\text{A.2})$$

$$\text{Sus}(A, B) \rightarrow (\neg \text{Bel}(A, B) \wedge \neg \text{Bel}(A, \neg B)) \quad (\text{A.3})$$

$$\text{Do}(A, B^{\text{act}}) \rightarrow \text{Acc}(A, B) \quad (\text{A.4})$$

$$\text{Bel}(A, B) \leftrightarrow (\text{Acc}(A, B) \wedge \neg \text{Acc}(A, \neg B)) \quad (\text{A.5})$$

$$\text{Dis}(A, B) \leftrightarrow (\neg \text{Acc}(A, B) \wedge \text{Acc}(A, \neg B)) \quad (\text{A.6})$$

$$\text{Bel}(A, B) \leftrightarrow \text{Dis}(A, \neg B) \quad (\text{A.7})$$

$$\text{Do}(A, B^{\text{act}}) \rightarrow (\text{Bel}(A, B) \vee \text{Sus}(A, B)) \quad (\text{A.8})$$

$$\text{Sus}(A, B) \leftrightarrow \text{Sus}(A, \neg B) \quad (\text{A.9})$$

$$\text{Ign}(A, B) \leftrightarrow \text{Ign}(A, \neg B) \quad (\text{A.10})$$

$$\text{Sus}^o(A, B) = \min \{ \text{Acc}^o(A, B), \text{Acc}^o(A, \neg B) \} \quad (\text{A.11})$$

*Now, here is the problem. People often have inconsistent commitments. The persons whose beliefs are consistent is, in fact, a rarity. [sic] Yet it is absurd to suppose that a person who has inconsistent commitments is thereby committed to everything. If, by oversight, I believe both that I will give a talk on campus at noon, but also that I will be in town at noon, this hardly commits me to believing that the Battle of Hastings was in 1939.*

Paraconsistent belief revision, Graham Priest

# 2

## Belief Revision

Even though belief is a matter of degree, the binary belief (or opinion) plays a crucial role in our daily life. Doyle<sup>22</sup> who could be considered as the founder of belief revision follows Dennett in distinguishing between binary judgmental assertion (opinions) and graded underlying feelings (belief). In many cases, we assert the conclusion of an argument that its premises are all believed, though we find the conclusion unlikely or

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<sup>22</sup> Hansson, S.O., 2011, Logic of Belief Revision, retrieved from '<http://plato.stanford.edu/entries/logic-belief-revision/>' [Accessed: 12.02.2015]

counterintuitive. We cannot describe this assertion without distinguishing degrees of Belief from opinion (or Belief).

Doyle in his paper where he introduced his Truth Maintenance System, explains why 'opinion revision system' might be the more accurate name:<sup>23</sup>

*I have used the term 'belief' freely in this paper, so much so that one might think the title 'Belief Revision System' more appropriate, if no less ambitious, than 'Truth Maintenance System.' Belief, however, for many people carries with it a concept of grading, yet the TMS has no non-trivial grading of beliefs. Perhaps a more accurate label would be 'opinion revision system', where I follow Dennett in distinguishing between binary judgemental assertions (opinions) and graded underlying feelings (beliefs).*

In belief revision or Truth Maintenance System (TSM), as it was called in AI and computer science, we start with beliefs or opinions per se. The epistemic state is represented by a set of sentences which the agent believes. These beliefs are not only those beliefs that we believe in, or explicitly believe. We should also consider beliefs which we are implicitly committed to believe. Obviously, in belief revision, the qualitative notion of belief plays the key role.

Briefly, the epistemic state of an agent could be represented by a belief set; this belief set is a set of sentences that an agent is committed to believe. As the (rational) agent should believe all logical consequences of her beliefs, all those consequences belong to her belief set as well<sup>24</sup>. The above property, the epistemic closure, could be based on classic or non-classic logic.

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<sup>23</sup> Doyle, J., 1979, A Truth Maintenance System, *Artificial Intelligence* 12, 231-272: P. 268

<sup>24</sup> Hansson, S.O., 2011, Logic of Belief Revision, retrieved from '<http://plato.stanford.edu/entries/logic-belief-revision/>' [Accessed: 13.02.2015] Hansson has cited from Isaac Levi how he has drawn this distinction

There are various belief revision systems. Theories differ in their assumption about the consistency of the belief set, the definition of epistemic closure, and allowing multiple belief sets. The question is how we could represent suspended judgment in belief revision. One may find two promising approaches: Indeterministic belief revision and paraconsistent belief revision.

Indeterministic belief revision allows representing the epistemic state by multiple (possible) belief sets while it applies the classic logic to define the epistemic closure for each possible belief set. Paraconsistent logic, on the contrary, disagrees that the belief set should be consistent, and it applies paraconsistent logic to define the epistemic closure. It might look implausible, but it is not. Priest says that there are good grounds for supposing that an ideally rational agent *must* have inconsistent beliefs<sup>25</sup>. It seems that he is in some respects right. It seems having inconsistent belief or information, is the reason for many inquiries. I prefer to replace the word *belief* with *accepted propositions* or *endorsed proposition* because of the definitions of belief (1.9) and disbelief (1.10), generally, I prefer to avoid the notion of 'inconsistent belief' here. A belief set, and a belief base should be consistent, but I need to capture the intuition that a rational agent might have the same non-neutral doxastic attitude toward two conflicting propositions. As these two theories, Indeterministic and Paraconsistent Belief Revision, are both a contradiction-tolerant model, they look plausible. However, they need some amendments for the sake of uniform terminology.

There are two critical questions to answer. If we work with Indeterministic belief revision, we need to find a way to define the acceptance set based on the set of all possible belief sets. When we work with paraconsistent belief set, we need a solution for defining the epistemic closure.

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<sup>25</sup> Priest, G. 2001. *Paraconsistent Belief revision*, Theoria 67: 214-228. P: 218

In this chapter, I start with the traditional belief revision (AGM), and I follow the inquiry with Indeterministic Belief Revision (IndBR), and eventually paraconsistent belief revision (Qualitative acceptance revision QAR)<sup>26</sup>.

## 1 Traditional Belief Revision (TBR)

We start with traditional belief revision. It provides enough intuition about how a belief revision theory might look like. There are two fundamental assumptions with regards to the project, the formal representation of suspended judgment. First, the belief set should be consistent. Second, we apply classic logic. From the second assumption follows that the *ex contradictione quodlibet* (ECQ) or the principle of explosion is valid in TBR.

A reason that some think that a belief set should be consistent is ECQ. It says that all sentences (everything) follow from a single contradiction. If an agent believes a contradiction or two contradiction sentences, then she is committed to believe all possible sentences (the whole language) and believing everything is absurd. As TBR applies classic logic, it endorses that a contradiction leads the agent to the triviality. Paraconsistent logic rejects ECQ.<sup>27</sup> An unwanted epistemic result of TBR is that all inconsistent epistemic states are the same!

Traditional belief revision assumes that a belief set should be consistent and after learning new information, we should stay consistent. In many cases, it seems plausible. For instance, imagine that an agent believed that all animals taste with their mouthpart

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<sup>26</sup> We could make more than one paraconsistent belief revision as there are many paraconsistent logics. See Priest, G, Routley R., and Norman J. (eds.), 1989. *Paraconsistent Logic: Essays on the Inconsistent*, Munich: Philosophia Verlag

<sup>27</sup> Priest, G., Tanaka, K., Weber, Z. 2016(last revision). *Paraconsistent Logic*. SEP. [Access 16 Jan 2017]. Retrieved from <https://plato.stanford.edu/archives/win2016/entries/logic-paraconsistent/>

( $B$ ). Surprisingly, she learns that a butterfly tastes by touching something with its feet, so she learns that  $\neg B$ . After learning  $\neg B$ , she should revise her belief set which contains  $B$ . She should give up her belief that all animals taste with their mouthpart. If she adds  $\neg B$  without giving up  $B$ , then she is committed to believe two contradicting sentences and consequently their conjunctions which is a contradiction ( $B \wedge \neg B$ ). If she believes a contradiction, then, based on *ex contradictione quodlibet* (ECQ) that anything could be inferred from contradiction, she should believe all sentences.

## 1.1 Logical consequence operator

Let  $L$  be the set of all sentences and  $L_i$  be a subset of  $L$ . Then the function  $Cn(L_i)$  gives the set of all logical consequences of  $L_i$ . The function  $Cn$  is a *Tarskian* logical consequence function if it satisfies the following three conditions:

$$\text{Inclusion} \quad L_i \subseteq Cn(L_i) \quad (1)$$

$$\text{Monotony} \quad L_i \subseteq L_j \rightarrow (Cn(L_i) \subseteq Cn(L_j)) \quad (2)$$

$$\text{Idempotence} \quad Cn(L_i) = Cn(Cn(L_i)) \quad (3)$$

As per (1), the set of all logical consequence of  $L_i$ , contains all members of  $L_i$ . (2) says that the logical consequence of a set is subset of the logical consequence of its superset. (3) says that  $Cn$  is an idempotence operator and the set of logical consequence of a set which is closed under logical operation,  $Cn(Cn(L_i))$ , is equivalent to the set  $Cn(L_i)$ . In other words, if a set is a deductive closure then the iteration of its logical consequence remains invariant under iteration.

## 1.2 Belief set

Now, one can define the belief set based on a Tarskian logical consequence function:

*Let  $L$  be the set of all sentences or the whole language and  $K$  be a subset of  $L$ , then  $K$  is a belief set if it is consistent and closed under logical consequences.*

If an agent believes all sentences in  $K$ , then she is committed to believe  $Cn(K)$ . When  $B \in K$ , it means  $B$  is accepted in  $K$ , and when  $\neg B \in K$ , it means that  $B$  is rejected in  $K$ . In contrast to the main line, there is another variant, which is presented by Pearce and Rautenberg. The idea is that believing  $\neg B$  does not entail the rejection of  $B$ <sup>28</sup>. As it allows to believe a proposition without rejecting its negation. I do not investigate on all possible variant in belief revision. As per (A.7) believing a proposition entails disbelief of its negation. Notice that the definition of belief is accepting a proposition and not accepting its negation. Therefore, one cannot believe and disbelieve a proposition at the same time. Let's go back to the TBR.

A set of propositions  $K$  is a belief set if and only if it satisfies two conditions:

$$\text{Consistency} \quad \neg(K \vdash \perp) \quad (4)$$

$$\text{Epi. Closure} \quad (K \vdash B) \rightarrow (B \in K) \quad (5)$$

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<sup>28</sup> Gardenfors, P., Rott, H., 1995, *Belief Revision*, in Handbook of Logic in Artificial Intelligence and Logic Programming, Volume 4, Edited by Gabbay, M., Hogger, C. J., Robinson J.A., Clarendon Press: P. 47 The main idea is that the negative information should be distinguished from positive information. They have started the paper by Gardenfors epistemic modelling of intuitionistic propositional logic, and the definition of proposition as an element of a class of function from  $K$  to  $K$ . As I understood, the key assumption in Gardenfors paper is  $\neg B$  is accepted if and only if the acceptance of  $B$  leads to contradiction. Pearce and Rautenberg have presented a model that it 'tolerate conflicting epistemic attitudes, towards propositions without leading to epistemic absurdity'. The paper which you find the idea is Pearce D., Rautenberg W. 1991. *Propositional logic based on the dynamics of disbelief*. In: Fuhrmann A., Morreau M. (eds) *The Logic of Theory Change*. Lecture Notes in Computer Science (Lecture Notes in Artificial Intelligence), vol 465. Springer, Berlin, Heidelberg

The reason for the assumption (4), which asserts that the set should be consistent, is that based on *ex contradictione quodlibet* ( $\{B, \neg B\} \models C$ , for all  $B$  and  $C$ ), from an inconsistent belief set anything follows, you can infer all sentences. Notice that as per (5), *Epistemic closure*, the agent is committed to believe all sentences in  $L$ , and it is irrational to believe everything.

### 1.3 Belief base

Notice that it is possible to have two different sets with the same logical consequences, for example, the logical consequences of  $K_i = \{B, B \supset C\}$  and  $K_j = \{B, C\}$  are the same ( $Cn(K_i) = Cn(K_j) = K$ ). We call a set  $K_j$  a belief base for  $K$ , iff  $Cn(K_j) = K$ . Obviously, based on the logic that we apply or our philosophical account of epistemic closure, the relationship between belief set and belief base varies.

There are various inconsistent *belief bases* and only one inconsistent *belief set*. Again, as per the definition of belief, I prefer to say that a belief base should not be inconsistent, and an acceptance base can be inconsistent.

### 1.4 Update rules

AGM or traditional belief revision works based on a belief set, and the agent revises her belief when she receives new information. A system that after receiving new information loses minimum information to stay consistent is called by Doyle a *Problem-solver* or a *Truth Maintenance System*<sup>29</sup>. The *problem* means *incompatible input*. The *problem* appears because new information is not compatible with what agent believes. In Doyle's words:

*How a problem solver revises its beliefs influences how it acts. Problem solvers typically revise their beliefs when new information (such as the expected effect of an action just taken or an observation just made) contradicts previous*

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<sup>29</sup> Doyle, J., 1979, A Truth Maintenance System, *Artificial Intelligence* 12, 231-272. P: 231

*beliefs. These inconsistencies may be met by rejecting the belief that the action occurred or that the observation occurred*<sup>30</sup>.

The new information does not always contradict our old beliefs. Also, sometimes we simply retract our belief without learning new information. This case happens mostly when we learn that the source of knowledge is not reliable. The most complicated belief change is in fact revision, where we ask ourselves how we should restore consistency again<sup>31</sup>. In general, there are three different changes in our belief set: Expansion, Contraction, and Revision. I explain them in detail.

### 1.4.1 Expansion

When we learn new information, which is not inconsistent with our beliefs, we simply add the sentence to our belief set. It is called *expansion*. For example, I learned that Kiwi is an animal that lives in New Zealand. I did not know that before, so I simply add this sentence to my belief set.

If one learns new information  $B$ , and she observes that  $B$  is compatible with her belief set,  $K$ , then  $K + B = Cn(K \cup \{B\})$  is her new belief set. Besides, if  $K_i$  is a belief base for  $K$ , then  $K_i \cup \{B\}$  is a base for  $K + B$ . Notice that during expansion she does not revise any old belief and she does not, lose information. This is one of the core assumptions in TBR.

Checking consistency is NP-complete. When we have a huge amount of data, and we receive new information, it is hard to check the consistency<sup>32</sup>. Even in our daily life

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<sup>30</sup> Doyle, J., 1979, A Truth Maintenance System, *Artificial Intelligence* 12, 231-272. P: 259

<sup>31</sup> Concerning to the belief change, I find TBR unsatisfactory and unrealistic. Sometimes the process of revision takes time. During that process, we suspend our judgment. When we focus on belief for representing our epistemic states, then in the dynamic we should define learning as believing! The noticeable part of learning is wondering and suspending our judgment without making new hasty belief.

<sup>32</sup> More information about NP-complete problems:

we learn and add new information to our belief set, and after a while, we learn that we are inconsistent. When there is a huge amount of data, it is likely to have an inconsistent belief set. This likelihood of having inconsistent beliefs is another motivation for accepting a paraconsistent logic and working with a paraconsistent belief revision model<sup>33</sup>.

### 1.4.2 Contraction

Contraction is the second possible change. Sometimes we retract a belief from our belief set. Assume that I believe that there is an interior pathway between the building A and the building V in Konstanz University. (Someone told me that from the fourth floor you could reach to the building V from the building A) However, unfortunately, I cannot find any way at the fourth floor to reach the building A. This experience cannot convince me that there *is* or there *is not* any internal way from V to A. So, I retract my belief that there is an internal way between building A and V at Konstanz University.<sup>34</sup>

If an agent believes  $B$ , and then she learns that  $B$  could be true or false and it is not only true (like having conflicting evidence for and against  $B$ ), then she should retract  $B$  from her belief set. As the agent, should retract the least information from her database, after the contraction, she should be certain that her belief set is the largest subset of her last belief set which  $B$  does not belong to it. This constraint is a core assumption in TBR. There is a problem: this policy does not support the unique

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Weisstein, E. W. *NP-Complete Problem*. MathWorld--A Wolfram Web Resource.

<http://mathworld.wolfram.com/NP-CompleteProblem.html>

More information about Non-monotonic reasoning and NP-completeness of consistency check:

Strasser, C. and Antonelli, 2001. (2016 last revision) Aldo, *Non-monotonic Logic*, SEP, Retrieved from <https://plato.stanford.edu/archives/win2016/entries/logic-nonmonotonic/>:

"...in order to determine whether a default is triggered by a pair of sets of sentences, one has to perform a consistency check, and such checks are not computable."

<sup>33</sup> See section "Motivations>Artificial intelligence>automated reasoning and >Belief revision" in Priest, G., Tanaka, K., Weber, Z. 2016(last revision). *Paraconsistent Logic*. SEP. [Access 16 Jan 2017]. Retrieved from <https://plato.stanford.edu/archives/win2016/entries/logic-paraconsistent/>

<sup>34</sup> Imagine an agent who believes  $B$  because of the evidence  $E$ , loses her evidence  $E$ , then she should ignore her belief without being in the state of disbelief. A similar concept with *contraction*, could be found in Law. *Repeal without replacement* provide an intuitive understanding of contraction.

outcome, it needs another constraint. In some cases, there are more than one logical possible retraction. For instance, assume  $K = Cn(\{B, C, B \wedge C\})$ . If an agent wants to retract  $B \wedge C$ , she has at least two options: retracting  $B$  or  $C$ . Notice that by deleting just  $B \wedge C$ , the set  $\{B, C\}$  entails  $B \wedge C$  as its logical consequence.

The agent might think that  $B$  is more important (preferable or valuable) than  $C$ , and it worth to keep  $B$  instead of  $C$ . Therefore, she keeps  $B$ . In traditional belief revision, the agent can compare all sentences, and she could order them based on their importance or based on how much they are entrenched.

There are some postulates for contraction in TBR. Let  $K \dot{-} B$  be the contraction of  $K$  by  $B$ . Then  $K \dot{-} B$  should satisfies all six following postulates:

$$\text{Closure} \quad K \dot{-} B = Cn(K \dot{-} B) \quad (6)$$

$$\text{Success} \quad B \notin Cn(\emptyset) \rightarrow B \notin Cn(K \dot{-} B) \quad (7)$$

$$\text{Inclusion} \quad K \dot{-} B \subseteq K \quad (8)$$

$$\text{Vacuity} \quad B \notin Cn(K) \rightarrow (K \dot{-} B = K) \quad (9)$$

$$\text{Extensionality} \quad B \leftrightarrow C \in Cn(\emptyset) \rightarrow (K \dot{-} B = K \dot{-} C) \quad (10)$$

$$\text{Recovery} \quad K \subseteq (K \dot{-} B) + B \quad (11)$$

The *Closure* (6) says that after a contraction, the belief set should satisfy the epistemic closure (5). The *Success* postulate (7) says that if we retract a sentence and the sentence is not a tautology, then it should not belong to the new belief set. (8) is *Inclusion*. A contraction should not produce new information, and the new belief set

should be a subset of the old belief set. Vacuity (9) says that the contraction of a sentence which does not belong to the old belief set, could not make a new belief set. The postulate (10) guarantees that the contraction of two equivalent sentences reaches to the same belief set. And eventually, the recovery postulate, maybe the most controversial postulate, says that the contraction and then expansion of a sentence reaches us to the old belief set or a richer one.

There are various counterexamples against recovery, and it is still open for new developments. As a counterexample, imagine an agent believes that 'Bats are warm-blooded, and they are birds.' Then the agent learns that they are not warm-blooded. She retracts both because she knows if it is a bird, then it is warm-blooded as well. Again, the agent learns that Bats are mammals and she tends to add the sentence that 'bats are warm-blooded.' to her belief set. Based on postulates (6) - (11), the agent should undo and reach to her last belief set. In other words, she should believe that a bat is a bird and therefore a warm-blooded animal.<sup>35</sup> The structure of an example could be the following: Adding  $B$  to a belief set which contains  $B \rightarrow C$  leads to believing  $C$ ; naturally when we retract  $B$ , we tend to retract  $C$  as well, because we believed it because of  $B$ ; while, TBR suggests to believe  $C$  even after retraction of  $B$ )

The operation  $\dot{\div}$  is called *Partial meet contraction* if and only if it satisfies postulates (6) -(11).

$$\text{Conjunctive Inclusion} \quad B \notin (K \dot{\div} B \wedge C) \rightarrow (K \dot{\div} B \wedge C) \subseteq K \dot{\div} B \quad (12)$$

$$\text{Conjunctive overlap} \quad (K \dot{\div} B) \cap (K \dot{\div} C) \subseteq K \dot{\div} (B \wedge C) \quad (13)$$

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<sup>35</sup> For more similar examples see the section *Recovery and its avoidance* in Hansson, S.O., 2011, *Logic of Belief Revision*, retrieved from 'http://plato.stanford.edu/entries/logic-belief-revision/' [Accessed: 13.02.2015]

For see an interesting idea to solve the problem see Rott, H. and Pagnucco, M. 2000, *Severe Withdrawal (and recovery)*, Journal of philosophical Logic, 29, 501-547

(12), *Conjunctive Inclusion*, pronounces that if a sentence does not belong to the contraction of the conjunction of two sentences, then the contraction of that sentence is a superset of the contraction of the conjunction of that sentence with another sentence. The reason is clear. The contraction by  $B \wedge C$  could lead us to three possible results. I) Retracting  $B$  and  $C$  both, II) retracting just  $B$ , or III) retracting just  $C$ . If  $B$  does not belong to the contraction by  $B \wedge C$ , then there are two possibilities number (I) and (II) because by retracting just  $C$ ,  $B$  is still there. In both cases the consequence of the conditional in (12) will hold.

(13), *Conjunctive overlap* pronounces that the intersection of the contraction by  $B$  and the contraction by  $C$ , is a subset of the contraction by  $B \wedge C$ .

#### 1.4.2.1 Entrenchment

If an agent should retract a believed proposition from her belief set, she should retract the least important one. For example, I am in the library, and I believe that Ali has a black headphone ( $B$ ). And, I believe that I see the world like other people, if I see something white, other people will see it white as well ( $C$ ). My friend, Nils comes toward me and says: a moment ago, I saw Ali in the library with his white headphone, which he always wears. I like to buy the same brand... Now, I realize that Nils believes that Ali's headphone is white! I think that I should retract  $B$  or  $C$ . Because  $B$  and  $C$  entail that Nils should observe that the headphone is white. It seems rational to retract  $B$ , instead of  $C$ ; because  $C$  is a general hypothesis and it is more valuable than  $B$ .

Rott defines epistemic entrenchment by the following statement:

*A sentence  $a$  is epistemically less entrenched in a belief state  $k$  than a sentence  $b$ , if and only if a person in belief state  $k$  who is forced to give up either  $a$  or  $b$ , will give up  $a$  and hold on  $b$ <sup>36</sup>.*

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<sup>36</sup> Rott, H., 1991, *Preferential Belief Change Using Generalized Epistemic Entrenchment*, P. 1

Why are some beliefs more important or entrenched than other beliefs? Let me explain it in Gärdenfors and Makinson words:

*Certain pieces of our knowledge and beliefs about the world are more important than others when planning future actions, conducting scientific investigations, or reasoning in general.<sup>37</sup>*

Let  $Val(B)$  be the value or the importance of  $B$ . Then  $Val(B) < Val(C)$ , means  $B$  is less entrenched than  $C$ .  $Val(B) \leq Val(C)$  means  $C$  is as least as  $B$  entrenched. And  $Val(B) \asymp Val(C)$  means they are equally entrenched.

There are five postulates for epistemic entrenchment:

$$\text{Dominance} \quad B \vdash C \rightarrow Val(B) \leq Val(C) \quad (14)$$

$$\text{Transitivity} \quad (Val(B) \leq Val(C) \wedge Val(C) \leq Val(D)) \rightarrow (Val(B) \leq Val(D)) \quad (15)$$

$$\text{Conjunctiveness} \quad Val(B) \leq Val(B \wedge C) \vee Val(C) \leq Val(B \wedge C) \quad (16)$$

$$\text{Minimality} \quad B \notin K \rightarrow \forall C (Val(B) \leq Val(C)) \quad (17)$$

$$\text{Maximality} \quad \forall B (Val(B) \leq Val(C)) \rightarrow C \in \text{Cn}(\emptyset) \quad (18)$$

The postulate (14) is Dominance. If  $C$  is the logical consequence of  $B$ , then  $B$  is equal or less entrenched than  $C$ . If an agent is about to retract  $C$ , then she has to retract also  $B$ . Because retracting  $C$  brings  $B$  again to the belief set, and it means the problem is not solved. The agent should retract  $B$ . For example,  $B \vee C$  is equally or more entrenched than  $B$ . It seems somehow counterintuitive, because we think  $B$

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<sup>37</sup> Gärdenfors P. and Makinson D. Revisions of knowledge systems using epistemic entrenchment. In Moshe Y. Vardi, editor, Proceedings of the Second Conference on Theoretical Aspects of Reasoning About Knowledge, pages 83–95, Monterey, California, March 1988. Morgan Kaufmann. P:88

contains more information than  $B \vee C$ . Number (15) the transitivity guarantees that epistemic entrenchment is transitive. As per the (14) we can infer, that  $Val(B \wedge C) \leq Val(B)$  and  $Val(B \wedge C) \leq Val(C)$ . From this inference and (16) we can conclude that  $Val(B) = Val(B \wedge C) \vee Val(C) = Val(B \wedge C)$ . It simply guarantees that after the contraction, one of the sentences  $B$  or  $C$  will be retracted from belief set. As per these three postulates (14) -(16), there is not any room for indecision.<sup>38</sup> For any arbitrary sentences, we are always able to decide to give up  $B$  or  $C$  or both. Because for any two non-tautology sentences, we have  $Val(B) \leq Val(B \wedge C) \vee Val(C) \leq Val(B \wedge C)$ . In case of  $Val(B) \leq Val(B \wedge C)$  and  $Val(C) \not\leq Val(B \wedge C)$ , we reach to  $Val(B) < Val(C)$  and we should retract  $B$ . In the case of  $Val(B) \leq Val(B \wedge C) \wedge Val(C) \leq Val(B \wedge C)$  we reach to  $Val(C) \asymp Val(B)$ , and we should retract both.

(17) is minimality, and it says that if a sentence is not in our belief set, then it is less or equal entrenched than all sentences in our belief set. Rott mentioned an unpleasant result of this postulate which you do not have any graduation for sentences that they do not belong to your belief set<sup>39</sup>. Moreover, the last postulate, (18), states that a tautology is equal or more entrenched than all sentences in our belief set. Again, Rott argues that as per this postulate 'it is forbidden to assign to some sentence a degree of epistemic entrenchment which is as high as the degree of logical truth.'<sup>40</sup>

The following is the definition of the entrenchment-based contraction:

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<sup>38</sup> Rott, H., 1991, *Preferential Belief Change Using Generalized Epistemic Entrenchment*, Journal of Logic, Language and Information, P:45-78, P: 48: '...it is forbidden to stay undecided in the comparison of two arbitrary sentences...'

<sup>39</sup> Ibid: '...it is forbidden to have a gradation of the sentences which are not included in the current belief set K...'

<sup>40</sup> Ibid

$$C \in (K \dot{-} B) \leftrightarrow ((C \in K \wedge (\text{Val}(B) < \text{Val}(B \vee C)) \vee B \in \text{Cn}(\emptyset))) \quad (19)$$

In the right side of the above formula, notice that from  $\text{Val}(B) < \text{Val}(B \vee C)$  follows  $\text{Val}(B) \neq \text{Val}(B \vee C)$ . It means that  $C$  and its logical consequence  $B \vee C$  belong to the belief set  $K \dot{-} B$ . Notice if  $\text{Val}(B) \neq \text{Val}(B \vee C)$  then we should retract  $B$  and  $C$  both.

### 1.4.3 Revision

The last possible epistemic change is the revision. Sometimes we cannot simply add what we have learned because our belief set will be inconsistent. We should revise our belief set. For example, I learn that 'Butterflies taste with their feet'  $C$ . I believed that butterflies taste with their proboscis and I believed that Butterflies do not taste with their feet  $\neg C$ . After learning  $C$ , first, I should retract a sentence which it is inconsistent with my new information  $\neg C$  from my belief set, and then, I can add what I have learnt  $C$ .

By applying asterisk  $*$  as a symbol for revision, we have  $K * C = (K \dot{-} \neg C) + C$ . This formula is called *Levi identity*, which it shows the connection between contraction and revision.

$$\textit{Levi Identity} \quad K * C = (K \dot{-} \neg C) + C \quad (20)$$

Besides, there is another formula which it is called Harper identity:

$$\textit{Harper Identity} \quad K \dot{-} C = (K * \neg C) \cap C \quad (21)$$

The Levi identity (20) and the Harper identity (21) reflect the relationship between revision and contraction. Like *partial meet contraction*, we have the *partial meet revision* by following six postulates:

$$\text{Closure} \quad K * B \text{ is a belief set} \quad (22)$$

$$\text{Success} \quad B \in (K * B) \quad (23)$$

$$\text{Inclusion} \quad K * B \subseteq K + B \quad (24)$$

$$\text{Vacuity} \quad \neg B \notin K \rightarrow (K * B = K + B) \quad (25)$$

$$\text{Extensionality} \quad (B \leftrightarrow C) \rightarrow (K * B = K * C) \quad (26)$$

$$\text{Consistency} \quad \neg(B \vdash \perp) \rightarrow \neg(K * B \vdash \perp) \quad (27)$$

The closure postulate, (22), is clear. Success, (23), says that if a sentence is not a tautology, then the sentence belongs to the revision of the belief set with that sentence. (24) states that a revision retracts and adds a sentence but expansion always add a sentence to the belief set. Therefore, the outcome of revising by a sentence is always a subset of the expansion by the same sentence.

Vacuity (25) pronounces that if a sentence does not belong to a belief set, then the revision of the belief set by that sentence is equal to the expansion of the belief set with that sentence. Extensionality, (26) says the revision of a set, with two logically equivalent sentences, leading to the similar belief set. Moreover, (27) says the revision of a belief set with a consistent sentence, is a consistent set and if we consider (22), it

means they both guarantees that the new set is a belief set and it satisfies (4) and (5). Besides, we have two supplementary postulates:

$$\text{Superexpansion} \quad K * (B \wedge C) \subseteq (K * B) + C \quad (28)$$

$$\text{Subexpansion} \quad \neg B \in \text{Cn}(K * C) \rightarrow ((K * C) + B \subseteq K * (B \wedge C)) \quad (29)$$

The formula (28), Superexpansion, and (29), Subexpansion, are related to the contraction supplementary postulates. Let me finish this section by Hansson words. He explains the relationship between them by the following text:

*Let \* be the partial meet revision defined from the partial meet contraction  $\div$  via the Levi identity. Then \* satisfies superexpansion if and only if  $\div$  satisfies conjunctive overlap. Furthermore, \* satisfies subexpansion if and only if  $\div$  satisfies conjunctive inclusion<sup>41</sup>.*

#### 1.4.4 Two general rules

In addition to the condition (4) that we should stay Consistent, and the condition (5) that we should accept all logical consequences of our belief set, which are two constraints of belief set, there are two constraints for belief change. Imagine there are  $n$  possible ways of updating a belief set after receiving new information and we need to pick the best possible update. I call each possible way  $K_{new}^n$  like  $K_{new}^1$ ,  $K_{new}^2$ , and etc. Let's  $Val(B)$  be the epistemic value of  $B$ , which shows how much a belief is entrenched or is epistemically valuable, and  $K_{new}$  be the best possible update. Then two constraints of belief updates are the following formulae:

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<sup>41</sup> Hansson, S.O., 2011, Logic of Belief Revision, retrieved from '<http://plato.stanford.edu/entries/logic-belief-revision/>' [Accessed: 26.01.2017]

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$$\text{Preservation} \quad \forall n \quad |K_{new}^n \cap K| \leq |K_{new} \cap K| \quad (30)$$

$$\text{Entrenchment} \quad \forall n \quad |K_{new}^n \cap K| = |K_{new} \cap K| \rightarrow Val(K - K_{new}^n) < Val(K - K_{new})^{42} \quad (31)$$

The principle (30) says that we should minimize the amount of information loss.  $K_{new}$  preserved equally or more than any other possible belief updates. (31) says that if we should retract a belief, we should retract the least important one because we should keep more entrenched beliefs<sup>43</sup>. Therefore,  $K_{new}$  is epistemically retracts less valuable sentences. Above formulae (30) and (31) are not necessary for understanding of this section. They are only two general condition for belief change.

## 2 The representation of doxastic attitudes in TBR.

As TBR represents our epistemic states by a belief set, the question is how we could represent unknown, disbelieved, and suspended sentences by just a simple belief set. The disbelief set could be made based on the belief set: the set of the negation of all believed sentences is the disbelief set.

### 2.1 Disbelief set

As per (A.7)<sup>44</sup>, an agent disbelieves  $B$  if and only if the agent believes  $\neg B$ . For each belief set,  $K$ , we can define a set of sentences  $D$  such that for all sentences in  $K$ , the negation of the sentence is in  $D$ . For example, if the belief set is  $K = \{B, C\}$ , we have  $D = \{\neg B, \neg C\}$ . By assuming this symmetry between belief and disbelief (A.7), the

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<sup>42</sup> The value of a set is the sum of the value of its members

<sup>43</sup> Rott, H. 2000. *Two Dogmas of Belief Revision*, The Journal of Philosophy, 97(9), 503-522  
Rott considers both principles are dogmas.

<sup>44</sup>  $Bel(A, \neg B) \leftrightarrow Dis(A, B)$

representation of one of them,  $K$  or  $D$ , is enough to represent the other one. A model which does not reflect this relationship, probably could not endorse (A.7).

The definition of a disbelief set (32) and the (33) shows the relationship between belief and disbelief set are the following:

$$D = \{B \mid \neg B \in K\} \quad (32)$$

$$\forall B (B \in K \leftrightarrow \neg B \in D) \quad (33)$$

## 2.2 Suspension and ignorance

A sentence, which belongs to  $L$ , is believed if it belongs to the belief set  $K$ . If its negation belongs to the belief set, then it is disbelieved. If the sentence and its negation do not belong to the belief set, then it is suspended, or the agent is ignorant about the sentence. How to know the epistemic attitude towards a sentence which is not believed or disbelieved? Is it suspension or ignorance? By suspended sentence, I mean a sentence which is not believed or disbelieved but the agent has conflicting evidence for and against it. Imagine an agent receive information about a sentence, and she add it to her belief set, then she receives information against it, while still she finds first received information reliable. Then retract the sentence without believing or disbelieving the sentence. On the other side, there are sentences which are unknown. The agent does not have any information about them. For making a distinction I think TBR cannot distinguish them. One may say, one way to distinguish suspension and ignorance, is defining  $L$  as a set of sentences such that the sentence or its negation is accepted by the agent<sup>45</sup>. Then  $L$  contains all believed, disbelieved and suspended sentences (or there is no unknown sentence in the language). Now, any sentence that does not belong to  $L$  is unknown to the agent. Another way for distinguishing

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<sup>45</sup> I think if we look at the Ranking theory as the general account of TBR, then this is how it defines the language. For ranking theory, the language contains suspended, believed and disbelieved sentences. The interpretation of suspension is an open topic. We will come back to the issue when we are working on Ranking theory.

suspension and ignorance is by another alternative definition of  $L$  such that  $L$  is the set of all sentences such that the sentence or its negation is not acceptable for the agent. Then  $L$  contains all unknown, believed, and disbelieved sentences; and any sentence that does not belong to  $L$  is suspended. Unfortunately, TBR does not say anything about this issue explicitly. We could infer from the definition of contraction and expansion, that any sentence that belongs to  $L$  and it is retracted is an unknown sentence for the agent. Besides, expansion says that we learn new information that we did not know, and we add it to our belief set. Therefore,  $L$  is the set of unknown, believed, and disbelieved sentences. These solutions clearly do not make sense, because it is not how TBR was introduced. We know that  $L$  is simply the set of all sentences and above suggestions are meaningless.

Let me illustrate the problem of representing suspended and unknown sentences by an example:

Ali does not have any idea what Phaistos Disc<sup>46</sup> is. He just knows that it is a historical object in the museum. He does not believe that Phaistos Disc was from Crete. Also, he does not disbelieve that Phaistos Disc was from Crete. We can say that he has non-belief attitude towards the sentence that Phaistos Disc was from Crete.



1. Phaistos Disc

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<sup>46</sup> Surprisingly I learned that this disk might be a fake historical object.

On the other side, we have Sara. She is an archeologist, and she researches on this object. There is a controversial debate whether Phaistos Disc was from Crete or not. Sara can easily convince Ali that it was from Crete or it was not. Sara is capable because she has convincing evidence for both sides. Ali and Sara are epistemically different. Sara suspends her judgment whether Phaistos Disc was from Crete or not, and Ali is ignorant that it was from Crete or not. Is it possible to represent this distinction in traditional belief revision?

Unfortunately, no. Because there is no way to define them both at the same time by our TBR model. For each sentence in the language, there are three possibilities: it is in the belief set (belief), its negation is in the belief set (disbelief), or the sentence or its negation is not in belief set (ignorance or suspension of judgment). We need four possible states for each sentence in our model to distinguish suspension of judgment and ignorance. The sentences that are out of the belief and disbelief set could play one role, and it is not sufficient.

TBR like many other simple theories in formal epistemology could not make any room for suspension and ignorance at the same time. It considers suspension and ignorance the same thing.

### 3 System of Spheres<sup>47</sup> and Indeterministic Belief Revision

Grove proposed the systems of spheres which is related to the TBR postulates for the belief change. Let  $M_L$  be the set of all possible belief sets  $k_i$ . For any belief set  $K$ , define  $[K]$  as  $\{k_i \in M_L: K \subseteq k_i\}$  which are all interpretations that make  $K$  true. If it is an inconsistent belief set, then  $[K_{\perp}] = \{k_{\perp}\}$ . In the same way and for the simplicity, for any sentence like  $B$ , we could write  $[B]$  instead of  $[\{B\}]$ .

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<sup>47</sup> Grove, A., 1988, *Two Modelling for Theory Change*, Journal of Philosophical Logic, Vol. 17, No. 2, P: 157-170

For any subset  $S$  of  $M_L$ , which could be called an sphere,  $t(S)$  gives the intersection of  $\bigcap\{x \in S\}$ . Notice that  $t(S)$  is a belief set. In case of  $S = \emptyset$ ,  $t(S)$  will be an inconsistent belief set.

There are five properties of the function  $t: 2^{M_L} \rightarrow \mathbf{K}$ :

$$t([K]) = K \quad (34)$$

$$\neg(t(S) \vdash \perp) \leftrightarrow \neg(S = \emptyset) \quad (35)$$

$$\forall B \forall S \subseteq M_L \quad (t(S) \cap [B]) = t(S)/B \quad (36)$$

$$\forall S_i, S_j \subseteq M_L \quad (S_i \subseteq S_j \rightarrow t(S_j) \subseteq t(S_i)) \quad (37)$$

$$\forall K_i, K_j \in \mathbf{K} \quad (K_i \subseteq K_j \rightarrow [K_j] \subseteq [K_i]) \quad (38)$$

(34) says that the set of formulas of all interpretation of a belief set is that belief set. (35) says that the formulas of  $S$  are consistent if and only if it is not the empty set. The formula ' $t(S)/B$ ' in (36), means the smallest belief set containing both  $t(S)$  and  $B$ , which is  $\{C: B \rightarrow C \in T\}$  or  $\text{Cn}(t(S) \cup \{B\})$ . (37) says that for the formulas of the superset of any sphere is a subset of the formulas in the sphere itself. (38) says that the set of the interpretations of a belief set is subset of the set of the interpretations of its superset.

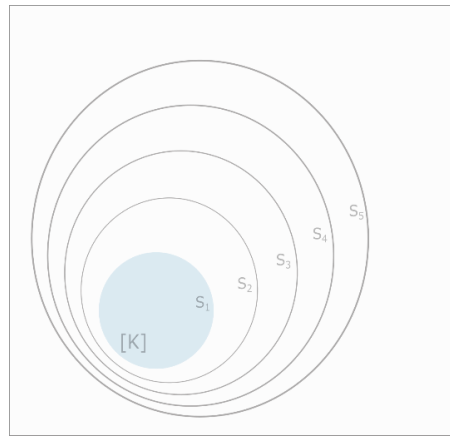
Let  $\mathbf{S}$  be the the collection of some subsets of  $M_L$ , then we call  $\mathbf{S}$  a system of spheres centered on  $X$  if it satisfies the following conditions:

$$U, V \in \mathbf{S} \rightarrow (U \subseteq V \vee V \subseteq U) \quad (39)$$

$$\forall U \in \mathbf{S} (X \subseteq U) \tag{40}$$

$$M_L \in \mathbf{S} \tag{41}$$

$$\forall B, \exists U \in \mathbf{S}, \forall V \in \mathbf{S} ((U \cap [B] \neq \emptyset \wedge V \cap [B] \neq \emptyset) \rightarrow U \subseteq V) \tag{42}$$



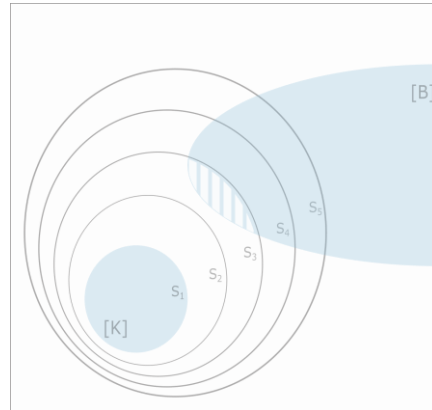
2. A system of spheres centered on  $[K]$

The first condition (39) says that for any two spheres in a system of spheres, one of them is the subset of the other. The second condition (40) is about the center of the system. If a system of spheres centered on  $X$ , then  $X$  is the subset of all spheres in the system. (41) simply shows that  $M_L$  always belongs to the system. The last formula (42) says that if the intersection of all interpretations of a sentence and a sphere in  $\mathbf{S}$  is not empty set, then there is the smallest sphere in  $\mathbf{S}$  that intersect with all interpretations of that sentence <sup>48</sup>.

The revision of a belief set  $K$  with  $B$  belongs to the intersection of  $[B]$  and  $[K]$ . The diagram (3) illustrates how the system of spheres explains revision. The revision belongs to the stripes area.

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<sup>48</sup> Grove, A., 1988, *Two Modelling for Theory Change*, Journal of Philosophical Logic, Vol. 17, No. 2, P: 159

3. The diagram of  $K * B$ 

### 3.1 Indeterministic Belief Revision: from spheres to ellipses

Traditional belief revision is deterministic. For a belief set and a given input  $B$ , we have one outcome. Indeterministic belief revision proposes that we should allow for there being several equally reasonable revisions of a theory (belief set) with a given proposition. Consider  $K$  as a belief set, and  $B$  a proposition. The agent is going to revise  $K$  with  $B$ . Each fallback is a subset of  $K$  which it is consistent with  $B$  (we can call it a  $B$ -permitting subset of  $K$ ). As we said in the last section, we have a family of spheres, or possible fallbacks such that they are nested. Lewis's sphere semantics for counterfactuals is centered on a single world, and in grove's model, they are centered on a theory. In diagram (3), the stripe area is the strongest  $B$ -permitting fallback of  $K$  expanded by  $B$ .

#### 3.1.1 relational belief revision

Indeterministic belief revision proposes a relational belief revision instead of functional belief revision. In TBR, connectedness is guaranteed by (14) - (19). If we give up connectedness, which it means some sentences are incomparable, then the belief revision will not be functional anymore. Instead of a family of spheres, as we have in TBR, we will have a family of ellipses. When we give up connectedness, we have a



$$\begin{aligned} \text{Revision by conjunction} \quad & ( K_i \in K_m * C \wedge \neg B \notin K_i ) \rightarrow ( K_i + B \in K_m * \\ & (B \wedge C) ) \end{aligned} \quad (48)$$

$$\begin{aligned} & (K_i \in K_m * C \wedge \forall K_j (K_j \in K_m * \\ & B \vee C \rightarrow \neg C \notin K_j)) \rightarrow \\ & \exists K_j (K_j \in K_m * B \vee C \wedge K_i = K_j + C) \end{aligned} \quad (49)$$

Seriality (43) says that for any belief set and a proposition, there is a set which is the possible revision of that set by the proposition. Success, (44), says that any possible revision of  $K_m$  with  $B$ , should have  $B$  as its member. The formula (45) is expansion, and it says that if a belief set is consistent with a proposition, then the revision of that set with that proposition is simply their expansion or  $K_i = K_m + B = K_m * B$ .

Strong consistency postulate (46) says that in the case of revising a set by a consistent proposition, the result is consistent as well. Substitutivity or extensionality, (47) says that if two propositions are equivalent, then the revision of a belief set with those propositions reach the same outcome.

The formula (48) is the revision by conjunction. The expansion of  $K_i$  a possible belief set of  $K_m$  with  $C$  while  $\neg B$  does not belong to  $K_i$ , is the possible belief set of the revision of  $K_m$  with  $B \wedge C$ . Obviously as  $K_i$  is the revision of  $K_m$  by  $C$ , we know that  $C$  is its member (success postulate), so, when we expand it with  $B$ , as it is compatible with  $B$ , we reach to a new set that  $B$  and  $C$  belongs to that set. (48) says that this set is the possible revision of the main belief set with the conjunction of  $B$  and  $C$ .

The last postulate (49), says that if all possible revision of a belief set with the disjunction of  $B$  and  $C$  does not have  $\neg C$  as their member, then among them there is a belief set which its expansion with  $C$  is the revision of the first belief set with  $C$ .

By simple step, we can define functional belief revision based on this relational belief revision. The belief revision is *functional* when  $(K_i \in K_m * B \wedge K_j \in K_m * B) \rightarrow K_i = K_j$ . It means, every two possible revisions of  $K_m$  with  $B$  are the same and there is just one outcome for the revision function of a belief set and an input proposition.

### 3.1.2 Epistemic Entrenchment

There are five postulates for IndBR epistemic entrenchment. The connectedness cannot be derived from these postulates. In case of  $Val(B) \leq Val(C)$  we say that  $C$  epistemically is as entrenched as  $B$ . The axioms (50)- (54) is the rationality requirement for a logically omniscient agent.

$$\text{Dominance} \quad B \vdash C \rightarrow Val(B) \leq Val(C) \quad (50)$$

$$\text{Transitivity} \quad (Val(B) \leq Val(C) \wedge Val(C) \leq Val(D)) \rightarrow Val(B) \leq Val(D) \quad (51)$$

$$\text{Conjunctive} \quad (Val(B) \leq Val(C) \wedge Val(B) \leq Val(D)) \rightarrow Val(B) \leq Val(C \wedge D) \quad (52)$$

*Closure*

$$\text{Bottom} \quad (\perp \notin K_m \rightarrow B \notin K_m) \leftrightarrow (Val(B) \leq Val(\perp)) \quad (53)$$

$$\text{Top} \quad Val(T) \leq Val(B) \rightarrow \vdash B \quad (54)$$

(50) and (51) are respectively like (14) and (15). Conjunctive closure (52), differ from (16) and it is the reason that the connectedness need not hold anymore. As per (52) if a proposition is less entrenched than two other propositions, then it is less entrenched than their conjunction. We could not derive that for any two propositions one of them is equally entrenched as their conjunction. This postulate says that to have (16) for two propositions, the agent should be able to compare them (or they should

be connected). By adding the following postulate IndBR epistemic entrenchment is equivalent to the TBR epistemic entrenchment:

$$\text{Connectedness} \quad \text{Val}(B) \leq \text{Val}(C) \vee \text{Val}(C) \leq \text{Val}(B) \quad (55)$$

By accepting (55), we could infer (16). For the proof, first, assume  $\text{Val}(B) \leq \text{Val}(C)$  then you have (16), after that, assume  $\text{Val}(C) \leq \text{Val}(B)$  and again you can infer (16), therefore their disjunction gives us (16) as well.

Any two possible belief sets of an agent, have one of these three possible relationships: one of them may be the subset of the other one. This case cannot happen in Indeterministic belief revision, because of preservation, if one possible belief set is a superset of another possible belief set, then the agent should take the superset and ignore the smaller one. Second, they are disjoint sets. This cannot happen because all possible belief sets contain all tautologies. Third, they are overlapping sets. In this case, two possible belief sets intersect, and there are some sentences that they belong just to one of the possible belief sets. So, if an agent has more than one possible belief sets, then for any two possible belief sets of the agent, their difference cannot be the empty set. Now, by this description, I illustrate that indeterministic belief revision allows conflicting possible belief sets.

As an example, imagine  $B \in K_i \wedge B \notin K_j$  and  $C \notin K_i \wedge C \in K_j$ . If the agent revises them with  $\neg B \vee \neg C$ , then she could have  $\neg C \in K_i \wedge C \in K_j$  and  $B \in K_i \wedge \neg B \in K_j$ . Therefore, an agent has contradicting possible belief sets. Notice that, from this result, it does not follow that  $\neg C$  and  $C$  are not connected. For any possible belief set a proposition and its negation as per the revision postulate should be connected. The proof is simple. According to the strong consistency, a proposition does not belong to the belief set, or its negation does not belong. Therefore, at least is as entrenched as the contradiction. If the other one belongs, then we could compare, if not, they are

equally entrenched. Obviously, an agent always can compare the epistemic entrenchment of a proposition and its negation.

An agent may find it possible to believe a proposition in a possible belief set and disbelieve it in the other one. Could we call it suspended judgment?

## 4 The representation of suspended judgment in IndBR

Imagine we have a set of possible belief sets. How could we say that a proposition, in general, is believed or not? In the example, we had  $\neg C \in K_i \wedge C \in K_j$ , is  $C$  suspended? May I say that it is believed and disbelieved at the same time!! I prefer to say *no*. so, for the sake of uniform terminology, it seems compelling to say that every member of a possible belief set as an accepted proposition. Then, we could say that  $C$  and  $\neg C$  are accepted. And as per the definition of acceptance, accepting two contradicting propositions at the same time, is permitted.

### 4.1 Accepted proposition

It seems that we could have a definition for acceptance in IndBR: A proposition is accepted if and only if there is a possible belief set that contain that proposition.

$$\text{IndBR acceptance} \quad \text{Acc}(B) \leftrightarrow \exists K_i (B \in K_i) \quad (56)$$

In other words, (56) says that a proposition is accepted if and only if the agent can accept it in a consistent acceptable theory. Or if it is possible to believe a proposition, then it is accepted.

In the above example, imagine  $E$  is unknown or  $\neg E \notin K_i \wedge E \notin K_j$ . The proposition like  $C \wedge E$  is not accepted, and its negation is accepted in  $K_i$  because  $\neg C \in K_i$ . It follows that  $C \wedge E$  is disbelieved.

## 4.2 Basic doxastic attitudes

Four doxastic states belief, disbelief, suspension, and ignorance could be defined as follows:

$$\text{IndBR Belief} \quad \text{Bel}(B) \leftrightarrow \exists K_i (B \in K_i) \wedge \neg \exists K_j (\neg B \in K_j) \quad (57)$$

$$\text{IndBR Disbelief} \quad \text{Dis}(B) \leftrightarrow \neg \exists K_i (B \in K_i) \wedge \exists K_j (\neg B \in K_j) \quad (58)$$

$$\text{IndBR Ignorance} \quad \text{Ign}(B) \leftrightarrow \neg \exists K_i (B \in K_i) \wedge \neg \exists K_j (\neg B \in K_j) \quad (59)$$

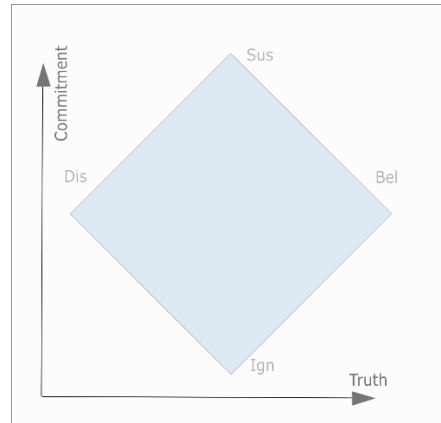
$$\text{IndBR Suspension} \quad \text{Sus}(B) \leftrightarrow \exists K_i (B \in K_i) \wedge \exists K_j (\neg B \in K_j) \quad (60)$$

All above formulas are clear enough. Imagine an agent suspends  $B$  and does not know about  $E$ , what should be her doxastic attitude about  $B \wedge E$ ? We could find the answer according (56) -(60) also we could do it with an epistemic bi-lattice.

## 4.3 Epistemic bi-lattice

We can apply a bi-lattice to explain how this model works. There are two axes: the horizontal axis which is the truth axis, and the vertical axis which is the commitment axis. In the truth axis from left to right, we had  $\text{Dis}(B) <_{\text{truth}} \text{Ign}(B), \text{Sus}(B) <_{\text{truth}} \text{Bel}(B)$ . It simply means that if an agent disbelieves  $B$ , she thinks that the proposition is not true. If she thinks that it is unknown, then at least it could be possibly true, therefore its place is after disbelief. The same could be said for suspension. And in the right side, an agent who believes  $B$ , thinks that  $B$  is true. In the vertical axis, which is commitment axis we have another inequality  $\text{Ign}(B) <_{\text{Comm}} \text{Bel}(B), \text{Dis}(B) <_{\text{Comm}} \text{Sus}(B)$ . When an agent does not know about  $B$ , she does not have any commitment. When she suspends, she is fully

committed to both side. And when she believes or disbelieves, she is committed just to one side. In this text, in general, four basic doxastic states are illustrated by the following diagram:



6. Epistemic Bi-lattice

For the conjunction of two propositions, we could apply minimum (meet). For example, if  $C$  is believed and  $E$  is unknown, their conjunction  $C \wedge E$  is unknown. If one of them is disbelieved, their conjunction is disbelieved. For the disjunction of two propositions we could apply maximum (join). For example, if  $C$  is believed and  $E$  is unknown, their disjunction  $C \vee E$  is believed. It is easy to observe the validity of the following table:

$\wedge$	Bel	Dis	Ign	Sus	$\vee$	Bel	Dis	Ign	Sus	$\neg$
Bel	Bel	Dis	Ign	Sus	Bel	Bel	Bel	Bel	Bel	Dis
Dis	Dis	Dis	Dis	Dis	Dis	Bel	Dis	Ign	Sus	Bel
Ign	Ign	Dis	Ign	Dis*	Ign	Bel	Ign	Ign	Bel**	Ign
Sus	Sus	Dis	Dis*	Sus	Sus	Bel	Sus	Bel**	Sus	Sus

Table 1

There are four dark squares which demand more clarification. Two squares (\*), say that the conjunction of a suspended proposition and an unknown proposition is a disbelieved proposition. We could simply investigate this claim in two different ways. First, we apply (56) -(60) to prove it. Assume that  $B$  is suspended and  $E$  is unknown. We know that  $\exists K_i (B \in K_i) \wedge \exists K_j (\neg B \in K_j)$ . If  $K_j$  contains  $\neg B$ , we are certain that  $\neg(B \wedge E)$  belongs to  $K_j$ . Therefore,  $\exists K_j \neg(B \wedge E) \in K_j$ . On the other side as there is no possible belief set that contains  $E$ , we could infer that there is no possible belief set that contains  $B \wedge E$  as well. Therefore,  $B \wedge E$  is disbelieve or in the other words  $\neg \exists K_i (B \wedge E) \in K_i \wedge \exists K_j \neg(B \wedge E) \in K_j$ . Second, we could reach to the same conclusion with the bi lattice. As  $B$  is suspended, it means that  $Dis(B) <_{truth} Sus(B)$ , if we add the conjunction of  $Ign(E)$  we have  $Dis(B) \wedge Ign(E) <_{truth} Sus(B) \wedge Ign(E)$  we have (a)  $Dis(B \wedge E) <_{truth} Sus(B) \wedge Ign(E)$ . On the other hand, we have  $Ign(E) <_{comm} Dis(E)$ , and in the same way  $Ign(E) \wedge Sus(B) <_{comm} Dis(E) \wedge Sus(B)$ , and eventually we reach to the second part (b)  $Ign(E) \wedge Sus(B) <_{comm} Dis(B \wedge E)$ . From (a) and (b), we have could conclude that  $Dis(B \wedge E) <_{comm} Ign(E) \wedge Sus(B) <_{comm} Dis(B \wedge E)$ . It means that the conjunction of a suspended and an unknown proposition is disbelieved.  $Ign(E) \wedge Sus(B) \rightarrow Dis(B \wedge E)$ .<sup>50</sup> The argument for two squares \*\* is the same.

#### 4.4 Some unwanted results

As per the above theory and the way that we define doxastic states (56) -(60) we could reach to some epistemic states that they are not intuitively convincing. In our example, we observed that if  $B$  is suspended and  $E$  is unknown, their conjunction  $B \wedge E$

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<sup>50</sup> The same idea nad theories could be find here

Anderson, A., Belnap, N., Dunn, M., 1992, *Entailment, the logic of relevance and necessity*, Princeton UP, P: 506

Also Belnap's paper by following name: *A useful four-valued logic: How a computer should think*. The section in the book has the same title. Section 81. Belnap introduce this four-valued logic for an agent who has inconsistent information. They are not logical values, they are epistemic values, because Belnap defines {T}=told true, and {F}=told false and so on.

is disbelieved. If  $B$  is suspended  $\neg B$  is also suspended. If  $E$  is unknown, then  $\neg E$  is unknown. With same calculation that we did for  $B \wedge E$  we could say that  $\neg B \wedge E$ ,  $B \wedge \neg E$ ,  $\neg B \wedge \neg E$  all are disbelieved because one of them is suspended and the other one is unknown. Now, we have four propositions  $B \wedge E$ ,  $\neg B \wedge E$ ,  $B \wedge \neg E$ ,  $\neg B \wedge \neg E$  that they are all disbelieved but their disjunction is believed  $(B \wedge E) \vee (\neg B \wedge E) \vee (B \wedge \neg E) \vee (\neg B \wedge \neg E) = \text{T}$  because it is a tautology. It seems that the bi-lattice does not work properly. I think that believing the disjunction of some disbelieved propositions is an unwanted result but believing the disjunction of two suspended proposition is not irrational. My first attempt to find a solution by working with Indeterministic belief revision and a bi-lattice, does not work properly and it is not satisfying. I tried to do some amendments, then I thought I can start with an acceptance base instead of possible belief sets, and I found it less complicated and more plausible. I think that we could do better.

## 5 Qualitative acceptance Revision

As TBR and IndBR were not satisfactory, I made a new theory which I call it *Qualitative acceptance Revision*. The idea is that instead of more than one belief set; we could start with an acceptance base. However, first I need to mention some similar ideas in the literature for more clarification.

For representing suspension, there are some suitable ideas in the literature by people who adhere paraconsistent belief revision. Paraconsistent Belief revision accepts that sometimes it is rational to have an inconsistent belief set. My theory differs from Paraconsistent Belief Revision (PBR). I prefer to call my theory qualitative acceptance revision because I think an agent should not believe a contradiction and also an agent should not believe a proposition and its negation at the same time. So, there is a clear distinction between Qualitative acceptance Revision (QAR) and Paraconsistent Belief Revision. Let's start with some argument by Priest about

Paraconsistent Belief Revision and then discuss the difference between QAR and PBR. Priest gives some reasons why a rational agent must have an inconsistent belief:<sup>51</sup>

*Further, there are, in fact, good grounds for supposing that an ideally rational agent must have inconsistent beliefs. Such an agent would not believe something unless the evidence supposed its truth. Hence, every one of their beliefs,  $a_1, \dots, a_n$ , is rationally grounded. But the rational agent also knows that no one is perfect and that the evidence is overwhelming that everyone has false beliefs (rational agents included: rationality does not entail infallibility). Hence, they believe  $\neg (a_1 \wedge \dots \wedge a_n)$ . So, their beliefs are inconsistent.*

I prefer to paraphrase the above text by replacing belief with acceptance. I have two reasons: first, belief is stronger than acceptance. Accepting two contradicting propositions does not seem irrational. In many cases in our daily life, we find two people that they disagree each other, and we think that they are both right and their argument looks convincing. So, we accept both ideas without believing one of them. Second, I prefer to apply a uniform terminology, and it seems, mostly, what Priest calls belief, is, at least in my terminology, acceptance<sup>52</sup>.

I do not agree with Priest in some cases. First, a rational agent should not *believe* a contradiction. Second, a rational agent should not *believe* two contradiction propositions at the same time. My claim is that a rational agent could *accept* two contradiction propositions.

I call my theory *qualitative acceptance* revision because it seems to me that the model is like our daily epistemic activities, and the model is intuitively comprehensible.

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<sup>51</sup> Priest, G. 2001. *Paraconsistent Belief revision*, Theoria 67: 214-228. P: 218

<sup>52</sup> Maybe I can add the third reason. While belief update works on a fixed conceptual framework, acceptance update can change the conceptual framework or set of possibilities.

There are at least two differences between qualitative acceptance revision (QAR) and indeterministic belief revision (IndBR): first, in QAR, the epistemic state of an agent will be represented by an acceptance base which is not necessarily consistent, and it is not closed under logical consequences. Second, the definition of an accepted proposition in QAR is different from IndBR. Moreover, it does not produce the same problem as IndBR.

## 5.1 Acceptance base

When an agent learns that  $B$ , she had to believe also  $B \vee E$  and  $B \vee \neg E$ . TRB and IndBR, treat  $B$ ,  $B \vee E$ , and  $B \vee \neg E$  equally. Epistemically they are different. Therefore, I thought that starting with an acceptance base (or even a belief base) is a better strategy. On the other side, I thought that we, at least sometimes, have some contradicting accepted propositions. If we represent the epistemic state of an agent with an acceptance base, it does not lead to triviality. The only key question is how the closure should be defined. Let's start with the acceptance base:

An acceptance base  $G$  is a set of propositions which an agent accepts as her basic accepted propositions<sup>53</sup>. To put it differently, an acceptance base  $G$  is a set of propositions which their epistemic value is more than zero. The acceptance base  $G$  is not necessarily a consistent set:

$$G = \{B \mid \text{val}(B) > 0\} \tag{61}$$

An acceptance base is the result of an inquiry. An inquiry is also a set of received information through the investigation. Sometimes, we receive contradicting information. The positive side of this definition is that we could distinguish various inconsistent acceptance sets. We could not distinguish various inconsistent belief set because they were all the same (**L**).

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<sup>53</sup> Any other accepted proposition is derived from this set.

## 5.2 Inferable sets<sup>54</sup>

The closure does not hold for the acceptance base. But, it could be applied to any consistent subset of the acceptance set. In daily life and even in scientific investigations, we do the same thing. A physicist accepts that ‘light is particle’ and ‘light is not a particle.’ Depending on the situation, she applies one of them and not both at the same time. So, an argument based on an acceptance base is valid, if and only if it could be derived from a consistent subset of the acceptance set. The outcome of an argument is conditionally accepted.

We could have some possible belief sets or inferable sets<sup>55</sup> (which hold consistency and closure) based on any acceptance set. To that end, first, we need to define  $G_i$ , maximally consistent subsets of  $G$ . It plays a crucial role in QAR. The set  $G_i$ , is a maximally consistent subset of  $G$ . We could call it a maximally consistent inferable acceptance base:

$$(G_i \subseteq G) \wedge \neg (G_i \vdash \perp) \wedge \forall B [(B \in G \wedge G_i \cup \{B\} \not\vdash \perp) \rightarrow B \in G_i]^{56} \quad (62)$$

For every  $G_i$ , let's define its correspondence possible belief set  $K_i = \text{Cn}(G_i)$ . As the logical consequence function is Tarskian and satisfies (1) - (3), we could conclude that a possible belief set satisfies (4) and (5) exactly as a belief set does.

$$K_i = \text{Cn}(G_i) \quad (63)$$

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<sup>54</sup> I realized that ‘derivable sets’ could be also an alternative term.

<sup>55</sup> I prefer to avoid this term, but I think it provide some intuition. Please do not consider the word belief as epistemologists consider.

<sup>56</sup> We could define the degree of triviality as the number of maximally consistent subset of the acceptance set.

The formula (63) says that  $K_i$  is the set of the logical consequence of  $G_i$ . As  $G_i$  is consistent, the set  $K_i$  could be a possible legitimate belief set which satisfies (4) and (5). It might be helpful if you like to compare this theory with IndBR. In both theories, we have a set of possible belief sets, and we define accepted proposition based on this set of possible belief sets.

### 5.3 Accepted in

A proposition  $B$  is accepted in  $G_i$ , if and only if it follows from  $G_i$ . Let  $Acc(B, G_i)$  be 'B is accepted in  $G_i$ '. We say  $B$  is accepted in  $G_i$  if and only if it follows from  $G_i$ :

$$Acc(B, G_i) \equiv G_i \vdash B \quad (64)$$

### 5.4 Assessable proposition

How does an agent know that a proposition is accepted in  $G$ ? We need to define the concept of assessability to define accepted proposition in QAR. A proposition is an assessable proposition if and only if the agent can decide in all  $G_i$  whether it is true or false. In other words, a proposition is assessable if and only if from every  $G_i$  the proposition or its negation could be inferred. Let  $Ass(B, G_i)$  be  $B$  is assessable in  $G_i$  then:

$$Ass(B, G_i) \leftrightarrow (Acc(B, G_i) \vee Acc(\neg B, G_i)) \quad (65)$$

Now we could define assessable proposition as a proposition that is assessable in any maximally consistent acceptance base. It means that the agent could always decide that it is true or false in a specific maximally consistent acceptance base:

$$Ass(B, G) \leftrightarrow \forall G_i Ass(B, G_i) \quad (66)$$

## 5.5 Accepted proposition

I found that if we agree that being assessable is a necessary condition for the accepted proposition, then we will not have the same problem that we had in IndBR<sup>57</sup>.

$$Acc(B, G) \rightarrow Ass(B, G) \quad (67)$$

And eventually the definition of the accepted proposition:

$$Acc(B, G) \leftrightarrow Ass(B, G) \wedge \exists G_i Acc(B, G_i) \quad (68)$$

(68) says that a proposition is acceptable if and only if it is assessable and there is a  $G_i$  that it is accepted in  $G_i$ . If we compare it with (56), we could see that there is a big difference between the definition of acceptance in QAR and IndBR.

## 5.6 Basic doxastic attitudes

We have four doxastic states. We could define them by the following formulas:

$$\text{NAR Belief} \quad Bel(B) \leftrightarrow \forall G_i Acc(B, G_i) \quad (69)$$

$$\text{NAR Disbelief} \quad Dis(B) \leftrightarrow \forall G_i Acc(\neg B, G_i) \quad (70)$$

$$\text{NAR Ignorance} \quad Ign(B) \leftrightarrow \exists G_i \neg Ass(B, G_i) \quad (71)$$

$$\text{NAR Suspension} \quad Sus(B) \leftrightarrow \exists G_i Acc(B, G_i) \wedge \exists G_i Acc(\neg B, G_i) \wedge Ass(B, G) \quad (72)$$

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<sup>57</sup> If we do the same thing in IndBR then they are somehow equivalent. The definition of acceptance in IndBR looks nice and I had to present my observation.

(72) looks complicated, but it simply says that a proposition is suspended if and only if it is not disbelieved, it is not believed, and it is not unknown or ignorance<sup>58</sup>.

## 5.7 Conditionally accepted propositions

We could find some propositions that have a more complicated situation. I can illustrate these propositions by an example. Imagine  $G = \{B, \neg B, B \rightarrow C\}$ . There are two maximally consistent inferable subsets of  $G$ . The first one is  $G_i = \{B, B \rightarrow C\}$  and the second one is  $G_j = \{\neg B, B \rightarrow C\}$ . The proposition  $C$  is accepted in  $G_i$  and it is not accepted in  $G_j$ . Also, it is not assessable, because its negation  $\neg C$  is not accepted anywhere. Obviously  $C$  is not believed, is, not disbelieved, is not unknown, and is not suspended. On the other hand, however, the proposition  $\neg C$  is not accepted anywhere, it is not unknown or ignorance (not assessable)! One might think that this is a big disadvantage, I think this is not. The proposition  $C$  is a conditionally accepted proposition and its acceptance depends on what agent ignore in his acceptance base. We could call it accepting-by-retracting or accepting-by-ignoring.

Let  $Acc(B|C, G)$  be  $B$  in  $G$  under the condition  $C$  is acceptable. Then

$$Acc(B|C, G) \leftrightarrow \forall G_i (Acc(C, G_i) \rightarrow Acc(B, G_i)) \quad ^{59} \quad (73)$$

A proposition  $B$  is accepted in  $G$  under the condition  $C$  if and only if  $B$  is accepted in all maximally consistent subset of  $G$  which  $C$  is accepted<sup>60</sup>.

## 5.8 Epistemic change

During an inquiry, an agent might learn and accept a proposition, and she might decide to contract her accepted proposition. We do not need to define revision

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<sup>58</sup> Ignorance, unknown, non-assessable are interchangeable.

<sup>59</sup> According (73), we have various kinds of conditionals in NAR.

<sup>60</sup> Another difference between IndBR and QAR is that in IndBR if an agent accepts a proposition under the condition  $C$  and accept its negation under condition not- $C$ , then she suspends it. The same can't be hold in NAR.

because when an agent accepts a proposition, she should not be necessarily worried about inconsistency.

### 5.8.1 Expansion

If an agent learns that  $B$ , then she simply adds it to her acceptance base. Even if the proposition is accepted, she should add it. Because as per (68), an agent might accept a proposition which does not belong to the acceptance set.

$$G + B = G \cup \{B\} \quad (74)$$

### 5.8.2 Contraction

Contracting the proposition  $B$  could be reduce to the question of which subset of  $G$  should be the new acceptance set. Maybe someone asks why not just add  $\neg B$  to the acceptance set. It does not work because if an agent accepts  $B$  and then she thinks that  $B$  is not acceptable, then it is not suspended. Suspension is accepting a proposition and its negation. Anyway, the contraction does not entail accepting new proposition<sup>61</sup>.

$$G \div B \subseteq G \quad (75)$$

Obviously, after contracting  $B$ , the sentence should not be acceptable anymore.

$$\neg Acc(B, G \div B) \quad (76)$$

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<sup>61</sup> We could define the contraction by following:  $(G \cup \{\neg B\})_i - \{\neg B\}$ .

The contraction should not retract propositions that do not play any role in accepting  $B$ . In other words, the agent should be conservative to keep accepted propositions as much as she can.

$$\forall H \subseteq G [(H \not\vdash \perp \wedge \neg \text{Acc}(B, H)) \rightarrow \neg(G \dot{-} B \subset H)] \quad (77)$$

In other words, the formula (77) says that if we could add a proposition from  $G$  to the outcome  $G \dot{-} B$ , and still  $B$  is not acceptable, then that proposition belongs to the outcome:

$$(C \in G \wedge \neg \text{Acc}(B, G \dot{-} B \cup \{C\})) \rightarrow C \in G \dot{-} B \quad (78)$$

(77) and (78) are equivalent. This constraint is like (6).

Now, we need another constraint which relates to the epistemic value. If an agent could have more than one possible acceptance base which satisfies (75)-(78), then she should choose one which is more valuable. The epistemic value of the outcome is higher than all other possible outcomes:

$$\text{Val}(H) = \sum_{B \in H} \text{Val}(B) \quad (79)$$

$$\forall H \subseteq G [(H \not\vdash \perp \wedge \neg \text{Acc}(B, H)) \rightarrow \text{Val}(G \dot{-} B) > \text{Val}(H)] \quad (80)$$

The epistemic value of a set of propositions is the sum of its members<sup>62</sup>.

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<sup>62</sup> Notice that we could make even more satisfying theory of acceptance revision. For example, we could in the same way define conditional contraction (not accepting  $B$  under condition  $C$ ). It has the same constraints. QAR is very fruitful theory, it is not just suitable for representing suspended judgment, it seems interesting for people who do research into conditional or four-valued logic. The best way to calculate the outcome of epistemic value is introduced in the last chapter.

Besides, if we apply the degree of vagueness (number of the inferable sets of an acceptance base), then someone might be interested to find the relationship between the vagueness and contraction. Is a less vague outcome always better than the vaguer one? Is the goal of contraction, having a less vague outcome? The answer is not. Sometimes an agent might prefer a vaguer acceptance base, because she like to keep some propositions even if it cost higher degree of vagueness. It all depends on the epistemic value of the sentences in the acceptance base.

The epistemic value of a proposition which is in the acceptance base, shows the quantitative acceptance. This chapter is all about the qualitative nature of acceptance and all doxastic attitudes. Fortunately, QAR works better than TBR when it comes to the representation of suspended judgment. Let’s compare it to my former proposal, IndBR+Belnap proposal. Here is how QAR works when there are two propositions:

$\wedge$	Bel	Dis	Ign	Sus	$\vee$	Bel	Dis	Ign	Sus	$\neg$
Bel	Bel	Dis	Ign	Sus	Bel	Bel	Bel	Bel	Bel	Dis
Dis	Dis	Dis	Dis	Dis	Dis	Bel	Dis	Ign	Sus	Bel
Ign	Ign	Dis	*	**	Ign	Bel	Ign	*	**	Ign
Sus	Sus	Dis	***	****	Sus	Bel	Sus	***	****	Sus

Table 2

If two propositions are believed, then each is accepted in all inferable bases, therefore their conjunction is believed as well. If a proposition  $B$  is believed, and  $C$  is disbelieved, then in all inferable bases,  $B$  and  $\neg C$  are accepted, therefore their conjunction  $B \wedge \neg C$  is not accepted, and  $\neg B \vee \neg C$  is accepted in all inferable bases

because  $\neg C$  is accepted. If both propositions are disbelieved, then their negation are believed, therefore their conjunction and disjunction is disbelieved.

Generally, disbelief is dominant when it comes to conjunction. The conjunction of a disbelieved proposition with another proposition is always disbelieved. Besides, belief is dominant when it comes to disjunction. The disjunction of a believed proposition with another proposition is believed.

There are eight cells that they need specific argument and clarification. The cells \*, is the conjunction and disjunction of two propositions  $B$  and  $C$  which are both unknown (ignorance). When a proposition is unknown, then there is at least one inferable base which the proposition is not assessable:  $\exists G_i \neg Ass(B, G_i)$ . There are two cases: (I)  $\neg Ass(B \wedge C, G_i)$ , from this follows that  $B \wedge C$  is unknown as well. Or (II)  $Ass(B \wedge C, G_i)$ , then necessarily  $Acc(\neg(B \wedge C), G_i)$ , because if  $Acc(B \wedge C, G_i)$ , then  $Acc(B, G_i)$ , but  $B$  was not assessable. Could  $Acc(\neg(B \wedge C), G_i)$  happen? Yes, a simple example is the case that  $B \wedge C = \perp$ . Therefore, for two unknown propositions, their conjunction could be an unknown proposition or a disbelieved proposition. With same pattern, one can prove that their disjunction is either an unknown proposition or a believed proposition.

The cells \*\*\*\*, when both propositions are suspended, their conjunction could be suspended or disbelieved. The result could not be a believed proposition, because then both propositions should be believed as well. The result could not be an unknown proposition, because for any inferable base, both propositions are assessable, therefore their conjunction is assessable as well. The result could be a suspended proposition, a simple example is the case that both propositions are equivalent. Also, the result could be a disbelieved proposition, when  $B \wedge C = \perp$ . With same argument, the disjunction of two suspended propositions is suspended or believed.

The cells \*\* and \*\*\* are the same: one proposition is suspended, and the other proposition is unknown. Their conjunction could not be a believed proposition, because then they must be believed. The result could be a disbelieved, suspended or

unknown proposition. For a disbelieved proposition, an example is  $B \wedge C = \perp$ . For having an unknown proposition as the result, let  $G_i$  be the inferable set that  $B$  is not assessable in it. Then, if  $C$  is accepted in  $G_i$ , then their conjunction cannot be assessable. If  $B \wedge C$  is accepted, then  $B$  is accepted and therefore is assessable. If  $\neg(B \wedge C)$  is accepted, then  $\neg B$  should be accepted and therefore be assessable, which cannot happen again. Therefore, if there is an inferable set which  $B$  is not assessable and  $C$  is accepted in the inferable set, then  $B \wedge C$  is unknown. A simple example could be an acceptance base containing only  $C$  and  $\neg C$ . The inferable base which contain  $C$ , make  $B \wedge C$  unknown. At the end, there are cases that the conjunction of an unknown proposition and a suspended proposition is suspended. As an example, let the acceptance base contains  $\neg C$  and  $B \wedge C$ . Then there are two inferable bases, one contains only  $\neg C$ , which entails  $\neg(B \wedge C)$ , and another inferable set contains  $B \wedge C$  which entails  $B \wedge C$ . So, the conjunction of a suspended proposition and an unknown proposition, could be a disbelieved, suspended, or unknown proposition. Same argument for their disjunction could be applied. Their disjunction could be a believed, a suspended or an unknown proposition.

$\wedge$	Bel	Dis	Ign	Sus	$\vee$	Bel	Dis	Ign	Sus	$\neg$
Bel	Bel	Dis	Ign	Sus	Bel	Bel	Bel	Bel	Bel	Dis
Dis	Dis	Dis	Dis	Dis	Dis	Bel	Dis	Ign	Sus	Bel
Ign	Ign	Dis	I/D	D/S/I	Ign	Bel	Ign	I/B	B/S/I	Ign
Sus	Sus	Dis	D/S/I	S/D	Sus	Bel	Sus	B/S/I	S/B	Sus

Table 3

In contrast with IndBR that the conjunction of a suspended and unknown proposition is only a disbelieved proposition, in QAR, their conjunction could be an unknown, a suspended or a disbelieved proposition. Thus, we will not reach to same unwanted result as we observed in IndBR. Suspended judgment could be represented suitably in qualitative acceptance revision, QAR, without any specific unwanted result.

The research about the representations of suspended judgment, could be divided into two parts: the representation of qualitative doxastic attitudes including qualitative suspended judgment, and the representation of quantitative doxastic attitudes. The first part is finished here. Now, I begin the second part, which is about quantitative doxastic attitudes and their relationship with qualitative doxastic attitudes.

*On Mondays, Wednesdays, and Fridays, I call myself a probabilist (much like Earman). In broad outline I agree with probabilism's key tenets: that (1) an agent's beliefs come in degrees, which we may call credences; and that (2) these credences are rationally required to conform to the probability calculus...*

*But on the contrary days of the week I am more critical of probabilism. A number of well-known arguments are offered in its support, but each of them is inadequate.*

Arguments For\_Or Against\_Probabilism? Alan Hajek

# 3

## Bayesian Epistemology

I am confident that tomorrow our 24-7-days library is open. I am less confident that it is rainy. Thus, tomorrow I go to the library, but I am not certain that I take an umbrella tomorrow. I know that tomorrow the library is not, necessarily, open. There are some possibilities that, for some legitimate reasons, it might be closed. For instance, they might have decided to repair something. Notice that this kind of possibilities does not affect my thought to check again and again their website for some possible new

announcements. I can say that I believe that tomorrow the library is open (Qualitative belief). Moreover, I believe with .68 that it is rainy (Quantitative belief); however, concerning the library, I can also say that I believe .99 that tomorrow the library is open. When an agent has a qualitative belief, one can conclude that she has a quantitative belief. But from having a quantitative belief does not follow that the agent has qualitative belief.

Let go back to (A.4) and (A.8). (A.4) says if an agent is doing the action  $B^{act}$ , like endorsing  $B$  when somebody state it; then he has accepted  $B$ . (A.8) says that from accepting follows that the proposition  $B$  is believed or it is suspended. The question is what about degrees of belief? How could we know or measure her degrees of belief? One<sup>63</sup> may believe that our betting behavior and our degree of belief are connected. By saying that our degree of confidence or belief could be measured by our betting ratio. If an agent accepts a bet at odds of 1:4 that  $B$  is true, then her degree of belief is at least .2. If it is her maximum odds, then .2 is her degree of belief.

The connection between our betting behavior and our degree of belief naturally leads us to the mathematics of gambling: probability theory. This theory could help us to model our degrees of belief. The idea that our uncertainty or degree of belief about the occurrence of an event can be expressed by a probability function, could be traced back to Pascal.<sup>64</sup> Therefore, the probability is not all about frequencies and statistics. We can talk about our judgments by using probabilities, e.g., I believe that tomorrow is rainy with .6. Subjectivists go one step further by saying that 'Probability is the degree of belief.' Even when an agent says that the probability of having *Tail* when we flip a fair coin is .5, she is talking about her degrees of belief.

Not everyone is happy with the subjectivist interpretation. There are different interpretations of probability. The probability of having *Tail* is the same for all agents. But different agents have different probabilities for Wesley So being the next chess

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<sup>63</sup> Like Ramsey or de Finetti, see

Talbott, W., *Bayesian Epistemology*, The Stanford Encyclopedia of Philosophy

<sup>64</sup> Gabby, M., Hartmann, S., Wood, J., *Handbook of the Logic*. 2011. Volume 10, Inductive Logic, The development of subjective bayesianism, by James M. Joyce, North Holland

champion of United States of America. Why should we assume that they are categorically the same? We can trace back this issue to the debate between Conceptualists, like de Morgan and Quetelet, who adhered that probability is strength of belief; and Materialists, with the idea that probability is a measure of the relative frequency for repeating events. Peirce was against conceptualists, and he believed that applying the odds of the event for measuring the degree of belief is legitimate if it is determined by the objective relative frequency of the event<sup>65</sup>.

We have somehow the same debate in the 20<sup>th</sup> century but with different umbrella terms: Subjectivists vs. Objectivists. De Finetti, is a typical subjectivist who has presented a completely subjective interpretation of probability. He describes how he found this interpretation convincing:

*... while reading a book by Czuber, Wahrscheinlichkeitsrechnung ... that book briefly pointed to the various conceptions of probability, ... I cannot recall well the content of the book ... It seems to me that it mentioned De Morgan as the representative of the subjectivistic point of view... I seemed to realize that every other definition was meaningless... Although I found it natural to give to the distinct sides of an apparently "perfect" die the same probability, I could not see how one could give an objective meaning to probability on these grounds. The physical symmetry of the die looked to me as a circumstance that could explain why each individual attributes the same probability to the various sides of the die.*

*But — I thought — there could be a thousand reasons to make an exception. For example, if one discovered that there was an imperfection in the die, or if one were influenced by the fact that after casting the die, frequencies distant from 1/6 have been recorded, ascribing this to some alleged imperfection of the die rather than to chance. Examples of this sort can be subjectively interpreted as situations in which one tries to give*

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<sup>65</sup> Burch R., *Charles Sanders Peirce*, 2014. Retrieved from SEP, '<http://plato.stanford.edu/entries/peirce/>' [Accessed 20.11.2014]

*an objective justification for subjective opinions...a subjective component is always present...*<sup>66</sup>

As you can observe, Subjectivists in their specific interpretation of probability maintain that 'probability is degree of belief,' or 'degree of confidence' or partial belief of suitable agent<sup>67</sup>. One of their reason is that *the subjective component* is always present.

We could adopt various approaches to apply the probabilistic model for representing degrees of belief. The main line in subjective probability is Bayesian epistemology. Bayesian epistemology spells out that a rational or ideal agent should obey probability laws (Kolmogorov axioms). Also, Bayesian epistemologists agree that an agent's epistemic state could be represented by a single probability function. At this point, the subjective probability camp is split into two camps. The dilemma is that our probabilities should be sharp, or it is rational to have imprecise, indeterminate, or interval-valued probabilities. Some think that indeterminacy is compatible with rationality<sup>68</sup>. The first camp, which adheres to the idea that 'perfect rationality requires one to have sharp probabilities,' is the received view. It is important to mention that sharp does not necessarily entail uniqueness<sup>69</sup>. It could be observed when an agent thinks that there are two sharp possible probabilities or degree of beliefs for a proposition and both are acceptable. Like a biased coin and some evidence that propose it will be .6 tail, and another reliable source which says .54 tail. Also, we could make examples based on having various and legitimate reference class for our probability. The second camp says that the probability might be imprecise. Like a coin that all agent knows is that it is not unfair more than .8 tail or head. Then her interval

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<sup>66</sup> De Finetti, B., 2008. *Philosophical Lectures on Probability*, Edited by Mura, A., Springer

<sup>67</sup> Hajek, A., 2011. *Interpretations of Probability*, Retrieved from SEP, <http://plato.stanford.edu/entries/probability-interpret/> [Accessed 20.11.2014]

<sup>68</sup> Walley, P., 1992. *Statistical Reasoning with Imprecise Probabilities*. London, Chapman and Hall. (1.1.4. arguments for imprecise probabilities)

<sup>69</sup> Elga, A., 2010. *Subjective probability should be sharp*. Philosophers' Imprint, Volume 10

is [.2, .8]. There is no precise probability here. Also, the agent might even find .41 it comes head more plausible than .21 head, because .21 is closer to the .2.

In this chapter, I investigate into Bayesian epistemology (sharp) and in the next chapter Dempster-Shafer theory of evidence (kind of imprecise probability). DS could be observed as a generalization of the probability theory. It can represent ignorance or lack of knowledge.

First, I explain how Bayesian Epistemology works; then I show why it is not sufficient for representing the suspension of judgment because it cannot distinguish suspension from ignorance because one has to say that BE cannot represent ignorance, or BE necessitates equal distribution of probability for all possibilities, and this move, or generally, any kinds of distribution brings extra information without justification and then we will have a similar probability function for two different doxastic attitudes. I explain in detail these issues in this chapter.

## 1 Preliminary

In this section, I introduce BE norms and their corresponding arguments. BE proposes a model to represent epistemic states and some norms to illustrate the rationality constraints. There are two kinds of norms: synchronic norms and diachronic norms. Synchronic norms (five norms) concern the rationality constraints of a belief function at a given time, and diachronic norms (one norm) concern the rationality constraint of a belief function at different times. Besides, there are various arguments for rationality constraints, but I introduce the main line which is the Dutch book argument.<sup>70</sup> The Dutch Book argument (DBA) has convinced many formal

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<sup>70</sup> For the origin of the term, Dutch Book: Wakker,P., '*History of the term Dutch Book*', Retrieved from, 'people.few.eur.nl/wakker/miscella/dutchbk.htm [Accessed 21.11.2014]: 'someone told me around 1987 that the term Dutch Book had been invented when Dutch insurance companies for ship in the 19<sup>th</sup> century would organize and combine insurance in such a way as to make money whatever contingency occurred...'

epistemologists that rational credences are probabilities. For example, Hartmann has pointed out that: ‘Dutch Book Arguments provide an important defense of the thesis that rational credences are probabilities. An agent’s credences are identified with her betting prices; it is then shown that she is susceptible to sure losses iff these prices do not conform to Kolmogorov's axioms.’<sup>71</sup>

DBA could be traced back to Ramsey classic works in ‘Truth and Probability,’ which was an argument against Keynes work, ‘A Treatise on Probability.’ It is another classic example of debate between subjectivists and objectivists. Ramsey died on 1930 (at the age of 26), and his manuscripts were acquired by Nicholas Rescher<sup>72</sup>.

Three years later, 1933, Kolmogorov, in his book, ‘Foundation of the Theory of Probability,’ presented the first axiomatization of probability theory<sup>73</sup>. Thanks to Kolmogorov axioms, we can define a suitable or rational agent, as an agent that her degree of credence or belief should satisfy the axioms of probability<sup>74</sup>. This criterion is the first synchronic norm as coherence constraint.

For qualitative belief, I apply  $Bel(B)$ , and I apply  $Bel^\circ(B)$  to illustrate quantitative belief. BE start with a belief function at the give time t:

$$Bel_t^\circ: 2^W \rightarrow [0, 1] \quad (1)$$

$2^W$ , is the power set of  $W$ , set of all possibilities, and the algebra of propositions which is closed under conjunctions, disjunctions and negations. The function  $Bel_t^\circ$  is BE model. For simplicity when time is not our concern I write  $Bel^\circ$ . Also, when I need to

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<sup>71</sup> Hartmann, S., *Bayesian Epistemology*, 2010. A entry in ‘A Companion to Epistemology’ edited by Dancy, J., Sosa, E., and Steup, M., Wiley-Blackwell

<sup>72</sup> Autograph manuscripts documenting works of Ramsey: digital.library.pitt.edu (search keyword: ‘Ramsey’)

<sup>73</sup> Kolmogorov in his book mentions Von Mises and Brenstein who propose postulational system not by basic concept of probability, but by other concepts.

<sup>74</sup> Vineberg, S., 2011. Dutch Book Argument, Retrieved from SEP, <http://plato.stanford.edu/entries/dutch-book> [accessed 21.11.2014]

mention a theory I distinguish them by subscripts like:  $Bel_{BE}^\circ$  or  $Bel_{DS}^\circ$ . In each chapter, the default is the theory that I am presenting. Here by  $Bel^\circ$  I mean  $Bel_{BE}^\circ$ .

## 1.1 Probabilism

Probabilism is the coherence constraint which says that an agent should obey probability laws<sup>75</sup>:

$$Bel^\circ(B) \geq 0 \quad (2)$$

$$Bel^\circ(W) = 1 \quad (3)$$

$$B \cap C = \emptyset \rightarrow Bel^\circ(B \cup C) = Bel^\circ(B) + Bel^\circ(C) \quad (4)$$

(2) says that our degree of belief could not be a negative number. The formula (3) says that the probability of  $W$ , or all tautologies, is 1. The third axiom (4), says that the probability of a disjunction of two mutually exclusive propositions is equal to the sum of their probabilities. The third axiom says that our probability theory is additive.

## 1.2 Regularity

The second synchronic norm is the regularity norm. If a proposition is not a tautology, then its degree of belief could always be less than 1. As per (A.7), a proposition which is not a contradiction could not have zero as its degree of belief.

$$\neg(\vdash B) \rightarrow Bel^\circ(B) < 1 \quad (5)$$

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<sup>75</sup> Kolmogorov, A. N., Foundations of the theory of probability, translated by Morrison, N., Chelsea Publishing Company, New York

### 1.3 Principle of Indifference

The principle of indifference addresses the problem of the prior degree of belief when an agent does not have evidence. The principle proposes a uniform distribution over possibilities. Castell<sup>74</sup> explains this principle as follow:

*Each member of a set of propositions should be assigned the same probability (of truth) in the absence of any reason to assign them different probabilities.*

The principle of indifference forces the agent to assign equal distribution of the degree of belief when she does not have any information. From the Principle of indifference follows that if an agent has no reason to prefer a possibility over another, then she should assign the same degree to all of them. We know that as per (2)-(4) the sum of all degrees of possibilities should be 1.

Let  $W$  be the set of all possibilities  $\{w_i\}$ . The lemma derived from the principle of indifference is the following formula:

$$(\forall_{w_i, w_j} Bel^\circ(w_i) = Bel^\circ(w_j)) \rightarrow Bel^\circ(w_i) = \frac{1}{|W|} \quad (6)$$

The problem is that without information about possibilities, and using the principle of indifference, we encounter paradoxes like Bertrand paradox. However, there are proposals to solve these paradoxes, but always there are more paradoxes.<sup>76</sup>

Concerning our project, representing suspension and ignorance, those paradoxes are not the main problem. The problem is that BE cannot represent ignorance. The following example shows the problem. Imagine a coin is about to be tossed, and there is no information that the coin is fair or not. Let  $B$  be the proposition that 'it lands head

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<sup>76</sup> Talbott, W. *Bayesian Epistemology*, 2008. Retrieved from SEP, '<http://plato.stanford.edu/entries/epistemology-bayesian/>' [Accessed 14.12.2014]

up'. As the agent does not have any information about the coin, it might be even a tail-tail or head-head coin. As there is no information in favor or against  $B$  and its negation, therefore as per (6) they have the same degree of belief .5. This degree of belief is equal to the degree of belief of an agent who knows that the coin is fair!

## 1.4 Principal Principle

The next norm is the Principal Principle. Imagine a fair coin was scheduled to be tossed at noon yesterday. We do not have any information about the outcome. To what degree should you believe that the coin falls head? The answer is simply .5.

Let  $Ch(B)$  be the chance of  $B$ . Then principal principle is:

$$Bel^\circ(B|E \wedge Ch(B) = a) = a \quad (7)$$

(7) shows the relation between objective chance and credence. It says that our degree of belief is equal to our objective chance if we accept that the chance function is valid and  $E$  is compatible evidence with the chance function and admissible. This norm reverses the direction of inference regarding the subjective and objective understanding of probability. Peirce, as an objectivist, had proposed that applying probability for the degree of belief, is legitimate, if it ultimately rests on quantities obtained adopting an objectivist understanding of probability<sup>77</sup>. Objectivists like Peirce have their own problems. For example, suppose you live in Monaco, and you meet a person who lives in this city. If you know that, in Monaco, one in three people is a millionaire, then you cannot say that your degree of belief that the person is a millionaire is 1/3. But if you consider other classes which she belongs (and not just living in Monaco), like she is a teacher, your probability would be different. The

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<sup>77</sup> Burch R., *Charles Sanders Peirce*, 2014. Retrieved from SEP, '<http://plato.stanford.edu/entries/peirce/>' [Accessed 25.12.2014]

problem is called the problem of the reference class. In our example, the person belongs to different reference classes, and in many cases, the frequency information could be related to the incompatible reference class, such the set of students, illegal workers, etc. There are some solutions about the reference class. For instance, Reichenbach has stated that:

*If we are asked to find the probability holding for an individual future event, we must first incorporate the case in a suitable reference class. An individual thing or event may be incorporated in many reference classes... We then proceed by considering the narrowest reference class for which suitable statistics can be compiled*<sup>78</sup>.

Solutions including Reichenbach were not convincing.<sup>79</sup> The question of reference class problem is still open.

In contrast to frequentists, Lewis has reversed the direction of inference and in his paper 'A Subjectivist's Guide to Objective Chance':

*Given two kinds of probability, credence and chance, we can have hybrid probabilities of probabilities. (no second order probabilities', which suggests one kind of probability self-applied.) ... to the believer in chance, chance is a proper subject to have beliefs about. Propositions about chance will enjoy various degrees of belief, and other propositions will be believed to various degrees conditionally upon them.*<sup>80</sup>

Here, the agent should not have any inadmissible information about  $B$ . Information about  $B$  is inadmissible if it is about  $B$  and it has effect on  $B$ 's beyond chance. One cannot apply Principal Principle when there is an inadmissible

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<sup>78</sup> Pollock, J., 1990, *Nomic Probability and The Foundations of Induction*, OUP, P. 110

<sup>79</sup> Thorn, P., 2012, *Two problem of Direct Inference*, *Erkenntnis* 76(3)

<sup>80</sup> Lewis, D., 1980, *A subjectivist's guide to Objective Chance*, in *Studies in Inductive Logic and Probability*, Volume II, edited by Jeffrey, R., University of California Press, P. 263

information.<sup>81</sup> Lewis, at the end of his paper about Principal Principle, mentions a Humean speculation:

*...according to our broadly Humean<sup>82</sup> speculation (and the Principal Principle) if I were perfectly reasonable and knew all about the course of history up to now (no matter what that course of history actually is, and no matter what time is now) then there would be only one credence function I could have. Any other would be unreasonable...I shall not attempt to decide between Humean and the anti-Humean variants of my approach to credence and chance. The Principal Principle does not.*

As far as I can understand, the Humean argument says that our degree of belief is equal to a unique credence function if the agent knows all about the course of history. So, if the agent knows only what the chance of a proposition is, then the degree of belief is equal to the objective chance. If we have any evidence which it is beyond the objective chance, then we could not apply the Principal Principle. In other words, the inadmissible information breaks the connection between credence and chance. Lewis's words:

*The power of the Principal Principle depends entirely on how much is admissible. If nothing is admissible it is vacuous. If everything is admissible it is inconsistent ... I have no definition of admissibility to offer...I suggest that two different sorts of information are generally admissible... The first kind of admissible proposition is the proposition that is related to past*

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<sup>81</sup> Loewer, B., 2000, Lewis theory of Objective chance, Retrieved from his homepage, [rci.rutgers.edu/~loewer/papers/lewis-theory-of-objective-chance.pdf](http://rci.rutgers.edu/~loewer/papers/lewis-theory-of-objective-chance.pdf)

<sup>82</sup> Loewer, B., *Humean Supervenience*, 1996, *Philosophical Topics* 24, P. 101-127: Humean thesis is the doctrine that all there is to the world is a vast mosaic of local matters of particular fact, just one little thing and then another... we have geometry: a system of external relation of spatiotemporal distances between points...and at these points we have local qualities: perfectly natural intrinsic properties which need nothing bigger than a point at which to be instantiated. For short, we have an arrangement of qualities. All else, superveniences on that.

*events or that is about historical information. The second kind of admissible proposition is about hypothetical information*<sup>83</sup>.

*Admissibility* plays a crucial role in Principal Principle. In literature, you can find some chance-credence principle which is not related to the admissibility<sup>84</sup>. Also, there are some accounts that captures the intuition behind this principle. For example, Spohn<sup>85</sup> proposes projectionist approach, and in this interpretation, one can see Principal Principle as an extreme application of reflection principle (I will discuss the reflection principle). Also, there is an interesting argument that the term inadmissible in the Principal Principle implicitly applies the principle of indifference.<sup>86</sup>

## 1.5 Unwanted result

Regarding our project, the formal representation of suspended judgment, a comparison between the principle of indifference and objective chance is important. Because an agent applies the latter when there is a chance function which means she has some information, while she applies the former when there is no information. When the chance of a proposition is .5, and all information is admissible, then the Principal Principle says the degree of belief is .5. Unfortunately, if an agent does not have any information about the proposition, based on the principle of indifference assign the same degree of belief.

Be represent doxastic attitudes of two agent with two different doxastic attitudes with the same function. It seems that .5 is a suitable option for a suspended proposition and it is not a proper to assign .5 as the degree of belief of an unknown proposition.

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<sup>83</sup> Lewis, D., 1980, *A subjectivist's guide to Objective Chance*, in *Studies in Inductive Logic and Probability*, Volume II, edited by Jeffrey, R., University of California Press, P. 272

<sup>84</sup> Hall, N. 1994, *Correcting The guide to Objective Chance*, *Mind* Volume 103, OUP, P 507-517

<sup>85</sup> Spohn W., 2010, *Chance and Necessity*. In: Eells E., Fetzer J. (eds) *The Place of Probability in Science*. Boston Studies in the Philosophy of Science, vol 284. Springer, Dordrecht

<sup>86</sup> Hawthorne, J., Landers, J., Wallmann, Ch., Williamson, J., 2014, *The principal Principle implies the Principle of Indifference*, Retrieved from Academia.edu, 'http://academia.edu/9693231/the\_Principal\_Principle\_Implies\_the\_Principle\_of\_Indifference' [Accessed 28.12.2014]

The principle of indifference cannot help BE to represent ignorance, and it leads to the same problem that TBR had: BE fails to distinguish suspension from ignorance.

## 1.6 Reflection principle

The next norm is the reflection principle which is a diachronic norm. This principle is introduced by Van Fraassen in his paper 'Belief and the Will'. By adopting the Dutch Book strategy, he illustrated that for any proposition  $B$  and any future time  $t$ , an agent's current probability for  $B$  conditional on later assigning its probability  $a$  is itself  $a$ :<sup>87</sup>

$$Bel_t^\circ(B | \text{my degree of belief in } B \text{ at } t^+ \text{ is } a) = a \quad (8)$$

## 1.7 Conditionalization

Another diachronic norm is Bayesian conditionalization which says that an agent's degree of belief at  $t^+$  after receiving new total evidence  $E_{t^+}$  should follow the following norm:

$$Bel_{t^+}^\circ(B) = Bel_t^\circ(B | E_{t^+}) \quad (9)$$

Above norm could be called simple conditionalization. It has an important property: If an agent updates her belief function with a proposition  $B$ , then the degree of belief of  $\neg B$  will be 0, and it remain always zero after updating with any other proposition, simple conditionalization maintains certainty and is cumulative. We cannot use simple conditionalization for learning, because learning does not always increase certainty, sometimes it increases uncertainty.

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<sup>87</sup> Van Fraassen, B., 1984, *Belief and the Will*, Journal of Philosophy, Volume 81. Issue 5, 235

As the new information might be a partial belief, we need to find a conditionalization when the evidence does not give a proposition with certainty. The new degree of belief with Jeffrey conditionalization which is the generalization of (9):

$$Bel_{new}^{\circ}(B) = Bel_{old}^{\circ}(B|C).Bel_{new}^{\circ}(C) + Bel_{old}^{\circ}(B|\neg C).Bel_{new}^{\circ}(\neg C) \quad (10)$$

The common assumption in both cases is that the new information is part of the algebra. Therefore, simple and general (Jeffrey) conditionalization, cannot explain learning in general sense, because sometime an agent learns how to grasp a concept.

## 1.8 Crucial questions

I explained what BE is about and how it works, now, I am going to answer some questions:

- A. Could we represent the degrees of suspension and ignorance in BE?
- B. How does the discussion about the suspension change the Dutch book argument, as an argument for probabilism, which justifies BE based on the betting behavior of a rational agent?
- C. What is the relationship between the degrees of belief and four basic qualitative doxastic attitudes?

## 2 Representing the degree of suspension and ignorance

Regarding the first question (A), the short answer is no. The long answer: the principle of indifference proposes to represent ignorance (having no evidence) with a uniform distribution. If an agent does not know, and she thinks they are equally probable, then her degree of belief should be .5. Unfortunately, also, BE proposes to

assign the same degree if she had information from two highly reliable but contradicting sources that tomorrow it is rainy (for example a weather forecast website says that it is rainy, and a local TV weather forecast says that it is not rainy). As we could observe, in two epistemically different situations, BE proposes to assign the same probability .5. In the first case, the agent is ignorant, and in the second case, she suspends her judgment. Also, we expect that in the state of non-belief, when the degree of suspension is raising, then the degree of ignorance should be decreasing. Unfortunately, the belief (probability) function gives us no extra information to distinguish them. In the first chapter, I introduce some standards for degrees of suspended judgment. The second standard says that if the agent does not believe and does not disbelieve a proposition, then the higher degree of suspension entails the lower degree of ignorance and vice versa. Unfortunately, BE cannot reflect that property. If we take .5 as the highest degree of suspension, highly conflicting evidence in favor of the proposition and its negation, then we must assign the same degree .5 for the case that the agent does not know anything about the proposition because as per the principle of indifference she does not have any evidence to assign different degrees to the proposition and its negation.

There are some proposals for representing suspended judgment in a probabilistic way (accepting probability axioms). One of them is representing suspended judgment with special kinds of intervals  $[0, a]$  while  $a$  is a small number close to zero, and it says the agent tend to disbelieve the proposition and her degree of belief is vague over zero and some number close to zero. This proposal was introduced by Van Fraassen.<sup>88</sup> The first problem with this representation is that after some conditionalizations the interval  $[0, a]$  can turn to disbelief, by having 0 as the degree of belief, but it cannot turn to belief (a number like  $b$  as the degree of belief), because zero after all conditionalization will remain zero. The suspended is not neutral and is more incline to disbelief. Another

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<sup>88</sup> Van Fraassen, B., 1998, *The Agnostic Subtly Probabilified*. *Analysis*, 58(3), 212-220

problem is that the suspension of proposition should entail the suspension of its negation, as we discussed in the first chapter about the symmetry between suspension of a proposition and suspension of its negation; and this criterion does not hold here, because if the degree of belief to the proposition  $B$  is  $[0, a]$ , then the degree of belief for its negation  $\neg B$  is  $[1 - a, 1]$ , and obviously as per the proposed definition by Van Fraassen  $\neg B$  is not suspended.<sup>89</sup>

Another proposal could be representing the degree of suspension and ignorance by retracting the principle of indifference. From having no reason to assign propositions different probabilities, it does not follow that they should have the same probabilities. If there is a reason to assign similar probabilities, then the agent should assign similar probabilities. If the agent does not have any reason to assign different probabilities and does not have any reason to assign similar probabilities, then her probabilities should be vague over  $[0,1]$  or in other word she should not assign any probability.

Moreover, from rejecting the principle of indifference from BE follows that ignorance cannot be represented.

There are various ways to establish probabilism, and Dutch book argument is the most important one. I show that Dutch book argument is flawed.

### 3 Dutch book argument for probabilism

How could probabilism<sup>90</sup> (three basic probability axioms (2) - (4)) be established, or why should our degrees of belief satisfy probabilism? DBA argues that if our degree of belief violates the probability laws, then we would be vulnerable to accept a bet with the sure loss. Also, if we obey, there is no sure loss.

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<sup>89</sup> For more information about the proposal and the criticism see Hájek, A., 1998, *Agnosticism Meets Bayesianism*. *Analysis*, 58(3), 199-206

<sup>90</sup> In this text probabilism and Bayesianism are interchangeable

Proof. Imagine there is a bet which pays 1 euro if  $B$ , otherwise nothing. The price that an agent accepts as a fair price for trading this bet (buying or selling), shows her degree of belief. For example, if I find .70 euro a fair price, then my degree of belief is .70. Also, if my degree of belief of  $B$  is  $a$ , then I should find  $a$  a fair price for  $B$ .

First axiom. Assume that an agent violates the first axiom (2),  $Bel^\circ(B) \geq 0$ . She believes  $B$  with degree  $-a$  (which  $a > 0$ ). Then she is ready to sell a bet that pays 1 euro if  $B$ , otherwise zero. Obviously, she loses at least  $a$  euro in any case.

Second axiom. Assume that an agent violates the second axiom which is normalization,  $Bel^\circ(W) = 1$ , then she is in one of two following states (i).  $Bel^\circ(W) < 1$  or (ii).  $Bel^\circ(W) > 1$ . (i) if she believes all tautologies with a degree which is less than one, then she accepts to sell a bet on  $W$  to pay 1 euro if  $W$ , otherwise zero. As  $W$  is always true, she loses her money again. (ii) in this case you buy a bet to receive one euro while you paid more than one euro.

Third axiom. Assume that  $B$  and  $C$  are incompatible. The third axiom states that our degree of belief should be additive:  $B \cap C = \emptyset \rightarrow Bel^\circ(B \cup C) = Bel^\circ(B) + Bel^\circ(C)$ . If an agent violates this axiom, then she has one of following belief functions:

$$(i) \quad Bel^\circ(B \cup C) > Bel^\circ(B) + Bel^\circ(C)$$

$$(ii) \quad Bel^\circ(B \cup C) < Bel^\circ(B) + Bel^\circ(C)$$

In the first case (i), the agent is ready to buy  $B \cup C$  and to sell  $B$  and  $C$ . In this trade, she pays  $Bel^\circ(B \cup C) - (Bel^\circ(B) + Bel^\circ(C))$ , and whatever happens, she losses this money because if  $B \cup C$  is winning then she receives 1 euro but, she should pay one euro, because of  $B$  or  $C$ . And she just loses the initial price. If  $B \cup C$  is losing, then she receives nothing, and she should pay nothing. Again, the initial price is gone. In the second case (ii), the argument is the same, we just need to replace buy with sell and v.v. Converse Dutch book argument could be proved as well. It says that if we obey

probability axioms, then there is no Dutch book. I just need to talk about the DBA and its problems.

### 3.1 Czech book argument

DBA indicates that there are some rational constraints on our belief functions. The main idea is that an agent should not accept a bet or a set of bets if it is guaranteed that she loses the bet in all possible worlds that might come true. The term ‘sure loss’ plays a crucial role in DBA, because it is the only motivation in DBA to accept probabilism. Beside this, there is an important assumption in the argument. If an agent thinks that a price is fair, then she is ready to buy or sell it at that price. It creates an opportunity to make a parallel argument which we could call the Czech book argument. The Czech book argument says that if an agent violates probability laws, then there is a bet which the agent accepts, and she wins in all possible outcomes, and her sure win is guaranteed. For proof, we just need to replace buy with sell and v.v. in DBA. This parallel weakens the power of DBA because it says that violating probability axioms could lead to a sure win and as per the Czech argument one can propose to violate probability axioms.

Hajek<sup>91</sup> proposed a solution to this problem. He said if we replace fair with fair or favorable, then the problem does not exist anymore. He has mentioned that ‘bets that you consider fair are not the only one that you accept. You also accept bets that you consider favorable, that is better than fair.’<sup>92</sup> He says if we replace fair with fair-or-favorable, then the revised version of Dutch book argument will hold, and the revised version of Czech book argument will not work anymore. Let us look at the outcome of the replacement. First Dutch book argument:

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<sup>91</sup> Hajek considers that Ramsey has presented the same idea, because in ‘Truth and Probability’, Ramsey leaves open the possibility that some or all of bets could be consider better than fair by the agent. DBA is formulated by various subjectivist by focusing just on fair prices and not favorable.

<sup>92</sup> Hajek, A., *Arguments For-Or Against-Probabilism?*, in ‘Degrees of Belief’ Edited by Huber, F., Schmidt-Petri C., Synthese 342, Springer, P 229

DBA (*Revised*): If the agent violates probability axioms, there is a set of bets, which the agent considers each of them *fair-or-favorable*, which collectively the sure loss is guaranteed.

Converse DBA (*Revised*): If the agent obeys probability axioms, there is not a set of bets, which the agent considers each of them *fair-or-favorable*, which collectively the sure loss is guaranteed.

And Czech book argument:

CBA (*Revised*): If the agent violates probability axioms, there is a set of bets, which the agent considers each of them *fair-or-favorable*, which collectively guarantees sure gain.

Converse CBA (*Revised*): If the agent obeys probability axioms, there is not a set of bets, which the agent considers each of them *fair-or-favorable*, which collectively guarantees sure gain.

### 3.1.1 Ignorance

Here, I think we have another problem which is connected to our project: formal representation of suspended judgment. If we accept that an agent should follow some norms because of avoiding sure loss or gaining sure gain, then an ignorant person could always win in a group of individuals by exploiting the situation to her advantage. Imagine there are three agents. Two of them believe  $B$  with degree .4 and .7 respectively. Then the third agent can easily win in all possible world by buying a bet from the first agent and selling it to the second agent. The only thing that she need is setting her degree of belief in between somewhere like .55. Then she finds both trade not just fair but favorable. The person could be called an opportunist. It seems in a group of agents the ignorance could be more successful than others in terms of success. She applies kind of centralist policy to set her degrees of belief. Her norm is

following: my degree of belief is the average of the degree of belief of others. Epistemically, the opportunist accepts the testimony as the only source of knowledge. However, this problem, it seems, is related to a group of rational agents, and one may say it is not against DBA, still, I think justifying probabilism needs more discussion. We have other serious problems.

### 3.2 Package principle and betting on our actions

One of the assumptions in DBA is that if an agent accepts to trade a bet on  $B$  to win 1 euro for  $b$  euro and accept to trade  $C$  for  $c$  euro, then she accepts to trade them both for  $b + c$ . This assumption is not acceptable. An agent might accept  $B$  and  $C$  separately but avoid trading them together. Also, an agent might accept to buy two bets and not each separately. There are examples: I accept to buy a bet that I sit on the floor for ten seconds in the next minute for .90 euro to win 1 euro. Also, I accept to buy that I do not sit on the floor for ten second in the next minute for .90 euro to win 1 euro. I am certain that if I buy this bet, the outcome is under my control and my sure gain is somehow guaranteed. But if I buy both with that price I win maximum 1 euro while I paid 1.80 euro. Betting on our action contradict the idea that our fair price shows our degree of belief. Notice that our action is a subset of events and we used to bet on our  $b + c$ . In addition, the DBA for reflection principle by Van Fraassen cannot be held without accepting that an agent could bet on her actions, the proposition like 'Van Fraassen will fully believe that H' means he is betting on his action: here believing is the action.<sup>93</sup> As I mentioned betting on our action provides a class of counterexample against applying DBA for probabilism.

The example for avoiding buying two bets separately and accepting the package is interesting. Imagine someone asks me to buy a bet that Lake Konstanz is the largest lake in Europe. I do not have any evidence and knowledge about this proposition. I

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<sup>93</sup> Van Fraassen, B., 1984, *Belief and the Will*, Journal of Philosophy, Volume 81. Issue 5, 235

think I cannot buy with any price that it is the largest or it is not. But, I accept to buy with the price 1 euro both together.

Both of above problematic examples share the same assumption: An agent cannot avoid trading a bet. An agent is always ready to decide to sell or buy a bet! I think it is not simply irrational. I think a rational agent should avoid acting in the case of being ignorant. Notice that for accepting any bet on  $B$ , I should be sure that I am capable to know that  $B$  is true. It is not always possible.<sup>94</sup>

How are these issues connected to our project? An agent, who is completely ignorant, should stay undecided, and she should not trade any bet. If an agent does not know that a coin is fair or not, then she should not trade a bet that it is head by paying 50 cents to win 1 euro. Unfortunately, BE proposes to accept this trade. A person who knows that the coin is fair could legitimately trade that bet. BE sees both cases the same.

## 4 Quantitative notion of belief and basic doxastic attitudes

What is the relationship between our degree of belief and the qualitative notion of belief? The Lockean thesis gives us an answer. It says one should believe a proposition  $B$  just in case one's degree of belief for  $B$  is sufficiently high. Foley defined and called it the Lockean Thesis<sup>95</sup>:

*Add the idea that belief-talk is a simple way of categorizing our degree of confidence in the truth of a proposition. To say that we believe a proposition is just to say that we are sufficiently confident of its truth for our attitude to be one of belief. Then it is epistemically rational for us to believe a proposition just in case*

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<sup>94</sup> The same objection but with different approach is presented in: Dokic, J., Engel, P., *Frank Ramsey-truth and success*, P. 70

<sup>95</sup> Foley, R., 1992, *The Epistemology of Belief and the Epistemology of Degrees of Belief*. *American Philosophical Quarterly*, 29(2), 111-124. Retrieved from <http://www.jstor.org/stable/20014406>

*it is epistemically rational for us to have sufficiently high degree of confidence in it, sufficiently high to make our attitude towards it one of belief. I will call this way of thinking about the relationship between the rationality of beliefs and the rationality of degrees of belief "the Lockean thesis."*

If we represent degrees of belief with probability theory which  $Bel^\circ(B) + Dis^\circ(B) = 1$ , then the threshold should be more than .5, otherwise we might have a proposition which is believed and disbelieved at the same time and it is not acceptable.

Unfortunately, the Lockean thesis in the context mentioned above leads to an unwanted paradox called 'Lottery paradox'. Imagine the threshold that an agent thinks her degree of belief is high enough to believe, is  $1 - \varepsilon$  which  $\varepsilon \in (0,1]$  (notice that we do not even need to restrict the threshold more than .5, and it works with any threshold), and she is going to a lottery to buy a ticket. There are a thousand tickets, and one of them will win. Her threshold is .9. Therefore, she disbelieves that every ticket  $t_i$  win, because  $Dis^\circ(B_{t_i}) = .999 > .9$  and  $Bel^\circ(B_{t_i}) = .001 < .9$ . The agent disbelieves that every ticket is winning:  $\forall t_i Dis(B_{t_i})$ . But surprisingly she believes that the disjunction of all those disbelieved proposition  $Bel(\bigvee t_i B_{t_i})$  because  $Dis^\circ(\bigvee t_i B_{t_i}) = 1 > .9$ . It says that the conjunction of disbelieved propositions is believed by the agent. It seems irrational. (It is possible to show that the conjunction of believed propositions is disbelieved). As we observe BE could not even successfully represent belief and degrees of belief, and we could not expect to have a plausible representation of suspended judgment.

As BE cannot represent acceptance and it is all about belief and degrees of belief, I think the Lockean thesis, which has our intuitive support, does not work properly here.

## 5 BE cannot distinguish suspension and ignorance

BE cannot represent the degree of suspension and ignorance at the same time. Therefore, BE proposes a rational agent to have the same betting behavior when she

suspends a proposition and when she is ignorant and does not have any information about the proposition. Same representation leads to having same betting behavior while a rational agent should avoid any decision when she is ignorant (betting on the coin which it might be fair or biased). As the problem of DBA is connected to the representation of quantitative suspension of judgment and ignorance, it seems that other similar approaches in BE encounter the same problems. We observed that BE could not represent the degree of suspension and ignorance. We should find another way to represent the degrees of belief.

Besides, it is not possible to explain how the qualitative and quantitative notion of belief is connected. Also, BE is all about degrees of belief and disbelief, while I need to find a way to represent acceptance because after defining acceptance I can define all four basic epistemic states.

*Bernoulli's notion of non-additive probabilities was the ancestor of what we now call belief functions. ... It was also clearly Bernoulli's idea of probability assessment through combining weights of evidence that motivated Shafer to recast Dempster's theory of random sampling into a theory of evidence and to represent evidence using support functions.*

Classic Works of the DS Theory of Belief Functions  
Roland Yager, Liping Liu

# 4

## Dempster-Shafer theory

For representing all possible quantitative doxastic attitudes, we need a theory which could give us four basic functions for ignorance, disbelief, belief, and suspended judgment. Dempster-Shafer theory of evidence makes room for representing the degrees of ignorance and belief.

Bayesian epistemology represents ignorance by a uniform distribution, which essentially mixes the lack-of-belief or non-belief with disbelief<sup>96</sup>, but Dempster-Shafer

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<sup>96</sup> Liu, L., Yager, R.R., 2008, Classic Works of the Dempster Shafer Theory of Belief Functions, Springer, p. 7

theory of evidence could distinguish these two epistemic states by giving us two different belief functions. Among various theories in formal epistemology, DS is distinctive for many reasons; e.g., it is a generalized version of Bayesian epistemology, and it could represent the degree of ignorance formally. For any proposition, there is a way to represent the degree of belief and degree of ignorance, and we could see it as a kind of upper and lower probability for any proposition<sup>97</sup>.

Besides, in this chapter, I try to find a solution for the lottery paradox. If we find a solution for representing all doxastic attitudes in DS (the relationship between the quantitative and qualitative notion of belief, disbelief, Suspension and ignorance.), then it will be a solution not only for DS but also for Bayesian functions as well because BE probability functions are a kind of DS functions.

I begin by explaining DS theory of evidence, and then I explain how the degrees of belief, disbelief, suspension, and ignorance are representable in DS. Also, I explain a new technique to define four epistemic states based on the degrees of belief and how it relates to the problem of combining evidence which is still an open issue.

## 1 Preliminary

### 1.1 Set of possibilities

DS is a theory of evidence. Sources of knowledge play the key role to define the set of possibilities and the degrees of belief. Every source suggests a set of possibilities, which DS calls a focal set; then the union of all focal sets is the subset of the set of possibilities. A source of knowledge may not only give information about the degree of belief but also gives information about the set of possibilities. If the suggested set of

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<sup>97</sup> When DS introduced by Dempster, he considered it as a lower and upper probability theory, but nowadays this is not valid anymore and some people think that calling DS a lower and upper probability model might be even misleading. I think DS is not about ambiguity as upper and lower probability like to be.

possibilities is not a subset of the agent's set of possibilities, then the agent should change her set of possibilities.

A possibility is not maximally specific because, during the process of inquiry, the agent might split it into two possibilities or even merge two possibilities. In other words, focal sets change during the inquiry. The union of all suggested sets of possibilities is the fully believed proposition which holds according to all sources of knowledge or suggested sets of possibilities. The smallest fully believed proposition is unique because the union of all sets of suggested possibilities is unique. The set of possibilities  $W$  contains all the smallest believed proposition by any source of knowledge. So, the set of possibilities is the union of all suggested sets of possibilities that the agent considers that they deserve inquiry.

Some propositions, set of possibilities, may have zero as their degree of belief. DS does not consider regularity as it was considered in BE; otherwise, one could say that an agent with various independent<sup>98</sup> sources of evidence receive information about various suggested sets of possibilities or focal elements and the union of all suggested sets of possibilities is the set of possibilities. However, we need to define it in a way that suits DS.

Concerning the definition, a question arises: If a source of knowledge gives the union of all suggested sets of possibilities by another source of knowledge, could we call it an informative source? As the set of possibilities is unknown in an infallible sense, it seems that the source does not have any content and it is endorsing just a tautology. This issue remains a problem in DS theory of evidence. In the coming section, I illustrate by an example how DS represents the information from various sources of knowledge.

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<sup>98</sup> It is not easy to define independent source. In Dempster words in Liu, L., Yager, R.R., 2008, *Classic Works of the Dempster Shafer Theory of Belief Functions*, Springer, p. 68:

...The mechanism adopted here assumes independence of the sources, a concept whose real-world meaning is not so easily described as its mathematical definition.

The function which represents evidence is called a mass function or a basic probability assignment.

## 1.2 Mass Function

Based on the sources of knowledge and the degree of certainty, we could introduce the mass function. The mass function shows the intrinsic evidence which the agent has for each member of the power set of the set of possibilities, or propositions. The mass of every proposition,  $B$ , does not say anything about the mass of its subsets or supersets. The mass of  $B$  is the credence of the proposition  $B$  that the agent does not know how it should be distributed among its possibilities. Here is the distinction between DS and BE. For BE, there is always a default distribution. DS does not use the principle of indifference; therefore, there is no default distribution. Mass function assigns 0 to a proposition that no sources of knowledge suggest it as the set of possibilities. Also, DS normalizes the evidential support of each proposition. The sum of all masses should be 1 because DS needs to assign 1 as the degree of belief of the set of possibilities  $W$  as the infallible knowledge. The definition of belief function as I explain later illustrates why DS needs the normalization. After normalization the degree of belief to the union of all suggested set of possibilities will be also 1. Also, one can explain DS by starting with belief function and then mass function. However, starting with mass function is more intuitive.

A function  $M: 2^W \rightarrow [0,1]$  is a mass function if and only if:

$$\sum \{M(B) \mid B \subseteq W\} = 1 \tag{1}$$

The mass function is formed based on various sources of knowledge<sup>99</sup>. Every source can give the agent a mass number about its suggested set of possibilities or the focal element. A set  $D$  is a focal element if and only if its mass is more than zero.<sup>100</sup> I recommend interpreting a focal element as a suggested set of possibilities because intuitively it makes DS understandable, and it helps us to see the relationship among RT and BR and DS. Let's see how Liu and Yager define the mass:<sup>101</sup>

*The basic probability number  $m(A)$  allocated to a focal element  $A$  is not further divided into smaller chunks allocated to proper subsets of  $A$ .*<sup>102</sup>

Let's illustrate the issue with an example to explain how DS represent evidence as it is received while BE cannot do the same thing. Imagine an agent is on an island, and there are four Panthers on the island namely, two blue Panthers, and two pink panthers. They are made by an entertainment company to entertain people with a game. The tails of one of the blue Panthers and one of the pink panthers are long. So, there are four Panthers: a blue panther with a long tail,  $\{w_1\} = B \wedge C$ , a blue panther with short tail,  $\{w_2\} = B \wedge \neg C$ , a pink panther with long tail,  $\{w_3\} = \neg B \wedge C$ , and a pink panther with short tail,  $\{w_4\} = \neg B \wedge \neg C$ . The panthers draw the logo of the company on the doors, and the person is lucky if she sees the panther while she is drawing. Otherwise, she should seek relevant information to form a belief about who did that painting. One day, when she is going out, she sees the shape, the logo, on her door! She asks people to seek some information. She finds five sources. Let  $B$  be 'it was a blue panther', and  $C$  be 'it had a long tail'. Source<sub>1</sub> says, 'it was a blue panther'  $B$ , source<sub>2</sub> says, 'it was a blue panther with a long tail',  $B \wedge C$ ,

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<sup>99</sup> A source of knowledge could be a sensor, an observation, a person, a TV channel or something else. Please notice that it is not like traditional epistemology.

<sup>100</sup> Shafer, G., 2008, *Belief function and parametric models*, P: 276 in Liu, L., Yager, R.R., 2008, *Classic Works of the Dempster Shafer Theory of Belief Functions*, Springer, p. 265-290

<sup>101</sup> Liu, L., Yager, R.R., 2008, *Classic Works of the Dempster Shafer Theory of Belief Functions*, Springer: p. 3

<sup>102</sup> Basic probability number or assignment and mass are interchangeable terms.

source<sub>3</sub> says, ‘it had a long tail, or it was a pink panther’; equivalently, ‘it was not the blue panther with short tail’,  $\neg B \vee C$ , source<sub>4</sub> says, ‘it was a pink and it had a long tail’,  $\neg B \wedge C$ , and source<sub>5</sub> says ‘it was made by one of those four panthers’. How could she represent the information from various sources? What should be her degree of belief? And what should she believe?

Each source gives information with different certainty about their information. Let’s say Source<sub>1</sub> have .60 certainty about the report; Source<sub>2</sub> have .90 certainty; Source<sub>3</sub> .90; Source<sub>4</sub> .80; and Source<sub>5</sub> says 1. It seems that the information of Source<sub>5</sub> is not valuable because the agent does not learn something new. So, she ignores that information. DS normalizes the information in a way that the sum of all sources is equal to 1 and assigning all uncertainties to  $W$ . So, the interpretation of the information by Source<sub>1</sub> is ‘it is  $B$  60%, and 40% it may be  $W$  ( $B$  or  $\neg B$ ). Therefore, the relative normalized firmness of information from various sources are as following:  $M(B) = .15$ ;  $M(B \wedge C) = .225$ ;  $M(\neg B \vee C) = .225$ ;  $M(\neg B \wedge C) = .2$ ; and  $M(W) = .2$  relatively certain about its report.

As we know BE cannot distinguish the case that the agent has one source that says ‘it was a blue panther’; from the case with two sources that one source says, ‘it was blue panther with long tail’ and another source that says, ‘it was a blue panther with short tail’. BE assigns the same degree of belief .5 to blue panther with a long tail and blue panther with a short tail in both cases! In our example, BE suggests acting the same by using the principle of indifference for source 1, 3, 4, 5. As we always see the agent receives information as a support of a proposition and the proposition is not always a singleton (with one possibility) but BE suggests distributing that degree among the possibilities in the proposition. In our example with five sources, BE suggests having the following degree of belief without representing the evidential support as it received by the agent:

$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$
.425	.125	.325	.125

Table 1

DS, in contrast, by having basic probability assignment shows the information precisely as it received. We see every source is restricting her possibilities to a subset of all four possibilities. Source<sub>1</sub> restricts the answers to  $B \wedge C$  or  $B \wedge \neg C$  (blue panthers), source<sub>2</sub> restricts it to  $B \wedge C$  (blue panther with long tail), source<sub>3</sub> restricts it to  $\neg B \wedge C$ ,  $\neg B \wedge \neg C$  or  $B \wedge C$  (pink or long tail,  $\neg B \vee C$ ), source<sub>4</sub> restricts it to  $(\neg B \wedge C)$ , and source<sub>5</sub> does not restrict it.<sup>103</sup> In other word, every source gives information about the possible set of possibilities.<sup>104</sup>

As it was mentioned, a source of knowledge could give information to support even more than one possibility. For instance, if someone says that it was a blue panther, then she is saying that I should restrict my possibilities to two possible answers:  $B \wedge C$  or  $B \wedge \neg C$ . Also, sometimes a source says what is not the case. Like source<sub>4</sub> says that it was not the blue panther with a short tail  $\neg(B \wedge \neg C)$ . Source<sub>5</sub> seems not restricting the set of possibilities, but it does. It restricts the set of possibilities to the set of all possibilities. Notice a source might change the space of probability by changing the set of possibilities. Imagine the source<sub>6</sub> who is saying it was not a panther! This information changes the conceptual framework.

For any set of possibilities  $W$ , the power set  $2^W$  is the set of all propositions or set of all possible evidence that one can receive from a source of knowledge. In the above example, there are four possible worlds:  $\{w_1\} = B \wedge C$ ,  $\{w_2\} = B \wedge \neg C$ ,  $\{w_3\} = \neg B \wedge C$ ,

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<sup>103</sup> Sometimes we know how much we do not know. Therefore, it is necessary to accept that we should avoid normalize the degree of belief as BE suggest and on contrary to BE we could assign that degree to the degree of ignorance.

<sup>104</sup> Shafer calls the union of all suggested set of possibilities, *Core*. The smallest proposition which is certainly believed. It is not necessarily the set of all possibilities.

$\{w_4\} = \neg B \wedge \neg C$ . The agent may receive information that restricts her possibilities in 16 different ways. In the example, she received evidences in favor of five suggested set of possibilities  $\{w_1\}$ ,  $\{w_3\}$ ,  $\{w_1, w_2\}$ ,  $\{w_1, w_3, w_4\}$ , and  $\{w_1, w_2, w_3, w_4\}$ :

$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$
	*		*		*		
$\{w_2, w_3\}$	$\{w_2, w_4\}$	$\{w_3, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$
					*		*

Table 2

Now, DS assigns the normalized degree to each proposition as its intrinsic evidential support: Source<sub>1</sub> is .15; Source<sub>2</sub> is .225; Source<sub>3</sub> is .225; Source<sub>4</sub> is .2, and Source<sub>5</sub> is .2 relatively certain about its report.

$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$
	.225		.2		.15		
$\{w_2, w_3\}$	$\{w_2, w_4\}$	$\{w_3, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$
					.225		.2

Table 3

This example shows how DS makes its basic probability assignment or mass function and represent intrinsic evidential support of each proposition. Let's go further

step by step and introduce all functions. By introducing DS's Belief function, the contrast between BE and DS will appear.

### 1.3 Belief Function

The degree of belief of a proposition,  $B$ , is the degree of commitment that the real world is in  $B$  or as I will explain later the degree of belief is the sum of the masses of its subsets.

There are some texts that start explaining DS by belief function instead of mass function. There is no difference between these two approaches because both approaches are mathematically equivalent. We can define the mass function based on belief function and vice versa. Shafer showed that the two definitions are equivalent<sup>105</sup>. I will explain how one can find the mass function or basic probability assignment of a belief function.

Let  $W$  be a set of all possibilities. The function  $Bel^\circ: 2^W \rightarrow [0,1]$ , is a belief function if and only if it satisfies the following three axioms:

$$Bel^\circ(\emptyset) = 0^{106} \tag{2}$$

$$Bel^\circ(W) = 1 \tag{3}$$

$$Bel^\circ\left(\bigcup_{i=1}^n B_i\right) \geq \sum_{\substack{I \subseteq \{1,2,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|+1} Bel\left(\bigcap_{i \in I} B_i\right) \tag{4}$$

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<sup>105</sup> Shafer in his book, A mathematical theory of evidence, chapter II, showed it. We should just add  $m(\emptyset)=0$ . Then three axioms of belief function, is equivalent with the definition of mass function plus  $m(\emptyset)=0$ .

<sup>106</sup> In this chapter  $Bel$  without subscript indicates  $Bel_{DS}$ .

The first axiom (2) says that the degree of belief for empty set is zero. The second axiom (3) says that the degree of belief for  $W$  is 1 and it means the agent believes that the answer is in  $W$ .

The third axiom (4) is inequality of the union of propositions. In Bayesian probability, for two propositions we have:  $Bel^\circ(B \cup C) = Bel^\circ(B) + Bel^\circ(C) - Bel^\circ(B \cap C)$  ; for three propositions:  $Bel^\circ(B \cup C \cup D) = Bel^\circ(B) + Bel^\circ(C) + Bel^\circ(D) - Bel^\circ(B \cap C) - Bel^\circ(B \cap D) - Bel^\circ(C \cap D) + Bel^\circ(B \cap C \cap D)$  and so on. In DS, it is the same, but with inequality: for two propositions:  $Bel^\circ(B \cup C) \geq Bel^\circ(B) + Bel^\circ(C) - Bel^\circ(B \cap C)$  ; for three propositions:  $Bel^\circ(B \cup C \cup D) \geq Bel^\circ(B) + Bel^\circ(C) + Bel^\circ(D) - Bel^\circ(B \cap C) - Bel^\circ(B \cap D) - Bel^\circ(C \cap D) + Bel^\circ(B \cap C \cap D)$  and so on.

In case of  $C \cap D = \emptyset$  , in Bayesian epistemology  $Bel^\circ(B \cup C) = Bel^\circ(B) + Bel^\circ(C)$ , and in DS,  $Bel^\circ(B \cup C) \geq Bel^\circ(B) + Bel^\circ(C)$ . As you see, DS is not additive. I will explain how this property makes room for representing ignorance.

The bridge between the mass function and belief function is the following:

$$Bel^\circ(B) = \sum_{B \cap C = C} M(C) \quad (5)$$

$B \cap C = C$  or  $C \subseteq B$  both are equivalent in the above formula (5). I preferred the above form because of its symmetry with the definition of plausibility function which I explain later.

Now we could go back to our example and write the belief function based on the mass function.

	$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$
$M$	0	.225	0	.2	0	.15	0	0
$Bel^\circ$	0	.225	0	.2	0	.375	.425	.225
	$\{w_2, w_3\}$	$\{w_2, w_4\}$	$\{w_3, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$
$M$	0	0	0	0	0	.225	0	.2
$Bel^\circ$	.2	0	.2	.575	.375	.650	.2	1

Table 4

There are two ways to define basic probability assignment (mass function) based on a belief function. The first definition is a recursive function:

$$M^\circ(\emptyset) = 0, M^\circ(B) = Bel^\circ(B) - \sum \{M^\circ(C) \mid C \subset B\} \tag{6}$$

Alternatively, the equivalent definition is the following function:

$$M^\circ(B) = \sum \{(-1)^{|A-B|} Bel^\circ(B) \mid B \subseteq A\} \tag{7}$$

Also, a belief function is a BE (probability) function if the mass of propositions except the singletons are zero, or every proposition that has a mass does not contain

more than one possibility. In this case, the belief function will be additive because there is no undistributed evidential support.

## 1.4 Plausibility Function

The plausibility function illustrates the maximum degree of certainty which could be assigned coherently as per agent's evidence.

The degree of plausibility of a proposition is one minus its degree of disbelief (8). We also could define the plausibility function based on the mass function (9):

$$Pla^\circ(B) = 1 - Bel^\circ(\neg B) = 1 - Dis^\circ(B) \quad (8)$$

$$Pla^\circ(B) = \sum_{B \cap C \neq \emptyset} M(C) \quad (9)$$

One can add another formula to this section. As the above formulas are saying, the degree of plausibility is always more or equal to the degree of belief:

$$Pla^\circ(B) \geq Bel^\circ(B) \quad (10)$$

The difference between the degree of plausibility and belief is that higher degree of belief  $Bel^\circ(B)$ , means higher evidential support. The plausibility function is the degree of all evidential support that are not distributed, and they are compatible with the proposition. For example, for a coin that one does not know to be fair or biased, the degree of plausibility that it is head is 1, and the degree of plausibility that is not head is also 1. So, plausibility function says what is the maximum legitimate degree of certainty as per evidence (mass function). Notice, the degree of plausibility of  $B$  for an ignorant agent is 1, but it does not mean that she accepts  $B$ . Therefore, we cannot

interpret the degree of plausibility as the degree of acceptance. The belief function is closer to the notion of degree of acceptance than the plausibility function.

## 2 The degrees of suspension, disbelief, and ignorance

The degree of disbelief to  $B$ ,  $Dis^\circ(B)$ , is equal to the degree of belief to its negation. For example, the degree of disbelief to  $\{w_2, w_3, w_4\}$  is the degree of belief to  $\{w_1\}$  which it is .225 . Or the degree of disbelief to  $\{w_3\}$  is the degree of belief to  $\{w_1, w_2, w_4\}$  which is .375. The degree of ignorance or vagueness for any proposition is one minus the degree of belief and disbelief. Also, if we recall the definition of suspension of judgment in belief revision; accepting a proposition and its negation at same time, then maybe it seems plausible to say that the degree of conflict in DS is the degree of suspension. The degree of conflict or suspension is the minimum of the degree of belief and disbelief (believing its negation).

$$Dis^\circ(B) = Bel^\circ(\neg B) \quad (11)$$

$$Ign^\circ(B) = 1 - (Bel^\circ(B) + Dis^\circ(B)) \quad (12)$$

$$Sus^\circ(B) = \min \{Bel^\circ(B), Dis^\circ(B)\} \quad (13)$$

So, the above formulae are representing quantitative ignorance, suspension, and disbelief. The degree of disbelief could be calculated by the degree of belief in the negation of the proposition.  $Dis^\circ(B) = Bel^\circ(\neg B)$ . The degree of disbelief shows the degree of evidential support of the negation of the proposition. It seems that BE and DS hold this formula both but DS does not hold what is valid in BE:  $Dis_{BE}^\circ(B) = 1 - Bel_{BE}^\circ(B) = Bel_{BE}^\circ(\neg B)$  and on the contrary DS does not hold:  $Dis_{DS}^\circ(B) \neq 1 - Bel_{DS}^\circ(B)$ .

(12) says the degree of ignorance is equal to 1 minus the all evidential support for and against the proposition. So, the degree of evidence *about* (for and against) a

proposition plus degree of ignorance is 1. When there is no information, the degree of ignorance is 1. When the agent knows she is fully informed, like ‘the coin is fair’, then her degree of ignorance is zero. When the degree of ignorance is zero, the belief function is like a simple BE function.

(13) says what degrees of suspension mean. The degree of suspension is the degree of conflicting evidential support. Therefore, the minimum degree of belief and disbelief shows the conflicting evidential support. The highest degree of suspension is .5, and the minimum is 0. When the degree of suspension is zero, the degree of ignorance is one because  $Sus^\circ(B) = \min\{Bel^\circ(B), Dis^\circ(B)\} = 0$ , therefore  $Bel^\circ(B) = Dis^\circ(B) = 0$ , and consequently  $Ign^\circ(B) = 1 - (Bel^\circ(B) + Dis^\circ(B)) = 1 - 0 = 1$ . Also, when the degree of belief and disbelief are equal  $Bel^\circ(B) = Dis^\circ(B)$  then higher degree of suspension necessitate lower degree of ignorance and vice versa. This property meets one of the standards that I explained in the first chapter.

DS is different from BE is a significant way. DS could distinguish the quantitative notion of ignorance from suspension. This finding is valuable for the project.

The table in the next page illustrates all four epistemic functions with our example.

	$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$
$M$	0	.225	0	.2	0	.15	0	0
$Bel^\circ$	0	.225	0	.2	0	.375	.425	.225
$Dis^\circ$	1	.2	.650	.375	.575	.2	0	.2
$Ign^\circ$	0	.575	.350	.425	.425	.425	.575	.575
$Sus^\circ$	0	.2	0	.2	0	.2	0	.2
	$\{w_2, w_3\}$	$\{w_2, w_4\}$	$\{w_3, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$
$M$	0	0	0	0	0	.225	0	.2
$Bel^\circ$	.2	0	.2	.575	.375	.650	.2	1
$Dis^\circ$	.225	.425	.375	0	.2	0	.225	0
$Ign^\circ$	.575	.575	.425	.425	.425	.350	.575	0
$Sus^\circ$	.2	0	.2	0	.2	0	.2	0

Table 5

### 3 Qualitative doxastic attitudes

Now we need to observe what might differ in DS regarding the relationship between quantitative and qualitative notions of belief. As we can represent all four degrees of suspension, ignorance, belief, and disbelief; one may expect that the Lockean thesis should look different here. We need to find a proper approach.

The first difference is that in BE there is a threshold for defining belief; and in DS, as we need to define four basic epistemic states, in contrast, the acceptance threshold plays the primary role.

The second difference is about the difference between acceptance and belief. The acceptance threshold does not need to produce a consistent set of accepted propositions, while the belief threshold needs to give a consistent belief set. Therefore, the acceptance threshold could be any number between zero and one. Notice that belief threshold in BE needs to be more than .5.

The third difference is that the acceptance threshold is context sensitive (I think it is the same for belief). An agent may endorse a proposition in a context while she rejected to endorse it in another context. I think that a project about finding the proper threshold in different contexts is a key project to find the connection between knowledge and our actions.<sup>107</sup>

It seems rational to say that if an agent accepts  $B$ , then she should accept any proposition of which its degree of belief is more than or equal to  $B$ , in other words:

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<sup>107</sup> For example, I noticed that when we learn lots of completely new information (while we have no relevant information about it in our database), then it is rational to increase the threshold, and when we learn lots of information which contradict our old information, then it is rational to decrease the threshold. In the first case, the new information says that the context is changing and in the second case the new information says that our conceptual framework does not work properly. Finding the right account about how threshold is connected to our acceptance change and the change in our set of possibilities is an interesting project.

$$(Bel^\circ(C) \geq Bel^\circ(B) \wedge Acc(B)) \rightarrow Acc(C) \quad (14)$$

(14) creates a new possibility to avoid defining a number as a threshold which is not related to the propositions and their degrees of belief. An agent who accepts a proposition is restricting her options for the threshold. It does not need to be specific and exact. What accepting a proposition implies is that the threshold is less than its degree of belief. Let  $cr$  be the threshold; critical point, for accepting a proposition then

$$Acc(B) \rightarrow (cr \leq Bel^\circ(B)) \quad (15)$$

If the degree of belief for a proposition is more than or equal to the critical point  $cr$ , then it is an accepted proposition and as it is accepted, the proposition is believed if its negation is not accepted, and the proposition is suspended if its negation is accepted. The critical point might be any degree between zero and one. The milestones of the interdependent project about the legitimate critical point could be divided into two parts: a) finding general criteria (like (14) - (15)). b) finding the acceptable  $I_{cr}$  subset of  $[0,1]$  that satisfies the criteria. The first part is the philosophical or epistemological step, and the second part is the mathematical step. Notice that there is always at least one critical point which satisfies the condition. Zero is always a legitimate threshold because all propositions will be accepted and suspended, and there is no believed proposition except  $W$ . ( $0 \in I_{cr}$ )

For any belief function  $Bel^\circ$ , and threshold  $cr$ , there is a set of believed propositions, which are accepted, and the negation of which is not accepted. A legitimate threshold should produce a consistent belief set. That set could be called a suggested set of possibilities, or suggested proposition by the agent. This set could be called a belief set if it is consistent and the intersection of each two believed propositions belongs to the set. (or it is closed under logical consequence).

If a threshold is legitimate, then the set of believed propositions as per the threshold is closed under logical consequence. Alternatively, if a threshold is legitimate, then it produces a belief set.

### 3.1 From belief function to the quaternary epistemic states

The critical point plays a central role in the quaternarization of degrees of belief. We could define belief, suspended judgment, disbelief, and ignorance based on the critical point. For defining basic doxastic attitudes, we need to define what an *accepted proposition* is. An accepted proposition indicates that the agent is capable of accepting the proposition because she finds the degree of belief firm enough to accept that proposition. The agent might find the negation of that proposition also firm enough to accept.

Accepted (believable) proposition: A proposition is *accepted* if and only if its degree of belief is more than the critical point  $Cr$ .

$$Acc(B) \leftrightarrow (cr \leq Bel^\circ(B)) \quad (16)$$

Based on this definition we could define the basic doxastic attitudes as per the relationship between acceptance and basic doxastic attitudes. A proposition,  $B$ , is *believed* if and only if  $B$  is an *accepted* (believable) proposition and  $\neg B$  is not an *accepted* proposition. (A5)  $Bel(A, B) \leftrightarrow (Acc(A, B) \wedge \neg Acc(A, \neg B))$ . A proposition is *disbelieved* if and only if its negation is believed.

An essential property of the above definition is that from accepting a proposition does not follow that its negation is necessarily not accepted. It makes room for defining suspended judgments. The proposition,  $B$ , is *suspended* if and only if the proposition and its negation are accepted; in other words, their degree of suspension is more than the critical point. And eventually Ignorance: An agent is ignorant about the proposition,

$B$  , if and only if the proposition and its negation both are not accepted. Or the maximum degree of belief and disbelief is less than the critical point.

$$Bel(B) \equiv (Bel^\circ(B) \geq cr \wedge Bel^\circ(\neg B) < cr) \quad (17)$$

$$Dis(B) \equiv (Bel^\circ(B) < cr \wedge Bel^\circ(\neg B) \geq cr) \quad (18)$$

$$Sus(B) \equiv (Bel^\circ(B) \geq cr \wedge Bel^\circ(\neg B) \geq cr) \quad (19)$$

$$Ign(B) \equiv (Bel^\circ(B) < cr \wedge Bel^\circ(\neg B) < cr) \quad (20)$$

Notice that if a proposition  $B$  is believed then as per (17) necessarily  $Bel^\circ(B) > Dis^\circ(B)$ .

For finding thresholds which produce a belief set we assume that the agent wants:

a) to accept all propositions of which their degrees of belief are more than the critical point

b) not to accept any proposition of which their degrees of belief is less than the critical point.

c) to believe the intersection of two believed propositions (close under logical consequence)

d) to have a consistent set of belief

Every set of believed propositions produces a set of possibilities, or proposition, which is the intersection of all believed propositions. Let's call it a core belief and the set  $CB_{cr}$  be the core belief of the set of all believed propositions as per the critical point  $cr$ . Core belief is

$$(Bel^\circ(B) \geq cr \wedge Dis^\circ(B) < cr) \leftrightarrow ((B \cap CB_{cr}) = CB_{cr}) \quad (21)$$

(21) says that the intersection of all believed proposition is equal to the core belief. (notice that the core belief is the smallest believed proposition).

A threshold is legitimate if the intersection of all believed propositions as per that threshold (which is the core belief), is believed, and any superset of this core belief is believed as well. Let  $Leg(cr)$  means  $cr$  is legitimate:

$$Leg(cr) \rightarrow \forall B, C [(Bel(B) \wedge Bel(C)) \rightarrow Bel(B \cap C)] \quad (22)$$

$$Leg(cr) \rightarrow \forall B [Bel(B) \rightarrow Bel(B \cup C)] \quad (23)$$

(22) guarantees that the third (c) criteria for a legitimate threshold are held. (23) says if a proposition is believed all supersets of the proposition should be believed as well. If a threshold produces an inconsistent believed propositions then according to (22) the threshold is legitimate if the empty set  $\emptyset$ , which the intersection of two believed contradicting propositions, is believed. No threshold can satisfy it. Because for any threshold between zero and one the empty set is disbelieved ( $>0$ ) or is suspended (0). Therefore (22) and (23) are enough for finding the legitimate thresholds.

$$\{\forall B, C [Bel(B) \rightarrow Bel(B \cup C)] \wedge \forall B, D [(Bel(B) \wedge Bel(D)) \rightarrow Bel(B \cap D)]\} \leftrightarrow Leg(cr) \quad (24)$$

Still, there is a way to find a better and simpler criterion for a legitimate threshold:

*If the set of believed propositions as per the threshold  $cr$ , contains the intersection of its members then it is a legitimate threshold.*

Let  $K_{\geq cr}$  be the set of believed proposition when the threshold is  $cr$ , then the above definition says:

$$Leg(cr) \leftrightarrow \left( \bigcap K_{cr} \right) \in K_{cr} \quad (25)$$

One needs to find all thresholds which satisfy (25).

I think by following steps, the agent can reach all legitimate thresholds:

a) An ordered set of degrees of belief:

$$Deg = \{b_i = Bel^\circ(B_i) \mid B_i \in 2^W\} \text{ (ordered set)}$$

For every  $Bel^\circ(B_i)$  and  $Bel^\circ(B_{i+1})$  all thresholds in  $(Bel^\circ(B_i), Bel^\circ(B_{i+1})]$  produce the same set of believed propositions because what is accepted according to  $Bel^\circ(B_{i+1})$  as the threshold, is also accepted in any number in interval, and every proposition which is not accepted according to  $Bel^\circ(B_{i+1})$  is not accepted for any number in the interval.

b) For every proposition  $B_i$  define the set of believed propositions:

$$K_{\geq Bel^\circ(B_i)} = \{C \mid Bel^\circ(C) \geq Bel^\circ(B_i) \wedge Dis^\circ(C) < Bel^\circ(B_i)\}$$

Introducing the set of believed propositions for all thresholds in the set  $Deg$ .

c)  $K_{\geq Bel^\circ(B_i)}$  is consistent and deductively closed if and only if  $\bigcap_{C \in K_{\geq Bel^\circ(B_i)}} C \in K_{\geq Bel^\circ(B_i)}$  and it means that  $Bel^\circ(B_i)$  is a legitimate threshold.

d) If  $K_{\geq B}$  is consistent then all numbers between  $(b_{i-1}, b_i]$  are legitimate.  $Bel^\circ(B_i) = b_i$ . The reason is that they produce the same set of believed proposition as  $Bel^\circ(B_i)$  does.

Above instruction is a solution for finding all legitimate thresholds because for finding the legitimate thresholds one needs just to check all numbers that are assigned as a degree of belief to a proposition. For any number in the interval between every two number in the ordered set, the result of believed propositions does not change because the set of all accepted propositions and the set of all not accepted propositions remain the same.

Let us find legitimate thresholds for the example (finding the panther):

	$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$
$Bel^\circ$	0	.225	0	.2	0	.375	.425	.225
$Dis^\circ$	1	.2	.650	.375	.575	.2	0	.2
	$\{w_2, w_3\}$	$\{w_2, w_4\}$	$\{w_3, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_1, w_2, w_3, w_4\}$
$Bel^\circ$	.2	0	.2	.575	.375	.650	.2	1
$Dis^\circ$	.225	.425	.375	0	.2	0	.225	0

Table 6

a)  $Deg = \{0, .2, .225, .375, .425, .575, .650, 1\}$

b) Set of believed propositions for every  $b_i$  in  $Deg$ .

- $K_0 = \{C \mid Bel^\circ(C) \geq 0 \wedge Dis^\circ(C) < 0\} = \{\emptyset\}$  All propositions are suspended
- $K_{.2} = \{C \mid Bel^\circ(C) \geq .2 \wedge Dis^\circ(C) < .2\} =$   
 $\{\{w_1\}, \{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$
- $K_{.225} = \{C \mid Bel^\circ(C) \geq .225 \wedge Dis^\circ(C) < .225\} =$   
 $\{\{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$
- $K_{.375} = \{C \mid Bel^\circ(C) \geq .375 \wedge Dis^\circ(C) < .375\} =$   
 $\{\{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$
- $K_{.425} = \{C \mid Bel^\circ(C) \geq .425 \wedge Dis^\circ(C) < .425\} =$   
 $\{\{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$
- $K_{.575} = \{C \mid Bel^\circ(C) \geq .575 \wedge Dis^\circ(C) < .575\} =$   
 $\{\{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$
- $K_{.650} = \{C \mid Bel^\circ(C) \geq .650 \wedge Dis^\circ(C) < .650\} =$   
 $\{\{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$
- $K_1 = \{C \mid Bel^\circ(C) \geq 1 \wedge Dis^\circ(C) < 1\} =$   
 $\{\{w_1, w_2, w_3, w_4\}\}$

c) Consistent suggested core beliefs:

- $K_0 = \{C \mid Bel^\circ(C) \geq 0 \wedge Dis^\circ(C) < 0\} = \{\emptyset\}$  All propositions are suspended

$\bigcap_{B \in K_0} B = \emptyset$  and  $\emptyset \in K_0$  Therefore 0 is legitimate

As it was expected zero is always a legitimate threshold.

$$\bullet K_{.2} = \{C \mid Bel^\circ(C) \geq .2 \wedge Dis^\circ(C) < .2\} = \\ \{\{w_1\}, \{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$$

$\bigcap_{B \in K_{.2}} B = \{\{w_1\}\}$  and  $\{w_1\} \in K_{.2}$  Therefore .2 is legitimate

$$\bullet K_{.225} = \{C \mid Bel^\circ(C) \geq .225 \wedge Dis^\circ(C) < .225\} =$$

$$\{\{w_1\}, \{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_4\}, \{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$$

$\bigcap_{B \in K_{.225}} B = \{\{w_1\}\}$  and  $\{w_1\} \in K_{.225}$  Therefore .225 is legitimate

$$\bullet K_{.375} = \{C \mid Bel^\circ(C) \geq .375 \wedge Dis^\circ(C) < .375\} =$$

$$\{\{w_1, w_2\}, \{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_1, w_2, w_4\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$$

$\bigcap_{B \in K_{.375}} B = \{\{w_1\}\}$  and  $\{w_1\} \notin K_{.375}$  Therefore .375 is not legitimate

$$\bullet K_{.425} = \{C \mid Bel^\circ(C) \geq .425 \wedge Dis^\circ(C) < .425\} =$$

$$\{\{w_1, w_3\}, \{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$$

$\bigcap_{B \in K_{.425}} B = \{\{w_1, w_3\}\}$  and  $\{w_1, w_3\} \notin K_{.425}$  Therefore .425 is not legitimate

$$\bullet K_{.575} = \{C \mid Bel^\circ(C) \geq .575 \wedge Dis^\circ(C) < .575\} =$$

$$\{\{w_1, w_2, w_3\}, \{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$$

$\bigcap_{B \in K_{.575}} B = \{\{w_1, w_3\}\}$  and  $\{w_1, w_3\} \notin K_{.575}$  Therefore .575 is not legitimate

$$\bullet K_{.650} = \{C \mid Bel^\circ(C) \geq .650 \wedge Dis^\circ(C) < .650\} =$$

$$\{\{w_1, w_3, w_4\}, \{w_1, w_2, w_3, w_4\}\}$$

$\bigcap_{B \in K_{.650}} B = \{\{w_1, w_3, w_4\}\}$  and  $\{w_1, w_3, w_4\} \in K_{.650}$  Therefore .650 is legitimate

$$\bullet K_1 = \{C \mid Bel^\circ(C) \geq 1 \wedge Dis^\circ(C) < 1\} = \{\{w_1, w_2, w_3, w_4\}\}$$

$\bigcap_{B \in K_1} B = \{\{w_1, w_2, w_3, w_4\}\}$  and  $\{\{w_1, w_2, w_3, w_4\}\} \in K_1$  Therefore 1 is legitimate

So zero and one are always legitimate

d) Legitimate thresholds:

$$\text{Legitimate thresholds} = [0, .225] \cup (.575, 1]$$

An impressive result of this account is that suspending all proposition by taking zero as the acceptance threshold always is legitimate. Also, taking one as the threshold and believing the only  $W$  is always legitimate.

In the above example, if the agent takes .650 as her threshold, then she believes that it was not the blue panther with a short tail  $\neg(B \wedge \neg C)$ . This proposition expresses the most valuable information according to .650 as her acceptance threshold.

It could be observed that a threshold like .5 does not work because the intersection of two believed propositions is not believed while .6 is a legitimate threshold and the intersection of believed propositions is believed. It is easy to make new examples and observe the result of this account. Intuitively I find this proposal rich and convincing. Let's go further by making an epistemic logic based on this proposal to compare it to Belnap's model which we discussed in chapter two.

### 3.2 Truth table (as a four-valued logic)

We could make a four-valued (logic) based on this proposal. Let's  $CB$  be the smallest believed proposition, or the intersection of all believed proposition, based on a legitimate threshold, then:

	$\neg$
<i>Bel</i>	<i>Dis</i>
<i>Dis</i>	<i>Bel</i>
<i>Ign</i>	<i>Ign</i>
<i>Sus</i>	<i>Sus</i>

Table 7

$\wedge$	<i>Bel</i>	<i>Sus</i>	<i>Ign</i>	<i>Dis</i>
<i>Bel</i>	<i>Bel</i>	<i>Sus</i>	<i>Ign</i>	<i>Dis</i>
<i>Sus</i>	<i>Sus</i>	<i>Dis</i> ( $\emptyset$ ), <i>Sus</i>	*	<i>Dis</i>
<i>Ign</i>	<i>Ign</i>	*	<i>Dis</i> ( $\emptyset$ ), <i>Ign</i>	<i>Dis</i>
<i>Dis</i>	<i>Dis</i>	<i>Dis</i>	<i>Dis</i>	<i>Dis</i>

Table 8

$\vee$	<i>Bel</i>	<i>Sus</i>	<i>Ign</i>	<i>Dis</i>
<i>Bel</i>	<i>Bel</i>	<i>Bel</i>	<i>Bel</i>	<i>Bel</i>
<i>Sus</i>	<i>Bel</i>	<i>Bel</i> (CB), <i>Sus</i>	*	<i>Bel</i>
<i>Ign</i>	<i>Bel</i>	*	<i>Bel</i> (CB), <i>Ign</i>	<i>Sus</i>
<i>Dis</i>	<i>Bel</i>	<i>Sus</i>	<i>Ign</i>	<i>Dis</i>

Table 9

$\rightarrow$	<i>Bel</i>	<i>Sus</i>	<i>Ign</i>	<i>Dis</i>
<i>Bel</i>	<i>Bel</i>	<i>Sus</i>	<i>Bel</i>	<i>Bel</i>
<i>Sus</i>	<i>Bel</i>	<i>Bel(CB), Sus</i>	*	<i>Sus</i>
<i>Ign</i>	<i>Bel</i>	*	<i>Bel(CB), Ign</i>	<i>Ign</i>
<i>Dis</i>	<i>Bel</i>	<i>Bel</i>	<i>Bel</i>	<i>Bel</i>

Table 10

*Dis*( $\emptyset$ ), *Sus*, means it is disbelieved if it is an empty set, and it is suspended otherwise. *Dis*( $\emptyset$ ), *Ign*, means it is disbelieved if it is an empty set, and it is unknown otherwise.

*Bel*(*CB*), *Sus* means it is believed if it is the core belief, and it is suspended otherwise. *Bel*(*CB*), *Ign* means it is believed if it is the core belief, and it is unknown otherwise.

Let's prove one of the outcomes to show how it could be proved. The intersection of two suspended propositions; *B* and *C*, could be a disbelieved proposition or a suspended proposition (proof): for proving that a disbelieved proposition or a suspended proposition, one needs only an example. If  $B \wedge C = \emptyset$ , then *Dis*( $B \wedge C$ ) because its degree of belief is zero. Also  $B \wedge C$  could be a suspended proposition, a simple example is  $C = B$ , then  $B \wedge C = B$ , which means  $B \wedge C$  is suspended, because *B* is suspended. Now, I should prove that their intersection cannot be a believed proposition or an unknown proposition. First, the outcome cannot be a believed proposition, because if  $B \wedge C$  is believed, then *B* would be believed as well (which is not). In other words, if  $Bel^o(B \wedge C) \geq cr$ , then  $Bel^o(B) \geq cr$ . Second, it cannot be an unknown proposition, because if it is unknown, then  $Bel^o(B \vee C) \leq cr$ , and

consequently  $Bel^o(B) \leq cr$  (which is not). Therefore,  $B \wedge C$  cannot be a believed or an unknown proposition.

In contrast to Belnap's model, by having the acceptance threshold concerning a belief function and a legitimate function the agent can have only three of all four epistemic states. I explain the issue in the next section.

### 3.3 Qualitative Ignorance and suspended judgment

Unfortunately, DS could not represent all doxastic attitudes, because there is an important problem. For any legitimate acceptance threshold, one may have one of three epistemic states that two of them are belief and disbelief. I mean, it is impossible to have any unknown proposition when there is a proposition which is suspended. Also if the agent suspend a proposition, she cannot consider any other proposition as an unknown proposition.

Proof. Imagine  $B$  is suspended and  $C$  is unknown according to the legitimate critical point  $cr$ , then

$$Bel^o(B) \geq cr, Bel^o(\neg B) \geq cr, Bel^o(C) < cr, Bel^o(\neg C) < cr.$$

- I) One can prove that  $B \cup C$  is believed.  $Bel^o(B \cup C) \geq cr$ , because  $B \subset B \cup C$ , and  $Bel^o(B \cup C) \geq Bel^o(B) \geq cr$ . Also  $Bel^o(\neg B \cap \neg C) < cr$ , because  $(\neg B \cap \neg C) \subset \neg C$ , and  $Bel^o(\neg B \cap \neg C) < Bel^o(C) < cr$ .
- II) Also  $B \cup \neg C$  is believed.  $Bel^o(B \cup \neg C) \geq cr$ , because  $B \subset B \cup \neg C$ , and  $Bel^o(B \cup \neg C) \geq Bel^o(B) \geq cr$ . Also  $Bel^o(\neg B \cap C) < cr$ , because  $(\neg B \cap C) \subset C$ , and  $Bel^o(\neg B \cap C) < Bel^o(C) < cr$ .

Moreover, as per (I) and (II),  $Bel(B \cup \neg C)$  and  $Bel(B \cup C)$ , one can conclude that  $Bel(B)$  because the threshold was legitimate, therefore the intersection of two believed proposition should be believed. But as we assumed  $B$  was suspended,

$Sus(B)$ , and a proposition cannot be suspended and believed at the same time as I discussed in the first chapter. Therefore, the assumption that a legitimate proposition can produce a suspended proposition and an unknown proposition at the same time is wrong. Therefore, for a legitimate threshold it is impossible to have a suspended proposition and an unknown proposition at the same time.

I think the above consequence of the Acceptance threshold proposal is not quite plausible. The way that DS works cannot do better. Every model in formal epistemology has its limitation and problems. The transition from mass function and the belief function make this problem. In the last chapter, I introduce quantitative acceptance revision model and this acceptance threshold proposal works properly there.

#### **4 Suspension/Ignorance and the rule of combination**

It seems DS can represent a quantitative and qualitative suspension of judgment (with some problems). There is another problem when we see how DS propose to combine evidence. One can expect that an agent should suspend her judgment when she combines two highly conflicting evidence, unfortunately, DS cannot provide a solution for this problem. Finding a satisfying rule of combination still is an open problem. I think this project provides clear and clean interpretation because of distinguishing suspension and ignorance.

DS in its classic interpretation cannot distinguish suspension and ignorance, that is why I tried to give a satisfying interpretation of DS. In the next section, one can see how classic DS propose to combine information and make a new mass function when there is information from various sources. I explain the rule of combination which are proposed by Dempster, Yager, Inagaki and Dubois and Prade and I illustrates why they are not working and how the problem is just about the distinction between suspension and ignorance.

#### 4.1 The Dempster rule of combination.

It is essential to know how to combine information from two independent sources of knowledge, which give two mass or belief functions. If an agent receives information from two different sources what should she do? The first thought may be the following: If an agent learns to believe  $B$  and learns to believe  $C$ , where  $C$  is compatible with  $B$ , then she should conclude that  $B \cap C$  should be believed. If  $B$  and  $C$  are not compatible, then she should believe only all tautologies. Dempster rule capture the intuition when it comes to the notion of belief, but it fails to keep the valuable information during the combination of evidence. Receiving conflicting information does not always mean that that information is not valuable. Let's see how Dempster's rule works and fails.

Dempster's rule of combination is a three-step process: 1. The intersection of focal elements, which defines where masses should be distributed. Here it ignores all incompatible focal elements. 2. Multiplication of corresponding basic probability numbers  $M_1 M_2$ , which distributes non-normalized masses 3. Normalization, which makes the result a new mass function.

The combination of two pieces of evidence could be calculated by the following equation:

$$M_{12}(B) = \left( \sum_{C \cap D = B} M_1(C) \cdot M_2(D) \right) / \left( \sum_{C \cap D \neq \emptyset} M_1(C) \cdot M_2(D) \right) \quad (26)$$

This rule of combination is not what we like. The reason is simple; it ignores all conflict. Let's propositions  $B$  and  $D$  be the focal elements of  $M_1$ , and  $C$  and  $D$  be a focal element in  $M_2$ , while  $B$  and  $C$  and  $D$  are mutually exclusive, then in the combined mass function, the mass of  $B$  and  $C$  are both zero, because  $\sum_{E \cap F = B} M_1(E) \cdot M_2(F)$  because for any two propositions that their intersection is  $B$  or  $C$ , the mass of one of them is zero.

An example can illustrate the problem. Imagine there are three possibilities:  $w_1$ ,  $w_2$ ,  $w_3$ , and two mass functions,  $M_1$  and  $M_2$ , and their corresponding belief function,  $Bel_1^\circ$  and  $Bel_2^\circ$ :

	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$	$\{w_1, w_2, w_3\}$
$M_1$	.9	.1	0	0	0	0	0
$Bel_1^\circ$	.9	.1	0	1	.9	.1	1
$M_2$	0	.1	.9	0	0	0	0
$Bel_2^\circ$	0	.1	.9	.1	.9	1	1

Table 11

If the agent takes  $cr = .3$  as the threshold, then she thinks as per the first source that she should believe  $\{w_1\}$  as core belief or smallest believed proposition and she should believe its supersets. Then  $\{w_2, w_3\}$  is disbelieved. Also, she thinks that as per the second source she should believe  $\{w_3\}$  as the core belief and all its supersets while she disbelieves  $\{w_1, w_2\}$ . First thing that we expect is that as  $\{w_1\}$  and  $\{w_3\}$  are not compatible, therefore, the qualitative conclusion of the combination should not propose to believe them. Dempster's rule does that. The combination  $M_{12}$  is:

	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$	$\{w_1, w_2, w_3\}$
$M_1$	.9	.1	0	0	0	0	0
$Be_1^\circ$	.9	.1	0	1	.9	.1	1
$M_2$	0	.1	.9	0	0	0	0

$Bel_2^o$	0	.1	.9	.1	.9	1	1
$M_{12}$	0	1	0	0	0	0	0
$Bel_{12}^o$	0	1	0	1	0	1	1

Table 12

Dempster's rule satisfy the constraint that  $\{w_1\}$  and  $\{w_3\}$  should not be believed in the conclusion but it has a negative result when we look at what it proposes to believe:  $\{w_2\}$ ! The proposition  $\{w_2\}$  is disbelieved according to the both sources, and after the combination it is believed.

I think the agent, which considers  $Cr = .3$  as the threshold, should suspend  $\{w_1\}$  and  $\{w_3\}$  after the combination. Because in comparison to  $\{w_2\}$ , it has more support, at least one source is .9 certain that  $\{w_1\}$  should be believed.

Another problem is that the combination ignores the conflicting focal elements and it does not matter two sources are highly conflicting or not. For example, even each source assign .9999999 as the mass of  $\{w_1\}$  and  $\{w_3\}$  and .0000001 as the mass of  $\{w_2\}$ , the combination is the same! Or .3 as the mass of  $\{w_1\}$  and  $\{w_3\}$  and .7 as the mass of  $\{w_2\}$ , again the result is the same. It is not realistic.

The problem may be the first step of the combination: intersection. The result always assigns degrees to the intersection of focal elements. We need a rule of combination that, based on the reliability of sources of knowledge, makes a proper result. When two entirely reliable sources are giving two highly conflicting evidence, then the result should be a suspended judgment. When they are not reliable, and they are independent, then it seems reasonable to ignore the conflicting data, but the problem is that we could not simply assign 1 to their intersection. When two sources

of knowledge that we do not know whether or not they are reliable, are giving highly conflicting information, then it is reasonable to assume that they are not reliable. So, we could ignore the conflict, and we apply the intersection of their suggested focal elements.

Shafer introduced the method of discounting a belief function and explained its importance in the combination of highly incompatible belief function<sup>108</sup>. Let  $1 - a$  be the degree of reliability (or trust as Shafer called it) such that  $0 \leq a \leq 1$  and  $Bel^a(B) = (1 - a) \cdot Bel(B)$ . For combining two or more than two *belief* functions (not mass function), we could apply the following rule:

$$Bel^\circ(B) = 1/n ( Bel^{a_1^\circ}(B) + \dots + Bel^{a_n^\circ}(B) ) \quad (27)$$

Notice that in Dempster's rule even if an agent receives the information from 10 sources, nine compatible sources and one highly incompatible source, the result is the same as having two conflicting sources. By the discounting method, the result is different and consequently more plausible.

One of the problems with discounting method is that the order of combination changes the result. Imagine an agent use this method for two belief functions  $Bel^{a_1^\circ}$  and  $Bel^{a_2^\circ}$ . Then, she receives  $Bel^{a_3^\circ}$ . All with the same degree of reliability. The result for A will be  $Bel^{a_1^\circ}(B)/4 + Bel^{a_2^\circ}(B)/4 + Bel^{a_3^\circ}(B)/2$ . If she receives  $Bel^{a_1^\circ}$  and  $Bel^{a_3^\circ}$  and then  $Bel^{a_2^\circ}$ , the result is different  $Bel^{a_1^\circ}(B)/4 + Bel^{a_3^\circ}(B)/4 + Bel^{a_2^\circ}(B)/2$ . The reason behind the problem of discounting method is that the average operator is not *associative*.

## 4.2 Yager's rule

Yager proposed a new rule to avoid Dempster rule's problem. He focused on the normalization and the redistribution of masses. He did not change the structure of the

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<sup>108</sup> Shafer, G., 1976, *A mathematical theory of evidence*, p:251

rule completely. Yager introduced  $Q$ , the ground probability mass assignment. The difference between  $Q$  and  $M$  is the normalization:

$$Q(B) = \sum_{C \cap D = B} M_1(C).M_2(D) \quad (28)$$

In Yager's rule, the ground probability mass assignment of the empty set is equal or more than zero. (in DS traditional model it is always zero).

$$Q(\emptyset) = \sum_{C \cap D = \emptyset} M_1(C).M_2(D) \quad (29)$$

So  $Q(\emptyset)$  could be zero or more than zero.  $0 \leq Q(\emptyset)$ . (28) proposes that we should assign the degree of conflict to  $Q(\emptyset)$ .

Next step is just adding this degree  $Q(\emptyset)$  to the set of all possibilities:

$$M(W) = Q(W) + Q(\emptyset) \quad (30)$$

From Yager's point of view,  $M(W)$  is the degree of ignorance. The degree of belief based on Yager, could be calculate as per  $Q$  as the mass of propositions (except  $W$ ) and  $M$  as the mass of  $W$ .

	$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_2, w_3\}$	$\{w_1, w_2, w_3\}$
$M_1$	0	.9	.1	0	0	0	0	0
$M_2$	0	0	.1	.9	0	0	0	0
$Q_{12}$ Yager	.99	0	.01	0	0	0	0	0
$M_{12}$ Yager	0	0	.01	0	0	0	0	.99
$Bel_{12}^{\circ}$ Yager	0	0	.01	0	.01	0	.01	1
$Bel_{12}^{\circ}$ Dempster	0	0	1	0	1	0	1	1

Table 13

Yager’s rule cannot help so much. It has the same problem that DS had because still  $\{w_2\}$  has the highest degree of belief among proper subsets of  $W$ . According to the threshold  $Cr = .3$  it is unknown or ignorance, but for  $Cr = .001$  it should be believed while  $\{w_1\}$  or  $\{w_3\}$  are unknown! It ignores the conflicting evidence as Dempster’s rule does. We need a combination rule that propose to suspend a proposition in case of highly conflicting evidence for and against it.

Yager’s rule is better than Dempster’s rule because it does not give too much credit to the intersection of focal element. Still, the distinction between suspension and ignorance is missing, and it is easy to see how the mistake of taking *having conflicting evidence* and *having no evidence* the same epistemic state, is problematic in formal epistemology.

### 4.3 Inagaki's unified combination rule

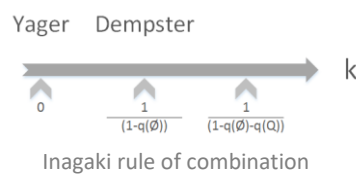
Inagaki unifies Dempster's and Yager's rule via defining unified combination rule by applying Yager's ground probability mass assignment. Take  $Q(\emptyset)$  as the degree of conflict, then

$$M(B) = (1 + k \cdot Q(\emptyset)) \cdot Q(B) \tag{31}$$

$$B \neq \emptyset; B \neq W; 0 \leq k \leq \left(\frac{1}{1 - Q(\emptyset) - Q(W)}\right)$$

$$M(W) = (1 + k \cdot Q(\emptyset)) \cdot Q(W) + (1 + k \cdot Q(\emptyset) - k) \cdot Q(\emptyset) \tag{32}$$

Based on what one takes as  $k$ , the result will be different.



When  $k$  is in its highest degree then

$$M(B) = ((1 - Q(\emptyset))/(1 - Q(W) - Q(\emptyset))) \cdot Q(\emptyset)$$

$$M(W) = Q(W)$$

These rules, Dempster, Yager and even the extreme rule, could not solve our problem because they follow the first step, they work on the intersection of focal elements.

### 4.4 Dubois and Prade’s disjunctive consensus rule

From Dubois and Prade’s point of view, above rules work for a set of sources that they are reliable, and we retract the conflicting evidence. Dubois and Prade distinct disjunctive and conjunctive pooling.

Conjunctive pooling: if the sources are completely reliable and properly interpreted, then there is no room for conflicting evidence, we should ignore them because it could be self-contradictory to claim that those sources are reliable.

Disjunctive pooling: if the sources are not completely reliable, but we have no information about their reliability then we can assume that one of those sources tells the truth without specifying which one<sup>109</sup>.

$$Q(B) = \sum_{C \cup D = B} M_1(C).M_2(D) \tag{33}$$

As an example, let us see how this rule treats example mentioned above.

	{w <sub>1</sub> }	{w <sub>2</sub> }	{w <sub>3</sub> }	{w <sub>1</sub> , w <sub>2</sub> }	{w <sub>1</sub> , w <sub>3</sub> }	{w <sub>2</sub> , w <sub>3</sub> }	{w <sub>1</sub> , w <sub>2</sub> , w <sub>3</sub> }
M <sub>1</sub>	0.9	.1	0	0	0	0	0
M <sub>2</sub>	0	.1	.9	0	0	0	0
M <sub>12</sub>	0	.01	0	.09	.81	.09	0

Table 14

<sup>109</sup> Dubois D., Prade, h., 1992, On the combination of evidence in various mathematical frameworks, in Reliability data collection and analysis, by Flamm, J. and Luisi, T., 1992: page 213-241

Again, this rule does not change so much, because still, like other rules, the degree of belief for  $\{w_1\}$  is zero. There is a problem. I think Yager's rule and Dubois and Prade's rule has the same nature toward conflicting data, but the later gives us better and more elaborate results however it cannot distinguish ignorance and suspension during the combination.

## 5 Basic doxastic attitudes in DS

Dempster-Shafer theory of evidence can represent the degree of suspension and ignorance, but it fails to keep their natural value when it combines two sources of evidence. It is not fully successful to represent doxastic attitudes qualitatively however it is more successful than BE.

The definition of degrees of suspension and ignorance works properly. The degree of suspension is the degree of conflicting belief, and the degree of ignorance is one minus the sum of the degree of belief and disbelief.

Regarding the relationship between belief function and basic qualitative doxastic attitudes, I have proposed a proposal. If the agent tries to distinguish doxastic attitudes, she should answer the acceptance threshold question: At which degree of belief I should accept a proposition? Finding the legitimate threshold is what an agent should do in a context and according to her belief function. The goal is trying to have a consistent and closed belief set. This proposal works, but it has its limits. Briefly, I can say concerning the quaternary judgmental assertions (qualitative belief, disbelief, suspension, and ignorance) an agent cannot suspend a proposition while she finds another proposition unknown and vice versa.

Unfortunately, DS does not give a sensible rule of combination for combining two pieces of evidence. I showed that proposed rules of combination that work based on the intersection of focal elements could not be plausible. Disjunctive pooling method

is a more elaborate rule of combination than DS and Yage's rule, but it is still in Dempster's camp because it cannot distinguish suspended judgment from ignorance during combining evidence. The rule of combination after years still is an open question, and it needs a separate project to be discussed. Still, there is not a satisfactory rule of combination to keep the evidential support of proposition as it is naturally expected. I showed that neglecting the distinction between suspension and ignorance plays is the reason that there is not the satisfactory rule of combination.

*...Probably the best thing to say is that the philosopher inquires into the rational laws of the dynamics of belief. However, the term “rational” is equally ambiguous between a normative and a descriptive perspective.*

The Laws of Belief, Wolfgang Spohn

# 5

## Ranking theory

By applying ranking theory, we can represent quantitative and qualitative notions of belief, suspension of judgment<sup>110</sup>, ignorance, belief change (like belief revision), conditionals, conditionalization (like general conditionalization in subjective probability). It entails all Belief Revision postulates for contraction, expansion, and

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<sup>110</sup> The representation of suspended judgment need some improvement which I will present. This theory is open for more improvement and new achievement in Formal Epistemology.

revision<sup>111</sup> and it is fruitful as much as a subjective probability because it is capable of representing degrees of belief and conditionalization. For understanding its motivation and its philosophical background, we could come back to the history of probability by looking at Bernoulli<sup>112</sup> on non-additive probability, or Joseph Butler<sup>113</sup> who had the same ideas, that if  $Bel_{Bern}^{\circ}(B|C) > 0$ , then  $Bel_{Bern}^{\circ}(\neg B|C) = 0$  which is not like the familiar complementation principle of negation.<sup>114</sup>

$$Bel_{Bern}^{\circ}(B|C) > 0 \rightarrow Bel_{Bern}^{\circ}(\neg B|C) = 0 \quad (1)$$

This theory and its capability of defining the degree of unopinionatedness (neutrality), is an exceptional property among other theories. During the inquiry into Ranking theory (RT), I noticed that the following questions play crucial roles: A) what are ranks? A narrower question could be: do ranks relate to qualitative epistemic updates, or do they relate to the quantitative epistemic changes? B) Could RT represent the grading of suspension as the degree of contradiction (minimum degree of belief and disbelief)? What about the degree of ignorance? Could we consider the degree of neutrality as the degree of suspension? C) what does suspension mean in RT and what RT can say about qualitative ignorance? In the end, I introduce a General Ranking Theory which is capable of being a basic qualitative model of the qualitative acceptance theory. The complete qualitative model of qualitative acceptance revision is presented in the last chapter.

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<sup>111</sup> See Spohn, W. 2014. *AGM, ranking theory, and the many ways to cope with examples*. In David Makinson on Classical Methods for Non-Classical Problems (pp. 95-118). Springer Netherlands. Not only it entails belief revision postulates for expansion, contraction and expansion.

<sup>112</sup> Spohn, W. 2009, *A survey of Ranking Theory*, in Huber, F., and Schmidt-Petri, C., (editors), *Degrees of Belief*, Dordrecht, Springer, P. 186

<sup>113</sup> Cohen, J. 1980. *Some historical Remarks on the Baconian conception of probability*. *Journal of the History of ideas* 41, 219-231: P 224

<sup>114</sup>  $Bel_{BE}^{\circ}(B|C) > 0$ , then  $Bel_{BE}^{\circ}(\neg B|C) = 1 - Bel_{BE}^{\circ}(B|C) \geq 0$

## 1 Classic Ranking Theory

Ranking theory, like subjective probability, takes the object of belief to be a proposition, set of possibilities. The objects of all doxastic attitudes, namely belief, disbelief, and suspension of judgment is a proposition, are a set of centered possible worlds, or a set of possibilities<sup>115</sup>:

*These objects are pure contents, i.e., propositions. To be a bit more explicit: We assume a non-empty set  $W$  of mutually exclusive and jointly exhaustive possible worlds or possibilities, as I prefer to say, for avoiding the grand associations of the term ‘world’ and for allowing to deal with de se attitudes and related phenomena (where doxastic alternatives are considered to be centered worlds rather than worlds). And we assume an algebra  $A$  of subsets of  $W$ , which we call propositions. All the functions we shall consider for representing doxastic attitudes will be functions defined on that algebra  $A$ .*

### 1.1 Negative Ranking function

Assume a non-empty set of possible worlds that they are mutually exclusive and jointly exhaustive. Take each of them as a possibility. Then define  $\mathcal{A}$  as an algebra of subsets of  $W$ . Each member (like  $B$ ) of  $\mathcal{A}$  is a proposition.

Let  $\mathcal{A}$  be an algebra over  $W$ . Then  $\kappa$  is a negative ranking function for  $\mathcal{A}$ , if and only if,  $\kappa$  is a function from  $\mathcal{A}$  into  $R^* = R^+ \cup \{\infty\}$  such that for all  $B, C \in \mathcal{A}$ :

$$\kappa(W) = 0 \tag{2}$$

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<sup>115</sup> Spohn, W. 2009, *A survey of Ranking Theory*, in Huber, F., and Schmidt-Petri, C., (editors), *Degrees of Belief*, Dordrecht, Springer, p: 4

$$\kappa (\emptyset) = \infty \quad (3)$$

$$\kappa (B \cup C) = \min\{\kappa (B), \kappa (C)\} \quad (4)$$

The function  $\kappa$  is a grading of disbelief<sup>116</sup>. It is the reason that  $\kappa$  is called a negative ranking function.  $\kappa (A) = 0$  means  $A$  is not disbelieved. Notice that not-disbelieving does not entail believing. This simple move, helps us to represent at least three epistemic states because a proposition and its negation might be not-disbelieved, and it means they are not believed or disbelieved. I discuss this issue later.

Besides, as we have  $\kappa (B \cup \neg B) = \kappa (W) = \min\{\kappa (B), \kappa (\neg B)\} = 0$ , one can conclude that

$$\kappa (B) = 0 \vee \kappa (\neg B) = 0 \quad (5)$$

## 1.2 Positive Ranking function

It is possible to model our doxastic states based on degrees of belief instead of degrees of disbelief. We can define a positive ranking function by the following:

Let  $\mathcal{A}$  be an algebra over  $W$ . Then  $\pi$  is a positive ranking function for  $\mathcal{A}$  iff  $\pi$  is a function from  $\mathcal{A}$  into  $R^* = R^+ \cup \{\infty\}$  such that for all  $B, C \in \mathcal{A}$ :

$$\pi (W) = \infty \quad (6)$$

$$\pi (\emptyset) = 0 \quad (7)$$

$$\pi (B \cap C) = \min\{\pi (B), \pi (C)\} \quad (8)$$

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<sup>116</sup> Spohn, W., 2012, *Laws of Belief, ranking theory and its applications*, OUP, P: 70

The function  $\pi$  is the grading of belief.  $\pi(B) = 0$  means that  $B$  is not believed. (6) says that the agent should believe tautologies, and (7) says that her degree of belief for all contradictions is zero. And (8) says that the degree of belief of a conjunction of two propositions is their minimum degree of belief.

As we have  $\pi(B \cap \neg B) = \pi(\emptyset) = \min\{\pi(B), \pi(\neg B)\} = 0$ , Therefore,

$$\pi(B) = 0 \vee \pi(\neg B) = 0 \tag{9}$$

### 1.3 Two-sided ranking function

The third function is a two-sided ranking function. Let  $\mathcal{A}$  be an algebra over  $\mathcal{W}$ . Then  $\tau$  is a two-sided ranking function for  $\mathcal{A}$  iff  $\tau$  is a function from  $\mathcal{A}$  into  $R^* = R^+ \cup \{\infty\}$  such that there is a negative ranking function  $\kappa$  and its positive counterpart  $\pi$  for which for all  $B \in \mathcal{A}$ :

$$\tau(B) = \kappa(\neg B) - \kappa(B) = \pi(B) - \kappa(B) \tag{10}$$

So one can conclude that:

$$\tau(B) + \tau(\neg B) = 0 \tag{11}$$

(11) asserts that the two-sided rank of a proposition plus the two-sided rank of its negation is zero.

### 1.4 Conditional Rank

Let  $\kappa$  be a negative ranking function for  $\mathcal{A}$ , and  $\kappa(B) < \infty$  or  $\mathcal{A}$  is regular. Then the conditional rank of  $B \in \mathcal{A}$  on given  $\mathcal{A}$  is defined as:

$$\kappa(B|C) = \kappa(B \cap C) - \kappa(B) \quad (12)$$

According to (12), we have

$$\kappa(B|C) = 0 \vee \kappa(\neg B|C) = 0 \quad (13)$$

Also, we can say  $\kappa(B \cap C) = \kappa(B) + \kappa(B|C)$ . As we have  $B = (B \cap C) \cup (B \cap \neg C)$ , we have  $\kappa(B) = \min\{\kappa(B \cap C), \kappa(B \cap \neg C)\}$ .

Besides, the positive conditional rank as well as the negative conditional rank could be defined:  $\pi(B|C) = \pi(\neg C \cup B) - \pi(\neg C)$ .

The general conditionalization like Jeffrey conditionalization in subjective probability<sup>117</sup>, for epistemic update or revision, could be defined as follow:

## 1.5 Spohn Conditionalization

Let  $\kappa$  be a negative ranking function for  $\mathcal{A}$  and  $B \in \mathcal{A}$  such that  $\kappa(B), \kappa(\neg B) < \infty$ , and  $n \in \mathbb{N}^+$ . Then the  $B \rightarrow n$  – conditionalization of  $\kappa$  is defined by

$$\kappa_{B \rightarrow n}(w) = \begin{cases} \kappa(w|B) & ; \quad w \in B \\ \kappa(w|\neg B) + n & ; \quad w \notin B \end{cases} \quad (14)$$

One could see the conditionalization as a three step process: first the negative rank of all possible worlds in the proposition decrease by subtracting  $\kappa(B)$  because  $\kappa(w|B)$  is equal to  $\kappa(w \cap B) - \kappa(B) = \kappa(w) - \kappa(B)$  in this step the negative rank of  $B$  will be zero. In the second step, the negative rank of all possible worlds in  $\neg B$  will decrease by subtracting  $\kappa(\neg B)$  (like what happened for possible worlds in  $B$ ) but

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<sup>117</sup> The relation between simple conditionalization and Jeffrey conditionalization in BE is like the relation between simple conditionalization and Spohn conditionalization in RT.

in the third step it increases by adding  $n$ . This three step guarantees that the negative rank of  $B$  is zero and the negative rank of its negation is  $n$ .

## 1.6 Regularity

A negative ranking function  $\kappa$  for  $\mathcal{A}$  is called regular iff  $\kappa(B) < \infty$  for all non-empty  $B \in \mathcal{A}$ :

$$\kappa \text{ is regular} \leftrightarrow \forall B \in \mathcal{A} ((B \neq \emptyset) \rightarrow \kappa(B) < \infty) \quad (15)$$

## 1.7 Qualitative belief and degree of unopinionatedness

Based on the definition of a two-sided ranking function, one can identify three kinds of values for every given  $\tau(B)$ :

$\tau(B) > 0$ , which means the agent believes  $B$ .

$\tau(B) < 0$ , which means the agent disbelieves  $B$ .

$\tau(B) = 0$ , which means the agent suspends her judgment towards  $B$ .

Here, RT gives a clear relationship between qualitative and quantitative belief, disbelief, and suspension. But why the agent should suspend only when  $\tau(B) = 0$ , Spohn in his book says: <sup>118</sup>

*It may seem unfair that the range of belief extends to all positive reals (or integers) and the range of disbelief to all negative reals (or integers), whereas there is only one way to be neutral, namely by assigning rank 0. Why should neutrality not comprise a larger range of ranks? We could just as well distinguish some positive rank (or some positive number)  $z$  and define the closed interval  $[z, -z]$  as the range of*

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<sup>118</sup> Spohn, W., 2012, *The Laws of Belief, ranking theory and its applications*, OUP, p. 76, 77

*neutrality. So  $\tau(B) > z$ , express belief in  $B$ ,  $\tau(B) > -z$ , express disbelief in  $B$ , and everything in between express suspense of judgment...*

Moreover, here there is something even more exciting regarding our project.<sup>119</sup>

*... how exactly the parameter  $z$  is fixed depends on how strictly we want to understand belief in the given context. The crucial point is that, however we fix the parameter  $z$ , we always get the formal structure of belief we want to have... The study of belief is the study of that ranking structure.*

The definitions of belief, disbelief, and suspension need amendment. The suspension set could be established based on the degree of neutrality:  $S = \{B \mid -z \leq \tau(B) \leq z\}$ . If  $-z \leq \tau(B) \leq z$ , then  $-z \leq \tau(\neg B) \leq z$ , so fortunately, it endorses one of the axioms (A.9) that I like to hold in this project  $Sus(A, B) \leftrightarrow Sus(A, \neg B)$ . In the other word, if  $B \in S$ , then  $\neg B \in S$ .

## 1.8 Doxastic attitudes in RT

The relationship between two-sided ranking function and doxastic states is the following:

$$\tau(B) > z \equiv Bel(B) \tag{16}$$

$$\tau(B) < -z \equiv Dis(\neg B) \tag{17}$$

$$-z \leq \tau(B) \leq z \equiv Sus(B) \tag{18}$$

I think (18) should be the state of suspension and not ignorance because there are cases that RT suggest assigning zero as the negative rank, like a fair coin. Also, it seems

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<sup>119</sup> Ibid

to me that RT suggests assigning no rank to a proposition which is unknown (my interpretation). If there is a subset of  $\mathcal{W}$  which is not a member of the algebra and does not have any rank, then one can call it unknown proposition. I discuss it in the coming section about qualitative belief, suspension, disbelief and ignorance.

Every negative ranking function (or two-sided ranking function), illustrates a unique belief set which is closed under logical consequence. The set of all believed propositions is the belief set:

$$K = \{B \mid \tau(B) > z; z > 0\} \quad (19)$$

Moreover, we could define the disbelief ( $D$ ) and suspension set ( $S$ ):

$$D = \{B \mid \tau(B) < -z\} \quad (20)$$

$$S = \{B \mid z \geq \tau(B) \geq -z\} \quad (21)$$

The proof of closure is simple. Take  $B$  and  $C$  as believed propositions, then their conjunction belongs to  $K$  because we have  $\tau(B) > z$ , and  $\tau(C) > z$  and it means

$\kappa(\neg B) > z$  and  $\kappa(\neg C) > z$ , on the other side we have  $\kappa(\neg B \cup \neg C) = \min\{\kappa(\neg B), \kappa(\neg C)\}$ ,

Moreover, as the negative ranks of  $\neg B$  and  $\neg C$  are both more than  $z$ , we can conclude that

$\kappa(\neg B \cup \neg C) > z$ , and it entails that  $\tau(B \cap C) > z$  or  $B \cap C$  is believed:  $Bel(B \cap C)$ .

## 1.9 What are Ranks?

The account that negative ranks are the grading of disbelief, and positive ranks are the grading of belief is the classic interpretation of RT. We could rewrite ranking function with different notations to reflect this interpretation:

$$Bel_{RT}^{\circ}(B) = \pi(B) = \kappa(\neg B) \quad (22)$$

$$Dis_{RT}^{\circ}(B) = \kappa(B) \quad (23)$$

So one can conclude that

$$Bel_{RT}^{\circ}(B) = Dis_{RT}^{\circ}(\neg B) \quad (24)$$

As believing a proposition is equivalent with disbelieving its negation, one can legitimately expect the same property when she thinks of quantitative belief. (24) endorses this intuition.

One of the most remarkable properties of RT is the relationship between the degree of belief and disbelief. If we know an agent's degree of belief of a proposition, then we cannot always know her degree of disbelief. For example, if we know that the agent believes  $k(B)$  with the degree of 0, then there is no way to know what the degree of disbelief  $k(\neg B)$  is. If we know that the degree of belief is more than zero  $k(B) > 0$ , then we could be certain that its degree of disbelief is zero  $k(\neg B) = 0$ . It seems that RT differs from BE with regards to the relationship between degrees of belief and disbelief. It is always possible to know the degree of disbelief in BE when we know the degree of belief. RT and DS are similar regarding to this issue. Notice BE does not have this property. However, I like this property of RT, but the classic interpretation of RT is not satisfying. There are reasons to avoid the grading-of-disbelief interpretation of negative ranks. In the next section, I explain the problem.

### 1.9.1 The quantitative nature of ranks

I think it is simplistic to say that the ranks are grading of belief and disbelief. One criticism of this interpretation is that an agent can legitimately have a positive degree of belief and disbelief at the same time. For instance, in some degree I believe that in the next hour it is rainy because of the weather forecast that I watched on TV, also in some degree I disbelieve because sky is clear blue. I think that It is not necessary to assign zero to the degree of belief or disbelief as (1) is proposing (also (5) the law of negation endorses).<sup>120</sup> It seems to me that RT focuses, in return, on the difference between the degree of belief and disbelief. It ignores the contradicting evidence and considers which one is preferable. Therefore, if a proposition and its negation are not both preferable, then RT says: it is suspended. This resemblance between ranks and the degree of preferability is clear to me, but their exact relationship is a conundrum.

It seems natural to expect the ranking theory to represent information which is received as a probability function: a fair coin, a dice, or even statistical information. What should be the negative rank of a proposition that the coin comes up head? Ranking theory says zero. What should be the negative rank of the proposition that a dice come up 5? Again, it should be zero! The two-sided rank should be zero as well.

I think the preferability is the key to a plausible interpretation. Let's begin with the definition of preferability and the degree of preferability:

The degree of preferability of a proposition is the degree of belief to the proposition minus its degree of disbelief.

$$Pref^\circ(B) = Bel^\circ(B) - Dis^\circ(B) \quad (25)$$

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<sup>120</sup> I do not mean that ranking theory is not contradiction-tolerant, RT, Qualitatively, is a contradiction-tolerant theory.

The first thing that connects the quantitative notion of preferability and the qualitative notion of belief is the expectation that if a proposition is believed, then its degree of preferability should be positive.

$$Bel(B) \rightarrow Pref^\circ(B) > 0 \tag{26}$$

What are properties that  $Pref^\circ$  and the two-sided ranking function  $\tau$ , both have in common? The first property is that from believing a proposition follows that  $\tau$  is positive (16) says that  $\tau(B) > z$  and  $z > 0$  therefore,  $Bel(B) \rightarrow \tau(B) > 0$ . Another property is the relation between the degree of preferability of a proposition and its negation:

$$Pref^\circ(B) + Pref^\circ(\neg B) = 0 \tag{27}$$

(27) is similar to (11) which says  $\tau(B) + \tau(\neg B) = 0$ .<sup>121</sup> Unfortunately the, preferability function is not exactly the two-sided ranking function. Let me illustrate the situation with two examples. Imagine we have four possibilities and  $Bel_{BE}^\circ$  is a Bayesian function as follow:

	$w_1$	$w_2$	$w_3$	$w_4$
$Bel_{BE}^\circ$	.6	0	.1	.3
$Dis_{BE}^\circ$	.4	1	.9	.7

Table 1

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<sup>121</sup> The belief function can be Bayesian or DS. In both cases, the preferability function holds this property.

How can an agent translate this information into a ranking function? Could the preferability function help?

	$w_1$	$w_2$	$w_3$	$w_4$
$Pref^\circ(B)$	.2	-1	-.8	-.4

Table 2

The degree of preferability may help the agent to have an ordered set of possibilities with ordinal numbers. The property of the two-sided ranking function which is important is that for all propositions  $B$  and  $C$ , the two-sided ranking function is such that  $\tau(B \cup C) \geq \tau(B)$ . Therefore, if a proposition is believed, each superset of the proposition is believed as well. This property is important for the closure. All preferability functions have this property as well: for all  $B$  and  $C$  the preferability function holds that  $Pref^\circ(B \cup C) \geq Pref^\circ(B)$ . Therefore, if one defines a preferability threshold for defining believed propositions, then for any believed proposition all supersets of the proposition are believed as well.

Besides, there is another exciting property that two-sided ranking function and the preferability function have in common: it is impossible to have two possibilities with positive two-sided rank or with positive preferability degrees. The proof is simple, if the two-sided ranks of two possibilities are positive,  $\tau(\{w_i\}) > 0$  and  $\tau(\{w_j\}) > 0$ , then from  $\tau(\{w_i\}) > 0$  one can conclude that  $\kappa(W - \{w_i\}) > \kappa(\{w_i\})$  and  $\kappa(W - \{w_i\}) > 0$ .

$$\kappa(W - \{w_i\}) = \min (\kappa(\{w_n\} | w_n \in W - \{w_i\})) > 0 \text{ and as } w_j \in (W - \{w_i\}),$$

$\kappa(\{w_j\}) > 0$  and therefore, it is impossible to have two possibilities with positive two-sided ranks.

For proving that two possibilities with a positive degree of preferability, assume that  $Pref^\circ(\{w_i\}) > 0$ , and  $Pref^\circ(\{w_j\}) > 0$ . Then from  $Pref^\circ(\{w_i\}) > 0$  one can conclude that  $Bel^\circ(\{w_i\}) > Bel^\circ(W - \{w_j\}) = Bel^\circ(\{w_j\}) + Bel^\circ(W - \{w_j\} - \{w_j\})$ , therefore

$$Bel^\circ(\{w_i\}) - Bel^\circ(\{w_j\}) > Bel^\circ(W - \{w_j\} - \{w_j\}) \quad (I)$$

By same argument, one can conclude that

$$Bel^\circ(\{w_j\}) > Bel^\circ(W - \{w_i\}) = Bel^\circ(\{w_i\}) + Bel^\circ(W - \{w_j\} - \{w_j\}),$$
 therefore

$$Bel^\circ(\{w_j\}) - Bel^\circ(\{w_i\}) > Bel^\circ(W - \{w_j\} - \{w_j\}) \quad (II)$$

Moreover, (I) and (II) cannot be true at the same time. Therefore, it is impossible to have two possibilities with positive degrees of preferability.

The preferability function which is made based on a probability function is not precisely a two-sided rank because it cannot hold the most essential property of two-sided ranks: For all degree of neutrality,  $z$ , if two propositions are believed or if the two-sided rank of two propositions are more than  $z$ , then the two-sided rank of the intersection of those propositions is more than  $z$  as well:  $\forall z ((\tau(B) \geq z \wedge \tau(C) \geq z) \rightarrow \tau(B \cap C) \geq z)$ . This property guarantees the closure. Unfortunately, one cannot say the same thing about the preferability function. The second example illustrates why it does not work:

	$w_1$	$w_2$	$w_3$	$w_4$	$w_5$
$Bel_{BE}^\circ$	.2	.2	.2	.2	.2
$Dis_{BE}^\circ$	.8	.8	.8	.8	.8
$Pref^\circ$	-.6	-.6	-.6	-.6	-.6

Table 3

One can observe that  $Pref^\circ(\{w_4, w_5\}) = Pref^\circ(\{w_3, w_5\}) = -.2$ , and

$Pref^\circ(\{w_1, w_2, w_3\}) = Pref^\circ(\{w_1, w_2, w_5\}) = .2$ , but their intersection:

$$Pref^\circ(\{w_1, w_2, w_3\} \cap \{w_1, w_2, w_5\}) = Pref^\circ(\{w_1, w_2\}) = -.2$$

The intersection of two propositions with positive degrees of preferability has a negative degree of preferability. In RT the intersection of two propositions with a positive two-sided rank always has a positive two-sided rank. From having a positive two-sided rank of the proposition  $B$  and the proposition  $C$ ,  $\tau(B) > 0, \tau(C) > 0$ , follows that the negative ranks of  $\neg B$  and  $\neg C$  are both positive ( $\kappa(\neg B) > 0$  and  $\kappa(\neg C) > 0$ ). And consequently  $\tau(B \cap C) > 0$  because  $\kappa(\neg B \cup \neg C) = \min\{\kappa(\neg B), \kappa(\neg C)\} > 0$ . Therefore, the degree of preferability cannot be the two-sided ranking function.

If an agent receives information that all possibilities in her set of possibilities are equally probable, then what is her negative ranking function? As they are equally probable, then their negative ranks should be the same, and as at least one possibility should have zero as its negative rank, therefore, their negative ranks are all zero. Therefore, RT proposes to assign zero as their negative rank. Does this translation give us some hints to find the relationship between two-sided rank and preferability function?

Let's see how RT works when an agent is in a lottery. Imagine there are eight tickets and the probability of each to be the winning ticket is  $1/8$ . Then RT suggests that the negative rank of each possibility (ticket) should be zero. Consequently, the two-sided rank of all propositions is zero and they are all suspended. Is there any other possible translation by RT? I think no. I think this is the best translation of the information and the best equivalent ranking function.

Now, imagine the same lottery, but two tickets are two times more probable than other tickets. The probability of two tickets,  $T_1, T_2$  are each .2, and other six possibilities are .1 probable. What should be the equivalent negative ranking function?

If we construct an ordered set of possibilities with an ordinal number (the degree of preferability), then two tickets are more preferable than others (-.6 their degree of preferability while for other possibilities it is -.7). Then the agent must suspend two tickets with .2 as their probability and disbelieves all other possibilities, and she must disbelieve that one of the tickets  $T_3$  or  $T_4$  or...  $T_8$  is winning, but the proposition that  $T_1$  or  $T_2$  is winning is less probable than ( $T_3$  or  $T_4$  or...  $T_8$ ) is winning!

As it can be observed the relationship between BE and RT is a mystery, and still, it is an open question. From Spohn's point of view they are two distinct doxastic modes:

*I am reluctant to opt for interactionism. My experience rather is that belief and probability are like oil and water; they do not mix easily. So, the dualistic alternative to interactionism is separatism, the view that there are indeed two distinct doxastic modes. You may be described as being in the one or as being in the other, perhaps on different occasions; but there is no good way to mediate between or combine the two modes. I sense the absurdity of this position; therefore, I am not determinately promoting it. However, it is obvious that I have so far adopted methodological separatism, as one might call it.*

If ranking theory and statistical information are two distinct doxastic modes, then it seems, receiving statistical information cannot change an agent's ranking function.

### 1.9.2 The qualitative nature of ranks

A qualitative approach to the nature of ranks is an account of how qualitative epistemic updates produce ranks. Ranks illustrate the quantitative nature of our epistemic states from qualitative updates.

I try to illustrate by an example how a qualitative approach works in contrast with the quantitative approach. The question of the nature of ranks and its relation to statistical information remains open.

One interpretation of RT says that the nature of coming information is quantitative, and it mean that the agent should fix her ranks based on the coming quantitative information. One can ask someone, who accept this interpretation, how should one translate other quantitative information, like statistical information, to a ranking function? I could not find a systematic way to answer this question. The best answer in the literature is that they are two distinct doxastic modes. Another interpretation says that the agent receives only qualitative information, and then she assigns ranks to her possibilities. I explain how the second interpretation may look alike and how it works.

In the qualitative interpretation of ranks, the ranks are the degree of contradicting information. Ranks are the production of changing the order of possibilities. Every change produces a change in the ranks of possibilities. When I made this interpretation of the ranking theory, I observed that this interpretation could explain what the set of possibilities is,  $W$ , and what does it means when an agent considers an algebra over  $W$ .

The set of possibilities  $W$  is the set of possibilities that their place *could* be known by some imaginable information. The algebra over  $W$  shows that the agent expects to receive information about propositions in the algebra. For example, imagine that  $W = \{w_1, w_2, w_3\}$ , then the agent believes all information can change the order of these three possibilities. If she considers  $\mathcal{A} = \{\emptyset, \{w_1\}, \{w_2, w_3\}, \{w_1, w_2, w_3\}\}$ , then it means she expects to receive information about  $\{w_1\}, \{w_2, w_3\}$  and all information will change the order of these two sets. In other words, for the agent the proposition  $\{w_2, w_3\}$  is like a possibility. Every proposition in an algebra that its proper subsets are

not a member of the algebra are atoms. For example, atoms of  $\mathcal{A}$  are  $\{w_1\}, \{w_2, w_3\}$ , and atoms of the algebra  $2^W$  are  $\{w_1\}, \{w_2\}, \{w_3\}$ .

It is better to say that every ranking function produces an ordered set of atoms (instead of possibilities) because the algebra which the function is defined over it, is not always  $2^W$ .

We assume that the agent receives qualitative information about propositions. There are two possible qualitative epistemic changes: An agent might learn to suspend a proposition, or an agent learns to believe a proposition (or disbelieve its negation). So, the question is how each qualitative epistemic change affects ranks.

If an agent learns that she should suspend the proposition  $B$ , then it means that she should suspend  $\neg B$  as well, therefore the negative rank of  $B$  and  $\neg B$  should be less than or equal to  $z$ . If they are suspended, then the agent does not need to change anything. For simplicity, I introduce a model when  $z = 0$ . If the proposition is not suspended and the degree of neutrality is zero, then the agent should change her ranking function as the following formula suggest:

$$Sus(B) \rightarrow \kappa_{new}(w_i) = \kappa_{old}(w_i) - \min\{\kappa_{old}(w_j) | w_j \in B\} \quad (28)$$

(28) guarantees that the minimum negative rank of possibilities in a proposition which should be suspended is equal to 0. As the suspension of a proposition  $Sus(B)$ , necessitates the suspension of its negation  $Sus(B) \equiv Sus(\neg B)$ , therefore, the same rank assignment should be done for  $\neg B$ .

One can define the conditional suspension like  $Sus(B|C)$ , and it means that after suspension the negative rank of  $B \cap C$  is equal to the negative rank of  $C$ .  
 $(\kappa_{new}(B \cap C) = \kappa_{old}(C))$

The second possible qualitative epistemic update happens when an agent learns that she should believe a proposition. An agent assigns the ranks after believing  $B$  by the following formula:

$$Bel(B) \rightarrow \begin{cases} (\forall w_i \in B; \kappa_{new}(w_i) = \kappa_{old}(w_i) - \min\{\kappa_{old}(w_j) \mid w_j \in B\}) \\ (\forall w_i \notin B; \kappa_{new}(w_i) = \kappa_{old}(w_i) + 1)^{122} \end{cases} \quad (29)$$

(29) says that if an agent believes  $B$ , then she should assign zero as the rank of the proposition, and she should add one degree to the rank of its negation. Therefore, believing  $B$  produce the same degree that a conditionalization like  $B \rightarrow (\kappa(\neg B) + 1)$  does. The above formulae (28) and (29) need an amendment for all possible degree of neutrality.

Now, two interpretations of ranks are introduced, and it is time for one of the most critical question: RT and degrees of suspension. Sections 1.9.1 and 1.9.2 are not connected to other sections in this chapter, because the interpretations of ranking theory do not play any role in other sections specially when it comes to the general ranking theory which I explain at the end of this chapter.

## 1.10 Grading of suspension

If one sees the ranks as the degrees of belief or disbelief, then one should agree that RT cannot represent the degree of suspension because the degree of contradiction (conflicting evidence from two different sources of knowledge) is always zero. (1) and (3.5) are both illustrating this point: a positive (more than zero) negative rank of a proposition necessitates zero as the negative rank of its negation.

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<sup>122</sup> An alternative account could be adding the rank of the believed proposition to its negation as the degree of surprise. I avoid it because of the simplicity of the example. My main intention is giving an intuition by giving a simple idea.

I think one could say that ranking functions are a preferability-like functions. By adopting this approach, one can say that ranking theory is a contradiction-tolerance model. I mean however RT does not represent the degree of conflict, but it is not like DS rule of combination that simply delete conflicting evidence. I discuss the qualitative suspension in the next section. As RT represents qualitative suspended judgment, I like to check whether RT represents the degrees of suspension or not.

One of the things in RT that might grasp one's attention is the unopinionatedness threshold or the degree of neutrality. Could this threshold be the degree of suspension?

The threshold  $z$  can't be the degree of suspension, because the degree of suspension at least can be different for every proposition. If we agree that  $z$  is the degree of suspension, then we should agree that all propositions have the same degree of suspension. Also, the degree of belief (two-sided rank) which is ignored because of  $z$ <sup>123</sup>, cannot represent the degree of suspension, because then all propositions which their two-sided rank is more than  $z$  would have the same degree of suspension. The second reason to reject this idea is that  $z$  does not relate to the degree of contradictory information.

The threshold  $z$  is the threshold of unopinionatedness, and it says that an agent ignores<sup>124</sup> ranks which are less than the threshold and what makes a proposition to be believed or disbelieved depends on the unopinionatedness threshold. Notice that there is a difference between the unopinionatedness threshold and the degree of unopinionatedness. We need to be careful about it. For every proposition the degree of unopinionatedness, or the degree to which the agent ignores, is different. If the two-sided rank of a proposition is 10, and  $z = 6$ , then the proposition is believed, and the degree of unopinionatedness is 6. If the two-sided rank of a proposition is 4 and  $z = 6$ , then the degree of unopinionatedness of the proposition is 4. The degree of

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<sup>123</sup>  $\tau(B) > z \rightarrow Sus^o(B) = z; \tau(B) \leq z \rightarrow Sus^o(B) = \tau(B)$

<sup>124</sup> 'ignore' means the rank does not lead to a belief because it is not high enough (like Lockean thesis)

unopinionatedness is the two-sided rank that an agent ignores. As we observe, however, the unopinionatedness threshold is the same for all propositions, but the degree of unopinionatedness is different. The threshold  $z$  is the maximum possible degree of unopinionatedness for any proposition.

It seems RT cannot represent degrees of suspension and degrees of ignorance, but RT can represent the degrees of unopinionatedness. So, RT is introducing new quantitative epistemic state which is opinionatedness, which is noticeable and unique. Because of this property, it is even harder to find the relationship between RT and BE. What is the equivalent quantitative opinionatedness function in BE? Let's finish this section with the claim that RT cannot represent quantitative ignorance and suspension.

### 1.11 Qualitative Suspension and Ignorance

RT can produce a suspension set, a belief set, and a disbelief set from a negative ranking function. So, every proposition with a negative rank belongs to one of the above sets. We have the right terminology for the property: assessable proposition. A ranking function and the unopinionatedness threshold generate the set of assessable propositions from a ranking function. The assessability set is the set of all propositions that the proposition or its negation is accepted. So, if a subset  $D$  of  $\mathcal{W}$  is not a member of  $\mathcal{A}$  and the negative ranking function is  $k: \mathcal{A} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ , then  $D$  does not have any rank and it is unknown.

$$\neg Ass(B) \equiv Ign(B) \equiv B \notin \mathcal{A}^{125} \tag{30}$$

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<sup>125</sup> Belief means accepting a proposition and not accepting its negation; disbelief means believing the negation of the proposition; suspension means accepting the proposition and accepting its negation. According to these definitions, assessable proposition is a proposition which is believed, disbelieved, or suspended. This definition is equivalent with the following definition: a proposition is assessable  $Ass(B)$  if the proposition is accepted or its negation is accepted.

$$Ign = \{B | Ign(B)\} \quad (31)$$

So, briefly, the observation shows that RT can represent the degree of belief and disbelief if one interprets ranking functions as the degree of belief and disbelief. Alternatively, RT can represent the degree of preferability and opinionatedness (I prefer this interpretation). Also, RT can represent the qualitative belief, disbelief, suspension, and ignorance cleanly and distinctively.

The beauty of RT is that it is the only model which its classic version can distinguish assessable propositions from non-assessable propositions. If a theory can distinguish assessable from non-assessable propositions, then that theory can distinguish ignorance from suspension. If we agree that a proposition is the set of possibilities, then RT introduced two kinds of propositions: propositions which are in the algebra and they have ranks, and propositions which the ranking function does not assign any number to it. A proposition without any rank is an unknown proposition.<sup>126</sup> So, all propositions with rank, are assessable; and it means they are believed, disbelieved or are suspended. But it is not enough to say that all unknown propositions are propositions without rank. We need a way to have some unknown propositions with positive degree of belief; propositions which have some evidential support, but it is not enough support to make them an accepted proposition. In the next section, I introduce a general ranking theory to show how by some changes, one can illustrate qualitative and quantitative suspension at the same time. I have an idea to represent the degrees of suspension in RT.

## 2 General Ranking Theory

I made a general ranking theory to represent the degree of conflicting evidence or suspended judgment by allowing multiple ranking functions. RT can represent qualitative suspension, but it cannot represent quantitative suspension (it cannot

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<sup>126</sup> The same argument does not work for BE because the Lockean thesis does not work properly in BE.

because always the rank of the proposition or its negation is zero, and there is no degree of conflicting evidence). In General Ranking Theory (GRT) the agent can have a positive rank for a proposition in one function, and a positive rank in another ranking function. This generalization is connected to the Qualitative acceptance Revision (QAR) as a generalization of belief revision that I explained in chapter 2. The basic ranking function constructs an acceptance base (which can be inconsistent or consistent), and then a general ranking function will be calculated based on this acceptance base and its corresponding basic ranking function.

## 2.1 Basic ranking function

Let  $W$  be a set of possibilities and  $\theta: 2^W \rightarrow \mathbb{R}^+ \cup \{\infty\}$  be a (positive) basic ranking function such that

$$\theta(\emptyset) = 0 \tag{32}$$

$$\theta(W) = \infty \tag{33}$$

The first difference between RT and GRT is that GRT assigns basic ranking function to all proposition or set of possibilities, but RT assigns ranks only to propositions which are a member of the algebra. According to the basic ranking function and an acceptance threshold, one can define an acceptance base.

## 2.2 Acceptance base

Then the acceptance base  $G$  is the set of all propositions that their basic rank is more than the acceptance threshold  $Cr_{Acc}$ .

$$G = \{B | \theta(B) \geq Cr_{Acc}\} \tag{34}$$

By introducing  $\theta$  as a basic ranking function; I separate propositions with basic rank, that their basic rank is positive; from propositions that their basic rank is zero and they do not have explicit evidential support. For example, if an agent learns that tomorrow it is rainy ( $B$ ) with basic rank 12, without any other information about  $\neg B$ , then her basic rank that tomorrow is not sunny would be 0. Now, if the acceptance threshold is 8, then  $B$  is in the acceptance base. By introducing  $G$  as the acceptance base, I separate propositions the basic rank of which is high enough to be used in an argument. In other words, the acceptance base is the set of propositions from which the agent can infer other propositions by applying classic logic.

Notice that the acceptance base may be inconsistent. The acceptance base is not necessarily closed under logical consequence. Therefore, there is more than one possible inconsistent acceptance base. This is the very important property of this model. GRT can represent various inconsistent agents. As people are inconsistent in different ways, it is irrational to say that all inconsistent agents are the same as we see in traditional belief revision theories. A model cannot represent suspended judgment if it cannot represent inconsistency.

### 2.3 Assessable propositions

The next step is about reasoning from an acceptance base. If the acceptance base is consistent, then all logical consequences of the acceptance set are valid. If the acceptance set is not consistent, then the logical consequence of the acceptance set is not valid, and the agent should not draw any conclusion from the acceptance base because if she does, then she must consider all propositions to be acceptable and this is irrational. The agent should draw a conclusion from a consistent subset of  $G$ , and what he infers, is conditional; unless it is derivable from all maximally consistent subset of  $G$ . One can call every maximally consistent subset of  $G$ , an inferable acceptance base; because as it is consistent, one can draw conclusion from it. The minimum number of inferable acceptance bases is one.

For every acceptance base  $G$ , There is a set  $I$  of subsets  $G_i$  of  $G$  such that

$$I = \{G_i | (G_i \subseteq G) \wedge \neg (G_i \vdash \perp) \wedge \forall B [(B \in G \wedge G_i \cup \{B\} \not\vdash \perp) \rightarrow B \in G_i]\} \quad (35)$$

These subsets  $G_i$ , are inferable acceptance bases. The assessable propositions are:

$$Ass(B) \equiv \forall G_i (G_i \vdash B \vee G_i \vdash \neg B) \quad (36)$$

For any  $G$  there is a set  $Ass_G$  of all assessable propositions that is also the smallest subalgebra  $\mathcal{A}$  that contains all members of  $G$ . For the proof see below.

$$Ass_G = \mathcal{A} = \{B | Ass(B)\} \quad (37)$$

For proving the set of all assessable propositions is an algebra we need to prove that i) the empty set is assessable and therefore always is a member of any set of assessable propositions,  $\forall G; \emptyset \in Ass_G$ , and ii) if a proposition is assessable then its negation is also assessable and belongs to the set of assessable propositions,  $B \in Ass_G \rightarrow \neg B \in Ass_G$ , and iii) the intersection of every two assessable propositions is assessable as well and eventually,

$$(B \in Ass_G \wedge C \in Ass_G) \rightarrow (B \cap C) \in Ass_G.$$

- i) For any acceptance base, the empty set  $\emptyset$  is always assessable because for all inferable acceptance base its negation  $W$  is derivable. Therefore,  $\forall G; \emptyset, W \in Ass_G$
- ii) For any assessable proposition, we have  $Ass(B) \equiv \forall G_i (G_i \vdash B \vee G_i \vdash \neg B)$ , therefore one can conclude that  $\neg B$  is assessable because of  $Ass(B) \equiv Ass(\neg B) \equiv \forall G_i (G_i \vdash B \vee G_i \vdash \neg B)$
- iii) Imagine that  $B$  and  $C$  are both assessable, therefore we have  $Ass(B) \equiv \forall G_i (G_i \vdash B \vee G_i \vdash \neg B)$  and  $Ass(C) \equiv \forall G_i (G_i \vdash C \vee G_i \vdash$

$\neg C$ ). As  $B$  and  $C$  are both assessable in all inferable sets, one can conclude that  $B \cap C$  is assessable as well:  $Ass(B \cap C) \equiv \forall G_i (G_i \vdash B \cap C \vee G_i \vdash \neg(B \cap C))$ ,  $G_i \vdash B \cap C$  if  $G_i \vdash B$  and  $G_i \vdash C$ , otherwise  $G_i \vdash \neg(B \cap C)$ .

Thus, the set of all assessable propositions,  $Ass_G$ , is an algebra.

## 2.4 General ranking function

Let  $\mathcal{A}$  be the set of all assessable propositions, then the function  $\mu: 2^W \rightarrow \mathbb{R}^+ \cup \{\infty\}$  is a (positive) general ranking function such that

$$B \notin \mathcal{A} \rightarrow \mu(B) = \theta(B) \quad (38)$$

$$B \in \mathcal{A} \rightarrow \mu(B) = \max \{\mu(B|G_i)\} \quad (39)$$

Where

$$\mu(B|G_i) = \max \{\mu(B|S) \mid S \subseteq G_i\} \quad (40)$$

, and

$$\mu(B|S) = \begin{cases} 0 & S \not\vdash B \\ \min\{\theta(D) \mid D \in S\} & S \vdash B \end{cases} \quad (41)$$

(38) says that the positive general rank of a proposition which is not assessable is equal to zero. It guarantees that the general rank of all propositions which are not assessable is less than the acceptance threshold. Notice that when the acceptance threshold is zero, all propositions are assessable. Therefore, there is no problem with this definition when the acceptance threshold is zero.

(39)-(41) assign the highest degree of (direct or indirect) evidential support to the proposition. For illustrating how  $\mu(B|S)$  works, imagine  $\theta(B) = 5$ ,  $\theta(D \cup \neg B) = 12$  and  $S = \{B, D \cup \neg B\}$ . Then as per (41)  $\mu(D|S) = 5$  and  $\mu(\neg D|S) = 0$ . And if  $G_i = \{B, D \cup \neg B, C \cap D\}$  while  $\theta(C \cap D) = 8$ , then  $\mu(D|G_i) = 8$  because there are two ways to derive  $B$  from  $G_i$ , and one gives it 8, and the other 5. (40) finds the conditional rank of the proposition in given  $G_i$  and (39) assign the maximum rank among various conditional ranks to the proposition. Now, if  $G = \{B, \neg B, D \cup \neg B, C \cap D\}$  then as per (39) the general rank will be 8 as well. What (39)-(41) is doing is finding the strongest argument to defend a proposition. The strongest argument of  $B$  is the smallest subset of the acceptance base which  $B$  is derivable, and the minimum basic rank of its proposition is higher than other subsets.

Using the minimum function in (41) is like using minimum for a logical value of the intersection of two propositions. If one assigns values zero and one to two propositions, then the value of their intersection  $C \cap D$  is the minimum value. Imagine  $C$  is true or its value is one, and  $D$  is false, and its value is zero, then the value of their intersection is the minimum value which is zero, and it means  $C \cap D$  is false. What (41) is suggesting is using the same behavior for calculating the conditional rank.

The reason that I call it a general ranking theory is that for all acceptance bases, consistent or inconsistent, every inferable set produces a standard positive ranking function. Moreover, consequently, when  $G$ , the acceptance base, is consistent, then the general ranking function over the assessable set produces a classic positive ranking function. Therefore, the general ranking theory allows the agent to have multiple standard ranking functions.

I am going to prove that the set of assessable propositions of a consistent acceptance base  $G$  produces a standard ranking function as per (39)-(41). This proof also shows that every inferable acceptance base will generate a standard ranking function: as  $G$  is consistent, there is only one inferable acceptance base which is  $G$

itself. Let  $G$  be the consistent set of propositions  $G = \{B_1, B_2, \dots, B_n\}$ . The set of assessable propositions is the set of propositions that  $\text{Ass}_G = \{D \mid G \vdash D \text{ or } G \vdash \neg D\}$ . As the standard ranking function is defined over the algebra, I introduce a sub-function  $\mu_{ass}$  of general ranking function which its domain is the assessable set. The function  $\mu_{ass}: \text{Ass}_G \rightarrow \mathbb{R}^+ \cup \{\infty\}$  is a standard ranking function if it satisfies three axioms: (i)  $\mu_{ass}(W) = \infty$ , (ii)  $\mu_{ass}(\emptyset) = 0$ , (iii)  $\mu_{ass}(B \cap C) = \min\{\mu_{ass}(B), \mu_{ass}(C)\}$ . For simplicity, in this proof, I use the same notation  $\mu$  for this sub-function.

Proving (i) as per (33) we have  $\theta(W) = \infty$ , and as per (39) as  $W \in \mathcal{A}$ , therefore  $\mu(W) = \max\{\mu(W|G_i)\} = \max\{\mu(W|G)\}$ . And  $\mu(W|G)$  as per (40) is  $\mu(W|G) = \max\{\mu(W|S) \mid S \subseteq G\}$ .  $\mu(W|S)$  can be calculated by (41).  $\mu(B|S) = \begin{cases} 0 & S \not\vdash B \\ \min\{\theta(D) \mid D \in S\} & S \vdash B \end{cases}$ . We need to calculate all proper subsets of  $G$  including  $\{W\}$ .

$$\mu(W|\{W\}) = \begin{cases} 0 & S \not\vdash W \\ \min\{\theta(D) \mid D \in \{W\}\} = \theta(W) = \infty & S \vdash W \end{cases}$$

Therefore  $\mu(W|\{W\}) = \infty$  and  $\mu(W|G) = \max\{\mu(W|S) \mid S \subseteq G\} = \mu(W|\{W\}) = \infty$ . Thus  $\mu(W) = \infty$ .

For proving (ii) as there is no subset  $S$  of  $G$  such that  $S \vdash \emptyset$ , therefore  $\mu(\emptyset|S) = 0$  for all  $S$  and  $\mu(\emptyset) = \mu(\emptyset|G) = 0$ .

For proving (iii)  $\mu(B \cap C) = \min\{\mu(B), \mu(C)\}$ , assume that  $B$  and  $C$  are assessable and their general rank is  $\mu(B|G)$  and  $\mu(C|G)$ . We need to prove that  $\mu(B \cap C) = \min\{\mu(B), \mu(C)\}$ . Imagine  $\mu(B|G) = \max\{\mu(B|S) \mid S \subseteq G\} = \mu(B|S_B)$ , and  $S_B$  provides the best argument for  $B$  and consequently makes its maximum rank and  $S_C$  provides the best argument for  $C$ . If I prove that  $S_B \cup S_C$  is the best argument which provides the maximum rank for  $B \cap C$  then (iii) is proved because

$$\begin{aligned} \mu(B \cap C|G) &= \mu(B \cap C|S_B \cup S_C) = \min\{\theta(D) \mid D \in S_B \cup S_C\} \\ &= \min\{\min\{\theta(D) \mid D \in S_B\}, \min\{\theta(D) \mid D \in S_C\}\} \\ &= \min\{\mu(B), \mu(C)\} \end{aligned}$$

Assume that  $S_m \neq S_B \cup S_C$  and  $\mu(B \cap C | S_m) > \mu(B \cap C | S_B \cup S_C)$ . Therefore  $\min\{\theta(D) | D \in S_m\} > \min\{\theta(D) | D \in S_B \cup S_C\}$ . As  $S_m \vdash B \cap C$ , one can conclude that  $S_m \vdash B$  as well. So, (a)  $\mu(B | S_m) > \mu(B | S_B \cup S_C)$ . Also (b)  $\mu(B | S_B \cup S_C) = \mu(B | S_B)$  because  $S_B$  provides the maximum rank among subsets of  $S_B \cup S_C$ . From (a) and (b) follows that  $\mu(B | S_m) > \mu(B | S_B)$  and it is against the definition of  $S_B$  such that  $S_B$  provides the best argument for  $B$  and the maximum rank. So, the assumption that  $\mu(B \cap C | S_m) > \mu(B \cap C | S_B \cup S_C)$  is wrong, therefore  $S_B \cup S_C$  is the best argument for  $B \cap C$ , and consequently  $\mu(B \cap C | G) = \min\{\mu(B), \mu(C)\}$ .

So, if the acceptance base is consistent then the general ranking function is a standard positive ranking function, and it means every conditional general ranking function given an inferable set, produce a standard ranking function as well, and an inconsistent acceptance base produces conditional general ranking functions which are standard ranking functions.

If one defines accepted proposition based on the general rank, then she can define all doxastic attitudes. Here is the suggestion:

## 2.5 Accepted propositions

A proposition is accepted in  $G_i$  if and only if its conditional rank  $\mu(B | G_i)$  is more than the acceptance threshold, and it is generally an accepted proposition if and only if its general rank  $\mu$  is more than the acceptance threshold:

$$Acc(B, G_i) \equiv \mu(B | G_i) \geq Cr_{Acc} \quad (42)$$

$$Acc(B) \equiv \mu(B) \geq Cr_{Acc} \quad (43)$$

(42) is saying that a proposition is accepted if and only if it is accepted in all inferable acceptance bases and the maximum function in (39) guarantees that property.

As general rank show evidential support of a proposition, therefore if a proposition is assessable and have enough evidential support; then it should be accepted as well.

## 2.6 Belief, disbelief, suspension, and ignorance

For any proposition one of the following properties is valid and therefore for any  $\mu$  and a proposition  $B$  we have:

$$Bel(B) \equiv (\mu(B) \geq Cr_{Acc} \wedge \mu(\neg B) < Cr_{Acc}) \tag{44}$$

$$Dis(B) \equiv Bel(\neg B) = (\mu(\neg B) \geq Cr_{Acc} \wedge \mu(B) < Cr_{Acc}) \tag{45}$$

$$Sus(B) \equiv (\mu(B) \geq Cr_{Acc} \wedge \mu(\neg B) \geq Cr_{Acc}) \tag{46}$$

$$Ign(B) \equiv (\mu(B) < Cr_{Acc} \wedge \mu(\neg B) < Cr_{Acc}) \tag{47}$$

Let me illustrate this model by an example. Imagine that someone is at a party and it is dark at night. Under the light of candles, she sees a coat, and she is thinking about its color. She considers four possibilities:  $W_1$  it is dark blue.  $W_2$  it is light blue (not dark and blue).  $W_3$  it is dark, and it is not blue, and  $W_4$  It is light, and it is not blue. Let  $B$  be ‘it is blue’ and  $D$  be ‘it is dark’. The following table is the basic ranking function  $\theta$ :

$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$
0	2	1	11	7	18	5	7
$\{w_1, w_2, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_3, w_4\}$	$\{w_2, w_4\}$	$\{w_2, w_3\}$

Table 4

∞	4	6	2	5	1	0	2
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Gray cells are accepted propositions when  $Cr_{Acc} = 10$ . The corresponding subalgebra or the set of all assessable propositions and their general ranks  $\mu$ , and their epistemic states are:

	$\emptyset$	$\{w_1\}$	$\{w_2\}$	$\{w_3\}$	$\{w_4\}$	$\{w_1, w_2\}$	$\{w_1, w_3\}$	$\{w_1, w_4\}$
$\theta$	0	2	1	11	7	18	5	7
$\mu$	0	2	1	11	7	18	5	7
<i>ES</i>	<i>Dis</i>	<i>Ign</i>	<i>Ign</i>	<i>Sus</i>	<i>Dis</i>	<i>Sus</i>	<i>Ign</i>	<i>Ign</i>
	$\{w_1, w_2, w_3, w_4\}$	$\{w_2, w_3, w_4\}$	$\{w_1, w_3, w_4\}$	$\{w_1, w_2, w_4\}$	$\{w_1, w_2, w_3\}$	$\{w_3, w_4\}$	$\{w_2, w_4\}$	$\{w_2, w_3\}$
$\theta$	∞	4	6	2	5	1	0	2
$\mu$	∞	4	6	18	18	11	0	0
<i>ES</i>	<i>Bel</i>	<i>Ign</i>	<i>Ign</i>	<i>Sus</i>	<i>Bel</i>	<i>Sus</i>	<i>Ign</i>	<i>Ign</i>

Table 5

### 2.7 General positive and negative ranking function.

From a positive ranking function  $\mu$ , a negative ranking function  $\nu$  can be induced.

$$\nu(B) = \mu(\neg B) \tag{48}$$

### 3 Comparison between RT and GRT

Ranking theory can represent the qualitative belief, disbelief, suspension, and ignorance; but it cannot represent degrees of belief, disbelief, suspension, and ignorance. There are two reasons for this claim. First, there is no room for having a positive degree of belief and disbelief at the same time. Moreover, second, a ranking function share the same property the preferability function had in DS. The degree of Preferability is the difference between the degree of belief and disbelief of a proposition. One can observe that by considering the symmetry between the two-sided rank of a proposition and its negation. General ranking theory can represent all four qualitative epistemic states<sup>127</sup>, and it can model the degree of belief, disbelief, and suspension for all propositions. GRT disagree with DS that one knows how much she does not know.

If one uses  $\mathbf{1}$  and  $\mathbf{0}$  as the domain of basic ranking function ( $\mathbf{1}$  will also be used for  $W$ ), and  $\mathbf{1}$  as the acceptance threshold, then the general ranking function will be a model for Qualitative acceptance Revision. Its proof is simple: first, the acceptance base, is the set of propositions that their mass is  $\mathbf{1}$ . Also, the process of defining inferable sets is exactly like acceptance revision and they produce the same assessable bases; because the general ranking function assign  $\mathbf{0}$  to an unknown (not assessable) proposition and its negation. Also, it assigns  $\mathbf{1}$  to a proposition or its negation (or both), if it is assessable as per (38)-(41). Here like acceptance revision, a proposition is accepted if and only if it is assessable (the general rank of the proposition and its negation are not  $\mathbf{0}$ ), and there is an inferable base that the proposition could be derived from it (general rank assigns  $\mathbf{1}$  in this case).

In the last chapter, I Compare all theories which I discussed in the previous chapters: their strength and shortcoming.

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<sup>127</sup> RT cannot represent unknown propositions which their evidential support is inadequate, while General Ranking Theory can distinguish unknown propositions with no evidential support from unknown propositions with an inadequate evidential support.

# 6

## Acceptance Revision

I discussed various theories in formal epistemology to see whether they could represent a qualitative and quantitative suspension of judgment or not. Also, I made some changes to represent suspended judgment by distinguishing suspended judgment from ignorance. There are problems like the lottery paradox or the preface paradox in formal epistemology which is connected to suspended judgment, and I tried to show how those amendments deal with the problems. In this chapter while I am

explaining my findings, I introduce a theory (acceptance revision) as a conclusion, which can represent the qualitative and quantitative doxastic attitudes, namely belief, disbelief, ignorance, and suspension. In the end, all theories will be compared in one table.

## 1 Definitions and relationships among doxastic attitudes

In the first chapter the definitions of belief, disbelief, suspension, and ignorance, based on the notion of acceptance, were introduced:

$$\text{Bel}(A, B) \equiv (\text{Acc}(A, B) \wedge \neg \text{Acc}(A, \neg B)) \quad (1)$$

$$\text{Dis}(A, B) \equiv (\neg \text{Acc}(A, B) \wedge \text{Acc}(A, \neg B)) \quad (2)$$

$$\text{Sus}(A, B) \equiv (\text{Acc}(A, B) \wedge \text{Acc}(A, \neg B)) \quad (3)$$

$$\text{Ign}(A, B) \equiv (\neg \text{Acc}(A, B) \wedge \neg \text{Acc}(A, \neg B)) \quad (4)$$

From (1) to (4) one can derive  $\text{Bel}(A, B) \leftrightarrow \text{Dis}(A, \neg B)$ ,  $\text{Sus}(A, B) \leftrightarrow \text{Sus}(A, \neg B)$ ,  $\text{Ign}(A, B) \leftrightarrow \text{Ign}(A, \neg B)$ ,  $(\text{Sus}(A, B) \vee \text{Ign}(A, B)) \rightarrow (\neg \text{Bel}(A, B) \wedge \neg \text{Bel}(A, \neg B))$ .

Also, one can define other doxastic attitudes like doubt. Let  $\text{Dou}(A, B)$  be  $A$  doubts that  $B$ , then it means  $\text{Acc}(A, \neg B)$ . So, if an agent doubt  $B$ , it means she accept  $\neg B$ . Consequently, doubt is disbelieving or suspending a proposition:  $\text{Dou}(A, B) = \text{Dis}(A, B) \vee \text{Sus}(A, B)$ .

## 2 TBR, IndBR, and NAR

After the first chapter, the research into various theories in formal epistemology was started. I began with Traditional Belief Revision (TBR or AGM) and then

Indeterministic Belief Revision as two qualitative theories. TBR represents the agent's epistemic states by a belief set. The problem with TBR is that one cannot distinguish suspension from ignorance because for any belief set and a proposition, there are three doxastic attitudes: the proposition is believed, disbelieved, or neither believed nor disbelieved (non-belief). In the second step, I observed Indeterministic Belief Revision (IBR) which represent agent's epistemic states by a set of possible belief sets. It was not satisfying because if an agent could have more than one possible belief set, it then follows that a proposition is believed if it is believed in a possible belief set and is not disbelieved in other belief sets. Therefore, a proposition is accepted if it is accepted in a belief set. The unwanted result is that a proposition that is unknown in some belief sets (not believed or disbelieved) can be accepted because of just one belief set. It leads to at least one counterintuitive result: the disjunction of some disbelieved propositions can be believed. The definition of the accepted proposition in IndBR that cause the problem is the following: a proposition is accepted if and only if there is a belief set that contains the proposition. For avoiding the problem, I started to make a theory which I call it Qualitative acceptance Revision by giving up the idea that epistemic state should be presented by one or multiple possible belief sets.

In comparison to TBR and IndBR; QAR represents the epistemic states with an acceptance base instead of a belief set or a set of belief sets (5). From any consistent subset of the acceptance base (6), an agent can derive some results. The outcome is not necessarily an accepted proposition, but it is a conditionally accepted proposition (7). A proposition is a conditionally accepted proposition if and only if it is derivable from a consistent subset of acceptance base (7). Let's  $G$  be an acceptance base,  $G_i$  be inferable base, and  $Acc(B, G_i)$  be  $B$  is conditionally accepted in  $G_i$ , then

$$G = \{B | val(B) > Cr\} \tag{5}$$

$$(G_i \subseteq G) \wedge (G_i \not\vdash \perp) \wedge \forall B [(B \in G \wedge G_i \cup \{B\} \not\vdash \perp) \rightarrow B \in G_i] \quad (6)$$

$$Acc(B, G_i) \equiv G_i \vdash B \quad (7)$$

According to (6), the inferable bases  $G_i$  are the largest consistent subsets of  $G$ . A proposition is called an assessable proposition in  $G_i$  if and only if the proposition or its negation is derivable from  $G_i$ . A proposition is an assessable proposition if and only if it is assessable for all  $G_i$ .

$$Ass(B, G_i) \equiv (Acc(B, G_i) \vee Acc(\neg B, G_i)) \quad (8)$$

$$Ass(B, G) \equiv \forall G_i Ass(B, G_i) \quad (9)$$

A proposition  $B$ , is accepted, if and only if it is assessable in  $G$  and there is an inferable base that  $B$  is derivable.

$$Acc(B, G) \equiv Ass(B, G) \wedge \exists G_i Acc(B, G_i) \quad (10)$$

$$Acc(B, G) \rightarrow Ass(B, G) \quad (11)$$

From (10) one can conclude that every accepted proposition is assessable. As we have the definition of acceptance, we could define all four basic doxastic attitudes. Notice that a proposition may be unknown while it is conditionally accepted.

### 3 BE and DS

In chapter 3, I started to explain the quantitative epistemic states. The basic theory is Bayesian Epistemology with the central idea that the probability of a proposition is the degree of belief, and an agent should obey probability laws (axioms). One of the

most well-known arguments for probabilism is the Dutch Book Argument. Unfortunately, there is a counter-argument which is called Czech Book Argument, and it neutralizes the DBA. Also, the package principle is needed for the Dutch Book argument, and it does not work in all cases in particular in the case where an agent is going to bet on her action. After some observations about Bayesian Epistemology, I came back to the main question: can Bayesian Epistemology represent suspended judgment? The first option which comes to mind is .5, by saying that if an agent assigns .5 as the degree of the proposition and its negation, then it is suspended. The problem is that the principle of indifference gives the same probability to a proposition that the agent does not have any information for distributing her degrees of belief. Thus, BE cannot distinguish between ignorance and suspension. Both propositions, the unknown and the suspended proposition, receive .5 as its degrees of belief.

There is a proposal to represent suspension by interval probability  $[0, a]$ . The problem with this account is that if an agent suspends her judgment, then according to the conditionalization she cannot believe it by any update. So, the proposal to show suspension by  $[0, a]$  such that  $a$  is a small number, does not work, because it assumes that if she suspends her judgment toward a proposition, she also tends to disbelieve it. But, it follows from suspending a proposition that the agent suspends its negation as well. So, the agent has to suspend the negation while she tends to disbelieve as well, and obviously this cannot happen in BE.

Besides, there is another problem in BE. When one asks about the relationship between quantitative epistemic states and qualitative epistemic states in BE, the answer is Lockean thesis: an agent believes a proposition if her degree of belief is high enough. This proposal does not work for BE because it leads to the Lottery paradox.

As BE does not provide enough space for representing all basic doxastic states, I continued research into a new theory: the Dempster Shafer theory of evidence. In comparison to BE, DS is not additive. Therefore, the sum of the degree of belief of a

proposition and its negation is not necessarily one. I noticed that this property allows us to represent degrees of suspension and ignorance as well as degrees of belief and disbelief in DS.

### 3.1 Definition of quantitative doxastic attitudes

In standard DS, there is no function for degrees of suspension and ignorance, but it is possible to introduce them. I define the degree of disbelief, suspension, and ignorance by following formulae:

$$Dis^\circ(B) = Bel^\circ(\neg B) \quad (12)$$

$$Ign^\circ(B) = 1 - (Bel^\circ(B) + Dis^\circ(B)) \quad (13)$$

$$Sus^\circ(B) = \min \{Bel^\circ(B), Dis^\circ(B)\} \quad (14)$$

(12) and (13) are standard and valid in the entire dissertation. In DS, all quantitative epistemic states are representable.

### 3.2 Acceptance threshold

I found that for any belief function, there is an acceptance-threshold that defines accepted propositions, and therefore all qualitative epistemic states are representable as well. The threshold is context sensitive, and it is not always the same as the Lockean thesis suggests. For every proposition, there are some legitimate thresholds that designate whether a proposition is accepted or not. By the below algorithm, one can find the legitimate thresholds:

- a) Set of degrees of belief:  $Deg = \{x_i = Bel^\circ(B) \mid B \in 2^W\}$  (ordered set)
- b) For every proposition  $B$  define the set of the believed proposition:

$$K_{\geq B} = \{C \mid Bel^\circ(C) \geq Bel^\circ(B) \wedge Dis^\circ(C) < Bel^\circ(B)\}$$

Notice that  $Bel^\circ(B)$  is the threshold.

- c)  $K_{\geq B}$  is consistent if and only if  $\bigcap_{C \in K_{\geq B}} C \in K_{\geq B}$
- d) If  $K_{\geq B}$  is consistent then all numbers between  $(x_{i-1}, x_i]$  are legitimate.  
 $Bel^\circ(B) = x_i$ .

Moreover, the definitions of qualitative doxastic attitudes are:

$$Bel(B) \equiv (Bel^\circ(B) \geq cr \wedge Bel^\circ(\neg B) < cr) \quad (15)$$

$$Dis(B) \equiv (Bel^\circ(B) < cr \wedge Bel^\circ(\neg B) \geq cr) \quad (16)$$

$$Sus(B) \equiv (Bel^\circ(B) > cr \wedge Bel^\circ(\neg B) \geq cr) \quad (17)$$

$$Ign(B) \equiv (Bel^\circ(B) < cr \wedge Bel^\circ(\neg B) < cr) \quad (18)$$

The idea is that an agent should believe the logical consequences of her belief set. The consistency of the belief set plays the crucial role to find the legitimate threshold.

### 3.3 Unwanted results

This account seems plausible, however there is a small problem. For every function and legitimate threshold, an agent cannot suspense a proposition while she finds another proposition unknown. In other words, for any legitimate threshold a proposition can be believed, disbelieved, or suspended; or the proposition can be believed, disbelieved, or unknown. (15)-(18) are standard definition in the entire text.

## 4 RT and GRT

After DS, it was time to work on Ranking Theory which has a qualitative and quantitative representation of belief without leading to the lottery paradox.

RT works based on the negative ranks which are grades of disbelief. I realized that this interpretation of ranking functions is wrong because RT proposes that the degree of belief of a proposition or the degree of belief of its negation is zero. It seems that an agent cannot have degree of belief, unless she believes or suspends the proposition. Also, for any two-sided ranking function, the sum of the two-sided rank of a proposition and its negation is zero. It seems that the negative ranking function is not about the grading of disbelief, but it is about something like the difference between the degree of belief and disbelief; or the degree of preferability.

Another problem with RT is that there is no way to represent the degree of suspension or the degree of contradiction (or conflicting evidence). Always the negative rank of a proposition or its negation is zero. Briefly, RT can express the order of preferability of propositions, and qualitative belief, disbelief, suspension, and ignorance. I introduced a general ranking theory GRT that can represent the degree of suspension as well. Finding GRT helped me to construct a quantitative version of QAR (in chapter two), which I am going to explain now as the complete result of this text.

## 5 Acceptance revision

I tried to construct a theory to represent all basic qualitative and quantitative doxastic attitudes based on degrees of acceptance. Its mass function, is similar to the mass function in DS (but not the same function), concerning closure, it is like a paraconsistent theory in formal epistemology, concerning the degree of acceptance, it is similar to ranking theory because of using the minimum function. However, one can observe these similarities, Acceptance revision is a new theory. I cannot say it works properly in all situations, but I think I did my best to capture all intuitions. Here is the complete theory:

## 5.1 Mass function

The function  $m^\circ: 2^W \rightarrow [0,1]$ , mass function, assigns masses to propositions such that:

$$m^\circ(\emptyset) = 0 \tag{19}$$

$$m^\circ(W) = 1 \tag{20}$$

The idea behind introducing  $m^\circ$  is that an agent can rationally have evidential support for a proposition as well as its negation at the same time. In other words, by assigning real numbers to all propositions, the degree of contradiction or the degree of suspended judgement will be representable.  $m^\circ$  does not have normalization like the mass function in DS.

An agent can revise her mass function by assigning new numbers to every proposition, and it does not need any special calculation.<sup>128</sup>

## 5.2 Acceptance base

An agent accepts all propositions that their masses are greater than or equal to the acceptance threshold  $Cr_{Acc}$ . The set  $G$  is an acceptance base that the masses of all propositions in  $G$  are greater than or equal to the acceptance threshold:

$$G = \{B | m^\circ(B) \geq Cr_{Acc}\} \tag{21}$$

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<sup>128</sup> One could make a more detailed version of revision for an acceptance revision, but I prefer to keep it as simple as I can. There are two ways of revision. Revision about the evidence and the reliability of sources, and the revision concerning the acceptance threshold.

I apply the term Acceptance base because first, the propositions in  $G$  are not necessarily believed; second, it is not closed under logical consequence operations; and third, it is not necessarily consistent. These three properties allow us to have various inconsistent acceptance base while all inconsistent belief sets are the same in AGM.

However, there are cases where the mass of a proposition  $B$  is smaller than the acceptance threshold, but it is derivable from the acceptance base. So, there are two different accepted propositions: a) accepted propositions that are explicitly accepted in the acceptance base and b) accepted propositions that are derivable from the acceptance base. Now, I need to reach the same set of accepted propositions in the quantitative version that I reached in the qualitative version in chapter two, NAR. So, all accepted proposition based on the acceptance base, should have a degree of acceptance which is more than the threshold.

The second possible revision is the revision of the acceptance threshold. So, an agent may revise her epistemic state by updating her mass function or her acceptance threshold.

### 5.3 Acceptance function

For finding all accepted propositions based on a given acceptance base  $G$ , I introduce a quantitative acceptance function. Two important properties of the acceptance function are a) finding all accepted propositions and assigning a degree of acceptance which is more than the acceptance threshold. b) finding all propositions that are not accepted and assigning a degree less than the acceptance threshold.

Let  $G$  be an acceptance base, as per (6) every  $G_i$  is one of the largest consistent subsets of the acceptance base. If the acceptance base is consistent, then  $G_i$  is unique. As per (7) if a proposition is derivable from  $G_i$ , then it is accepted in  $G_i$ . And if a proposition  $B$  or its negation  $\neg B$  is accepted in  $G_i$ , then the proposition is assessable in  $G_i$  (as per (8)). If a proposition is assessable in all  $G_i$ , then it is assessable. An

accepted proposition was defined in (10) as a proposition which is assessable and there is at least one  $G_i$ , where the proposition is accepted. The set of all assessable propositions is an algebra as it was proved in the chapter five (5.37)

$$\text{Ass}_G \equiv \mathcal{A} = \{B \mid \text{Ass}(B)\}.$$

### 5.3.1 Unknown propositions or non-assessable propositions

A proposition which is not assessable is ignorance (or an unknown proposition). Therefore, its degree of acceptance should be less than the acceptance threshold. Assigning the mass of an unknown proposition as the degree of acceptance guarantees that the degree of acceptance remains less than the acceptance threshold.<sup>129</sup> Let  $\text{Acc}^\circ: 2^W \rightarrow [0,1]$

$$B \notin \mathcal{A} \rightarrow \text{Acc}^\circ(B) = m^\circ(B) \tag{22}$$

### 5.3.2 Assessable propositions

As in the qualitative version of acceptance revision, the acceptance degree for assessable proposition should be found in a given inferable base. Then the degree of acceptance comes from the best inferable base. The degree of acceptance of any proposition  $B$  in an inferable set  $G_i$  depends on how  $B$  could be derived.  $B$  might be derived in more than one way. Imagine  $G_i = \{B, B \vee C, \neg C, D, W\}$ , then there are two ways to derive  $B$ . Let's call every subset of  $G_i$  that entails  $B$ , a ground of  $B$  in  $G_i$ . And let  $S_k$  be one of the smallest grounds of  $B$  in  $G_i$ , i.e. there is no ground  $S_j$  of  $B$  in  $G_i$  such that  $S_j \subset S_k$ . (In the example,  $\{B \vee C, \neg C\}$  is one the smallest ground of  $B$ , but

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<sup>129</sup> If their mass was more than threshold, then they were explicitly in the acceptance set, and every proposition  $B$  which is in the acceptance set is necessarily assessable, because as per (6) for every inferable set and the proposition  $B$  is in the inferable set, or its negation is derivable, therefore it is not in the set.

$\{B \vee C, \neg C, D\}$  is not.) The degree of acceptance of  $B$  given  $S_k$ , as one of the smallest grounds of  $G_i$ , is

$$Acc_{S_k}^\circ(B) = \min\{m^\circ(D) \mid D \in S_k\} \quad (23)$$

The motivation behind this calculation was explained in GRT. However, the way that one should calculate the degree of acceptance in an inferable base is different. Let  $\{S_1, \dots, S_m\}$  be the set of all smallest grounds of  $B$  in  $G_i$ , then the degree of acceptance in an inferable base is

$$Acc_{G_i}^\circ(B) = \sum_{I \subseteq \{S_1, \dots, S_m\}} (-1)^{|I|+1} \prod_{S_j \in I} Acc_{S_j}^\circ(B) \quad (24)$$

(24) guarantees that the degree of acceptance of  $W$  is always 1 and for other propositions is always less or equal to one. Also, it says that if an agent has more than one argument that a proposition is true, then her degree of acceptance is more than the case that he has only one of those arguments. (24) may seem a complicated formula but it has a simple and intuitive meaning. Imagine there are two distinctive grounds for a proposition given an inferable set. Then each ground makes the agent more confident that the proposition which is accepted. Assume that first and second ground provide respectively .6 and .7 degree of acceptance. Then the degree of acceptance given the inferable set will be  $.6 + (1-.6) * .7 = .6 + .28 = .88$ . If there were three grounds respectively .6, .7, and .4; then it will be  $(.6 + (1-.6) * .7) + (1 - (.6 + (1-.6) * .7)) * .4 = .880 + .048 = .928$ . In other word, every ground makes a percentage of the degree of ignorance to the degree of acceptance.

The degree of acceptance of  $B$  when  $B$  is assessable, is its maximum degree in all inferable sets:

$$Acc^\circ(B) = \max \{ Acc_{G_j}^\circ(B) \mid \text{all inferable base } G_j \} \quad (25)$$

The formula (25) says that the degree of acceptance comes from the best inferable base. Acceptance function assigns the degree of acceptance based on (22) and (25).

#### 5.4 Accepted propositions

Every proposition whose degree of acceptance is higher than the acceptance threshold, is accepted:

$$Acc(B) \equiv Acc^\circ(B) \geq Cr_{Acc} \quad (26)$$

#### 5.5 Qualitative doxastic attitudes

Moreover, other doxastic attitudes are evaluable based on acceptance:

$$Bel(B) \equiv ( Acc^\circ(B) \geq Cr_{Acc} \wedge Acc^\circ(\neg B) < Cr_{Acc} ) \quad (27)$$

$$\begin{aligned} Dis(B) &\equiv Bel(\neg B) \quad (28) \\ &= ( Acc^\circ(\neg B) \geq Cr_{Acc} \wedge Acc^\circ(B) < Cr_{Acc} ) \end{aligned}$$

$$Sus(B) \equiv ( Acc^\circ(B) \geq Cr_{Acc} \wedge Acc^\circ(\neg B) \geq Cr_{Acc} ) \quad (29)$$

$$Ign(B) \equiv ( Acc^\circ(B) < Cr_{Acc} \wedge Acc^\circ(\neg B) < Cr_{Acc} ) \quad (30)$$

#### 5.6 Quantitative doxastic attitudes

And degrees of belief, disbelief, suspension and ignorance are:

$$Bel^\circ(B) = Acc^\circ(B) - Acc^\circ(\neg B) \quad (31)$$

$$Dis^\circ(B) \equiv Acc^\circ(\neg B) - Acc^\circ(B) \quad (32)$$

$$Sus^\circ(B) \equiv \min \{Acc^\circ(B), Acc^\circ(\neg B)\} \quad (33)$$

$$Ign^\circ(B) \equiv 1 - (Acc^\circ(B) + Acc^\circ(\neg B)) \quad (34)$$

Acceptance revision proposes to work on acceptance instead of belief to represent all doxastic attitudes and this approach works better.

## 6 Comparison

I discussed eight theories in the entire dissertation which I introduced three of them: Traditional Belief Revision or (TBR), Indeterministic Belief Revision (IBR), Bayesian Epistemology (BE), Dempster Shafer theory of Evidence (DS), Ranking Theory (RT). And Qualitative acceptance revision (QAR), General Ranking Theory (GRT), acceptance revision (AC). My goal was finding some ways to have a clear distinction between suspension and ignorance and represent all doxastic attitudes. The following table compare all theories.

	TBR	IBR	BE	DS	RT	NAR	GRT	AC
<i>Bel</i>	√	≠	×	≠	√	√	√	√
<i>Dis</i>	√	≠	×	≠	√	√	√	√
<i>Sus</i>	×	≠	×	≠	√	√	√	√
<i>Ign</i>	×	≠	×	≠	√	√	√	√
<i>Bel<sup>o</sup></i>	×	×	≠	√	≠	×	√	√
<i>Dis<sup>o</sup></i>	×	×	≠	√	≠	×	√	√
<i>Sus<sup>o</sup></i>	×	×	×	√	×	×	√	√
<i>Ign<sup>o</sup></i>	×	×	×	√	×	×	≠	√

Representable: √ | Not representable: × | representable but needs amendment: ≠

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