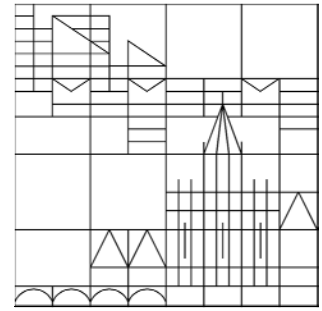


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Representation of Classical Solutions to a Linear Wave Equation with Pure Delay

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Abstract

For a wave equation with pure delay, we study an inhomogeneous initial-boundary value problem in a bounded 1D domain. Under smoothness assumptions, we prove unique existence of classical solutions for any given finite time horizon and give their explicit representation. Continuous dependence on the data in a weak extrapolated norm is also shown.

1 Introduction

The wave equation is a typical linear hyperbolic second-order partial differential equation which naturally arises when modeling phenomena of continuum mechanics such as sound, light, water or other kind of waves in acoustics, (electro)magnetics, elasticity and fluid dynamics, etc. (cf. [6, 13]). Providing a rather adequate description of physical processes, partial differential equations, or equations with distributed parameters in general, have found numerous applications in mechanics, medicine, ecology, etc. Introducing after-effects such as delay into these equations has gained a lot of attention over several past decades. See e.g., [2, 3, 7, 8]. Mathematical treatment of such systems requires additional carefulness since distributed systems with delay often turn out to be even ill-posed (cf. [4, 5, 12]).

In the present paper, we consider an initial-boundary value problem for a general linear wave equation with pure delay and constant coefficients in a bounded interval subject to non-homogeneous Dirichlet boundary conditions. To solve the equation, we employ Fourier's separation method as well as the special functions referred to as delay sine and cosine functions which were introduced in [9, 10]. We prove the existence of a unique classical solution on any finite time interval, show its continuous dependence on the data in a very weak extrapolated norm, give its representation as a Fourier series and prove its absolute and uniform convergence under certain conditions on the data.

2 Equation with pure delay

For $T > 0$, $l > 0$, we consider the following linear wave equation in a bounded interval $(0, l)$ with a single delay being a second order partial difference-differential equation for an unknown function η

$$\partial_{tt}\eta(t, x) = a^2\partial_{xx}\eta(t - \tau, x) + b\partial_x\eta(t - \tau, x) + d\eta(t - \tau, x) + g(t, x) \text{ for } (t, x) \in (0, T) \times (0, l) \quad (2.1)$$

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subject to non-homogeneous Dirichlet boundary conditions and initial conditions

$$\begin{aligned}\eta(t, 0) &= \theta_1(t), & \eta(t, l) &= \theta_2(t) \text{ for } t > -\tau, \\ \eta(t, x) &= \psi(t, x) \text{ for } (t, x) \in (-\tau, 0) \times (0, l).\end{aligned}\tag{2.2}$$

Since we are interested in studying classical solutions, the following compatibility conditions are required to assure for smoothness of the solution on the boundary of time-space cylinder

$$\psi(t, 0) = \theta_1(t), \quad \psi(t, l) = \theta_2(t) \text{ for } t > -\tau.$$

Definition 2.1. Under a classical solution to the problem (2.1), (2.2) we understand a function $\eta \in \mathcal{C}^0([-\tau, T] \times [0, l])$ which satisfies $\partial_{tt}\eta, \partial_{tx}\eta, \partial_{xx}\eta \in \mathcal{C}^0([-\tau, 0] \times [0, l])$ as well as $\partial_{tt}\eta, \partial_{tx}\eta, \partial_{xx}\eta \in \mathcal{C}^0([0, T] \times [0, l])$ and, being plugged into Equations (2.1), (2.2), turns them into identity.

Remark 2.2. The previous definition does not impose any continuity of time derivatives in $t = 0$. If the continuity or even smoothness are desired, additional compatibility conditions on the data, including g , are required.

Let $\|\cdot\|_{k,2} := \|\cdot\|_{H^{k,2}((0,l))}$, $k \in \mathbb{N}_0$, denote the standard Sobolev norm (cf. [1]) and $\|\cdot\|_{-k,2} := \|\cdot\|_{H^{-k,2}((0,l))}$ denote the norm of corresponding negative Sobolev space. We introduce the norm $\|\cdot\|_X := \sqrt{\sum_{k=0}^{\infty} \|\cdot\|_{-k,2}^2}$ and define the Hilbert space X as a completion of $L^2((0, l))$ with respect to $\|\cdot\|_X$. Obviously, $X \hookrightarrow (\mathcal{D}((0, l)))'$, i.e., X can be continuously embedded into the space of distributions.

With this notation, we easily see that $\mathcal{A} := a^2\partial_x^2 + b\partial_x + d$ (with ∂_x denoting the distributional derivative) is a bounded linear operator on X since

$$\begin{aligned}\|\mathcal{A}\|_{L(X)} &= \sup_{\|u\|_X=1} \|\mathcal{A}u\|_X = \sup_{\|u\|_X=1} \sqrt{\sum_{k=0}^{\infty} \|a^2\partial_x^2 u + b\partial_x u + du\|_{-k,2}^2} \\ &\leq \sup_{\|u\|_X=1} \sum_{k=0}^{\infty} (a^2\|u\|_{-k-2,2} + b\|u\|_{-k-1,2} + d\|u\|_{-k,2}) \\ &\leq \sup_{\|u\|_X=1} (a^2 + b + c)\|u\|_X = a^2 + b + d.\end{aligned}$$

Theorem 2.3. There exists a constant $C > 0$, dependent only on a, b, d, l, τ, T , such that the estimate

$$\begin{aligned}\max_{t \in [0, T]} (\|\eta(\cdot, t)\|_X^2 + \|\eta_t(\cdot, t)\|_X^2) &\leq C (\|\psi(0, \cdot)\|_X^2 + \|\psi_t(0, \cdot)\|_X^2) + C \int_{-\tau}^0 (\|\psi(s, \cdot)\|_X^2 + \|\psi_t(s, \cdot)\|_X^2) dt + \\ &\quad C \int_0^T (\|g(s, \cdot)\|_X^2 + |\theta_1(s)|^2 + |\theta_2(s)|^2) ds\end{aligned}$$

holds true for any classical solution of Equations (2.1), (2.2).

Proof. Let η be the classical solution to Equations (2.1), (2.2). We define

$$w(t, x) := \begin{cases} \eta(t, x) & \text{for } (t, x) \in [-\tau, 0] \times [0, l], \\ \eta(t, x) - \theta_1(t) - \frac{x}{l}(\theta_2(t) - \theta_1(t)) & \text{for } (t, x) \in (0, T] \times [0, l]. \end{cases}$$

Then w satisfies homogeneous Dirichlet boundary conditions and solves the equation

$$\partial_{tt}w(t, \cdot) = \mathcal{A}w(t - \tau, \cdot) + f(t, \cdot) \quad (2.3)$$

in the extrapolated space X with

$$f(t, \cdot) = g(t, \cdot) + b(\theta_2(t) - \theta_1(t)) + \theta_1(t) + \frac{dx}{dt}(\theta_2(t) - \theta_1(t)).$$

We multiply the equation with $w_t(t, \cdot)$ in the scalar product of X and use Young's inequality to get the estimate

$$\begin{aligned} \partial_t \|w_t(t, \cdot)\|_X^2 &= \langle \mathcal{A}w(t - \tau, \cdot), w_t(t, \cdot) \rangle_X + \langle f(t, \cdot), w_t(t, \cdot) \rangle_X \\ &\leq \|w(t - \tau, \cdot)\|_X^2 + \left(1 + \|\mathcal{A}\|_{L(X)}^2\right) \|w_t(t, \cdot)\|_X^2 + \|f(t, \cdot)\|_X^2. \end{aligned} \quad (2.4)$$

As in [11], we introduce the history variable

$$z(s, t, x) := w(t - s, x) \text{ for } (s, t, x) \in [0, \tau] \times [0, T] \times [0, l]$$

and obtain

$$z_t(s, t, x) + z_s(s, t, x) = 0 \text{ for } (s, t, x) \in (0, \tau) \times (0, T) \times (0, l).$$

Multiplying these identities with $w(t, \cdot)$ in X and performing a partial integration, we find

$$\partial_t \int_0^\tau \|z(s, t, \cdot)\|_X^2 ds = - \int_0^\tau \partial_s \|z(s, t, \cdot)\|_X^2 ds = \|w(t, \cdot)\|_X^2 - \|w(t - \tau, \cdot)\|_X^2. \quad (2.5)$$

Adding equations (2.4) and (2.5) to the trivial identity

$$\partial_t \|w(t, \cdot)\|_X^2 \leq \|w(t, \cdot)\|_X^2 + \|w_t(t, \cdot)\|_X^2,$$

we obtain

$$\partial_t \left(\|w(t, \cdot)\|_X^2 + \|w_t(t, \cdot)\|_X^2 + \int_{-\tau}^0 \|w(s, t, \cdot)\|_X^2 ds \right) \leq (2 + \|\mathcal{A}\|_{L(X)}) \|w(t, \cdot)\|_X^2 + \|w_t(t, \cdot)\|_X^2 + \|f(t, \cdot)\|_X^2.$$

Thus, we have shown

$$\partial_t E(t) \leq (2 + \|\mathcal{A}\|_{L(X)}) E(t) + \|f(t, \cdot)\|_X^2, \quad (2.6)$$

where

$$E(t) := \|w(t, \cdot)\|_X^2 + \|w_t(t, \cdot)\|_X^2 + \int_{-\tau}^0 \|z(s, t, \cdot)\|_X ds.$$

From Equation (2.6) we conclude

$$E(t) \leq E(0) + (2 + \|\mathcal{A}\|_{L(X)}) \int_0^t E(s) ds + \int_0^t \|f(s, \cdot)\|_X^2 ds.$$

Using now the integral form of Gronwall's inequality, we obtain

$$\begin{aligned} E(t) &\leq E(0) + \int_0^t \|f(s, \cdot)\|_X^2 ds + \int_0^t e^{(2 + \|\mathcal{A}\|_{L(X)})(t-s)} \left(E(0) + \int_0^s \|f(\xi, \cdot)\|_X^2 d\xi \right) ds \\ &\leq \tilde{C} \left(E(0) + \int_0^T \|f(s, \cdot)\|_X^2 ds \right) \end{aligned} \quad (2.7)$$

for certain $\tilde{C} > 0$. Taking into account

$$\begin{aligned} c_1 (\|w(t, \cdot)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2) &\leq \|\eta(t, \cdot)\|_X^2 \leq C_1 (\|w(t, \cdot)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2), \\ c_2 (\|f(t, \cdot)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2) &\leq \|g(t, \cdot)\|_X^2 \leq C_2 (\|f(t, \cdot)\|_X^2 + |\theta_1(t)|^2 + |\theta_2(t)|^2) \end{aligned}$$

for some constants $c_1, c_2, C_1, C_2 > 0$ and exploiting the definition of $E(t)$, the proof is a direct consequence of Equation (2.7). \square

Corollary 2.4. Solutions of Equation (2.1), (2.2) are unique. The solution map

$$(\psi, g, \theta_1, \theta_2) \mapsto \eta$$

is well-defined, linear and continuous in the norms from Theorem 2.3.

Remark 2.5. It was essential to consider the weak space X . If the space corresponding to the usual wave equation is used, i.e., $(\eta, \eta_t) \in H_0^1((0, l)) \times L^2((0, l))$, there follows from [5] that Equation (2.1), (2.2) is an ill-posed problem due to the lack of continuous dependence on the data even in the homogeneous case.

Next, we want to establish conditions on the data allowing for the existence of a classical solution. Performing the substitution

$$\xi(t, x) := e^{-\frac{b}{2a^2}x} \eta(t, x) \text{ for } (t, x) \in [-\tau, T] \times [0, l] \quad (2.8)$$

with a new unknown function ξ (cp. [11]), the initial boundary value problem (2.1), (2.2) can be written in the following simplified form with a self-adjoint operator on the right-hand side

$$\partial_{tt}\xi(t, x) = a^2 \partial_{xx}\xi(t - \tau, x) + c\xi(t - \tau, x) + f(t, x) \text{ for } (t, x) \in (0, T) \times (0, l) \quad (2.9)$$

with $c := d - \frac{b^2}{4a^2}$ complemented by the following boundary and initial conditions

$$\xi(t, 0) = \mu_1(t), \xi(t, l) = \mu_2(t) \text{ for } t > -\tau \text{ with } \mu_1(t) := \theta_1(t), \quad \mu_2(t) := e^{\frac{b}{2a^2}l} \theta_2, \quad (2.10)$$

$$\xi(t, x) = \varphi(t, x) \text{ for } (t, x) \in (-\tau, 0) \times (0, l) \text{ with } \varphi(t, x) := e^{\frac{b}{2a^2}x} \psi(t, x) \quad (2.11)$$

and

$$f(t, x) := e^{\frac{b}{2a^2}x} g(t, x) \text{ for } (t, x) \in [0, T] \times [0, l].$$

The solution will be determined in the form

$$\xi(t, x) = \xi_0(t, x) + \xi_1(t, x) + G(t, x).$$

Here, G is an arbitrary function with $\partial_{tt}G, \partial_{tx}G, \partial_{xx}G \in \mathcal{C}^0([-\tau, T] \times [0, l])$ satisfying the boundary conditions

$$G(t, 0) = \mu_1(t), \quad G(t, l) = \mu_2(t).$$

Assuming $\mu_1, \mu_2 \in \mathcal{C}^2([-\tau, T])$, we let

$$G(t, x) := \mu_1(t) + \frac{x}{l} (\mu_2(t) - \mu_1(t)) \text{ for } (t, x) \in [-\tau, T] \times [0, l]. \quad (2.12)$$

- ξ_0 solves the homogeneous equation

$$\partial_{tt}\xi_0(t, x) = a^2 \partial_{xx}\xi_0(t - \tau, x) + c\xi_0(t - \tau, x) \quad (2.13)$$

subject to homogeneous boundary and non-homogeneous initial conditions

$$\begin{aligned} \xi_0(t, 0) &\equiv 0, \xi_0(t, l) = 0 \text{ in } (-\tau, T), \\ \xi_0(t, x) &= \Phi(t, x) \text{ for } (t, x) \in (-\tau, 0) \times (0, l) \text{ with } \Phi(t, x) := \varphi(t, x) - G(t, x). \end{aligned} \quad (2.14)$$

In particular, with the function G selected as in Equation (2.12), we obtain

$$\Phi(t, x) = \varphi(t, x) - \mu_1(t) - \frac{x}{l} (\mu_2(t) - \mu_1(t)) \text{ for } (t, x) \in [-\tau, 0] \times [0, l]. \quad (2.15)$$

- ξ_1 solves the non-homogeneous equation

$$\partial_{tt}\xi_1(t, x) = a^2 \partial_{xx}\xi_1(t - \tau, x) + c\xi_1(t - \tau, x) + F(t, x) \text{ for } (t, x) \in (0, T) \times (0, l) \quad (2.16)$$

with

$$F(t, x) := a^2 \partial_{xx}G(t - \tau, x) + cG(t - \tau, x) - \partial_{tt}G(t, x) \text{ for } (t, x) \in [0, T] \times [0, l] \quad (2.17)$$

subject to homogeneous boundary and initial conditions. For G from Equation (2.12), we have

$$F(t, x) = f(t, x) + c \left(\mu_1(t - \tau) + \frac{x}{l} (\mu_2(t - \tau) - \mu_1(t - \tau)) \right) - \left(\ddot{\mu}_1(t) + \frac{x}{l} (\ddot{\mu}_2(t) - \ddot{\mu}_1(t)) \right). \quad (2.18)$$

3 Homogeneous equation

In this section, we obtain a formal solution to the initial-boundary value problem (2.13) with initial and boundary conditions given in Equations (2.10), (2.11). We exploit Fourier's separation method to determine ξ_0 in the product form $\xi_0(t, x) = T(t)X(x)$. After plugging this ansatz into Equation (2.13), we find

$$X(x)\ddot{T}(t) = a^2X''(x)T(t - \tau) + cX(x)T(t - \tau).$$

Hence,

$$X(x) \left(\ddot{T}(t) - cT(t - \tau) \right) = a^2X''(x)T(t - \tau).$$

By formally separating the variables, we deduce

$$\frac{X''(x)}{X(x)} = \frac{\ddot{T}(t) - cT(t - \tau)}{a^2T(t - \tau)} = -\lambda^2.$$

Thus, the equation can be decoupled as follows

$$\ddot{T}(t) + (a^2\lambda^2 - c)T(t - \tau) = 0, \quad X''(x) + \lambda^2X(x) = 0. \quad (3.1)$$

These are linear second order ordinary (delay) differential equations with constant coefficients.

Due to the zero boundary conditions for ξ_0 , the boundary conditions for the second equation in (3.1) will also be homogeneous, i.e.,

$$X(0) = 0, \quad X(l) = 0.$$

Therefore, we obtain a Sturm & Liouville problem admitting non-trivial solutions only for the eigen-numbers

$$\lambda^2 = \lambda_n^2 = \left(\frac{\pi n}{l}\right)^2 \text{ for } n \in \mathbb{N}$$

and the corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{\pi n}{l}x\right) \text{ for } n \in \mathbb{N}.$$

Assuming

$$\left(\frac{\pi a}{l}\right)^2 - c > 0,$$

we denote

$$\omega_n = \sqrt{\left(\frac{\pi n}{a}\right)^2 - c} \text{ for } n \in \mathbb{N}$$

and consider the first equation in (3.1), i.e.,

$$\ddot{T}(t) + \omega_n^2 T(t - \tau) = 0 \text{ for } n \in \mathbb{N}. \quad (3.2)$$

The initial conditions for each of the equations in (3.2) can be obtained by expanding the initial data into a Fourier series with respect to the eigenfunction basis of the second equation in (3.1)

$$\begin{aligned} \Phi(t, \cdot) &= \sum_{n=1}^{\infty} \Phi_n(t) \sin\left(\frac{\pi n}{l}x\right) \text{ with } \Phi_n(t) = \frac{2}{l} \int_0^l (\varphi(t, s) - G(t, s)) \sin\left(\frac{\pi n}{l}s\right) ds, \\ \partial_t \Phi(t, \cdot) &= \sum_{n=1}^{\infty} \dot{\Phi}_n(t) \sin\left(\frac{\pi n}{l}x\right) \text{ with } \dot{\Phi}_n(t) = \frac{2}{l} \int_0^l (\partial_t \varphi(t, s) - \partial_t G(t, s)) \sin\left(\frac{\pi n}{l}s\right) ds \end{aligned} \quad (3.3)$$

for $t \in [-\tau, T]$. Let us further determine the solution of the Cauchy problem associated with each of the equations in (3.2) subject to the initial conditions from Equation (3.3).

First, we briefly present some useful results from the theory of second order delay differential equations with pure delay obtained in [9]. The authors considered a linear homogeneous second order delay differential equation

$$\ddot{x}(t) + \omega^2 x(t - \tau) = 0 \text{ for } t \in (0, \infty), \quad x(t) = \beta(t) \text{ for } t \in [-\tau, 0]. \quad (3.4)$$

They introduced two special functions referred to as delay cosine and sine functions. Exploiting these functions, a unique solution to the initial value problem (3.4) was obtained.

Definition 3.1. Delay cosine is the function given as

$$\cos_\tau(\omega, t) = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \leq t < 0, \\ 1 - \omega^2 \frac{t^2}{2!}, & 0 \leq t < \tau, \\ \vdots & \vdots \\ 1 - \omega^2 \frac{t^2}{2!} + \omega^4 \frac{(t-\tau)^4}{4!} - \dots + (-1)^k \omega^{2k} \frac{(t-(k-1)\tau)^{2k}}{(2k)!}, & (k-1)\tau \leq t < k\tau \end{cases} \quad (3.5)$$

with $2k$ -order polynomials on each of the intervals $(k-1)\tau \leq t < k\tau$ continuously adjusted at the nodes $t = k\tau$, $k \in \mathbb{N}_0$.

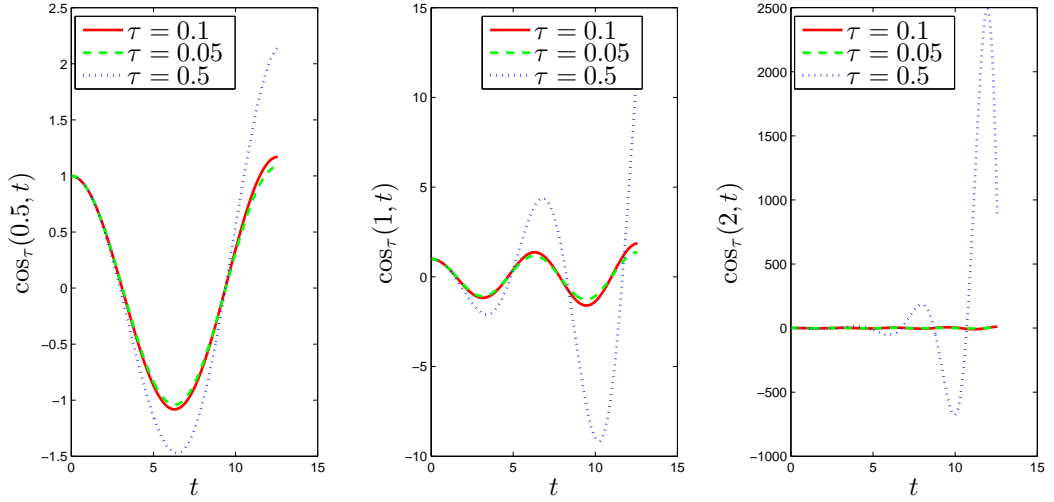


Figure 1: Delay cosine function

Definition 3.2. Delay sine is the function given as

$$\cos_\tau(\omega, t) = \begin{cases} 0, & -\infty < t < -\tau, \\ \omega(1 + \tau), & -\tau \leq t < 0, \\ \omega(1 + \tau) - \omega^3 \frac{t^3}{3!}, & 0 \leq t < \tau, \\ \vdots & \vdots \\ \omega(1 + \tau) - \omega^3 \frac{t^3}{3!} + \dots + (-1)^k \omega^{2k+1} \frac{(t-(k-1)\tau)^{2k+1}}{(2k+1)!}, & (k-1)\tau \leq t < k\tau \end{cases} \quad (3.6)$$

with $(2k+1)$ -order polynomials on each of the intervals $(k-1)\tau \leq t < k\tau$ continuously adjusted at the nodes $t = k\tau$, $k \in \mathbb{N}_0$.

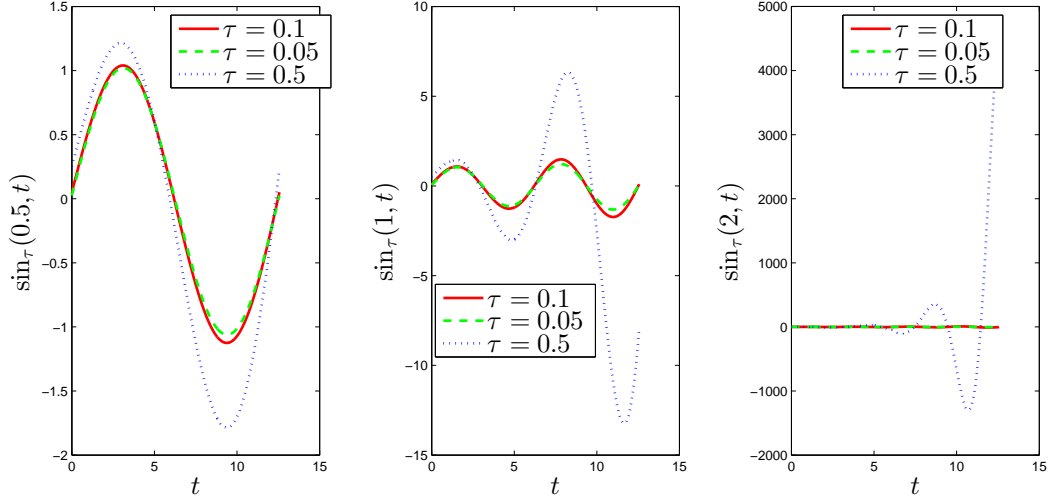


Figure 2: Delay sine function

There has further been proved that delay cosine uniquely solves the linear homogeneous second order ordinary delay differential equation with pure delay subject to the unit initial conditions $x \equiv 1$ in $[-\tau, 0]$ and the delay sine in its turn solves Equation (3.4) subject to the initial conditions $x(t) = \omega(t + \tau)$ for $t \in [-\tau, 0]$.

Using the fact above, the solution of the Cauchy problem was represented in the integral form. In particular, the solution x to the homogeneous delay differential equation (3.4) with the initial conditions $x \equiv \beta$ in $[-\tau, 0]$ for an arbitrary $\beta \in \mathcal{C}^2([-\tau, 0])$ was shown to be given as

$$x(t) = \beta(-\tau) \cos_\tau(\omega, t) + \frac{1}{\omega} \dot{\beta}(-\tau) \sin_\tau(\omega, t) + \frac{1}{\omega} \int_{-\tau}^0 \sin_\tau(\omega, t - \tau - s) \ddot{\beta}(s) ds. \quad (3.7)$$

Turning back to the delay differential equation (3.2) with the initial conditions (2.4), we obtain their unique solution in the form

$$T_n(t) = \Phi_n(-\tau) \cos_\tau(\omega_n, t) + \frac{1}{\omega_n} \dot{\Phi}_n(-\tau) \sin_\tau(\omega_n, t) + \frac{1}{\omega_n} \int_{-\tau}^0 \sin_\tau(\omega_n, t - \tau - s) \ddot{\Phi}(s) ds. \quad (3.8)$$

Thus, assuming sufficient smoothness of the data to be specified later, the solution ξ_0 to the homogeneous equation (2.13) satisfying homogeneous boundary and non-homogeneous initial conditions $\xi \equiv \Phi$ in $[-\tau, 0] \times [0, l]$ reads as

$$\begin{aligned} \xi_0(t, x) = \sum_{n=1}^{\infty} \left(\Phi_n(-\tau) \cos_\tau(\omega_n, t) + \frac{1}{\omega_n} \dot{\Phi}_n(-\tau) \sin_\tau(\omega_n, t) + \right. \\ \left. \frac{1}{\omega_n} \int_{-\tau}^0 \sin_\tau(\omega_n, t - \tau - s) \ddot{\Phi}(s) ds \right) \sin\left(\frac{\pi n}{l} x\right), \quad (3.9) \\ \Phi_n(t) = \frac{2}{l} \int_0^l (\varphi(t, s) - G(t, s)) \sin\left(\frac{\pi n}{l} s\right) ds \text{ for } n \in \mathbb{N}. \end{aligned}$$

4 Non-homogeneous equation

Next, we consider the non-homogeneous equation (2.16) with the right-hand side from Equation (2.18) subject to homogeneous initial and boundary conditions

$$\partial_{tt}\xi_1(t, x) = a^2\partial_{xx}\xi_1(t - \tau, x) + c\xi_1(t - \tau, x) + F(t, x) \text{ for } (t, x) \in (0, T) \times (0, l),$$

where

$$F(t, x) = f(t, x) + c\left(\mu_1(t - \tau) + \frac{x}{l}(\mu_2(t - \tau) - \mu_1(t - \tau))\right) - \left(\ddot{\mu}_1(t) - \frac{x}{l}(\ddot{\mu}_2(t) - \ddot{\mu}_1(t))\right).$$

The solution will be constructed as a Fourier series with respect to the eigenfunctions of the Sturm & Liouville problem from the previous section, i.e.,

$$\xi_1(t, x) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{\pi n}{l}x\right). \quad (4.1)$$

Plugging the ansatz from (4.1) into Equation (2.6) and comparing the time-dependent Fourier coefficients, we obtain a system of countably many second order delay differential equations

$$\ddot{T}_n(t) + \omega_n T_n(t - \tau) = F_n(t) \text{ for } t \in (0, T) \text{ with } F_n(t) = \frac{2}{l} \int_0^l F(t, s) \sin\left(\frac{\pi n}{l}s\right) ds. \quad (4.2)$$

In [9], the initial value problem for the non-homogeneous delay differential equation

$$\ddot{x}(t) + \omega^2 x(t - \tau) = f(t) \text{ for } t \geq 0$$

with homogeneous initial conditions $x \equiv 0$ in $[-\tau, 0]$ was shown to be uniquely solved by

$$x(t) = \int_0^t \sin_{\tau}(\omega, t - \tau - s) f(s) ds. \quad (4.3)$$

Exploiting Equation (4.3), the equations in (4.2) subject to zero initial conditions are uniquely solved by

$$T_n(t) = \int_0^t \sin_{\tau}(\omega_n, t - \tau - s) F_n(s) ds. \quad (4.4)$$

Therefore, the non-homogeneous partial delay differential equation with homogeneous initial and boundary condition formally reads as

$$\xi_1(t, x) = \sum_{n=1}^{\infty} \left(\int_0^t \sin_{\tau}(\omega_n, t - \tau - s) F_n(s) ds \right) \sin\left(\frac{\pi n}{l}x\right) \text{ for } (t, x) \in [0, T] \times [0, l]. \quad (4.5)$$

5 General case solution

The solution in the general case can thus formally be represented as the following series

$$\begin{aligned} \xi(t, x) = & \sum_{n=1}^{\infty} \left(\Phi_n(-\tau) \cos_{\tau}(\omega_n, t) + \frac{1}{\omega_n} \dot{\Phi}_n(-\tau) \sin_{\tau}(\omega_n, t) + \right. \\ & \left. \frac{1}{\omega_n} \int_{-\tau}^0 \sin_{\tau}(\omega_n, t - \tau - s) \ddot{\Phi}_n(s) ds \right) \sin\left(\frac{\pi n}{l}x\right) + \\ & \sum_{n=1}^{\infty} \left(\int_0^t \sin_{\tau}(\omega_n, t - \tau - s) F_n(s) ds \right) \sin\left(\frac{\pi n}{l}x\right) + G(t, x). \end{aligned} \quad (5.1)$$

5.1 Convergence of the Fourier series

Theorem 5.1. Let $T > 0$, $\tau > 0$ and $m := \lceil \frac{T}{\tau} \rceil$. Further, let the data functions φ , μ_1 , μ_2 and f be such that their Fourier coefficients Φ_n and F_n given in Equations (3.3) and (4.5) satisfy the conditions

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(|\Phi_n(-\tau)| + |\dot{\Phi}_n(-\tau)| \right) n^{2m+3+\alpha} = 0, \quad \lim_{n \rightarrow \infty} \max_{s \in [-\tau, 0]} |\ddot{\Phi}_n| n^{2m+3+\alpha} = 0, \\ \lim_{n \rightarrow \infty} \max_{k=1, \dots, m} \max_{t \in [(k-1)\tau, \max\{k\tau, T\}]} |F_n(t)| n^{2k+3+\alpha} = 0 \end{aligned} \quad (5.2)$$

for an arbitrary, but fixed $\alpha > 0$. Let

$$\left(\frac{\pi}{l} a \right)^2 > c.$$

Then the classical solution to problem (2.9)–(2.11) can be represented as an absolutely and uniformly convergent Fourier series given in Equation (5.1). The latter series is a twice continuously differentiable function with respect to both variables. Its derivatives of order less or equal two with respect to t and x can be obtained by a term-wise differentiation of the series and the resulting series are also absolutely and uniformly convergent in $[0, T] \times [0, l]$.

Proof. We regroup the series from Equation (5.1) into the following sum

$$\xi(t, x) = S_1(t, x) + S_2(t, x) + S_3(t, x) + G(t, x),$$

where

$$\begin{aligned} S_1(t, x) &= \sum_{n=1}^{\infty} A_n(t) \sin\left(\frac{\pi n}{l} x\right), \quad S_2(t, x) = \sum_{n=1}^{\infty} B_n(t) \sin\left(\frac{\pi n}{l} x\right), \quad S_3(t, x) = \sum_{n=1}^{\infty} C_n(t) \sin\left(\frac{\pi n}{l} x\right), \\ A_n(t) &= \Phi_n(-\tau) \cos_{\tau}(\omega_n, t) + \frac{1}{\omega_n} \dot{\Phi}_n(-\tau) \sin_{\tau}(\omega_n, t), \\ B_n(t) &= \frac{1}{\omega_n} \int_{-\tau}^0 \sin_{\tau}(\omega_n, t - \tau - s) \ddot{\Phi}_n ds, \\ C_n(t) &= \frac{1}{\omega_n} \int_{-\tau}^0 \sin_{\tau}(\omega_n, t - \tau - s) F_n(s) ds \end{aligned}$$

and

$$\omega_n = \sqrt{\left(\frac{\pi n}{l} a\right)^2 - c} \text{ for } n \in \mathbb{N}.$$

1. First, we consider the coefficient functions A_n . For an arbitrary $t \in [0, T]$ with $(k-1)\tau \leq t < k\tau$, we find

$$\begin{aligned} A_n(t) &= \Phi_n(-\tau) \cos_{\tau}(\omega_n, t) + \frac{1}{\omega_n} \dot{\Phi}_n(-\tau) \sin_{\tau}(\omega_n, t) \\ &= \left(1 - \left(\frac{\pi n}{l} a\right)^2 \frac{t^3}{2!} + \dots + (-1)^k \left(\frac{\pi n}{l} a\right)^{2k} \frac{(t-(k-1)\tau)^{2k}}{(2k)!} \right) \Phi_n(-\tau) + \\ &\quad \left((1+\tau) - \left(\frac{\pi n}{l} a\right)^2 \frac{t^2}{3!} + \dots + (-1)^k \left(\frac{\pi n}{l} a\right)^{2k} \frac{(t-(k-1)\tau)^{2k+1}}{(2k+1)!} \right) \dot{\Phi}_n(-\tau). \end{aligned}$$

If $\Phi_n(-\tau)$ and $\dot{\Phi}_n(-\tau)$, $n \in \mathbb{N}$, are such that the condition

$$\lim_{n \rightarrow \infty} \left(|\Phi_n(-\tau)| + |\dot{\Phi}_n(-\tau)| \right) n^{2k+3+\alpha} = 0$$

holds true, the series S_1 as well as its derivatives of order less or equal 2 converge absolutely and uniformly. Note that a single differentiation with respect to x corresponds, roughly speaking, to a multiplication with n .

2. Next, we consider the coefficients B_n . For an arbitrary $t \in [0, T]$ with $(k-1)\tau \leq t < k\tau$, we perform the substitution $t - \tau - s = \xi$ and exploit the mean value theorem to estimate

$$\begin{aligned} |B_n(t)| &= \left| \frac{1}{\omega_n} \int_{t-\tau}^t \sin_\tau(\omega_n, \xi) \ddot{\Phi}_n(t - \tau - \xi) d\xi \right| \\ &\leq \tau \max_{-\tau \leq s \leq 0} |\ddot{\Phi}_n(s)| \max_{j=k-1, k} \max_{t-\tau \leq s \leq t} \left| (s - \tau) - \left(\frac{\pi n}{l} a\right)^2 \frac{s^3}{3!} + \dots \right. \\ &\quad \left. + (-1)^j \left(\frac{\pi n}{l} a\right)^{2j} \frac{(s - (j-1)\tau)^{2j+1}}{(2j+1)!} \right|. \end{aligned}$$

Applying the theorem on differentiation under the integral sign to B_n and taking into account that $\sin_\tau\left(\frac{\pi n}{l} a, \cdot\right)$ is twice weakly differentiable in $[0, \infty)$, namely: $\sin_\tau\left(\frac{\pi n}{l} a, \cdot\right) \in W_{\text{loc}}^{2, \infty}((0, \infty))$, its derivatives are polynomials of order lower than those of $\sin_\tau\left(\frac{\pi n}{l} a, \cdot\right)$ and their convolution with $\ddot{\Phi}_n$ is continuous, analogous estimates can be obtained for $\dot{\Phi}_n$ and $\ddot{\Phi}_n$ which, in their turn, also follow to be continuous functions.

Now, if the condition

$$\lim_{n \rightarrow \infty} \max_{s \in [-\tau, 0]} |\ddot{\Phi}_n| n^{2m+3+\alpha} = 0$$

is satisfied, the series S_2 as well as its derivatives of order less or equal 2 converge absolutely and uniformly.

3. Finally, we look at the Fourier coefficients C_n . Again, for an arbitrary period of time $t \in [0, T]$ with $(k-1)\tau \leq t < k\tau$, $0 \leq k \leq m$, we substitute $t - \tau - \xi = s$. Once again, using the mean value theorem, we estimate

$$\begin{aligned} |C_n(t)| &= \left| \frac{1}{\omega_n} \int_{t-\tau}^t \sin_\tau\left(\frac{\pi n}{l} a, \xi\right) F_n(t - \tau - \xi) d\xi \right| \\ &\leq \tau \max_{t-\tau \leq s \leq t} |\ddot{\Phi}_n(s)| \max_{j=k-1, k} \max_{t-\tau \leq s \leq t} \left| (s - \tau) - \left(\frac{\pi n}{l} a\right)^2 \frac{s^3}{3!} + \dots \right. \\ &\quad \left. + (-1)^j \left(\frac{\pi n}{l} a\right)^{2j} \frac{(s - (j-1)\tau)^{2j+1}}{(2j+1)!} \right|. \end{aligned}$$

As before, C_n can be shown to be twice continuously differentiable. If now

$$\lim_{n \rightarrow \infty} \max_{k=1, \dots, m} \max_{t \in [(k-1)\tau, \max\{k\tau, T\}]} |F_n(t)| n^{2k+3+\alpha} = 0 \quad (5.3)$$

is satisfied, then both S_3 and its derivatives of order less or equal 2 converge absolutely and uniformly.

Since all three conditions are guaranteed by the assumptions of the Theorem due to the fact $k \leq m$, the proof is finished. \square

Remark 5.2. From the practical point of view, the rapid decay condition on the Fourier coefficients of the data given in Equation (5.2) mean a sufficiently high Sobolev regularity of the data and corresponding higher order compatibility conditions at the boundary of $(0, l)$ (cf. [11]).

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