

# Dynamics of a noninteracting colloidal fluid in a quenched Gaussian random potential: a time-reversal-symmetry-preserving field-theoretic approach

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**Abstract.** We develop a field-theoretic perturbation method preserving the fluctuation–dissipation relation (FDR) for the dynamics of the density fluctuations of a noninteracting colloidal gas plunged in a quenched Gaussian random field. It is based on an expansion about the Brownian noninteracting gas and can be considered and justified as a low-disorder or high-temperature expansion. The first-order bare theory yields the same memory integral as the mode-coupling theory (MCT) developed for (ideal) fluids in random environments, apart from the bare nature of the correlation functions involved. It predicts an ergodic dynamical behavior for the relaxation of the density fluctuations, in which the memory kernels and correlation functions develop long-time algebraic tails. An FDR-consistent renormalized theory is also constructed from the bare theory. It is shown to display a dynamic ergodic–nonergodic transition similar to the one predicted by the MCT at the level of the density fluctuations, but, at variance with the MCT, the transition does not fully carry over to the self-diffusion,

which always reaches normal diffusive behavior at long time, in agreement with known rigorous results.

**Keywords:** diffusion in random media, mode coupling theory, memory effects, Brownian motion

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## 1. Introduction

In a number of circumstances, simple fluids may generically develop slow and complex dynamics. For instance, glassy dynamics unfolds in the low-temperature and high-density regimes corresponding to supercooled or overcompressed liquid states. It is characterized by a considerable slowing-down of the structural relaxation, eventually leading to the fluid falling out of equilibrium at the glass transition [1–3]. Another example is provided by fluids in quenched-random environments, with either geometric or energetic disorder. Their single-particle dynamics is often characterized by diffusion anomalies, possibly leading to diffusion-localization transitions or other types of nonergodic behaviors [4–6].

A versatile framework to investigate such problems on unified grounds from first principles is provided by the mode-coupling theory (MCT), more specifically a self-consistent current-relaxation theory, as termed by Götze [7, 8]. In its first few years, this very scheme could indeed be successively applied to liquid helium at zero temperature [9, 10], to noninteracting electrons in a random impurity potential [11–13], to the random Lorentz gas [14–16], and to simple glassforming liquids [17, 18].

It is in the field of glassy dynamics that the MCT has had the strongest influence. It was indeed quickly realized that the theory seems to satisfactorily capture many non-trivial aspects of the dynamics of simple glassforming liquids, at least on a qualitative or semi-quantitative level [19, 20]. This triggered and shaped an intensive experimental and computational effort and stimulated numerous further theoretical developments. A difficulty has however been nagging all along, for the main results of the MCT essentially follow from the analysis of a predicted sharp transition between a fluid-like ergodic state and a glass-like nonergodic one. In fact, such a kinetic transition is absent in the actual dynamics of glassforming liquids, and it must be interpreted as giving rise to a dynamical crossover in the moderately supercooled or overcompressed regimes in order to make contact between observations and theory.

From this unsettling situation and the need to clear it up emerged an interest for theoretical approaches in which the MCT, or a MCT-like theory, would be the outcome of a well-defined and controlled approximation scheme, amenable to systematic corrections and improvements. Indeed, the original derivation of the MCT within the Mori–Zwanzig projection-operator formalism does not really lend itself to such a

program, although proposed extensions exist [21–26]. By contrast, field-theoretic approaches appear as methods of choice for such a purpose, and a number of them have accordingly been developed [27–37]. In particular, the most recent studies have paid special attention to the symmetries of the dynamical action, from which crucial equilibrium results readily stem, such as the fluctuation–dissipation relation (FDR) [32–38]. It is actually one of the great strengths of field theories to offer command on these aspects.

In the present work, we follow the lead of the latter studies, but, instead of glass-forming liquids, we focus on noninteracting fluids in quenched-random environments. This indeed appears as an interesting new window on the use of field theory and its relation with MCT, complementary to what has already been done. Note that, within MCT, the presence of interactions does not actually lead to any particular technical difficulty [39–42]. However, for more general considerations, it clearly seems advisable to first isolate the effects of disorder from those of interactions, hence the present restriction to noninteracting systems. In this respect, it should be borne in mind that the dynamics of a pure noninteracting gas, while essentially trivial in a particle-based formalism, is not so simple from a field-theoretic perspective [38].

Before being more specific about our approach, it is worth mentioning that fluids in quenched-random environments have recently received renewed attention, thanks to ingenious experimental developments leading to novel realizations of such systems. Important examples, further investigated by computer simulations, include colloids and aerosols in optical speckle patterns [43–54], binary mixtures of superparamagnetic particles squeezed between glass slides [55–58], and colloids diffusing over rough randomly packed colloidal monolayers [59]. Therefore, beyond purely technical considerations, it also seems timely to try and achieve further theoretical progress in this field.

In practice, we here study the equilibrium dynamics of the density fluctuations of a gas of noninteracting Brownian particles plunged in a random external potential-energy landscape with Gaussian statistics. This specific nature of the disorder indeed appears as particularly well suited for our initial field-theoretic developments, being itself formulated as a very simple and nonsingular field theory.

The time evolution of the density fluctuations is governed by the so-called Dean–Kawasaki (DK) equation (generalized to include the random potential), a nonlinear Langevin equation for the density field with a multiplicative thermal noise [60, 61]. Using the functional formalism of Martin–Siggia–Rose–Janssen–de Dominicis (MSRJD) [62–64], this equation can be turned into a dynamical action functional. As alluded above, it was recently recognized that such an action possesses properties of time-reversal (TR) invariance under specific sets of field transformations, intimately connected to the FDR [32, 65]. These TR symmetries can play the role of guiding principles as to how to develop perturbation theories consistent with the FDR at each order of expansion. Indeed, a difficulty that defeats too naive approaches is that the Gaussian and non-Gaussian components of the action are not separately invariant under these field transformations [31, 32]. One such FDR-preserving theory for the full DK equation (with interactions and without random potential) has recently been developed via the linearization of one of the TR transformations, called the  $\mathcal{U}$ -transformation, at the expense of introducing a new set of conjugated fields. Further details can be found in [37].

Another TR transformation, known as the  $\mathcal{T}$ -transformation, suggests an expansion about the pure noninteracting system as a possible approach. The action is thus decomposed into its free and disorder-induced components, then the latter is treated perturbatively around the former which is non-Gaussian. This procedure can actually be considered and motivated as a weak-disorder or high-temperature expansion. The rationale behind this scheme is that the corresponding nonlinear TR field transformation leaves separately invariant the two decomposed parts of the action. Consistency with the FDR however requires the free part of the action to be treated *exactly*. Notwithstanding its non-Gaussianity due to the multiplicative nature of the thermal noise, this is made possible thanks to the special form of its cubic nonlinearity (quadratic in the noise-response field) and to causality. This aspect is a novel feature of this perturbation method. It is also advantageous that one is freed from introducing extra fields into the problem.

The present method *per se* is a bare perturbation theory, that is, the perturbative corrections are naturally expressed in terms of the bare correlation and response functions. It is not a loop expansion, and it would be a challenge to develop the two-particle-irreducible effective action method for the strongly non-Gaussian noninteracting gas. Note that the approach could also be applied to the full DK equation, with a perturbative treatment of the interactions. This will be examined separately in the future.

We now summarize the main results of our work. The first-order bare theory (FOBT) gives a dynamical equation for the density correlation function that can be put in the same form as that of the self-consistent MCT developed by one of us [39–42], albeit with a memory term written in terms of the bare correlation function (see equations (9.29) and (9.18) below). From this equation, one can compute the mean-squared displacement (MSD) and characterize the long-time tails that develop due to the quenched randomness. The corresponding dynamics is found to always remain ergodic, until the theory breaks down at too strong disorder.

A first-order renormalized theory (FORT) has also been developed out of the bare perturbation theory. It is self-consistently derived from a second-order bare calculation, with empirical adjustments constrained by the requirements of consistency with the FDR and with the FOBT, and eventually singled out through numerical considerations. This theory is distinct from the MCT, but shows some structural similarity with it. In particular, a self-closed dynamical equation for the density correlation function is again obtained (see equations (10.17)–(10.19) below). However, its predictions noticeably improve upon those of the MCT. Indeed, an ergodicity-breaking transition is still predicted for the density fluctuations, but, at variance with the MCT, it does only partially carry over to the MSD, which always reaches a normal diffusive behavior at long time, in agreement with the known rigorous results [66]. Note that, if the Brownian dynamics is replaced with a Newtonian energy-conserving one, then a diffusion-localization transition does occur [67–69], as found in the MCT. Therefore, the confrontation of the MCT and of the present theory might well represent a first step towards an understanding of the phenomenon of avoided or rounded kinetic transitions.

The paper is organized as follows. In sections 2 and 3, we present the time evolution equation for the density fluctuations of Brownian particles in a frozen Gaussian random potential, and the corresponding dynamical action. The time-reversal symmetries of the action and the resulting FDRs are contained in section 4. Section 5

describes the FDR-preserving perturbation expansion method about the non-Gaussian pure noninteracting state. We write down in section 6 the nonperturbative form of the dynamical equations for the correlation and response functions. Section 7 recalls the solution for the pure noninteracting reference state. Sections 8–10 present the main results of the paper, namely, the first-order perturbation corrections to the simple free diffusion. Summary and outlook are given in the last section.

## 2. Time-evolution equation for the density fluctuations of colloidal particles moving in a (random) external potential

In the present work, we investigate a situation where  $N$  colloidal particles in a volume  $V$ , hence the average fluid density  $\rho_0 = N/V$ , move in a (random) external potential. The particle positions are denoted by  $\{\mathbf{r}_i\}$ ,  $i = 1, 2, \dots, N$ . As a first step, the derivation of the time-evolution equation for the density fluctuations of these particles is required. This task can be carried out in a rather general way, following an approach due to Dean [60]. We consider the case of interacting particles, as this does not introduce any particular difficulty at this stage.

The motion of the individual particles is assumed to be described by the overdamped Langevin equation,

$$\dot{\mathbf{r}}_i(t) = \frac{D_0}{T} \mathbf{F}_i(t) + \mathbf{f}_i(t), \quad (2.1)$$

where  $D_0$  is the bare diffusion coefficient,  $T$  is the temperature of the system (the Boltzmann constant  $k_B$  is set to unity throughout), and  $\mathbf{f}_i(t)$  is a Gaussian thermal noise with zero mean and variance

$$\langle f_i^\alpha(t) f_j^\beta(t') \rangle = 2D_0 \delta_{ij} \delta_{\alpha\beta} \delta(t - t'), \quad (2.2)$$

$\alpha$  and  $\beta$  denoting vector components in Cartesian coordinates. The force  $\mathbf{F}_i(t)$  acting on the  $i$ th particle is given by

$$\mathbf{F}_i(t) = \mathbf{F}_i^{\text{int}}(t) + \mathbf{F}_i^{\text{ext}}(t), \quad (2.3)$$

where

$$\mathbf{F}_i^{\text{int}}(t) = -\frac{\partial}{\partial \mathbf{r}_i(t)} \sum_{j=1}^N u(|\mathbf{r}_i(t) - \mathbf{r}_j(t)|) \quad (2.4)$$

is due to the interactions between the fluid particles with pair potential  $u(r)$  (for simplicity,  $\nabla u(0) = \mathbf{0}$  is assumed), and

$$\mathbf{F}_i^{\text{ext}}(t) = -\frac{\partial v(\mathbf{r}_i(t))}{\partial \mathbf{r}_i(t)} \quad (2.5)$$

derives from the external potential with one-body potential energy  $v(\mathbf{r})$ .

The microscopic fluid density is defined as

$$\rho(\mathbf{r}, t) \equiv \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i(t)) = \sum_{i=1}^N \rho_i(\mathbf{r}, t), \quad (2.6)$$

where we introduced the single-particle densities,  $\rho_i(\mathbf{r}, t) \equiv \delta(\mathbf{r} - \mathbf{r}_i(t))$ ,  $i = 1, 2, \dots, N$ . Its fluctuations about the average fluid density are denoted by

$$\delta\rho(\mathbf{r}, t) \equiv \rho(\mathbf{r}, t) - \rho_0. \quad (2.7)$$

In order to derive the dynamical equation for  $\rho(\mathbf{r}, t)$ , we follow the Itô prescription. Consider the following set of stochastic equations for the variables  $x_a(t)$  (using the summation convention),

$$\frac{dx_a(t)}{dt} = h_a + g_{ab}\xi_b(t), \quad (2.8)$$

where the correlation of the Gaussian white noise  $\xi_b(t)$  is defined as

$$\langle \xi_b(t)\xi_{b'}(t') \rangle = \delta_{bb'}\delta(t - t'). \quad (2.9)$$

The Itô chain rule then gives the stochastic equation for a variable  $y[\{x\}]$  in the form

$$\frac{dy(t)}{dt} = \frac{dx_a(t)}{dt} \frac{\partial y}{\partial x_a} + \frac{1}{2} \frac{\partial^2 y}{\partial x_a \partial x_b} g_{ac} g_{cb}. \quad (2.10)$$

Using this rule, we get the dynamical equation

$$\begin{aligned} \partial_t \rho(\mathbf{r}, t) &= D_0 \nabla^2 \rho(\mathbf{r}, t) - \sum_{i=1}^N \dot{\mathbf{r}}_i(t) \cdot \nabla \rho_i(\mathbf{r}, t) \\ &= D_0 \nabla^2 \rho(\mathbf{r}, t) - \sum_{i=1}^N \nabla \rho_i(\mathbf{r}, t) \cdot \left[ \frac{D_0}{T} \mathbf{F}_i(t) + \mathbf{f}_i(t) \right]. \end{aligned} \quad (2.11)$$

One can express the force contributions in equation (2.11) in terms of the fluid density, as

$$\begin{aligned} - \sum_{i=1}^N \nabla \rho_i(\mathbf{r}, t) \cdot \mathbf{F}_i^{\text{int}}(t) &= \nabla \cdot \left[ \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i(t)) \frac{\partial}{\partial \mathbf{r}_i(t)} \int d\mathbf{r}' u(|\mathbf{r}_i(t) - \mathbf{r}'|) \sum_{j=1}^N \delta(\mathbf{r}' - \mathbf{r}_j(t)) \right] \\ &= \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \int d\mathbf{r}' u(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}', t) \right] \end{aligned} \quad (2.12)$$

and

$$- \sum_{i=1}^N \nabla \rho_i(\mathbf{r}, t) \cdot \mathbf{F}_i^{\text{ext}}(t) = \nabla \cdot \left[ \sum_{i=1}^N \delta(\mathbf{r} - \mathbf{r}_i(t)) \frac{\partial v(\mathbf{r}_i(t))}{\partial \mathbf{r}_i(t)} \right] = \nabla \cdot [\rho(\mathbf{r}, t) \nabla v(\mathbf{r})]. \quad (2.13)$$

Also, the thermal noise defined as  $\boldsymbol{\eta}(\mathbf{r}, t) \equiv - \sum_{i=1}^N \rho_i(\mathbf{r}, t) \mathbf{f}_i(t)$  keeps a Gaussian character with zero mean and correlations given by

$$\begin{aligned}
\langle \eta^\alpha(\mathbf{r}, t) \eta^\beta(\mathbf{r}', t') \rangle &= \sum_{i=1}^N \sum_{j=1}^N \delta(\mathbf{r} - \mathbf{r}_i(t)) \delta(\mathbf{r}' - \mathbf{r}_j(t')) \langle f_i^\alpha(t) f_j^\beta(t') \rangle \\
&= 2D_0 \rho(\mathbf{r}, t) \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t').
\end{aligned} \tag{2.14}$$

Substituting equations (2.12)–(2.14) into equation (2.11), one obtains the desired dynamical equation,

$$\begin{aligned}
\partial_t \rho(\mathbf{r}, t) &= D_0 \nabla^2 \rho(\mathbf{r}, t) + \frac{D_0}{T} \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \int d\mathbf{r}' u(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}', t) \right] \\
&\quad + \frac{D_0}{T} \nabla \cdot [\rho(\mathbf{r}, t) \nabla v(\mathbf{r})] + \nabla \cdot \left[ \sqrt{\rho(\mathbf{r}, t)} \boldsymbol{\xi}(\mathbf{r}, t) \right],
\end{aligned} \tag{2.15}$$

where  $\boldsymbol{\xi}(\mathbf{r}, t)$  is a Gaussian thermal noise with zero mean and variance

$$\langle \xi^\alpha(\mathbf{r}, t) \xi^\beta(\mathbf{r}', t') \rangle = 2D_0 \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t'). \tag{2.16}$$

Equation (2.15) can be expressed in terms of a free-energy density functional  $\mathcal{F}[\rho; v]$  as

$$\partial_t \rho(\mathbf{r}, t) = \frac{D_0}{T} \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \left. \frac{\delta \mathcal{F}[\rho; v]}{\delta \rho(\mathbf{r})} \right|_{\rho(\mathbf{r}, t)} \right] + \nabla \cdot \left[ \sqrt{\rho(\mathbf{r}, t)} \boldsymbol{\xi}(\mathbf{r}, t) \right], \tag{2.17}$$

where

$$\mathcal{F}[\rho; v] = \mathcal{F}_{\text{id}}[\rho] + \mathcal{F}_{\text{int}}[\rho] + \mathcal{F}_{\text{ext}}[\rho; v], \tag{2.18a}$$

$$\mathcal{F}_{\text{id}}[\rho] = T \int d\mathbf{r} \rho(\mathbf{r}) [\ln(\rho(\mathbf{r})/\rho_0) - 1], \tag{2.18b}$$

$$\mathcal{F}_{\text{int}}[\rho] = \frac{1}{2} \int \int d\mathbf{r} d\mathbf{r}' u(|\mathbf{r} - \mathbf{r}'|) \delta\rho(\mathbf{r}) \delta\rho(\mathbf{r}'), \tag{2.18c}$$

$$\mathcal{F}_{\text{ext}}[\rho; v] = \int d\mathbf{r} v(\mathbf{r}) \delta\rho(\mathbf{r}). \tag{2.18d}$$

The Fokker–Planck equation for equation (2.17) reads

$$\frac{\partial}{\partial t} P[\rho, t; v] = -D_0 \int d\mathbf{r} \frac{\delta}{\delta \rho(\mathbf{r})} \nabla \cdot \rho(\mathbf{r}) \nabla \left[ \frac{\delta}{\delta \rho(\mathbf{r})} + \frac{1}{T} \frac{\delta \mathcal{F}[\rho; v]}{\delta \rho(\mathbf{r})} \right] P[\rho, t; v]. \tag{2.19}$$

Evidently, the equilibrium Boltzmann distribution  $P_{\text{eq}}[\rho; v] \propto \exp(-\mathcal{F}[\rho; v]/T)$  is a stationary solution of this equation.

Finally, within the functional formalism of MSRJD [62–64], the time evolution described by equations (2.15)–(2.17) can be recast into a dynamical generating functional

$$Z[l, \hat{l}; v] = \int D\rho \int D\hat{\rho} J(\rho) e^{S[\rho, \hat{\rho}; v]} e^{\int_{\mathbf{r}, t} [\rho(\mathbf{r}, t) l(\mathbf{r}, t) + \hat{\rho}(\mathbf{r}, t) \hat{l}(\mathbf{r}, t)]}, \tag{2.20}$$

where the action  $S[\rho, \hat{\rho}; v]$  takes the form

$$S[\rho, \hat{\rho}; v] = \int_{\mathbf{r}, t} \left\{ i\hat{\rho}(\mathbf{r}, t) \left( \partial_t \rho(\mathbf{r}, t) - \frac{D_0}{T} \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \frac{\delta \mathcal{F}[\rho; v]}{\delta \rho(\mathbf{r})} \Big|_{\rho(\mathbf{r}, t)} \right] \right) - D_0 \rho(\mathbf{r}, t) [\nabla \hat{\rho}(\mathbf{r}, t)]^2 \right\}, \quad (2.21)$$

with  $\int_{\mathbf{r}, t} \equiv \int d\mathbf{r} \int dt$ . Here, the thermal average has already been performed, and the term proportional to  $\rho(\nabla \hat{\rho})^2$  comes from the average over the multiplicative thermal noise. The Jacobian  $J(\rho)$  guarantees that the normalization condition  $Z[l = 0, \hat{l} = 0; v] = 1$ , of critical importance in applications of the formalism to quenched-disordered systems, indeed holds. In the Itô discretization scheme,  $J(\rho)$  becomes a constant and can be absorbed into the functional measure. From the knowledge of  $Z[l, \hat{l}; v]$ , the time-dependent correlation functions of the fields  $\rho$  and  $\hat{\rho}$  can be straightforwardly obtained as functional derivatives with respect to  $l$  and  $\hat{l}$  at  $l = 0, \hat{l} = 0$ . More generally, dynamical quantities averaged over the thermal noise can be evaluated with respect to the action  $S[\rho, \hat{\rho}; v]$  as

$$\langle A[\rho, \hat{\rho}] \rangle = \int_{\rho, \hat{\rho}} A[\rho, \hat{\rho}] e^{S[\rho, \hat{\rho}; v]}, \quad (2.22)$$

where  $\int_{\rho, \hat{\rho}} \equiv \int D\rho \int D\hat{\rho}$  and  $\langle \dots \rangle$  generically denotes a thermal average.

### 3. Noninteracting Brownian gas in a Gaussian random potential

We may now specialize the above equations in accordance with the aim of the present study, which is to investigate the effect of a quenched-random environment on the dynamics of colloids. To this end, we consider what appears to be the simplest non-trivial case. First, in most of this work, we will simply ignore the particle interactions and set  $u(r) = 0$  for all  $r$ , in order to merely focus on the aspect of quenched disorder. Second, the one-body potential energy function  $v(\mathbf{r})$ , from which the external potential is built, should be sampled from a convenient functional probability space. A natural option is to turn to a homogeneous and isotropic Gaussian random field, whose statistical properties are fully encoded in its mean, which can be set to zero without loss of generality, and its covariance. Therefore, we shall assume Gaussian statistics for  $v(\mathbf{r})$ , with

$$\overline{v(\mathbf{r})} \equiv 0, \quad \overline{v(\mathbf{r})v(\mathbf{r}')} \equiv w\Phi(|\mathbf{r} - \mathbf{r}'|), \quad (3.1)$$

where  $\overline{\dots}$  denotes an average over the random-field distribution. The normalized random-field covariance  $\Phi(r)$  obeys  $\Phi(0) = 1$ , so that  $w$  appears as a straightforward measure of the disorder strength. It will determine the behavior of the system and should be compared with the typical thermal energy fluctuations, a purpose readily served by a single dimensionless control parameter representing the relative disorder strength,  $\lambda \equiv w/T^2$ .

Although we choose to introduce Gaussian statistics for the external potential from the outset, it might be useful to recall that this represents a common assumption in a number of simple circumstances of interest. For instance, a standard argument based

on the central limit theorem and used in a variety of related problems [70–72] states that the one-body potential generated by a statistically homogeneous frozen matrix of randomly placed interaction sites is expected to develop Gaussian statistics under suitable conditions, as it is a sum of a large number of random fluid-matrix pair interactions in the thermodynamic limit. It should nevertheless be stressed that, although this argument can be made rigorous in some special limits [73, 74], it can lead to difficulties in more generic cases [75, 76]. Other possible situations expected to yield Gaussian random fields are associated with linear combinations of random Fourier modes [77, 78] or with coarse-graining of a random field, be it Gaussian or not, over extended enough regions [79]. The latter approach is practically relevant to polarizable colloids in speckle patterns, in the regime where the effective external potential results from the integrated effect of the random light intensity field over the whole volume of a particle [43, 45, 47].

Before considering the dynamics, a few structural properties of the system should be derived. From a configurational point of view, one actually deals with an ideal gas in an external potential. Its one-particle configurational integral is readily shown to be self-averaging, with the nonrandom limit

$$\lim_{V \rightarrow +\infty} \frac{1}{V} \int_V d\mathbf{r} e^{-v(\mathbf{r})/T} = \overline{e^{-v(\mathbf{r})/T}} = e^{\lambda/2}. \quad (3.2)$$

Therefore, for any single realization of  $v(\mathbf{r})$  in the thermodynamic limit, one straightforwardly gets

$$\langle \rho(\mathbf{r}) \rangle = \rho_0 e^{-\lambda/2} e^{-v(\mathbf{r})/T}, \quad (3.3a)$$

$$\langle \rho(\mathbf{r}) \rho(\mathbf{r}') \rangle = \rho_0 e^{-\lambda/2} e^{-v(\mathbf{r})/T} \delta(\mathbf{r} - \mathbf{r}') + \rho_0^2 e^{-\lambda} e^{-[v(\mathbf{r})+v(\mathbf{r}')]/T}, \quad (3.3b)$$

where the normalization factors precisely stem from the one-particle configurational integral. Computing now disorder averaged quantities, one gets  $\overline{\langle \rho(\mathbf{r}) \rangle} = \rho_0$ , as it should, and

$$C_{\text{st}}(|\mathbf{r} - \mathbf{r}'|) \equiv \overline{\langle \delta\rho(\mathbf{r}) \delta\rho(\mathbf{r}') \rangle} = \rho_0 \delta(\mathbf{r} - \mathbf{r}') + \rho_0^2 \left[ e^{\lambda\Phi(|\mathbf{r}-\mathbf{r}'|)} - 1 \right], \quad (3.4a)$$

$$C_{\text{d}}(|\mathbf{r} - \mathbf{r}'|) \equiv \overline{\langle \delta\rho(\mathbf{r}) \rangle \langle \delta\rho(\mathbf{r}') \rangle} = \rho_0^2 \left[ e^{\lambda\Phi(|\mathbf{r}-\mathbf{r}'|)} - 1 \right], \quad (3.4b)$$

where  $C_{\text{st}}(r)$  denotes the static density correlation function and  $C_{\text{d}}(r)$  the so-called disconnected density correlation function. In reciprocal space, the same density correlations are described in terms of the static and disconnected structure factors,  $S_k^{\text{st}}$  and  $S_k^{\text{d}}$ . They are obtained by Fourier transforming  $C_{\text{st}}(r)$  and  $C_{\text{d}}(r)$ , respectively, and normalizing by  $\rho_0$ . Since

$$C_{\text{st}}(|\mathbf{r} - \mathbf{r}'|) - C_{\text{d}}(|\mathbf{r} - \mathbf{r}'|) = \rho_0 \delta(\mathbf{r} - \mathbf{r}'), \quad (3.5)$$

the structure factors obey  $S_k^{\text{st}} = 1 + S_k^{\text{d}}$ . Note that, in fact, both equalities generically hold for a noninteracting gas in any type of homogeneous and isotropic random environment. The actual dependence of the above structural quantities on  $|\mathbf{r} - \mathbf{r}'|$  or on the wavevector modulus  $k$  follows from this property of homogeneity and isotropy.

We now turn to dynamics. Since  $Z[l = 0, \hat{l} = 0; v]$  is normalized to 1, hence independent from any specific random potential-energy realization, the noise-averaged dynamical quantities given by equation (2.22) can be further disorder-averaged as [80]

$$\overline{\langle A[\rho, \hat{\rho}] \rangle} = \int_{\rho, \hat{\rho}} A[\rho, \hat{\rho}] \overline{e^{S[\rho, \hat{\rho}; v]}} \equiv \int_{\rho, \hat{\rho}} A[\rho, \hat{\rho}] e^{S_{\text{eff}}[\rho, \hat{\rho}]} \equiv \langle A[\rho, \hat{\rho}] \rangle_{\text{eff}}, \quad (3.6)$$

where the effective action  $S_{\text{eff}}[\rho, \hat{\rho}]$  generically consists of two terms,

$$S_{\text{eff}}[\rho, \hat{\rho}] = S_{\text{bulk}}[\rho, \hat{\rho}] + S_{\text{dis}}[\rho, \hat{\rho}]. \quad (3.7)$$

The first one is that part of  $S[\rho, \hat{\rho}; v]$  that does not explicitly involve  $v(\mathbf{r})$  and is therefore left unaffected by the disorder average. It generically reads

$$S_{\text{bulk}}[\rho, \hat{\rho}] = \int_{\mathbf{r}, t} \left\{ i\hat{\rho}(\mathbf{r}, t) \left( \partial_t \rho(\mathbf{r}, t) - \frac{D_0}{T} \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \Big|_{\rho(\mathbf{r}, t)} \right] \right) - D_0 \rho(\mathbf{r}, t) [\nabla \hat{\rho}(\mathbf{r}, t)]^2 \right\}, \quad (3.8)$$

with  $\mathcal{F}_{\text{bulk}}[\rho] \equiv \mathcal{F}_{\text{id}}[\rho] + \mathcal{F}_{\text{int}}[\rho]$ , and describes the dynamics of a bulk fluid in the absence of an external field. In the present noninteracting case (we have set  $u(r) = 0$  for all  $r$ ),  $\mathcal{F}_{\text{bulk}}[\rho]$  reduces to  $\mathcal{F}_{\text{id}}[\rho]$ , and  $S_{\text{bulk}}[\rho, \hat{\rho}]$  to

$$S_{\text{free}}[\rho, \hat{\rho}] = \int_{\mathbf{r}, t} \left\{ i\hat{\rho}(\mathbf{r}, t) (\partial_t - D_0 \nabla^2) \rho(\mathbf{r}, t) - D_0 \rho(\mathbf{r}, t) [\nabla \hat{\rho}(\mathbf{r}, t)]^2 \right\}, \quad (3.9)$$

which rules ‘free’ dynamics in the absence of disorder and interactions. Note that  $S_{\text{free}}[\rho, \hat{\rho}]$  is *non-Gaussian* and possesses a cubic nonlinearity arising from the multiplicative thermal noise. Since the quenched randomness has Gaussian statistics, one can readily perform the disorder average on the remaining factor in  $e^{S[\rho, \hat{\rho}; v]}$ ,

$$\overline{\exp \left( -\frac{D_0}{T} \int_{\mathbf{r}, t} i\hat{\rho}(\mathbf{r}, t) \nabla \cdot [\rho(\mathbf{r}, t) \nabla v(\mathbf{r})] \right)} \equiv e^{S_{\text{dis}}[\rho, \hat{\rho}]}, \quad (3.10)$$

and obtain the second term,

$$S_{\text{dis}}[\rho, \hat{\rho}] = \frac{1}{2} \lambda D_0^2 \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} [\nabla^\alpha \nabla^\beta \Phi(|\mathbf{r} - \mathbf{r}'|)] [\rho(\mathbf{r}, t) \nabla^\alpha \hat{\rho}(\mathbf{r}, t)] [\rho(\mathbf{r}', t') \nabla'^\beta \hat{\rho}(\mathbf{r}', t')], \quad (3.11)$$

where the summation convention is implied for the Cartesian indices (this will systematically be the case in the following) and the  $\nabla'$  operator acts on  $\mathbf{r}'$ . As is common with quenched-random systems [80], the disorder-induced contribution becomes *nonlocal* in time after disorder averaging, i.e. it does not only couple the fields at any given time, but also between different time slices. In fact,  $S_{\text{dis}}[\rho, \hat{\rho}]$  represents an effective time-persistent dynamical interaction between the fluid particles induced by the presence of the quenched random potential. It displays both cubic and quartic nonlinearities in  $\hat{\rho}(\mathbf{r}, t)$  and  $\delta\rho(\mathbf{r}, t)$ . Through integration by parts, it can be rewritten as

$$S_{\text{dis}}[\rho, \hat{\rho}] = -\frac{1}{2} \lambda \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} \Phi(|\mathbf{r} - \mathbf{r}'|) \Lambda(\mathbf{r}, t) \Lambda(\mathbf{r}', t'), \quad (3.12)$$

where we introduce the composite response field

$$\Lambda(\mathbf{r}, t) \equiv D_0 \nabla \cdot [\rho(\mathbf{r}, t) \nabla \hat{\rho}(\mathbf{r}, t)]. \quad (3.13)$$

The latter leads to the physical response function, as discussed in the next section.

#### 4. Physical response function, time-reversal symmetry, and fluctuation–dissipation relation

We now define our main quantities of interest and discuss some crucial relations between them. For the sake of generality, we retain interactions between the colloids, as they barely add any additional complexity.

A fundamental feature of the fluid systems when studied at the level of the density field is that the physical response function  $R(\mathbf{r}, t; \mathbf{r}', t')$ , pertaining to the change of the local average density under a small external field coupled to the density fluctuation, differs from the ordinary response function (to the thermal noise)  $G(\mathbf{r}, t; \mathbf{r}', t')$ , because of the multiplicative nature of the noise in the original Langevin equation, equation (2.15) [31]. The main quantities of interest are thus the density correlation function  $C(\mathbf{r}, t; \mathbf{r}', t')$  and the above response functions, defined as

$$C(\mathbf{r}, t; \mathbf{r}', t') = \langle \delta\rho(\mathbf{r}, t) \delta\rho(\mathbf{r}', t') \rangle_{\text{eff}}, \quad (4.1a)$$

$$G(\mathbf{r}, t; \mathbf{r}', t') = -i \langle \rho(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t') \rangle_{\text{eff}}, \quad (4.1b)$$

$$\begin{aligned} R(\mathbf{r}, t; \mathbf{r}', t') &= \frac{i}{T} \langle \rho(\mathbf{r}, t) \Lambda(\mathbf{r}', t') \rangle_{\text{eff}} \\ &= -\frac{\rho_0 D_0}{T} \nabla^2 G(\mathbf{r}, t; \mathbf{r}', t') + i \frac{D_0}{T} \langle \rho(\mathbf{r}, t) \nabla' \cdot [\delta\rho(\mathbf{r}', t') \nabla' \hat{\rho}(\mathbf{r}', t')] \rangle_{\text{eff}}. \end{aligned} \quad (4.1c)$$

It is also useful to introduce the so-called connected density correlation function,

$$F(\mathbf{r}, t; \mathbf{r}', t') = C(\mathbf{r}, t; \mathbf{r}', t') - C_d(|\mathbf{r} - \mathbf{r}'|). \quad (4.2)$$

Note that the physical response function involves the composite response field, equation (3.13), hence has two contributions: one is simply proportional to the noise-response function, while an additional ‘anomalous’ term arises from the multiplicative thermal noise. Due to the explicit appearance of the temperature  $T$  in the expression of the physical response function  $R(\mathbf{r}, t; \mathbf{r}', t')$ , it is found convenient to instead use the function  $\bar{R}(\mathbf{r}, t; \mathbf{r}', t')$  defined as

$$\bar{R}(\mathbf{r}, t; \mathbf{r}', t') \equiv T R(\mathbf{r}, t; \mathbf{r}', t') = i \langle \rho(\mathbf{r}, t) \Lambda(\mathbf{r}', t') \rangle_{\text{eff}}. \quad (4.3)$$

Causality commands that the response functions obey

$$G(\mathbf{r}, t; \mathbf{r}', t') = 0, \quad \bar{R}(\mathbf{r}, t; \mathbf{r}', t') = 0, \quad t \leq t'. \quad (4.4)$$

In terms of the fields, this means

$$\langle \rho(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t') \rangle_{\text{eff}} = 0, \quad \langle \rho(\mathbf{r}, t) \Lambda(\mathbf{r}', t') \rangle_{\text{eff}} = 0, \quad t \leq t'. \quad (4.5)$$

Moreover, the normalization condition on the dynamical generating functional in the MSRJD formalism results in additional causality constraints, among which

$$\langle \hat{\rho}(\mathbf{r}, t) \rangle_{\text{eff}} = 0, \quad \langle \hat{\rho}(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t') \rangle_{\text{eff}} = 0, \quad \langle \Lambda(\mathbf{r}, t) \rangle_{\text{eff}} = 0, \quad \langle \Lambda(\mathbf{r}, t) \Lambda(\mathbf{r}', t') \rangle_{\text{eff}} = 0. \quad (4.6)$$

As with static quantities, the actual spatial dependence of the above correlation and response functions is on  $|\mathbf{r} - \mathbf{r}'|$ , because of the homogeneity and isotropy of the random field. When time-translation invariance additionally holds, we will therefore write  $C(\mathbf{r}, t; \mathbf{r}', t') \equiv C(|\mathbf{r} - \mathbf{r}'|, t - t')$ ,  $G(\mathbf{r}, t; \mathbf{r}', t') \equiv G(|\mathbf{r} - \mathbf{r}'|, t - t')$ , and  $\bar{R}(\mathbf{r}, t; \mathbf{r}', t') \equiv \bar{R}(|\mathbf{r} - \mathbf{r}'|, t - t')$ .

Equilibrium dynamics is known to possess time-reversal symmetry. This symmetry is reflected in the invariance (up to irrelevant boundary terms) of the effective action, equation (3.7), under special field transformations with time reversal [32, 38, 65].

The approach developed in the present work is motivated by the invariance of the action under the so-called  $\mathcal{T}$ -transformation [32, 38, 65], as shown in appendix A. This transformation reads

$$\mathcal{T} : \begin{cases} \rho(\mathbf{r}, t) \rightarrow \rho(\mathbf{r}, -t), \\ \hat{\rho}(\mathbf{r}, t) \rightarrow \hat{\rho}(\mathbf{r}, -t) + i h(\mathbf{r}, -t), \end{cases} \quad (4.7)$$

with the function  $h(\mathbf{r}, t)$  defined through the equation

$$D_0 \nabla \cdot [\rho(\mathbf{r}, t) \nabla h(\mathbf{r}, t)] = \partial_t \rho(\mathbf{r}, t), \quad (4.8)$$

which can be solved in Fourier space [38].

This definition implies the relation

$$\mathcal{T} \Lambda(\mathbf{r}, t) = \Lambda(\mathbf{r}, -t) - i \partial_t \rho(\mathbf{r}, -t) \quad (4.9)$$

for the composite response field. Thus, with the identification of the physical response function  $\bar{R}(\mathbf{r}, t; \mathbf{r}', t')$  in equation (4.3), the FDR is immediately obtained from equations (4.7) and (4.9). Indeed, from the Ward–Takahashi identities [81]

$$\langle \rho(\mathbf{r}, t) \rho(\mathbf{r}', t') \rangle_{\text{eff}} = \langle [\mathcal{T} \rho(\mathbf{r}, t)] [\mathcal{T} \rho(\mathbf{r}', t')] \rangle_{\text{eff}} = \langle \rho(\mathbf{r}, -t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}}, \quad (4.10a)$$

$$\langle \rho(\mathbf{r}, t) \Lambda(\mathbf{r}', t') \rangle_{\text{eff}} = \langle [\mathcal{T} \rho(\mathbf{r}, t)] [\mathcal{T} \Lambda(\mathbf{r}', t')] \rangle_{\text{eff}} = \langle \rho(\mathbf{r}, -t) \Lambda(\mathbf{r}', -t') \rangle_{\text{eff}} - i \partial_{t'} \langle \rho(\mathbf{r}, -t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}}, \quad (4.10b)$$

follows the relation

$$\bar{R}(\mathbf{r}, t; \mathbf{r}', t') = \bar{R}(\mathbf{r}, -t; \mathbf{r}', -t') + \partial_{t'} C(\mathbf{r}, t; \mathbf{r}', t'), \quad (4.11)$$

i.e. with time-translation invariance,

$$\bar{R}(|\mathbf{r} - \mathbf{r}'|, t - t') - \bar{R}(|\mathbf{r} - \mathbf{r}'|, t' - t) = -\partial_t C(|\mathbf{r} - \mathbf{r}'|, t - t'). \quad (4.12)$$

For future reference, we note that causality, equation (4.4), and the FDR, equation (4.12), imply

$$\int_{-\infty}^{+\infty} dt' \bar{R}(|\mathbf{r} - \mathbf{r}'|, t - t') = C_{\text{st}}(|\mathbf{r} - \mathbf{r}'|) - C_{\text{d}}(|\mathbf{r} - \mathbf{r}'|), \quad (4.13)$$

where the equilibrium relations  $C(|\mathbf{r} - \mathbf{r}'|, 0) = C_{\text{st}}(|\mathbf{r} - \mathbf{r}'|)$  and  $C(|\mathbf{r} - \mathbf{r}'|, t \rightarrow +\infty) = C_{\text{d}}(|\mathbf{r} - \mathbf{r}'|)$  have been used.

Another field transformation exists, that leaves the effective action invariant. It will play a minor role in the present work, but should be mentioned for completeness and because it might be of general interest in dynamical studies of random-field systems. Interestingly, it does not involve a time reversal in its primary formulation and therefore holds in generic out-of-equilibrium situations. However, it can be usefully specialized to equilibrium dynamics through composition with the  $\mathcal{T}$ -transformation.

Thus, guided by [65], we show in appendix A that  $S_{\text{eff}}[\rho, \hat{\rho}]$  is invariant under the  $\mathcal{U}'$ -transformation defined as

$$\mathcal{U}' : \begin{cases} \rho(\mathbf{r}, t) & \rightarrow \rho(\mathbf{r}, t), \\ \hat{\rho}(\mathbf{r}, t) & \rightarrow -\hat{\rho}(\mathbf{r}, t) + 2i \int_{\mathbf{r}', t'} K_{\lambda}^{-1}(\mathbf{r}, t; \mathbf{r}', t') \text{DET}([\rho], \mathbf{r}', t'). \end{cases} \quad (4.14)$$

The functional  $\text{DET}([\rho], \mathbf{r}, t)$  represents the deterministic nonrandom part of the density evolution equation and here reads (see equation (2.17))

$$\text{DET}([\rho], \mathbf{r}, t) = \partial_t \rho(\mathbf{r}, t) - \frac{D_0}{T} \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \Big|_{\rho(\mathbf{r}, t)} \right]. \quad (4.15)$$

The kernel  $K_{\lambda}^{-1}(\mathbf{r}, t; \mathbf{r}', t')$  is the inverse of the density-dependent symmetric kernel characterizing both the Gaussian noise and disorder in the system,

$$\begin{aligned} K_{\lambda}(\mathbf{r}, t; \mathbf{r}', t') &\equiv \{ \nabla^{\alpha} \nabla'^{\beta} [2D_0 \rho(\mathbf{r}, t) \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') + \lambda D_0^2 \rho(\mathbf{r}, t) \rho(\mathbf{r}', t') \nabla^{\alpha} \nabla'^{\beta} \Phi(|\mathbf{r} - \mathbf{r}'|)] \} \\ &\equiv K_0(\mathbf{r}, t; \mathbf{r}', t') + \lambda \Delta K(\mathbf{r}, t; \mathbf{r}', t'), \end{aligned} \quad (4.16)$$

and is accordingly defined through

$$\delta(\mathbf{r} - \mathbf{r}'') \delta(t - t'') = \int_{\mathbf{r}', t'} K_{\lambda}(\mathbf{r}, t; \mathbf{r}', t') K_{\lambda}^{-1}(\mathbf{r}', t'; \mathbf{r}'', t''). \quad (4.17)$$

The composition of  $\mathcal{T}$  and  $\mathcal{U}'$  yields the  $\mathcal{U}$ -transformation. It obviously leaves the action invariant, since  $\mathcal{T}$  and  $\mathcal{U}'$  separately do, and involves a time reversal inherited from  $\mathcal{T}$ . As shown in appendix A, it reads

$$\mathcal{U} : \begin{cases} \rho(\mathbf{r}, t) & \rightarrow \rho(\mathbf{r}, -t), \\ \hat{\rho}(\mathbf{r}, t) & \rightarrow -\hat{\rho}(\mathbf{r}, -t) + \frac{i}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \Big|_{\rho(\mathbf{r}, -t)} \\ & + i\lambda D_0 \int_{\mathbf{r}', t'} K_{\lambda}^{-1}(\mathbf{r}, -t; \mathbf{r}', -t') \nabla' \cdot \left\{ \rho(\mathbf{r}', -t') \nabla' \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r}' - \mathbf{r}''|) \mathcal{T}\text{DET}([\rho], \mathbf{r}'', t'') \right\}, \end{cases} \quad (4.18)$$

with

$$\mathcal{T}\text{DET}([\rho], \mathbf{r}, t) = \partial_t \rho(\mathbf{r}, -t) - \frac{D_0}{T} \nabla \cdot \left[ \rho(\mathbf{r}, -t) \nabla \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \Big|_{\rho(\mathbf{r}, -t)} \right]. \quad (4.19)$$

It is clear that, in the absence of a random field ( $\lambda = 0$ ), this transformation reduces to the  $\mathcal{U}$ -transformation as defined in [32] for bulk fluids, hence the shared naming. It becomes nonlocal in time in the presence of a random field. Note that, in principle, the integral over  $t''$  of the total time derivative  $\partial_{t''} \rho(\mathbf{r}'', -t'')$  contained in  $\mathcal{T}\text{DET}([\rho], \mathbf{r}'', t'')$  vanishes in an equilibrium setting, but we found that explicitly keeping such terms makes some calculations in appendix A more straightforward.

As the  $\mathcal{T}$ -transformation, the  $\mathcal{U}$ -transformation can be used to derive relations between response functions and correlations. In particular, as shown in appendix A, the Ward–Takahashi identity

$$\langle \rho(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t') \rangle_{\text{eff}} = \langle [\mathcal{U}\rho(\mathbf{r}, t)] [\mathcal{U}\hat{\rho}(\mathbf{r}', t')] \rangle_{\text{eff}} \quad (4.20)$$

leads to the following decomposition of the noise-response function,

$$\begin{aligned} & G(|\mathbf{r} - \mathbf{r}'|, t - t') + G(|\mathbf{r} - \mathbf{r}'|, t' - t) \\ &= \int_{\mathbf{r}''} C(|\mathbf{r} - \mathbf{r}''|, t - t') Q^{-1}(|\mathbf{r}'' - \mathbf{r}'|) + \Delta C^{\text{nG}}(|\mathbf{r} - \mathbf{r}'|, t - t') + \Delta C^{\text{dis}}(|\mathbf{r} - \mathbf{r}'|, t - t'). \end{aligned} \quad (4.21)$$

Here,  $Q^{-1}(|\mathbf{r} - \mathbf{r}'|)$  is the functional inverse of the static density correlation function of the bulk fluid if its free energy is restricted to its Gaussian approximation,  $\Delta C^{\text{nG}}(|\mathbf{r} - \mathbf{r}'|, t - t')$  originates in the non-Gaussian nature of  $\mathcal{F}_{\text{bulk}}[\rho]$  due to  $\mathcal{F}_{\text{id}}[\rho]$ , and  $\Delta C^{\text{dis}}(|\mathbf{r} - \mathbf{r}'|, t - t')$  is a disorder-induced contribution. Their detailed expressions can be found in appendix A.

For systems with a Gaussian bulk free energy and no random potential,  $Q^{-1}(|\mathbf{r} - \mathbf{r}'|) = C^{-1}(|\mathbf{r} - \mathbf{r}'|, 0)$ , while  $\Delta C^{\text{nG}}(|\mathbf{r} - \mathbf{r}'|, t - t')$  and  $\Delta C^{\text{dis}}(|\mathbf{r} - \mathbf{r}'|, t - t')$  vanish. One then recovers the familiar Deker–Haake–Miyazaki–Reichman (DHMR) linear relation between the noise-response function and the density correlation function (Deker and Haake first considered the case of additive noise [82], then Miyazaki and Reichman extended the result to multiplicative noise [31]).

In the following, it will be found convenient to work in reciprocal space, i.e. with correlation and response functions Fourier transformed with respect to their spatial variations. Thus, in Fourier space, equations (4.12) and (4.21) take the form (setting  $t' = 0$ )

$$\bar{R}_k(t) - \bar{R}_k(-t) = -\partial_t C_k(t), \quad (4.22a)$$

$$G_k(t) + G_k(-t) = \frac{C_k(t)}{Q_k} + \Delta C_k^{\text{nG}}(t) + \Delta C_k^{\text{dis}}(t). \quad (4.22b)$$

For noninteracting colloids,  $Q_k = \rho_0$ .

## 5. Expansion around the disorder-free dynamics

We now describe the main theoretical development at the heart of the present work, which is a perturbative expansion dictated by the  $\mathcal{T}$ -transformation, equation (4.7). The key point here is that the two contributions  $S_{\text{free}}[\rho, \hat{\rho}]$  (to which  $S_{\text{bulk}}[\rho, \hat{\rho}]$  reduces in the noninteracting case) and  $S_{\text{dis}}[\rho, \hat{\rho}]$  to the effective action  $S_{\text{eff}}[\rho, \hat{\rho}]$  are separately invariant under this transformation, as shown in appendix A. Therefore, with a due account of this property, it should be possible to lay out a scheme that preserves the FDR, which precisely stems from the  $\mathcal{T}$ -transformation, order by order.

The present perturbative approach first involves an expansion in terms of  $S_{\text{dis}}[\rho, \hat{\rho}]$  about the free dynamics ruled by  $S_{\text{free}}[\rho, \hat{\rho}]$ , as

$$\langle A[\rho, \hat{\rho}] \rangle_{\text{eff}} = \int_{\rho, \hat{\rho}} A[\rho, \hat{\rho}] e^{S_{\text{free}}[\rho, \hat{\rho}]} e^{S_{\text{dis}}[\rho, \hat{\rho}]} = \int_{\rho, \hat{\rho}} A[\rho, \hat{\rho}] e^{S_{\text{free}}[\rho, \hat{\rho}]} \sum_{n_{\text{dis}}=0}^{\infty} \frac{S_{\text{dis}}[\rho, \hat{\rho}]^{n_{\text{dis}}}}{n_{\text{dis}}!}. \quad (5.1)$$

This step can be seen as a weak-disorder or high-temperature expansion, since  $S_{\text{dis}}[\rho, \hat{\rho}]$  is proportional to the relative disorder strength  $\lambda = w/T^2$ . Defining the average over the free part of the action as

$$\langle A[\rho, \hat{\rho}] \rangle_{\text{f}} \equiv \int_{\rho, \hat{\rho}} A[\rho, \hat{\rho}] e^{S_{\text{free}}[\rho, \hat{\rho}]}, \quad (5.2)$$

we thus have

$$\langle A[\rho, \hat{\rho}] \rangle_{\text{eff}} = \sum_{n_{\text{dis}}=0}^{\infty} \frac{1}{n_{\text{dis}}!} \langle A[\rho, \hat{\rho}] S_{\text{dis}}[\rho, \hat{\rho}]^{n_{\text{dis}}} \rangle_{\text{f}}. \quad (5.3)$$

Now, the free part of the action has a non-Gaussian cubic nonlinearity due to the multiplicative thermal noise. In order to maintain the invariance of  $S_{\text{free}}[\rho, \hat{\rho}]$  under the  $\mathcal{T}$ -transformation and preserve the FDR order by order, this nonlinearity should be treated *exactly*. It turns out that this can be readily achieved thanks to the causality conditions and the presence of two  $\hat{\rho}$  fields in this cubic contribution. Indeed, splitting the free part of the action into its Gaussian and non-Gaussian components,  $S_0[\rho, \hat{\rho}]$  and  $S_{\text{m}}[\rho, \hat{\rho}]$ , respectively, with

$$S_0[\rho, \hat{\rho}] \equiv \int_{\mathbf{r}, t} \{ i\hat{\rho}(\mathbf{r}, t) (\partial_t - D_0 \nabla^2) \delta\rho(\mathbf{r}, t) - D_0 \rho_0 [\nabla \hat{\rho}(\mathbf{r}, t)]^2 \}, \quad (5.4)$$

$$S_{\text{m}}[\rho, \hat{\rho}] \equiv -D_0 \int_{\mathbf{r}, t} \delta\rho(\mathbf{r}, t) [\nabla \hat{\rho}(\mathbf{r}, t)]^2, \quad (5.5)$$

one can rewrite the averages over the free dynamics as

$$\langle B[\rho, \hat{\rho}] \rangle_{\text{f}} = \int_{\rho, \hat{\rho}} B[\rho, \hat{\rho}] e^{S_0[\rho, \hat{\rho}]} e^{S_{\text{m}}[\rho, \hat{\rho}]} = \langle B[\rho, \hat{\rho}] e^{S_{\text{m}}[\rho, \hat{\rho}]} \rangle_0 = \sum_{n_{\text{m}}=0}^{\infty} \frac{1}{n_{\text{m}}!} \langle B[\rho, \hat{\rho}] S_{\text{m}}[\rho, \hat{\rho}]^{n_{\text{m}}} \rangle_0, \quad (5.6)$$

where  $\langle \dots \rangle_0$  denotes the Gaussian average defined as

$$\langle B[\rho, \hat{\rho}] \rangle_0 \equiv \int_{\rho, \hat{\rho}} B[\rho, \hat{\rho}] e^{S_0[\rho, \hat{\rho}]}. \quad (5.7)$$

The key observation is that, due to the twice faster increase of the number of  $\hat{\rho}$  fields with  $n_{\text{m}}$ , the summation in equation (5.6) will be rapidly *terminated* at a low order. Indeed, consider a generic product of  $\delta\rho$  and  $\hat{\rho}$  fields or space derivatives thereof. If it has an odd number of factors, its Gaussian average trivially vanishes. If its number of factors is even, one can use Wick's theorem to decompose its Gaussian average as a sum of products of two-point averages. Then, if the number of noise-response fields exceeds the number of density fields (necessarily, by at least two), each term in the sum will unavoidably have a factor of the form  $\langle \hat{\rho}(\mathbf{r}_i, t_i) \hat{\rho}(\mathbf{r}_j, t_j) \rangle_0$ . Such factors identically vanish due to causality (equation (4.6) also holds with the Gaussian action  $S_0$ ), hence the whole Gaussian average vanishes. For instance, one generically gets

$$\langle \delta\rho(\mathbf{r}_i, t_i) \delta\rho(\mathbf{r}_j, t_j) \hat{\rho}(\mathbf{r}_k, t_k) \hat{\rho}(\mathbf{r}_l, t_l) \rangle_0 \neq 0, \quad (5.8)$$

$$\langle \delta\rho(\mathbf{r}_i, t_i) \delta\rho(\mathbf{r}_j, t_j) \hat{\rho}(\mathbf{r}_k, t_k) \hat{\rho}(\mathbf{r}_l, t_l) \hat{\rho}(\mathbf{r}_m, t_m) \hat{\rho}(\mathbf{r}_n, t_n) \rangle_0 = 0. \quad (5.9)$$

Now, if  $B[\rho, \hat{\rho}]$  is such a generic product with  $p$  density fields and  $q$  response fields, then the term of order  $n_m$  in equation (5.6) also involves such a product, with  $p + n_m$  density fields and  $q + 2n_m$  response fields. As just shown, its Gaussian average vanishes if  $q + 2n_m > p + n_m$ , i.e.  $n_m > p - q$ . It is precisely this simplification that makes possible an exact treatment of the cubic nonlinearity due to the multiplicative thermal noise. Indeed, if  $p \geq q$ , the expansion in equation (5.6) terminates at most at  $n_m = p - q$ , while for  $p < q$ , the first term of equation (5.6) already vanishes and one gets  $\langle B[\rho, \hat{\rho}] \rangle_f = 0$ .

Note that the bound on  $n_m$  is the same for all terms in equation (5.3). Indeed,  $S_{\text{dis}}[\rho, \hat{\rho}]$  given by equation (3.11) can be rewritten as

$$\begin{aligned} S_{\text{dis}}[\rho, \hat{\rho}] = & \frac{1}{2} \lambda D_0^2 \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} [\nabla^\alpha \nabla^\beta \Phi(|\mathbf{r} - \mathbf{r}'|)] \\ & \times [\rho_0^2 + 2\rho_0 \delta\rho(\mathbf{r}, t) + \delta\rho(\mathbf{r}, t) \delta\rho(\mathbf{r}', t')] [\nabla^\alpha \hat{\rho}(\mathbf{r}, t)] [\nabla'^\beta \hat{\rho}(\mathbf{r}', t')], \end{aligned} \quad (5.10)$$

where the factor 2 in the integrand comes from the exchange symmetry between the dummy indices  $\mathbf{r}, t$  and  $\mathbf{r}', t'$ . So, if  $A[\rho, \hat{\rho}]$  is a product of  $p$  density fields and  $q$  response fields, then the term of order  $n_{\text{dis}}$  in equation (5.3) involves products of  $p$  to  $p + 2n_{\text{dis}}$  density fields and  $q + 2n_{\text{dis}}$  response fields. Following the above argument, its Gaussian average vanishes if  $n_m > p + 2n_{\text{dis}} - q - 2n_{\text{dis}} = p - q$  (the bound is imposed by the product with the largest number of density fields), independent of  $n_{\text{dis}}$ . Accordingly, for  $p < q$ , one also gets  $\langle A[\rho, \hat{\rho}] \rangle_{\text{eff}} = 0$ .

Further simplifications might occur in the computation of Gaussian-averaged products when space-time points are repeated. Indeed, through equation (4.5), which also holds with the Gaussian action  $S_0$ , causality directly sets  $\langle \rho(\mathbf{r}_i, t_i) \hat{\rho}(\mathbf{r}_i, t_i) \rangle_0 = 0$ . Less directly, the time ordering in equation (4.5) also imposes the vanishing of certain products of two-point averages with loop-like time dependence. For instance, one gets

$$\langle \rho(\mathbf{r}_i, t_i) \hat{\rho}(\mathbf{r}_j, t_j) \rangle_0 \langle \rho(\mathbf{r}_j, t_j) \hat{\rho}(\mathbf{r}_i, t_i) \rangle_0 = 0, \quad (5.11)$$

$$\langle \rho(\mathbf{r}_i, t_i) \hat{\rho}(\mathbf{r}_j, t_j) \rangle_0 \langle \rho(\mathbf{r}_j, t_j) \hat{\rho}(\mathbf{r}_k, t_k) \rangle_0 \langle \rho(\mathbf{r}_k, t_k) \hat{\rho}(\mathbf{r}_i, t_i) \rangle_0 = 0. \quad (5.12)$$

These equalities typically lead to a reduction in the number of terms in the expansion of Gaussian averages. Occasionally, they result in a truncation of equation (5.6) below the above-mentioned threshold.

These crucial features of the theory were first pointed out by Andreanov *et al* [32] and discussed in detail by Velenich *et al* [38], who demonstrated how they can be used to exactly compute arbitrary multi-point correlation functions in the noninteracting Brownian gas without external field. In this respect, the present work is, to the best of our knowledge, the first nontrivial extension of this early study, aiming at including the effect of a Gaussian quenched-random potential on the gas.

## 6. Dynamical equations for the correlation and response functions

It remains to derive the dynamical equations for the correlation and response functions, to which the above perturbation scheme will be applied. To this end, the following identities can be used, which are easily proved by functional integration by parts:

$$\left\langle \frac{\delta S_{\text{eff}}}{\delta \hat{\rho}(1)} \hat{\rho}(2) \right\rangle_{\text{eff}} = -\delta(12), \quad (6.1a)$$

$$\left\langle \frac{\delta S_{\text{eff}}}{\delta \hat{\rho}(1)} \Lambda(2) \right\rangle_{\text{eff}} = -\rho_0 D_0 \nabla^2 \delta(12), \quad (6.1b)$$

$$\left\langle \frac{\delta S_{\text{eff}}}{\delta \hat{\rho}(1)} \rho(2) \right\rangle_{\text{eff}} = 0. \quad (6.1c)$$

In the above expressions and in the following, the notation 1, 2, 3, etc, is used to refer to space-time points, in order to shorten the equations. Specifically, we set  $1 = (\mathbf{r}, t)$  and  $2 = (\mathbf{r}', t')$ , then  $i = (\mathbf{r}_i, t_i)$ ,  $i \geq 3$ . Since

$$\frac{\delta S_{\text{eff}}}{\delta \hat{\rho}(1)} = i(\partial_t - D_0 \nabla^2) \delta \rho(1) + 2\Lambda(1) - \lambda D_0 \nabla \cdot \left( \rho(1) \int_3 [\nabla \Phi(13)] \Lambda(3) \right), \quad (6.2)$$

where  $\Phi(13) \equiv \Phi(|\mathbf{r} - \mathbf{r}_3|)$ , one obtains the exact equations

$$(\partial_t - D_0 \nabla^2) G(12) = \delta(12) - \lambda D_0 \nabla \cdot \left( \int_3 [\nabla \Phi(13)] \langle \rho(1) \Lambda(3) \hat{\rho}(2) \rangle_{\text{eff}} \right), \quad (6.3a)$$

$$(\partial_t - D_0 \nabla^2) \bar{R}(12) = -\rho_0 D_0 \nabla^2 \delta(12) + \lambda D_0 \nabla \cdot \left( \int_3 [\nabla \Phi(13)] \langle \rho(1) \Lambda(3) \Lambda(2) \rangle_{\text{eff}} \right), \quad (6.3b)$$

$$(\partial_t - D_0 \nabla^2) C(12) = 2\bar{R}(21) - i\lambda D_0 \nabla \cdot \left( \int_3 [\nabla \Phi(13)] \langle \rho(1) \Lambda(3) \rho(2) \rangle_{\text{eff}} \right). \quad (6.3c)$$

These equations show an evident hierarchical structure, which calls for a perturbative study building on an expansion scheme such as the one developed in the previous section. In appendix A, we report an alternative derivation of equation (6.3c) based on the  $\mathcal{T}$ - and  $\mathcal{U}$ -transformations.

Substituting  $\rho(i) = \rho_0 + \delta\rho(i)$  and removing terms that vanish due to the various simple causality conditions, the multi-point averages in equation (6.3) can be simplified to (with the summation convention for the Cartesian indices)

$$\langle \rho(1) \Lambda(3) \hat{\rho}(2) \rangle_{\text{eff}} = D_0 \nabla_3^\gamma \langle \delta\rho(1) \delta\rho(3) [\nabla_3^\gamma \hat{\rho}(3)] \hat{\rho}(2) \rangle_{\text{eff}}, \quad (6.4a)$$

$$\langle \rho(1) \Lambda(3) \Lambda(2) \rangle_{\text{eff}} = D_0^2 \nabla_2^\beta \nabla_3^\gamma \langle \delta\rho(1) [\delta\rho(3) \rho_0 + \rho_0 \delta\rho(2) + \delta\rho(3) \delta\rho(2)] [\nabla_3^\gamma \hat{\rho}(3)] [\nabla_2^\beta \hat{\rho}(2)] \rangle_{\text{eff}}, \quad (6.4b)$$

$$\langle \rho(1) \Lambda(3) \rho(2) \rangle_{\text{eff}} = -i\rho_0 [\bar{R}(13) + \bar{R}(23)] + D_0 \nabla_3^\gamma \langle \delta\rho(1) [\rho_0 + \delta\rho(3)] \delta\rho(2) [\nabla_3^\gamma \hat{\rho}(3)] \rangle_{\text{eff}}. \quad (6.4c)$$

## 7. Zeroth-order theory: disorder-free case

In the absence of a random potential ( $\lambda = 0$ ), the particle system is a noninteracting Brownian gas, whose properties are very well known [38].

In Fourier space, the equations of motion simply reduce to (setting  $t' = 0$ )

$$(\partial_t + \Gamma_k)G_k^0(t) = \delta(t), \quad (7.1a)$$

$$(\partial_t + \Gamma_k)\bar{R}_k^0(t) = \rho_0\Gamma_k\delta(t), \quad (7.1b)$$

$$(\partial_t + \Gamma_k)C_k^0(t) = 2\bar{R}_k^0(-t), \quad (7.1c)$$

where  $\Gamma_k \equiv D_0k^2$  and the superscript 0 on the correlation and response functions denotes the absence of a random potential. The solutions are given by

$$G_k^0(t) = \theta(t)e^{-\Gamma_k t}, \quad (7.2a)$$

$$\bar{R}_k^0(t) = \theta(t)\rho_0\Gamma_k e^{-\Gamma_k t}, \quad (7.2b)$$

$$C_k^0(t) = \rho_0 e^{-\Gamma_k |t|}, \quad (7.2c)$$

where we used the static input for the density correlation function  $C_k^0(0) = \rho_0$ , since  $C(|\mathbf{r} - \mathbf{r}'|, 0) = \langle \delta\rho(\mathbf{r})\delta\rho(\mathbf{r}') \rangle = \rho_0\delta(\mathbf{r} - \mathbf{r}')$  for the noninteracting system in the absence of an external random potential. This  $\rho_0$  factor in  $C_k^0(t)$  is also the one required for consistency with the FDR.

For future use, it is interesting to note that this free dynamics can be fully characterized through suitable specializations of the definitions and symmetry-derived relations given in section 4. Indeed, it appears as the equilibrium dynamics for which equations (4.1c), (4.22a), and (4.22b), reduce to

$$\bar{R}_k^0(t) = \rho_0\Gamma_k G_k^0(t), \quad (7.3a)$$

$$\bar{R}_k^0(t) - \bar{R}_k^0(-t) = -\partial_t C_k^0(t), \quad (7.3b)$$

$$G_k^0(t) + G_k^0(-t) = \frac{C_k^0(t)}{\rho_0}, \quad (7.3c)$$

thereby demonstrating that the three functions of interest are directly related in a simple but fundamental way. In this respect, it should be fully appreciated that the considered dynamics involves both multiplicative noise and a non-Gaussian free energy. Therefore, the absence of an anomalous contribution to the physical response function in equation (7.3a) and the validity of the DHMR linear relation shown by equation (7.3c) are nontrivial observations. They result from a specific interplay of both aspects and from the cancellation effects discussed in section 5.

Regarding this, it might be useful to briefly show how the field-theoretic calculation unfolds in the present simple case. This serves as a preparation for the more complicated random-field situation and as a confirmation of the identity between the

correlation and response functions of the disorder-free noninteracting gas and those of the Gaussian theory based on  $S_0[\rho, \hat{\rho}]$ . To this end, we introduce the compact notations

$$\delta\rho(i) \equiv i, \quad \hat{\rho}(i) \equiv \hat{i}, \quad \nabla_i^\mu \hat{\rho}(i) \equiv \hat{i}^\mu, \quad (7.4)$$

to be used for the evaluation of averages here, in the next section, and in appendices C–E. With them, the cubic thermal noise term, equation (5.5), can be written as

$$S_m[\rho, \hat{\rho}] \equiv -D_0 \int_4 4\hat{4}^\delta \hat{4}^\delta, \quad (7.5)$$

and  $e^{S_m[\rho, \hat{\rho}]}$  in equation (5.6) expanded accordingly,

$$e^{S_m[\rho, \hat{\rho}]} = 1 - D_0 \int_4 4\hat{4}^\delta \hat{4}^\delta + \frac{1}{2} D_0^2 \int_4 \int_5 45\hat{4}^\delta \hat{4}^\delta \hat{5}^\varepsilon \hat{5}^\varepsilon + \dots \quad (7.6)$$

From equation (4.1b), we get  $G^0(12) = -i\langle \rho(1)\hat{\rho}(2) \rangle_f = -i\langle \delta\rho(1)\hat{\rho}(2) \rangle_f \equiv -i\langle 1\hat{2} \rangle_f$ . With one density field and one noise-response field, the expansion (5.6) terminates at its first term and

$$\langle 1\hat{2} \rangle_f = \langle 1\hat{2} \rangle_0. \quad (7.7)$$

The anomalous term in  $\bar{R}^0(12)$  reads  $iD_0 \nabla_2^\beta \langle \rho(1)\delta\rho(2)\nabla_2^\beta \hat{\rho}(2) \rangle_f = iD_0 \nabla_2^\beta \langle \delta\rho(1)\delta\rho(2)\nabla_2^\beta \hat{\rho}(2) \rangle_f \equiv iD_0 \nabla_2^\beta \langle 12\hat{2}^\beta \rangle_f$ , and the Gaussian expansion of  $\langle 12\hat{2}^\beta \rangle_f$  is

$$\langle 12\hat{2}^\beta \rangle_f = \langle 12\hat{2}^\beta \rangle_0 - D_0 \int_4 \langle 124\hat{2}^\beta \hat{4}^\delta \hat{4}^\delta \rangle_0 = 0, \quad (7.8)$$

where we used  $\langle 2\hat{2}^\beta \rangle_0 = \langle 4\hat{4}^\delta \rangle_0 = 0$  and  $\langle 2\hat{4}^\delta \rangle_0 \langle 4\hat{2}^\beta \rangle_0 = 0$ . Therefore,  $\bar{R}^0(12) = -\rho_0 D_0 \nabla^2 G^0(12)$  as expected. Finally,  $C^0(12) = \langle \delta\rho(1)\delta\rho(2) \rangle_f \equiv \langle 12 \rangle_f$  expands to

$$\langle 12 \rangle_f = \langle 12 \rangle_0 - D_0 \int_4 \langle 124\hat{4}^\delta \hat{4}^\delta \rangle_0 + \frac{1}{2} D_0^2 \int_4 \int_5 \langle 1245\hat{4}^\delta \hat{4}^\delta \hat{5}^\varepsilon \hat{5}^\varepsilon \rangle_0 = \langle 12 \rangle_0, \quad (7.9)$$

where we used  $\langle 4\hat{4}^\delta \rangle_0 = \langle 5\hat{5}^\varepsilon \rangle_0 = 0$  and  $\langle 4\hat{5}^\varepsilon \rangle_0 \langle 5\hat{4}^\delta \rangle_0 = 0$ .

## 8. First-order perturbation calculation

We may now perturbatively compute the three-point averages in equation (6.3) and obtain the first-order corrections to the free dynamics due to the random potential.

Applying equation (5.3) to the different terms in the simplified equation (6.4), one gets

$$\langle \delta\rho(1)\delta\rho(3)[\nabla_3^\gamma \hat{\rho}(3)]\hat{\rho}(2) \rangle_{\text{eff}} \equiv \langle 13\hat{3}^\gamma \hat{2} \rangle_{\text{eff}} = \langle 13\hat{3}^\gamma \hat{2} \rangle_f + O(\lambda), \quad (8.1a)$$

$$\langle \delta\rho(1)\delta\rho(3)[\nabla_3^\gamma \hat{\rho}(3)][\nabla_2^\beta \hat{\rho}(2)] \rangle_{\text{eff}} \equiv \langle 13\hat{3}^\gamma \hat{2}^\beta \rangle_{\text{eff}} = \langle 13\hat{3}^\gamma \hat{2}^\beta \rangle_f + O(\lambda), \quad (8.1b)$$

$$\langle \delta\rho(1)\delta\rho(2)[\nabla_3^\gamma \hat{\rho}(3)][\nabla_2^\beta \hat{\rho}(2)] \rangle_{\text{eff}} \equiv \langle 12\hat{3}^\gamma \hat{2}^\beta \rangle_{\text{eff}} = \langle 12\hat{3}^\gamma \hat{2}^\beta \rangle_f + O(\lambda), \quad (8.1c)$$

$$\langle \delta\rho(1)\delta\rho(3)\delta\rho(2)[\nabla_3^\gamma\hat{\rho}(3)][\nabla_2^\beta\hat{\rho}(2)] \rangle_{\text{eff}} \equiv \langle 132\hat{3}^\gamma\hat{2}^\beta \rangle_{\text{eff}} = \langle 132\hat{3}^\gamma\hat{2}^\beta \rangle_f + O(\lambda), \quad (8.1d)$$

$$\langle \delta\rho(1)\delta\rho(2)[\nabla_3^\gamma\hat{\rho}(3)] \rangle_{\text{eff}} \equiv \langle 12\hat{3}^\gamma \rangle_{\text{eff}} = \langle 12\hat{3}^\gamma \rangle_f + O(\lambda), \quad (8.1e)$$

$$\langle \delta\rho(1)\delta\rho(3)\delta\rho(2)[\nabla_3^\gamma\hat{\rho}(3)] \rangle_{\text{eff}} \equiv \langle 132\hat{3}^\gamma \rangle_{\text{eff}} = \langle 132\hat{3}^\gamma \rangle_f + O(\lambda), \quad (8.1f)$$

where we used the compact notations introduced above. For a first-order calculation, it is enough to compute the first term in the right-hand side of each line in equation (8.1), since the contributions in which the three-point averages appear in equation (6.3) already involve  $\lambda$  as a prefactor.

With equation (5.6), the free averages are turned into Gaussian averages defined through equation (5.7). As discussed in section 5, the number of useful terms in equation (5.6) is *a priori* determined by the number of  $\delta\rho$  and  $\hat{\rho}$  fields in the quantity to be averaged, through the requirements of causality.

The first average in equation (8.1) is thus obtained as

$$\langle 13\hat{3}^\gamma\hat{2} \rangle_f = \langle 13\hat{3}^\gamma\hat{2} \rangle_0 = \langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{2} \rangle_0, \quad (8.2a)$$

where we used  $\langle \hat{3}^\gamma\hat{2} \rangle_0 = 0$  and  $\langle 3\hat{3}^\gamma \rangle_0 = 0$ . Similarly, the second and third are

$$\langle 13\hat{3}^\gamma\hat{2}^\beta \rangle_f = \langle 13\hat{3}^\gamma\hat{2}^\beta \rangle_0 = \langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{2}^\beta \rangle_0, \quad (8.2b)$$

$$\langle 12\hat{3}^\gamma\hat{2}^\beta \rangle_f = \langle 12\hat{3}^\gamma\hat{2}^\beta \rangle_0 = \langle 1\hat{2}^\beta \rangle_0 \langle 2\hat{3}^\gamma \rangle_0. \quad (8.2c)$$

The average  $\langle 132\hat{3}^\gamma\hat{2}^\beta \rangle_f$  is shown to vanish,

$$\langle 132\hat{3}^\gamma\hat{2}^\beta \rangle_f = \langle 132\hat{3}^\gamma\hat{2}^\beta \rangle_0 - D_0 \int_4 \langle 1324\hat{3}^\gamma\hat{2}^\beta\hat{4}^\delta\hat{4}^\delta \rangle_0 = 0, \quad (8.2d)$$

since  $\langle 3\hat{4}^\delta \rangle_0 \langle 4\hat{3}^\gamma \rangle_0 = 0$  and  $\langle 3\hat{2}^\beta \rangle_0 \langle 2\hat{4}^\delta \rangle_0 \langle 4\hat{3}^\gamma \rangle_0 = 0$ . One analogously gets

$$\langle 12\hat{3}^\gamma \rangle_f = \langle 12\hat{3}^\gamma \rangle_0 - D_0 \int_4 \langle 124\hat{3}^\gamma\hat{4}^\delta\hat{4}^\delta \rangle_0 = -2D_0 \int_4 \langle 1\hat{4}^\delta \rangle_0 \langle 2\hat{4}^\delta \rangle_0 \langle 4\hat{3}^\gamma \rangle_0. \quad (8.2e)$$

Note that the effect of the multiplicative noise enters in equation (8.2e), making a nonperturbative contribution from the point of view of the free dynamics. Finally, one computes the remaining average as

$$\begin{aligned} \langle 132\hat{3}^\gamma \rangle_f &= \langle 132\hat{3}^\gamma \rangle_0 - D_0 \int_4 \langle 1324\hat{3}^\gamma\hat{4}^\delta\hat{4}^\delta \rangle_0 + \frac{1}{2}D_0^2 \int_4 \int_5 \langle 13245\hat{3}^\gamma\hat{4}^\delta\hat{4}^\delta\hat{5}^\varepsilon\hat{5}^\varepsilon \rangle_0 \\ &= \langle 1\hat{3}^\gamma \rangle_0 \langle 32 \rangle_0 + \langle 13 \rangle_0 \langle 2\hat{3}^\gamma \rangle_0, \end{aligned} \quad (8.2f)$$

since one gets  $\langle 13245\hat{3}^\gamma\hat{4}^\delta\hat{4}^\delta\hat{5}^\varepsilon\hat{5}^\varepsilon \rangle_0 = 0$  due to the repeated space-time points.

Taking the necessary spatial derivatives of the nonvanishing terms, one finally gets

$$\nabla_3^\gamma \langle 13\hat{3}^\gamma\hat{2} \rangle_f = \nabla_3^\gamma [\langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{2} \rangle_0] = \nabla_3^\beta [\langle 1\hat{3}^\beta \rangle_0 \langle 3\hat{2} \rangle_0], \quad (8.3a)$$

$$\nabla_2^\beta \nabla_3^\gamma \langle 13\hat{3}^\gamma\hat{2}^\beta \rangle_f = \nabla_2^\beta \nabla_3^\gamma [\langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{2}^\beta \rangle_0] = \nabla_3^\gamma [\langle 1\hat{3}^\gamma \rangle_0 \nabla_2^\beta \langle 3\hat{2} \rangle_0] = \nabla_3^\beta [\langle 1\hat{3}^\beta \rangle_0 \nabla_3^\gamma \langle 3\hat{2} \rangle_0], \quad (8.3b)$$

$$\nabla_2^\beta \nabla_3^\gamma \langle 12\hat{3}^\gamma \hat{2}^\beta \rangle_f = \nabla_2^\beta \nabla_3^\gamma [\langle 1\hat{2}^\beta \rangle_0 \langle 2\hat{3}^\gamma \rangle_0] = \nabla_2^\beta [\langle 1\hat{2}^\beta \rangle_0 \nabla_3^2 \langle 2\hat{3} \rangle_0] = \nabla_2^\beta [\langle 1\hat{2}^\beta \rangle_0 \nabla_2^2 \langle 2\hat{3} \rangle_0], \quad (8.3c)$$

$$\begin{aligned} \nabla_3^\gamma \langle 12\hat{3}^\gamma \rangle_f &= \nabla_3^\gamma \left[ -2D_0 \int_4 \langle 1\hat{4}^\delta \rangle_0 \langle 2\hat{4}^\delta \rangle_0 \langle 4\hat{3}^\gamma \rangle_0 \right] = -2D_0 \int_4 \langle 1\hat{4}^\delta \rangle_0 \langle 2\hat{4}^\delta \rangle_0 \nabla_3^2 \langle 4\hat{3} \rangle_0 \\ &= -2D_0 \int_4 \langle 1\hat{4}^\beta \rangle_0 \langle 2\hat{4}^\beta \rangle_0 \nabla_4^2 \langle 4\hat{3} \rangle_0 \end{aligned} \quad (8.3d)$$

$$\nabla_3^\gamma \langle 132\hat{3}^\gamma \rangle_f = \nabla_3^\gamma [\langle 1\hat{3}^\gamma \rangle_0 \langle 32 \rangle_0 + \langle 13 \rangle_0 \langle 2\hat{3}^\gamma \rangle_0] = \nabla_3^\beta [\langle 1\hat{3}^\beta \rangle_0 \langle 32 \rangle_0 + \langle 13 \rangle_0 \langle 2\hat{3}^\beta \rangle_0]. \quad (8.3e)$$

In the final expressions, all dummy Cartesian indices have been uniformly denoted by  $\beta$ .

With these results, the dynamical equations can be written down, up to the first order of the disorder-strength expansion. Restoring the explicit field notation, they read :

$$(\partial_t - D_0 \nabla^2)G(12) = \delta(12) - \lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\beta \{ \langle \delta\rho(1) \nabla_3^\beta \hat{\rho}(3) \rangle_0 \langle \delta\rho(3) \hat{\rho}(2) \rangle_0 \} \right), \quad (8.4a)$$

$$\begin{aligned} (\partial_t - D_0 \nabla^2)\bar{R}(12) &= -\rho_0 D_0 \nabla^2 \delta(12) \\ &\quad - i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\beta \{ \langle \delta\rho(1) \nabla_3^\beta \hat{\rho}(3) \rangle_0 [i\rho_0 D_0 \nabla_3^2 \langle \delta\rho(3) \hat{\rho}(2) \rangle_0] \} \right) \\ &\quad - i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_2^\beta \{ \langle \delta\rho(1) \nabla_2^\beta \hat{\rho}(2) \rangle_0 [i\rho_0 D_0 \nabla_2^2 \langle \delta\rho(2) \hat{\rho}(3) \rangle_0] \} \right), \end{aligned} \quad (8.4b)$$

$$\begin{aligned} (\partial_t - D_0 \nabla^2)C(12) &= 2\bar{R}(21) - \lambda \rho_0 D_0 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] [\bar{R}^0(13) + \bar{R}^0(23)] \right) \\ &\quad - i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\beta \{ \langle \delta\rho(1) \nabla_3^\beta \hat{\rho}(3) \rangle_0 \langle \delta\rho(3) \delta\rho(2) \rangle_0 + \langle \delta\rho(1) \delta\rho(3) \rangle_0 \langle \delta\rho(2) \nabla_3^\beta \hat{\rho}(3) \rangle_0 \} \right) \\ &\quad + 2\lambda D_0^2 \nabla^\alpha \left( \int_3 \int_4 [\nabla^\alpha \Phi(13)] \langle \delta\rho(1) \nabla_4^\beta \hat{\rho}(4) \rangle_0 \langle \delta\rho(2) \nabla_4^\beta \hat{\rho}(4) \rangle_0 [i\rho_0 D_0 \nabla_4^2 \langle \delta\rho(4) \hat{\rho}(3) \rangle_0] \right). \end{aligned} \quad (8.4c)$$

In these evolution equations, there are four space-time integrals in which the time integral can actually be detached from the corresponding space integral. We shall refer to these situations as isolated time integrals, which are due to the nonlocality in time induced by the quenched randomness. Indeed, they appear when a space-time integral acts on a variable which is present both in the time-independent random-field covariance and in a single time-dependent response function. Then, the time integral obviously acts on the response function only. We will next focus on these isolated time integrals to structure our analysis.

In our derivation, two of these isolated time integrals are directly obtained as  $\int dt_3 \bar{R}^0(13)$  and  $\int dt_3 \bar{R}^0(23)$ . They correspond to the first integral in equation (8.4c) and originate from the first term in equation (6.4c). We have purposefully arranged the above formulas to make the two others specifically appear as  $\int dt_3 [i\rho_0 D_0 \nabla_2^2 \langle \delta\rho(2) \hat{\rho}(3) \rangle_0]$  and  $\int dt_3 [i\rho_0 D_0 \nabla_4^2 \langle \delta\rho(4) \hat{\rho}(3) \rangle_0]$ , in the last integrals of equations (8.4b) and (8.4c), respectively. Indeed, although it might look like there are two distinct types of isolated

time integrals, our claim is that the difference is only superficial. To see this, it must be kept in mind that, within the zeroth-order theory, there is no distinction between  $\bar{R}^0(12)$  and  $-\rho_0 D_0 \nabla^2 G^0(12)$ . Therefore, one can safely replace  $i\rho_0 D_0 \nabla_2^2 \langle \delta\rho(2) \hat{\rho}(3) \rangle_0$  and  $i\rho_0 D_0 \nabla_4^2 \langle \delta\rho(4) \hat{\rho}(3) \rangle_0$  with  $\bar{R}^0(23)$  and  $\bar{R}^0(43)$  in the corresponding integrals. A direct hint in favor of this substitution is provided by a third appearance of this specific combination,  $i\rho_0 D_0 \nabla_3^2 \langle \delta\rho(3) \hat{\rho}(2) \rangle_0$ , in the first integral of equation (8.4b). Indeed, it is only when it is interpreted as  $\bar{R}^0(32)$  that the equations for  $G$ ,  $\bar{R}$ , and  $C$ , share the typical structure of the Schwinger–Dyson equation with the same self-energy. Accordingly, we translate equation (8.4) as

$$(\partial_t - D_0 \nabla^2)G(12) = \delta(12) + \lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\beta \left\{ [\nabla_3^\beta G^0(13)] G^0(32) \right\} \right), \quad (8.5a)$$

$$\begin{aligned} (\partial_t - D_0 \nabla^2)\bar{R}(12) &= -\rho_0 D_0 \nabla^2 \delta(12) + \lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\beta \left\{ [\nabla_3^\beta G^0(13)] \bar{R}^0(32) \right\} \right) \\ &\quad + \lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_2^\beta \left\{ [\nabla_2^\beta G^0(12)] \bar{R}^0(23) \right\} \right), \end{aligned} \quad (8.5b)$$

$$\begin{aligned} (\partial_t - D_0 \nabla^2)C(12) &= 2\bar{R}(21) - \lambda \rho_0 D_0 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] [\bar{R}^0(13) + \bar{R}^0(23)] \right) \\ &\quad + \lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\beta \left\{ [\nabla_3^\beta G^0(13)] C^0(32) + C^0(13) [\nabla_3^\beta G^0(23)] \right\} \right) \\ &\quad - 2\lambda D_0^2 \nabla^\alpha \left( \int_3 \int_4 [\nabla^\alpha \Phi(13)] [\nabla_4^\beta G^0(14)] [\nabla_4^\beta G^0(24)] \bar{R}^0(43) \right). \end{aligned} \quad (8.5c)$$

Note that, when the bare perturbation expansion is pushed to the second order, one can actually recognize the first-order expansion of  $\bar{R}$  precisely at the places where the proposed substitution is possible, as seen in the derivations of equations (D.12) and (E.15) in appendices D and E. As a corollary, the first-order renormalized theory deriving from the second-order bare theory also features isolated time integrals that are mere integrals of the now renormalized density response function, as seen in equations (D.13) and (E.16). These observations clearly lend further support to the above substitutions. More broadly, they hint at the possibility of a generic reduction of the isolated time integrals to integrals of the physical response function within the present framework, although a formal proof hereof is currently lacking.

Finally, once an isolated time integral is expressed as an integral of the physical response function, any reference to the corresponding space-time point can be fully eliminated, thanks to equations (4.13) and (3.5) giving

$$\int_{-\infty}^{+\infty} dt' \bar{R}(|\mathbf{r} - \mathbf{r}'|, t - t') = \rho_0 \delta(\mathbf{r} - \mathbf{r}'). \quad (8.6)$$

This relation has for sole basic ingredients the exact FDR and the exact equilibrium statistical mechanics of ideal gases. It thus holds nonperturbatively as well as at any order in  $\lambda$  of the present FDR-preserving perturbation scheme. Although technically unrelated to the substitutions advocated above, it acts as a natural continuation

thereof, making the structure of the dynamical equations immediately simpler. Thus, specializing equation (8.6) to the equilibrium free dynamics with  $\bar{R} \equiv \bar{R}^0$ , one eventually gets from equation (8.5) (after some rearrangements using integrations by parts and space-translation invariance to make all spatial derivatives act on  $1 = (\mathbf{r}, t)$ ),

$$(\partial_t - D_0 \nabla^2) G(12) = \delta(12) - \lambda D_0^2 \int_3 \nabla^\alpha ([\nabla^\alpha \nabla^\beta \Phi(13)][\nabla^\beta G^0(13)]) G^0(32), \quad (8.7a)$$

$$\begin{aligned} (\partial_t - D_0 \nabla^2) \bar{R}(12) &= -\rho_0 D_0 \nabla^2 \delta(12) - \lambda D_0^2 \int_3 \nabla^\alpha ([\nabla^\alpha \nabla^\beta \Phi(13)][\nabla^\beta G^0(13)]) \bar{R}^0(32) \\ &\quad + \lambda \rho_0 D_0^2 \nabla^\alpha \nabla^\beta ([\nabla^\alpha \Phi(12)][\nabla^\beta G^0(12)]), \end{aligned} \quad (8.7b)$$

$$\begin{aligned} (\partial_t - D_0 \nabla^2) C(12) &= 2\bar{R}(21) - \lambda \rho_0^2 D_0 \nabla^2 \Phi(12) \\ &\quad - \lambda D_0^2 \int_3 \nabla^\alpha ([\nabla^\alpha \nabla^\beta \Phi(13)][\nabla^\beta G^0(13)]) C^0(32) \\ &\quad + \lambda D_0^2 \int_3 \nabla^\alpha \nabla^\beta ([\nabla^\alpha \nabla^\beta \Phi(13)] C^0(13)) G^0(23) \\ &\quad + 2\lambda \rho_0 D_0^2 \int_3 \nabla^\alpha \nabla^\beta ([\nabla^\alpha \Phi(13)][\nabla^\beta G^0(13)]) G^0(23). \end{aligned} \quad (8.7c)$$

One sees that the time integral  $\int dt_3 \bar{R}^0(23)$  has generated a mere time-persistent term in the equation for the density correlation function (the contribution from  $\int dt_3 \bar{R}^0(13)$  vanishes by isotropy of the random field), while the last term in equation (8.7b) is now evidently local in time. The last term in equation (8.7c) is entirely due to the multiplicative thermal noise (see the comment about equation (8.2e) above).

The latter equations will be the basis for all developments in the remainder of this work.

## 9. Equilibrium dynamics: first-order bare theory

By itself, equation (8.7) forms an FOBT for the equilibrium dynamics of a noninteracting Brownian gas plunged in a quenched Gaussian random field. After Fourier transformation, under the assumption of time-translation invariance, one gets the following equilibrium dynamical equations (setting  $t \geq t' = 0$ ),

$$(\partial_t + \Gamma_k) G_k(t) = \delta(t) - \int_0^t ds \Sigma_k^0(t-s) G_k^0(s), \quad (9.1a)$$

$$(\partial_t + \Gamma_k) \bar{R}_k(t) = \rho_0 \Gamma_k \delta(t) - \int_0^t ds \Sigma_k^0(t-s) \bar{R}_k^0(s) + L_k^0(t), \quad (9.1b)$$

$$(\partial_t + \Gamma_k) C_k(t) = \lambda \rho_0^2 \Gamma_k \Phi_k - \int_0^t ds \Sigma_k^0(t-s) C_k^0(s) + N_k^0(t). \quad (9.1c)$$

There appear three memory kernels. The memory functions  $\Sigma_k^0(t)$  and  $L_k^0(t)$  are explicitly given by

$$\Sigma_k^0(t) = \lambda D_0^2 \int_{\mathbf{q}} \mathbf{q} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p^0(t), \quad (9.2)$$

$$L_k^0(t) = \lambda \rho_0 D_0^2 \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p^0(t), \quad (9.3)$$

where  $\int_{\mathbf{q}} \equiv \int d\mathbf{q}/(2\pi)^d$  and  $\mathbf{p} \equiv \mathbf{k} - \mathbf{q}$ . Note that the kernel  $L_k^0(t)$  would be absent in the usual case of a Langevin equation with additive thermal noise. It is hence associated with the anomalous part of the physical response function, arising from the multiplicative nature of the basic stochastic equation for the density variable.

In fact, one can further investigate the origin of  $L_k^0(t)$  by going back to the initial dynamics. Indeed, within the operator formalism of Martin, Siggia, and Rose [62, 83], an evolution equation for  $\bar{R}(\mathbf{r}, t; \mathbf{r}', t')$  can be obtained from equation (2.15), through multiplication by  $i\Lambda(\mathbf{r}', t')$  and double-averaging over thermal fluctuations and disorder. The contribution of the random forces deriving from the external potential then reads ( $\nabla$  and  $\nabla'$  act on  $\mathbf{r}$  and  $\mathbf{r}'$ , respectively)

$$i \frac{D_0}{T} \overline{\langle \nabla \cdot [\rho(\mathbf{r}, t) \nabla v(\mathbf{r})] \Lambda(\mathbf{r}', t') \rangle} = i \frac{D_0^2}{T} \nabla \cdot \overline{\langle \rho(\mathbf{r}, t) \nabla' \cdot [\rho(\mathbf{r}', t') \nabla' \hat{\rho}(\mathbf{r}', t')] \rangle \nabla v(\mathbf{r})}, \quad (9.4)$$

where a realization-dependent physical response function is clearly visible. Now, we may split  $\rho(\mathbf{r}', t')$  as  $\langle \rho(\mathbf{r}') \rangle + \rho(\mathbf{r}', t') - \langle \rho(\mathbf{r}') \rangle$ , where  $\langle \rho(\mathbf{r}') \rangle$  corresponds to the static density profile induced by the random field and  $\rho(\mathbf{r}', t') - \langle \rho(\mathbf{r}') \rangle$  to the thermal fluctuations about this profile. Focusing on the first contribution, one gets

$$i \frac{D_0^2}{T} \nabla \cdot \overline{\langle \rho(\mathbf{r}, t) \nabla' \cdot [\langle \rho(\mathbf{r}') \rangle \nabla' \hat{\rho}(\mathbf{r}', t')] \rangle \nabla v(\mathbf{r})} = i \frac{D_0^2}{T} \nabla^\alpha \nabla'^\beta \overline{[\nabla'^\beta \langle \rho(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t') \rangle] [\nabla^\alpha \{ \langle \rho(\mathbf{r}') \rangle v(\mathbf{r}) \}]}, \quad (9.5)$$

where a noise-response function appears. If the latter is evaluated with respect to the free dynamics, in the spirit of the present FOBT, the averages factorize and one obtains

$$\begin{aligned} i \frac{D_0^2}{T} \nabla^\alpha \nabla'^\beta \overline{[\nabla'^\beta \langle \rho(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t') \rangle] [\nabla^\alpha \{ \langle \rho(\mathbf{r}') \rangle v(\mathbf{r}) \}]} \\ = i \frac{D_0^2}{T} \nabla^\alpha \nabla'^\beta \left\{ [\nabla'^\beta \langle \rho(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t') \rangle] [\nabla^\alpha \overline{\langle \rho(\mathbf{r}') \rangle v(\mathbf{r})}] \right\} \\ = \lambda \rho_0 D_0^2 \nabla^\alpha \nabla'^\beta \left( [\nabla'^\beta G^0(\mathbf{r}, t; \mathbf{r}', t')] [\nabla^\alpha \Phi(|\mathbf{r} - \mathbf{r}'|)] \right) = L^0(\mathbf{r}, t; \mathbf{r}', t'), \end{aligned} \quad (9.6)$$

where the real-space expression for  $L^0(\mathbf{r}, t; \mathbf{r}', t')$  is read off equation (8.7b). In these final steps, we used equation (3.3a) to compute the disorder average over the Gaussian random field, and translational invariance to replace  $\nabla'^\beta$  with  $-\nabla^\beta$ . Eventually, it thus appears that the kernel  $L_k^0(t)$  arises, at least in part, from the interplay of the multiplicative nature of the thermal noise and of the static density heterogeneities imprinted in the fluid by the random external potential. Note that, if one repeats all these steps in the case of the density correlation function, i.e. starting with equation (2.15) multiplied by  $\rho(\mathbf{r}', t')$  and double-averaged, one obtains the term  $-\lambda \rho_0^2 D_0 \nabla^2 \Phi(|\mathbf{r} - \mathbf{r}'|)$  of

equation (8.7c), which gives  $\lambda\rho_0^2\Gamma_k\Phi_k$  in equation (9.1c). Indeed, to show this, one begins with

$$\frac{D_0}{T}\overline{\langle\nabla\cdot[\rho(\mathbf{r},t)\nabla v(\mathbf{r})]\rho(\mathbf{r}',t')\rangle}=\frac{D_0}{T}\nabla\cdot\overline{\langle\rho(\mathbf{r},t)\rho(\mathbf{r}',t')\rangle\nabla v(\mathbf{r})}, \quad (9.7)$$

which, after replacing  $\rho(\mathbf{r}',t')$  with its thermal average, becomes

$$\frac{D_0}{T}\nabla\cdot\overline{\langle\rho(\mathbf{r},t)\langle\rho(\mathbf{r}')\rangle\rangle\nabla v(\mathbf{r})}=\frac{D_0}{T}\nabla\cdot\overline{\langle\rho(\mathbf{r},t)\rangle[\nabla\{\langle\rho(\mathbf{r}')\rangle v(\mathbf{r})\}]}. \quad (9.8)$$

Then, if  $\rho(\mathbf{r},t)$  is set to evolve according to the free dynamics, one gets

$$\frac{D_0}{T}\nabla\cdot\overline{\langle\rho(\mathbf{r},t)\rangle_t[\nabla\{\langle\rho(\mathbf{r}')\rangle v(\mathbf{r})\}]}=\frac{D_0}{T}\nabla\cdot\left\{\langle\rho(\mathbf{r},t)\rangle_t[\nabla\overline{\langle\rho(\mathbf{r}')\rangle v(\mathbf{r})}]\right\}=-\lambda\rho_0^2D_0\nabla^2\Phi(|\mathbf{r}-\mathbf{r}'|), \quad (9.9)$$

as announced. As discussed below, this contribution is clearly an outgrowth of the disorder-induced static density profile.

The kernel  $N_k^0(t)$  originally consists of three integrals,

$$\begin{aligned} N_k^0(t) &= -\lambda D_0^2 \int_{-\infty}^0 ds \int_{\mathbf{q}} \mathbf{q} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p^0(t-s) C_k^0(s) \\ &\quad + \lambda D_0^2 \int_{-\infty}^0 ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{q} [\mathbf{k} \cdot \mathbf{q} \Phi_q] C_p^0(t-s) G_k^0(-s) \\ &\quad + 2\lambda\rho_0 D_0^2 \int_{-\infty}^0 ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p^0(t-s) G_k^0(-s), \end{aligned} \quad (9.10)$$

but actually reduces to a local function of time if one uses the identities (7.3) to rearrange this expression. Indeed, using equation (7.3c) to distribute the last integral over the first two, one gets

$$N_k^0(t) = \lambda D_0^2 \int_{-\infty}^0 ds \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] [p^2 G_p^0(t-s) C_k^0(s) + C_p^0(t-s) k^2 G_k^0(-s)], \quad (9.11)$$

which, with equation (7.3a) followed by equation (7.3b), leads to

$$N_k^0(t) = \frac{\lambda D_0}{\rho_0} \int_{-\infty}^0 ds \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \partial_s [C_p^0(t-s) C_k^0(s)] = \lambda D_0 \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] C_p^0(t). \quad (9.12)$$

The memory functions have to be related with one another, in order for equation (9.1) to obey the FDR. Using  $\rho_0\Gamma_p G_p^0(t) = \overline{R}_p^0(t)$  to rewrite

$$\Sigma_k^0(t) = \lambda \frac{D_0}{\rho_0} \int_{\mathbf{q}} \frac{\mathbf{q} \cdot \mathbf{p}}{p^2} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \overline{R}_p^0(t), \quad (9.13)$$

$$L_k^0(t) = \lambda D_0 \int_{\mathbf{q}} \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \overline{R}_p^0(t), \quad (9.14)$$

and forming the combination  $\rho_0\Sigma_k^0(t) - L_k^0(t)$ , one immediately finds that the kernels obey the FDR-like relation

$$\rho_0 \Sigma_k^0(t) - L_k^0(t) = \partial_t N_k^0(t) \quad (9.15)$$

as a mere corollary of the FDR  $\bar{R}_p^0(t) = -\partial_t C_p^0(t)$ .

Another interesting rearrangement of equation (9.10) through  $\rho_0 \Gamma_k G_k^0(t) = \bar{R}_k^0(t)$  is [31, 82]

$$N_k^0(t) = - \int_{-\infty}^0 ds \Sigma_k^0(t-s) C_k^0(s) + \int_{-\infty}^0 ds D_k^0(t-s) \bar{R}_k^0(-s), \quad (9.16)$$

where the new kernel  $D_k^0(t)$  consists of two parts:

$$D_k^0(t) \equiv M_k^0(t) + \frac{2}{\rho_0 \Gamma_k} L_k^0(t), \quad (9.17)$$

$$M_k^0(t) \equiv \frac{\lambda D_0}{\rho_0} \int_{\mathbf{q}} (\hat{\mathbf{k}} \cdot \mathbf{q})^2 \Phi_q C_p^0(t). \quad (9.18)$$

Here,  $\hat{\mathbf{k}}$  denotes the unit vector  $\mathbf{k}/k$  and  $M_k^0(t)$  turns out to be the MCT memory kernel [39–42], albeit in its ‘bare’ form (see below). It is then straightforward to show that  $M_k^0(t)$  is related to  $N_k^0(t)$  and  $L_k^0(t)$  (after using  $\rho_0 G_p^0(t) = C_p^0(t)$  in equation (9.3)) as

$$M_k^0(t) = \frac{1}{\rho_0} \left( N_k^0(t) - \frac{L_k^0(t)}{\Gamma_k} \right). \quad (9.19)$$

In combination with equations (9.15) and (9.17), this immediately leads to

$$D_k^0(t) = \frac{1}{\rho_0} \left( N_k^0(t) + \frac{L_k^0(t)}{\Gamma_k} \right), \quad (9.20)$$

$$\partial_t D_k^0(t) = \Sigma_k^0(t) + \frac{1}{\rho_0 \Gamma_k} (\partial_t - \Gamma_k) L_k^0(t). \quad (9.21)$$

Again, one can see that the presence of  $L_k^0(t)$  deeply changes the structure of the dynamics. Indeed, if the kernel  $L_k^0(t)$  were absent, one would simply get  $D_k^0(t) = M_k^0(t) = N_k^0(t)/\rho_0$  with the familiar relation  $\partial_t D_k^0(t) = \Sigma_k^0(t)$ , as found in the case of Langevin dynamics with additive thermal noise [82, 84].

We now consider some key features of these dynamical equations.

### 9.1. Consistency with the FDR

The present perturbation expansion is dictated by the time reversal invariance of the effective dynamical action. It is hence guaranteed to preserve the FDR at each order of the expansion. This is confirmed by explicitly showing that the above first-order dynamical equations for  $\bar{R}_k(t)$  and  $C_k(t)$  are indeed consistent with the FDR.

Taking the time derivative of the FDR,  $\bar{R}_k(t) = -\theta(t) \partial_t C_k(t)$ , one gets

$$\partial_t \bar{R}_k(t) = -\delta(t) \partial_t C_k(0) - \theta(t) \partial_t^2 C_k(t). \quad (9.22)$$

With the second derivative of  $C_k(t)$  obtained from equation (9.1c),

$$\partial_t^2 C_k(t) = -\Gamma_k \partial_t C_k(t) - \Sigma_k^0(t) C_k^0(0) - \int_0^t ds \Sigma_k^0(t-s) \partial_s C_k^0(s) + \partial_t N_k^0(t). \quad (9.23)$$

Equation (9.22) takes the form

$$\left(\partial_t + \Gamma_k\right) \bar{R}_k(t) = -\delta(t) \partial_t C_k(0) - \int_0^t ds \Sigma_k^0(t-s) \bar{R}_k^0(s) + L_k^0(t), \quad (9.24)$$

where equations (4.22a) and (9.15) ( $C_k^0(0) = \rho_0$ ) have been used.

Comparing equation (9.24) with equation (9.1b), we see that the dynamics obeys the FDR under the condition  $\rho_0 \Gamma_k = -\partial_t C_k(0) = \Gamma_k [C_k(0) - \lambda \rho_0^2 \Phi_k]$ , where the last equality follows from equation (9.1c) at  $t = 0$ , knowing that  $N_k^0(0) = \lambda \rho_0 D_0 \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{q} \Phi_q = 0$  by isotropy. This requires that

$$C_k(0) = \rho_0 + \lambda \rho_0^2 \Phi_k. \quad (9.25)$$

## 9.2. Presence of a static nonvanishing component

Since the memory terms in equation (9.1) only involve the bare correlation and response functions that are exponentially relaxing in time, the present FOBT does not sustain the possibility of a transition to a kinetically generated nonergodic state driven by the Gaussian random potential. This feature is at variance with the self-consistent MCT predictions [42].

Yet, it follows from equation (9.1c) that the density correlation function  $C_k(t)$  does exhibit a *disorder-induced* time-persistent component,

$$C_k(t \rightarrow +\infty) = \lambda \rho_0^2 \Phi_k. \quad (9.26)$$

This contribution is of a strictly *static* nature and must be distinguished from a kinetically generated nonergodicity parameter such as predicted by the MCT, for instance.

## 9.3. Disorder-induced static structure factors

We are examining the equilibrium dynamics, hence the initial condition for the density correlation function  $C_k(0)$  should yield the equilibrium static structure factor of the fluid  $S_k^{\text{st}}$ , through the relation  $C_k(0) = \rho_0 S_k^{\text{st}}$ . The latter acquires a *disorder-induced* contribution in the presence of the Gaussian random potential and equation (9.25) gives  $S_k^{\text{st}} = 1 + \lambda \rho_0 \Phi_k$ .

The time-persistent component of the density correlation function  $C_k(t \rightarrow +\infty)$  should similarly be related to the *disorder-induced* disconnected static structure factor  $S_k^{\text{d}}$  through  $C_k(t \rightarrow +\infty) = \rho_0 S_k^{\text{d}}$ , and one gets  $S_k^{\text{d}} = \lambda \rho_0 \Phi_k$  from equation (9.26).

Both expressions for  $S_k^{\text{st}}$  and  $S_k^{\text{d}}$  agree to first order with the exact static results, equation (3.4). In particular, the equality  $S_k^{\text{st}} = 1 + S_k^{\text{d}}$  is obeyed, ensuring the validity of the crucial relation (3.5).

In summary, it comes out of these first three points that the present FOBT is plainly consistent both with the FDR and with the equilibrium static results at the

same level of approximation. Hence, it manifestly fulfills all the basic requirements for a *bona fide* theory of equilibrium dynamics.

Once  $C_k(t \rightarrow +\infty)$  is linked to the disconnected static structure factor, it might be subtracted from the density correlation function to get its connected component,  $F_k(t) = C_k(t) - \rho_0 S_k^d$ , according to equation (4.2). Rewriting equation (9.1c),  $F_k(t)$  obeys

$$\left(\partial_t + \Gamma_k\right) F_k(t) = - \int_0^t ds \Sigma_k^0(t-s) C_k^0(s) + N_k^0(t), \quad (9.27)$$

with  $F_k(0) = \rho_0$ , based on equations (9.25) and (9.26). It is clear that  $F_k(t)$  can only relax to zero.

#### 9.4. Bare mode-coupling equations for the connected density correlation function

We may further try and simplify the dynamical equations for the connected density correlation function  $F_k(t)$ . With the FDR for the kernels, equation (9.15),  $\Sigma_k^0(t)$  is straightforwardly eliminated from equation (9.27), to get

$$\left(\partial_t + \Gamma_k\right) F_k(t) = - \frac{1}{\rho_0} \int_0^t ds N_k^0(t-s) \partial_s C_k^0(s) - \frac{1}{\rho_0} \int_0^t ds L_k^0(t-s) C_k^0(s), \quad (9.28)$$

where  $N_k^0(0) = 0$  is used in an integration by parts. With the diffusion equation  $\partial_s C_k^0(s) = -\Gamma_k C_k^0(s)$ , which follows from equation (7.3), one can put the above equation into the form

$$\left(\partial_t + \Gamma_k\right) F_k(t) = - \int_0^t ds M_k^0(t-s) \partial_s C_k^0(s), \quad (9.29)$$

where we used equation (9.19). The explicit expression for the kernel  $M_k^0(t)$  is found in equation (9.18). Note that there is a significant qualitative difference between the present use of equation (7.3) and the previous ones. Indeed, up to now, these equations were invoked to make substitutions within the kernels only, while here, a change in the formal structure of equation (9.28), hence of equation (9.1c), is achieved.

Apart from the bare nature of the memory term, equations (9.29) and (9.18) have the same form as those of the self-consistent MCT developed by one of us for the study of fluids in random environments [39–42]. Indeed, using the present notations, the latter read for a noninteracting Brownian gas:

$$\left(\partial_t + \Gamma_k\right) F_k(t) = - \int_0^t ds \mathcal{M}_k(t-s) \partial_s F_k(s), \quad (9.30a)$$

$$\mathcal{M}_k(t) = \frac{D_0}{\rho_0^2} \int_{\mathbf{q}} (\hat{\mathbf{k}} \cdot \mathbf{q})^2 S_q^d F_p(t), \quad (9.30b)$$

with  $F_k(0) = \rho_0$  and  $S_q^d$  the exact disconnected structure factor. These equations can immediately be brought forth from the former through a simple ad hoc renormalization scheme in which the linearized disconnected structure factor  $\rho_0 \lambda \Phi_q$  is replaced with its exact value  $S_q^d$  and the bare density correlation function  $C_k^0(t)$  is replaced with the

connected density correlation function  $F_k(t)$  (note that both  $C_k(t)$  and  $F_k(t)$  reduce to  $C_k^0(t)$  in the absence of disorder).

### 9.5. Mean-squared displacement and related quantities

In many studies, the interest mostly revolves around the mean-squared displacement (MSD) of a particle. Thus, this is a quantity of choice to investigate here.

Since we are dealing with a noninteracting gas, the connected density correlation function  $F_k(t)$  coincides with the self intermediate scattering function (with an additional  $\rho_0$  factor). The MSD  $\Delta(t)$  can therefore be obtained through the standard low- $k$  expansion

$$F_k(t) = \rho_0 \left[ 1 - \frac{k^2 \Delta(t)}{2d} + O(k^4) \right], \quad (9.31)$$

where  $d$  is the space dimension.

Equation (9.29) (since equations (9.27), (9.28), and (9.29), are fully equivalent, the choice of the starting equation is immaterial) can be straightforwardly integrated to get

$$F_k(t) = C_k^0(t) \left[ 1 - \int_0^t ds C_k^0(s)^{-1} \int_0^s du M_k^0(s-u) \partial_u C_k^0(u) \right], \quad (9.32)$$

which, in the low- $k$  limit, yields

$$\frac{\Delta(t)}{2dD_0} = t - \int_0^t ds \int_0^s du M_0^0(u), \quad (9.33)$$

with

$$M_0^0(t) = \lim_{k \rightarrow 0} M_k^0(t) = \frac{\lambda D_0}{d\rho_0} \int_{\mathbf{q}} q^2 \Phi_{\mathbf{q}} C_{\mathbf{q}}^0(t). \quad (9.34)$$

Equation (9.19) implies  $M_0^0(t) = -\lim_{k \rightarrow 0} [L_k^0(t)/(\rho_0 \Gamma_k)]$ , since  $\lim_{k \rightarrow 0} N_k^0(t) = 0$ . Thus, we observe that the diffusion of a particle is fully determined by the small-wavevector behavior of the sole kernels  $M_k^0(t)$  or  $L_k^0(t)$ . With a direct integration of equation (2.1) leading to

$$\frac{\Delta(t)}{2dD_0} = t + \frac{D_0}{2dT^2} \int_0^t ds \int_0^s du \overline{\langle \mathbf{F}_i(s) \mathbf{F}_i(u) \rangle} = t + \frac{D_0}{dT^2} \int_0^t ds \int_0^s du \overline{\langle \mathbf{F}_i(s) \mathbf{F}_i(u) \rangle}, \quad (9.35)$$

these low- $k$  kernels are immediately recognized as approximations for the force autocorrelation function.

Using the diffusion equation  $C_{\mathbf{q}}^0(t) = -\partial_t C_{\mathbf{q}}^0(t)/(D_0 q^2)$  in equation (9.34) to perform the inner time integration in equation (9.33), one alternatively obtains

$$\frac{\Delta(t)}{2dD_0} = \left( 1 - \frac{\lambda}{d} \right) t + \int_0^t ds m^0(s), \quad (9.36)$$

with

$$m^0(t) = \frac{\lambda}{\rho_0 d} \int_{\mathbf{q}} \Phi_{\mathbf{q}} C_{\mathbf{q}}^0(t) = \frac{\lambda}{\rho_0 d} \int_{\mathbf{r}} \Phi(r) C^0(r, t). \quad (9.37)$$

The second expression involving the bare diffusion kernel  $C^0(r, t) = \rho_0(4\pi D_0 t)^{-d/2} e^{-r^2/(4D_0 t)}$  results from Parseval's theorem.

From these relations, expressions for the time-dependent diffusion coefficient  $D(t) = \dot{\Delta}(t)/(2d)$  and the velocity autocorrelation function  $Z(t) = \ddot{\Delta}(t)/(2d)$  immediately follow, which read

$$\frac{D(t)}{D_0} = 1 - \int_0^t ds M_0^0(s) = 1 - \frac{\lambda}{d} + m^0(t), \quad (9.38)$$

$$\frac{Z(t)}{D_0} = -M_0^0(t) = \dot{m}^0(t). \quad (9.39)$$

These results show that the present FOBT fully agrees with earlier perturbative calculations at the same order [76] in predicting for the long-time diffusion coefficient

$$\frac{D_\infty}{D_0} = \lim_{t \rightarrow +\infty} \frac{D(t)}{D_0} = 1 - \frac{\lambda}{d}. \quad (9.40)$$

They also unambiguously demonstrate the breakdown of the approach at strong disorder, since negative values of  $D_\infty$ , hence of  $\Delta(t)$ , are obtained when  $\lambda$  exceeds the space dimension  $d$ . Correspondingly, anomalies (nonmonotonicity, overshoot above the initial value) appear in the density correlation functions at low  $k$  when this threshold is approached.

## 9.6. Asymptotic analysis and long-time tails

Making use of the explicit forms of  $C_k^0(t)$  or  $C^0(r, t)$  in equations (9.18), (9.34), and (9.37), the presence of long-time tails in the problem is straightforwardly demonstrated, since one obtains for the memory kernels

$$M_k^0(t) \sim \frac{D_0 k^2 \lambda \Phi_k}{(4\pi D_0 t)^{d/2}}, \quad t \rightarrow +\infty, k \neq 0, \quad (9.41a)$$

$$M_0^0(t) \sim \frac{2\pi D_0 \lambda \Phi_0}{(4\pi D_0 t)^{d/2+1}}, \quad t \rightarrow +\infty, \quad (9.41b)$$

$$m^0(t) \sim \frac{\lambda \Phi_0}{d(4\pi D_0 t)^{d/2}}, \quad t \rightarrow +\infty. \quad (9.41c)$$

The qualitative behavior of the velocity autocorrelation function  $Z(t)$  (see equation (9.39)), which is thus found negative, linear in the disorder strength, and relaxing as  $-t^{-(d/2+1)}$ , is exactly the same as in the Brownian random Lorentz gas [85]. More generally, these results are in agreement with previous phenomenological calculations [86].

In order to discuss the correlation functions, equation (9.32) is first explicitly written as

$$F_k(t) = \rho_0 e^{-D_0 k^2 t} \left[ 1 + D_0 k^2 \int_0^t ds \int_0^s du M_k^0(u) e^{D_0 k^2 u} \right], \quad (9.42)$$

then, after an integration by parts,

$$F_k(t) = \rho_0 \left[ e^{-D_0 k^2 t} + D_0 k^2 \int_0^t ds M_k^0(s) (t-s) e^{-D_0 k^2 (t-s)} \right]. \quad (9.43)$$

Standard analysis based on Laplace transforms then allows one to obtain

$$F_k(t) \sim \rho_0 \frac{\lambda \Phi_k}{(4\pi D_0 t)^{d/2}}, \quad t \rightarrow +\infty, k \neq 0. \quad (9.44)$$

For completeness, we also report the short-time expansions,

$$M_k^0(t) \sim \frac{D_0 \lambda}{d} \left[ \int_{\mathbf{q}} q^2 \Phi_q - D_0 t \int_{\mathbf{q}} q^2 (k^2 + q^2) \Phi_q \right], t \rightarrow 0, \quad (9.45a)$$

$$m^0(t) \sim \frac{\lambda}{d} \left[ 1 - D_0 t \int_{\mathbf{q}} q^2 \Phi_q \right], t \rightarrow 0, \quad (9.45b)$$

$$F_k(t) \sim \rho_0 \left[ 1 - D_0 k^2 t + \frac{(D_0 t)^2}{2} k^2 \left( k^2 + \frac{\lambda}{d} \int_{\mathbf{q}} q^2 \Phi_q \right) \right], t \rightarrow 0. \quad (9.45c)$$

## 9.7. Explicit example

In order to report complete solutions of the FOBT, we have to particularize the covariance of the Gaussian random field. Since it allows one to analytically perform the wave-vector integrals appearing in the definitions of  $M_k^0(t)$  and  $m^0(t)$ , a Gaussian covariance,

$$\Phi(r) = e^{-r^2/(2R^2)}, \quad \Phi_k = (2\pi R^2)^{d/2} e^{-k^2 R^2/2}, \quad (9.46)$$

where  $R$  controls the range of the random-field correlations, appears as a particularly convenient choice. One then obtains (see appendix B)

$$M_k^0(t) = \frac{\lambda}{2} \cdot \frac{2D_0}{R^2} \cdot \frac{1 + 2D_0 t/R^2 + k^2 R^2 (2D_0 t/R^2)^2}{(1 + 2D_0 t/R^2)^{d/2+2}} \cdot \exp \left[ -\frac{k^2 R^2 (2D_0 t/R^2)}{2(1 + 2D_0 t/R^2)} \right], \quad (9.47)$$

$$m^0(t) = \frac{\lambda}{d} \cdot \frac{1}{(1 + 2D_0 t/R^2)^{d/2}}. \quad (9.48)$$

With these formulas, the MSD can be expressed in closed form, and reads

$$\frac{\Delta(t)}{dR^2} = \frac{2D_0 t}{R^2} \left( 1 - \frac{\lambda}{d} \right) + \frac{\lambda}{d} \begin{cases} 2 \left( \sqrt{1 + 2D_0 t/R^2} - 1 \right) & \text{if } d = 1, \\ \ln(1 + 2D_0 t/R^2) & \text{if } d = 2, \\ \frac{2}{d-2} \left[ 1 - \frac{1}{(1 + 2D_0 t/R^2)^{d/2-1}} \right] & \text{if } d \geq 3. \end{cases} \quad (9.49)$$

In these expressions, the natural units of length and time,  $R$  and  $\tau = R^2/(2D_0)$ , respectively, have been made evident. The time  $\tau$  merely is the time at which the characteristic lengthscale of free diffusion  $\sqrt{2D_0 t}$  reaches the correlation length of the disorder.

The effect of the relative disorder strength  $\lambda$  on the time dependence of the MSD is shown in figure 1 for space dimensions  $d=1$  and  $d=3$ . Note that the theory is clearly pushed well beyond its range of validity, since results up to  $\lambda = d$  and slightly above, where its breakdown is obvious, are shown for completeness. The curve at  $\lambda = d$  emphasizes the transient between the short- and long-time normal diffusive regimes.

The correlation functions can be computed by a direct numerical integration of equation (9.43) with  $M_k^0(t)$  given by equation (9.47). Figure 2 shows the typical behavior of  $F_k(t)/\rho_0$  versus time obtained from this numerical solution with  $d=1$  and  $d=3$ ,  $kR = \pi/3$ , for different values of the relative disorder strength  $\lambda$ . In this log-log plots, the algebraic tail  $F_k(t)/\rho_0 \sim \lambda[R^2/(2D_0t)]^{d/2}e^{-k^2R^2/2}$  is clearly visible as a linear asymptote at long times.

## 10. Equilibrium dynamics: first-order renormalized theory

So far, a bare perturbation theory has been discussed, where the corrections due to the disorder were expressed in terms of the bare correlation and response functions. We now consider renormalized theories, where the bare correlation and response functions are replaced with renormalized ones in a self-consistent manner and based on the exact second order perturbation calculation.

Out of the bare perturbation expansion up to the second order (see appendices C–E), one identifies equations (C.13), (D.15), and (E.19), as a set of first-order renormalized dynamical equations, which should obey the FDR and reproduce the bare theory (we set  $t \geq t' = 0$ ):

$$(\partial_t + \Gamma_k)G_k(t) = \delta(t) - \int_0^t ds \Sigma_k(t-s)G_k(s), \quad (10.1a)$$

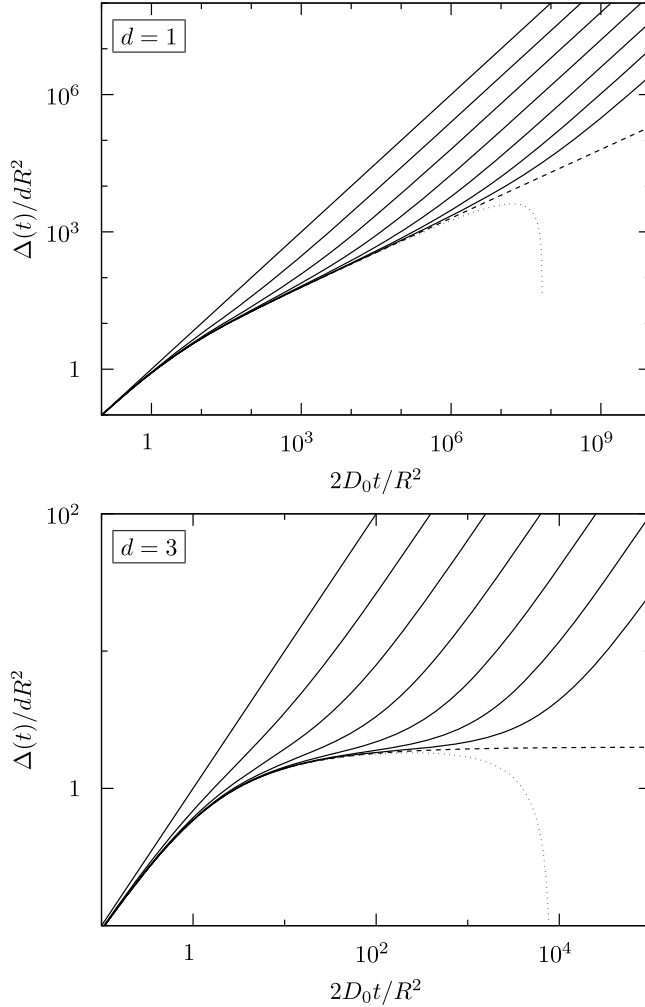
$$(\partial_t + \Gamma_k)\bar{R}_k(t) = \rho_0\Gamma_k\delta(t) - \int_0^t ds \Sigma_k(t-s)\bar{R}_k(s) + L_k(t), \quad (10.1b)$$

$$(\partial_t + \Gamma_k)F_k(t) = - \int_0^t ds \Sigma_k(t-s)F_k(s) + N_k(t). \quad (10.1c)$$

In the latter equation, the static time-persistent part of the density correlation function has been absorbed into the connected density correlation function, according to its definition (4.2). Clearly, these equations are structurally similar to the bare equations (9.1a), (9.1b), and (9.27). At this stage, the explicit expressions for the memory kernels  $\Sigma_k(t)$ ,  $L_k(t)$ , and  $N_k(t)$ , are left unspecified.

The relations between these kernels should be constrained by the FDR, equation (4.22a), also expressible as

$$\bar{R}_k(t) = -\theta(t)\partial_t F_k(t). \quad (10.2)$$



**Figure 1.** Time evolution of the mean-squared displacement in a noninteracting Brownian gas exposed to a Gaussian random field with Gaussian covariance (specified in equation (9.46)) in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom), according to the first-order bare theory. From left to right, top to bottom:  $\lambda = 0$ ,  $\lambda = d(1 - 1/4^n)$  with  $n = 1, 2, \dots, 6$ ,  $\lambda = d$  (dashed line), and  $\lambda = d(1 + 1/4^n)$  with  $n = 6$  (dotted line, unphysically diverging to  $-\infty$ ).

Taking the time derivative of the above and using equation (10.1c), one gets

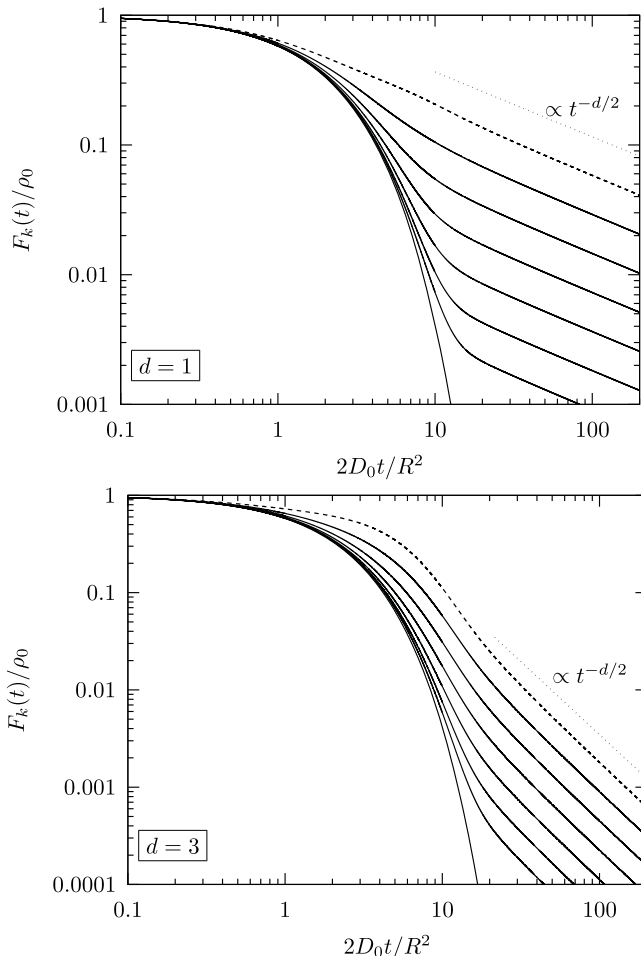
$$(\partial_t + \Gamma_k)\bar{R}_k(t) = -\delta(t)\partial_t F_k(0) - \int_0^t ds \Sigma_k(t-s)\bar{R}_k(s) + \rho_0 \Sigma_k(t) - \partial_t N_k(t). \quad (10.3)$$

Comparing with equation (10.1b), one sees that the FDR demands the two relations

$$\partial_t F_k(0) = -\rho_0 \Gamma_k, \quad (10.4a)$$

$$\partial_t N_k(t) = \rho_0 \Sigma_k(t) - L_k(t). \quad (10.4b)$$

Setting  $t = 0$  in equation (10.1c), one gets the initial condition  $N_k(0) = 0$  from equation (10.4a) and  $F_k(0) = \rho_0$ , to be used in equation (10.4b). Therefore, one should have



**Figure 2.** Time evolution of the connected density correlation function in a noninteracting Brownian gas exposed to a Gaussian random field with Gaussian covariance in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom), according to the first-order bare theory. The wavevector is  $kR = \pi/3$ . From left to right, bottom to top:  $\lambda = 0$ ,  $\lambda = d/2^n$  with  $n = 6, 5, \dots, 1$ ,  $\lambda = d$  (dashed line). The dotted line illustrates the long-time decay  $F_k(t)/\rho_0 \propto t^{-d/2}$ .

$$N_k(t) = \int_0^t ds [\rho_0 \Sigma_k(s) - L_k(s)]. \quad (10.5)$$

Using equation (10.4b) to eliminate  $\Sigma_k(t)$  in equation (10.1c), the latter becomes, after an integration by parts,

$$(\partial_t + \Gamma_k)F_k(t) = -\frac{1}{\rho_0} \int_0^t ds N_k(t-s) \partial_s F_k(s) - \frac{1}{\rho_0} \int_0^t ds L_k(t-s) F_k(s). \quad (10.6)$$

This equation, a renormalized version of equation (9.28), is clearly reminiscent of those that can be obtained with standard projection-operator techniques in the memory-function formalism [87]. However, one can interestingly note that it mixes two types of convolution integrals which are usually found to be mutually exclusive and only converted into one another by making use of special rearrangements [21, 88, 89].

Another possibility to eliminate  $\Sigma_k(t)$  is through mere Laplace transforms of equation (10.1). One then obtains nonlinear relations (once the kernels are specified) expressing the physical response and correlation functions in terms of the noise-response function, as

$$\bar{R}_k(t) = \rho_0 \Gamma_k G_k(t) + \int_0^t ds L_k(t-s) G_k(s), \quad (10.7)$$

$$F_k(t) = \rho_0 G_k(t) + \int_0^t ds N_k(t-s) G_k(s). \quad (10.8)$$

These expressions will be extremely useful in the following to perform first-order consistent substitutions, i.e. replacements of one function with another that entail corrections strictly beyond the first order.

We might now close the set of dynamical equations with explicit expressions for the first-order renormalized kernels, which are self-consistently determined from a second-order bare perturbation calculation.

### 10.1. Native first-order renormalized theory

We first consider the FORT that derives in the most literal way from the second-order bare theory. For this reason, we choose to term it native.

As seen in appendices C–E, one gets (equations (C.14) and (D.16))

$$\Sigma_k(t) = \lambda D_0^2 \int_{\mathbf{q}} \mathbf{q} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t), \quad (10.9)$$

$$L_k(t) = \lambda \rho_0 D_0^2 \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t). \quad (10.10)$$

As for the remaining kernel  $N_k(t)$ , it is in principle given by equation (E.20). However, as pointed out there, it is very likely that this expression does not fully comply with the requirements of a *bona fide* equilibrium dynamics. Yet, a possible workaround is to force consistency with the FDR, through the use of equation (10.4b). One then gets

$$\partial_t N_k(t) = -\lambda D_0 \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] [\rho_0 D_0 p^2 G_p(t)], \quad (10.11)$$

hence

$$N_k(t) = -\lambda D_0 \int_0^t ds \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] [\rho_0 D_0 p^2 G_p(s)]. \quad (10.12)$$

In appendix E, we check the suitability of this step, by showing that, thanks to equations (10.7), (10.8), and (10.1a), equation (E.20) can indeed be rewritten within its order of validity in  $\lambda$ , such that it agrees with equation (10.12) to first order in  $\lambda$ . Note that, in the present scheme, the explicit expression of  $N_k(t)$  is actually not needed for the computation of the three functions of interest. Indeed, with the above form of  $\Sigma_k(t)$ , the

dynamical equation for the noise-response function  $G_k(t)$  is self-closed. Its solution can be fed into the dynamical equation for the physical response function, equation (10.1b), or equivalently equation (10.7), to obtain  $\bar{R}_k(t)$ , from which  $F_k(t)$  is retrieved by integration of the FDR, equation (10.2). Self-consistency implies that this solution for  $F_k(t)$  be the same as that from equations (10.1c), (10.6), or (10.8).

Unfortunately, this theory as it stands does not appear to bring one very far. Indeed, our numerical attempts at computing  $G_k(t)$ , which is the required first step, faced instabilities that seem to prevent the application of the theory beyond rather modest disorder strengths (with a Gaussian random field covariance, equation (9.46), spurious divergences occur for  $\lambda/d > 0.272$  in  $d = 1$  and  $\lambda/d > 0.345$  in  $d = 3$ , i.e. significantly below the threshold  $\lambda/d = 1$  beyond which the FOBT produces blatantly unphysical results). Note that these calculations were based on computing the integrated response function  $H_k(t) = \int_0^t ds G_k(s)$  as an intermediate [90], whose evolution equation obtained from equations (10.1a) and (10.9) reads

$$(\partial_t + \Gamma_k)H_k(t) = 1 - \int_0^t ds S_k(t-s) \partial_s H_k(s), \quad S_k(t) = \lambda D_0^2 \int_{\mathbf{q}} \mathbf{q} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_{\mathbf{q}}] H_p(t). \quad (10.13)$$

This is exactly the type of nonlinear integro-differential equation met within the MCT, for which a well-established and efficient iterative numerical solution scheme has been developed long ago [91]. It usually shows remarkable stability, provided the underlying equations are themselves stable. Therefore, this suggests that the instabilities are intrinsic to the above renormalized equations, which in particular fail to guarantee that the kernels  $\Sigma_k(t)$  and  $S_k(t)$  are nonnegative functions of time, while this is the case for overdamped dynamics with the standard MCT kernels.

## 10.2. Modified first-order renormalized theory

In order to try and overcome these difficulties, one might exploit the freedom offered by the first-order consistent substitutions to generate variants of the theory, at the cost of an increased degree of empiricism in its derivation. Since  $\Sigma_k(t)$ ,  $L_k(t)$ , hence  $\partial_t N_k(t)$  via equation (10.4b), naturally acquire the character of response functions within the native FORT, we focused on the possibilities provided by equation (10.7) to replace  $G_p(t)$  with  $\bar{R}_p(t)/(\rho_0 \Gamma_p)$  in equations (10.9), (10.10), or (10.11). By separately making one or the other choice for two kernels, the third one being fixed by equation (10.4b), one obtains eight FDR-consistent theories in total, including the native one above entirely based on  $G_p(t)$ .

With respect to the criteria of consistency with the FDR and with the FOBT, these eight theories are all equally possible and valid by construction. Therefore, if one of them is to be favoured, this has to be based on arguments of a different nature. Since we identified difficulties with the native theory through numerical considerations, we shall pursue this line of reasoning here. We already know that the instabilities of the native theory will be present in two other variants of the FORT, for their  $\Sigma_k(t)$  is also given by equation (10.9).

After trying to numerically solve the dynamical equations for the eight variants of the theory, we find that one of them clearly stands out. Indeed, for some relevant choices of parameters, it appears unique in its ability to deliver physically acceptable numerical

results. This is particularly the case in the regime of sizeable disorder strengths, corresponding to  $\lambda/d > 1$ . The theory in question, to which we shall refer as the modified FORT, is the one entirely based on  $\bar{R}_p(t)$ , i.e. with

$$\Sigma_k(t) = \lambda \frac{D_0}{\rho_0} \int_{\mathbf{q}} \frac{\mathbf{q} \cdot \mathbf{p}}{p^2} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \bar{R}_p(t), \quad (10.14)$$

$$L_k(t) = \lambda D_0 \int_{\mathbf{q}} \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \bar{R}_p(t), \quad (10.15)$$

$$\partial_t N_k(t) = -\lambda D_0 \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \bar{R}_p(t), \quad (10.16)$$

hence

$$N_k(t) = \lambda D_0 \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] F_p(t), \quad (10.17)$$

where we used the FDR and  $\int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{q} \Phi_q = 0$  by isotropy. Note that these expressions achieve consistency with the FDR in a most natural way, since equation (10.4b) merely appears as a trivial corollary of equation (10.2).

Beyond the numerical arguments, some aspects of the theory discussed previously might actually be seen as further hints in favor of these equations. For instance, in our physical interpretation of  $L_k(t)$  at the bare level (see equations (9.4)–(9.6)), the kernel is proposed to initially involve a composite response field, as precisely does  $\bar{R}_k(t)$ . Also, the straightforward appearance of the combination  $\rho_0 \Gamma_p G_p(s)$  in equation (10.11) of the native theory suggests that a substitution by  $\bar{R}_p(t)$  might be in order, as we repeatedly assumed at the bare level (see the transition from equations (8.5) to (8.7)). On the other hand, the second-order result of appendix C does not provide one with any obvious reason to favour equation (10.14) over equation (10.9), since both expressions are seen to remain approximate at this order.

At the level of the response functions, it is now the dynamical equation for  $\bar{R}_k(t)$ , equation (10.1b), which is self-closed. However, from a physical point of view, the closed coupled set consisting of equations (10.1a) and (10.7) looks more telling, as it shows a mixed feedback scheme that might be pictorial of dynamics with multiplicative noise. Indeed, on the one hand, equation (10.7) formally represents the density response function as a mere byproduct of the noise-response function, in line with the fact that fluctuations and dynamics do fundamentally come to the system precisely through thermal noise. But, on the other hand, the couplings and memory effects represented by  $\Sigma_k(t)$  and  $L_k(t)$  are ruled by the density response function itself, as a reflection of the density dependence of the multiplicative thermal noise.

Formally, it is still possible to close equation (10.1a) and have the modified theory rest upon the mere determination of  $G_k(t)$ , as does the native one. Indeed, equations (10.7) and (10.15) can be recursively used to express  $\bar{R}_k(t)$  as an infinite sum of integrals of all orders in the disorder strength and involving  $G_k(t)$  only. A similar series expansion can be derived for  $F_k(t)$ , based on equations (10.8) and (10.17). When injected

into equations (10.14)–(10.17), these expressions characterize the present approach as some kind of resummation scheme beyond the native FORT.

Thanks to the FDR, the dynamical equations for the density correlation function, equations (10.1c) and (10.6), are self-closed as well. In particular, the latter can be usefully written as

$$(\partial_t + \Gamma_k)F_k(t) = -\frac{1}{\rho_0} \int_0^t ds N_k(t-s) \partial_s F_k(s) - \frac{1}{\rho_0} \int_0^t ds \partial_{t-s} \Lambda_k(t-s) F_k(s), \quad (10.18)$$

where  $N_k(t)$  is given by equation (10.17) and  $\Lambda_k(t)$  follows from equation (10.15) and the FDR, equation (10.2), as

$$\Lambda_k(t) = -\lambda D_0 \int_{\mathbf{q}} \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} [\mathbf{k} \cdot \mathbf{q} \Phi_q] F_p(t). \quad (10.19)$$

For definiteness, we recall the initial condition  $F_k(0) = \rho_0$ . Interestingly, these equations are clearly distinct from those obtained within the MCT, equation (9.30), but they belong to the same class of self-consistent nonlinear problems and can be analytically studied [7, 8, 92] and numerically solved [91] by the same means.

Therefore, we might now discuss the main features of their solutions, considering again the case of a Gaussian random-field covariance, equation (9.46), for the purpose of illustration.

### 10.3. Numerical solution of the modified first-order renormalized theory

The evolution of the correlation function  $F_k(t)/\rho_0$  with increasing disorder strength  $\lambda$  is displayed in figure 3 for a representative wavevector  $kR = \pi/3$  in space dimensions  $d = 1$  and  $d = 3$ . First, as one would expect, the dynamics simply slows down as  $\lambda$  increases, and a long-time relaxation tail gradually develops. Then, at a threshold  $\lambda_c(d)$ , obeying  $\lambda_c(d) < d$  (the importance of this inequality will be manifest later), the dynamics becomes nonergodic, i.e. a time-persistent plateau starts to continuously grow from zero with increasing positive  $\lambda - \lambda_c(d)$ , reflecting a partial arrest of the relaxation of the density fluctuations. The so-called nonergodicity parameter  $F_k(t \rightarrow +\infty)/\rho_0$ , corresponding to the height of this plateau, is solution of the nonlinear equation

$$\frac{F_k(t \rightarrow +\infty)}{\rho_0} = \frac{N_k(t \rightarrow +\infty)}{\rho_0 \Gamma_k + N_k(t \rightarrow +\infty) + \Lambda_k(t \rightarrow +\infty) - \Lambda_k(0)}, \quad (10.20)$$

where  $N_k(t \rightarrow +\infty)$  and  $\Lambda_k(t \rightarrow +\infty)$  are linear functionals of  $F_k(t \rightarrow +\infty)$ , as prescribed by equations (10.17) and (10.19). The wavevector dependence of  $F_k(t \rightarrow +\infty)/\rho_0$  is shown in figure 4 for the values of the disorder strength corresponding to nonergodic states in figure 3.

The details of the critical dynamics near the threshold are illustrated by figure 5. The long-time relaxation tail is seen to be algebraic,  $F_k(t)/\rho_0 \propto t^{-1/2}$ , independently of the space dimension. It lasts longer and longer as  $\lambda_c(d)$  is approached from below, and gradually recedes, giving way to the time-persistent plateau, as  $\lambda_c(d)$  is left from above. These evolutions are symmetric on both sides of  $\lambda_c(d)$ , with a diverging characteristic timescale  $\propto [\lambda - \lambda_c(d)]^{-2}$ . In the partially arrested state, the nonergodicity parameter grows  $\propto [\lambda - \lambda_c(d)]$  to leading order.

In most respects, this scenario is the same as the one found within the MCT [42]. This similarity can be traced back to the linearity of the kernels with the density correlation functions, which generically enforces continuous ergodicity-breaking transitions, if any [7, 8]. Such a linearity is an expected generic feature of MCT-like approaches to fluids in random fields, which has been found in all previous studies, either strictly [11–16] or to leading order in the strong disorder regime [39–42]. There is however one important difference with regard to the behavior of the nonergodicity parameter. Indeed, within the MCT, the evolution of the latter with increasing disorder strength mainly consists of the continuous broadening of a low-wavevector peak with maximum  $F_0(t \rightarrow +\infty)/\rho_0 = 1$ , which appears with a vanishing width at the ergodicity-breaking transition (this behavior is illustrated for the case of a fluid in a random porous solid in [41] and [93]). This implies the existence of a localization length in the nonergodic phase, which diverges as the transition is approached from above. There is no such thing in the present theory, as readily seen in figure 4. This difference can be traced back to the contrasting low-wavevector behaviors of the kernels in the two theories. Here, both  $N_k(t)$  and  $\Lambda_k(t)$  are  $O(k^2)$ , so that  $F_k(t \rightarrow +\infty)/\rho_0$  in equation (10.20) does not have to go to one as  $k \rightarrow 0$ , while it does have to in the MCT, where  $\mathcal{M}_k(t)$  is  $O(k^0)$  (see equation (9.30b)) and  $F_k(t \rightarrow +\infty)/\rho_0 = \mathcal{M}_k(t \rightarrow +\infty)/[\Gamma_k + \mathcal{M}_k(t \rightarrow +\infty)]$ .

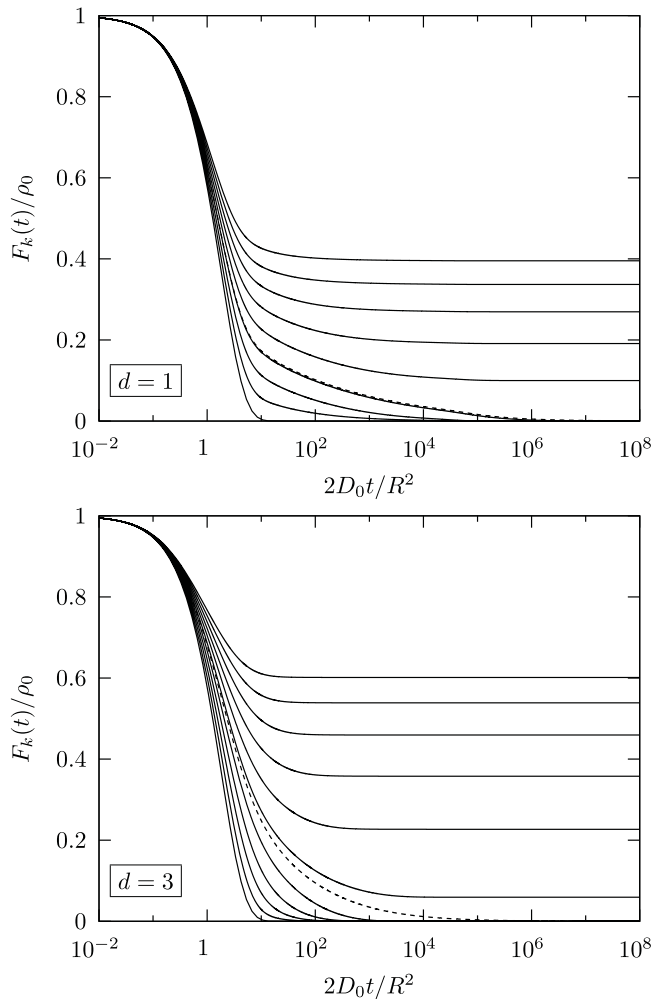
The absence of a localized state in the nonergodic phase is readily seen in the full wavevector dependence of the dynamics, as reported in figure 6. Indeed, as the density correlation functions relax toward their infinite-time limits, a peak forms on top of the nonergodicity parameter curve at low wavevectors, which becomes narrower and narrower with time. From equation (9.31), it is clear that this peak relates to the diffusional properties of the fluid and that its vanishing width with increasing time implies a diverging mean-squared displacement (MSD), hence a delocalized state. In passing, note that an occasional slight inaccuracy of the theory can be spotted in the top panel of figure 6. Indeed, at low wavevectors (below  $kR \simeq 0.4$ ), the nonergodicity parameter is reached from below, meaning a slightly nonmonotonic behavior of the density correlation function. Quantitatively, the phenomenon is very small, but, in principle, it violates the property that autocorrelation functions be completely monotone functions of time for overdamped dynamics.

The above reasoning is confirmed by a direct computation of the MSD. Using the low- $k$  expansion (9.31) in equation (10.6), knowing that  $\lim_{k \rightarrow 0} N_k(t) = 0$  and  $L_k(t) = O(k^2)$ , one generically obtains

$$\frac{\Delta(t)}{2dD_0} = t + \int_0^t ds \int_0^s du \lim_{k \rightarrow 0} \frac{L_k(u)}{\rho_0 \Gamma_k}, \quad (10.21)$$

which again connects the low-wavevector limit of  $L_k(t)$  to the force autocorrelation function through equation (9.35). Then, within the modified FORT, where  $L_k(t) = \partial_t \Lambda_k(t)$  and  $\Lambda_k(t)$  is given by equation (10.19), this can be rewritten as

$$\frac{\Delta(t)}{2dD_0} = \left(1 - \frac{\lambda}{d}\right) t + \int_0^t ds m(s), \quad (10.22)$$

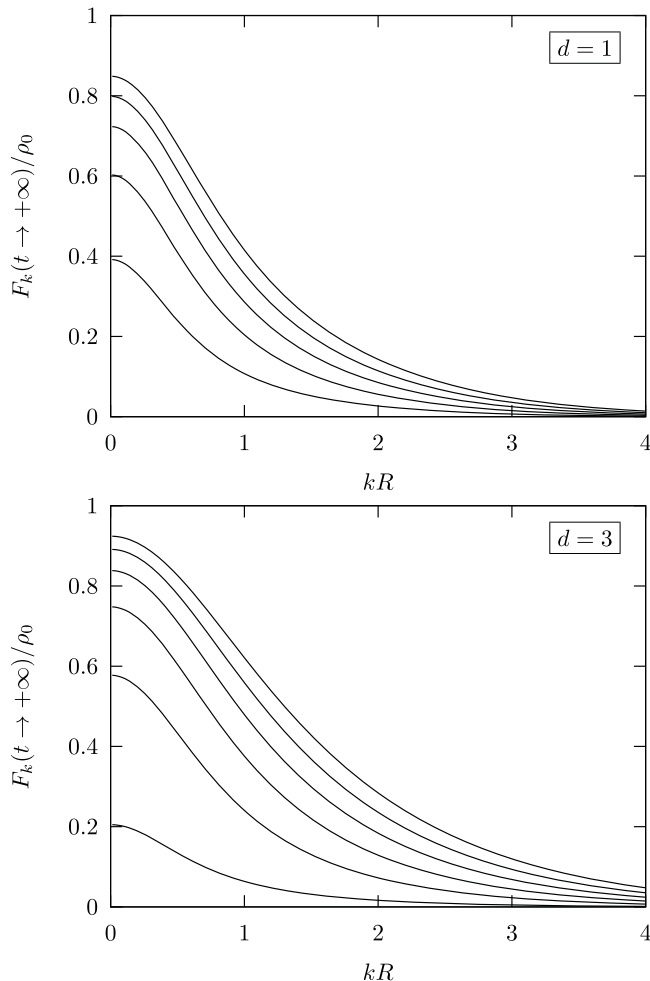


**Figure 3.** Time evolution of the connected density correlation function in a noninteracting Brownian gas immersed in a Gaussian random field with Gaussian covariance in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom), according to the modified first-order renormalized theory. The wavevector is  $kR = \pi/3$ . From left to right, bottom to top:  $\lambda = 0, 0.25, 0.5, \dots, 1.75, 2$ , for  $d = 1$ ;  $\lambda = 0, 0.5, 1, \dots, 4.5, 5$ , for  $d = 3$ . Ergodicity is broken for  $\lambda$  larger than  $\lambda_c(d = 1) = 0.76359$  and  $\lambda_c(d = 3) = 2.34686$ . The corresponding critical density correlation functions are reported with dashed lines.

with

$$m(t) = \frac{\lambda}{\rho_0 d} \int_{\mathbf{q}} \Phi_{\mathbf{q}} F_{\mathbf{q}}(t) = \frac{\lambda}{\rho_0 d} \int_{\mathbf{r}} \Phi(\mathbf{r}) F(\mathbf{r}, t), \quad (10.23)$$

which is an obvious renormalized version of equation (9.37). The corresponding results for the influence of the relative disorder strength on the time dependence of the MSD are shown in figure 7. Remarkably, it is found that a normal diffusive behavior is reached at long times for all disorder strengths, even those leading to nonergodic states. This feature is definitely at variance with the MCT predictions, where the ergodicity-breaking transition is also a diffusion-localization transition [42], and in complete agreement with the known rigorous results [66].



**Figure 4.** Wavevector dependence of the nonergodicity parameter of a noninteracting Brownian gas plunged in a Gaussian random field with Gaussian covariance in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom), according to the modified first-order renormalized theory. From bottom to top:  $\lambda = 1, 1.25, 1.5, 1.75, 2$ , for  $d = 1$ ;  $\lambda = 2.5, 3, 3.5, 4, 4.5, 5$ , for  $d = 3$ .

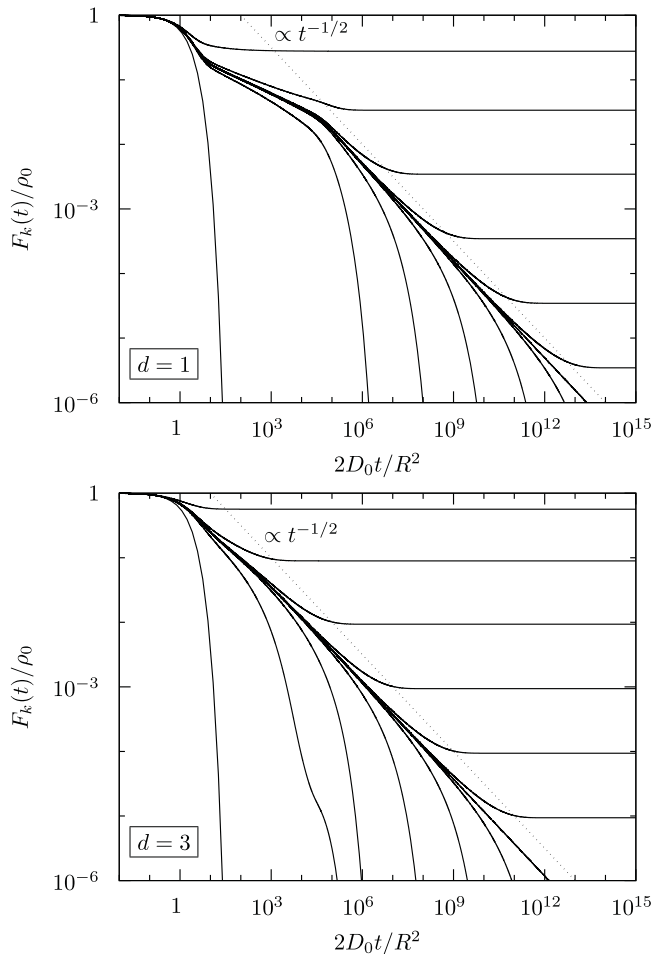
Further insight into this finding can be gained by considering the time-dependent diffusion coefficient  $D(t)$ , given by

$$\frac{D(t)}{D_0} = \frac{\dot{\Delta}(t)}{2dD_0} = 1 - \frac{\lambda}{d} + m(t), \quad (10.24)$$

or, more specifically, its long-time limit  $D_\infty = \lim_{t \rightarrow +\infty} D(t)$ . It is plotted in figure 8 for  $d = 1$  and  $d = 3$ . For  $\lambda < \lambda_c(d)$ , the system is ergodic,  $m(t)$  vanishes at long times because  $F_q(t)$  does for all  $q$ , and equation (9.40) from the FOBT is recovered. On the other hand, for  $\lambda > \lambda_c(d)$ , ergodicity is broken and one gets

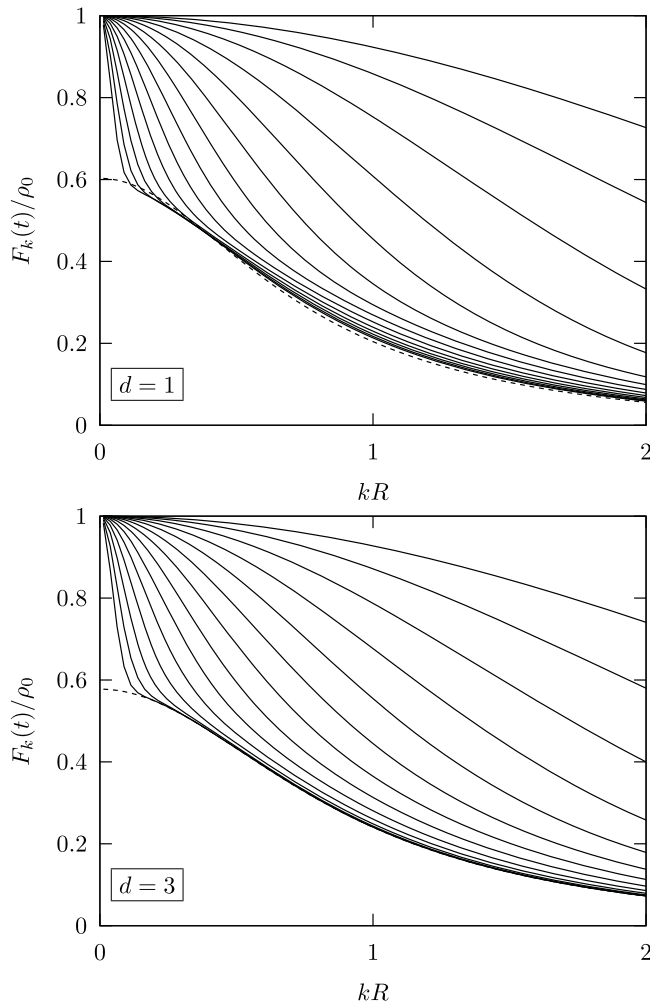
$$\frac{D_\infty}{D_0} = 1 - \frac{\lambda}{d} + \frac{\lambda}{\rho_0 d} \int_{\mathbf{q}} \Phi_{\mathbf{q}} F_{\mathbf{q}}(t \rightarrow +\infty) > 1 - \frac{\lambda}{d}. \quad (10.25)$$

As shown by the numerical results, the additional nonergodic contribution strongly restricts the decrease of  $D_\infty$  with the disorder strength compared to the ergodic



**Figure 5.** Time evolution of the connected density correlation function in a noninteracting Brownian gas in a Gaussian random field with Gaussian covariance in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom), according to the modified first-order renormalized theory. The wavevector is  $kR = \pi/3$ . From left to right, bottom to top:  $\lambda = 0, 0.9\lambda_c, 0.99\lambda_c, 0.999\lambda_c, 0.9999\lambda_c, 0.99999\lambda_c, \lambda_c, 1.00001\lambda_c, 1.0001\lambda_c, 1.001\lambda_c, 1.01\lambda_c, 1.1\lambda_c, 2\lambda_c$ , with  $\lambda_c(d = 1) = 0.76359$  and  $\lambda_c(d = 3) = 2.34686$ . The dotted line illustrates the long-time critical decay  $F_k(t)/\rho_0 \propto t^{-1/2}$ .

regime. The breakdown of equation (9.40) at  $\lambda = d$  is therefore avoided, so that  $D_\infty$  remains strictly positive. Note that this obviously requires the condition  $\lambda_c(d) < d$ . Unfortunately, this mechanism generates a corner singularity in  $D_\infty$  at  $\lambda_c(d)$ , as a result of the leading linear growth of  $F_q(t \rightarrow +\infty)$  above  $\lambda_c(d)$ . This is clearly a spurious feature of the present theory, as no such corner exists in the known exact results for  $D_\infty$  in  $d = 1$  and  $d = 2$  [76] and there is no obvious reason why this should be different in other space dimensions. As for the aspect of quantitative accuracy, comparison with the law  $D_\infty/D_0 = e^{-\lambda/d}$ , which is known to be exact in  $d = 1$  and a good approximation in  $d = 3$  [76], immediately shows that there is room for improvement. For completeness, we also report an analytic result from the MCT with an additional hydrodynamic approximation [16, 41],  $D_\infty/D_0 = 1 - (e^\lambda - 1)/d$ , which shows its predicted vanishing of the diffusion coefficient. Note that this expression is based on an exact treatment of



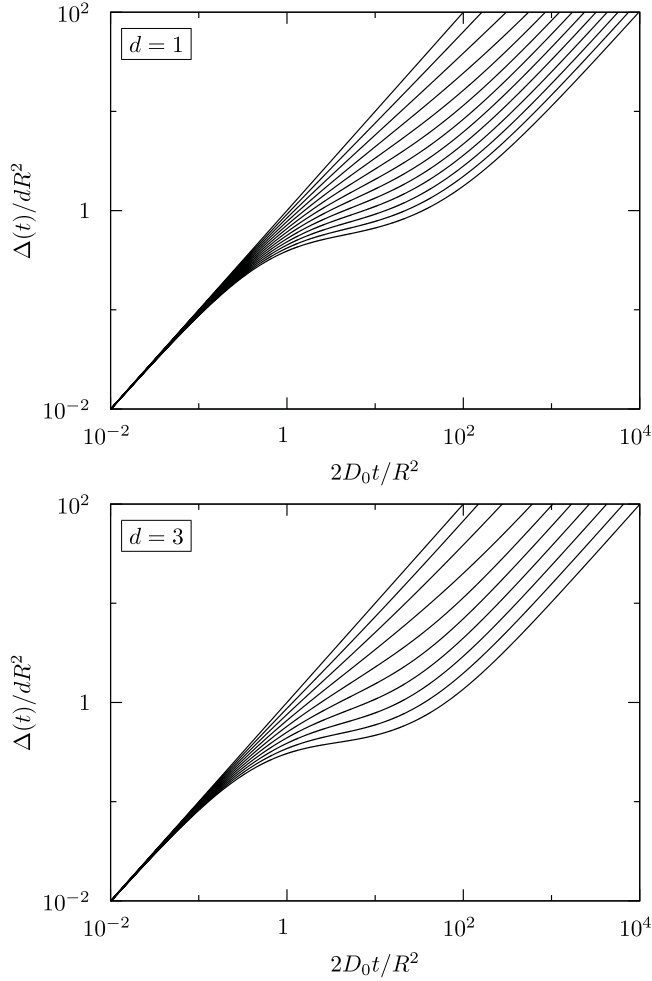
**Figure 6.** Wavevector dependence of the connected density correlation function at fixed times in a noninteracting Brownian gas exposed to a Gaussian random field with Gaussian covariance in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom), according to the modified first-order renormalized theory. For  $d = 1$ ,  $\lambda = 1.25$ ; for  $d = 3$ ,  $\lambda = 3$ . In both cases,  $\lambda > \lambda_c(d)$  and the system is nonergodic. From top to bottom:  $2D_0t/R^2 = 2^n \times 10^{-8}$ ,  $n = 24, \dots, 37$ . The nonergodicity parameter is shown as a dashed line.

the static correlations. If the structure factors are truncated to linear order in  $\lambda$ , this version of the MCT simply reproduces  $D_\infty/D_0$  from the FOBT in the ergodic phase.

#### 10.4. Relation between the present theory and the MCT

We close this section by considering how the present FORT can be related to the MCT. Indeed, as mentioned in introduction, a major motivation for the development of field-theoretic approaches to particle dynamics came from the search of an improved derivation of the MCT, with better controlled approximations. It is thus interesting to see where the present results stand from this perspective.

With equations (10.6) or (10.18), which are evocative of the memory-function formalism, and the closures (10.15), (10.17), and (10.19), the modified FORT manifestly



**Figure 7.** Time evolution of the mean-squared displacement in a noninteracting Brownian gas in a Gaussian random field with Gaussian covariance in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom), according to the modified first-order renormalized theory. From left to right, top to bottom:  $\lambda = 0, 0.5, 1, \dots, 5.5, 6$ , for  $d = 1$ ;  $\lambda = 0, 1, 2, \dots, 8, 9$ , for  $d = 3$ .

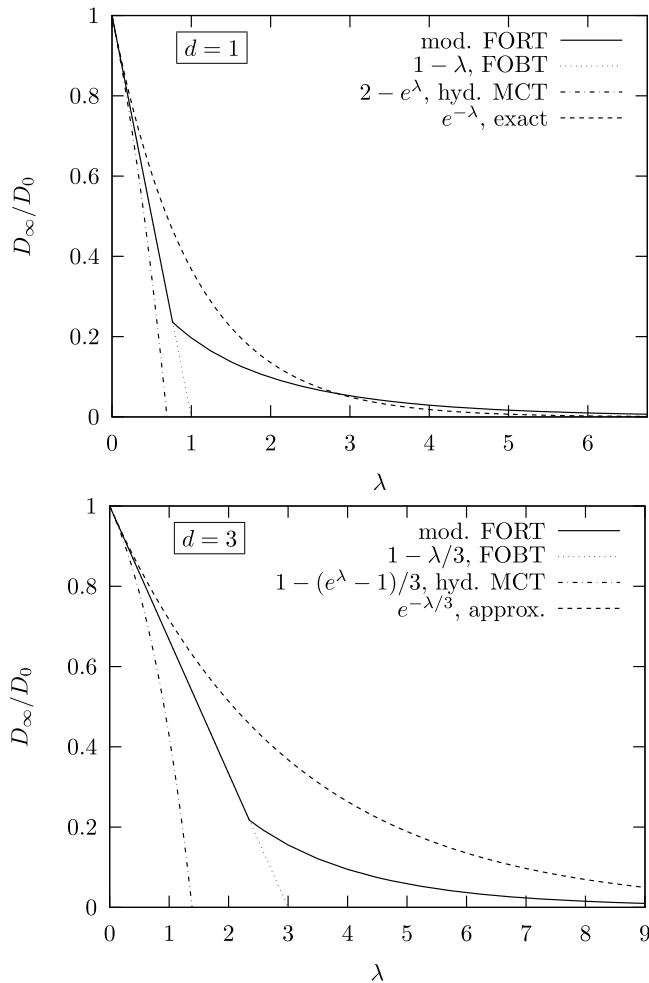
appears as an FDR-consistent MCT-like theory, in the sense that it relies on closed self-consistent dynamical equations for the density correlation function only.

In order to actually get the MCT equations from the present framework, one needs additional manipulations. In particular, the derivation of the bare MCT from the FOBT in the previous section shows that the use of first-order consistent substitutions has to be pushed further. Thus, keeping equation (10.17) for  $N_k(t)$ , equation (10.8) is invoked to set

$$L_k(t) = \lambda D_0^2 \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] F_p(t), \quad (10.26)$$

instead of equations (10.10) or (10.15). One then has the equality

$$\frac{1}{\rho_0} \left[ N_k(t) - \frac{L_k(t)}{\Gamma_k} \right] = \frac{\lambda D_0}{\rho_0} \int_{\mathbf{q}} (\hat{\mathbf{k}} \cdot \mathbf{q})^2 \Phi_q F_p(t) \equiv M_k(t), \quad (10.27)$$



**Figure 8.** Disorder-strength dependence of the long-time diffusion coefficient of a noninteracting Brownian gas plunged in a Gaussian random field with Gaussian covariance in space dimensions  $d = 1$  (top) and  $d = 3$  (bottom). Continuous line: modified first-order renormalized theory; the ergodicity-breaking transition at  $\lambda_c(d)$  is signalled by a corner singularity where the slope of  $D_\infty/D_0$  is discontinuous. Dotted line: first-order bare theory,  $D_\infty/D_0 = 1 - \lambda/d$ . Dashed line:  $D_\infty/D_0 = e^{-\lambda/d}$ ; this expression is exact in  $d = 1$  and a good approximation in  $d = 3$ . Dash-dotted line: mode-coupling theory with hydrodynamic approximation,  $D_\infty/D_0 = 1 - (e^\lambda - 1)/d$ .

which reproduces the MCT kernel  $\mathcal{M}_k(t)$ , equation (9.30b), with the linearized disconnected structure factor  $S_k^d = \lambda\rho_0\Phi_k$ . Therefore, it suffices to eventually replace  $F_k(s)$  with  $-\partial_s F_k(s)/\Gamma_k$  in the second convolution integral of equation (10.6) to get the MCT equations (9.30). The first-order compatibility of the latter substitution follows from the combination of the FDR, of equation (10.7), and of equation (10.8), or, more directly, from equations (10.1c) or (10.6). With this last step, however, the structure of the theory is changed and not only the details of the kernels. The equivalence of equations (10.1c) and (10.6) through the FDR corollary for the kernels, equation (10.4b), is broken, with issues for the consistency of the theory. For instance, changing equation (10.4b) to

restore this equivalence would in turn compromise the consistency of equations (10.1b) and (10.1c) with the FDR.

There would be no such difficulty if the relations

$$\overline{R}_k(t) = \rho_0 \Gamma_k G_k(t), \quad (10.28)$$

$$F_k(t) = \rho_0 G_k(t), \quad (10.29)$$

complemented by the FDR, held exactly. They amount to a mere truncation of both equations (10.7) and (10.8) to their first term and actually coincide with the defining equations of the dynamics of the noninteracting Brownian gas without field, equation (7.3). Generically, equation (10.28) holds for dynamics with additive noise, and equation (10.29), the DHMR relation, for nondisordered systems with Gaussian free energy and either additive or multiplicative noise [31, 82]. Their simultaneous validity for the noninteracting Brownian gas without field, a non-Gaussian system with multiplicative noise, stems from special circumstances described in section 5. Those obviously do not survive in the presence of a Gaussian random field (otherwise, the dynamics should be the same with and without field), as shown by equation (4.22b), in particular.

We thus conclude that the modified FORT is of a fundamentally distinct nature from the MCT.

## 11. Summary and outlook

The time evolution of the density fluctuations in a system of colloidal (Brownian) particles is characterized by a Langevin equation with multiplicative thermal noise, which drives the system into an equilibrium state governed by a highly non-Gaussian free-energy density functional. The multiplicative nature of the time-evolution equation at the density level generates unique dynamical features compared to the usual cases of Langevin equations with additive noise. Indeed, the corresponding free action is a non-Gaussian cubic field theory, and the physical response function is not the same as the usual noise-response function, but is given by a three-point function. It results that the direct loop expansion for the action fails to satisfy the FDR at each order [31]. These features pose a theoretical challenge as to how one can develop an FDR-compatible perturbation theory for the equilibrium dynamics. A profound resolution of this issue has recently been proposed, based on the TR symmetry of the action, i.e. its invariance properties under certain field transformations when time is reversed [32, 37]. This TR symmetry can indeed dictate perturbation theories that preserve the FDR.

In the present work, we have developed one such FDR-preserving perturbation theory to study the equilibrium dynamics of the density fluctuations of a noninteracting Brownian gas embedded in a frozen random potential-energy landscape with Gaussian statistics. Technically, it is quite different from previous work on bulk interacting liquids by one of us and others [32–34, 37], as it is motivated by the  $\mathcal{T}$ - rather than the  $\mathcal{U}$ -transformation, does not require the introduction of extra fields into the problem, and does not rely on a loop expansion. In practice, the present perturbation theory

involves a double expansion: (i) an expansion about the dynamics of the pure system, in terms of the disorder-induced contribution to the dynamical action, then (ii) an expansion in terms of the cubic contribution generated by the multiplicative thermal noise in the free dynamics. The first expansion can be seen as a weak-disorder or high-temperature expansion, since the disorder-induced part of the action is proportional to  $\lambda \equiv w/T^2$ ,  $w$  being the strength of the Gaussian random potential. An essential and novel aspect of the present perturbation theory is the *nonperturbative* (exact) nature of the second expansion. Indeed, the TR symmetry requires that the second expansion be carried out exactly. This is made possible by the form of the cubic term (containing two noise-response fields as factors) and by the causality requirements on the vanishing of averages involving hatted variables. The latter lead to a quick termination of the second expansion at each order of the first one.

We carried out a first-order calculation within this FDR-preserving perturbation scheme. The corresponding results, the first-order bare theory, consist of a set of dynamical equations for the correlation and response functions, which was explicitly checked to be consistent with the FDR, as intended. Using the properties of the dynamics of the pure noninteracting Brownian gas, the equation for the density correlation function can be rearranged as a MCT equation,

$$(\partial_t + \Gamma_k) F_k(t) = - \int_0^t ds M_k^0(t-s) \partial_s C_k^0(s), \quad M_k^0(t) = \frac{\lambda D_0}{\rho_0} \int_{\mathbf{q}} (\hat{\mathbf{k}} \cdot \mathbf{q})^2 \Phi_q C_p^0(t), \quad (11.1)$$

albeit with the memory integral expressed in terms of the bare density correlation function. Apart from this, the equation is the same as in the self-consistent MCT developed by one of us [39–42]. The bare theory allows one to compute the MSD, for which we recover results from earlier calculations at the same order [76], and to characterize the disorder-induced tails that develop in the long-time dynamics. The latter reproduce in detail the behavior found in the Brownian random Lorentz gas, thereby confirming the universal behavior of the persistent correlations induced by quenched disorder [85, 86]. Finally, the bare theory is clearly found to break down at too strong disorder, when  $\lambda$  exceeds the space dimension  $d$ . Below this threshold, the dynamics always remains ergodic.

From the second-order bare perturbation expansion, we also developed a first-order renormalized theory, constrained to obey the FDR. Out of different candidates, all consistent to first order, it is singled out as the only one delivering useful numerical results (without response functions that blow up, for instance) over a significant range of disorder strengths. It turns out that this theory is distinct from the MCT, but might be described as MCT-like, in the sense that the dynamical equation for the density correlation function is also self-closed:

$$(\partial_t + \Gamma_k) F_k(t) = - \frac{1}{\rho_0} \int_0^t ds N_k(t-s) \partial_s F_k(s) - \frac{1}{\rho_0} \int_0^t ds \partial_{t-s} \Lambda_k(t-s) F_k(s), \quad (11.2a)$$

$$N_k(t) = \lambda D_0 \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] F_p(t), \quad \Lambda_k(t) = -\lambda D_0 \int_{\mathbf{q}} \frac{\mathbf{k} \cdot \mathbf{p}}{p^2} [\mathbf{k} \cdot \mathbf{q} \Phi_q] F_p(t). \quad (11.2b)$$

Interestingly, its predictions somewhat improve upon those of the MCT. Indeed, in both cases, an ergodicity-breaking transition occurs in the dynamics of the density

fluctuations at strong enough disorder, but, in the present theory, it does not lead to a diffusion-localization transition in the MSD, at variance with the MCT. This is in agreement with known rigorous results, stating that normal diffusion is always obtained at long time for Brownian dynamics [66]. The reason for these contrasting predictions can be traced back to the distinct low-wavevector asymptotics of the two theories. Actually, the low-wavevector behavior of the single-particle MCT kernel has repeatedly been found to be a source of difficulties in the theory and is usually considered as spurious [11–13, 93–95]. It is therefore promising that the present approach seems to naturally circumvent this issue. It remains that the sharp ergodicity-breaking transition and the corresponding singularity in the long-time diffusion coefficient certainly are artifacts of the self-consistent theory. Indeed, the exact expressions of  $D_\infty$  are known in  $d = 1$  and  $d = 2$  [76]. They are infinitely differentiable functions of the relative disorder strength, and the same can naturally be expected in other space dimensions. An ergodicity-breaking transition would have to display quite unusual characteristics to be consistent with such a behavior. However, recent computer simulations in  $d = 1$  have evidenced strong transient, but long-lived, nonergodic effects in the system at hand [96]. In this respect, the theoretical predictions do not appear as an unreasonable first approximation.

In the present work, we took the initiative of developing a perturbative expansion method about the highly non-Gaussian pure noninteracting state. Compared to the maturity of the fully renormalized theories such as the loop expansion, such approaches are still at an early stage. It would be important for the future to gain a better understanding of their working principles. For instance, it would be useful to put the somewhat ad hoc arguments used in the derivation of the first-order renormalized theory on firm theoretical grounds. This would allow one to further investigate non-equilibrium phenomena, where by definition the equilibrium theorems cannot be used as guides. In this respect, we note that, in principle, the prediction of an ergodicity-breaking transition in the equilibrium theory calls for a reassessment within an out-of-equilibrium two-time formalism. Finally, it would be most interesting to apply the present perturbation scheme to the interacting Dean–Kawasaki equation (with or without the random potential). This would certainly enrich our current perspective on the use of field theory in particle-system dynamics, its relation with the MCT, and the possibilities to go beyond the latter.

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## Appendix A. Symmetries of the effective dynamical action for colloids in a Gaussian random field

In this Appendix, we provide the technical proofs for the invariance properties quoted in section 4, together with some of their implications.

### A.1. The $\mathcal{T}$ -transformation

We first show the invariance of  $S_{\text{bulk}}[\rho, \hat{\rho}]$ ,  $S_{\text{dis}}[\rho, \hat{\rho}]$ , and  $S_{\text{eff}}[\rho, \hat{\rho}]$ , under the  $\mathcal{T}$ -transformation, equation (4.7).

With integrations by parts and the definition of the composite response field, equation (3.8) is easily rewritten as

$$S_{\text{bulk}}[\rho, \hat{\rho}] = \int_{\mathbf{r}, t} \hat{\rho}(\mathbf{r}, t) [i\partial_t \rho(\mathbf{r}, t) + \Lambda(\mathbf{r}, t)] - \int_{\mathbf{r}, t} \Lambda(\mathbf{r}, t) \frac{i}{T} \left. \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \right|_{\rho(\mathbf{r}, t)}. \quad (\text{A.1})$$

The structure of the first term clearly calls for a field transformation of the form of equation (4.9), requiring equation (4.8). With the explicit application of the field transformation, one indeed finds

$$\begin{aligned} S_{\text{bulk}}[\mathcal{T}\rho, \mathcal{T}\hat{\rho}] &= \int_{\mathbf{r}, t} [\hat{\rho}(\mathbf{r}, -t) + ih(\mathbf{r}, -t)] \Lambda(\mathbf{r}, -t) - \int_{\mathbf{r}, t} [\Lambda(\mathbf{r}, -t) - i\partial_t \rho(\mathbf{r}, -t)] \frac{i}{T} \left. \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \right|_{\rho(\mathbf{r}, -t)} \\ &= \int_{\mathbf{r}, t} [\hat{\rho}(\mathbf{r}, t) + ih(\mathbf{r}, t)] \Lambda(\mathbf{r}, t) - \int_{\mathbf{r}, t} [\Lambda(\mathbf{r}, t) + i\partial_t \rho(\mathbf{r}, t)] \frac{i}{T} \left. \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \right|_{\rho(\mathbf{r}, t)}, \end{aligned} \quad (\text{A.2})$$

where the second line merely follows from the change of variable  $t \rightarrow -t$  in the integrals. Integrations by parts restore the initial form of the first integral and, recognizing the chain rule in the second one, one gets for now

$$S_{\text{bulk}}[\mathcal{T}\rho, \mathcal{T}\hat{\rho}] = S_{\text{bulk}}[\rho, \hat{\rho}] + \frac{1}{T} \int_t \partial_t \mathcal{F}[\rho(\mathbf{r}, t)]. \quad (\text{A.3})$$

We may repeat the calculation for  $S_{\text{dis}}[\rho, \hat{\rho}]$  as given by equation (3.12). One first gets

$$\begin{aligned} S_{\text{dis}}[\mathcal{T}\rho, \mathcal{T}\hat{\rho}] &= -\frac{1}{2} \lambda \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} \Phi(|\mathbf{r} - \mathbf{r}'|) [\Lambda(\mathbf{r}, -t) - i\partial_t \rho(\mathbf{r}, -t)] [\Lambda(\mathbf{r}', -t') - i\partial_{t'} \rho(\mathbf{r}', -t')] \\ &= -\frac{1}{2} \lambda \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} \Phi(|\mathbf{r} - \mathbf{r}'|) [\Lambda(\mathbf{r}, t) + i\partial_t \rho(\mathbf{r}, t)] [\Lambda(\mathbf{r}', t') + i\partial_{t'} \rho(\mathbf{r}', t')], \end{aligned} \quad (\text{A.4})$$

with again the change of variables  $t \rightarrow -t$ ,  $t' \rightarrow -t'$  in the integrals to obtain the second line. Then, the result can be rearranged as

$$\begin{aligned} S_{\text{dis}}[\mathcal{T}\rho, \mathcal{T}\hat{\rho}] &= S_{\text{dis}}[\rho, \hat{\rho}] - i\lambda \int_t \partial_t \left[ \int_{\mathbf{r}} \int_{\mathbf{r}', t'} \Phi(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}, t) \Lambda(\mathbf{r}', t') \right] \\ &\quad + \frac{1}{2} \lambda \int_t \partial_t \int_{t'} \partial_{t'} \left[ \int_{\mathbf{r}} \int_{\mathbf{r}'} \Phi(|\mathbf{r} - \mathbf{r}'|) \rho(\mathbf{r}, t) \rho(\mathbf{r}', t') \right]. \end{aligned} \quad (\text{A.5})$$

Since the differences  $S_{\text{bulk}}[\mathcal{T}\rho, \mathcal{T}\hat{\rho}] - S_{\text{bulk}}[\rho, \hat{\rho}]$  and  $S_{\text{dis}}[\mathcal{T}\rho, \mathcal{T}\hat{\rho}] - S_{\text{dis}}[\rho, \hat{\rho}]$  are mere integrals of total time derivatives, both  $S_{\text{bulk}}[\rho, \hat{\rho}]$  and  $S_{\text{dis}}[\rho, \hat{\rho}]$  are invariant under the  $\mathcal{T}$ -transformation at equilibrium. This obviously implies the invariance of  $S_{\text{eff}}[\rho, \hat{\rho}]$ .

## A.2. The $\mathcal{U}$ - and $\mathcal{U}'$ -transformations

A shared feature of the present theory and of the theory of Langevin processes with colored noise developed in [65] is that the dynamical action is a sum of quadratic and linear terms in the hatted variables, as a consequence of the Gaussianity of the noise and/or disorder. In the latter work, a symmetry of the action was unveiled, which can actually be related to this observation. We show that a similar one holds in the present case as well.

Denoting the thermal-noise contribution to the effective dynamical action as

$$S_{\text{noise}}[\rho, \hat{\rho}] = -D_0 \int_{\mathbf{r}, t} \rho(\mathbf{r}, t) [\nabla \hat{\rho}(\mathbf{r}, t)]^2, \quad (\text{A.6})$$

and adding it to the random-field term  $S_{\text{dis}}[\rho, \hat{\rho}]$ , restoration of the noise variance, equation (2.16), and integrations by parts can be used to get

$$S_{\text{noise}}[\rho, \hat{\rho}] + S_{\text{dis}}[\rho, \hat{\rho}] = -\frac{1}{2} \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} K_\lambda(\mathbf{r}, t; \mathbf{r}', t') \hat{\rho}(\mathbf{r}, t) \hat{\rho}(\mathbf{r}', t'), \quad (\text{A.7})$$

where the density-dependent symmetric kernel  $K_\lambda(\mathbf{r}, t; \mathbf{r}', t')$  is given by equation (4.16). Now, the remaining part of the action, which only involves the deterministic nonrandom part of the density evolution equation defined in equation (4.15) and thus reads

$$S_{\text{bulk}}[\rho, \hat{\rho}] - S_{\text{noise}}[\rho, \hat{\rho}] = \int_{\mathbf{r}, t} i \hat{\rho}(\mathbf{r}, t) \text{DET}([\rho], \mathbf{r}, t), \quad (\text{A.8})$$

can be rewritten as

$$S_{\text{bulk}}[\rho, \hat{\rho}] - S_{\text{noise}}[\rho, \hat{\rho}] = i \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} K_\lambda(\mathbf{r}, t; \mathbf{r}', t') \hat{\rho}(\mathbf{r}, t) \int_{\mathbf{r}'', t''} K_\lambda^{-1}(\mathbf{r}', t'; \mathbf{r}'', t'') \text{DET}([\rho], \mathbf{r}'', t''), \quad (\text{A.9})$$

through injection of equation (4.17) and minor reorganizations. It results that

$$S_{\text{eff}}[\rho, \hat{\rho}] = -\frac{1}{2} \int_{\mathbf{r}, t} \int_{\mathbf{r}', t'} K_\lambda(\mathbf{r}, t; \mathbf{r}', t') \hat{\rho}(\mathbf{r}, t) \left\{ \hat{\rho}(\mathbf{r}', t') - 2i \int_{\mathbf{r}'', t''} K_\lambda^{-1}(\mathbf{r}', t'; \mathbf{r}'', t'') \text{DET}([\rho], \mathbf{r}'', t'') \right\}. \quad (\text{A.10})$$

This expression is manifestly invariant under the  $\mathcal{U}'$ -transformation, equation (4.14), thanks to the symmetry of  $K_\lambda(\mathbf{r}, t; \mathbf{r}', t')$ .

Although one can directly use equation (4.14) to compose  $\mathcal{U}'$  with  $\mathcal{T}$ , we find it useful to first reorganize  $\mathcal{U}'\hat{\rho}(\mathbf{r}, t)$ . Indeed, this allows one to isolate contributions with distinct physical origins and facilitates comparisons with previous results. Once the explicit expression of  $\text{DET}([\rho], \mathbf{r}, t)$  is restored, equation (4.8) and a single integration by parts lead to

$$\mathcal{U}'\hat{\rho}(\mathbf{r}, t) = -\hat{\rho}(\mathbf{r}, t) - 2iD_0 \int_{\mathbf{r}', t'} \rho(\mathbf{r}', t') [\nabla' K_\lambda^{-1}(\mathbf{r}, t; \mathbf{r}', t')] \cdot \left( \nabla' \left[ h(\mathbf{r}', t') - \frac{1}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r}')} \Big|_{\rho(\mathbf{r}', t')} \right] \right). \quad (\text{A.11})$$

Treating the integral in the same way as  $S_{\text{noise}}[\rho, \hat{\rho}]$ , one then gets

$$\mathcal{U}'\hat{\rho}(\mathbf{r}, t) = -\hat{\rho}(\mathbf{r}, t) - \text{i} \int_{\mathbf{r}', t'} \int_{\mathbf{r}'', t''} K_{\lambda}^{-1}(\mathbf{r}, t; \mathbf{r}', t') K_0(\mathbf{r}', t'; \mathbf{r}'', t'') \left[ h(\mathbf{r}'', t'') - \frac{1}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r}'')} \Big|_{\rho(\mathbf{r}'', t'')} \right], \quad (\text{A.12})$$

where  $K_0(\mathbf{r}, t; \mathbf{r}', t')$  is nothing but  $K_{\lambda}(\mathbf{r}, t; \mathbf{r}', t')$  at  $\lambda = 0$ . Using equation (4.16), it can be replaced with  $K_{\lambda}(\mathbf{r}, t; \mathbf{r}', t') - \lambda \Delta K(\mathbf{r}, t; \mathbf{r}', t')$  to obtain

$$\begin{aligned} \mathcal{U}'\hat{\rho}(\mathbf{r}, t) = & -\hat{\rho}(\mathbf{r}, t) - \text{i}h(\mathbf{r}, t) + \frac{\text{i}}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \Big|_{\rho(\mathbf{r}, t)} \\ & + \text{i}\lambda \int_{\mathbf{r}', t'} \int_{\mathbf{r}'', t''} K_{\lambda}^{-1}(\mathbf{r}, t; \mathbf{r}', t') \Delta K(\mathbf{r}', t'; \mathbf{r}'', t'') \left[ h(\mathbf{r}'', t'') - \frac{1}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r}'')} \Big|_{\rho(\mathbf{r}'', t'')} \right]. \end{aligned} \quad (\text{A.13})$$

It remains to use the expression of  $\Delta K(\mathbf{r}, t; \mathbf{r}', t')$  to eventually get

$$\begin{aligned} \mathcal{U}'\hat{\rho}(\mathbf{r}, t) = & -\hat{\rho}(\mathbf{r}, t) - \text{i}h(\mathbf{r}, t) + \frac{\text{i}}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r})} \Big|_{\rho(\mathbf{r}, t)} \\ & + \text{i}\lambda D_0 \int_{\mathbf{r}', t'} K_{\lambda}^{-1}(\mathbf{r}, t; \mathbf{r}', t') \nabla' \cdot \left\{ \rho(\mathbf{r}', t') \nabla' \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r}' - \mathbf{r}''|) \text{DET}([\rho], \mathbf{r}'', t'') \right\} \end{aligned} \quad (\text{A.14})$$

after a last pair of integrations by parts. This formula can be used as an alternative to the second line of equation (4.14).

We may now compose  $\mathcal{U}'$  and  $\mathcal{T}$  to get the  $\mathcal{U}$ -transformation with time reversal. Since the application of  $\mathcal{T}$  to equation (4.8) gives  $D_0 \nabla \cdot [\rho(\mathbf{r}, -t) \nabla \mathcal{T}h(\mathbf{r}, t)] = \partial_t \rho(\mathbf{r}, -t)$ , hence  $\mathcal{T}h(\mathbf{r}, t) = -h(\mathbf{r}, -t)$ , the function  $h(\mathbf{r}, t)$  disappears when  $\mathcal{T}$  is applied to  $\mathcal{U}'\hat{\rho}(\mathbf{r}, t)$ , giving equation (4.18) as the final result.

Obviously, if  $\mathcal{U}'\hat{\rho}(\mathbf{r}, t)$  from equation (4.14) is left untouched, the expression

$$\mathcal{U}\hat{\rho}(\mathbf{r}, t) = -\hat{\rho}(\mathbf{r}, -t) - \text{i}h(\mathbf{r}, -t) + 2\text{i} \int_{\mathbf{r}', t'} K_{\lambda}^{-1}(\mathbf{r}, -t; \mathbf{r}', -t') \mathcal{T} \text{DET}([\rho], \mathbf{r}', t') \quad (\text{A.15})$$

is a valid replacement for the second line of equation (4.18).

### A.3. Implications of the $\mathcal{U}$ -transformation

As the  $\mathcal{T}$ -transformation, the  $\mathcal{U}$ -transformation can be used to derive equilibrium relations between correlations and responses.

In particular, a generalized form of the FDR can be obtained for the noise-response function. Indeed, expanding the Ward–Takahashi identity equation (4.20), one gets

$$\begin{aligned} G(|\mathbf{r} - \mathbf{r}'|, t - t') + G(|\mathbf{r} - \mathbf{r}'|, t' - t) = & \left\langle \rho(\mathbf{r}, t) \frac{1}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r}')} \Big|_{\rho(\mathbf{r}', t')} \right\rangle_{\text{eff}} \\ & + \lambda D_0 \left\langle \rho(\mathbf{r}, t) \int_{\mathbf{r}'', t''} K_{\lambda}^{-1}(\mathbf{r}', t'; \mathbf{r}'', t'') \nabla'' \cdot \left\{ \rho(\mathbf{r}'', t'') \nabla'' \int_{\mathbf{r}''', t'''} \Phi(|\mathbf{r}'' - \mathbf{r}'''|) \text{DET}([\rho], \mathbf{r}''', t''') \right\} \right\rangle_{\text{eff}}, \end{aligned} \quad (\text{A.16})$$

where we used time-translation invariance and the time-reversal symmetry of the correlations.

Using the explicit expression of  $\mathcal{F}_{\text{bulk}}[\rho]$ , the first average in the right-hand side of equation (A.16) can be rewritten as

$$\begin{aligned} \left\langle \rho(\mathbf{r}, t) \frac{1}{T} \frac{\delta \mathcal{F}_{\text{bulk}}[\rho]}{\delta \rho(\mathbf{r}')} \Big|_{\rho(\mathbf{r}', t')} \right\rangle_{\text{eff}} &= \left\langle \rho(\mathbf{r}, t) \left[ \frac{\delta \rho(\mathbf{r}', t')}{\rho_0} + \frac{1}{T} \int_{\mathbf{r}''} u(|\mathbf{r}' - \mathbf{r}''|) \delta \rho(\mathbf{r}'', t') \right] \right\rangle_{\text{eff}} \\ &+ \left\langle \rho(\mathbf{r}, t) \left[ \ln \left( 1 + \frac{\delta \rho(\mathbf{r}', t')}{\rho_0} \right) - \frac{\delta \rho(\mathbf{r}', t')}{\rho_0} \right] \right\rangle_{\text{eff}}. \end{aligned} \quad (\text{A.17})$$

The first term is due to the Gaussian part of the free energy,

$$\mathcal{F}_{\text{bulk,G}}[\rho] = \frac{T}{2} \int_{\mathbf{r}} \int_{\mathbf{r}'} Q^{-1}(|\mathbf{r} - \mathbf{r}'|) \delta \rho(\mathbf{r}) \delta \rho(\mathbf{r}'), \quad (\text{A.18})$$

where

$$Q^{-1}(|\mathbf{r} - \mathbf{r}'|) = \frac{\delta(\mathbf{r} - \mathbf{r}')}{\rho_0} + \frac{u(|\mathbf{r} - \mathbf{r}'|)}{T} \quad (\text{A.19})$$

is the functional inverse of the static density correlation function in the Gaussian theory defined by  $\mathcal{F}_{\text{bulk,G}}[\rho]$ . One can thus write

$$\left\langle \rho(\mathbf{r}, t) \left[ \frac{\delta \rho(\mathbf{r}', t')}{\rho_0} + \frac{1}{T} \int_{\mathbf{r}''} u(|\mathbf{r}' - \mathbf{r}''|) \delta \rho(\mathbf{r}'', t') \right] \right\rangle_{\text{eff}} = \int_{\mathbf{r}''} C(|\mathbf{r} - \mathbf{r}''|, t - t') Q^{-1}(|\mathbf{r}'' - \mathbf{r}'|). \quad (\text{A.20})$$

The second term, which we shall denote by  $\Delta C^{\text{nG}}(|\mathbf{r} - \mathbf{r}'|, t - t')$ , arises from the non-Gaussian nature of  $\mathcal{F}_{\text{id}}[\rho]$ . As such, it already appears in the absence of a random field.

The second average in the right-hand side of equation (A.16) manifestly arises from the presence of the quenched random potential (it has  $\lambda$  as a prefactor). Accordingly, we shall denote it by  $\Delta C^{\text{dis}}(|\mathbf{r} - \mathbf{r}'|, t - t')$ , for which we could not find any obvious simpler expression.

Combining these notations, equation (4.21) is finally obtained.

As an interesting consistency check, it is also possible to get the dynamical equations for the density correlation function, equations (6.1e) and (6.3c), directly from the  $\mathcal{T}$ - and  $\mathcal{U}$ -transformations. Indeed, consider the Ward–Takahashi identity

$$\left\langle \left[ \int_{\mathbf{r}'', t''} K_{\lambda}(\mathbf{r}, t; \mathbf{r}'', t'') \hat{\rho}(\mathbf{r}'', t'') \right] \rho(\mathbf{r}', t') \right\rangle_{\text{eff}} = \left\langle \left[ \mathcal{U} \int_{\mathbf{r}'', t''} K_{\lambda}(\mathbf{r}, t; \mathbf{r}'', t'') \hat{\rho}(\mathbf{r}'', t'') \right] [\mathcal{U} \rho(\mathbf{r}', t')] \right\rangle_{\text{eff}}. \quad (\text{A.21})$$

The direct application of the  $\mathcal{U}$ -transformation, equation (A.15), gives

$$\begin{aligned} &\mathcal{U} \int_{\mathbf{r}'', t''} K_{\lambda}(\mathbf{r}, t; \mathbf{r}'', t'') \hat{\rho}(\mathbf{r}'', t'') \\ &= \int_{\mathbf{r}'', t''} K_{\lambda}(\mathbf{r}, -t; \mathbf{r}'', -t'') [-\hat{\rho}(\mathbf{r}'', -t'') - ih(\mathbf{r}'', -t'')] + 2i\mathcal{T}\text{DET}([\rho], \mathbf{r}, t). \end{aligned} \quad (\text{A.22})$$

Using

$$\int_{\mathbf{r}'', t''} K_{\lambda}(\mathbf{r}, t; \mathbf{r}'', t'') \hat{\rho}(\mathbf{r}'', t'') = -2\Lambda(\mathbf{r}, t) + \lambda D_0 \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \Lambda(\mathbf{r}'', t'') \right] \quad (\text{A.23})$$

and

$$\int_{\mathbf{r}'', t''} K_\lambda(\mathbf{r}, t; \mathbf{r}'', t'') h(\mathbf{r}'', t'') = -2\partial_t \rho(\mathbf{r}, t) + \lambda D_0 \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \partial_{t''} \rho(\mathbf{r}'', t'') \right], \quad (\text{A.24})$$

this becomes

$$\mathcal{U} \int_{\mathbf{r}'', t''} K_\lambda(\mathbf{r}, t; \mathbf{r}'', t'') \hat{\rho}(\mathbf{r}'', t'') = 2i \mathcal{T} \text{DET}([\rho], \mathbf{r}, t) + 2\Lambda(\mathbf{r}, -t) - 2i \partial_t \rho(\mathbf{r}, -t) - \lambda D_0 \nabla \cdot \left[ \rho(\mathbf{r}, -t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \{ \Lambda(\mathbf{r}'', -t'') - i \partial_{t''} \rho(\mathbf{r}'', -t'') \} \right]. \quad (\text{A.25})$$

The Ward–Takahashi identity, equation (A.21), now explicitly reads

$$\begin{aligned} & -2 \langle \Lambda(\mathbf{r}, t) \rho(\mathbf{r}', t') \rangle_{\text{eff}} + \lambda D_0 \left\langle \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \Lambda(\mathbf{r}'', t'') \right] \rho(\mathbf{r}', t') \right\rangle_{\text{eff}} \\ &= 2i \langle \mathcal{T} \text{DET}([\rho], \mathbf{r}, t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}} + 2 \langle \Lambda(\mathbf{r}, -t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}} - 2i \langle \partial_t \rho(\mathbf{r}, -t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}} \\ & - \lambda D_0 \left\langle \nabla \cdot \left[ \rho(\mathbf{r}, -t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \{ \Lambda(\mathbf{r}'', -t'') - i \partial_{t''} \rho(\mathbf{r}'', -t'') \} \right] \rho(\mathbf{r}', -t') \right\rangle_{\text{eff}}. \end{aligned} \quad (\text{A.26})$$

The  $\mathcal{T}$ -transformation gives

$$\langle \mathcal{T} \text{DET}([\rho], \mathbf{r}, t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}} = \langle \text{DET}([\rho], \mathbf{r}, t) \rho(\mathbf{r}', t') \rangle_{\text{eff}}, \quad (\text{A.27})$$

$$\langle \Lambda(\mathbf{r}, t) \rho(\mathbf{r}', t') \rangle_{\text{eff}} = \langle \Lambda(\mathbf{r}, -t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}} - i \langle \partial_t \rho(\mathbf{r}, -t) \rho(\mathbf{r}', -t') \rangle_{\text{eff}}, \quad (\text{A.28})$$

and

$$\begin{aligned} & \left\langle \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \Lambda(\mathbf{r}'', t'') \right] \rho(\mathbf{r}', t') \right\rangle_{\text{eff}} \\ &= \left\langle \nabla \cdot \left[ \rho(\mathbf{r}, -t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \{ \Lambda(\mathbf{r}'', -t'') - i \partial_{t''} \rho(\mathbf{r}'', -t'') \} \right] \rho(\mathbf{r}', -t') \right\rangle_{\text{eff}}. \end{aligned} \quad (\text{A.29})$$

Therefore, one gets

$$\left\langle \left\{ i \text{DET}([\rho], \mathbf{r}, t) + 2\Lambda(\mathbf{r}, t) - \lambda D_0 \nabla \cdot \left[ \rho(\mathbf{r}, t) \nabla \int_{\mathbf{r}'', t''} \Phi(|\mathbf{r} - \mathbf{r}''|) \Lambda(\mathbf{r}'', t'') \right] \right\} \rho(\mathbf{r}', t') \right\rangle_{\text{eff}} = 0, \quad (\text{A.30})$$

which is nothing but

$$\left\langle \frac{\delta S_{\text{eff}}}{\delta \hat{\rho}(\mathbf{r}, t)} \rho(\mathbf{r}', t') \right\rangle_{\text{eff}} = 0. \quad (\text{A.31})$$

## Appendix B. Calculation of the memory kernel $M_k^0(t)$ for a Gaussian covariance

We analytically compute the memory kernel  $M_k^0(t)$  given in equation (9.18) for the Gaussian random potential with Gaussian covariance:

$$M_k^0(t) = A_k \int d\mathbf{q} (\mathbf{k} \cdot \mathbf{q})^2 e^{-q^2 R^2/2} e^{-D_0(\mathbf{k}-\mathbf{q})^2 t}, \quad A_k \equiv \frac{\lambda D_0 R^d}{k^2 (2\pi)^{d/2}}. \quad (\text{B.1})$$

The integral can be arranged as

$$M_k^0(t) = A_k e^{-D_0 k^2 t} \int d\mathbf{q} (\mathbf{k} \cdot \mathbf{q})^2 e^{-(R^2/2 + D_0 t)q^2 + 2D_0 t \mathbf{k} \cdot \mathbf{q}}. \quad (\text{B.2})$$

Completing the square in the argument of the exponential, we have

$$M_k^0(t) = A_k e^{-\frac{D_0 k^2 R^2 t}{R^2 + 2D_0 t}} \int d\mathbf{q} (\mathbf{k} \cdot \mathbf{q})^2 e^{-(R^2/2 + D_0 t) \left( \mathbf{q} - \frac{D_0 t}{R^2/2 + D_0 t} \mathbf{k} \right)^2}. \quad (\text{B.3})$$

Now, shifting the integration variable via  $\mathbf{u} \equiv \mathbf{q} - \frac{D_0 t}{R^2/2 + D_0 t} \mathbf{k}$ , we get

$$M_k^0(t) = A_k e^{-\frac{D_0 k^2 R^2 t}{R^2 + 2D_0 t}} \int d\mathbf{u} \left[ (\mathbf{k} \cdot \mathbf{u})^2 + 2\mathbf{k} \cdot \mathbf{u} \frac{k^2 D_0 t}{R^2/2 + D_0 t} + \left( \frac{k^2 D_0 t}{R^2/2 + D_0 t} \right)^2 \right] e^{-(R^2/2 + D_0 t) \mathbf{u}^2}. \quad (\text{B.4})$$

By isotropy, the first term  $(\mathbf{k} \cdot \mathbf{u})^2$  can be replaced with  $k^2 \mathbf{u}^2/d$  and the second term involving  $\mathbf{k} \cdot \mathbf{u}$  vanishes. We thus have

$$M_k^0(t) = A_k e^{-\frac{D_0 k^2 R^2 t}{R^2 + 2D_0 t}} \int d\mathbf{u} \left[ \frac{k^2 \mathbf{u}^2}{d} + \left( \frac{2k^2 D_0 t}{R^2 + 2D_0 t} \right)^2 \right] e^{-(R^2/2 + D_0 t) \mathbf{u}^2}. \quad (\text{B.5})$$

Using the integration formulas

$$\int d\mathbf{u} e^{-\alpha \mathbf{u}^2} = \left( \frac{\pi}{\alpha} \right)^{d/2}, \quad \int d\mathbf{u} \mathbf{u} \mathbf{u}^2 e^{-\alpha \mathbf{u}^2} = \frac{d}{2\alpha} \left( \frac{\pi}{\alpha} \right)^{d/2}, \quad (\text{B.6})$$

we obtain

$$M_k^0(t) = A_k e^{-\frac{D_0 k^2 R^2 t}{R^2 + 2D_0 t}} \left[ \frac{k^2}{R^2 + 2D_0 t} + \left( \frac{2k^2 D_0 t}{R^2 + 2D_0 t} \right)^2 \right] \left( \frac{2\pi}{R^2 + 2D_0 t} \right)^{d/2}. \quad (\text{B.7})$$

Putting the explicit expression for  $A_k$ , we have the final expression for the memory kernel,

$$M_k^0(t) = \lambda D_0 R^d e^{-\frac{D_0 k^2 R^2 t}{R^2 + 2D_0 t}} \frac{R^2 + 2D_0 t + (2D_0 t)^2 k^2}{(R^2 + 2D_0 t)^2} \left( \frac{1}{R^2 + 2D_0 t} \right)^{d/2}, \quad (\text{B.8})$$

which is equation (9.47).

### Appendix C. Renormalized equation for the noise-response function

The full dynamical equation for the noise-response function  $G(12)$  is given by equation (6.3a) and, after simplification, reads

$$(\partial_t - D_0 \nabla^2) G(12) = \delta(12) - \lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \nabla^\gamma \Phi(13)] \langle 13 \hat{3}^\gamma \hat{2} \rangle \right). \quad (\text{C.1})$$

Since there is no risk of confusion in these appendices, we shall here denote the averages over the effective action simply as  $\langle \dots \rangle$ .

It is straightforward to calculate the multi-point average up to the first order, as

$$\langle 13\hat{3}^\gamma\hat{2} \rangle = \langle 13\hat{3}^\gamma\hat{2} \rangle_f + \langle 13\hat{3}^\gamma\hat{2}S_{\text{dis}} \rangle_f + O(\lambda^2). \quad (\text{C.2})$$

The first term corresponds to equation (8.2a), and the first-order average involves  $S_{\text{dis}}[\rho, \hat{\rho}]$ , given by equation (5.10) and rewritten as

$$S_{\text{dis}}[\rho, \hat{\rho}] = \frac{1}{2}\lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] [\rho_0^2 + 2\rho_0 \delta\rho(6) + \delta\rho(6)\delta\rho(9)] [\nabla_6^a \hat{\rho}(6)] [\nabla_9^b \hat{\rho}(9)], \quad (\text{C.3})$$

where  $a$  and  $b$  denote summed-upon Cartesian indices. With these expressions, one readily obtains

$$\langle 13\hat{3}^\gamma\hat{2} \rangle = \langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{2} \rangle_0 + \frac{1}{2}\lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1369\hat{3}^\gamma\hat{6}^a\hat{9}^b\hat{2} \rangle_0 + O(\lambda^2), \quad (\text{C.4})$$

hence

$$\begin{aligned} \langle 13\hat{3}^\gamma\hat{2} \rangle &= \langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{2} \rangle_0 \\ &+ \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] [\langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{2} \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0 \langle 3\hat{2} \rangle_0 \\ &+ \langle 1\hat{6}^a \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 9\hat{2} \rangle_0] + O(\lambda^2). \end{aligned} \quad (\text{C.5})$$

On the other hand, one has the following (first-order) result for the noise-response function itself,

$$\begin{aligned} \langle 1\hat{3}^\gamma \rangle &= \langle 1\hat{3}^\gamma \rangle_f + \langle 1\hat{3}^\gamma S_{\text{dis}} \rangle_f + O(\lambda^2) \\ &= \langle 1\hat{3}^\gamma \rangle_0 + \frac{1}{2}\lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 169\hat{6}^a\hat{9}^b\hat{3}^\gamma \rangle_0 + O(\lambda^2) \\ &= \langle 1\hat{3}^\gamma \rangle_0 + \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0 + O(\lambda^2). \end{aligned} \quad (\text{C.6})$$

Equivalently, one can express the bare response in terms of the renormalized one as

$$\langle 1\hat{3}^\gamma \rangle_0 = \langle 1\hat{3}^\gamma \rangle - \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1\hat{6}^a \rangle \langle 6\hat{9}^b \rangle \langle 9\hat{3}^\gamma \rangle + O(\lambda^2). \quad (\text{C.7})$$

Likewise, one has

$$\langle 3\hat{2} \rangle_0 = \langle 3\hat{2} \rangle - \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 3\hat{6}^a \rangle \langle 6\hat{9}^b \rangle \langle 9\hat{2} \rangle + O(\lambda^2). \quad (\text{C.8})$$

Substituting these expressions into equation (C.5), one straightforwardly obtains

$$\langle 13\hat{3}^\gamma\hat{2} \rangle = \langle 1\hat{3}^\gamma \rangle \langle 3\hat{2} \rangle + \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1\hat{6}^a \rangle \langle 6\hat{3}^\gamma \rangle \langle 3\hat{9}^b \rangle \langle 9\hat{2} \rangle + O(\lambda^2). \quad (\text{C.9})$$

The dynamical equation for  $G(12)$  is then given by (up to second order in  $\lambda$ )

$$\begin{aligned} (\partial_t - D_0 \nabla^2)G(12) &= \delta(12) - \lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \nabla^\gamma \Phi(13)] \langle 1\hat{3}^\gamma \rangle \langle 3\hat{2} \rangle \right) \\ &- \lambda^2 D_0^4 \nabla^\alpha \left( \int_3 \int_6 \int_9 [\nabla^\alpha \nabla^\gamma \Phi(13)] [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1\hat{6}^a \rangle \langle 6\hat{3}^\gamma \rangle \langle 3\hat{9}^b \rangle \langle 9\hat{2} \rangle \right). \end{aligned} \quad (\text{C.10})$$

Note that equation (C.6) takes the form of the Schwinger–Dyson equation,

$$G = G_0 + G_0 \cdot \Sigma[G] \cdot G = G_0 + G_0 \cdot \Sigma^0[G_0] \cdot G_0 + \dots, \quad (\text{C.11})$$

and equation (C.10) would equivalently take the form

$$G_0^{-1} \cdot G = I + \Sigma[G] \cdot G, \quad (\text{C.12})$$

where  $G_0^{-1}$  is given by  $G_0^{-1}(12) = (\partial_t - D_0 \nabla^2)\delta(12)$ .

The Fourier-transformed dynamical equation for  $G$  is eventually given by (up to first order)

$$(\partial_t + \Gamma_k)G_k(t - t') = \delta(t - t') - \int_{t'}^t ds \Sigma_k(t - s)G_k(s - t'), \quad (\text{C.13})$$

$$\Sigma_k(t) = \lambda D_0^2 \int_{\mathbf{q}} \mathbf{q} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t). \quad (\text{C.14})$$

## Appendix D. Renormalized equation for the physical response function

The full dynamical equation for the physical response function is given by equation (6.3b), leading to

$$\begin{aligned} (\partial_t - D_0 \nabla^2)\bar{R}(12) &= -\rho_0 D_0 \nabla^2 \delta(12) \\ &+ \lambda D_0^3 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_2^\beta \nabla_3^\gamma [\rho_0 \langle 13\hat{3}^\gamma \hat{2}^\beta \rangle + \rho_0 \langle 12\hat{3}^\gamma \hat{2}^\beta \rangle + \langle 132\hat{3}^\gamma \hat{2}^\beta \rangle] \right). \end{aligned} \quad (\text{D.1})$$

The first two multi-point averages have already been computed (see equation (C.9)):

$$\langle 13\hat{3}^\gamma \hat{2}^\beta \rangle = \langle 1\hat{3}^\gamma \rangle \langle 3\hat{2}^\beta \rangle + \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1\hat{6}^a \rangle \langle 6\hat{3}^\gamma \rangle \langle 3\hat{9}^b \rangle \langle 9\hat{2}^\beta \rangle + O(\lambda^2), \quad (\text{D.2})$$

$$\langle 12\hat{3}^\gamma \hat{2}^\beta \rangle = \langle 1\hat{2}^\beta \rangle \langle 2\hat{3}^\gamma \rangle + \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1\hat{6}^a \rangle \langle 6\hat{2}^\beta \rangle \langle 2\hat{9}^b \rangle \langle 9\hat{3}^\gamma \rangle + O(\lambda^2). \quad (\text{D.3})$$

The last one is obtained up to the first order as (see equation (8.2d) for the first term)

$$\begin{aligned} \langle 132\hat{3}^\gamma \hat{2}^\beta \rangle &= \langle 132\hat{3}^\gamma \hat{2}^\beta \rangle_f + \langle 132\hat{3}^\gamma \hat{2}^\beta S_{\text{dis}} \rangle_f + O(\lambda^2) \\ &= \lambda \rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1236\hat{2}^\beta \hat{3}^\gamma \hat{6}^a \hat{9}^b \rangle_0 + O(\lambda^2) \\ &= \lambda \rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] [\langle 1\hat{2}^\beta \rangle_0 \langle 2\hat{6}^a \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0 + \langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{6}^a \rangle_0 \langle 6\hat{2}^\beta \rangle_0 \langle 2\hat{9}^b \rangle_0 \\ &\quad + \langle 1\hat{6}^a \rangle_0 \langle 6\hat{2}^\beta \rangle_0 \langle 2\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{2}^\beta \rangle_0 \langle 2\hat{9}^b \rangle_0] + O(\lambda^2), \end{aligned} \quad (\text{D.4})$$

where we used  $\int_9[\nabla_6^a\nabla_6^b\Phi(69)]\langle 6\hat{9}^b\rangle_0 = 0$  by isotropy.

Now, recall that the physical response function  $\bar{R}(12)$  is related to the noise-response function  $G(12)$  as shown by equation (4.1c), hence

$$\bar{R}(12) = i\rho_0 D_0 \nabla^2 \langle 1\hat{2} \rangle + iD_0 \nabla_2^\beta \langle 12\hat{2}^\beta \rangle. \quad (\text{D.5})$$

The three-point average  $\langle 12\hat{2}^\beta \rangle$  is evaluated up to the first order as

$$\begin{aligned} \langle 12\hat{2}^\beta \rangle &= \langle 12\hat{2}^\beta \rangle_f + \langle 12\hat{2}^\beta S_{\text{dis}} \rangle_f + O(\lambda^2) \\ &= \lambda\rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 126\hat{2}^\beta \hat{6}^a \hat{9}^b \rangle_0 + O(\lambda^2) \\ &= \lambda\rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 1\hat{6}^a \rangle_0 \langle 6\hat{2}^\beta \rangle_0 \langle 2\hat{9}^b \rangle_0 + O(\lambda^2). \end{aligned} \quad (\text{D.6})$$

One thus has

$$\bar{R}(23) = i\rho_0 D_0 \nabla_2^2 \langle 2\hat{3} \rangle + iD_0 \nabla_3^\gamma \langle 23\hat{3}^\gamma \rangle, \quad (\text{D.7})$$

$$\bar{R}(32) = i\rho_0 D_0 \nabla_3^2 \langle 3\hat{2} \rangle + iD_0 \nabla_2^\beta \langle 32\hat{2}^\beta \rangle, \quad (\text{D.8})$$

with

$$\langle 23\hat{3}^\gamma \rangle = \lambda\rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 2\hat{6}^a \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0 + O(\lambda^2), \quad (\text{D.9})$$

$$\langle 32\hat{2}^\beta \rangle = \lambda\rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 3\hat{6}^a \rangle_0 \langle 6\hat{2}^\beta \rangle_0 \langle 2\hat{9}^b \rangle_0 + O(\lambda^2). \quad (\text{D.10})$$

Crucially, the above integrals can be straightforwardly recognized in the first two terms of the right-hand side of equation (D.4). These terms are thus associated with the first-order expansion of  $\bar{R}$  and should be accounted for accordingly in the renormalization process.

Hence, from equations (D.2), (D.3), (D.4), (D.9), and (D.10), the following first-order renormalized expressions result:

$$\langle 13\hat{3}^\gamma \hat{2}^\beta \rangle = \langle 1\hat{3}^\gamma \rangle \langle 3\hat{2}^\beta \rangle, \quad \langle 12\hat{3}^\gamma \hat{2}^\beta \rangle = \langle 1\hat{2}^\beta \rangle \langle 2\hat{3}^\gamma \rangle, \quad \langle 132\hat{3}^\gamma \hat{2}^\beta \rangle = \langle 1\hat{3}^\gamma \rangle \langle 32\hat{2}^\beta \rangle + \langle 1\hat{2}^\beta \rangle \langle 23\hat{3}^\gamma \rangle. \quad (\text{D.11})$$

They provide one with the first-order renormalization

$$\begin{aligned} &D_0^3 \nabla_2^\beta \nabla_3^\gamma [\rho_0 \langle 13\hat{3}^\gamma \hat{2}^\beta \rangle + \rho_0 \langle 12\hat{3}^\gamma \hat{2}^\beta \rangle + \langle 132\hat{3}^\gamma \hat{2}^\beta \rangle] \\ &= D_0^2 \nabla_3^\gamma [\langle 1\hat{3}^\gamma \rangle (\rho_0 D_0 \nabla_2^\beta \langle 3\hat{2}^\beta \rangle + D_0 \nabla_2^\beta \langle 32\hat{2}^\beta \rangle)] + D_0^2 \nabla_2^\beta [\langle 1\hat{2}^\beta \rangle (\rho_0 D_0 \nabla_3^\gamma \langle 2\hat{3}^\gamma \rangle + D_0 \nabla_3^\gamma \langle 23\hat{3}^\gamma \rangle)] \\ &= D_0^2 \nabla_3^\gamma [\langle 1\hat{3}^\gamma \rangle (\rho_0 D_0 \nabla_2^2 \langle 3\hat{2} \rangle + D_0 \nabla_2^\beta \langle 32\hat{2}^\beta \rangle)] + D_0^2 \nabla_2^\beta [\langle 1\hat{2}^\beta \rangle (\rho_0 D_0 \nabla_3^2 \langle 2\hat{3} \rangle + D_0 \nabla_3^\gamma \langle 23\hat{3}^\gamma \rangle)] \\ &= D_0^2 \nabla_3^\gamma [\langle 1\hat{3}^\beta \rangle (\rho_0 D_0 \nabla_3^2 \langle 3\hat{2} \rangle + D_0 \nabla_2^\gamma \langle 32\hat{2}^\gamma \rangle)] + D_0^2 \nabla_2^\beta [\langle 1\hat{2}^\beta \rangle (\rho_0 D_0 \nabla_2^2 \langle 2\hat{3} \rangle + D_0 \nabla_3^\gamma \langle 23\hat{3}^\gamma \rangle)] \\ &= -iD_0^2 \{ \nabla_3^\beta [\langle 1\hat{3}^\beta \rangle \bar{R}(32)] + \nabla_2^\beta [\langle 1\hat{2}^\beta \rangle \bar{R}(23)] \}. \end{aligned} \quad (\text{D.12})$$

Therefore, one obtains the first-order renormalized dynamical equation

$$\begin{aligned} (\partial_t - D_0 \nabla^2) \bar{R}(12) &= -\rho_0 D_0 \nabla^2 \delta(12) \\ &\quad - i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \{ \nabla_3^\beta [\langle 1\hat{3}^\beta \rangle \bar{R}(32)] + \nabla_2^\beta [\langle 1\hat{2}^\beta \rangle \bar{R}(23)] \} \right), \end{aligned} \quad (\text{D.13})$$

or, through elimination of the isolated time integral thanks to the FDR,

$$\begin{aligned} (\partial_t - D_0 \nabla^2) \bar{R}(12) &= -\rho_0 D_0 \nabla^2 \delta(12) \\ &- i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \nabla^\beta \Phi(13)] \langle 1\hat{3}^\beta \rangle \bar{R}(32) \right) + i\lambda \rho_0 D_0^2 \nabla^\alpha \nabla^\beta ([\nabla^\alpha \Phi(12)] \langle 1\hat{2}^\beta \rangle). \end{aligned} \quad (\text{D.14})$$

In Fourier space, this equation takes the form

$$(\partial_t + \Gamma_k) \bar{R}_k(t - t') = \rho_0 \Gamma_k \delta(t - t') - \int_{t'}^t ds \Sigma_k(t - s) \bar{R}_k(s - t') + L_k(t - t'), \quad (\text{D.15})$$

where the kernel  $\Sigma_k(t)$  is given in equation (C.14). As for the new kernel  $L_k(t)$  arising from the composite nature of the physical response function, it can be obtained from equation (D.14) as

$$L_k(t) = \lambda \rho_0 D_0^2 \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t). \quad (\text{D.16})$$

## Appendix E. Renormalized equation for the correlation function

From equation (6.3c), the full dynamical equation for the correlation function is given by

$$\begin{aligned} (\partial_t - D_0 \nabla^2) C(12) &= 2\bar{R}(21) - \lambda \rho_0 D_0 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] [\bar{R}(13) + \bar{R}(23)] \right) \\ &- i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\gamma [\rho_0 \langle 12\hat{3}^\gamma \rangle + \langle 123\hat{3}^\gamma \rangle] \right). \end{aligned} \quad (\text{E.1})$$

We calculate the multi-point averages up to the first order in  $\lambda$  with the bare perturbation expansion. The three-point average is given by

$$\langle 12\hat{3}^\gamma \rangle = \langle 12\hat{3}^\gamma \rangle_f + \langle 12\hat{3}^\gamma S_{\text{dis}} \rangle_f + O(\lambda^2). \quad (\text{E.2})$$

The first term corresponds to equation (8.2e). The first-order contribution reads

$$\begin{aligned} \langle 12\hat{3}^\gamma S_{\text{dis}} \rangle_f &= \frac{1}{2} \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] [2\rho_0 \langle 126\hat{3}^\gamma \hat{6}^a \hat{9}^b \rangle_0 + \langle 1269\hat{3}^\gamma \hat{6}^a \hat{9}^b e^{-D_0 \int_4 4\hat{4}^\delta \hat{4}^\delta} \rangle_0] \\ &= \frac{1}{2} \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \left[ 2\rho_0 \langle 126\hat{3}^\gamma \hat{6}^a \hat{9}^b \rangle_0 - D_0 \int_4 \langle 12469\hat{3}^\gamma \hat{4}^\delta \hat{4}^\delta \hat{6}^a \hat{9}^b \rangle_0 \right] \\ &= \lambda \rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] [\langle 1\hat{6}^a \rangle_0 \langle 2\hat{9}^b \rangle_0 + \langle 1\hat{9}^b \rangle_0 \langle 2\hat{6}^a \rangle_0] \langle 6\hat{3}^\gamma \rangle_0 \\ &\quad - 2\lambda D_0^3 \int_4 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \{ \langle 1\hat{4}^\delta \rangle_0 [\langle 4\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0] \langle 2\hat{4}^\delta \rangle_0 \\ &\quad + \langle 1\hat{4}^\delta \rangle_0 \langle 4\hat{3}^\gamma \rangle_0 [\langle 2\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{4}^\delta \rangle_0] + [\langle 1\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{4}^\delta \rangle_0] \langle 4\hat{3}^\gamma \rangle_0 \langle 2\hat{4}^\delta \rangle_0 \} \\ &\quad - 2\lambda D_0^3 \int_4 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \{ \langle 1\hat{4}^\delta \rangle_0 [\langle 4\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0] [\langle 2\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0] \\ &\quad + [\langle 1\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0] [\langle 4\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0] \langle 2\hat{4}^\delta \rangle_0 + [\langle 1\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0] \langle 4\hat{3}^\gamma \rangle_0 [\langle 2\hat{9}^b \rangle_0 \langle 9\hat{4}^\delta \rangle_0] \}. \end{aligned} \quad (\text{E.3})$$

In the right-hand side of this equation, the first line is part of the second-order contribution to the time-persistent term (whose first-order expression is  $-\lambda\rho_0^2 D_0 \nabla^2 \Phi(12)$ , see equation (8.7c)). The next three terms are those that contribute to renormalize equation (8.2e), whereas the last contributions belong to the second-order renormalization. Therefore, apart from the time-persistent terms, one has the renormalized expression

$$\langle 12\hat{3}^\gamma \rangle = -2D_0 \int_4 \langle 1\hat{4}^\delta \rangle \langle 4\hat{3}^\gamma \rangle \langle 2\hat{4}^\delta \rangle + O(\lambda). \quad (\text{E.4})$$

Now, the four-point average in equation (E.1) is given by

$$\langle 123\hat{3}^\gamma \rangle = \langle 123\hat{3}^\gamma \rangle_f + \langle 123\hat{3}^\gamma S_{\text{dis}} \rangle_f + O(\lambda^2). \quad (\text{E.5})$$

The first term is already computed in equation (8.2f). The first-order contribution consists of three Gaussian averages:

$$\begin{aligned} \langle 123\hat{3}^\gamma S_{\text{dis}} \rangle_f &= \frac{1}{2} \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \\ &\times \{ \rho_0^2 \langle 123\hat{3}^\gamma \hat{6}^a \hat{9}^b \rangle_0 + 2\rho_0 \langle 1236\hat{3}^\gamma \hat{6}^a \hat{9}^b e^{-D_0 \int_4 4\hat{4}^\delta \hat{4}^\delta} \rangle_0 + \langle 12369\hat{3}^\gamma \hat{6}^a \hat{9}^b e^{-D_0 \int_4 4\hat{4}^\delta \hat{4}^\delta} \rangle_0 \}. \end{aligned} \quad (\text{E.6})$$

These averages are straightforward to compute. The first one is given by

$$\frac{1}{2} \rho_0^2 \langle 123\hat{3}^\gamma \hat{6}^a \hat{9}^b \rangle_0 = \rho_0^2 (\langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{6}^a \rangle_0 \langle 2\hat{9}^b \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 2\hat{3}^\gamma \rangle_0). \quad (\text{E.7})$$

It is also part of the time-persistent contribution at second order. Combining this term with the previous one of the same nature in equation (E.3) and using the procedure introduced in section 8 to eliminate the isolated time integrals, one gets

$$\begin{aligned} & -i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\gamma \left[ \lambda \rho_0^2 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \right. \right. \\ & \left. \left. \times \{ [\langle 1\hat{6}^a \rangle_0 \langle 2\hat{9}^b \rangle_0 + \langle 1\hat{9}^b \rangle_0 \langle 2\hat{6}^a \rangle_0] \langle 6\hat{3}^\gamma \rangle_0 + \langle 1\hat{3}^\gamma \rangle_0 \langle 3\hat{6}^a \rangle_0 \langle 2\hat{9}^b \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 2\hat{3}^\gamma \rangle_0 \} \right] \right) \\ & = -\rho_0^2 D_0 \nabla^2 \left[ \frac{\lambda^2 \Phi(12)^2}{2} \right]. \end{aligned} \quad (\text{E.8})$$

Adding the first-order term  $-\rho_0^2 D_0 \nabla^2 [\lambda \Phi(12)]$ , one recognizes the second-order expansion of  $-D_0 \nabla^2 C_d(12)$ , with  $C_d(12) \equiv C_d(|\mathbf{r} - \mathbf{r}'|)$ , according to the exact static equilibrium calculation, equation (3.4b). Such an identification is actually required to insure consistency between statics and dynamics.

The remaining Gaussian averages in equation (E.6) are given by

$$\begin{aligned} \rho_0 \langle 1236\hat{3}^\gamma \hat{6}^a \hat{9}^b e^{-D_0 \int_4 4\hat{4}^\delta \hat{4}^\delta} \rangle_0 &= -\rho_0 D_0 \int_4 \langle 12346\hat{3}^\gamma \hat{4}^\delta \hat{4}^\delta \hat{6}^a \hat{9}^b \rangle_0 \\ &= -2\rho_0 D_0 \int_4 \{ \langle 1\hat{3}^\gamma \rangle_0 [\langle 3\hat{4}^\delta \rangle_0 \langle 4\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 2\hat{4}^\delta \rangle_0 + \langle 3\hat{4}^\delta \rangle_0 \langle 4\hat{9}^b \rangle_0 \langle 2\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0 + \langle 3\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0 \langle 4\hat{9}^b \rangle_0 \langle 2\hat{4}^\delta \rangle_0] \\ &+ \langle 1\hat{4}^\delta \rangle_0 [\langle 4\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 3\hat{4}^\delta \rangle_0 + \langle 4\hat{9}^b \rangle_0 \langle 3\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0] \langle 2\hat{3}^\gamma \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0 \langle 4\hat{9}^b \rangle_0 \langle 3\hat{4}^\delta \rangle_0 \langle 2\hat{3}^\gamma \rangle_0 \\ &+ \langle 1\hat{4}^\delta \rangle_0 [\langle 4\hat{6}^a \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0] \langle 2\hat{4}^\delta \rangle_0 \\ &+ \langle 1\hat{4}^\delta \rangle_0 [\langle 4\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 6\hat{4}^\delta \rangle_0 + \langle 4\hat{9}^b \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{4}^\delta \rangle_0] \langle 2\hat{6}^a \rangle_0 \\ &+ \langle 1\hat{6}^a \rangle_0 [\langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{4}^\delta \rangle_0 \langle 4\hat{9}^b \rangle_0 + \langle 6\hat{4}^\delta \rangle_0 \langle 4\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0] \langle 2\hat{4}^\delta \rangle_0 \}. \end{aligned} \quad (\text{E.9})$$

and

$$\begin{aligned}
\frac{1}{2}\langle 12369\hat{3}^\gamma\hat{6}^a\hat{9}^b e^{-D_0 \int_4 4\hat{4}^\delta} \rangle_0 &= \frac{1}{2}\langle 12369\hat{3}^\gamma\hat{6}^a\hat{9}^b \rangle_0 \\
&= \langle 13 \rangle_0 [\langle 2\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0] + \langle 16 \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 9\hat{6}^a \rangle_0 \langle 2\hat{3}^\gamma \rangle_0 \\
&\quad + \langle 1\hat{3}^\gamma \rangle_0 [\langle 3\hat{9}^b \rangle_0 \langle 9\hat{6}^a \rangle_0 \langle 62 \rangle_0 + \langle 2\hat{6}^a \rangle_0 \langle 39 \rangle_0 \langle 6\hat{9}^b \rangle_0 + \langle 2\hat{6}^a \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 69 \rangle_0] \\
&\quad + [\langle 1\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0] \langle 32 \rangle_0 + \langle 1\hat{6}^a \rangle_0 [\langle 3\hat{9}^b \rangle_0 \langle 69 \rangle_0 \langle 2\hat{3}^\gamma \rangle_0 + \langle 39 \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 2\hat{3}^\gamma \rangle_0] \\
&\quad + [\langle 16 \rangle_0 \langle 3\hat{6}^a \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 63 \rangle_0] \langle 2\hat{9}^b \rangle_0 \langle 9\hat{3}^\gamma \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 [\langle 3\hat{9}^b \rangle_0 \langle 92 \rangle_0 + \langle 2\hat{9}^b \rangle_0 \langle 93 \rangle_0].
\end{aligned} \tag{E.10}$$

We now need to calculate the correlation function itself up to the first order of the bare perturbation expansion:

$$\begin{aligned}
\langle 13 \rangle &= \langle 13 \rangle_f + \langle 13S_{\text{dis}} \rangle_f + O(\lambda^2) \\
&= \langle 13 \rangle_0 + \frac{1}{2}\lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \left\{ \rho_0^2 \langle 13\hat{6}^a\hat{9}^b \rangle_0 - 2\rho_0 D_0 \int_4 \langle 1346\hat{4}^\delta\hat{4}^\delta\hat{6}^a\hat{9}^b \rangle_0 + \langle 1369\hat{6}^a\hat{9}^b \rangle_0 \right\} + O(\lambda^2) \\
&= \langle 13 \rangle_0 + \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \left\{ \rho_0^2 \langle 1\hat{6}^a \rangle_0 \langle 3\hat{9}^b \rangle_0 \right. \\
&\quad \left. - 2\rho_0 D_0 \int_4 (\langle 1\hat{4}^\delta \rangle_0 [\langle 4\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 3\hat{4}^\delta \rangle_0 + \langle 4\hat{9}^b \rangle_0 \langle 3\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0] + \langle 1\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0 \langle 4\hat{9}^b \rangle_0 \langle 3\hat{4}^\delta \rangle_0 \right. \\
&\quad \left. + \langle 16 \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 9\hat{6}^a \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 39 \rangle_0 \langle 6\hat{9}^b \rangle_0 + \langle 1\hat{6}^a \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 69 \rangle_0 \right\} + O(\lambda^2)
\end{aligned} \tag{E.11}$$

and

$$\begin{aligned}
\langle 32 \rangle &= \langle 32 \rangle_0 + \lambda D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \left\{ \rho_0^2 \langle 3\hat{6}^a \rangle_0 \langle 2\hat{9}^b \rangle_0 \right. \\
&\quad \left. - 2\rho_0 D_0 \int_4 (\langle 3\hat{4}^\delta \rangle_0 [\langle 4\hat{6}^a \rangle_0 \langle 6\hat{9}^b \rangle_0 \langle 2\hat{4}^\delta \rangle_0 + \langle 4\hat{9}^b \rangle_0 \langle 2\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0] + \langle 3\hat{6}^a \rangle_0 \langle 6\hat{4}^\delta \rangle_0 \langle 4\hat{9}^b \rangle_0 \langle 2\hat{4}^\delta \rangle_0 \right. \\
&\quad \left. + \langle 26 \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 9\hat{6}^a \rangle_0 + \langle 2\hat{6}^a \rangle_0 \langle 39 \rangle_0 \langle 6\hat{9}^b \rangle_0 + \langle 2\hat{6}^a \rangle_0 \langle 3\hat{9}^b \rangle_0 \langle 69 \rangle_0 \right\} + O(\lambda^2).
\end{aligned} \tag{E.12}$$

The first term in each integral corresponds to the first-order contribution to the time-persistent part of the correlation function,  $\lambda\rho_0^2\Phi(13)$  in  $\langle 13 \rangle$  and  $\lambda\rho_0^2\Phi(32)$  in  $\langle 32 \rangle$ . It should be discarded to avoid double-counting with equation (E.8) and we accordingly define  $\langle 13 \rangle_c$  and  $\langle 32 \rangle_c$ , where c stands for connected, from equations (E.11) and (E.12) without this term.

Therefore, we identify the first-order renormalization for the average  $\langle 123\hat{3}^\gamma \rangle$ , apart from the time-persistent terms, as

$$\langle 123\hat{3}^\gamma \rangle = \langle 1\hat{3}^\gamma \rangle \langle 32 \rangle_c + \langle 13 \rangle_c \langle 2\hat{3}^\gamma \rangle - 2D_0 \int_4 \langle 1\hat{4}^\delta \rangle \langle 43\hat{3}^\gamma \rangle \langle 2\hat{4}^\delta \rangle. \tag{E.13}$$

The bare first-order expression for  $\langle 43\hat{3}^\gamma \rangle$  is given by (see equation (D.9))

$$\langle 43\hat{3}^\gamma \rangle = \lambda\rho_0 D_0^2 \int_6 \int_9 [\nabla_6^a \nabla_6^b \Phi(69)] \langle 4\hat{6}^a \rangle_0 \langle 6\hat{3}^\gamma \rangle_0 \langle 3\hat{9}^b \rangle_0 + O(\lambda^2), \tag{E.14}$$

and can be spotted in equation (E.9). Again, the corresponding term appears associated with the first-order expansion of  $\bar{R}$ , as in equation (D.4).

Collecting equations (E.4) and (E.13), we have

$$\begin{aligned}
& \nabla_3^\gamma [\rho_0 \langle 12\hat{3}^\gamma \rangle + \langle 123\hat{3}^\gamma \rangle] \\
&= \nabla_3^\gamma [\langle 1\hat{1}3^\gamma \rangle \langle 32 \rangle_c + \langle 13 \rangle_c \langle 2\hat{3}^\gamma \rangle] - 2 \int_4 \langle 1\hat{4}^\delta \rangle [\rho_0 D_0 \nabla_4^2 \langle 4\hat{3} \rangle + D_0 \nabla_3^\gamma \langle 43\hat{3}^\gamma \rangle] \langle 2\hat{4}^\delta \rangle \\
&= \nabla_3^\gamma [\langle 1\hat{1}3^\gamma \rangle \langle 32 \rangle_c + \langle 13 \rangle_c \langle 2\hat{3}^\gamma \rangle] + 2i \int_4 \langle 1\hat{4}^\delta \rangle \bar{R}(43) \langle 2\hat{4}^\delta \rangle,
\end{aligned} \tag{E.15}$$

where we used (see equation (D.5))  $\bar{R}(43) = i\rho_0 D_0 \nabla_4^2 \langle 4\hat{3} \rangle + iD_0 \nabla_3^\gamma \langle 43\hat{3}^\gamma \rangle$ .

We are now ready to write down the first-order renormalized dynamical equation for the density correlation function. It reads, ignoring  $2\bar{R}(21)$  and performing the first integral in the right-hand side of equation (E.1),

$$\begin{aligned}
(\partial_t - D_0 \nabla^2) C(12) &= -D_0 \nabla^2 C_d(12) - i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\gamma [\langle 1\hat{1}3^\gamma \rangle \langle 32 \rangle_c + \langle 13 \rangle_c \langle 2\hat{3}^\gamma \rangle] \right) \\
&\quad + 2\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \int_4 \langle 1\hat{4}^\gamma \rangle \bar{R}(43) \langle 2\hat{4}^\gamma \rangle \right),
\end{aligned} \tag{E.16}$$

or, integrating out the physical response function in the last term and introducing the connected density correlation function,

$$\begin{aligned}
& (\partial_t - D_0 \nabla^2) F(12) \\
&= -i\lambda D_0^2 \nabla^\alpha \left( \int_3 [\nabla^\alpha \Phi(13)] \nabla_3^\gamma [\langle 1\hat{1}3^\gamma \rangle \langle 32 \rangle_c + \langle 13 \rangle_c \langle 2\hat{3}^\gamma \rangle] \right) + 2\lambda \rho_0 D_0^2 \nabla^\alpha \left( \int_4 [\nabla^\alpha \Phi(14)] \langle 1\hat{4}^\gamma \rangle \langle 2\hat{4}^\gamma \rangle \right).
\end{aligned} \tag{E.17}$$

We then get the Fourier-transformed equation of motion,

$$\begin{aligned}
(\partial_t + \Gamma_k) F_k(t - t') &= -\lambda D_0^2 \int_{-\infty}^t ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t - s) F_k(s - t') \\
&\quad + \lambda D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{q} [\mathbf{k} \cdot \mathbf{q} \Phi_q] F_p(t - s) G_k(t' - s) \\
&\quad + 2\lambda \rho_0 D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t - s) G_k(t' - s),
\end{aligned} \tag{E.18}$$

which we rewrite as

$$(\partial_t + \Gamma_k) F_k(t - t') = - \int_{t'}^t ds \Sigma_k(t - s) F_k(s - t') + N_k(t - t'), \tag{E.19}$$

$$\begin{aligned}
N_k(t - t') &\equiv -\lambda D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t - s) F_k(t' - s) \\
&\quad + \lambda D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{q} [\mathbf{k} \cdot \mathbf{q} \Phi_q] F_p(t - s) G_k(t' - s) \\
&\quad + 2\lambda \rho_0 D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t - s) G_k(t' - s).
\end{aligned} \tag{E.20}$$

Note that, in the absence of simple relations between the response and correlation functions beyond the FDR, it is not guaranteed that the sum of three integrals in equation (E.20) actually reduces to a local function of time as posited in equation (E.19). For the same reason, it is not obvious that the time derivative of  $N_k(t - t')$ , given by

$$\begin{aligned} \partial_t N_k(t - t') &= \rho_0 \Sigma_k(t - t') - \lambda D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t - s) \bar{R}_k(t' - s) \\ &\quad - \lambda D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{q} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \bar{R}_p(t - s) G_k(t' - s) \\ &\quad + 2\lambda \rho_0 D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{p} [\mathbf{k} \cdot \mathbf{q} \Phi_q] [\partial_t G_p(t - s)] G_k(t' - s), \end{aligned} \quad (\text{E.21})$$

where the FDR and integrations by parts have been used, equals  $\rho_0 \Sigma_k(t - t') - L_k(t - t')$ , as required for consistency with the FDR. In fact, general arguments support the exact opposite [31, 32].

However, using the following first-order consistent substitutions (see equations (10.7), (10.8), and (10.1a))

$$F_k(t) = \rho_0 G_k(t) + O(\lambda), \quad (\text{E.22a})$$

$$\bar{R}_k(t) = \rho_0 D_0 k^2 G_k(t) + O(\lambda), \quad (\text{E.22b})$$

$$\partial_t G_k(t) = -D_0 k^2 G_k(t) + O(\lambda), \quad (\text{E.22c})$$

in the above integrals, one can easily show, with calculations similar to those performed in section 9, that these properties hold to first order. One can use, in particular, the identity  $k^2 \mathbf{q} \cdot \mathbf{p} + p^2 (\mathbf{k} \cdot \mathbf{q} + 2\mathbf{k} \cdot \mathbf{p}) = \mathbf{k} \cdot \mathbf{p} (k^2 + p^2)$ .

As an example, we may show that, within the first order, equation (E.20) is indeed compatible with equation (10.12), which is FDR-consistent. One first uses equation (E.22a) in (E.20) to get

$$N_k(t - t') = \lambda \rho_0 D_0^2 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] [p^2 G_p(t - s) G_k(t' - s) + G_p(t - s) k^2 G_k(t' - s)] + O(\lambda^2). \quad (\text{E.23})$$

Then, equation (E.22c) gives

$$N_k(t - t') = \lambda \rho_0 D_0 \int_{-\infty}^{t'} ds \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \partial_s [G_p(t - s) G_k(t' - s)] + O(\lambda^2), \quad (\text{E.24})$$

hence

$$N_k(t - t') = \lambda \rho_0 D_0 \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] G_p(t - t') + O(\lambda^2). \quad (\text{E.25})$$

Since  $G_p(0) = 1$  and  $\int_{\mathbf{q}} \mathbf{k} \cdot \mathbf{q} \Phi_q = 0$  by isotropy, this can be rewritten as

$$N_k(t - t') = \lambda \rho_0 D_0 \int_{t'}^t ds \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_q] \partial_s G_p(s - t') + O(\lambda^2), \quad (\text{E.26})$$

and, with one last use of equation (E.22c), one gets

$$N_k(t-t') = -\lambda D_0 \int_{t'}^t ds \int_{\mathbf{q}} [\mathbf{k} \cdot \mathbf{q} \Phi_{\mathbf{q}}] [\rho_0 D_0 p^2 G_p(s-t')] + O(\lambda^2). \quad (\text{E.27})$$

Truncated to first order, this is nothing but equation (10.12).

Finally,  $N_k(t-t')$  can also be written as

$$N_k(t-t') = - \int_{-\infty}^{t'} ds \Sigma_k(t-s) F_k(t'-s) + \int_{-\infty}^{t'} ds D_k(t-s) [\rho_0 \Gamma_k G_k(t'-s)], \quad (\text{E.28})$$

with

$$D_k(t) = M_k(t) + \frac{2}{\rho_0 \Gamma_k} L_k(t), \quad (\text{E.29})$$

where  $M_k(t)$  is the mode-coupling kernel defined in equation (10.27). One can thus readily transpose the discussion around equation (9.16) of the bare theory to the renormalized framework.

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