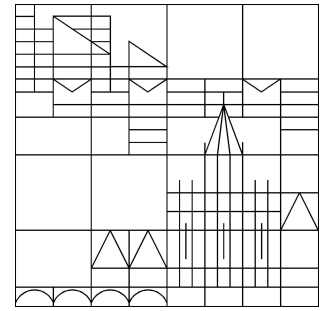


Universität Konstanz



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Oliver C. Schnürer
Felix Schulze
Miles Simon

Konstanzer Schriften in Mathematik

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STABILITY OF HYPERBOLIC SPACE UNDER RICCI FLOW

OLIVER C. SCHNÜRER, FELIX SCHULZE, AND MILES SIMON

ABSTRACT. We study the Ricci flow of initial metrics which are C^0 -perturbations of the *hyperbolic metric* on \mathbb{H}^n . If the perturbation is bounded in the L^2 -sense, and small enough in the C^0 -sense, then we show the following: In dimensions four and higher, the scaled Ricci harmonic map heat flow of such a metric converges smoothly, uniformly and exponentially fast in all C^k -norms and in the L^2 -norm to the hyperbolic metric as time approaches infinity. We also prove a related result for the Ricci flow and for the two-dimensional conformal Ricci flow.

1. INTRODUCTION

We investigate stability of hyperbolic space under Ricci flow

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} = -2 \operatorname{Ric}(g(t)) & \text{on } \mathbb{H}^n \times (0, \infty), \\ g(0) = g_0 & \text{on } \mathbb{H}^n. \end{cases}$$

As hyperbolic space expands under Ricci flow, it is convenient to consider the following modified Ricci flow

$$(1.2) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} = -2 \operatorname{Ric}(g(t)) - 2(n-1)g_{ij}(t) & \text{on } \mathbb{H}^n \times (0, \infty), \\ g(0) = g_0 & \text{on } \mathbb{H}^n. \end{cases}$$

The hyperbolic metric h of sectional curvature -1 is a stationary point to (1.2).

Note that by Lemma A.4, up to rescaling, this flow equation is equivalent to (1.1). As (1.1) and (1.2) are degenerate parabolic equations, we consider the following modified (or rescaled) Ricci harmonic map heat flow which is similar to DeTurck flow ([2])

$$(1.3) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} = -2 \operatorname{Ric}(g(t)) + \nabla_i V_j + \nabla_j V_i - 2(n-1)g_{ij}(t) & \text{on } \mathbb{H}^n \times (0, \infty), \\ g(0) = g_0 & \text{on } \mathbb{H}^n, \end{cases}$$

where $V_i = g_{ik} (g \Gamma_{rs}^k - h \Gamma_{rs}^k) g^{rs}$ and h is the hyperbolic metric on \mathbb{H}^n of sectional curvature equal to -1 . Once again, up to rescaling, this is equivalent to standard Ricci harmonic map heat flow.

We consider perturbations that are close to hyperbolic space.

Definition 1.1. Let g be a metric on \mathbb{H}^n . Let $\varepsilon > 0$. Then g is ε -close to h if

$$(1 + \varepsilon)^{-1} h \leq g \leq (1 + \varepsilon) h$$

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in the sense of matrices.

Let $(\lambda_i)_{1 \leq i \leq n}$ denote the eigenvalues of (g_{ij}) with respect to (h_{ij}) . Then this is equivalent to $(1 + \varepsilon)^{-1} \leq \lambda_i \leq 1 + \varepsilon$ for $1 \leq i \leq n$.

We denote with $\mathcal{M}^k(\mathbb{H}^n, I)$ the space of families $(g(t))_{t \in I}$ of sections in the space of Riemannian metrics on \mathbb{H}^n which are C^k on $\mathbb{H}^n \times I$. Similarly, we define \mathcal{M}^∞ , \mathcal{M}_{loc}^k and use $\mathcal{M}^k(\mathbb{H}^n)$ if the metric does not depend on t . We wish to point out that we use C^k on non-compact sets to denote the space, where covariant derivatives with respect to the hyperbolic metric h of order up to k are continuous and in L^∞ . We also use C_{loc}^k . For our convenience, we define $\mathcal{M}_0^\infty(\mathbb{H}^n, [0, \infty))$ to be the set of all metrics in $\mathcal{M}^0(\mathbb{H}^n, (0, \infty)) \cap \mathcal{M}_{loc}^0(\mathbb{H}^n, [0, \infty))$ which are smooth for positive times and, when restricted to time intervals of the form $[\delta, \infty)$, $\delta > 0$, are uniformly bounded in C^k for any $k \in \mathbb{N}$.

We use $|Z| \equiv {}^h|Z|$ to denote the norm of a tensor Z with respect to the hyperbolic metric h . Unless stated otherwise, $B_R(0)$ denotes a geodesic ball around a fixed point in hyperbolic space which we denote by 0. $\|\cdot\|_{L^2}$ denotes the L^2 -Norm with respect to the hyperbolic metric h . Sometimes, we write $x \rightarrow \infty$ instead of $|x| \rightarrow \infty$. Moreover, we use the Einstein summation convention and denote generic constants by c .

Our main theorem is

Theorem 1.2. *Let $n \geq 4$. For all $K > 0$ there exists $\varepsilon_1 = \varepsilon_1(n, K) > 0$ such that the following holds. Let $g_0 \in \mathcal{M}^0(\mathbb{H}^n)$ satisfy $\int |g_0 - h|^2 d\text{vol}_h \leq K$ and $\sup_{\mathbb{H}^n} |g_0 - h| \leq \varepsilon_1$. Then there exists a solution $g \in \mathcal{M}_0^\infty(\mathbb{H}^n, [0, \infty))$ to (1.3) such that*

$$\sup_{\mathbb{H}^n} |g(t) - h| \leq C(n, K) \cdot e^{-\frac{1}{4(n+2)}t}.$$

Moreover, $g(t) \rightarrow h$ exponentially in C^k as $t \rightarrow \infty$ for all $k \in \mathbb{N}$.

There might be different solutions to the ones obtained by our construction. The following theorem does not assume that the solution in question comes from the theorem above.

Theorem 1.3. *Let $n \geq 4$. For all $K > 0$ there exists $\varepsilon_1 = \varepsilon_1(n, K) > 0$ such that the following holds. Let $g \in \mathcal{M}_0^\infty(\mathbb{H}^n \times [0, \infty))$ be a solution to (1.3) that satisfies $\int_{\mathbb{H}^n} |g(0) - h|^2 d\text{vol}_h \leq K$, $\lim_{r \rightarrow \infty} \|g(0) - h\|_{L^\infty(\mathbb{H}^n \setminus B_r(0))} = 0$ and $\sup_{\mathbb{H}^n} |g(t) - h| \leq \varepsilon_1$ for all $t \geq 0$. Then*

$$\sup_{\mathbb{H}^n} |g(t) - h| \leq C(n, K) \cdot e^{-\frac{1}{4(n+2)}t}.$$

Moreover, $g(t) \rightarrow h$ exponentially in C^k as $t \rightarrow \infty$ for all $k \in \mathbb{N}$.

If $g(0) \in \mathcal{M}_{loc}^\infty(\mathbb{H}^n)$, solutions of (1.3) correspond to solutions of (1.2)

Theorem 1.4. *Let $n \geq 4$. Let g be a solution to (1.3) as in Theorem 1.2 or Theorem 1.3. Assume in addition that g is smooth. Then there exists a smooth family of diffeomorphisms of \mathbb{H}^n , $\varphi_0 = \text{id}_{\mathbb{H}^n}$, such that for $\tilde{g}(t) := \varphi_t^* g(t)$ the family $(\tilde{g}(t))_{t \geq 0}$ is a smooth solution to (1.2) satisfying*

$$\tilde{g}(t) \rightarrow (\varphi_\infty)^* h \quad \text{in } \mathcal{M}^\infty(\mathbb{H}^n) \quad \text{as } t \rightarrow \infty$$

for some smooth diffeomorphism φ_∞ of \mathbb{H}^n which satisfies $\varphi_t \rightarrow \varphi_\infty$ in $C^\infty(\mathbb{H}^n, \mathbb{H}^n)$ as $t \rightarrow \infty$ and, if $\lim_{r \rightarrow \infty} \|g(0) - h\|_{L^\infty(\mathbb{H}^n \setminus B_r(0))} = 0$,

$$|\varphi_\infty(x) - x| \rightarrow 0 \quad \text{as } |x| \rightarrow \infty.$$

Remark 1.5. All of the above results carry over directly if one replaces (\mathbb{H}^n, h) by any complete Riemannian manifold (M, h) , where h has sectional curvature equal to -1 .

Linearised stability of hyperbolic space under Ricci flow has been investigated before by V. Suneeta [10]. R. Ye considered stability of negatively curved manifolds on compact spaces in the paper [11]. H. Li and H. Yin [5] have shown stability of hyperbolic space in dimensions $n \geq 6$ under the assumptions that the deviation of the curvature of the initial metric from hyperbolic space decays exponentially and the initial metric is close to h (in the sense of Definition 1.1).

Similar results and methods to those found in this paper may be found in the authors' paper [7] addressing the stability of Euclidean space under Ricci flow. For further references, we refer to the introduction therein.

Here we outline the proofs of the main results of this paper, and explain the structure of these proofs and of the paper.

In the first part of the paper (chapters two and three) we consider the rescaled Ricci harmonic map heat flow.

There we prove short time existence using the same techniques as those presented in [8, 9, 7], see Theorem 2.4.

In chapter three, we show that the L^2 -norm of $g(t) - h$ is an exponentially decaying function of time (see Theorem 3.1, Corollary 3.2). This is the key ingredient to the proofs of our stability results. The calculations to prove this depend on an eigenvalue estimate for the Laplacian on hyperbolic domains due to H. McKean [6] and the closeness of the evolving metric to that of hyperbolic space. In contrast to the corresponding Euclidean result [7], we need strict monotonicity of our integral quantity to establish long time existence. The decay of the L^2 -norm implies that the C^0 -norm of $g(t) - h$ is exponentially decaying in time (Theorem 3.3). Interpolating between the C^0 -norm and C^k -norms, $k \in \mathbb{N}$, and using interior estimates, we see that all of the C^k -norms are exponentially decaying in time (for $t \geq 1$). This leads to long time existence and convergence.

In chapter four, we consider the related scaled Ricci flow $\tilde{g}(t)$ to the solution $g(t)$ obtained in chapters two and three. The two flows are related by time dependent diffeomorphisms $\varphi_t : \mathbb{H}^n \rightarrow \mathbb{H}^n$: $\tilde{g}(t) := \varphi_t^* g(t)$. As in the paper [7], we show that the estimates we obtained for $g(t)$ imply that $\tilde{g}(t) \rightarrow \psi^* h$ as $t \rightarrow \infty$ in the C^k -norms. Here ψ is a diffeomorphism, and this diffeomorphism is the C^k -limit of the time dependent diffeomorphisms φ_t which relate the two flows. We also show (as in [7]) that $\psi \rightarrow \text{id}$ as $|x| \rightarrow \infty$, if the initial metric g_0 satisfies $g_0 - h \rightarrow 0$ as $x \rightarrow \infty$ (see Theorem 4.2). The proofs of this chapter are the same (up to some minor modifications) as those of the paper [7].

In Appendix A we gather various standard results which we use in the paper.

In Appendix B we show that the arguments used here may be used in the Euclidean case to obtain analogous results (compare with [7]).

In Appendix C we address conformal Ricci flow in two dimensions and obtain a result similar to the two-dimensional result in [7] without assuming that $|g - h| \rightarrow 0$ near infinity.

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2. EXISTENCE

We first collect some evolution equations from [8], and then treat the question of short time existence.

In the following computations we always assume that in appropriate coordinates, we have at a fixed point and at a fixed time $h_{ij} = \delta_{ij}$ and $(g_{ij}) = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_i > 0$.

According to [8, Lemma 2.1], we get

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= g^{ab} \nabla_a^h \nabla_b^h g_{ij} - g^{kl} g_{ip} h^{pq} R_{jkql}(h) - g^{kl} g_{jp} h^{pq} R_{ikql}(h) \\ &\quad + \frac{1}{2} g^{ab} g^{pq} \left(\nabla_i^h g_{pa} \nabla_j^h g_{qb} + 2 \nabla_a^h g_{jp} \nabla_q^h g_{ib} - 2 \nabla_a^h g_{jp} \nabla_b^h g_{iq} \right. \\ &\quad \left. - 2 \nabla_j^h g_{pa} \nabla_b^h g_{iq} - 2 \nabla_i^h g_{pa} \nabla_b^h g_{jq} \right) \\ &\quad - 2(n-1)g_{ij}. \end{aligned}$$

Using that

$$R_{ijkl}(h) = -(h_{ik}h_{jl} - h_{il}h_{jk})$$

is the curvature tensor of hyperbolic space of sectional curvature -1 , we get

$$\begin{aligned} &-g^{kl} g_{ip} h^{pq} R_{jkql}(h) - g^{kl} g_{jp} h^{pq} R_{ikql}(h) - 2(n-1)g_{ij} \\ &= g^{kl} g_{ip} h^{pq} (h_{jq}h_{kl} - h_{jl}h_{kq}) + g^{kl} g_{jp} h^{pq} (h_{iq}h_{kl} - h_{il}h_{kq}) - 2(n-1)g_{ij} \\ &= 2(g^{kl} g_{ij} h_{kl} - h_{ij}) - 2(n-1)g_{ij} \\ &= 2(g^{kl} (h_{kl} - g_{kl})) g_{ij} + 2(g_{ij} - h_{ij}) \end{aligned}$$

and hence

Lemma 2.1. *A metric (g_{ij}) solving (1.3) fulfills*

$$\begin{aligned} \frac{\partial}{\partial t} g_{ij} &= g^{ab} \nabla_a^h \nabla_b^h g_{ij} + 2g_{ij} (g^{kl} (h_{kl} - g_{kl})) + 2(g_{ij} - h_{ij}) \\ &\quad + \frac{1}{2} g^{ab} g^{pq} \left(\nabla_i^h g_{pa} \nabla_j^h g_{qb} + 2 \nabla_a^h g_{jp} \nabla_q^h g_{ib} - 2 \nabla_a^h g_{jp} \nabla_b^h g_{iq} \right. \\ &\quad \left. - 2 \nabla_j^h g_{pa} \nabla_b^h g_{iq} - 2 \nabla_i^h g_{pa} \nabla_b^h g_{jq} \right). \end{aligned}$$

For later use, we derive the evolution equation of $|g - h|^2$.

Lemma 2.2. *Let $g \in \mathcal{M}^\infty(\mathbb{H}^n, (0, T))$, $T > 0$, be a solution to (1.3) which is ε -close to the hyperbolic metric h of sectional curvature -1 . Assume that $\varepsilon > 0$ is sufficiently small. Then*

$$(2.1) \quad \frac{\partial}{\partial t} |g - h|^2 \leq g^{ij} \nabla_i \nabla_j |g - h|^2 - (2 - \varepsilon) |\nabla(g - h)|^2 + (4 + \varepsilon) |g - h|^2,$$

where we write ε instead of $c(n)\varepsilon$ and ∇ instead of ${}^h\nabla$.

Proof. Note that the norm of a tensor Z of order m fulfills

$${}^h|Z|^2 \equiv |Z|^2 \leq (1 + \varepsilon) \cdot {}^g|Z|^2 \leq (1 + \varepsilon) \cdot {}^h|Z|^2.$$

Choose coordinates such that $h_{ij} = \delta_{ij}$ and $g_{ij} = \lambda_i \delta_{ij}$. We use $*$ similarly as in [3, Ch. 13] to denote contractions with respect to h , g or their inverses. Let $Z = g - h$. Lemma 2.1 yields

$$\begin{aligned} \frac{\partial}{\partial t} |Z|^2 &\equiv \frac{\partial}{\partial t} |g - h|^2 = 2 \sum_i (g_{ii} - h_{ii}) \left(\frac{\partial}{\partial t} g_{ii} \right) \\ &= g^{ij} \nabla_i \nabla_j |g - h|^2 - (2 - \varepsilon) |\nabla g|^2 \\ &\quad + 2 \sum_i (g_{ii} - h_{ii}) \left[2(g_{ii} - h_{ii}) - 2g_{ii} \sum_k (g^{kk} (g_{kk} - h_{kk})) \right] \\ &\quad + \sum_i (g_{ii} - h_{ii}) (\nabla g * \nabla g)_{ii} \\ &\leq g^{ij} \nabla_i \nabla_j |g - h|^2 - (2 - \varepsilon) |\nabla(g - h)|^2 \\ &\quad + 4 \sum_i (g_{ii} - h_{ii}) \left[(g_{ii} - h_{ii}) - g_{ii} \sum_k (g^{kk} (g_{kk} - h_{kk})) \right]. \end{aligned}$$

Let us examine the zeroth order term S of the above equation.

$$\begin{aligned} S &= 4 \sum_i (g_{ii} - h_{ii}) \left[(g_{ii} - h_{ii}) - g_{ii} \sum_k (g^{kk} (g_{kk} - h_{kk})) \right] \\ &= 4 \sum_i (\lambda_i - 1)^2 - 4 \sum_i \lambda_i (\lambda_i - 1) \sum_k \left(1 - \frac{1}{\lambda_k} \right) \\ &\leq (4 + \varepsilon) |Z|^2 - 4 \left(\sum_i (\lambda_i - 1) \right)^2. \end{aligned}$$

The claim follows. \square

We use this to show that we can solve Dirichlet problems for (1.3) on a short time interval. In the following we pick a point p_0 and fix it throughout. For simplicity of notation we will denote this point with 0 . All balls $B_R(0)$ are geodesic balls with respect to the hyperbolic metric h .

The following result also extends readily to (1.3) on all of \mathbb{H}^n provided that a non-compact maximum principle is applicable.

Corollary 2.3. *Let $g \in \mathcal{M}_0^\infty(B_R(0), [0, T])$, $0 < T < \infty$, be a solution to (1.3) on $B_R(0) \subset \mathbb{H}^n$ instead of \mathbb{H}^n with $g(t)|_{\partial B_R(0)} = h|_{\partial B_R(0)}$. Let $0 < \delta$. Then there*

exists $\varepsilon = \varepsilon(n, T, \delta) > 0$ such that $\sup_{B_R(0)} |g(0) - h| \leq \varepsilon$ implies

$$\sup_{B_R(0) \times [0, T]} |g - h| \leq \delta.$$

Proof. Assume without loss of generality that δ is smaller than ε in Lemma 2.2 and $\varepsilon \leq 1$. Fix $\varepsilon := \delta e^{-5T}$. Then Lemma 2.2 implies that $\sup_{B_R(0)} |g(t) - h| \leq \delta e^{-5(T-t)}$ as long as $\sup_{B_R(0)} |g(t) - h| \leq \varepsilon$. The result follows. \square

If solutions to (1.3) stay sufficiently close to the hyperbolic metric h of sectional curvature -1 , they exist for all times.

Theorem 2.4. *For all $n \in \mathbb{N}$ there exists a $\tilde{\delta}(n) > 0$ such that the following holds. Let $0 < \varepsilon < \delta \leq \tilde{\delta}(n)$. Then every metric $g_0 \in \mathcal{M}^0(\mathbb{H}^n)$ with $\|g_0 - h\|_{L^\infty} \leq \varepsilon$ has a δ -maximal solution $g \in \mathbb{M}_0^\infty(\mathbb{H}^n, [0, T_{g_0}))$ to (1.3), where $T_{g_0} > 0$ and $\|g(t) - h\|_{L^\infty} < \delta$ for all $t \in [0, T_{g_0})$. The solution is δ -maximal in the following sense. Either $T_{g_0} = \infty$ and $\|g(t) - h\|_{L^\infty} < \delta$ for all $0 \leq t$ or we can extend g to a solution on $\mathbb{H}^n \times [0, T_{g_0} + \tau)$, for some $\tau = \tau(n) > 0$, and $\|g(T_{g_0}) - h\|_{L^\infty} = \delta$.*

Proof. The proof follows directly from the corresponding proofs in [9, 8, 7]: We mollify g_0 and obtain g_0^i , $i \in \mathbb{N}$. Then we consider $g_0^{i,R} := \eta g_0^i + (1 - \eta)h$, $R \geq 1$, where $\eta : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a smooth function fulfilling $\eta = 1$ in $B_R(0)$, $\eta = 0$ outside $B_{2R}(0)$ and $|\nabla \eta| \leq 2/R$. Hence $\|g_0^{i,R} - h\|_{L^\infty} \leq \|g_0^i - h\|_{L^\infty} \leq \|g_0 - h\|_{L^\infty(\mathbb{H}^n)} \leq \varepsilon$. Arguing as in [9, 8, 7] (see Section 3 of [8]), and using that $g_0^{i,R}$ is $\tilde{\delta}(n)$ close to h , we see that there exist solutions $g^{i,R} \in \mathcal{M}^\infty(B_{3R}(0), [0, \tau])$ with $0 < \tau = \tau(n) < \infty$ to (1.3) on $B_{3R}(0)$ with $g^{i,R}(0) = g_0^{i,R}$ on $B_{3R}(0)$ and $g^{i,R} = h$ on $\partial B_{3R}(0) \times [0, \tau]$. From Lemma 5.1 of [9] we see that we have interior estimates of the form $|\nabla^j g^{i,R}|^2 \leq c/t^j$ on balls of radius $R/2$ for all $t \in [0, \tau]$ for all $j \in \mathbb{N}$. Taking a diagonal subsequence and using the Theorem of Arzelà-Ascoli, we obtain a solution $g \in \mathcal{M}^\infty(\mathbb{H}^n, (0, \tau])$. Furthermore, $g(t) \rightarrow g_0$ as $t \rightarrow 0$ in the C_{loc}^0 -norm as we see from Theorem 5.2 in [9].

If $\|g(t) - h\|_{L^\infty} < \delta$ for all $t \in [0, \tau]$, then we may repeat this argument in view of the fact that $\delta \leq \tilde{\delta}(n)$. By induction we obtain a solution $g \in \mathcal{M}_0^\infty(\mathbb{H}^n, [0, S))$ where either

- (a) $S = \infty$ and $\|g(t) - h\|_{L^\infty} < \delta$ for all $t > 0$, or
- (b) $S = N\tau$ for some $N \in \mathbb{N}$ and $\|g(t) - h\|_{L^\infty} < \delta$ for all $t \in [0, (N-1)\tau]$ but there exists at least one time $\tilde{t} \in ((N-1)\tau, N\tau]$ with $\|g(\tilde{t}) - h\|_{L^\infty} = \delta$.

In the case (a), we set $T_{g_0} = \infty$ and we are finished.

So assume we are in case (b) for the rest of the argument. In view of Lemma 2.2, the maximum principle, and the way we constructed our solutions, we see that in fact there is a first time $T_{g_0} \in ((N-1)\tau, N\tau]$ with $\|g(T_{g_0}) - h\|_{L^\infty} = \delta$ and $\|g(t) - h\|_{L^\infty} < \delta$ for all $t \in [0, T_{g_0})$. Using that $g(T_{g_0})$ is δ close to h and $\delta \leq \tilde{\delta}$ we may repeat the first part of the proof to obtain a solution defined on $[0, T_{g_0} + \tau]$. This completes the proof. \square

Proposition 2.5. *Assume everything is as in Theorem 2.4, and $\hat{T} > 0$ be given. If we choose $\varepsilon = \varepsilon(n, \delta, \hat{T}) > 0$ small enough in the above Theorem, then the solution $g \in \mathbb{M}_0^\infty(\mathbb{H}^n, [0, T_{g_0} + \tau))$ from Theorem 2.4 satisfies $T_{g_0} \geq \hat{T}$.*

Proof. By Corollary 2.3, we can choose $\varepsilon = \varepsilon(n, \delta, \hat{T})$ small enough so that $\|g(t) - h\|_{L^\infty} < \delta$ for all $t \in [0, \hat{T}] \cap [0, T_{g_0}]$. Theorem 2.4 yields a contradiction if $T_{g_0} < \hat{T}$. \square

3. CONVERGENCE

Convergence is based on a Lyapunov function.

Theorem 3.1. *Let $n \geq 4$. There exists $\delta_0 = \delta_0(n) > 0$ such that the following holds. Let $g \in \mathcal{M}^\infty(B_R, [0, T])$ be a solution to (1.3) with $g = h$ on $\partial B_R(0) \times [0, T]$ and assume that $\sup_{B_R(0) \times [0, T]} |g - h| \leq \delta_0$. Then we have*

$$\int_{B_R(0)} |g(t) - h|^2 d\text{vol}_h \leq e^{-\alpha t} \int_{B_R(0)} |g(0) - h|^2 d\text{vol}_h$$

for $\alpha(n) := (2(n-1)^2 - 17)/4 \geq \frac{1}{4}$.

Proof. Assume that δ_0 is such that g is $\varepsilon = \varepsilon(n)$ -close to h for some sufficiently small $\varepsilon > 0$. We compute, using Lemma 2.2,

$$\begin{aligned} \frac{\partial}{\partial t} \int_{B_R(0)} |Z|^2 d\text{vol}_h &\leq \int_{B_R(0)} g^{ij} \nabla_i \nabla_j |Z|^2 - (2 - \varepsilon) |\nabla Z|^2 + (4 + \varepsilon) |Z|^2 d\text{vol}_h \\ &= \int_{\partial B_R(0)} \nu_i g^{ij} \nabla_j |Z|^2 - \int_{B_R(0)} \nabla_j g^{jk} \nabla_k |Z|^2 d\text{vol}_h \\ &\quad + \int_{B_R(0)} -(2 - \varepsilon) |\nabla Z|^2 + (4 + \varepsilon) |Z|^2 d\text{vol}_h \\ &\leq \int_{B_r(0)} -(2 - \varepsilon) |\nabla |Z||^2 + (4 + \varepsilon) |Z|^2 d\text{vol}_h, \end{aligned}$$

where we used that $|\nabla_i g^{ij} \nabla_j |Z|^2| \leq \varepsilon |\nabla Z|^2$ and that on $\partial B_R(0)$ the gradient $\nabla |Z|^2$ is anti-parallel to the outer unit normal ν . Furthermore we apply Kato's inequality $|\nabla |Z||^2 \leq |\nabla Z|^2$ which is valid whenever $|Z| \neq 0$ and for Sobolev functions.

Using McKean's inequality [6] for the first eigenvalue

$$\sigma_1 \geq \frac{(n-1)^2}{4}$$

on hyperbolic domains we see

$$\begin{aligned} \frac{\partial}{\partial t} \int_{B_R(0)} |Z|^2 d\text{vol}_h &\leq \int_{B_R(0)} -(2 - \varepsilon) |\nabla |Z||^2 + (4 + \varepsilon) |Z|^2 d\text{vol}_h \\ &\leq \frac{8 - (n-1)^2 + \varepsilon}{2} \int_{B_R(0)} |Z|^2 d\text{vol}_h. \end{aligned}$$

Assuming that $\varepsilon < 1/2$, we can choose

$$\alpha := (2(n-1)^2 - 17)/4.$$

\square

Since for the proof of existence of a solution to (1.3) we use Dirichlet problems as above, this monotonicity extends to the constructed solutions on $\mathbb{H}^n \times [0, T)$: Let $g^{i,R}(t)$ be as in Theorem 2.4. Then we get

$$\|g^{i,R}(t) - h\|_{L^2(B_{3R}(0))}^2 \leq e^{-\alpha t} \|g_0^{i,R} - h\|_{L^2(B_{3R}(0))}^2 \leq e^{-\alpha t} \|g_0 - h\|_{L^2(\mathbb{H}^n)}^2$$

As $R \rightarrow \infty$, we obtain

Corollary 3.2. *Let $n \geq 4$ and $T > 0$ be given. Assume that $g_0 \in \mathcal{M}^\infty(\mathbb{H}^n)$ satisfies $\|g_0 - h\|_{L^2(\mathbb{H}^n)} < \infty$. Then there exists $\varepsilon_0 = \varepsilon_0(n, T)$ such that, if $\sup_{\mathbb{H}^n} |g_0 - h| \leq \varepsilon_0$ then a solution $g \in \mathcal{M}^\infty(\mathbb{H}^n, [0, T))$ to (1.3) with $g(\cdot, 0) = g_0(\cdot)$ exists and $\sup_{\mathbb{H}^n \times [0, T)} |g - h| \leq \delta_0$, where δ_0 is as in Theorem 3.1. Furthermore we have the estimate*

$$\|g(t) - h\|_{L^2(\mathbb{H}^n)}^2 \leq e^{-\alpha t} \|g_0 - h\|_{L^2(\mathbb{H}^n)}^2$$

for all $t \in [0, T)$, where $\alpha = \alpha(n) \geq \frac{1}{4}$.

Proof. Existence and closeness to h follow from Corollary 2.3, Proposition 2.5 and Theorem 2.4. \square

Using the gradient estimate we see that the exponential convergence of the L^2 -norm of $|g - h|$ also implies exponential convergence in the sup-norm, compare [7, Lemma 7.1].

Theorem 3.3. *Let $n \geq 4$. Assume that $g \in \mathcal{M}^\infty(\mathbb{H}^n, [0, T))$ is a solution to (1.3) with $\|g(0) - h\|_{L^2(\mathbb{H}^n)} =: K < \infty$, $\sup_{\mathbb{H}^n \times [0, T)} |g - h| \leq \delta_0$ and*

$$\|g(t) - h\|_{L^2(\mathbb{H}^n)}^2 \leq e^{-\alpha t} \|g(0) - h\|_{L^2(\mathbb{H}^n)}^2,$$

where δ_0 is as in Theorem 3.1. Then

$$(3.1) \quad \sup_{\mathbb{H}^n} |g(t) - h| \leq C(n, K) e^{-\beta t},$$

where $\beta = \frac{\alpha}{n+2} = \frac{2(n-1)^2 - 17}{4(n+2)} > 0$.

Proof. We can assume w.l.o.g. that $\delta_0 < 1$. We choose $\tau := \frac{n+1}{\alpha} \ln(\delta_0^{-1}) > 0$. Note that this implies

$$\sup_{\mathbb{H}^n} |g(t) - h| \leq e^{-\beta t}$$

for $t \in [0, \tau)$ and $\beta := \alpha/(n+1)$. By the interior estimates of the form $|\nabla^h g(t)| \leq c \cdot t^{-1/2}$, there exists a constant $C' = C'(n)$, such that

$$\left| \nabla^h g(\cdot, t) \right|^h \leq C'$$

for $t \in [\tau, T)$. Fix such a $t \in [\tau, T)$. Let $\gamma := \sup_{\mathbb{H}^n} |g(t) - h|$ and choose a point $p_0 \in \mathbb{H}^n$ such that $|g(p_0, t) - h(p_0)| \geq \frac{1}{2}\gamma$. By the gradient estimate, we have

$$|g(\cdot, t) - h| \geq \frac{1}{4}\gamma$$

on $B_{\gamma/(4C')}(p_0)$. This implies

$$\|g(t) - h\|_{L^2(\mathbb{H}^n)}^2 \geq \omega_n (C')^{-n} \left(\frac{\gamma}{4}\right)^{n+2},$$

where ω_n is the measure of the unit ball in \mathbb{R}^n . This yields

$$\gamma \leq 4(C')^{\frac{n}{n+2}} \left(\frac{K^2}{\omega_n} \right)^{\frac{1}{n+2}} e^{-\frac{\alpha}{n+2}t}.$$

Choosing $C(n, K) = 1 + 4(C')^{\frac{n}{n+2}} \left(\frac{K^2}{\omega_n} \right)^{\frac{1}{n+2}}$ we have (3.1). \square

This sup-estimate allows us to construct a solution which exists for all times.

Theorem 3.4. *Let $n \geq 4$. For all $K > 0$ there exists $\varepsilon_1 = \varepsilon_1(n, K) > 0$ such that the following holds. Let $g_0 \in \mathcal{M}^\infty(\mathbb{H}^n)$ satisfy $\|g_0 - h\|_{L^2(\mathbb{H}^n)} \leq K$ and $\sup_{\mathbb{H}^n} |g_0 - h| \leq \varepsilon_1$. Then there exists a solution $g \in \mathcal{M}^\infty(\mathbb{H}^n, [0, \infty))$ to (1.3) with $g(0) = g_0$ such that*

$$(3.2) \quad \sup_{\mathbb{H}^n} |g(t) - h| \leq C(n, K)e^{-\beta t}$$

for $\beta = \beta(n)$ as in Theorem 3.3.

Proof. According to Theorem 2.4, we obtain existence for all times if we can prove the estimate $\|g(t) - h\|_{L^\infty} \leq \tilde{\delta} = \tilde{\delta}_{\text{Thm. 2.4}}$ for all t for any a priori solution (that is, we must prove the estimate for all t that the solution is defined). Given any $T > 0$, we can choose $\varepsilon(n, T) > 0$ small enough so that such an estimate will hold, in view of Proposition 2.5 and Theorem 2.4 for $0 \leq t < T$. Theorem 3.1 implies integral bounds which combined with Theorem 3.3 yields $|g(t) - h| \leq \tilde{\delta}$ if $t \geq T$ and T is chosen sufficiently large. Choose T and $\varepsilon(n, T)$ so that both of these requirements are satisfied.

This implies long time existence.

Theorem 3.3 also implies (3.2) for $t \geq T$. Fixing $C(n, K)$ such that $C(n, K) \leq \delta \cdot e^{\beta T}$ we obtain (3.2) for all times. \square

By interpolation the exponential decay extends to higher derivatives of the evolving metric.

Theorem 3.5. *Let $n \geq 4$. Let $g_0 \in \mathcal{M}^\infty(\mathbb{H}^n)$ and $g \in \mathcal{M}^\infty(\mathbb{H}^n, [0, \infty))$ be as in Theorem 3.4. We have additionally*

$$\sup_{\mathbb{H}^n} \left| \nabla^j g(t) \right| \leq C(n, j, K, (\beta_j)) e^{-\beta_j t}$$

where $0 < \beta_j < \beta(n)$, $\beta(n)$ as in Theorem 3.3, is arbitrary. In particular,

$$\lim_{t \rightarrow \infty} \sup_{\mathbb{H}^n} \|g(t) - h\|_{C^k(\mathbb{H}^n)} = 0,$$

where $\|S\|_{C^k} := \sum_{i=0}^k \sup_{\mathbb{H}^n} |\nabla^i S|^2$.

Proof. From the interior estimates in Lemma A.1, we have $\sup_{\mathbb{H}^n} |\nabla^j g|^2(t) \leq c(n, j)/(t - L)$, for all $t \in [L, L + 1]$. In particular, $\sup_{\mathbb{H}^n} |\nabla^j g|^2(L + 1) \leq c(n, j)$. Hence, as $L > 0$ was arbitrary, we get

$$(3.3) \quad \sup_{\mathbb{H}^n} |\nabla^j g|^2(\cdot, t) \leq c(n, j)$$

for all $t \geq 1$. Interpolating on a ball of radius one (see Lemma A.6) gives us

$$\sup_{\mathbb{H}^n} |\nabla^j g|^2(t) \leq \tilde{c}(n, j) \left(\sup_{\mathbb{H}^n} |g(t) - h| \right)^{\frac{1}{2^j - 1}} \leq C(n, j, K) e^{-\frac{\beta}{2^j - 1} t}$$

in view of (3.3) and (3.1). Iterated interpolation yields the result, see e. g. [7]. \square

Proof of Theorem 1.2. As the decay of $|g(t) - h|$ as $t \rightarrow \infty$ obtained in this section does not depend on the smoothness of g_0 , we can approximate $g_0 \in \mathcal{M}^0(\mathbb{H}^n)$ and pass to a limit to obtain Theorem 1.2. \square

Proof of Theorem 1.3. Local closeness estimates (see Lemma A.2) show that

$$\lim_{r \rightarrow \infty} \|g(t) - h\|_{L^\infty(\mathbb{H}^n \setminus B_r(0))} = 0$$

is preserved during the flow, even uniformly on compact time intervals. Hence

$$\max\{|g(t) - h|^2 - \delta, 0\} \equiv (|g - h|^2 - \delta)_+$$

has compact support on $\mathbb{H}^n \times [0, K]$ for all $K < \infty$ and we may consider the integral $I_\delta := \|(|g - h|^2 - \delta)_+\|_{L^1(\mathbb{H}^n)}$ for any $\delta > 0$, which is similar to the integral $I_\delta^{m,p}$ defined in [7, Theorem 6.1] or to I_δ^p in Appendix B. The techniques of the proof of Theorem 3.1 and approximations as in [7, Theorem 6.1] imply for $R \gg 1$ that

$$I_\delta^R(t) := \|(|g(t) - h|^2 - \delta)_+\|_{L^1(B_R(0))} \leq e^{-\alpha t} \cdot I_\delta^R(0) \leq e^{-\alpha t} \cdot \|g(0) - h\|_{L^2(\mathbb{H}^n)}^2.$$

The rest of the proof is similar to the proof of Theorem 1.2. \square

4. GETTING BACK TO RICCI FLOW

Theorem 4.1. *Assume that $g \in \mathcal{M}^\infty(\mathbb{H}^n, [0, \infty))$ is the solution to (1.3) coming from Theorem 3.4. Then there exists a smooth map $\varphi : \mathbb{H}^n \times [0, \infty) \rightarrow \mathbb{H}^n$ such that $\varphi(\cdot, t) =: \varphi_t : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a diffeomorphism, $\varphi_0 = \text{id}$ and $\tilde{g}(\cdot, t) := (\varphi_t)^* g(\cdot, t)$ is a smooth solution to the scaled Ricci flow*

$$\frac{\partial}{\partial t} \tilde{g} = -2 \text{Ric} - 2(n-1)\tilde{g}$$

with $\tilde{g}_t \rightarrow g_0$ as $t \searrow 0$. Furthermore there exists a smooth diffeomorphism $\psi : \mathbb{H}^n \rightarrow \mathbb{H}^n$ with $\varphi_t \rightarrow \psi$ as $t \rightarrow \infty$ and $\tilde{g}_t \rightarrow \psi^* g_0$ as $t \rightarrow \infty$. Here convergence is in C^k on \mathbb{H}^n for all k .

Proof. This argument is the same as in Lemma 9.1 and Theorem 9.2 of [7] with some minor differences. We explain here where the argument of [7] must be modified in order for it to work in this case.

As explained in Lemma 9.1 in [7], we can construct smooth maps $\varphi : \mathbb{H}^n \times [0, \infty) \rightarrow \mathbb{H}^n$ such that

$$\begin{cases} \frac{\partial}{\partial t} \varphi^\alpha(x, t) = V^\alpha(\varphi(x, t), t), & (x, t) \in \mathbb{H}^n \times [0, \infty), \\ \varphi(x, 0) = x, & x \in \mathbb{H}^n, \end{cases}$$

where $V^\alpha(y, t) := -g^{\beta\gamma} ({}^g \Gamma_{\beta\gamma}^\alpha - {}^h \Gamma_{\beta\gamma}^\alpha)(y, t)$ and $\varphi_t := \varphi(\cdot, t) : \mathbb{H}^n \rightarrow \mathbb{H}^n$ are diffeomorphisms. (Compared to [7], we have changed the sign in the definition of V in order to correct a typo there.) This part of the proof is the same. A direct calculation shows that $\tilde{g}(t) := \varphi_t^* g(t)$ solves the scaled Ricci flow equation.

In Theorem 9.2 of [7] it is shown that $\varphi_t \rightarrow \varphi_\infty$ as $t \rightarrow \infty$ where $\varphi_\infty : \mathbb{H}^n \rightarrow \mathbb{H}^n$ is a smooth diffeomorphism, and the convergence is in C^k (for all k) on \mathbb{H}^n . The proof of this is carried out in three steps.

In step 1 it is shown that $|\frac{\partial}{\partial t}\varphi_t(x)| \leq \frac{1}{t^r}$ for some $r > 1$, for all $t \geq 1$, and $|\varphi_t(x) - x| \leq c$ for all t .

In step 2, the existence of a smooth function $\varphi_\infty : \mathbb{H}^n \rightarrow \mathbb{H}^n$ with $\varphi_t \rightarrow \varphi_\infty \equiv \psi$ as $t \rightarrow \infty$ is shown.

The proofs of steps one and two carry over to this situation without any changes.

In step 3, it is shown that φ_∞ is a diffeomorphism. This proof carries over with some minor modifications which we describe in the rest of the proof here.

Letting $l(t) := \varphi_t^*g(t)$, we know that l solves the *scaled Ricci flow* (1.2) on \mathbb{H}^n , and that

$$\sup_{\mathbb{H}^n}^{l(t)} |\text{Ric}(l(t)) + 2(n-1)l(t)| = \sup_{\mathbb{H}^n}^{g(t)} |\text{Ric}(g(t)) + 2(n-1)g(t)| \leq e^{-\beta t}.$$

for all $t > 0$ for a $\beta > 0$, in view of Theorem 3.5. Hence $l(t)$ converges locally uniformly (smoothly) to a smooth metric l_∞ on \mathbb{H}^n as explained in [7].

Choose geodesic coordinates for h centred at y in $B_1(y)$. Now using the definition of l , and the uniform convergence of l we get

$$\begin{aligned} \frac{1}{c}\delta_{\alpha\beta} \leq l_{\alpha\beta}(x, t) &= \frac{\partial\varphi_t^s}{\partial y^\alpha}(x, t) \frac{\partial\varphi_t^k}{\partial y^\beta}(x, t) g_{sk}(\varphi_t(x), t) \\ &\leq (1 + \tilde{\varepsilon}) \frac{\partial\varphi_t^s}{\partial y^\alpha} \frac{\partial\varphi_t^k}{\partial y^\beta}(x, t) h_{sk}(\varphi_t(x)) \\ &\leq c(1 + \tilde{\varepsilon}) (D\varphi_t)^T(x, t). \end{aligned}$$

In particular, we see that $\det(D\varphi_t)^2(x) \geq \frac{1}{(1+\tilde{\varepsilon})^{n_c}} > 0$ for $x \in B_1(y)$, where Df is the Jacobian of f . As explained in [7], this shows that φ_∞ is a diffeomorphism. \square

Theorem 4.2. *Let everything be as in the above Theorem 4.1, with the extra assumption that $\sup_{(\mathbb{H}^n \setminus B_r(0))} |g_0 - h| \rightarrow 0$ as $r \rightarrow \infty$. Then the diffeomorphism ψ appearing in the above Theorem satisfies $\psi \rightarrow \text{id}$ as $x \rightarrow \infty$ (in C^k for all k). In particular, for every $\eta > 0$, there exists an $R > 0$ such that*

$$\sup_{\mathbb{H}^n \setminus B_R(0)} |\varphi_t(x) - x| \leq \eta$$

for all t .

Proof. The proof is completely analogous to the proof of Lemma 9.3 in [7]. Let $\eta > 0$. From Lemma A.2 and the estimates of Theorem 3.4 we can choose $R > 0$ large so that

$$|g(t) - h| \leq \eta \text{ on } \mathbb{H}^n \setminus B_R(0)$$

for all $t > 0$. From the interior estimates of [9] (see Lemma A.1) we get

$$|\nabla^2 g| \leq \frac{c}{t}$$

for $t \in [0, 1]$, and hence

$$|\nabla^2 g| \leq \frac{c}{t}$$

for all $t \in [0, \infty)$, in view of Theorem 3.4 and interpolation with higher order derivatives, see Lemma A.1 and Lemma A.6. Interpolating between the C^0 -norm and the C^2 -norm on a ball of radius one for $t \leq 1$ (see Lemma A.6) we get

$$|\nabla g| \leq \frac{\sqrt{c}\sqrt{\eta}}{\sqrt{t}}$$

for all $t \in [0, \infty)$ on $\mathbb{H}^n \setminus B_R(0)$. Arguing as in Step 1 of the above Theorem gives us the result. \square

APPENDIX A. SCALING AND INTERIOR ESTIMATES

Lemma A.1. *Let $(\mathbb{H}^n, g(t))_{t \in [0, T]}$ be a solution to equation (1.3), with $|g(t) - h| \leq \varepsilon(n)$ for $\varepsilon(n) > 0$ small enough. Then*

$$\sup_{\mathbb{H}^n} \left| {}^h \nabla^j g(\cdot, t) \right|^2 \leq \frac{c(j, n)}{t^j}$$

for all $t \leq \min\{1, T\}$.

Proof. This is Theorem 4.3 in [9]. \square

Lemma A.2. *Fix a point $p_0 \in \mathbb{H}^n$ and let $g \in \mathcal{M}^\infty(\mathbb{H}^n, [0, T])$, where $T \in (0, \infty]$, be a solution to (1.3) such that*

$$\sup_{\mathbb{H}^n \setminus B_r(p_0)} |g(\cdot, 0) - h| \rightarrow 0 \text{ as } r \rightarrow \infty.$$

Then for every $0 < \tau < T$ and $0 < \varepsilon \leq 1$ there exist an $R_0 > 0$ such that

$$\sup_{(\mathbb{H}^n \setminus B_{R_0}(p_0)) \times [0, \tau]} |g - h| \leq \varepsilon.$$

Proof. Choose a smooth function $\eta : \mathbb{R} \rightarrow \mathbb{R}$, $0 \leq \eta \leq 1$, such that $\eta \equiv 1$ on $(-\infty, 1]$, $\eta \equiv 0$ on $[2, \infty)$ and $\eta' \leq 0$. We can furthermore assume that $|\eta''| \leq 8$ and $|\eta'|^2 \leq 16\eta$.

Let $\rho_p(\cdot)$ denote the distance to a point $p \in \mathbb{H}^n$ with respect to the hyperbolic metric, and define the cut-off function

$$\gamma_{p, R} := \eta\left(\frac{\rho_p}{R}\right).$$

Then we have, suppressing in the following the subscripts p and R ,

$$|\nabla \gamma| \leq \frac{|\eta'|}{R}, \quad |\nabla^2 \gamma| \leq \frac{C}{R} + \frac{C}{R^2}.$$

Define $\psi := \gamma|Z|^2$. Using Lemma 2.2 and the above estimates we see that

$$\begin{aligned} \frac{\partial}{\partial t} \psi &\leq g^{ij} \nabla_i \nabla_j \psi - 2g^{ij} \nabla_i \gamma \nabla_j |Z|^2 - |Z|^2 g^{ij} \nabla_i \nabla_j \gamma \\ &\quad - (2 - \varepsilon) \gamma |\nabla Z|^2 + (4 + \varepsilon) \psi \\ &\leq g^{ij} \nabla_i \nabla_j \psi + (4 + \varepsilon) \psi + \frac{\varepsilon}{R} + \frac{\varepsilon}{R^2}, \end{aligned}$$

where we used the estimates on the derivatives of γ and Kato's inequality. Note that the closeness assumption can be justified inductively using continuity on small time intervals and the following argument. Now choosing $\text{dist}(p, p_0)$ big enough and $R = \frac{1}{2} \text{dist}(p, p_0)$ we can integrate the inequality to yield the desired estimate. \square

Lemma A.3. *The scaled Ricci flow and the scaled Ricci harmonic map heat flow are related as follows. Assume $\varphi_t : \mathbb{H}^n \rightarrow \mathbb{H}^n$ solves*

$$\frac{\partial}{\partial t} \varphi_t(x, t) = -V(\varphi_t(x, t), t)$$

where the components of V are given by $V^\alpha := g^{\beta\gamma} \left(g \Gamma_{\beta\gamma}^\alpha - h \Gamma_{\beta\gamma}^\alpha \right)$ and that the $\varphi_t : \mathbb{H}^n \rightarrow \mathbb{H}^n$ are smooth and diffeomorphisms for all time. Let $(\mathbb{H}^n, g(t))_{t \in [0, T)}$ be a solution to the scaled Ricci harmonic map heat flow (1.3), $V_i = g_{i\alpha} V^\alpha$. Then $(\mathbb{H}^n, \tilde{g}(t))_{t \in [0, T)}$ solves the scaled Ricci flow (1.2), with $\tilde{g}(0) = g(0)$, where here $\tilde{g}(t) := \varphi_t^*(g(t))$.

Proof. For $\tilde{g}(t) := \varphi_t^* g(t)$, we get

$$\begin{aligned} \frac{\partial}{\partial t} (\tilde{g}(t)) &= (\varphi_t)^* \left(\frac{\partial}{\partial t} g \right) + \frac{\partial}{\partial s} \Big|_{s=0} (\varphi_{t+s}^* g(t)) \\ &= -2 \operatorname{Ric}(\tilde{g}(t)) - 2(n-1)\tilde{g}(t) + \varphi_t^* (\mathcal{L}_{V(t)} g(t)) - \mathcal{L}_{(\varphi_t^{-1})_* V(t)} (\varphi_t^* g(t)) \\ &= -2 \operatorname{Ric}(\tilde{g}(t)) - 2(n-1)\tilde{g}(t), \end{aligned}$$

where here $\mathcal{L}_W k$ is the Lie-derivative of k in the direction W (in coordinates $(\mathcal{L}_W k)_{ij} = {}^k \nabla_i W_j + {}^k \nabla_j W_i$), see [1, Chapter 2, Section 6]. \square

Lemma A.4. *The Ricci flow*

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}(g)$$

and the scaled Ricci flow (1.2) are equivalent in the following sense.

Let $(\mathbb{H}^n, \tilde{g}(\tilde{t}))_{\tilde{t} \in [0, \tilde{T})}$ be a solution to the scaled Ricci flow. Define $(\mathbb{H}^n, g(t))_{t \in [0, T)}$ by

$$g(\cdot, t) := (1 + 2(n-1)t) \tilde{g}(\cdot, \tilde{t}(t)),$$

where

$$\tilde{t}(t) := \frac{\log(1 + 2(n-1)t)}{2(n-1)}$$

$$\text{and } T := \frac{e^{2(n-1)\tilde{T}} - 1}{2(n-1)}.$$

Then $(\mathbb{H}^n, g(t))_{t \in [0, T)}$ solves the Ricci flow. Alternatively, let $(\mathbb{H}^n, g(t))_{t \in [0, T)}$ be a solution to the Ricci flow. Define $(\mathbb{H}^n, \tilde{g}(\tilde{t}))_{\tilde{t} \in [0, \tilde{T})}$ by

$$\tilde{g}(\cdot, \tilde{t}) := e^{-2(n-1)\tilde{t}} g(\cdot, t(\tilde{t})),$$

where $t(\tilde{t}) := \frac{e^{2(n-1)\tilde{t}} - 1}{2(n-1)}$ and $\tilde{T} := \frac{\log(1 + 2(n-1)T)}{2(n-1)}$. Then \tilde{g} solves the scaled Ricci flow.

Proof. We prove the first claim by calculating. The second claim is shown in a similar way. We calculate at t_0 , and let $\tilde{t}_0 := \frac{\log(1+2(n-1)t_0)}{2(n-1)}$.

$$\begin{aligned} \left(\frac{\partial}{\partial t}g\right)(\cdot, t_0) &= 2(n-1)\tilde{g}(\cdot, \tilde{t}_0) + \left(\frac{\partial}{\partial \tilde{t}}\tilde{g}\right)(\cdot, \tilde{t}_0) \\ &= 2(n-1)\tilde{g}(\cdot, \tilde{t}_0) - 2\text{Ric}(\tilde{g}, \tilde{t}_0) - 2(n-1)\tilde{g}(\cdot, \tilde{t}_0) \\ &= -2\text{Ric}(\tilde{g}(\tilde{t}_0)) \\ &= -2\text{Ric}(g(t_0)) \end{aligned}$$

where the last line follows from the fact, that the Ricci tensor is invariant under scaling of the metric. \square

Lemma A.5. *Let $u \in C^2$ on $[0, \infty)$ or \mathbb{R} . Then*

$$\|Du\|_{L^\infty}^2 \leq 32 \cdot \|u\|_{L^\infty} \cdot \|D^2u\|_{L^\infty}.$$

Proof. Assume without loss of generality that $Du(0) \geq \frac{1}{2}\|Du\|_{L^\infty} =: \frac{1}{2}M$. Then $Du(x) \geq \frac{1}{4}M$ for all $0 \leq x \leq \frac{M}{4\|D^2u\|_{L^\infty}}$. Hence

$$2\|u\|_{L^\infty} \geq \left|u\left(\frac{M}{4\|D^2u\|_{L^\infty}}\right) - u(0)\right| \geq \frac{M}{4} \cdot \frac{M}{4\|D^2u\|_{L^\infty}}.$$

The claim follows. \square

Lemma A.6. *Let B be a compact subset of a Riemannian manifold M . Assume that B has C^2 -boundary. Let $u \in C^2(M)$. Then*

$$\|\nabla u\|_{L^\infty(B)}^2 \leq c(B) \cdot \|u\|_{L^\infty(B)} \cdot \left(\|\nabla^2 u\|_{L^\infty(B)} + \|\nabla u\|_{L^\infty(B)}\right).$$

Proof. For every point $p \in B$ and every unit vector $\xi \in T_p M$ there exists a curve $\gamma : [0, \infty) \rightarrow B$, parametrised by arc-length, such that $\gamma(0) = p$, $\gamma'(0) \in \pm\{\xi\}$ and

$$\begin{aligned} |(u \circ \gamma)'(t)| &\leq |\nabla u(\gamma(t))|, \\ |(u \circ \gamma)''(t)| &\leq c(B) \cdot (|\nabla^2 u(\gamma(t))| + |\nabla u(\gamma(t))|). \end{aligned}$$

Note that $c(B)$ depends on the curvature of γ but can be chosen uniformly for all $p \in B$. Choosing p and ξ such that ∇u attains its maximum at p in direction ξ , Lemma A.5 yields the statement. \square

APPENDIX B. EUCLIDEAN SPACE

We consider the situation of the main theorem, Theorem 1.3, in [7]. Instead of a Lyapunov function involving $\varphi_m + \psi_m - 2n = \sum_{i=1}^n \frac{1}{\lambda_i^m} (\lambda_i^m - 1)^2$, however, we study a Lyapunov function involving $|g - h|^p$, $p \geq 2$. This simplifies the proof.

Recall that the Ricci harmonic map heat flow with Euclidean background metric is

$$\begin{aligned} \frac{\partial}{\partial t}g_{ij} &= g^{ab}\nabla_a\nabla_b g_{ij} + \frac{1}{2}g^{ab}g^{pq}(\nabla_i g_{pa}\nabla_j g_{qb} + 2\nabla_a g_{jp}\nabla_q g_{ib} \\ &\quad - 2\nabla_a g_{jp}\nabla_b g_{iq} - 2\nabla_j g_{pa}\nabla_b g_{iq} - 2\nabla_i g_{pa}\nabla_b g_{jq}), \end{aligned}$$

where ∇ denotes covariant differentiation w. r. t. the Euclidean metric h . Calculating as in Lemma 2.2, we see that

$$(B.1) \quad \frac{\partial}{\partial t} |g - h|^2 - g^{ij} \nabla_i \nabla_j |g - h|^2 \leq - \left(\frac{2}{1 + \varepsilon} - 9\varepsilon(1 + \varepsilon)^2 \right) |\nabla g|^2 \leq 0$$

if $0 < \varepsilon \leq \frac{1}{7}$. Note that there is no zeroth order term in the evolution equation on Euclidean space. Hence $|g(t) - h| \leq \varepsilon$ is preserved during the flow and we obtain long time existence, see [9]. Define

$$I_\delta^p(t) := \int_{\mathbb{R}^n} (|g - h|^p - \delta)_+.$$

Using (B.1), and calculating as in [7], we get

$$\frac{d}{dt} I_\delta^p(t) \leq - \int_{\{|g-h|^p > \delta\}} \frac{p}{2} \cdot \frac{2 - (11 + 9\varepsilon)(1 + \varepsilon)^2 \varepsilon}{1 + \varepsilon} \cdot |g - h|^{p-2} \cdot |\nabla g|^2 \leq 0$$

for $0 < \varepsilon \leq \frac{1}{8}$. The rest of the proof is similar to [7]. If we further restrict p to $2 \leq p < n$ then we can argue as in the paper [7] to prove Theorem 1.4 of that paper.

APPENDIX C. CONFORMAL RICCI FLOW IN TWO DIMENSIONS

Let us consider the Euclidean ball $B := B_1(0) \subset \mathbb{R}^2$ equipped with the metric $(g_{ij}) = (e^{f+u} \delta_{ij})$, where $f = \log 4 - 2 \log(1 - |x|^2)$ and $u = u(x, t)$. For $u \equiv 0$, we get hyperbolic space of sectional curvature equal to -1 . Consider rescaled Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} - 2g_{ij}.$$

As $R_{ij} = -\frac{1}{2} \delta_{ij} \Delta_\delta(u + f)$, this is equivalent to

$$(C.1) \quad \dot{u} = e^{-u-f} \Delta_\delta u + 2(e^{-u} - 1) = e^{-u} \Delta_h u + 2(e^{-u} - 1) = \Delta_g u + 2(e^{-u} - 1).$$

In contrast to Theorem 1.4, we do not have to assume that the eigenvalues (λ_i) of $g(0)$ with respect to h are close to one. This is similar to [7, Theorem A.1]. There, however, we had to assume that $\lambda_i(x, 0) \rightarrow 1$ for $|x| \rightarrow \infty$ in order to obtain convergence to \mathbb{R}^2 , see [7, Theorem A.2].

Theorem C.1. *Let $u_0 \in C^0(B)$ satisfy $\|u_0\|_{L^\infty} < \infty$. Then there exists $u \in C^\infty(B \times (0, \infty))$ solving (C.1) such that $u(\cdot, t) \rightarrow u_0$ in $C_{loc}^0(B)$ as $t \searrow 0$. As $t \rightarrow \infty$, $u(\cdot, t) \rightarrow 0$ exponentially in C^∞ w. r. t. the hyperbolic metric.*

If an arbitrary solution u is uniformly bounded for small times, we also get exponential convergence.

Proof. Assume $|u_0| \leq c_0$. Mollify and modify u_0 to u_0^i with $|u_0^i| \leq 2c_0$ and $u_0^i = 0$ near $\partial B_{1-\frac{1}{i}}(0)$ and $u_0^i = u_0$ on $B_{1-\frac{2}{i}}(0)$. We can construct solutions $u^i : B_{1-\frac{1}{i}}(0) \times [0, T_i)$ to C.1 with $u^i(\cdot, 0) = u_0$ on $B_{1-\frac{1}{i}}(0)$ and $u^i(\cdot, t) = 0$ on $\partial B_{1-\frac{1}{i}}(0)$ using the arguments presented in Chapter VI of [4]. These solutions remain bounded by $2c_0$ from the maximum principle. Hence, the arguments of Chapter VI of [4] imply that $T_i = \infty$.

Spatially constant barriers $b = b(t) = \log(1 + ae^{-2t})$, $a > -1$, solving (C.1) converge exponentially to 0 as $t \rightarrow \infty$. Hence the maximum principle applied to each u^i on $B_{1-\frac{1}{i}}(0)$ implies that the u_i remain uniformly bounded and go exponentially to zero (uniformly in i) as $t \rightarrow \infty$.

Now we address smooth convergence: Writing $l^i := e^{u^i}$ we obtain the evolution equation

$$\frac{\partial}{\partial t} l^i = (1/l^i) \Delta_h l^i - (1/(l^i)^2) |{}^h \nabla l^i|^2 + 2(1 - l^i).$$

We can assume without loss of generality, that $|l_i - 1| \leq \varepsilon$ for some small ε . The interior estimates of Lemma A.1 hold here as the equation for l^i has the same form as the equation studied in Theorem 4.3 of [9]. Hence, by taking a diagonal subsequence, we get a solution $l = e^u$ which approaches 1 exponentially. Interpolating between the C^0 -norm and C^k -norms and using Lemma A.1 again, we see that l approaches 1 in all C^k -norms exponentially. \square

To treat the question of uniqueness of such solutions we work in the unrescaled setting. Note that by Lemma A.4 this is equivalent to the rescaled equation. With respect to the hyperbolic metric h on \mathbb{H}^2 as a background metric, a solution $e^{u(p,t)} h$ to the Ricci flow satisfies

$$(C.2) \quad \dot{u} = e^{-u} \Delta_h u + 2e^{-u}.$$

We first prove a noncompact maximum principle.

Lemma C.2. *Let $v \in C^\infty(\mathbb{H}^2, [0, T])$ be a bounded solution to*

$$(C.3) \quad \dot{v} \leq a \Delta_h v + cv$$

with $a, c \in L^\infty(\mathbb{H}^2, [0, T])$, $a > 0$. If $v(\cdot, 0) \leq 0$ then $v(\cdot, t) \leq 0$ for all $t \in [0, T]$.

Proof. Pick a fixed point $p_0 \in \mathbb{H}^2$ and let $r(\cdot) := \text{dist}_h(\cdot, p_0)$. Then the function $\rho := \sqrt{r^2 + 1}$ is a smooth function on \mathbb{H}^2 with $\rho(p) \rightarrow \infty$ as $p \rightarrow \infty$ and

$$a \Delta_h \rho \leq C$$

for a constant $C > 0$. Let us first assume that v satisfies

$$\dot{v} \leq a \Delta_h v - c' v$$

with a function $c' \geq 0$. Then for any $\delta > 0$ the function

$$w := v - \delta \rho - 2\delta C t - \delta$$

satisfies at the first non-negative interior maximum

$$\dot{w} < a \Delta_h w.$$

Since $w(\cdot, t) \rightarrow -\infty$ as $p \rightarrow \infty$ an application of the maximum principle proves the estimate as $\delta \rightarrow 0$. In the general case let $|c(p, t)| \leq K$ and $v' := e^{-Kt} v$ which satisfies

$$\dot{v}' \leq a \Delta_h v' - (K - c)v'.$$

The previous estimate can be applied. \square

This gives us a uniqueness statement.

Lemma C.3. *Let $u, \tilde{u} \in C^\infty(\mathbb{H}^2, (0, T)) \cap C^0(\mathbb{H}^2, [0, T])$ be two bounded solutions of (C.2) s.t.*

$$u(\cdot, t) \rightarrow u_0 \text{ and } \tilde{u}(\cdot, t) \rightarrow u_0$$

uniformly as $t \rightarrow 0$ for some continuous function u_0 on \mathbb{H}^2 . Then $u \equiv \tilde{u}$.

Proof. Define for $\gamma > 0$

$$u_\gamma(p, t) := u(p, e^{-\gamma t}) + \gamma.$$

Then u_γ again solves (C.2) with initial values $u_0 + \gamma$. Since the initial values are attained uniformly we have $u_\gamma > \tilde{u}$ for a short time interval $[0, 2\delta]$, $\delta > 0$. By interior estimates as in Lemma A.1 the functions u_γ, \tilde{u} are bounded uniformly in C^∞ on time intervals $[\delta, T]$. Interpolating between the two solutions, we see that the difference satisfies an equation of the form (C.3), to which the noncompact maximum principle applies. Thus $u_\gamma > \tilde{u}$ for all $\gamma > 0$ and $\gamma \rightarrow 0$ gives the desired estimate. \square

If u_0 is uniformly continuous then also $u(\cdot, t)$ converges uniformly as $t \rightarrow 0$.

Lemma C.4. *Let $u \in C^\infty(\mathbb{H}^2, (0, T)) \cap C^0(\mathbb{H}^2, [0, T])$ be a bounded solution of (C.2) s.t. $u_0 := u(\cdot, 0)$ is uniformly continuous. Then $u(\cdot, t) \rightarrow u_0$ uniformly.*

Proof. Pick a point $p_0 \in \mathbb{H}^2$ and let $u_{p_0}(t)$ be the solution to (C.2), which is constant in space and has initial value $u(p_0)$. It is a direct computation to check that

$$\begin{aligned} \frac{\partial}{\partial t}(u - u_{p_0})^2 &= e^{-u} \Delta_h(u - u_{p_0})^2 - 2e^{-u} |\nabla^h u|^2 + 2(u - u_{p_0})(e^{-u} - e^{-u_{p_0}}) \\ &\leq e^{-u} \Delta_h(u - u_{p_0})^2 - 2e^{-u} |\nabla^h u|^2. \end{aligned}$$

By a similar argument as in Lemma A.2, but now for small radii, one obtains the desired closeness-estimate. \square

Corollary C.5. *Any bounded solution to (C.1) in $C^\infty(\mathbb{H}^2, (0, \infty)) \cap C^0(\mathbb{H}^2, [0, T])$ with initial data u_0 , which is uniformly continuous with respect to the hyperbolic metric, is unique. As $t \rightarrow \infty$, $u(\cdot, t) \rightarrow 0$ exponentially in C^∞ .*

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OLIVER SCHNÜRER: FACHBEREICH MATHEMATIK UND STATISTIK, UNIVERSITÄT KONSTANZ, 78457 KONSTANZ, GERMANY

E-mail address: `Oliver.Schnuerer@uni-konstanz.de`

FELIX SCHULZE: FREIE UNIVERSITÄT BERLIN, ARNIMALLEE 6, 14195 BERLIN, GERMANY

E-mail address: `Felix.Schulze@math.fu-berlin.de`

MILES SIMON: UNIVERSITÄT FREIBURG, ECKERSTRASSE 1, 79104 FREIBURG I. BR., GERMANY

E-mail address: `msimon@gmx.de`