

Duality theory for robust utility maximisation

Daniel Bartl¹ · Michael Kupper² · Ariel Neufeld³

Abstract

In this paper, we present a duality theory for the robust utility maximisation problem in continuous time for utility functions defined on the positive real line. Our results are inspired by – and can be seen as the robust analogues of – the seminal work of Kramkov and Schachermayer (*Ann. Appl. Probab.* 9:904–950, 1999). Namely, we show that if the set of attainable trading outcomes and the set of pricing measures satisfy a bipolar relation, then the utility maximisation problem is in duality with a conjugate problem. We further discuss the existence of optimal trading strategies. In particular, our general results include the case of logarithmic and power utility, and they apply to drift and volatility uncertainty.

Keywords Robust utility maximisation · Duality theory · Bipolar theorem · Drift and volatility uncertainty

Mathematics Subject Classification (2020) 91B16 · 93E20

JEL Classification C61 · G11

Daniel Bartl is grateful for financial support through the Austrian Science Fund (FWF) under project P28661 and the Vienna Science and Technology Fund (WWTF) under project MA16-021. Ariel Neufeld is grateful for financial support through Nanyang Assistant Professorship Grant (NAP Grant) *Machine Learning based Algorithms in Finance and Insurance*.

✉ A. Neufeld
ariel.neufeld@ntu.edu.sg

D. Bartl
daniel.bartl@univie.ac.at

M. Kupper
kupper@uni-konstanz.de

- ¹ Department of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
- ² Department of Mathematics and Statistics, University of Konstanz, Universitätsstrasse 10, 78464 Konstanz, Germany
- ³ Division of Mathematical Sciences, NTU Singapore, 21 Nanyang Link, 637371 Singapore, Singapore

1 Introduction

The goal of this paper is to develop a duality theory for the robust utility maximisation problem. Given a utility function $U : (0, \infty) \rightarrow \mathbb{R}$ which is nondecreasing and concave, one defines the robust utility maximisation problem as

$$u(x) := \sup_{g \in \mathcal{C}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)]. \quad (1.1)$$

Here \mathcal{P} denotes a set of possibly nondominated probability measures, and $\mathcal{C}(x) := x\mathcal{C}$ is a set of random variables. Financially speaking, the set \mathcal{P} represents the set of possible candidates for the real-world measure, which is not known to the portfolio manager who tries to solve the maximisation problem, whereas the set $\mathcal{C}(x)$ represents all the possible portfolio values at terminal time T available with initial capital $x > 0$. Note that if $\mathcal{P} = \{\mathbb{P}\}$ is a singleton, then the robust utility maximisation problem coincides with the *classical utility maximisation problem* and has the financial interpretation that the portfolio manager (believes to) know the real-world measure \mathbb{P} . The robust utility maximisation problem with respect to nondominated probability measures \mathbb{P} has been already widely studied. In the discrete-time setting, Bartl [1] and Bartl et al. [2] consider the robust utility maximisation problem with random endowment, Blanchard and Carassus [6] and Rásonyi and Meireles-Rodrigues [38] consider the robust utility maximisation problem for unbounded utility functions, whereas Neufeld and Šikić [29, 30] consider the case where there are convex [29] and non-convex [30] frictions in the market. In the continuous-time setting, Biagini and Pinar [5], Denis and Kervarec [10], Lin et al. [21], Matoussi et al. [23], Tevzadze et al. [42] and Uğurlu [43] analyse the robust utility maximisation problem under drift and/or volatility uncertainty, Liang and Ma [19], Neufeld and Nutz [28] consider drift, volatility and jump uncertainty, Fouque et al. [12], Ismail and Pham [14] and Pun [37] consider correlation and covariance uncertainty, Lin and Riedel [20] consider interest rate uncertainty, Chau and Rásonyi [8] analyse the robust utility maximisation problem when the market has transaction costs, Guo et al. [13] use a penalisation approach, Pham et al. [36] analyse a robust dynamic mean–variance approach, and Yang et al. [44] analyse the robust utility maximisation problem under constraints and borrowing costs.

To the best of our knowledge, the only paper so far which provides a duality theory for the robust utility maximisation problem with nondominated probability measures in a continuous-time setting is the one by Denis and Kervarec [10]. Indeed, [10] provides such a duality theory under drift and volatility uncertainty, but under the strong assumptions that the utility function is bounded and the trading strategies possess some continuity (as a functional of the stock price). In addition, the corresponding volatility matrix is required to be of diagonal form.

In order to get an idea how one could try to identify the dual problem for the robust utility maximisation problem, let us recall the main idea in the seminal paper by Kramkov and Schachermayer [18] which provides a complete duality theory in the classical case where $\mathcal{P} = \{\mathbb{P}\}$ is a singleton. Consider the conjugate function $V : (0, \infty) \rightarrow \mathbb{R}$ defined by

$$V(y) := \sup_{x \geq 0} (U(x) - xy), \quad y > 0.$$

In the classical case where $\mathcal{P} = \{\mathbb{P}\}$ is a singleton, Brannath and Schachermayer [7] developed a bipolar theorem on $L_+^0(\mathbb{P})$, despite the fact that $L_+^0(\mathbb{P})$ is generally not locally convex, by considering the dual pairing $(L_+^0(\mathbb{P}), L_+^0(\mathbb{P}))$ endowed with the bilinear map $\langle g, h \rangle := \mathbb{E}_{\mathbb{P}}[gh]$ (see also Žitković [45] for a conditional version). This bipolar theorem then allows identifying the dual optimisation problem and proving that the corresponding optimisation problems are conjugate. More precisely, let $\mathcal{C} \subseteq L_+^0(\mathbb{P})$ and define the polar set

$$\mathcal{D} := \{h \in L_+^0(\mathbb{P}) : \mathbb{E}_{\mathbb{P}}[gh] \leq 1 \text{ for all } g \in \mathcal{C}\}.$$

Then one can define the dual optimisation problem by

$$v(y) := \inf_{h \in \mathcal{D}} \mathbb{E}_{\mathbb{P}}[V(yh)].$$

By the bipolar relation that

$$\mathcal{D} = \{h \in L_+^0(\mathbb{P}) : \mathbb{E}_{\mathbb{P}}[gh] \leq 1 \text{ for all } g \in \mathcal{C}\}, \quad (1.2)$$

$$\mathcal{C} = \{g \in L_+^0(\mathbb{P}) : \mathbb{E}_{\mathbb{P}}[gh] \leq 1 \text{ for all } h \in \mathcal{D}\} \quad (1.3)$$

together with a minimax argument, Kramkov and Schachermayer [18, Theorem 3.1] proved that indeed u and v are conjugates, namely that

$$\begin{aligned} u(x) &= \inf_{y \geq 0} (v(y) + xy), & x > 0, \\ v(y) &= \sup_{x \geq 0} (u(x) - xy), & y > 0. \end{aligned} \quad (1.4)$$

However, in the robust analogue where \mathcal{P} is not a singleton, it is not clear how to find a suitable dual pairing $(\mathcal{X}, \mathcal{X}^*)$ for $\mathcal{X} \supseteq \mathcal{C}$ such that bipolar relations like (1.2) and (1.3) hold. Our approach is the following. Instead of working on an arbitrary measurable space (Ω, \mathcal{F}) , we impose that Ω is a Polish space endowed with its Borel σ -field. This allows us to use the natural dual pairing $(C_b(\Omega), \mathfrak{P}(\Omega))$ consisting of the bounded continuous functions $C_b := C_b(\Omega)$ together with the set of Borel probability measures $\mathfrak{P}(\Omega)$ on Ω . Given a set \mathcal{C} of nonnegative measurable functions defined on Ω , we then define its polar set by

$$\mathcal{D} := \{\mathbb{Q} \in \mathfrak{P}(\Omega) : \mathbb{E}_{\mathbb{Q}}[g] \leq 1 \text{ for all } g \in \mathcal{C}\}. \quad (1.5)$$

This allows us to formulate a bipolar relation on the subset C_b , namely we require that

$$\mathcal{D} = \{\mathbb{Q} \in \mathfrak{P}(\Omega) : \mathbb{E}_{\mathbb{Q}}[g] \leq 1 \text{ for all } g \in \mathcal{C} \cap C_b\}, \quad (1.6)$$

$$\mathcal{C} \cap C_b = \{g \in C_b^+ : \mathbb{E}_{\mathbb{Q}}[g] \leq 1 \text{ for all } \mathbb{Q} \in \mathcal{D}\}, \quad (1.7)$$

where $C_b^+ := \{g \in C_b(\Omega) : g \geq 0\}$. In our first result, we show that if \mathcal{C} is a set of nonnegative measurable functions and \mathcal{D} is a set of probability measures defined by

(1.5) such that the bipolar relations (1.6) and (1.7) hold, then the functions u and v defined by (1.1) and

$$v(y) := \inf_{\mathbb{Q} \in \mathcal{D}} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{Q}} \left[V \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right]$$

indeed satisfy both conjugate relations in (1.4); see Theorem 2.10 below. Here, we set $V(-\infty) := \infty$, and $\frac{d\mathbb{Q}}{d\mathbb{P}} := -\infty$ if \mathbb{Q} is not absolutely continuous with respect to \mathbb{P} .

At first glance, the bipolar relations (1.6) and (1.7) might seem restrictive. However, it turns out that, in fact, this bipolar relation is *naturally* satisfied in the context of drift and volatility uncertainty. Let $\Omega = C([0, T]; \mathbb{R}^d)$ and consider the set of probability measures $\mathcal{P} := \mathcal{P}_{\text{sem}}^{\text{ac}}(\Theta)$ for which the canonical process $(S_t)_{0 \leq t \leq T}$ is a semimartingale with differential characteristics taking values in a set $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$. Then we define

$$\mathcal{C} := \{g : \Omega \rightarrow [0, \infty] : \exists H \in \mathcal{H} \text{ such that } g \leq 1 + (H \cdot S)_T \text{ } \mathbb{P}\text{-a.s., } \forall \mathbb{P} \in \mathcal{P}\} \quad (1.8)$$

so that \mathcal{C} is the set of \mathcal{P} -quasi surely superreplicable claims, and \mathcal{D} is its polar set defined in (1.5). As admissibility condition on the set \mathcal{H} of hedging strategies, we require for each $H \in \mathcal{H}$ that the stochastic integral satisfies $H \cdot S \geq -c$ \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}$ for some constant $c > 0$ (where c can depend on both H and \mathbb{P}), similarly to the classical admissibility condition. This setting can be seen as the robust analogue to the setting of Kramkov and Schachermayer [18, Theorem 2.1]. We show in the proof of Theorem 3.4 below that for $\mathcal{P} := \mathcal{P}_{\text{sem}}^{\text{ac}}(\Theta)$ together with \mathcal{C} and \mathcal{D} defined in (1.8) and (1.5), the bipolar relations (1.6) and (1.7) are satisfied *automatically*.

Let us explain why (1.6) and (1.7) hold. To see that (1.6) holds, note that since \mathcal{C} is the set of superhedgeable claims, every element in \mathcal{D} satisfies $\mathbb{E}_{\mathbb{Q}}[(H \cdot S)_T] \leq 0$ for all $H \in \mathcal{H}$. In other words, \mathcal{D} can be seen as the set of separating measures (in the notion of Kabanov [16]). Moreover, since S has continuous sample paths, it is well known that the set of separating measures coincides with the set of local martingale measures; see e.g. Delbaen and Schachermayer [9, Lemma 5.1.3]. Having this in mind, the condition (1.6) means that the set of separating measures is already determined by the superhedgeable claims which are continuous. We show that this is indeed true due to the Polish structure of Ω ; see Propositions 5.7 and A.2.

To see that (1.7) holds, we need to show that every nonnegative and continuous bounded claim g can be superhedged. In the classical theory where $\mathcal{P} = \{\mathbb{P}\}$ is a singleton, this has been proved (even for measurable claims) by El Karoui and Quenez [11] and Kramkov [17] in the following way. First they show that there exists a nonnegative process $(Y_t)_{0 \leq t \leq T}$ with $Y_T = g$ such that Y is a \mathbb{Q} -supermartingale for every equivalent local martingale measure \mathbb{Q} . Then the optional decomposition theorem guarantees that $g = Y_T \leq 1 + (H \cdot S)_T$ for some hedging strategy H . Robust analogues of the results in [11] and [17] have recently been developed by Soner and Nutz [35], Neufeld and Nutz [25] and Nutz [34] and are compatible with the set $\mathcal{P} := \mathcal{P}_{\text{sem}}^{\text{ac}}(\Theta)$. This allows us to prove that indeed (1.7) holds; see Proposition 5.9 below.

After sketching how to verify the bipolar relation for drift and volatility uncertainty, let us continue to briefly sketch the main ingredients for the proofs of our abstract main results, Theorems 2.10 and 2.16. The proof of Theorem 2.10, which deals

with utility functions that are bounded from below, follows the lines of Kramkov and Schachermayer [18] and makes use of arguments from convex analysis. Here the bipolar relation of the trading outcomes and the pricing measures is crucial, and the Polish structure of the underlying space allows us to overcome difficulties arising from the non-dominated framework. Theorem 2.16 then extends the previously obtained results to utility functions which are unbounded (from both above and below), for instance logarithmic or power utility functions (see also Corollaries 3.6 and 3.7). This is done by means of an approximation which requires us to construct ‘optimal’ trading strategies. In contrast to the classical (dominated) framework, the existence of such strategies cannot be proved by tools such as the Komlós lemma, and we rely on the so-called medial limits instead.

The remainder of this paper is organised as follows. In Sect. 2, we present the main results in the abstract setting. In Sect. 3, we state our main results in the setting of drift and volatility uncertainty. The proofs of the results of Sect. 2 are provided in Sect. 4, and the proofs of the results of Sect. 3 are in Sect. 5.

2 Main results in the abstract setting

We fix a time horizon $T \in (0, \infty)$ and a Polish space Ω with its Borel σ -field \mathcal{F} . We denote by $\mathfrak{P}(\Omega)$ the set of all Borel probability measures endowed with the topology induced by weak convergence, making it a Polish space as well. Without further mention, we interpret $\mathfrak{P}(\Omega)$ as a subset of the set of all nonnegative finite measures. We denote by $\mathcal{F}^* := \bigcap_{\mathbb{P} \in \mathfrak{P}(\Omega)} \mathcal{F}^{\mathbb{P}}$ the universal σ -field. Moreover, we denote by C_b the set of all continuous functions from Ω to \mathbb{R} and by C_b^+ the subset of all nonnegative functions in C_b . We denote by \mathbb{S}_+^d the set of all symmetric positive semi-definite matrices in $\mathbb{R}^{d \times d}$ and by $\mathbb{S}_{++}^d \subseteq \mathbb{S}_+^d$ the set of all positive definite matrices in \mathbb{S}_+^d . We say that $A \leq B$ holds for $A, B \in \mathbb{R}^{d \times d}$ if $B - A$ is in \mathbb{S}_+^d .

We first fix a nonempty set $\mathcal{P} \subseteq \mathfrak{P}(\Omega)$, which represents the set of possible candidates for the unknown real-world measure, and a nonempty set

$$\mathcal{C} \subseteq \{X : \Omega \rightarrow [0, \infty] : X \text{ is } \mathcal{F}^* \text{-measurable}\}. \quad (2.1)$$

We set $\mathcal{C}(x) := x\mathcal{C}$ for every $x > 0$. Then we fix $\emptyset \neq \mathfrak{P} \subseteq \mathfrak{P}(\Omega)$ and define

$$\mathcal{D} := \{\mathbb{Q} \in \mathfrak{P} : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } X \in \mathcal{C}\}. \quad (2.2)$$

We set $\mathcal{D}(y) := y\mathcal{D}$ for every $y > 0$. In other words, we interpret \mathcal{D} , roughly speaking, as the polar set of \mathcal{C} . Typically, one chooses $\mathfrak{P} := \mathfrak{P}(\Omega)$ or

$$\mathfrak{P} := \{\mathbb{Q} \in \mathfrak{P}(\Omega) : \exists \mathbb{P} \in \mathcal{P} \text{ such that } \mathbb{Q} \approx \mathbb{P}\}; \quad (2.3)$$

see also Sects. 3 and 5. Note that a priori, there are no restrictions on the set \mathfrak{P} . If \mathfrak{P} is given by (2.3) as in Sect. 3, it corresponds to all semimartingale models \mathbb{Q} which are equivalent to at least one of the candidates $\mathbb{P} \in \mathcal{P}$.

We impose the following assumption, which is standard in the utility maximisation literature in mathematical finance; see e.g. Kramkov and Schachermayer [18].

Assumption 2.1 For every $\mathbb{P} \in \mathcal{P}$, there exists a $\mathbb{Q} \in \mathcal{D}$ such that $\mathbb{Q} \ll \mathbb{P}$.

A function $U : (0, \infty) \rightarrow [-\infty, \infty)$ is called a *utility function* if it is concave and nondecreasing. We fix a utility function U and define $U(0) := \lim_{x \downarrow 0} U(x)$. Moreover, we consider the conjugate function

$$\begin{aligned} V(y) &:= \sup_{x \geq 0} (U(x) - xy), & y > 0, \\ V(0) &:= \lim_{y \downarrow 0} V(y), \\ V(y) &:= +\infty, & y < 0. \end{aligned}$$

In the following two subsections, we present our main duality results for the robust utility maximisation problem in the abstract setting, where we distinguish the two cases $U(0) > -\infty$ and $U(0) = -\infty$.

2.1 Main result for utility functions bounded from below

This subsection provides our main result for utility functions U with $U(0) > -\infty$. More precisely, we impose the following condition on U .

Assumption 2.2 The utility function $U : [0, \infty) \rightarrow \mathbb{R}$ is real-valued.

The robust utility maximisation problem is then defined as the maximisation problem

$$u(x) := \sup_{g \in \mathcal{C}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)], \quad x > 0. \quad (2.4)$$

Remark 2.3 Note that Assumption 2.2 ensures that $U : [0, \infty) \rightarrow \mathbb{R}$ is continuous; see Rockafellar [39, Theorem 10.1]. Moreover, common utility functions defined on $(0, \infty)$ like the power utilities $U(x) := \frac{1}{p}x^p$, $p \in (0, 1)$, and the exponential utilities $U(x) = -e^{-\lambda x}$, $\lambda > 0$, satisfy Assumption 2.2.

We define the corresponding dual function

$$v(y) := \inf_{\mathbb{Q} \in \mathcal{D}(y)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[V \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right], \quad y > 0, \quad (2.5)$$

where we make the convention $\frac{d\mathbb{Q}}{d\mathbb{P}} := -\infty$ if \mathbb{Q} is not absolutely continuous with respect to \mathbb{P} . We impose the following standard condition (see e.g. [18]) on the robust utility maximisation problem.

Assumption 2.4 There exists $x_0 \in (0, \infty)$ such that $u(x_0) < \infty$.

For the second part of the main result of this subsection, we assume that medial limits exist.

Assumption 2.5 There exists a positive linear functional $\lim \text{med}: \ell^\infty \rightarrow \mathbb{R}$, called *medial limit*, satisfying $\liminf_{n \rightarrow \infty} \leq \lim \text{med}_{n \rightarrow \infty} \leq \limsup_{n \rightarrow \infty}$ and such that for any uniformly bounded sequence of universally measurable functions $X_n: \Omega \rightarrow \mathbb{R}$, $n \in \mathbb{N}$, the medial limit $X := \lim \text{med}_{n \rightarrow \infty} X_n$ is universally measurable and satisfies $\mathbb{E}_{\mathbb{P}}[X] = \lim \text{med}_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[X_n]$ for every $\mathbb{P} \in \mathfrak{P}(\Omega)$. Moreover, following Bartl et al. [2], we extend the definition of the $\lim \text{med}$ from ℓ^∞ to $[-\infty, \infty]^{\mathbb{N}}$ by setting

$$\lim \text{med}_{n \rightarrow \infty} x_n := \sup_{k \in \mathbb{N}} \inf_{m \in \mathbb{N}} \lim \text{med}_{n \rightarrow \infty} ((-m) \vee (x_n \wedge k)).$$

Remark 2.6 The existence of the medial limit is guaranteed under the usual ZFC axioms together with Martin's axiom; see Meyer [24] and Normann [31]. In the robust mathematical finance literature, the usage of medial limits appeared first in Nutz [32] to construct an (aggregated) stochastic integral simultaneously under a set of non-dominated probability measures. In Nutz [33], medial limits were applied to construct superhedging strategies in the quasi-sure setting in discrete time. Moreover, Bartl et al. [2] used medial limits in the context of robust utility maximisation on the real line in the discrete-time setting. Roughly speaking, medial limits turn out to be particularly useful in robust finance theory when dealing with a set of non-dominated probability measures, where classical limit arguments like the Komlós theorem cannot be applied; we refer to [32, 33, 2] for more details and properties regarding medial limits.

Under the condition that Assumption 2.5 holds, we define, for every $x > 0$,

$$\overline{\mathcal{C}}(x) := \{\lim \text{med } g_n : \Omega \rightarrow [0, \infty] : (g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(x)\}$$

and $\overline{\mathcal{C}} := \overline{\mathcal{C}}(1)$. Observe that $\overline{\mathcal{C}}(x) = x\overline{\mathcal{C}}$ for all $x > 0$ and we write $\overline{\mathcal{C}}(x)$. Moreover, we also consider the robust utility maximisation problem

$$\overline{u}(x) := \sup_{g \in \overline{\mathcal{C}}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)], \quad x > 0, \quad (2.6)$$

and impose the following conditions.

Assumption 2.7 There exists $x_0 \in (0, \infty)$ such that $\overline{u}(x_0) < \infty$.

Assumption 2.8 For every $x \in (0, \infty)$ and $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(x)$, the sequence of random variables

$$\max \left\{ U \left(g_n + \frac{1}{n} \right), 0 \right\}, \quad n \in \mathbb{N},$$

is uniformly integrable with respect to \mathbb{P} for all $\mathbb{P} \in \mathcal{P}$.

Remark 2.9 Whereas Assumptions 2.4 and 2.7 are standard in the mathematical finance literature, Assumption 2.8 is not common. However, note that every utility function U which is bounded from above automatically satisfies Assumptions 2.4,

2.7 and 2.8, no matter what \mathcal{C} and \mathcal{P} are. In addition, we show in Sect. 5.2 that in the setting of drift and volatility uncertainty (see Sect. 3), Assumptions 2.4, 2.7 and 2.8 are automatically satisfied for the logarithm and power utility functions.

Now we are ready to state our main result in the abstract setting for utility functions U satisfying Assumption 2.2, which can be seen as a robust version of the classical result of Kramkov and Schachermayer [18, Theorem 3.1]. Note that as in [18, Theorem 3.1], no asymptotic elasticity condition on U is needed.

Theorem 2.10 *Let U be a utility function satisfying Assumption 2.2, let \mathcal{C} , \mathcal{D} be as in (2.1) and (2.2), and let \mathcal{P} be a set of probability measures such that Assumptions 2.1 and 2.4 hold. Moreover, assume that*

- 1) *the sets \mathcal{P} , \mathcal{D} of probability measures are both convex and compact;*
- 2) *we have*

$$\mathcal{D} = \{\mathbb{Q} \in \mathfrak{P}: \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } X \in \mathcal{C} \cap C_b\};$$

- 3) *we have*

$$\{X \in C_b^+: \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } \mathbb{Q} \in \mathcal{D}\} = \mathcal{C} \cap C_b.$$

Then the following hold:

- (i) *u from (2.4) is nondecreasing, concave, and $u(x) \in \mathbb{R}$ for all $x > 0$.*
- (ii) *v from (2.5) is nonincreasing, convex and proper.*
- (iii) *The functions u and v are conjugates, i.e.,*

$$u(x) = \inf_{y \geq 0} (v(y) + xy), \quad x > 0,$$

$$v(y) = \sup_{x \geq 0} (u(x) - xy), \quad y > 0.$$

- (iv) *For every $x > 0$, we have*

$$u(x) := \sup_{g \in \mathcal{C}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)] = \sup_{g \in \mathcal{C}(x) \cap C_b} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)].$$

If we assume in addition that Assumptions 2.5, 2.7 and 2.8 hold, then we additionally obtain:

- (v) *For every $x > 0$, we have*

$$u(x) = \bar{u}(x).$$

- (vi) *For every $x > 0$, there exists $\hat{g} \in \bar{\mathcal{C}}(x)$ such that*

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(\hat{g})] = \sup_{g \in \bar{\mathcal{C}}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)] =: \bar{u}(x).$$

Remark 2.11 Item (ii) implies that $v(y) > -\infty$ for all $y \in [0, \infty)$ and that there exists $y_0 \in (0, \infty)$ such that $v(y) \in \mathbb{R}$ for all $y \geq y_0$.

2.2 Main result for utility functions unbounded from below

In this section, we provide our main result for utility functions U with $U(0) = -\infty$. More precisely, we impose the following condition on U .

Assumption 2.12 The utility function $U : [0, \infty) \rightarrow [-\infty, \infty)$ has $U(0) = -\infty$, $U|_{(0, \infty)}$ is real-valued, and we have

$$\lim_{n \rightarrow \infty} \frac{U(x_n)}{x_n} = 0$$

for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ with $\lim_{n \rightarrow \infty} x_n = \infty$.

Remark 2.13 Note that Assumption 2.12 and the fact that $U(0) := \lim_{x \downarrow 0} U(x)$ ensure that $U : [0, \infty) \rightarrow [-\infty, \infty)$ is continuous; see [39, Theorem 10.1]. Moreover, common utility functions on $(0, \infty)$ like the logarithmic utility $U(x) := \log x$ and the power utilities $U(x) := \frac{1}{p}x^p$, $p \in (-\infty, 0)$, satisfy Assumption 2.12.

In this section, we impose that medial limits exist (see Assumption 2.5) and consider the robust utility maximisation problem in (2.6). In addition, we assume the following.

Assumption 2.14 Let

$$V_1(y) := \sup_{x \geq 0} (U_1(x) - xy), \quad y > 0,$$

where $U_1(\cdot) := U(\cdot + 1)$, $x \geq 0$. Then for each $y > 0$ and each $\mathbb{P} \in \mathcal{P}$, there exists $\mathbb{Q} \in \mathcal{D}$ such that

$$\mathbb{E}_{\mathbb{P}} \left[\max \left\{ V_1 \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right), 0 \right\} \right] < \infty.$$

Remark 2.15 Although Assumption 2.14 is a priori not standard in the literature, we observe that it is a modest assumption. Indeed, every utility function U which is bounded from above automatically satisfies Assumption 2.14, no matter what \mathcal{C} and \mathcal{P} are (because V_1 is nonincreasing with $V_1(0) = U_1(\infty) = U(\infty)$). In addition, we show in Sect. 5.2 that in the setting of Sect. 3, Assumption 2.14 is automatically satisfied for the logarithm and power utility functions. Furthermore, Assumption 2.14 implies that $v(y) < \infty$ for all $y > 0$.

Theorem 2.16 *Let Assumption 2.5 hold. Let U be a utility function satisfying Assumption 2.12, let \mathcal{C} , \mathcal{D} be as in (2.1) and (2.2), and let \mathcal{P} be a set of probability measures such that Assumptions 2.1, 2.7, 2.8 and 2.14 hold. Moreover, assume that*

- 1) *the sets \mathcal{P} , \mathcal{D} of probability measures are both convex and compact;*
- 2) *we have*

$$\mathcal{D} = \{\mathbb{Q} \in \mathfrak{P} : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } X \in \mathcal{C} \cap \mathcal{C}_b\};$$

3) we have

$$\{X \in C_b^+ : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } \mathbb{Q} \in \mathcal{D}\} = \mathcal{C} \cap C_b.$$

Then the following hold:

- (i) \bar{u} from (2.6) is nondecreasing, concave, and $\bar{u}(x) \in \mathbb{R}$ for all $x > 0$.
- (ii) v from (2.5) is nonincreasing, convex, and $v(y) \in \mathbb{R}$ for all $y > 0$.
- (iii) The functions \bar{u} and v are conjugates, i.e.,

$$\bar{u}(x) = \inf_{y \geq 0} (v(y) + xy), \quad x > 0,$$

$$v(y) = \sup_{x \geq 0} (\bar{u}(x) - xy), \quad y > 0.$$

- (iv) For every $x > 0$, there exists $\hat{g} \in \bar{\mathcal{C}}(x)$ such that

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(\hat{g})] = \sup_{g \in \bar{\mathcal{C}}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)] =: \bar{u}(x).$$

3 Main results under drift and volatility uncertainty

The goal of this section is to show that the main assumptions imposed in Theorems 2.10 and 2.16, namely the bipolar relation of \mathcal{C} and \mathcal{D} and the convex-compactness assumption on \mathcal{P} and \mathcal{D} , are naturally fulfilled in the context of simultaneous drift and volatility uncertainty.

To that end, **in this section**, let $\Omega = C([0, T]; \mathbb{R}^d)$ with its Borel σ -field \mathcal{F} . We denote by $(S_t)_{0 \leq t \leq T}$ the canonical process on Ω , i.e., $S_t(\omega) = \omega(t)$. Moreover, let $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ be the raw filtration generated by S , i.e., $\mathcal{F}_t = \sigma(S_s, s \leq t)$, and denote by $\mathbb{F}^* = (\mathcal{F}_t^*)_{0 \leq t \leq T}$ the corresponding universal filtration.

Now consider the following sets of Borel probability measures on Ω which were introduced in Neufeld and Nutz [26]. We set

$$\mathcal{P}_{\text{sem}} := \{\mathbb{P} \in \mathfrak{P}(\Omega) : S \text{ is a semimartingale on } (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\},$$

$$\mathcal{P}_{\text{sem}}^{\text{ac}} := \{\mathbb{P} \in \mathcal{P}_{\text{sem}} : B^{\mathbb{P}} \ll dt, C^{\mathbb{P}} \ll dt \text{ } \mathbb{P}\text{-a.s.}\},$$

where $B^{\mathbb{P}}$ and $C^{\mathbb{P}}$ denote the first and second characteristic of the continuous semimartingale S under \mathbb{P} . For any Borel set $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$, we then define the set \mathcal{P} by

$$\mathcal{P} := \mathcal{P}_{\text{sem}}^{\text{ac}}(\Theta) := \{\mathbb{P} \in \mathcal{P}_{\text{sem}}^{\text{ac}} : (b^{\mathbb{P}}, c^{\mathbb{P}}) \in \Theta \text{ } \mathbb{P} \otimes dt\text{-a.e.}\}, \quad (3.1)$$

where $(b^{\mathbb{P}}, c^{\mathbb{P}}) = (\frac{dB^{\mathbb{P}}}{dt}, \frac{dC^{\mathbb{P}}}{dt})$ denote the differential characteristics of S under \mathbb{P} . We use the standard terminology to say that a property holds \mathcal{P} -q.s. if it holds true \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}$. **Throughout this section**, we fix a set $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$ and impose the following conditions.

Assumption 3.1 The set $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$ satisfies the following properties:

- Θ is convex and compact;
- there exists $\underline{c} \in \mathbb{S}_{++}^d$ such that $\underline{c} \leq c$ for all $c \in \text{proj}_c(\Theta) =: \Theta_c$, where

$$\text{proj}_c(\Theta) := \{c \in \mathbb{S}_+^d : \exists b \in \mathbb{R}^d \text{ such that } (b, c) \in \Theta\}.$$

Remark 3.2 Assumption 3.1 guarantees that each $c \in \Theta_c$ is in \mathbb{S}_{++}^d and in particular is invertible. Moreover, we have for each $c \in \Theta_c$ that both c and c^{-1} are bounded. The uniform ellipticity condition in Assumption 3.1, however, imposes that each $\mathbb{P} \in \mathcal{P}$ corresponds to a complete financial market. From a technical point of view, this condition allows us for each $\mathbb{P} \in \mathcal{P}$ to guarantee the existence of an equivalent martingale measure $\mathbb{Q} \in \mathcal{M} := \mathcal{P}_{\text{sem}}^{\text{ac}}(\tilde{\Theta})$, where $\tilde{\Theta} := \{0, \dots, 0\} \times \Theta_c \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$, and conversely for each $\mathbb{Q} \in \mathcal{M}$, there exists $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P} \approx \mathbb{Q}$; we refer to Proposition 5.2. We point out that a similar condition has also been imposed in Denis and Kervarec [10]; see Hypothesis (H) in their paper. This in turn allows us to identify $\mathcal{D} = \mathcal{M}$, which together with Proposition A.2 is the key property enabling us to show that the bipolar relation on the subset C_b introduced in (1.6) and (1.7) naturally holds in the context of drift and volatility uncertainty; we refer to Propositions 5.7 and 5.9.

Next, let us introduce a particular filtration $\mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ defined by

$$\mathcal{G}_t := \bigcap_{s>t} (\mathcal{F}_s^* \vee \mathcal{N}^{\mathcal{P}}), \quad 0 \leq t \leq T, \quad (3.2)$$

where $\mathcal{N}^{\mathcal{P}}$ is the collection of all sets which are \mathcal{F}_T - \mathbb{P} -null (i.e., subsets of \mathbb{P} -nullsets in \mathcal{F}_T) for all $\mathbb{P} \in \mathcal{P}$. A priori, the filtration \mathbb{G} looks unnatural. However, it will be helpful in the sequel to apply results from Neufeld and Nutz [25] and Nutz [34] where this filtration has been used; see also Remark 5.3. In addition, note that for every $\mathbb{P} \in \mathcal{P}$, the filtration \mathbb{G} satisfies $\mathbb{F} \subseteq \mathbb{G} \subseteq \mathbb{F}_+^{\mathbb{P}}$, where \mathbb{F}_+ denotes the right-continuous version of \mathbb{F} and $\mathbb{F}_+^{\mathbb{P}}$ denotes the usual \mathbb{P} -augmentation of \mathbb{F}_+ ; see also the following Remark 3.3.

Remark 3.3 It follows from [26, Proposition 2.2] that the process $(S_t)_{0 \leq t \leq T}$ is a (\mathbb{P}, \mathbb{F}) -semimartingale if and only if it is a $(\mathbb{P}, \mathbb{F}_+)$ -semimartingale, if and only if it is a $(\mathbb{P}, \mathbb{F}_+^{\mathbb{P}})$ -semimartingale. Moreover, the associated semimartingale characteristics with respect to these filtrations are the same. In particular, we see that (3.1) does not depend on the choice of the filtration \mathbb{G} , as long as $\mathbb{F} \subseteq \mathbb{G} \subseteq \mathbb{F}_+^{\mathbb{P}}$.

For any fixed $\mathbb{P} \in \mathfrak{P}(\Omega)$ such that S is a \mathbb{P} -semimartingale and any (predictable) process H which is (\mathbb{P}, S) -integrable in the semimartingale sense (see e.g. [15, Definition III.6.17]), denote by $\int H dS := H \cdot S := {}^{(\mathbb{P})}(H \cdot S)$ the usual stochastic integral under \mathbb{P} . Let \mathcal{H} be the set of all \mathbb{G} -predictable processes H which are (\mathbb{P}, S) -integrable in the semimartingale sense for all $\mathbb{P} \in \mathcal{P}$ and such that $H \cdot S \geq -c$ \mathbb{P} -a.s. for all $\mathbb{P} \in \mathcal{P}$ for some constant $c > 0$, where c may depend on H and \mathbb{P} . Finally, we specify the sets \mathcal{C} , \mathcal{D} appearing in Theorem 2.10 of the previous section.

We define

$$\begin{aligned} \mathcal{C} &:= \{X : \Omega \rightarrow [0, \infty] \mathcal{F}_T^* \text{-measurable} : \exists H \in \mathcal{H} \text{ with} \\ &\quad 1 + (H \cdot S)_T \geq X \text{ } \mathcal{P}\text{-q.s.}\}, \\ \mathfrak{P}_e(\mathcal{P}) &:= \{\mathbb{Q} \in \mathfrak{P}(\Omega) : \exists \mathbb{P} \in \mathcal{P} \text{ such that } \mathbb{Q} \approx \mathbb{P}\}, \\ \mathcal{D} &:= \{\mathbb{Q} \in \mathfrak{P}_e(\mathcal{P}) : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } X \in \mathcal{C}\}. \end{aligned} \quad (3.3)$$

Note that we choose $\mathfrak{P} := \mathfrak{P}_e(\mathcal{P})$ in the definition of \mathcal{D} . Moreover, for all $x, y > 0$, we define the sets $\mathcal{C}(x)$, $\bar{\mathcal{C}}(x)$ and $\mathcal{D}(y)$, as well as the functions $u(x)$, $\bar{u}(x)$ and $v(y)$ analogously to Sect. 2. Now we are able to state the main results of this section. We distinguish the two cases $U(0) > -\infty$ and $U(0) = -\infty$.

Theorem 3.4 *Let U be a utility function satisfying Assumption 2.2, and let \mathcal{P} , \mathcal{C} and \mathcal{D} defined in (3.1) and (3.3) be such that Assumptions 2.4 and 3.1 hold. Then:*

(I) *Items (i)–(iv) of Theorem 2.10 hold.*

If we assume in addition that Assumptions 2.5, 2.7 and 2.8 hold, then we additionally obtain:

(II) *Items (v) and (vi) of Theorem 2.10 hold.*

Theorem 3.5 *Let Assumption 2.5 hold. Let U be a utility function satisfying Assumption 2.12, and let \mathcal{P} , \mathcal{C} and \mathcal{D} defined in (3.1) and (3.3) be such that Assumptions 2.7, 2.8, 2.14 and 3.1 hold. Then:*

(I) *Items (i)–(iv) of Theorem 2.16 hold.*

The idea of the proofs of Theorems 3.4 and 3.5 is to verify the bipolar relation of \mathcal{C} and \mathcal{D} and the convex-compactness assumption on \mathcal{P} and \mathcal{D} so that one can apply Theorems 2.10 and 2.16, respectively. We refer to Sect. 5.1 for their proofs.

Finally, we should like to emphasise that Assumptions 2.4, 2.7, 2.8 and 2.14 are naturally satisfied in the setting of Sect. 3 by showing that they automatically hold true in the cases where $U(x) = \log x$, $U(x) = \frac{x^p}{p}$, $p \in (-\infty, 0) \cup (0, 1)$, and $U(x) = -e^{-\lambda x}$, $\lambda > 0$; see also Remarks 2.9 and 2.15. As in the previous results, we distinguish the two cases $U(0) > -\infty$ and $U(0) = -\infty$.

Corollary 3.6 *Let U be either a power utility $U(x) = \frac{x^p}{p}$ for some $p \in (0, 1)$ or an exponential utility function $U(x) = -e^{-\lambda x}$ for some $\lambda > 0$. Moreover, let \mathcal{P} , \mathcal{C} and \mathcal{D} defined in (3.1) and (3.3) be such that Assumption 3.1 holds. Then*

(I) *Items (i)–(iv) of Theorem 2.10 hold, and $v(y) \in \mathbb{R}$ for all $y > 0$.*

If we assume in addition that Assumption 2.5 holds, then we additionally obtain:

(II) *Items (v) and (vi) of Theorem 2.10 hold.*

Corollary 3.7 *Let Assumption 2.5 hold. Let U be either the log utility function $U(x) = \log x$ or a power utility $U(x) = \frac{x^p}{p}$ for some $p \in (-\infty, 0)$. Moreover, let*

\mathcal{P} , \mathcal{C} and \mathcal{D} defined in (3.1) and (3.3) be such that Assumption 3.1 holds. Then:

(I) Items (i)–(iv) of Theorem 2.16 hold.

The proofs of Corollaries 3.6 and 3.7 are provided in Sect. 5.2.

4 Proofs of Theorems 2.10 and 2.16

We start with two well-known results on the extension of utility functions defined on $(0, \infty)$, which we provide for the sake of completeness.

Lemma 4.1 *Let $U: (0, \infty) \rightarrow \mathbb{R}$ be a nondecreasing and concave function. Let $V: (0, \infty) \rightarrow (-\infty, \infty]$ be defined by*

$$V(y) := \sup_{x \geq 0} (U(x) - xy), \quad y > 0. \quad (4.1)$$

Moreover, define $\tilde{U}: \mathbb{R} \rightarrow [-\infty, \infty)$ and $\tilde{V}: \mathbb{R} \rightarrow (-\infty, \infty]$ by

$$\tilde{U}(x) := \begin{cases} U(x) & \text{for } x > 0, \\ \lim_{x \downarrow 0} U(x) & \text{for } x = 0, \\ -\infty & \text{for } x < 0, \end{cases} \quad \tilde{V}(y) := \begin{cases} V(y) & \text{for } y > 0, \\ \lim_{y \downarrow 0} V(y) & \text{for } y = 0, \\ \infty & \text{for } y < 0. \end{cases} \quad (4.2)$$

Define the function $\varphi: \mathbb{R} \rightarrow (-\infty, \infty]$ by $\varphi(x) = -\tilde{U}(-x)$, $x \in \mathbb{R}$. Then:

- (i) \tilde{U} is nondecreasing, concave, proper and upper semicontinuous.
- (ii) \tilde{V} is nonincreasing, convex, proper and lower semicontinuous.
- (iii) \tilde{V} is the convex conjugate of φ .
- (iv) We have for every $x > 0$ that $U(x) = \inf_{y \geq 0} (V(y) + xy)$.

Proof Item (i) and that \tilde{V} is nonincreasing follow directly from the definitions and assumptions together with [39, Theorem 10.1]. Therefore φ is convex, proper and lower semicontinuous. Hence the biconjugate theorem (see [39, Theorem 12.2]) ensures that the conjugate φ^* of φ is convex, proper and lower semicontinuous and that $\varphi^{**} = \varphi$. Therefore, to prove (ii) and (iii), it remains to show that $\varphi^* = \tilde{V}$. To that end, note that (4.2) implies for every $y \in \mathbb{R}$ that

$$\varphi^*(y) = \sup_{x \in \mathbb{R}} \left(xy - (-\tilde{U}(-x)) \right) = \sup_{x \in \mathbb{R}} (-xy + \tilde{U}(x)) = \sup_{x \geq 0} (-xy + \tilde{U}(x)).$$

As a consequence, we see that $\mathbb{R} \ni y \mapsto \varphi^*(y)$ is nonincreasing, that

$$\varphi^*(y) = \sup_{x \geq 0} (-xy + \tilde{U}(x)) = \sup_{x \geq 0} (x|y| + \tilde{U}(x)) = +\infty \quad \text{for any } y < 0,$$

and due to (4.1) that for any $y > 0$,

$$\varphi^*(y) = \sup_{x \geq 0} (-xy + \tilde{U}(x)) = \sup_{x \geq 0} (-xy + U(x)) = V(y). \quad (4.3)$$

Moreover, φ^* being nonincreasing implies that $\varphi^*(0) \geq \limsup_{y \downarrow 0} \varphi^*(y)$, and φ^* being lower semicontinuous implies that $\varphi^*(0) \leq \liminf_{y \downarrow 0} \varphi^*(y)$. Therefore we obtain from (4.3) that

$$\tilde{V}(0) = \lim_{y \downarrow 0} V(y) = \lim_{y \downarrow 0} \varphi^*(y) = \varphi^*(0).$$

This shows that $\varphi^* = \tilde{V}$.

Finally, to see that (iv) holds, note that the biconjugate theorem (see [39, Theorem 12.2]) and (iii) imply that

$$\sup_{y \in \mathbb{R}} (xy - \tilde{V}(y)) = \varphi^{**}(x) = \varphi(x) = -\tilde{U}(-x), \quad x \in \mathbb{R}.$$

Therefore, we deduce from (4.2) that for all $x > 0$,

$$\begin{aligned} U(x) &= \tilde{U}(x) = -\sup_{y \in \mathbb{R}} (-xy - \tilde{V}(y)) = \inf_{y \in \mathbb{R}} (xy + \tilde{V}(y)) \\ &= \inf_{y \geq 0} (xy + \tilde{V}(y)) = \inf_{y \geq 0} (xy + V(y)). \end{aligned} \quad \square$$

Lemma 4.2 *Let $V: (0, \infty) \rightarrow \mathbb{R}$ be a nonincreasing and convex function, and let $U: (0, \infty) \rightarrow [-\infty, \infty)$ be defined by*

$$U(x) := \inf_{y \geq 0} (V(y) + xy), \quad x > 0. \quad (4.4)$$

Moreover, define $\tilde{U}: \mathbb{R} \rightarrow [-\infty, \infty)$ and $\tilde{V}: \mathbb{R} \rightarrow (-\infty, \infty]$ by

$$\tilde{U}(x) := \begin{cases} U(x) & \text{for } x > 0, \\ \lim_{x \downarrow 0} U(x) & \text{for } x = 0, \\ -\infty & \text{for } x < 0, \end{cases} \quad \tilde{V}(y) := \begin{cases} V(y) & \text{for } y > 0, \\ \lim_{y \downarrow 0} V(y) & \text{for } y = 0, \\ \infty & \text{for } y < 0. \end{cases}$$

Define the function $\varphi: \mathbb{R} \rightarrow (-\infty, \infty]$ by $\varphi(x) = -\tilde{U}(-x)$, $x \in \mathbb{R}$. Then:

- (i) \tilde{U} is nondecreasing, concave, proper and upper semicontinuous.
- (ii) \tilde{V} is nonincreasing, convex, proper and lower semicontinuous.
- (iii) \tilde{V} is the convex conjugate of φ .
- (iv) We have for every $y > 0$ that $V(y) = \sup_{x \geq 0} (U(x) - xy)$.

Proof Item (ii) and that \tilde{U} is nondecreasing follow from their definitions and [39, Theorem 10.1]. Moreover, since for any $y \geq 0$, the function $(0, \infty) \ni x \mapsto V(y) + xy$ is continuous and affine, we get from (4.4) that \tilde{U} is concave and upper semicontinuous. As a consequence, we see that φ is a convex lower semicontinuous function. Moreover, (4.4) and the definitions of \tilde{U} , \tilde{V} imply that

$$\inf_{y \in \mathbb{R}} (\tilde{V}(y) + xy) = \inf_{y \geq 0} (\tilde{V}(y) + xy) = \tilde{U}(x) \quad \text{for any } x \in \mathbb{R}.$$

Therefore, we get that

$$\varphi(x) = -\tilde{U}(-x) = -\inf_{y \in \mathbb{R}} (\tilde{V}(y) - xy) = \sup_{y \in \mathbb{R}} (-\tilde{V}(y) + xy).$$

Hence we conclude that φ is the convex conjugate of \tilde{V} . In particular, as \tilde{V} is proper, we get from [39, Theorem 12.2] that φ and hence also \tilde{U} is proper. Moreover, by the biconjugate theorem (see [39, Theorem 12.2]), we have $\tilde{V} = \tilde{V}^{**} = \varphi^*$. Thus we see that (i)–(iii) hold.

Finally, (i)–(iii) and the biconjugate theorem (see [39, Theorem 12.2]) imply that for all $y > 0$,

$$\begin{aligned} V(y) &= \tilde{V}(y) = \tilde{V}^{**}(y) = \sup_{x \in \mathbb{R}} (xy - \tilde{V}^*(x)) = \sup_{x \in \mathbb{R}} (xy - \varphi^{**}(x)) \\ &= \sup_{x \in \mathbb{R}} (xy - \varphi(x)) = \sup_{x \in \mathbb{R}} (xy + \tilde{U}(-x)) = \sup_{x \in \mathbb{R}} (-xy + \tilde{U}(x)) \\ &= \sup_{x \geq 0} (-xy + \tilde{U}(x)) = \sup_{x \geq 0} (-xy + U(x)). \quad \square \end{aligned}$$

We also consider the robust maximisation problem, which will be useful in the sequel, given by

$$u_c(x) := \sup_{g \in (\mathcal{C}(x) \cap \mathcal{C}_b)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)], \quad x > 0.$$

Lemma 4.3 *Suppose $\{X \in \mathcal{C}_b^+ : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } \mathbb{Q} \in \mathcal{D}\} = \mathcal{C} \cap \mathcal{C}_b$ and Assumptions 2.1 and 2.4 hold. Then $(0, \infty) \ni x \mapsto u(x)$ and $(0, \infty) \ni x \mapsto u_c(x)$ are both finite-valued, nondecreasing and concave. In particular, with $u(0) := \lim_{x \downarrow 0} u(x)$ and $u_c(0) := \lim_{x \downarrow 0} u_c(x)$, both $[0, \infty) \ni x \mapsto u(x)$ and $[0, \infty) \ni x \mapsto u_c(x)$ are continuous.*

Moreover, if in addition Assumptions 2.5 and 2.7 hold, then $(0, \infty) \ni x \mapsto \bar{u}(x)$ is finite-valued, nondecreasing and concave. In particular, if $\bar{u}(0) := \lim_{x \downarrow 0} \bar{u}(x)$, then $[0, \infty) \ni x \mapsto \bar{u}(x)$ is continuous.

Proof First note that the assumptions ensure that the constant function 1 belongs to $\mathcal{C} \cap \mathcal{C}_b \subseteq \mathcal{C}$. This implies for every $x > 0$ that

$$u(x) = \sup_{g \in \mathcal{C}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)] \geq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(x)] = U(x) > -\infty. \quad (4.5)$$

Since U is concave and nondecreasing and $\mathcal{C}(x) = x\mathcal{C}$ for all $x > 0$, it immediately follows that u is concave and nondecreasing, too. Furthermore, u being nondecreasing, concave and Assumption 2.4 ensure that $u(x) < \infty$ for every $x > 0$ since any nondecreasing concave function is finite everywhere if it is finite in one point. Together with (4.5), we see that $u(x) \in \mathbb{R}$ for all $x > 0$. Finally, the continuity of u now follows from [39, Theorem 10.1]. Since $1 \in \mathcal{C} \cap \mathcal{C}_b$, the same arguments guarantee that the results also hold for u_c .

For the second part, note that Assumption 2.5 ensures that medial limits exist and hence \bar{u} is well defined. Using Assumption 2.7, the result for \bar{u} now follows by the same arguments. \square

From now on, we define, for $f \in \{u, u_c, \bar{u}\}$,

$$f(0) := \lim_{x \downarrow 0} f(x) \in [-\infty, \infty),$$

which are well defined by Lemma 4.3. We start with the proof of Theorem 2.10 where U satisfies Assumption 2.2. This will help us to prove Theorem 2.16 where U only satisfies Assumption 2.12.

Proof of Theorem 2.10 We begin with the first part (which does not involve \bar{u}). First, (i) has been proved in Lemma 4.3. We next prove (iii) and (iv). Note that the definition of V ensures for any $y > 0, x > 0, g \in \mathcal{C}(x), \mathbb{P} \in \mathcal{P}, \mathbb{Q} \in \mathcal{D}$ with $\mathbb{Q} \ll \mathbb{P}$ that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[V \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] &\geq \mathbb{E}_{\mathbb{P}} \left[U(g) - gy \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \\ &= \mathbb{E}_{\mathbb{P}}[U(g)] - y\mathbb{E}_{\mathbb{Q}}[g] \geq \mathbb{E}_{\mathbb{P}}[U(g)] - xy. \end{aligned} \quad (4.6)$$

This ensures for all $x, y > 0$ that

$$\sup_{g \in (\mathcal{C}(x) \cap C_b)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)] - xy \leq \sup_{g \in \mathcal{C}(x)} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g)] - xy \leq v(y), \quad (4.7)$$

which in turn implies that

$$\sup_{x > 0} (u_c(x) - xy) \leq \sup_{x > 0} (u(x) - xy) \leq v(y), \quad y > 0. \quad (4.8)$$

Moreover, (4.8) implies for every $y > 0$ that

$$u(0) = \lim_{x \downarrow 0} (u(x) - xy) \leq v(y), \quad (4.9)$$

and hence we obtain the weak duality

$$\sup_{x \geq 0} (u_c(x) - xy) \leq \sup_{x \geq 0} (u(x) - xy) \leq v(y), \quad y > 0. \quad (4.10)$$

To see the opposite inequalities, note that by the bipolar representation in assumptions (2) and (3), it holds for all $x > 0, y > 0$ and $g \in C_b^+$ that $g \in \mathcal{C}(x) \cap C_b$ if and only if $\sup_{\mathbb{Q} \in \mathcal{D}(y)} \mathbb{E}_{\mathbb{Q}}[g] \leq xy$, and hence we obtain for every $y > 0$ that

$$\begin{aligned} \sup_{x > 0} (u_c(x) - xy) &= \sup_{x > 0} \sup_{g \in \mathcal{C}(x) \cap C_b} \inf_{\mathbb{P} \in \mathcal{P}} (\mathbb{E}_{\mathbb{P}}[U(g)] - xy) \\ &= \sup_{g \in C_b^+} \inf_{\mathbb{P} \in \mathcal{P}} \inf_{\mathbb{Q} \in \mathcal{D}(y)} (\mathbb{E}_{\mathbb{P}}[U(g)] - \mathbb{E}_{\mathbb{Q}}[g]). \end{aligned} \quad (4.11)$$

Now, for every $g \in C_b^+$, the mapping

$$\mathcal{D} \times \mathcal{P} \ni (\mathbb{Q}, \mathbb{P}) \mapsto \mathbb{E}_{\mathbb{P}}[U(g)] - \mathbb{E}_{\mathbb{Q}}[g] \quad (4.12)$$

is convex, and also lower semicontinuous since $U(g)$ is bounded from below. Moreover, for every fixed $(\mathbb{Q}, \mathbb{P}) \in \mathcal{D} \times \mathcal{P}$, the mapping

$$C_b^+ \ni g \mapsto \mathbb{E}_{\mathbb{P}}[U(g)] - \mathbb{E}_{\mathbb{Q}}[g]$$

is concave. This, (4.12) and the assumption that both \mathcal{D} and \mathcal{P} are compact ensure that we can apply Sion's minimax theorem (see Sion [41, Theorem 4.2']) which establishes for every $y > 0$ that

$$\begin{aligned} & \sup_{g \in C_b^+} \inf_{\mathbb{P} \in \mathcal{P}, \mathbb{Q} \in \mathcal{D}(y)} (\mathbb{E}_{\mathbb{P}}[U(g)] - \mathbb{E}_{\mathbb{Q}}[g]) \\ &= \inf_{\mathbb{P} \in \mathcal{P}, \mathbb{Q} \in \mathcal{D}(y)} \sup_{g \in C_b^+} (\mathbb{E}_{\mathbb{P}}[U(g)] - \mathbb{E}_{\mathbb{Q}}[g]). \end{aligned} \quad (4.13)$$

Moreover, one can check for any fixed $\mathbb{P} \in \mathcal{P}$ and $\mathbb{Q} \in \mathcal{D}(y)$ with $\mathbb{Q} \ll \mathbb{P}$ that

$$\sup_{g \in C_b^+} (\mathbb{E}_{\mathbb{P}}[U(g)] - \mathbb{E}_{\mathbb{Q}}[g]) = \mathbb{E}_{\mathbb{P}} \left[\sup_{x > 0} \left(U(x) - \frac{d\mathbb{Q}}{d\mathbb{P}} x \right) \right] = \mathbb{E}_{\mathbb{P}} \left[v \left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right].$$

This, (4.11) and (4.13) demonstrate that for every $y > 0$,

$$\sup_{x > 0} (u_c(x) - xy) = v(y).$$

Therefore, the weak duality (4.10) and the fact that $u \geq u_c$ imply that

$$\sup_{x \geq 0} (u_c(x) - xy) = \sup_{x \geq 0} (u(x) - xy) = v(y), \quad y > 0. \quad (4.14)$$

Moreover, note that (4.14) together with Lemmas 4.3 and 4.1 shows that

$$u_c(x) = \inf_{y \geq 0} (v(y) + xy) = u(x), \quad x > 0,$$

which together with (4.14) proves that (iii) and (iv) hold. Furthermore, note that (i) and (iii) together with Lemma 4.1 imply that v is nonincreasing, convex and proper, which proves (ii). This finishes the first part of the proof.

To prove the second part of Theorem 2.10 (which involves \bar{u}), note that by the definition of $g \in \bar{\mathcal{C}}(x)$, there exists a sequence $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(x)$ with $g = \lim \text{med}_{n \rightarrow \infty} g_n$. Therefore, the definition of V and Fatou's lemma for the medial limit (see Bartl et al. [2, Lemma 3.8(v)]) imply that for any $y > 0$, $x > 0$, $g \in \bar{\mathcal{C}}(x)$, $\mathbb{P} \in \mathcal{P}$, $\mathbb{Q} \in \mathcal{D}$ with $\mathbb{Q} \ll \mathbb{P}$,

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[V \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] &\geq \mathbb{E}_{\mathbb{P}}[U(g)] - \mathbb{E}_{\mathbb{P}} \left[gy \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \\ &= \mathbb{E}_{\mathbb{P}}[U(g)] - y \mathbb{E}_{\mathbb{Q}} \left[\lim \text{med}_{n \rightarrow \infty} g_n \right] \\ &\geq \mathbb{E}_{\mathbb{P}}[U(g)] - y \lim \text{med}_{n \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[g_n] \\ &\geq \mathbb{E}_{\mathbb{P}}[U(g)] - xy. \end{aligned} \quad (4.15)$$

This, the fact that $u(x) \leq \bar{u}(x)$ as $\mathcal{C}(x) \subseteq \bar{\mathcal{C}}(x)$ for every $x > 0$, (4.14) and (4.9) (with u replaced by \bar{u}) show that

$$v(y) = \sup_{x \geq 0} (u(x) - xy) \leq \sup_{x \geq 0} (\bar{u}(x) - xy) \leq v(y), \quad y > 0,$$

which implies that

$$\sup_{x \geq 0} (u(x) - xy) = \sup_{x \geq 0} (\bar{u}(x) - xy) = v(y), \quad y > 0.$$

Combining this with Lemmas 4.3 and 4.1 shows that

$$\bar{u}(x) = \inf_{y \geq 0} (v(y) + xy) = u(x), \quad x > 0,$$

which proves (v).

Finally, to see that (vi) holds, we know from (v) that $\bar{u} = u$; hence for each $n \in \mathbb{N}$, there exists an element $g_n \in \mathcal{C}(x)$ such that

$$\bar{u}(x) \leq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(g_n)] + \frac{1}{n}. \quad (4.16)$$

Define

$$\widehat{g} := \lim \operatorname{med}_{n \rightarrow \infty} g_n \in \bar{\mathcal{C}}(x).$$

Since U is concave, we obtain by Jensen's inequality for medial limits (see [2, Lemma 3.8(iii)]) that

$$U(\widehat{g}) = U\left(\lim \operatorname{med}_{n \rightarrow \infty} g_n\right) \geq \lim \operatorname{med}_{n \rightarrow \infty} U(g_n).$$

By Assumption 2.8, the sequence $\max\{U(g_n), 0\}$, $n \in \mathbb{N}$, is uniformly integrable with respect to any $\mathbb{P} \in \mathcal{P}$. Therefore, Fatou's lemma for the medial limit (see [2, Lemma 3.8(v)]) and (4.16) ensure that

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(\widehat{g})] \geq \inf_{\mathbb{P} \in \mathcal{P}} \lim \operatorname{med}_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[U(g_n)] \geq \lim \operatorname{med}_{n \rightarrow \infty} \left(\bar{u}(x) - \frac{1}{n}\right) = \bar{u}(x).$$

This shows that (vi) holds and finishes the proof. \square

It remains to prove Theorem 2.16. To that end, **from now on**, we define, for every $n \in \mathbb{N}$,

$$U_n(x) := U\left(x + \frac{1}{n}\right), \quad x \geq 0,$$

$$V_n(y) := \sup_{x \geq 0} (U_n(x) - xy), \quad y \geq 0,$$

and define \bar{u}_n and v_n as in (2.6) and (2.5), but for U_n and V_n , respectively. Note that if U satisfies Assumption 2.12, then each U_n , $n \in \mathbb{N}$, is a utility function which satisfies Assumption 2.2; in particular, we can apply Theorem 2.10 to each U_n . This will be useful, by applying a limit argument, to prove Theorem 2.16.

Lemma 4.4 *Let the assumptions in Theorem 2.16 hold. Then for every $y > 0$, we have $\inf_{n \in \mathbb{N}} V_n(y) = V(y)$.*

Proof Since $U_n \geq U$, it follows from the definition that $V_n \geq V$ for each $n \in \mathbb{N}$, and hence we focus on showing that $\inf_{n \in \mathbb{N}} V_n \leq V$. To that end, fix some $y > 0$ and let $(x_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$ be such that for each $n \in \mathbb{N}$,

$$V_n(y) = \sup_{x \geq 0} \left(U\left(x + \frac{1}{n}\right) - xy \right) \leq U\left(x_n + \frac{1}{n}\right) - x_n y + \frac{1}{n}. \quad (4.17)$$

In particular, by monotonicity of U , we have that

$$\sup_{x \geq 0} (U(x) - xy) \leq \sup_{x \geq 0} \left(U\left(x + \frac{1}{n}\right) - xy \right) \leq U\left(x_n + \frac{1}{n}\right) - x_n y + \frac{1}{n}. \quad (4.18)$$

Now notice that U satisfying Assumption 2.12 enforces that $\liminf_{n \rightarrow \infty} x_n > 0$, since otherwise $\liminf_{n \rightarrow \infty} U(x_n + 1/n) - x_n y = U(0) = -\infty$, which contradicts (4.18) since $U|_{(0, \infty)}$ is real-valued by Assumption 2.12; indeed, if the right-hand side of (4.18) is $-\infty$, we get $U(1) - y \leq \sup_{x \geq 0} (U(x) - xy) \leq -\infty$. Therefore, without loss of generality, we may assume that $x_n > 0$ for each n . Moreover, we claim that $\limsup_{n \rightarrow \infty} x_n < \infty$. Indeed, if $\limsup_{n \rightarrow \infty} x_n = \infty$, there is a subsequence (which we still denote by $(x_n)_{n \in \mathbb{N}}$) such that $\lim_{n \rightarrow \infty} x_n = \infty$. Therefore, by concavity and monotonicity of U , we get that

$$\frac{U(x_n)}{x_n} \leq \frac{U(x_n + \frac{1}{n})}{x_n} \leq \left(\frac{U(x_n)}{x_n} + \frac{\partial_+ U(x_n)}{n x_n} \right),$$

where $\partial_+ U$ denotes the right derivative of U . Therefore, as U is nondecreasing and concave and satisfies Assumption 2.12, we obtain that

$$\lim_{n \rightarrow \infty} \frac{U(x_n + \frac{1}{n})}{x_n} = 0.$$

For any fixed $0 < \varepsilon < y$, we hence see for big enough n that

$$\left| \frac{U(x_n + \frac{1}{n})}{x_n} \right| \leq \varepsilon.$$

This ensures for any big enough n that

$$U\left(x_n + \frac{1}{n}\right) - x_n y = x_n \left(\frac{U(x_n + \frac{1}{n})}{x_n} - y \right) \leq x_n (\varepsilon - y) < 0.$$

This in turn implies that

$$\lim_{n \rightarrow \infty} U\left(x_n + \frac{1}{n}\right) - x_n y = -\infty,$$

which again contradicts (4.18) as above. So the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, and after passing to a subsequence, it has a limit $x \in (0, \infty)$. Then (4.17) yields

$$V(y) \geq U(x) - xy = \lim_{n \rightarrow \infty} \left(U\left(x_n + \frac{1}{n}\right) - x_n y \right) \geq \inf_{n \in \mathbb{N}} V_n(y),$$

which completes the proof. \square

Proof of Theorem 2.16 Recall that (i) has been proved in Lemma 4.3. Furthermore, since each U_n satisfies Assumption 2.2, Theorem 2.10 implies for every $n \in \mathbb{N}$ that

$$\begin{aligned} \bar{u}_n(x) &:= \inf_{y \geq 0} (v_n(y) + xy), & x > 0, \\ v_n(y) &:= \sup_{x \geq 0} (\bar{u}_n(x) - xy), & y > 0. \end{aligned} \quad (4.19)$$

Now we claim that $\bar{u}(x) = \inf_{n \in \mathbb{N}} \bar{u}_n(x)$ for each $x > 0$. Indeed, since $\bar{u}_n \geq \bar{u}$ by monotonicity, we only need to show that $\bar{u}(x) \geq \inf_{n \in \mathbb{N}} \bar{u}_n(x)$. To that end, fix $x > 0$. By Theorem 2.10 (v), we have $\bar{u}_n = u_n$; hence there exists for each n an element $g_n \in \mathcal{C}(x)$ such that

$$\bar{u}_n(x) \leq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U_n(g_n)] + \frac{1}{n} = \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[U\left(g_n + \frac{1}{n}\right) \right] + \frac{1}{n}. \quad (4.20)$$

Define

$$\hat{g} := \lim \text{med}_{n \rightarrow \infty} g_n \in \bar{\mathcal{C}}(x).$$

Since U is concave, we obtain by Jensen's inequality for medial limits (see [2, Lemma 3.8(iii)]) that

$$U(\hat{g}) = U\left(\lim \text{med}_{n \rightarrow \infty} \left(g_n + \frac{1}{n}\right)\right) \geq \lim \text{med}_{n \rightarrow \infty} U\left(g_n + \frac{1}{n}\right).$$

By Assumption 2.8, the sequence $\max\{U(g_n + 1/n), 0\}$, $n \in \mathbb{N}$, is uniformly integrable with respect to every $\mathbb{P} \in \mathcal{P}$. Therefore, Fatou's lemma for medial limits and (4.20) ensure that

$$\begin{aligned} \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(\hat{g})] &\geq \inf_{\mathbb{P} \in \mathcal{P}} \lim \text{med}_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}} \left[U\left(g_n + \frac{1}{n}\right) \right] \\ &\geq \lim \text{med}_{n \rightarrow \infty} \left(\bar{u}_n(x) - \frac{1}{n} \right) = \inf_{n \in \mathbb{N}} \bar{u}_n(x). \end{aligned}$$

This together with the fact that $\inf_{n \in \mathbb{N}} \bar{u}_n(x) \geq \bar{u}(x)$ shows that for every $x > 0$,

$$\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[U(\hat{g})] = \bar{u}(x) = \inf_{n \in \mathbb{N}} \bar{u}_n(x). \quad (4.21)$$

In particular, we see that (iv) holds.

Next, we claim that $\inf_{n \in \mathbb{N}} v_n(y) = v(y)$ for each $y > 0$. Indeed, by Lemma 4.4, we know that $\inf_{n \in \mathbb{N}} V_n(y) = V(y)$ for every $y > 0$, and since $n \mapsto V_n(y)$ is decreasing, Assumption 2.14 together with the monotone convergence theorem implies for every $y > 0$ that

$$\inf_{n \in \mathbb{N}} v_n(y) = \inf_{\mathbb{Q} \in \mathcal{D}, \mathbb{P} \in \mathcal{P}} \inf_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} \left[V_n \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] = \inf_{\mathbb{Q} \in \mathcal{D}, \mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[V \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] = v(y).$$

This, (4.21) and (4.19) ensure that for every $x > 0$,

$$\begin{aligned} \bar{u}(x) &= \inf_{n \in \mathbb{N}} \bar{u}_n(x) = \inf_{n \in \mathbb{N}} \inf_{y \geq 0} (v_n(y) + xy) \\ &= \inf_{y \geq 0} \left(\inf_{n \in \mathbb{N}} v_n(y) + xy \right) = \inf_{y \geq 0} (v(y) + xy). \end{aligned} \quad (4.22)$$

Furthermore, since we know from (4.19) that each v_n is nonincreasing and because $\inf_{n \in \mathbb{N}} v_n(y) = v(y)$, we see that also v is nonincreasing on $[0, \infty)$. In addition, as $n \mapsto v_n(y)$ is nonincreasing for each $y > 0$ and each v_n is convex, we conclude that also $v = \lim_{n \rightarrow \infty} v_n$ is convex. Moreover, by (4.22), we have for all $x > 0$, $y \geq 0$ that $\bar{u}(x) \leq v(y) + xy$, which together with Lemma 4.3 implies that $v(y) > -\infty$ for all $y \geq 0$. In addition, by Assumption 2.14, we get $v(y) < \infty$ for all $y > 0$. This implies that $v(y) \in \mathbb{R}$ for every $y > 0$ and hence proves (ii). Finally, we can apply Lemma 4.2 together with (4.22) to conclude that for every $y > 0$, we have

$$v(y) = \sup_{x \geq 0} (\bar{u}(x) - xy),$$

which together with (4.22) proves (iii) and finishes the proof. \square

5 Proofs of Theorems 3.4 and 3.5 and Corollaries 3.6 and 3.7

The idea of all the proofs here is to verify that the assumptions in Theorem 2.10 are satisfied. To that end, **throughout this section**, we put ourselves into the setting of Sect. 3 and refer by \mathcal{C} , \mathcal{D} , \mathcal{P} to the corresponding sets specified there in (3.1) and (3.3).

We recall the set of probability measures

$$\mathfrak{P}_e(\mathcal{P}) := \{\mathbb{Q} \in \mathfrak{P}(\Omega) : \exists \mathbb{P} \in \mathcal{P} \text{ such that } \mathbb{Q} \approx \mathbb{P}\}$$

and consider the following sets of probability measures:

$$\mathfrak{M}_e(\mathcal{P}) := \{\mathbb{Q} \in \mathfrak{P}_e(\mathcal{P}) : S \text{ is a } (\mathbb{Q}, \mathbb{F})\text{-local martingale}\},$$

$$\mathcal{M} := \{\mathbb{Q} \in \mathcal{P}_{\text{sem}}^{\text{ac}} : S \text{ is a } (\mathbb{Q}, \mathbb{F})\text{-local martingale with } c^{\mathbb{Q}} \in \Theta_c \mathbb{Q} \otimes dt\text{-a.e.}\}.$$

Remark 5.1 We have $\mathcal{M} = \mathcal{P}_{\text{sem}}^{\text{ac}}(\tilde{\Theta})$ for $\tilde{\Theta} := \{0, \dots, 0\} \times \Theta_c \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$. In addition, using Assumption 3.1, we show in Proposition 5.7 that in fact $\mathcal{M} = \mathfrak{M}_e(\mathcal{P})$.

5.1 Proof of Theorems 3.4 and 3.5

Lemma 5.2 *Let Assumption 3.1 hold. Then for each $\mathbb{P} \in \mathcal{P}$, there exists $\mathbb{Q} \in \mathcal{M}$ such that $\mathbb{Q} \approx \mathbb{P}$. Conversely, for each $\mathbb{Q} \in \mathcal{M}$, there exists $\mathbb{P} \in \mathcal{P}$ such that $\mathbb{P} \approx \mathbb{Q}$.*

Proof Let $\mathbb{P} \in \mathcal{P}$ and consider the canonical decomposition of S under \mathbb{P} ,

$$S_t = \int_0^t b_s^{\mathbb{P}} ds + M_t^{\mathbb{P}}, \quad 0 \leq t \leq T,$$

where $M^{\mathbb{P}}$ is a continuous local \mathbb{P} -martingale with $\frac{d\langle M^{\mathbb{P}} \rangle}{dt} = c^{\mathbb{P}}$. Then Assumption 3.1 guarantees that the stochastic process

$$Z_t := \mathcal{E}\left(\int_0^t -(c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dM_s^{\mathbb{P}}\right), \quad 0 \leq t \leq T, \tag{5.1}$$

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential, is well defined and e.g. by applying Novikov’s condition, one sees that Z defines a strictly positive continuous \mathbb{P} -martingale. This uses that Assumption 3.1 guarantees that $(c^{\mathbb{P}})^{-1}$ and $b^{\mathbb{P}}$ are bounded. Therefore, one can define a measure $\mathbb{Q} \approx \mathbb{P}$ using $(Z_t)_{0 \leq t \leq T}$ as density process. Moreover, Girsanov’s theorem and Remark 5.1 ensure that $\mathbb{Q} \in \mathcal{M}$.

Conversely, let $\mathbb{Q} \in \mathcal{M}$. By Neufeld and Nutz [26, Theorem 2.6], there exists an \mathbb{F} -predictable process c such that $c = c^{\mathbb{Q}} \mathbb{Q} \otimes dt$ -a.s. Consider the set

$$\Upsilon := \{(\omega, t) \in \Omega \times [0, T] : \exists b \in \mathbb{R}^d \text{ with } (b, c_t(\omega)) \in \Theta\}.$$

Since $\Theta \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$ is compact (and hence closed) by Assumption 3.1 and the map $(\Omega \times [0, T]) \times \mathbb{R}^d \ni (\omega, t, b) \mapsto (b, c_t(\omega)) \in \mathbb{R}^d \times \mathbb{S}_+^d$ is a Carathéodory function, the measurable implicit function theorem (see Rockafellar [40, Theorem 14.16]) ensures that $\Upsilon \subseteq \Omega \times [0, T]$ is an element of the \mathbb{F} -predictable σ -field and that there exists an \mathbb{F} -predictable \mathbb{R}^d -valued stochastic process $(b_t)_{t \in [0, T]}$ such that

$$(b_t(\omega), c_t(\omega)) \in \Theta \quad \text{for all } (\omega, t) \in \Upsilon.$$

Using $c = c^{\mathbb{Q}} \mathbb{Q} \otimes dt$ -a.s., $\Theta_c = \text{proj}_c(\Theta)$ and $\mathbb{Q} \in \mathcal{M}$ implies that Υ has $\mathbb{Q} \otimes dt$ -full measure. Next, similarly as above, Assumption 3.1 guarantees that the process

$$\tilde{Z}_t := \mathcal{E}\left(\int_0^t c_s^{-1} b_s dS_s\right), \quad 0 \leq t \leq T, \tag{5.2}$$

is well defined and a strictly positive continuous \mathbb{Q} -martingale. Hence one can define a measure $\mathbb{P} \approx \mathbb{Q}$ using $(\tilde{Z}_t)_{t \in [0, T]}$ as density process. Moreover, Girsanov’s theorem ensures that the process $M^{\mathbb{P}} := S - \int_0^\cdot b_s ds$ is a local \mathbb{P} -martingale. This in turn

shows that

$$S_t = S_t - \int_0^t b_s ds + \int_0^t b_s ds = M_t^{\mathbb{P}} + \int_0^t b_s ds, \quad t \in [0, T],$$

which implies that $\mathbb{P} \in \mathcal{P}$. □

As a consequence of the above lemma, we obtain the following observation.

Remark 5.3 Let Assumption 3.1 hold. Then Lemma 5.2 implies that the collection $\mathcal{N}^{\mathcal{P}}$ of all sets which are \mathcal{F}_T - \mathbb{P} -null for every $\mathbb{P} \in \mathcal{P}$ coincides with the analogous set $\mathcal{N}^{\mathcal{M}}$. In particular, recalling (3.2), we see that

$$\mathcal{G}_t = \bigcap_{s>t} (\mathcal{F}_s^* \vee \mathcal{N}^{\mathcal{M}}), \quad 0 \leq t \leq T.$$

Lemma 5.4 Let Assumption 3.1 hold. Then there exists $\widehat{c} \in \mathbb{S}_{++}^d$ which is diagonal and satisfies $\widehat{c} \leq c$ for all $c \in \Theta_c$.

Proof Due to Assumption 3.1, there exists $\underline{c} \in \mathbb{S}_{++}^d$ with $\underline{c} \leq c$ for all $c \in \Theta_c$. Let $\lambda_{\min}(\underline{c}) > 0$ be the smallest eigenvalue of \underline{c} . Then we define $\widehat{c} = (\widehat{c}^{ij})_{i,j \in \{1, \dots, d\}}$ by

$$\widehat{c}^{ij} := \lambda_{\min}(\underline{c}) \text{Id}_{\mathbb{R}^{d \times d}} = \begin{cases} \lambda_{\min}(\underline{c}) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Clearly, \widehat{c} is in \mathbb{S}_{++}^d and diagonal. Moreover, any eigenvalue of $\underline{c} - \widehat{c}$ is of the form $\lambda_i - \lambda_{\min}(\underline{c})$ for some eigenvalue λ_i of \underline{c} . This implies $\lambda_{\min}(\underline{c} - \widehat{c}) = 0$, which ensures that $\widehat{c} \leq \underline{c}$. □

Lemma 5.5 Let Assumption 3.1 hold and let $(H_t)_{t \in [0, T]}$ be a \mathbb{G} -predictable process. Then $(H_t)_{t \in [0, T]}$ is S -integrable with respect to \mathbb{P} for all $\mathbb{P} \in \mathcal{P}$ if and only if $(H_t)_{t \in [0, T]}$ is S -integrable with respect to \mathbb{Q} for all $\mathbb{Q} \in \mathcal{M}$.

Proof For the “only if” part, assume that $(H_t)_{t \in [0, T]}$ is S -integrable with respect to every $\mathbb{P} \in \mathcal{P}$ and let $\mathbb{Q} \in \mathcal{M}$. By Lemma 5.2, there exists $\mathbb{P} \in \mathcal{P}$ with $\mathbb{P} \approx \mathbb{Q}$. Let

$$S = S_0 + M^{\mathbb{P}} + \int_0^\cdot b_s^{\mathbb{P}} ds$$

be the canonical decomposition of S under \mathbb{P} , where $M^{\mathbb{P}}$ is a local \mathbb{P} -martingale with second differential characteristic $c^{\mathbb{P}} = (c^{ij, \mathbb{P}})_{i,j \in \{1, \dots, d\}}$. Note that we have $c^{\mathbb{Q}} = c^{\mathbb{P}} \mathbb{Q} \otimes dt$ -a.s. because $\mathbb{Q} \approx \mathbb{P}$. As $H = (H^{(1)}, \dots, H^{(d)})$ is S -integrable with respect to \mathbb{P} and $\mathbb{P} \approx \mathbb{Q}$, we thus have by Jacod and Shiryaev [15, Definition III.6.17] that \mathbb{Q} -a.s. (and \mathbb{P} -a.s.)

$$\int_0^T \sum_{i,j=1}^d H_s^{(i)} c_s^{ij, \mathbb{Q}} H_s^{(j)} ds = \int_0^T \sum_{i,j=1}^d H_s^{(i)} c_s^{ij, \mathbb{P}} H_s^{(j)} ds < \infty.$$

By [15, Theorem III.6.4], we obtain that H is S -integrable with respect to \mathbb{Q} .

Conversely, assume now that $(H_t)_{t \in [0, T]}$ is S -integrable with respect to every $\mathbb{Q} \in \mathcal{M}$ and let $\mathbb{P} \in \mathcal{P}$. Moreover, let

$$S = S_0 + M^{\mathbb{P}} + \int_0^\cdot b_s^{\mathbb{P}} ds$$

be the canonical decomposition of S under \mathbb{P} , where $M^{\mathbb{P}}$ is a local \mathbb{P} -martingale with second differential characteristic $c^{\mathbb{P}} = (c^{ij, \mathbb{P}})_{i, j \in \{1, \dots, d\}}$. By Lemma 5.2, there exists $\mathbb{Q} \in \mathcal{M}$ with $\mathbb{Q} \approx \mathbb{P}$. Moreover, due to Assumption 3.1, we know from Lemma 5.4 that there exists $\widehat{c} \in \mathbb{S}_{++}^d$ which is diagonal and satisfies $\widehat{c} \leq c$ for all $c \in \Theta_c$. Therefore, by [15, Theorem III.6.4], we have \mathbb{Q} -a.s. that

$$\begin{aligned} \sum_{i=1}^d \left(\widehat{c}^{ii} \int_0^T |H_s^{(i)}|^2 ds \right) &= \int_0^T \sum_{i, j=1}^d H_s^{(i)} \widehat{c}^{ij} H_s^{(j)} ds \\ &\leq \int_0^T \sum_{i, j=1}^d H_s^{(i)} c_s^{ij, \mathbb{Q}} H_s^{(j)} ds < \infty. \end{aligned}$$

This and the fact that $\widehat{c}^{ii} > 0$ for each i imply that each summand on the left-hand side is nonnegative and hence finite \mathbb{Q} -a.s. In particular, we have for each $i \in \{1, \dots, d\}$ that \mathbb{Q} -a.s. (and hence also \mathbb{P} -a.s.),

$$\int_0^T |H_s^{(i)}|^2 ds < \infty. \tag{5.3}$$

Moreover, the assumption that Θ is compact (and hence bounded) ensures that we have $\mathcal{K} := \sup_{(b, c) \in \Theta} [\|b\| + \|c\|] < \infty$. This, (5.3), the fact that $c^{\mathbb{P}} = c^{\mathbb{Q}} \mathbb{P} \otimes dt$ -a.s., the Cauchy–Schwarz inequality and [15, Theorem III.6.4] imply that \mathbb{P} -a.s. (and \mathbb{Q} -a.s.),

$$\begin{aligned} &\int_0^T \left| \sum_{i=1}^d H_s^{(i)} b_s^{i, \mathbb{P}} \right| ds + \int_0^T \sum_{i=1}^d H_s^{(i)} c_s^{ij, \mathbb{P}} H_s^{(j)} ds \\ &\leq \int_0^T \mathcal{K} \sqrt{\sum_{i=1}^d |H_s^{(i)}|^2} ds + \int_0^T \sum_{i=1}^d H_s^{(i)} c_s^{ij, \mathbb{Q}} H_s^{(j)} ds \\ &\leq \mathcal{K} \int_0^T \left(1 + \sum_{i=1}^d |H_s^{(i)}|^2 \right) ds + \int_0^T \sum_{i=1}^d H_s^{(i)} c_s^{ij, \mathbb{Q}} H_s^{(j)} ds < \infty. \end{aligned}$$

By [15, Definition III.6.17], we hence get that $(H_t)_{t \in [0, T]}$ is S -integrable with respect to \mathbb{P} , which finishes the proof. \square

The following lemma is one of the two main tools to verify that the bipolar relations of \mathcal{C} and \mathcal{D} assumed in Theorems 2.10 and 2.16 hold. It states that on

$\Omega = C([0, T]; \mathbb{R}^d)$, the set of separating measures, which coincides with the set of local martingale measures as S has continuous sample paths, is already characterised by the separation of continuous functions.

Lemma 5.6 *Let*

$$\Gamma := \{\gamma \in C_b(\Omega) : \text{there exists } H \in \mathcal{H} \text{ such that } \gamma \leq (H \cdot S)_T\},$$

and let $\mathbb{Q} \in \mathfrak{P}(\Omega)$ be such that $\mathbb{E}_{\mathbb{Q}}[\gamma] \leq 0$ for all $\gamma \in \Gamma$. Then $(S_t)_{0 \leq t \leq T}$ is a local (\mathbb{Q}, \mathbb{F}) -martingale.

Proof This follows directly from Proposition A.2 and Remark A.1, where the latter is a slight modification of [3, Proposition 5.5] and [4, Proposition 4.4]. \square

The following two results are consequences of Lemma 5.6.

Proposition 5.7 *We have that*

$$\mathfrak{M}_e(\mathcal{P}) = \mathcal{D} = \{\mathbb{Q} \in \mathfrak{P}_e(\mathcal{P}) : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } X \in \mathcal{C} \cap C_b\}.$$

In addition, if Assumption 3.1 holds, then we also have

$$\mathcal{M} = \mathfrak{M}_e(\mathcal{P}).$$

Proof Throughout this proof, let

$$(\mathcal{C} \cap C_b)^\circ := \{\mathbb{Q} \in \mathfrak{P}_e(\mathcal{P}) : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } X \in \mathcal{C} \cap C_b\}.$$

Now, to see that $\mathfrak{M}_e(\mathcal{P}) \subseteq \mathcal{D}$, let $\mathbb{Q} \in \mathfrak{M}_e(\mathcal{P})$ and $X \in \mathcal{C}$. Then there exists $\mathbb{P} \in \mathcal{P}$ with $\mathbb{P} \approx \mathbb{Q}$. This implies that there exists $H \in \mathcal{H}$ with $X \leq 1 + (H \cdot S)_T$ \mathbb{Q} -a.s. Therefore, since $H \cdot S$ is a \mathbb{Q} -supermartingale by the definition of the set \mathcal{H} , we conclude that $\mathbb{E}_{\mathbb{Q}}[X] \leq 1$. Further, $\mathcal{D} \subseteq (\mathcal{C} \cap C_b)^\circ$ follows directly from the definition of $(\mathcal{C} \cap C_b)^\circ$.

To see that $\mathfrak{M}_e(\mathcal{P}) \supseteq (\mathcal{C} \cap C_b)^\circ$, let $\mathbb{Q} \in (\mathcal{C} \cap C_b)^\circ$. By definition, $\mathbb{Q} \in \mathfrak{P}_e(\mathcal{P})$. Now, for each $\gamma \in \Gamma \subseteq C_b(\Omega)$, there exists $c \geq 0$ such that $c + \gamma \geq 0$. This implies that $\frac{1}{c}(c + \gamma) \in \mathcal{C} \cap C_b$. This in turn implies that $\mathbb{E}_{\mathbb{Q}}[\frac{1}{c}(c + \gamma)] \leq 1$ which is equivalent to $\mathbb{E}_{\mathbb{Q}}[\gamma] \leq 0$. By Lemma 5.6, we get that \mathbb{Q} is a local martingale measure for S . This and the fact that $\mathbb{Q} \in \mathfrak{P}_e(\mathcal{P})$ imply that $\mathbb{Q} \in \mathfrak{M}_e(\mathcal{P})$. Hence we have shown that

$$\mathfrak{M}_e(\mathcal{P}) = \mathcal{D} = (\mathcal{C} \cap C_b)^\circ.$$

Finally, if Assumption 3.1 holds, then $\mathcal{M} \subseteq \mathfrak{M}_e(\mathcal{P})$ follows directly from Lemma 5.2. Conversely, $\mathfrak{M}_e(\mathcal{P}) \subseteq \mathcal{M}$ follows by Girsanov's theorem for semimartingales [15, Theorem III.3.24] and the fact that a semimartingale with continuous sample paths is a local martingale if and only if its predictable finite-variation part vanishes. \square

Proposition 5.8 *Let Assumption 3.1 hold. Then both $\mathcal{P}, \mathcal{D} \subseteq \mathfrak{P}(\Omega)$ are convex and compact.*

Proof By definition, $\mathcal{M} = \mathcal{P}_{\text{sem}}^{\text{ac}}(\tilde{\Theta})$, where $\tilde{\Theta} := \{0, \dots, 0\} \times \Theta_c \subseteq \mathbb{R}^d \times \mathbb{S}_+^d$ and $\Theta_c := \text{proj}_c(\Theta) \subseteq \mathbb{S}_+^d$. Moreover, as Θ is convex and compact by assumption, so is $\tilde{\Theta}$. Therefore the compactness of \mathcal{P} and \mathcal{M} follows directly from Liu and Neufeld [22, Theorem 2.5], whereas the convexity of \mathcal{P} and \mathcal{M} follows by [15, Theorem III.3.40]. In addition, we know by Proposition 5.7 that $\mathcal{M} = \mathcal{D}$, which finishes the proof. \square

The following lemma is the second crucial tool to prove the bipolar relation between \mathcal{C} and \mathcal{D} . It heavily uses the fact that one can construct a process Y which is a \mathbb{Q} -supermartingale for every $\mathbb{Q} \in \mathcal{M}$, as well as the robust optional decomposition theorem.

Proposition 5.9 *Let Assumption 3.1 hold. Then we have that*

$$\{X \in C_b^+ : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } \mathbb{Q} \in \mathcal{D}\} = \mathcal{C} \cap C_b.$$

Proof By definition, $\{X \in C_b^+ : \mathbb{E}_{\mathbb{Q}}[X] \leq 1 \text{ for all } \mathbb{Q} \in \mathcal{D}\} \supseteq \mathcal{C} \cap C_b$. To see the opposite inclusion, let $X \in C_b^+$ be such that $\mathbb{E}_{\mathbb{Q}}[X] \leq 1$ for all $\mathbb{Q} \in \mathcal{D}$. Since X is nonnegative, bounded and continuous (and so Borel), and since by Neufeld and Nutz [27, Theorem 2.1] the set \mathcal{M} satisfies the so-called Condition (A) (see [27] or Nutz [34] for the precise definition), we can apply the same argument as in the proofs of [34, Theorem 3.2] and Neufeld and Nutz [25, Theorem 2.3] and use Remark 5.3 to obtain a \mathbb{G} -adapted nonnegative process $(Y_t)_{0 \leq t \leq T}$ with càdlàg sample paths which is a (\mathbb{Q}, \mathbb{G}) -supermartingale for every $\mathbb{Q} \in \mathcal{M}$ and satisfies

$$\begin{aligned} Y_0 &\leq \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[X], \\ Y_T &= X \quad \mathbb{Q}\text{-a.s. for all } \mathbb{Q} \in \mathcal{M}. \end{aligned} \tag{5.4}$$

Moreover, since the set \mathcal{M} is saturated (in the sense of [34], see also [34, Lemma 4.2]), the robust optional decomposition theorem [34, Theorem 2.4] ensures the existence of a \mathbb{G} -predictable process H such that H is S -integrable for all $\mathbb{Q} \in \mathcal{M}$ and

$$Y - (H \cdot S) \text{ is nonincreasing} \quad \mathbb{Q}\text{-a.s. for all } \mathbb{Q} \in \mathcal{M}. \tag{5.5}$$

Combining this, (5.4) and the fact that $\mathcal{M} = \mathcal{D}$ by Proposition 5.7 implies that

$$1 + (H \cdot S)_T \geq Y_0 + (H \cdot S)_T \geq Y_T = X \quad \mathbb{Q}\text{-a.s. for all } \mathbb{Q} \in \mathcal{M}. \tag{5.6}$$

Moreover, for any $\mathbb{Q} \in \mathcal{M}$, we use (5.4), (5.5), that $Y \geq 0$ is a \mathbb{Q} -supermartingale and that $\mathcal{M} = \mathcal{D}$ to see that

$$(H \cdot S)_t \geq Y_t - Y_0 \geq \mathbb{E}_{\mathbb{Q}}[X | \mathcal{G}_t] - 1 \geq -1 \quad \mathbb{Q}\text{-a.s.}$$

for all $t \in [0, T]$. Therefore, we conclude that $H \cdot S \geq -1$ \mathcal{M} -q.s., which by Lemma 5.2 implies that $H \cdot S \geq -1$ \mathcal{P} -q.s. This and Lemma 5.5 ensure that $H \in \mathcal{H}$.

Moreover, Lemma 5.2 and (5.6) ensure that

$$1 + (H \cdot S)_T \geq X \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P},$$

which by definition shows that $X \in \mathcal{C}$. As $X \in C_b^+$ by assumption, we get $X \in \mathcal{C} \cap C_b$. \square

Now we are able to finish the proof of Theorem 3.4.

Proof of Theorems 3.4 and 3.5 We verify that Assumption 2.1, the bipolar relation of \mathcal{C} and \mathcal{D} and the convex-compactness assumption on \mathcal{P} and \mathcal{D} are satisfied. To that end, note that Lemma 5.2 ensures that Assumption 2.1 holds. Moreover, Proposition 5.8 yields that \mathcal{P} and \mathcal{D} (with $\mathfrak{P} := \mathfrak{P}_e(\mathcal{P})$) are both convex and compact (compare 1) in Theorem 2.10). In addition, we get from Propositions 5.7 and 5.9 that the bipolar relation of \mathcal{C} and \mathcal{D} (see 2) and 3) in Theorem 2.10) hold. Therefore, the result now follows directly from Theorems 2.10 and 2.16, respectively. \square

5.2 Proofs of Corollaries 3.6 and 3.7

The idea for this is to verify in the setting of Sect. 3 that Assumptions 2.4 and 2.7, as well as Assumptions 2.8 and 2.14, hold for the specific utility functions. Then the result immediately follows from Theorems 3.4 and 3.5.

First, note that every utility function which is bounded from above automatically satisfies Assumptions 2.4 and 2.7, as well as Assumptions 2.8 and 2.14; see also Remarks 2.9 and 2.15. Therefore, we only have to focus on the utility functions $U(x) := \log x$ and $U(x) := \frac{x^p}{p}$, $p \in (0, 1)$.

Moreover, observe that due to Assumption 3.1, we have that

$$\mathcal{K} := 1 + \sup_{(b,c) \in \Theta} (\|b\| + \|c\| + \|c^{-1}\|) < \infty. \quad (5.7)$$

The following lemma will be used several times in this subsection.

Lemma 5.10 *Let Assumption 3.1 hold. Then for every $\mathbb{P} \in \mathcal{P}$, there is $\mathbb{Q} \in \mathfrak{M}_e(\mathcal{P})$ such that for every $\delta \in (0, \infty)$,*

$$\mathbb{E}_{\mathbb{Q}} \left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right)^\delta \right] < \infty.$$

Proof Let $\mathbb{P} \in \mathcal{P}$. Jensen's inequality, Lemma 5.2 and Proposition 5.7 ensure that the statement holds for $\delta \in (0, 1]$; hence we only need to focus on the case $\delta > 1$. Note that from the proof of Lemma 5.2, see (5.1) and (5.2), we know that there exists $\mathbb{Q} \in \mathfrak{M}_e(\mathcal{P})$ such that

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \mathcal{E} \left(\int_0^T (c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dS \right) \quad \mathbb{Q}\text{-a.s.},$$

where $(b^{\mathbb{P}}, c^{\mathbb{P}})$ denote the differential characteristics of S under \mathbb{P} . This and the fact that $c^{\mathbb{Q}} = c^{\mathbb{P}} \mathbb{Q} \otimes dt$ -a.s. imply for every $\delta > 1$ that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[\mathcal{E} \left(\int_0^T (c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dS \right)^{\delta} \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\exp \left(\int_0^T (c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dS - \frac{1}{2} \int_0^T \sum_{i,j=1}^d b_s^{i,\mathbb{P}} ((c_s^{\mathbb{P}})^{-1})^{ij} b_s^{j,\mathbb{P}} ds \right)^{\delta} \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\exp \left(\int_0^T \delta (c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dS - \frac{1}{2} \int_0^T \sum_{i,j=1}^d \delta^2 b_s^{i,\mathbb{P}} ((c_s^{\mathbb{P}})^{-1})^{ij} b_s^{j,\mathbb{P}} ds \right) \right. \\ &\quad \left. \times \exp \left(\frac{1}{2} (\delta^2 - \delta) \int_0^T \sum_{i,j=1}^d b_s^{i,\mathbb{P}} ((c_s^{\mathbb{P}})^{-1})^{ij} b_s^{j,\mathbb{P}} ds \right) \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[\mathcal{E} \left(\int_0^T \delta (c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dS \right) \exp \left(\frac{1}{2} (\delta^2 - \delta) \int_0^T \sum_{i,j=1}^d b_s^{i,\mathbb{P}} ((c_s^{\mathbb{P}})^{-1})^{ij} b_s^{j,\mathbb{P}} ds \right) \right]. \end{aligned}$$

This, the fact that we have (5.7) by Assumption 3.1 and the fact that S under \mathbb{Q} is a local martingale show that for every $\delta > 1$, we indeed have that

$$\begin{aligned} & \mathbb{E}_{\mathbb{Q}} \left[\mathcal{E} \left(\int_0^T (c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dS \right)^{\delta} \right] \\ & \leq \mathbb{E}_{\mathbb{Q}} \left[\mathcal{E} \left(\int_0^T \delta (c_s^{\mathbb{P}})^{-1} b_s^{\mathbb{P}} dS \right) \right] \exp \left(\frac{1}{2} (\delta^2 - \delta) T d^2 \mathcal{K}^3 \right) \\ & \leq \exp \left(\frac{1}{2} (\delta^2 - \delta) T d^2 \mathcal{K}^3 \right) < \infty. \quad \square \end{aligned}$$

Lemma 5.11 *Let Assumption 3.1 hold, and fix $x > 0$ and $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(x)$. Then for every $\mathbb{P} \in \mathcal{P}$ and every $\varepsilon \in (0, 1)$, we have that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} [(g_n)^{\varepsilon}] < \infty.$$

Proof Fix $\varepsilon \in (0, 1)$, $n \in \mathbb{N}$ and $\mathbb{P} \in \mathcal{P}$. By Lemma 5.10, there exists $\mathbb{Q} \in \mathfrak{M}_e(\mathcal{P})$ which satisfies for every $\delta \in (0, \infty)$ that $c(\delta) := \mathbb{E}_{\mathbb{Q}}[(\frac{d\mathbb{P}}{d\mathbb{Q}})^{\delta}] < \infty$. Therefore, Hölder’s inequality (applied to $p := \frac{1}{1-\varepsilon}$, $q := \frac{1}{\varepsilon}$) and the fact that $\mathcal{D} = \mathfrak{M}_e(\mathcal{P})$ by Proposition 5.7 ensure that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} [(g_n)^{\varepsilon}] & \leq \mathbb{E}_{\mathbb{Q}} \left[\frac{d\mathbb{P}}{d\mathbb{Q}} (g_n)^{\varepsilon} \right] \leq \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right)^{1/(1-\varepsilon)} \right]^{(1-\varepsilon)} (\mathbb{E}_{\mathbb{Q}} [g_n])^{\varepsilon} \\ & \leq c(1/(1-\varepsilon))^{1-\varepsilon} x^{\varepsilon} < \infty. \quad \square \end{aligned}$$

Lemma 5.12 *Let Assumption 3.1 hold and let $U(x) := \log x$. Then for every $x > 0$ and every $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(x)$, the sequence of random variables*

$$\max \left\{ \log \left(g_n + \frac{1}{n} \right), 0 \right\}, \quad n \in \mathbb{N},$$

is \mathbb{P} -uniformly integrable for every $\mathbb{P} \in \mathcal{P}$.

Proof Fix $\mathbb{P} \in \mathcal{P}$, let $\varepsilon \in (0, 1)$ and define the function $\Psi: [0, \infty) \rightarrow [0, \infty)$ by $\Psi(x) = \exp(\varepsilon x)$. Then by the de la Vallée-Poussin theorem, it suffices to show that

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} \left[\Psi \left(\max \left\{ \log \left(g_n + \frac{1}{n} \right), 0 \right\} \right) \right] < \infty.$$

Since $x \mapsto \Psi(x) = \exp(\varepsilon x)$ is increasing, we have for every $n \in \mathbb{N}$ that

$$\begin{aligned} \Psi \left(\max \left\{ \log \left(g_n + \frac{1}{n} \right), 0 \right\} \right) &= \max \left\{ \Psi \left(\log \left(g_n + \frac{1}{n} \right) \right), \Psi(0) \right\} \\ &= \max \left\{ \left(g_n + \frac{1}{n} \right)^{\varepsilon}, 1 \right\}. \end{aligned}$$

Hence it suffices to show that

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} \left[\left(g_n + \frac{1}{n} \right)^{\varepsilon} \right] < \infty.$$

But as $(g_n + \frac{1}{n})^{\varepsilon} \leq (g_n)^{\varepsilon} + (\frac{1}{n})^{\varepsilon}$ for each $n \in \mathbb{N}$, it is enough to show that

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} [(g_n)^{\varepsilon}] < \infty. \quad (5.8)$$

Lemma 5.11 now implies that (5.8) holds. \square

Lemma 5.13 *Let Assumption 3.1 hold and let $U(x) := \frac{x^p}{p}$ for some $p \in (0, 1)$. Then for every $x > 0$ and every $(g_n)_{n \in \mathbb{N}} \subseteq \mathcal{C}(x)$, the sequence of random variables*

$$\frac{(g_n + \frac{1}{n})^p}{p}, \quad n \in \mathbb{N},$$

is \mathbb{P} -uniformly integrable for every $\mathbb{P} \in \mathcal{P}$.

Proof Fix $\mathbb{P} \in \mathcal{P}$, let $\varepsilon \in (p, 1)$ and define the function $\Psi: [0, \infty) \rightarrow [0, \infty)$ by $\Psi(x) = x^{\varepsilon/p}$. Then by the de la Vallée-Poussin theorem, it suffices to show that

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} \left[\Psi \left(\frac{(g_n + \frac{1}{n})^p}{p} \right) \right] < \infty.$$

For this, since $\Psi(x) = x^{\varepsilon/p}$, it suffices to show that $\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}} [(g_n)^{\varepsilon}] < \infty$, which follows directly from Lemma 5.11. \square

Now recall that

$$V_1(y) := \sup_{x \geq 0} (U_1(x) - xy), \quad y > 0,$$

where $U_1(x) := U(x + 1)$, $x \geq 0$. Then we have the following result.

Lemma 5.14 *For each $p \in (0, 1)$ and $y > 0$, we have that*

$$V_{1,\log}(y) := \sup_{x \geq 0} (\log(x + 1) - xy) \leq \log \frac{1}{y} - 1 + y,$$

$$V_{1,p}(y) := \sup_{x \geq 0} \left(\frac{(x + 1)^p}{p} - xy \right) \leq \left(\frac{1}{p} - 1 \right) \left(\frac{1}{y} \right)^{\frac{p}{1-p}} + y.$$

Proof We start with the log-case. For every $y > 0$, let $\widehat{x}_{1,\log}(y) := \frac{1}{y} - 1$. Then one sees, using the first-order condition, that for every $y > 0$,

$$\begin{aligned} V_{1,\log}(y) &\leq \sup_{x \geq -1} (\log(x + 1) - xy) = \log(\widehat{x}_{1,\log}(y) + 1) - \widehat{x}_{1,\log}(y)y \\ &= \log \frac{1}{y} - 1 + y. \end{aligned}$$

To see the result for the power-case, we set $\widehat{x}_{1,p}(y) := y^{\frac{1}{p-1}} - 1$ for all $y > 0$. Then, using the first-order condition, we get for every $y > 0$ that

$$\begin{aligned} V_{1,p}(y) &\leq \sup_{x \geq -1} \left(\frac{(x + 1)^p}{p} - xy \right) = \frac{(\widehat{x}_{1,p}(y) + 1)^p}{p} - \widehat{x}_{1,p}(y)y \\ &= \left(\frac{1}{p} - 1 \right) \left(\frac{1}{y} \right)^{\frac{p}{1-p}} + y. \quad \square \end{aligned}$$

Lemma 5.15 *Let Assumption 3.1 hold. Then for every $y > 0$ and every $\mathbb{P} \in \mathcal{P}$, there exists $\mathbb{Q} \in \mathcal{D}$ such that*

$$\mathbb{E}_{\mathbb{P}} \left[\max \left\{ V_{1,\log} \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right), 0 \right\} \right] < \infty.$$

Proof Let $y > 0$ and $\mathbb{P} \in \mathcal{P}$. By Proposition 5.7, we know that $\mathcal{D} = \mathfrak{M}_e(\mathcal{P})$. Moreover, by Lemma 5.10, there exists $\mathbb{Q} \in \mathfrak{M}_e(\mathcal{P})$ which satisfies for every $\delta \in (0, \infty)$ that $c(\delta) := \mathbb{E}_{\mathbb{Q}}[(\frac{d\mathbb{P}}{d\mathbb{Q}})^\delta] < \infty$. This and Lemma 5.14 imply that

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\max \left\{ V_{1,\log} \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right), 0 \right\} \right] &= \mathbb{E}_{\mathbb{P}} \left[\max \left\{ \log \left(\frac{1}{y} \frac{d\mathbb{P}}{d\mathbb{Q}} \right) - 1 + y \frac{d\mathbb{Q}}{d\mathbb{P}}, 0 \right\} \right] \\ &\leq \max \left\{ \log \frac{1}{y}, 0 \right\} + \mathbb{E}_{\mathbb{P}} \left[\max \left\{ \log \frac{d\mathbb{P}}{d\mathbb{Q}}, 0 \right\} \right] \\ &\quad + y \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right]. \end{aligned}$$

Since $\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}] = 1$, it hence suffices to show that $\mathbb{E}_{\mathbb{P}}[\max\{\log \frac{d\mathbb{P}}{d\mathbb{Q}}, 0\}] < \infty$. To see this, note that the fact that $\log x \leq 1 + x$ for all $x \geq 0$ and Lemma 5.10 yield

$$\begin{aligned} \mathbb{E}_{\mathbb{P}} \left[\max \left\{ \log \frac{d\mathbb{P}}{d\mathbb{Q}}, 0 \right\} \right] &= \mathbb{E}_{\mathbb{Q}} \left[\frac{d\mathbb{P}}{d\mathbb{Q}} \max \left\{ \log \frac{d\mathbb{P}}{d\mathbb{Q}}, 0 \right\} \right] \\ &\leq \mathbb{E}_{\mathbb{Q}} \left[\frac{d\mathbb{P}}{d\mathbb{Q}} \left(1 + \frac{d\mathbb{P}}{d\mathbb{Q}} \right) \right] < \infty. \quad \square \end{aligned}$$

Lemma 5.16 *Let Assumption 3.1 hold. Then for every $y > 0$ and every $\mathbb{P} \in \mathcal{P}$, there exists $\mathbb{Q} \in \mathcal{D}$ such that*

$$\mathbb{E}_{\mathbb{P}} \left[\max \left\{ V_{1,p} \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right), 0 \right\} \right] < \infty.$$

Proof Let $y > 0$ and $\mathbb{P} \in \mathcal{P}$. By Proposition 5.7, we know that $\mathcal{D} = \mathfrak{M}_e(\mathcal{P})$. Moreover, by Lemma 5.10, there exists $\mathbb{Q} \in \mathfrak{M}_e(\mathcal{P})$ which satisfies for every $\delta \in (0, \infty)$ that $c(\delta) := \mathbb{E}_{\mathbb{Q}}[(\frac{d\mathbb{P}}{d\mathbb{Q}})^\delta] < \infty$. This, the fact that $\mathbb{E}_{\mathbb{P}}[\frac{d\mathbb{Q}}{d\mathbb{P}}] = 1$ and Lemma 5.14 yield

$$\begin{aligned} &\mathbb{E}_{\mathbb{P}} \left[\max \left\{ V_{1,p} \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right), 0 \right\} \right] \\ &= \left(\frac{1}{p} - 1 \right) \left(\frac{1}{y} \right)^{\frac{p}{1-p}} \mathbb{E}_{\mathbb{P}} \left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right)^{\frac{p}{1-p}} \right] + y \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] \\ &= \left(\frac{1}{p} - 1 \right) \left(\frac{1}{y} \right)^{\frac{p}{1-p}} \mathbb{E}_{\mathbb{Q}} \left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right)^{1+\frac{p}{1-p}} \right] + y \mathbb{E}_{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right] < \infty. \quad \square \end{aligned}$$

Lemma 5.17 *Let the utility function U be either $U(x) := \log x$ or $U(x) := \frac{x^p}{p}$ for some $p \in (0, 1)$, and let Assumption 3.1 hold. Then for every $x > 0$, we have that $u(x) < \infty$. If in addition Assumption 2.5 holds, we also have $\bar{u}(x) < \infty$.*

Proof By Lemma 5.10, we know that for every \mathbb{P} , there exists $\mathbb{Q}_{\mathbb{P}} \in \mathfrak{M}_e(\mathcal{P})$ which satisfies for every $\delta \in (0, \infty)$ that $c(\delta) := \mathbb{E}_{\mathbb{Q}_{\mathbb{P}}}[(\frac{d\mathbb{P}}{d\mathbb{Q}_{\mathbb{P}}})^\delta] < \infty$. By the weak duality obtained in (4.6) and (4.7), the fact that $V(y) \leq V_1(y)$ for every $y \geq 0$ and Lemmas 5.15 and 5.16, we see that for every $x > 0, y > 0$,

$$u(x) \leq v(y) + xy \leq \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[\max \left\{ V_1 \left(y \frac{d\mathbb{Q}_{\mathbb{P}}}{d\mathbb{P}} \right), 0 \right\} \right] + xy < \infty.$$

If Assumption 2.5 holds in addition, the same arguments for the weak duality with respect to \bar{u} derived in (4.15) show that $\bar{u}(x) < \infty$ for all $x > 0$. \square

We are now able to provide the proof of Corollaries 3.6 and 3.7.

Proof of Corollaries 3.6 and 3.7 We verify that Assumptions 2.4 and 2.7, as well as 2.8 and 2.14, hold for the specific utility functions.

First, note that the utility functions $U(x) = \frac{x^p}{p}$, $p \in (0, 1)$, and $U(x) = -e^{-\lambda x}$, $\lambda > 0$, satisfy Assumption 2.2, while $U(x) := \log x$ and $U(x) := \frac{x^p}{p}$, $p \in (-\infty, 0)$, satisfy Assumption 2.12. Moreover, every utility function which is bounded from above automatically satisfies Assumptions 2.4 and 2.7, as well as 2.8 and 2.14; see Remarks 2.9 and 2.15. Therefore, we only need to show that Assumptions 2.4 and 2.7, as well as 2.8 and 2.14, hold for $U(x) := \log x$ and $U(x) = \frac{x^p}{p}$, $p \in (0, 1)$.

To that end, note that for these utility functions, Lemma 5.17 guarantees that Assumptions 2.4 and 2.7 hold; Lemmas 5.15 and 5.16 show that Assumption 2.14 holds; and Lemmas 5.12 and 5.13 ensure that Assumption 2.8 holds. Therefore, Corollary 3.6 directly follows from Theorem 3.4 together with the fact that Assumption 2.14 implies $v(y) < \infty$ for every $y > 0$, and Corollary 3.7 directly follows from Theorem 3.5. \square

Appendix A: Continuous separation

Throughout this section, we work in the framework of Sect. 3. We recall that $\Omega := C([0, T]; \mathbb{R}^d)$ is endowed with its Borel σ -field \mathcal{F} . Moreover, we let $(S_t)_{0 \leq t \leq T}$ be the canonical process. In addition, \mathbb{F} is the raw filtration generated by S , and \mathbb{F}_+ the corresponding right-continuous version of \mathbb{F} .

Let us denote by $\mathcal{H}_{s,d}(\mathbb{F}_+)$ the set of all d -dimensional \mathbb{F}_+ -simple processes $H: \Omega \times [0, T] \rightarrow \mathbb{R}^d$ which have the form $H_t(\omega) := \sum_{\ell=1}^L h_\ell(\omega) \mathbf{1}_{(\tau_\ell(\omega), \tau_{\ell+1}(\omega)]}(t)$, where $L \in \mathbb{N}$, $0 \leq \tau_1 \leq \dots \leq \tau_{L+1} \leq T$ are \mathbb{F}_+ -stopping times and for each ℓ , $h_\ell := (h_\ell^{(1)}, \dots, h_\ell^{(d)}): \Omega \rightarrow \mathbb{R}^d$ is bounded and $\mathcal{F}_{\tau_\ell+}$ -measurable. Furthermore we define for every $m \in \mathbb{N}$ the set

$$\mathcal{H}_{s,d,m}(\mathbb{F}_+) := \{H \in \mathcal{H}_{s,d}(\mathbb{F}_+): H \cdot S \geq -m \text{ pointwise on } \Omega \times [0, T]\}.$$

Remark A.1 Recall the filtration \mathbb{G} and the set \mathcal{H} of strategies introduced in Sect. 3. Then the fact that $\mathbb{F}_+ \subseteq \mathbb{G}$ implies that $\mathcal{H}_{s,d,m}(\mathbb{F}_+) \subseteq \mathcal{H}$ for each $m \in \mathbb{N}$.

The following result slightly extends [3, Proposition 5.5] and [4, Proposition 4.4].

Proposition A.2 *Consider the set*

$$\Gamma_d := \{\gamma \in C_b(\Omega) : \text{there exist } m \in \mathbb{N} \text{ and } H \in \mathcal{H}_{s,d,m}(\mathbb{F}_+) \text{ with } \gamma \leq (H \cdot S)_T\}.$$

Moreover, let $\mathbb{Q} \in \mathfrak{P}(\Omega)$ be such that $\mathbb{E}_{\mathbb{Q}}[\gamma] \leq 0$ for all $\gamma \in \Gamma_d$. Then S is a local (\mathbb{Q}, \mathbb{F}) -martingale.

Proof First of all, note that $S = (S^{(1)}, \dots, S^{(d)})$ is a d -dimensional local (\mathbb{Q}, \mathbb{F}) -martingale if and only if each component $S^{(i)}$ is a local (\mathbb{Q}, \mathbb{F}) -martingale. In addition, since each $H \in \mathcal{H}_{s,d,m}(\mathbb{F}_+)$ is simple, the stochastic integral $H \cdot S$ is well defined and satisfies $H \cdot S = \sum_{i=1}^d H^{(i)} \cdot S^{(i)}$. Now, for every $i \in \{1, \dots, d\}$, let

$$\mathcal{H}_{s,1,m}^{(i)}(\mathbb{F}_+) := \{H \in \mathcal{H}_{s,1}(\mathbb{F}_+): H \cdot S^{(i)} \geq -m \text{ pointwise on } \Omega \times [0, T]\}.$$

Then for every $i \in \{1, \dots, d\}$, we see that any $H \in \mathcal{H}_{s,1,m}^{(i)}(\mathbb{F}_+)$ can be extended to an element $H_d := (H_d^{(1)}, \dots, H_d^{(d)}) \in \mathcal{H}_{s,d,m}(\mathbb{F}_+)$ by setting

$$(H_d^{(1)}, \dots, H_d^{(i)}, \dots, H_d^{(d)}) := (0, \dots, H, \dots, 0)$$

which satisfies for any $\gamma \in C_b(\Omega)$ that $\gamma \leq (H_d \cdot S)_T$ if and only if $\gamma \leq (H \cdot S^{(i)})_T$. Therefore, we conclude that for each $i \in \{1, \dots, d\}$, we have

$$\Gamma^{(i)} := \{\gamma \in C_b(\Omega) : \text{there exist } m \in \mathbb{N}, H \in \mathcal{H}_{s,1,m}^{(i)}(\mathbb{F}_+) \text{ with } \gamma \leq (H \cdot S^{(i)})_T\} \subseteq \Gamma_d.$$

As a consequence, it suffices to prove for each $i \in \{1, \dots, d\}$ that if $\mathbb{Q} \in \mathfrak{P}(\Omega)$ satisfies $\mathbb{E}_{\mathbb{Q}}[\gamma] \leq 0$ for all $\gamma \in \Gamma^{(i)}$, then $S^{(i)}$ is a local (\mathbb{Q}, \mathbb{F}) -martingale.

Therefore, we fix any component $\mathcal{S} := S^{(i)}$ and assume that $\mathbb{E}_{\mathbb{Q}}[\gamma] \leq 0$ for all $\gamma \in \Gamma^{(i)}$. We want to show that \mathcal{S} is a local (\mathbb{Q}, \mathbb{F}) -martingale with localising sequence

$$\tau_m := \inf\{t \geq 0 : |\mathcal{S}_t| \geq m\} \wedge T.$$

We follow the arguments in [3, Proposition 5.5] and [4, Proposition 4.4]. Fix $m \in \mathbb{N}$ and write $\tau := \tau_m$. We first show that \mathcal{S}^τ is an \mathbb{F} -supermartingale. To that end, let $0 \leq s < t \leq T$, and define, for every $0 < \varepsilon \leq 1$,

$$\begin{aligned} \sigma &:= \inf\{r \geq s : |\mathcal{S}_r| \geq m\} \wedge T, \\ \sigma_\varepsilon &:= \inf\{r \geq s : \mathcal{S}_r > m - \varepsilon \text{ or } \mathcal{S}_r \leq -m + \varepsilon\} \wedge T. \end{aligned}$$

Since both τ and σ are hitting times of a closed set and \mathcal{S} has continuous sample paths, they are \mathbb{F} -stopping times, whereas σ_ε are \mathbb{F}_+ -stopping times for all $0 < \varepsilon < 1$.

Now fix an arbitrary \mathcal{F}_s -measurable function $h : \Omega \rightarrow [0, 1]$. Notice that $\sigma = \tau$ on $\{\tau \geq s\}$, so that $\mathbf{1}_{\{\tau \geq s\}}(\mathcal{S}_t^\sigma - \mathcal{S}_s) = \mathcal{S}_t^\tau - \mathcal{S}_s^\tau$. Moreover, σ_ε increases to σ as ε tends to 0, and therefore $\mathcal{S}_t^{\sigma_\varepsilon} \rightarrow \mathcal{S}_t^\sigma$ as \mathcal{S} has continuous sample paths. Since additionally $|\mathcal{S}_t^{\sigma_\varepsilon} - \mathcal{S}_s| \leq 2m$, we have

$$\mathbb{E}_{\mathbb{Q}}[h(\mathcal{S}_t^\tau - \mathcal{S}_s^\tau)] = \mathbb{E}_{\mathbb{Q}}[h\mathbf{1}_{\{\tau \geq s\}}(\mathcal{S}_t^\sigma - \mathcal{S}_s)] = \lim_{\varepsilon \rightarrow 0} \mathbb{E}_{\mathbb{Q}}[h\mathbf{1}_{\{\tau \geq s\}}(\mathcal{S}_t^{\sigma_\varepsilon} - \mathcal{S}_s)]. \tag{A.1}$$

Recall that $g := h\mathbf{1}_{\{\tau \geq s\}} : \Omega \rightarrow [0, 1]$ is \mathcal{F}_s -measurable. Thus by [3, Lemma 5.3], there exists a sequence of continuous \mathcal{F}_s -measurable functions $g_k : \Omega \rightarrow [0, 1]$ which converge \mathbb{Q} -almost surely to g . Moreover, as $\mathcal{S} : \Omega \rightarrow C([0, T]; \mathbb{R})$ is continuous, we get from [3, Lemma 5.4] that for every $0 < \varepsilon < 1$, the function

$$\Omega \ni \omega \mapsto \mathcal{S}_{t \wedge \sigma_\varepsilon(\omega)}(\omega) \in \mathbb{R}$$

is lower semicontinuous. For every fixed $k \in \mathbb{N}$, it holds for $H := g_k \mathbf{1}_{((s, \sigma_\varepsilon \wedge t])}$ that

$$\Omega \ni \omega \mapsto (H \cdot \mathcal{S})_T(\omega) \in \mathbb{R} \text{ is lower semicontinuous.} \tag{A.2}$$

Moreover, the fact that $|\mathcal{S}_t - \mathcal{S}_s| \leq 2m$ on $\llbracket s, \sigma_\varepsilon \rrbracket$ and g_k has values in $[0, 1]$ implies that $H \cdot \mathcal{S} \in [-2m, 2m]$ and so

$$H \in \mathcal{H}_{s,1,2m}^{(i)}(\mathbb{F}_+).$$

In addition, observe that (A.2) ensures that there exists a sequence of bounded continuous functions $\gamma_n: \Omega \rightarrow \mathbb{R}$ such that $\gamma_n \leq (H \cdot \mathcal{S})_T$ and γ_n increases pointwise to $(H \cdot \mathcal{S})_T$. Therefore we have for each $n \in \mathbb{N}$ that $\gamma_n \in \Gamma^{(i)}$, and so by assumption we have for every $\varepsilon \in (0, 1)$, $k \in \mathbb{N}$ that

$$\mathbb{E}_{\mathbb{Q}}[g_k(\mathcal{S}_t^{\sigma_\varepsilon} - \mathcal{S}_s)] = \mathbb{E}_{\mathbb{Q}}[(H \cdot \mathcal{S})_T] = \sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{Q}}[\gamma_n] \leq 0.$$

We hence conclude from (A.1) that

$$\mathbb{E}_{\mathbb{Q}}[h(\mathcal{S}_t^\tau - \mathcal{S}_s^\tau)] = \lim_{\varepsilon \rightarrow 0} \mathbb{E}_{\mathbb{Q}}[h \mathbf{1}_{\{\tau \geq s\}}(\mathcal{S}_t^{\sigma_\varepsilon} - \mathcal{S}_s)] = \lim_{\varepsilon \rightarrow 0} \lim_{k \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[g_k(\mathcal{S}_t^{\sigma_\varepsilon} - \mathcal{S}_s)] \leq 0.$$

This in turn implies \mathbb{Q} -a.s. that $\mathbb{E}_{\mathbb{Q}}[\mathcal{S}_t^\tau | \mathcal{F}_s] \leq \mathcal{S}_s^\tau$, and so \mathcal{S}^τ is indeed a (\mathbb{Q}, \mathbb{F}) -supermartingale.

By similar arguments, one can also show that \mathcal{S}^τ is a (\mathbb{Q}, \mathbb{F}) -submartingale. Thus we conclude that \mathcal{S} is indeed a (\mathbb{Q}, \mathbb{F}) -martingale. \square

References

1. Bartl, D.: Exponential utility maximization under model uncertainty for unbounded endowments. *Ann. Appl. Probab.* **29**, 577–612 (2019)
2. Bartl, D., Cheridito, P., Kupper, M.: Robust expected utility maximization with medial limits. *J. Math. Anal. Appl.* **471**, 752–775 (2019)
3. Bartl, D., Kupper, M., Neufeld, A.: Pathwise superhedging on prediction sets. *Finance Stoch.* **24**, 215–248 (2020)
4. Bartl, D., Kupper, M., Prömel, D.J., Tangpi, L.: Duality for pathwise superhedging in continuous time. *Finance Stoch.* **23**, 697–728 (2019)
5. Biagini, S., Pinar, M.Ç.: The robust Merton problem of an ambiguity averse investor. *Math. Finance Econ.* **11**, 1–24 (2017)
6. Blanchard, R., Carassus, L.: Multiple-priors optimal investment in discrete time for unbounded utility function. *Ann. Appl. Probab.* **28**, 1856–1892 (2018)
7. Brannath, W., Schachermayer, W.: A bipolar theorem for $L_0^+(\mathbb{P})$. In: Azéma, J., et al. (eds.) *Séminaire de Probabilités XXXIII. Lecture Notes in Math.*, vol. 1709, pp. 349–354. Springer, Berlin (1999)
8. Chau, H.N., Rásonyi, M.: Robust utility maximisation in markets with transaction costs. *Finance Stoch.* **23**, 677–696 (2019)
9. Delbaen, F., Schachermayer, W.: *The Mathematics of Arbitrage*. Springer, Berlin (2006)
10. Denis, L., Kervarec, M.: Optimal investment under model uncertainty in nondominated models. *SIAM J. Control Optim.* **51**, 1803–1822 (2013)
11. El Karoui, N., Quenez, M.C.: Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control Optim.* **33**, 29–66 (1995)
12. Fouque, J.P., Pun, C.S., Wong, H.Y.: Portfolio optimization with ambiguous correlation and stochastic volatilities. *SIAM J. Control Optim.* **54**, 2309–2338 (2016)
13. Guo, I., Langrené, N., Loeper, G., Ning, W.: Robust utility maximization under model uncertainty via a penalization approach. Preprint (2020). [arXiv:1907.13345](https://arxiv.org/abs/1907.13345)
14. Ismail, A., Pham, H.: Robust Markowitz mean–variance portfolio selection under ambiguous covariance matrix. *Math. Finance* **29**, 174–207 (2019)
15. Jacod, J., Shiryaev, A.N.: *Limit Theorems for Stochastic Processes*, 2nd edn. Springer, Berlin (2003)

16. Kabanov, Yu.: On the FTAP of Kreps–Delbaen–Schachermayer. In: Kabanov, Yu., et al. (eds.) *Statistics and Control of Random Processes. The Liptser Festschrift. Proceedings of Steklov Mathematical Institute Seminar*, pp. 191–203. World Scientific, Singapore (1997)
17. Kramkov, D.O.: Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probab. Theory Relat. Fields* **105**, 459–479 (1996)
18. Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab.* **9**, 904–950 (1999)
19. Liang, Z., Ma, M.: Robust consumption–investment problem under CRRA and CARA utilities with time-varying confidence sets. *Math. Finance* **30**, 1035–1072 (2020)
20. Lin, Q., Riedel, F.: Optimal consumption and portfolio choice with ambiguous interest rates and volatility. *Econ. Theory* **71**, 1189–1202 (2021)
21. Lin, Q., Sun, X., Zhou, C.: Horizon-unbiased investment with ambiguity. *J. Econ. Dyn. Control* **114**, 103896 (2020)
22. Liu, C., Neufeld, A.: Compactness criterion for semimartingale laws and semimartingale optimal transport. *Trans. Am. Math. Soc.* **372**, 187–231 (2019)
23. Matoussi, A., Possamaï, D., Zhou, C.: Robust utility maximization in non-dominated models with 2BSDEs: the uncertain volatility model. *Math. Finance* **25**, 258–287 (2015)
24. Meyer, P.A.: Limites médiales, d’après Mokobodzki. In: Dellacherie, C., et al. (eds.) *Séminaire de Probabilités VII, Lecture Notes in Mathematics*, vol. 321, pp. 198–204. Springer, Berlin (1973)
25. Neufeld, A., Nutz, M.: Superreplication under volatility uncertainty for measurable claims. *Electron. J. Probab.* **18**(48), 1–14 (2013)
26. Neufeld, A., Nutz, M.: Measurability of semimartingale characteristics with respect to the probability law. *Stoch. Process. Appl.* **124**, 3819–3845 (2014)
27. Neufeld, A., Nutz, M.: Nonlinear Lévy processes and their characteristics. *Trans. Am. Math. Soc.* **369**, 69–95 (2017)
28. Neufeld, A., Nutz, M.: Robust utility maximization with Lévy processes. *Math. Finance* **28**, 82–105 (2018)
29. Neufeld, A., Šikić, M.: Robust utility maximization in discrete-time markets with friction. *SIAM J. Control Optim.* **56**, 1912–1937 (2018)
30. Neufeld, A., Šikić, M.: Nonconcave robust optimization with discrete strategies under Knightian uncertainty. *Math. Methods Oper. Res.* **90**, 229–253 (2019)
31. Normann, D.: Martin’s axiom and medial functions. *Math. Scand.* **38**, 167–176 (1976)
32. Nutz, M.: Pathwise construction of stochastic integrals. *Electron. Commun. Probab.* **17**(24), 1–7 (2012)
33. Nutz, M.: Superreplication under model uncertainty in discrete time. *Finance Stoch.* **18**, 791–803 (2014)
34. Nutz, M.: Robust superhedging with jumps and diffusion. *Stoch. Process. Appl.* **125**, 4543–4555 (2015)
35. Nutz, M., Soner, H.M.: Superhedging and dynamic risk measures under volatility uncertainty. *SIAM J. Control Optim.* **50**, 2065–2089 (2012)
36. Pham, H., Wei, X., Zhou, C.: Portfolio diversification and model uncertainty: a robust dynamic mean-variance approach. Preprint (2018). [arXiv:1809.01464](https://arxiv.org/abs/1809.01464)
37. Pun, C.S.: G -expected utility maximization with ambiguous equicorrelation. *Quant. Finance* **21**(3), 403–419 (2021)
38. Rásonyi, M., Meireles-Rodrigues, A.: On utility maximisation under model uncertainty in discrete-time markets. *Math. Finance* **31**, 149–175 (2021)
39. Rockafellar, R.T.: *Convex Analysis*. Princeton University Press, Princeton (1997)
40. Rockafellar, R.T., Wets, R.J.B.: *Variational Analysis*. Springer, Berlin (2009)
41. Sion, M.: On general minimax theorems. *Pac. J. Math.* **8**, 171–176 (1958)
42. Tevzadze, R., Toronjadze, T., Uzunashvili, T.: Robust utility maximization for a diffusion market model with misspecified coefficients. *Finance Stoch.* **17**, 535–563 (2013)
43. Uğurlu, K.: Robust utility maximization of terminal wealth with drift and volatility uncertainty. *Optimization*, 1–22 (2020)
44. Yang, Z., Liang, G., Zhou, C.: Constrained portfolio–consumption strategies with uncertain parameters and borrowing costs. *Math. Financ. Econ.* **13**, 393–427 (2019)
45. Žitković, G.: A filtered version of the bipolar theorem of Brannath and Schachermayer. *J. Theor. Probab.* **15**, 41–61 (2002)