Non-Markovian qubit dynamics in the presence of 1/f noise

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Within the lowest-order Born approximation, we calculate the exact dynamics of a qubit in the presence of 1/f noise, without Markov approximation. We show that the non-Markovian qubit time-evolution exhibits asymmetries and beatings that can be observed experimentally and cannot be explained within a Markovian theory. The present theory for 1/f noise is relevant for both spin- and superconducting qubit realizations in solid-state devices, where 1/f noise is ubiquitous.

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I. INTRODUCTION

Random telegraph noise has been encountered in a wide range of situations in many different areas of physics[1]. A typical example in condensed matter physics is that of a resistor coupled to an ensemble of randomly switching impurities, producing voltage fluctuations with a spectral density that scales inversely proportional with the frequency, hence the name "1/f noise". The quest to build and coherently control quantum two-level systems functioning as qubits in various solid state systems has once more highlighted the importance of understanding 1/f noise, being a limitation to the quantum coherence of such devices.

The description of low-frequency noise (such as 1/f noise) is complicated by the presence of long-time correlations in the fluctuating environment which prohibit the use of the Markov approximation. Only in few cases, non-Markovian effects have been taken into account exactly, e.g., for the relaxation of an atom to thermal equlibrium[2]. Here, we are interested in the decoherence and relaxation of a qubit, i.e., a single two-level system (spin 1/2). For the spin-boson model, i.e., a qubit coupled to a bath of harmonic oscillators, the dynamics has been calculated within a rigorous Born approximation without making a Markov approximation[3,4]. Here, we carry out a similar analysis for 1/f noise and find even stronger effects than in the spin-boson case (see Fig. 1).

Charge and to some extent (via the spin-orbit interaction) spin qubits in quantum dots[5] formed in semiconductor[6] or carbon[7] structures are subject to 1/f noise. In superconducting (SC) Josephson junctions, SC interference devices (SQUIDs), and SC qubits, 1/f noise has been extensively studied experimentally[8,9,10,11,12,13,14,15] and theoretically[16,17].

Even where the origin of 1/f noise is known, the induced decoherence is not fully understood. Most theoretical work is either restricted to longitudinal fluctuations or employs a Markov approximation. Here, we present a calculation of the qubit dynamics in the presence of 1/f noise which is exact within the lowest-order Born approximation. In particular, we make no use of a Markov approximation. In contrast to earlier calculations[18,19,21,22], we allow for arbitrary qubit Hamiltonians and include

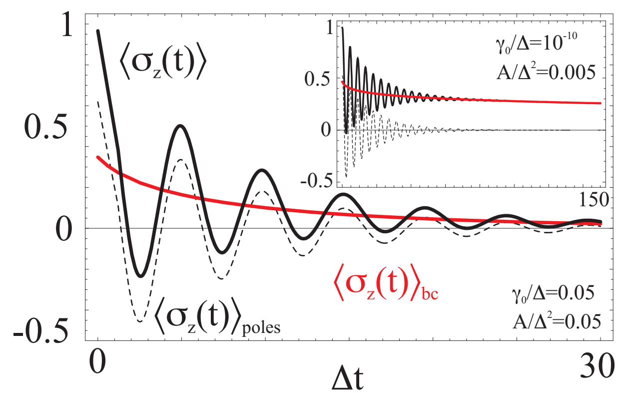

FIG. 1: (Color online) Non-Markovian time-evolution of the unbiased ($\epsilon = 0$) qubit (spin) z-component $\langle\sigma_z(t)\rangle$, for $A/\Delta^2 = 0.05$ and $\gamma_0/\Delta = 0.05$ (solid black line). The Markovian pole contribution $\langle\sigma_z(t)\rangle_{\rm poles}$ is plotted as a dashed line for comparison. The essential non-Markovian part is non-exponential and given by the branch cut contribution $\langle\sigma_z(t)\rangle_{\rm bc}$ (red solid line). Inset: Plot for $A/\Delta^2 = 0.005$ and $\gamma_0/\Delta = 10^{-10}$. Here, the essential non-Markovian part is the long-time asymmetry which carries information about the initial state.

transverse as well as longitudinal (phase) 1/f noise. Non-Gaussian 1/f noise originating from few fluctuators was studied in[23,24,25], while numerical studies using an adiabatic approximation were carried out in[26]. The coupling to a single fluctuator was also studied[27].

II. MODEL

We model the qubit (spin 1/2) coupled to a bath of two-level fluctuators with the Hamiltonian

$$H = H_S + H_B + H_{SB} \tag{1}$$

with

$$H_S = \Delta\sigma_x + \epsilon\sigma_z, \tag{2}$$
$$H_{SB} = \sigma_z X, \tag{3}$$

where σ_x and σ_z are Pauli matrices describing the qubit and $X = \sum_{i=1}^{N} v_i\sigma_z^i$ where σ_z^i operates on the i-th fluctuator. In a SC qubit, Δ and ϵ denote the tunneling and energy bias between the two qubit states. In a spin qubit, ϵ is the Zeeman splitting and Δ a transverse field. The bath Hamiltonian H_B need not be provided explicitly; it is sufficient to know the auto-correlator $C(t) = \langle X(0)X(t)\rangle$ of the bath operator $X(t)$, where $\langle \ldots \rangle = \mathrm{Tr}_B(\ldots \rho_B)$ denotes a trace over the bath degrees of freedom with the bath density matrix ρ_B. We can further assume that the fluctuators are unbiased, $\langle X(t)\rangle = 0$. For independent two-level fluctuators with switching rates γ_i, one obtains

$$C(t) = \sum_i v_i^2 \langle \sigma_i(t)\sigma_i(0)\rangle = \sum_i v_i^2 e^{-\gamma_i |t|}. \tag{4}$$

The noise spectral density is the Fourier transform

$$S(\omega) = \int_{-\infty}^{\infty} dt\, C(t) e^{-i\omega t} = \sum_i (2v_i^2\gamma_i)/(\gamma_i^2 + \omega^2). \tag{5}$$

While this correlator describes essentially classical bath dynamics (as is commonly assumed for 1/f noise), it should be emphasized that our model is *not* classical, because the $[H_{SB}, H_S] \neq 0$. In the case of a large number of fluctuators, the sum in $C(t)$ can be converted into an integral. For 1/f noise, one typically assumes a distribution of fluctuators of the form $P(v,\gamma) \propto 1/\gamma v^\beta$, where both v and γ are limited by upper and lower cut-offs[31]. The spectral density of the ensemble of fluctuators then becomes

$$S(\omega) \propto \int_{v_{\min}}^{v_{\max}} \int_{\gamma_0}^{\gamma_c} dv\, d\gamma P(v,\gamma) \frac{2v^2\gamma}{\gamma^2 + \omega^2}. \tag{6}$$

For $\gamma_0 = 0$ this yields 1/f noise of the form $S(\omega) \propto 1/|\omega|$. The divergence at low frequencies is cut off by the finite duration of a qubit measurement, if not by other effects at even shorter times. A low-frequency cut-off $\gamma_0 > 0$ yields

$$S(\omega) = 2\pi A \frac{\arctan(\omega/\gamma_0)}{\pi} \frac{1}{\omega}, \tag{7}$$

where A depends on the cut-offs and the exponent β. For $\gamma_0 \to 0$, we recover $S(\omega) \to 2\pi A/|\omega|$. Inverting the above Fourier transform, we obtain

$$C(t) = -A\, \mathrm{Ei}(-\gamma_0 |t|), \tag{8}$$

where Ei denotes the exponential integral function.

III. QUBIT DYNAMICS

The density matrix ρ of the total system, consisting of the qubit and the bath, obeys the Liouville equation, $\dot\rho(t) = -i[H, \rho(t)]$. The time evolution of the reduced density matrix of the qubit alone $\rho_S(t) = \mathrm{Tr}_B \rho$ is then determined by the generalized master equation (GME)[3,4]

$$\dot\rho_S(t) = -i[H_S, \rho_S(t)] - i\int_0^t \Sigma(t-t')\rho_S(t')dt', \tag{9}$$

where the self-energy superoperator $\Sigma(t)$ gives rise to memory effects, i.e., the time evolution of $\rho_S(t)$ depends on the state $\rho_S(t')$ at all earlier times $t' \leq t$. Therefore, the qubit dynamics is inherently non-Markovian. Expanding the right-hand side of the GME in orders of H_{SB} and only keeping the lowest (second) order, one obtains Σ in (lowest-order) Born approximation $\Sigma(t)\rho_S = -i\mathrm{Tr}_B[H_{SB}, e^{-itH_0}[H_{SB}, \rho_S \otimes \rho_B]e^{itH_0}]$, where $H_0 = H_S + H_B$.

Introducing the Bloch vector $\langle \boldsymbol\sigma(t)\rangle = \mathrm{Tr}_S \boldsymbol\sigma \rho_S(t)$, where $\boldsymbol\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli operators, we write the GME as a generalized Bloch equation

$$\langle \dot{\boldsymbol\sigma}\rangle = R * \langle \boldsymbol\sigma\rangle + \boldsymbol{k}, \tag{10}$$

where the star denotes convolution and[3,4]

$$R(t) = \begin{pmatrix} -\frac{E^2}{\Delta^2}\Gamma_1(t) & -\epsilon\delta(t) + \frac{E}{\Delta}K_y^+(t) & 0 \\ \epsilon\delta(t) - \frac{E}{\Delta}K_y^+(t) & -\Gamma_y(t) & -\Delta\delta(t) \\ 0 & \Delta\delta(t) & 0 \end{pmatrix} \tag{11}$$

with $E = \sqrt{\Delta^2 + \epsilon^2}$ and[3,4] $\Gamma_1(t) = (2\Delta/E)^2 \cos(Et)C'(t)$, $\Gamma_y(t) = (2\Delta/E)^2(1 + (\epsilon/\Delta)^2\cos(Et))C'(t)$, and $K_y^+(t) = (4\epsilon\Delta/E^2)\sin(Et)C'(t)$, where $C'(t)$ and $C''(t)$ denote the real and imaginary parts of $C(t)$. Since for 1/f noise, $C''(t) = 0$, we find $\mathbf{k}(t) = 0$[3,4]. As shown in[3,4], Eq.(10) can be solved by means of the Laplace transform (LT) $f(s) = \int_0^\infty f(t)e^{-ts}\,dt$, where

$$\langle \boldsymbol\sigma(s)\rangle = (s - R(s))^{-1} \left(\langle \boldsymbol\sigma(t=0)\rangle - \boldsymbol{k}(s)\right). \tag{12}$$

The LT $R(s)$ of $R(t)$, has entries according to Eq. (11), with $\delta(t)$ replaced by 1, and, for 1/f noise

$$\Gamma_1(s) = (2A/E^2)\Delta^2 \left(C(s+iE) + C(s-iE)\right), \tag{13}$$

$$\Gamma_y(s) = (2A/E^2)\left(2\Delta^2 C(s) + \epsilon^2\left(C(s+iE) + C(s-iE)\right)\right), \tag{14}$$

$$K_y^+(s) = i(2A/E^2)\Delta\epsilon\left(C(s+iE) - C(s-iE)\right), \tag{15}$$

where the LT of the correlator $C(t)$ in Eq. (4) is

$$C(s) = \frac{A}{s}\log\left(1 + s/\gamma_0\right). \tag{16}$$

We recover $\langle \boldsymbol\sigma(t)\rangle$ from $\langle \boldsymbol\sigma(s)\rangle$ by way of an inverse LT as carried out below, first for the special case of an unbiased qubit ($\epsilon = 0$) and then for the general case.

IV. UNBIASED QUBIT

We first assume that the qubit is prepared at time $t = 0$ in one of the eigenstates $|0\rangle = |\uparrow\rangle$ of σ_z, i.e., $\langle \boldsymbol\sigma\rangle = (0, 0, 1)$, and that the qubit is unbiased, $\epsilon = 0$. If the fluctuators were absent the qubit would undergo a precession about the x axis, $\langle \sigma_z(t)\rangle = \cos(\Delta t)$. Due to the presence of the fluctuators, we find (see also Appendix A)

$$\langle\sigma_z(s)\rangle = \frac{s^2 + 4A\log(1+s/\gamma_0)}{s\,(s^2+\Delta^2+4A\log(1+s/\gamma_0))}. \quad (17)$$

We expand $\langle\sigma_z(s)\rangle$ in leading order of A,

$$\langle\sigma_z(s)\rangle = \frac{s}{s^2+\Delta^2} + 4A\Delta^2\frac{\log(1+s/\gamma_0)}{s(s^2+\Delta^2)^2} + O(A^2). \quad (18)$$

The coherent spin oscillations in the time domain are obtained from the inverse LT, the so-called Bromwich integral[3,4] (see Fig. 2), $\langle\sigma_z(t)\rangle = \frac{1}{2\pi i}\lim_{\eta\downarrow 0}\int_{-i\infty+\eta}^{i\infty+\eta}\langle\sigma_z(s)\rangle e^{ts}\,ds$. The integral contour can be closed in the left complex half-plane $\mathrm{Re}(s) < 0$ (Fig. 2). The behavior of $\langle\sigma_z(t)\rangle$ is therefore given by the analytic structure of $\langle\sigma_z(s)\rangle$ in the left half-plane, see Fig. 2. In the absence of the fluctuating environment ($A=0$), $\langle\sigma_z(s)\rangle$ has two poles at $s=\pm i\Delta$ which yield $\langle\sigma_z(t)\rangle = \cos(\Delta t)$, as expected. The coupling to the environment has two effects: (i) a shift of the poles, and (ii) the appearance of a branch point (bp) due to the logarithm in Eq. (18) and the associated branch cut (bc) that we choose to lie on the real axis between $-\gamma_0$ and $-\infty$. Here, it should be noted that in the case of an unbiased qubit, the presence of 1/f noise does not lead to the appearance of a pole on the real axis, and thus there is only pure dephasing and no T_1 type decay (spin relaxation), in contrast to other types of environment[4]. The exact shift of the poles has been calculated numerically from Eq. (17). To lowest order in A, we find $\Delta_r \equiv \Delta_r' + i\Delta_r'' \simeq \Delta + \frac{A}{\Delta}\log\left(1+\frac{\Delta^2}{\gamma_0^2}\right) \pm 2i\frac{A}{\Delta}\arctan\frac{\Delta}{\gamma_0}$, where the real part Δ_r' is the renormalized frequency of the coherent oscillations, while the imaginary part Δ_r'' describes an exponential decay of those oscillations. If a Markovian approximation were made by setting $s=0$ in $\Gamma_1(s)$, $\Gamma_y(s)$, and $K_y^+(s)$, then the bc would be missed completely and only an exponential decay with a rate $2A/\gamma_0$ would be obtained. The Markov approximation is only justified if $\gamma_0 \gg \Delta$, i.e., if the bath dynamics is much faster than the system dynamics. Here, we entirely avoid making a Markov approximation.

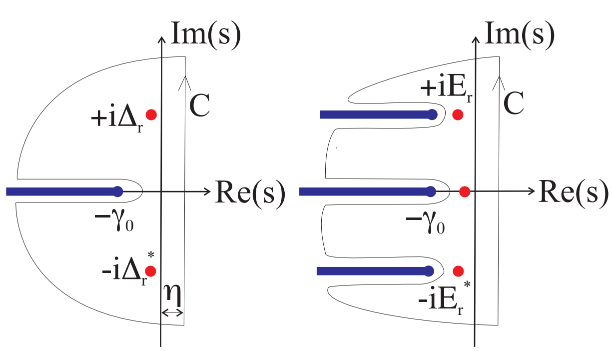

FIG. 2: (Color online) Analytic structure of $\langle\sigma_z(s)\rangle$ in the complex s plane, for (a) the unbiased case, $\epsilon = 0$ and (b) the biased case, $\epsilon \neq 0$. Red dots denote poles, blue lines branch cuts.

The Bromwich integral can then be divided into two parts, $\langle\sigma_z(t)\rangle = \langle\sigma_z(t)\rangle_{\rm poles} + \langle\sigma_z(t)\rangle_{\rm bc}$. The integration in the first term along the contour C, not including the line integrals along the bc (Fig. 2) yields the sums of the residues from the poles $\langle\sigma_z(t)\rangle_{\rm poles} = \frac{1}{2\pi i}\int_C ds\,\langle\sigma_z(s)\rangle e^{st} = r'\cos(\Delta_r' t)e^{-\Delta_r'' t} - r''\sin(\Delta_r' t)e^{-\Delta_r'' t}$, where $r' = 1-(2A/\Delta^2)\log(1+\Delta^2/\gamma_0^2) + O(A^2)$ and $r'' = (4A/\Delta^2)\arctan(\Delta/\gamma_0) + O(A^2)$. For $A=0$, this reduces to $\cos(\Delta t)$.

The branch-cut contribution to lowest order in A is

$$\langle\sigma_z(t)\rangle_{\rm bc} = \frac{4A}{\Delta^2}I_1(\gamma_0/\Delta, \Delta t) \quad (19)$$

with the integral $I_n(a,b) = \int_a^\infty dy\frac{e^{-by}}{y^n(y^2+1)^2}$, where we have used Eq. (18) and introduced dimensionless variables and where $a>0$ and $b\geq 0$. For $n=1$, we find (Fig. 3)

$$I_1(a,b) = \frac{1}{2}\mathrm{Re}\left[(ib+2)e^{-ib}(-i\pi + \mathrm{Ei}(ib-ab))\right] - \frac{1}{2}\frac{1}{1+a^2}e^{-ab} - \mathrm{Ei}(-ab). \quad (20)$$

For $a=\gamma_0/\Delta > 1$ and $b>0$ ($t>0$), the effect of the environment from the bc integral is exponentially suppressed: $I_1(a,b) < e^{-ab}/b$ and thus $|\langle\sigma_z(t)\rangle_{\rm bc}| < (4A/\Delta^3 t)e^{-\gamma_0 t}$. The physically more interesting regime is $a=\gamma_0/\Delta \ll 1$. Within this regime, we can distinguish two temporal regimes: short times $ab \ll 1$ ($t \ll \gamma_0^{-1}$) and long times $ab \gg 1$ ($t \gg \gamma_0^{-1}$). In the short-time case, the integral is cut off from above by a combination of the y^{-5} and the exponential factor. The effect of the latter can be approximated by cutting off the integral at $1/b$, with the result $I_1(a,b) \approx -I_1(1/b,0) + I_1(a,0)$, where $I_1(a,0) = -\frac{1}{2}(1+a^2)^{-1} + \frac{1}{4}\log(1+a^{-2})$ is the bc integral for $t=0$ ($b=0$). Note that $I_1(a,0) \geq 0$ due to the logarithmic term. In the long-time case, the integral is cut off by the exponential whereas the $(y^2+1)^2$ factor in the denominator becomes irrelevant, $I_1(a,b) \approx -\mathrm{Ei}(-ab)$.

At this point, the parameter that controls the strength of the non-Markovian effects due to 1/f noise can be identified as $\xi = (A/\Delta^2)\log(1+\Delta^2/\gamma_0^2)$. The regime of validity of the Born approximation (the only approximation required in this paper) is confined by the condition $\xi \ll 1$. The resulting damped qubit oscillation is plotted in Fig. 1 for $A/\Delta^2 = 0.05$ and $\gamma_0/\Delta = 0.05$ where $\xi \approx 0.1$. If the infrared cutoff is lowered, the non-Markovian effects due to 1/f noise become more pronounced. However, since the dependence on the infrared cutoff γ_0 is only logarithmic, the result does not change drastically even if γ_0 is much smaller than in our example, as long as A is chosen sufficiently small to ensure the validity of the Born approximation. E.g., for $\Delta \approx 10\,$GHz and $\gamma_0 \approx 1\,$Hz (cf. Ref. 10) then $\gamma_0/\Delta = 10^{-10}$. With[28] $A/\Delta^2 = 0.005$, one finds a long-lived asymmetry as shown in the inset of

Fig. (1). The intermediate asymptotics of this contribution is $\langle\sigma_z\rangle_{\rm bc} \approx \xi \approx 0.1$, while for longer times this contribution also decays logarithmically to zero. A similar long-time behavior has been found also for longitudinal coupling[10].

V. THE BIASED CASE

We again assume that the qubit prepared at time $t = 0$ in one of the eigenstates $|0\rangle = |\uparrow\rangle$ of σ_z, i.e., $\langle\boldsymbol{\sigma}\rangle = (0, 0, 1)$, but now the qubit is biased, $\epsilon \neq 0$. In the absence of the fluctuators ($A = 0$), the qubit would now undergo a precession about an axis in the xz plane with frequency $E/2\pi$, where $E = \sqrt{\Delta^2 + \epsilon^2}$. In this unperturbed situation, $\langle\sigma_z(s)\rangle$ has three poles at $s = \pm iE$ and $s = 0$, the former two giving rise to undamped oscillations of $\langle\sigma_z(t)\rangle$ with frequency $E/2\pi$ and amplitude Δ^2/E^2, while the latter allows for a non-vanishing stationary value ϵ^2/E^2 of $\langle\sigma_z(t)\rangle$ in the long-time limit.

Including 1/f noise we find in leading order in A (see Appendix A),

$$\langle\sigma_z(s)\rangle = \frac{s^2+\epsilon^2}{s(s^2+E^2)} + 4A\frac{\Delta^2}{E^2}\mathrm{Re}\left[\frac{\Delta^2}{(E^2+s^2)^2}C(s) + \frac{\epsilon^2}{s^2\,(s+iE)^2}C(s+iE)\right] + O(A^2). \quad (21)$$

Analogously to the unbiased case, the poles are shifted in the presence of the fluctuators. In leading order in A, we find three poles at $-E_r'' = -(4A\Delta^2/E^3)\arctan(E/\gamma_0)$, and $\pm iE_r = \pm iE \pm (iA\Delta^2/E^3)\log(1+E^2/\gamma_0^2) - (2A\Delta^2/E^3)\arctan(E/\gamma_0)$. From the shift of these poles (Fig. 2b), we obtain $\langle\sigma_z(t)\rangle_{\rm poles} = \frac{\Delta^2}{E^2}\cos(E_r't)e^{-E_r''t} + \frac{\epsilon^2}{E^2}e^{-2E_r''t}$. However, while in the unbiased case a Markovian treatment at least qualitatively describes the pole contribution correctly, in the biased case, there is another effect that is elusive in a Markovian analysis. As shown in Fig. 2b, there are three bp's in the biased case, lying at $-\gamma_0$ and $-\gamma_0 \pm iE$. We find that as the two poles near $\pm iE$ approach the bp's at $-\gamma_0 \pm iE$ as A is increased, these poles split into two poles. This behavior is illustrated in Fig. 4. The significance of this splitting is that it leads to beating patterns already in the pole part of $\langle\sigma_z(t)\rangle$, as shown in Fig. 5. It should be noted that, again, the precise value of γ_0 is not critical for the possibility to observe the effect, since γ_0 only enters in the argument of a logarithm; even a much smaller value of γ_0 can thus be compensated by only a slight increase of the system-environment coupling constant A.

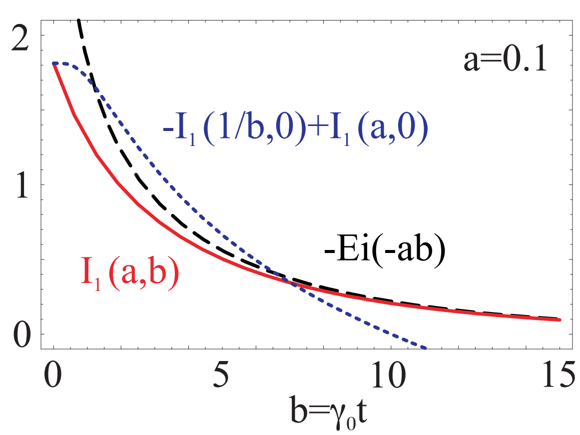

FIG. 3: (Color online) Branch cut integral function $I_1(a,b)$ (solid red line) and two asymptotes (black dashed and blue dotted lines).

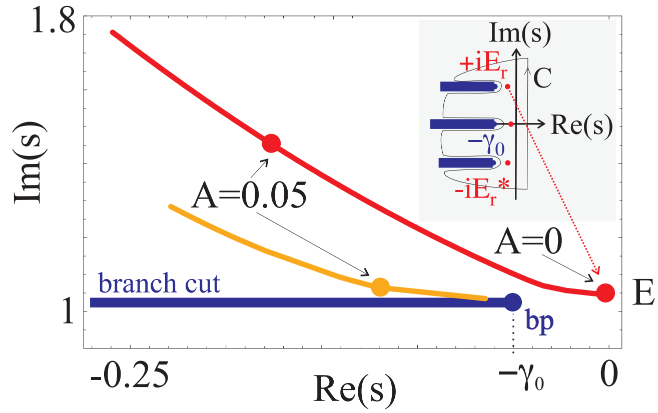

FIG. 4: (Color online) Shift and splitting of the poles of $\langle\sigma_z(s)\rangle$ for the biased system ($\epsilon = 0.3$ and $\gamma_0 = 0.05$). Shown is the pole located at $s = iE$ for the undamped system ($A = 0$), indicated as a red dot (see Inset b). The pole at $s = -iE$ behaves similarly. With increasing A the pole shifts toward the vicinity of the branch point (bp), where a second pole (orange dot) appears. Shown as red and orange dots are the two poles for $A = 0.05$. The splitting of the poles leads to a beating in $\langle\sigma_z(t)\rangle_{\rm poles}$, see Fig. 5. Inset: Analytic structure of $\langle\sigma_z(s)\rangle$ for $\epsilon \neq 0$, where red dots are poles, and blue lines are branch cuts (see Fig. 2).

The three bc's give rise to a contribution to $\langle\sigma_z(t)\rangle$,

$$\langle\sigma_z(t)\rangle_{\rm bc} = -\frac{4A\Delta^2}{E^4}\left[\frac{\Delta^2+\epsilon^2\cos(Et)}{E^2}I_1 + \frac{\epsilon^2}{E^2}\left(\sin(Et)I_2 - \cos(Et)I_3\right)\right], \quad (22)$$

where the functions I_n are as defined above and are evaluated at the arguments $a = \gamma_0/E$ and $b = Et$. For the unbiased case $\epsilon = 0$ and $E = \Delta$, one retrieves the previous result. The integrals I_2 and I_3 can be calculated in closed form, but will not be given here. The damped oscillations $\langle\sigma_z(t)\rangle$, consisting of both pole and bc contributions, are plotted in Fig. 5.

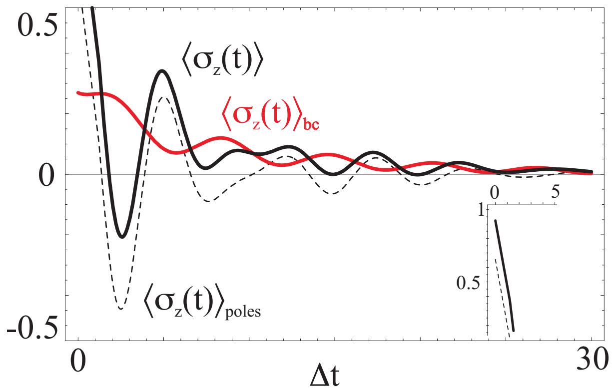

FIG. 5: (Color online) Oscillation $\langle\sigma_z(t)\rangle$ of the biased qubit for $\epsilon/\Delta = 0.3$, $A/\Delta^2 = 0.05$ and $\gamma_0/\Delta = 0.05$. The beating due to the splitting of the poles at $\pm iE$ can be observed in $\langle\sigma_z(t)\rangle_{\mathrm{poles}}$.

VI. COMPARISON WITH AN EXACTLY SOLVABLE CASE

The circumstance that in the case $\Delta = 0$ the coupling Hamiltonian between the system and the environment H_{SB} commutes with the system Hamiltonian H_S makes this special case exactly solvable[18,19,21,22]. A state prepared transverse to the common direction of the fixed precession axis and the fluctuating field, e.g., as $\langle\boldsymbol{\sigma}(t = 0)\rangle = (1, 0, 0)$, for low-frequency noise essentially leads to a Gaussian decay behavior $\langle\sigma_x(t)\rangle = \cos(\epsilon t)\exp\left(-ct^2\right)$. The Born approximation which we have employed here can only be expected to yield this result in lowest-order of the coupling constant, i.e.,

$$\langle\sigma_x(t)\rangle \simeq \cos(\epsilon t)\left(1 - ct^2 + O(c^2t^4)\right). \tag{23}$$

Here, we show that our result indeed has this form in the special case $\Delta = 0$.

To this end, we take the limit $\Delta \to 0$ in the propagator, Eq. (12), as shown in the Appendix A. We then find

$$\langle\sigma_x(s)\rangle = P_{xx}(s) = \frac{s + \Gamma_y(s)}{(s + \Gamma_y(s))^2 + \left(\epsilon - \tilde{K}_y^+(s)\right)^2}. \tag{24}$$

From Eq. (16) and omitting logarithmic corrections, we can use $C(s) \simeq A/s$, and thus $\Gamma_y(s) \simeq 4As/(s^2+\epsilon^2)$ and $\tilde{K}_y^+(s) \simeq 4A\epsilon/(s^2+\epsilon^2)$. Substituting this into Eq. (24) and expanding to lowest order in A, we find

$$\langle\sigma_x(s)\rangle \simeq \frac{s}{s^2+\epsilon^2} + A\frac{s(3\epsilon^2 - s^2)}{(s^2+\epsilon^2)^3}, \tag{25}$$

which equals the LT of Eq. (23) to lowest order, with the identification $c = A/2$. Therefore, our result is consistent with the known exact result for $\Delta = 0$, but, within the Born approximation, goes far beyond it, in that it includes arbitrary values of ϵ and Δ.

VII. DISCUSSION

We find the following essentially non-Markovian features in the decay of the z-component of the spin: (i) The spin decay is non-exponential and asymmetric. For relatively large infrared cutoff γ_0, there is an "initial loss" of coherence on a typical time scale $1/\gamma_0$, as seen in Figs. 1 and 5. More importantly, for the typical case of small γ_0, there is a long-time asymmetry favouring the qubit near its initial state. (ii) In the biased case, 1/f noise can lead to a two-frequency oscillation, exhibiting a characteristic beating pattern. Here, we have concentrated on the longitudinal component $\langle\sigma_z(t)\rangle$ of the qubit under the influence of both longitudinal and transverse 1/f noise. The transverse component $\langle\sigma_x(t)\rangle$ shows similar behavior. The predicted non-Markovian effects are observable in free induction decay (Ramsey fringe) experiments. Indeed, such asymmetries are clearly visible in superconducting qubits[10,29] Measurements on a superconducting flux qubit have shown deviations from the exponential decay and beatings[30]. The question whether these effects are due to the mechanisms described here or not require further investigation.

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APPENDIX A: FORM OF THE PROPAGATOR

The propagator (resolvent) for solving the generalized Bloch equation in Laplace space is defined in Eq. (12) as

$$P(s) = (s - R(s))^{-1}. \tag{A1}$$

Using the form of the relaxation matrix $R(s)$, we obtain the following expressions for the matrix elements of $P(s)$,

$$P_{xx}(s) = \frac{1}{D(s)}\left(s + \Gamma_y(s) + \frac{\Delta^2}{s}\right), \tag{A2}$$

$$P_{yy}(s) = \frac{1}{D(s)}\left(s + \frac{E^2}{\Delta^2}\Gamma_1(s)\right), \tag{A3}$$

$$P_{zz}(s) = \frac{1}{s} - \frac{\Delta^2}{s^2}P_{yy}(s), \tag{A4}$$

$$P_{xy}(s) = -P_{yx}(s) = -\frac{1}{D(s)}\left(\epsilon - \frac{E}{\Delta}K_y^+(s)\right), \tag{A5}$$

$$P_{xz}(s) = P_{zx}(s) = -\frac{\Delta}{s}P_{xy}(s), \tag{A6}$$

$$P_{yz}(s) = -P_{zy}(s) = -\frac{\Delta}{s}P_{yy}(s), \tag{A7}$$

with the definition

$$D(s) = \left(s + \Gamma_y(s) + \frac{\Delta^2}{s}\right)\left(s + \frac{E^2}{\Delta^2}\Gamma_1(s)\right) + \left(\epsilon - \frac{E}{\Delta}K_y^+(s)\right)^2. \tag{A8}$$

The solution in Laplace space is now obtained according to Eq. (12), with $\mathbf{k}=0$,

$$\langle\sigma_i(s)\rangle = \sum_{j=x,y,z} P_{ij}(s)\langle\sigma_j(t=0)\rangle. \tag{A9}$$

E.g., for $\langle\boldsymbol{\sigma}(t=0)\rangle=(0,0,1)$, we find $\langle\sigma_i(s)\rangle = P_{iz}(s)$. Using Eqs. (A4), (A6), and (A7), we recover the known results from Ref. 4 in the special case $\mathbf{k}=0$. The remaining matrix elements, Eqs. (A2), (A3), and (A5), allow us the use different initial conditions.

1. The case $\epsilon = 0$

For an unbiased qubit, $\epsilon=0$ and thus $E=\Delta$, so that the quantities discussed above are reduced to the form

$$D(s) = \left(s+\Gamma_y(s)+\frac{\Delta^2}{s}\right)(s+\Gamma_1(s)), \tag{A10}$$

$$P_{xx}(s) = (s+\Gamma_1(s))^{-1}, \tag{A11}$$

$$P_{yy}(s) = \frac{s+\Gamma_1(s)}{D(s)} = \left(s+\Gamma_y(s)+\frac{\Delta^2}{s}\right)^{-1}, \tag{A12}$$

$$P_{zz}(s) = (s+\Gamma_y(s))P_{yy}(s)/s, \tag{A13}$$

$$P_{yz}(s) = -P_{zy}(s) = -\Delta P_{yy}(s)/s, \tag{A14}$$

$$P_{xy}(s) = P_{yx}(s) = P_{xz}(s) = P_{zx}(s) = 0. \tag{A15}$$

2. The case $\Delta = 0$

In the case of a diagonal system Hamiltonian H_S, we set $\Delta=0$ and thus $E=\epsilon$, and

$$\Gamma_y(s) = \frac{E^2}{\Delta^2}\Gamma_1(s) = 2A\left(C(s+i\epsilon)+C(s-i\epsilon)\right), \tag{A16}$$

$$\tilde{K}_y^+(s) \equiv \frac{E}{\Delta}K_y^+(s) = 2iA\left(C(s+i\epsilon)-C(s-i\epsilon)\right), \tag{A17}$$

$$D(s) = (s+\Gamma_y(s))^2 + \left(\epsilon-\tilde{K}_y^+(s)\right)^2. \tag{A18}$$

With Eqs. (A2–A7), we obtain

$$P_{xx}(s) = P_{yy}(s) = \frac{s+\Gamma_y(s)}{D(s)}, \tag{A19}$$

$$P_{zz}(s) = \frac{1}{s}, \tag{A20}$$

$$P_{xy}(s) = -P_{yx}(s) = -\frac{\epsilon-\tilde{K}_y^+(s)}{D(s)}, \tag{A21}$$

$$P_{xz}(s) = P_{zx}(s) = P_{yz}(s) = P_{zy}(s) = 0. \tag{A22}$$

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