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Tilman Irmscher

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Global Existence in Nonlinear Hyperbolic Thermoelasticity with Radial Symmetry

Tilman Irmscher[1]

Department of Mathematics and Statistics, University of Konstanz
78457 Konstanz, Germany

Abstract: In this paper we consider a nonlinear system of hyperbolic thermoelasticity in two or three dimensions with Dirichlet boundary conditions in the case of radial symmetry. We prove the global existence of small, smooth solutions and the exponential stability.

Keywords: nonlinear hyperbolic thermoelasticity, second sound, exponential stability, radial symmetry.

AMS subject classification: 74 F 05, 74 H 40

1 Introduction

The equations of thermoelasticity are used to model the behaviour of elastic and heat conductive media. Let $u = u(t, x)$, $\vartheta = \vartheta(t, x)$, and $q = q(t, x)$ ($t \geqslant 0$, $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ bounded) be the displacement vector, the temperature difference to a fixed reference temperature, and the heat flux, respectively, then the *linear* differential equations for (u, ϑ, q) are first

$$u_{tt} - \alpha\Delta u + \beta\nabla\vartheta = 0 \quad \text{in } [0,\infty)\times(0,L), \tag{1.1a}$$

$$\zeta\vartheta_t + \gamma \operatorname{div} q + \beta \operatorname{div} u_t = 0 \quad \text{in } [0,\infty)\times(0,L), \tag{1.1b}$$

where (1.1a) is an equation of motion and (1.1b) describes the conservation of energy. The positive coefficients α, β, ζ, γ depend on the material. For a physical derivation of (1.1) we refer to [2].

These two equations have to be completed by a heat equation. We use Cattaneo's law of heat propagation

$$\tau q_t + q + \kappa\nabla\vartheta = 0 \quad \text{in } [0,\infty)\times(0,L) \tag{1.2}$$

with positive constants κ, τ. The system (1.1) - (1.2) is purely hyperbolic, but slightly damped, and it models thermal disturbances as wave-like pulses propagating with finite speed, the so-called second sound. For a review of recent literature to the system of hyperbolic thermoelasticity we refer to [4].

[1]E-mail: tilman.irmscher@web.de

If we use FOURIER's law

$$q + \kappa \nabla \vartheta = 0 \quad \text{in } [0,\infty) \times (0,L), \tag{1.3}$$

instead of (1.2) we get the (hyperbolic-)parabolic system of classical thermoelasticity including the paradox of infinite propagation speed of heat pulses.

The system of *nonlinear* parabolic thermoelasticity with DIRICHLET boundary conditions in two or three space dimensions has been investigated in [9] in view of global existence of small, smooth solutions and exponential decay. Therein, particularly radial symmetry has been studied. As proved in [13], these results can be carried over to some other boundary conditions.

Tarabek [14] als well as Racke [11] then used CATTANEO'S law of heat conduction instead of the classical (i. e. FOURIER's) law and discussed the now purly hyperbolic system in the one-dimensional, nonlinear case. It is also well known that under certain conditions the linear hyperbolic system in two or three – actually in all – space dimensions is exponentially stable, cf. [11, 12]. For the multidimensional nonlinear hyperbolic system there are no comparable results on the global existence or exponential stability. This work shall close this gap for space dimensions $n = 2, 3$ in the radially symmetric case.

We do not want to give a derivation of the nonlinear equations. We rather refer to the mentioned papers and the cited literature therein. Then we want to consider the following nonlinear differential equations for (u, ϑ, q):

$$u_{i|t|t} - A_{ij}(\nabla u, \vartheta, q) u_{j|k|k} + B_{ij}(\nabla u, \vartheta, q)\vartheta_{|j} = 0 \quad \text{in} \quad [0,\infty) \times \Omega, \tag{1.4a}$$

$$c(\nabla u, \vartheta, q)\vartheta_{|t} + g(\nabla u, \vartheta, q) q_{i|i} + B_{ij}(\nabla u, \vartheta, q) u_{i|j|t} = 0 \quad \text{in} \quad [0,\infty) \times \Omega, \tag{1.4b}$$

$$T_{ij}(\nabla u, \vartheta) q_{j|t} + q_i + K_{ij}(\nabla u, \vartheta)\vartheta_{|j} = 0 \quad \text{in} \quad [0,\infty) \times \Omega, \tag{1.4c}$$

with the initial data

$$u(0) = u^0, \quad u_t(0) = u^1, \quad \vartheta(0) = \vartheta^0, \quad q(0) = q^0, \tag{1.5}$$

and the DIRICHLET boundary conditions

$$u|_{\partial\Omega} = \vartheta|_{\partial\Omega} = 0. \tag{1.6}$$

It is self-evident that (1.4a) and (1.4c) hold for all $i = 1, ..., n$. Also note that we use the EINSTEIN summation convention, i. e. repeated indices are implicitly summed over. This shortens for example the product of matricies A, B to $(AB)_{ij} = A_{ik}B_{kj}$. Finally, we denote the partial derivative $\partial_i(...)$, and $\partial_t(...) = (...)_t$ with $(...)_{|i}$, and $(...)_{|t}$, respectively.

Remark 1.1. For more generality, one would use $C_{ijkl}(\nabla u, \vartheta, q)u_{j|k|l}$ in (1.4a) instead of the LAPLACIAN $A_{ij}(\nabla u, \vartheta, q)u_{j|k|k}$. However, this restriction has turned out to be technically very helpfull in [9] as well as in this paper.

The appearing coefficients are subject to the following conditions:

Assumption 1.2. *Let $A, B, c, g, T,$ and K be smooth functions. Assume $A, T,$ and K to be symmetrical matrices and that there are positiv contants $A_0, c_0, g_0, T_0, K_0,$ and ϱ such that*

$$A(\zeta,\eta,\chi)\xi\cdot\xi \geqslant A_0|\xi|^2, \qquad c(\zeta,\eta,\chi) \geqslant c_0, \qquad T(\zeta,\eta)\xi\cdot\xi \geqslant T_0|\xi|^2, \tag{1.7a}$$

$$B(\zeta,\eta,\chi)\xi\cdot\xi \neq 0, \qquad g(\zeta,\eta,\chi) \geqslant g_0, \qquad K(\zeta,\eta)\xi\cdot\xi \geqslant K_0|\xi|^2 \tag{1.7b}$$

for all $\zeta \in \mathbb{R}^{n\times n}, \eta \in \mathbb{R}, \chi \in \mathbb{R}^n$ with $|\zeta|, |\eta|, |\chi| < \varrho$, and $\xi \in \mathbb{R}^n \setminus \{0\}$.
Furthermore, we want to regard the nonlinear system as a perturbation of the isotropic linear one, i. e. some constants $\alpha, \beta, \zeta, \gamma, \tau,$ and κ exist with

$$A(0,0,0) = \alpha E_n, \qquad c(0,0,0) = \zeta, \qquad T(0,0) = \tau E_n,$$
$$B(0,0,0) = \beta E_n, \qquad g(0,0,0) = \gamma, \qquad K(0,0) = \kappa E_n.$$

Hence, we can rewrite (1.4) to

$$u_{tt} - \alpha\Delta u + \beta\nabla\vartheta = F, \tag{1.8a}$$

$$\zeta\vartheta_t + \gamma \operatorname{div} q + \beta \operatorname{div} u_t = G, \tag{1.8b}$$

$$\tau q_t + q + \kappa\nabla\vartheta = H, \tag{1.8c}$$

with

$$F := \big(A(\nabla u, \vartheta, q) - A(0,0,0)\big)\Delta u - \big(B(\nabla u, \vartheta, q) - B(0,0,0)\big)\nabla\vartheta, \tag{1.9a}$$

$$G := -\big(c(\nabla u, \vartheta, q) - c(0,0,0)\big)\vartheta_t - \big(g(\nabla u, \vartheta, q) - g(0,0,0)\big)\operatorname{div} q - \operatorname{tr}\Big(\big(B(\nabla u, \vartheta, q) - B(0,0,0)\big)\nabla u_t\Big), \tag{1.9b}$$

$$H := -\big(\tau(\nabla u, \vartheta) - \tau(0,0)\big)q_t - \big(K(\nabla u, \vartheta) - K(0,0)\big)\nabla\vartheta. \tag{1.9c}$$

Finally, we introduce for $j \in \mathbb{N}_0$ according to (1.5) the notation

$$u^j := \partial_t^j u(0), \quad \vartheta^j := \partial_t^j \vartheta(0).$$

Already in the parabolic case it has been proved that curl-free data give the exponential stability, cf. [9]. Therefore, we will now turn over to the radially symmetric case which ensures the rotation to vanish. Thus we will succeed for the first time in proving the existence of global solutions of the nonlinear initial boundary value problem (1.4) – (1.6) in hyperbolic thermoelasticity.

But the at first view obvious way, i. e. carrying over the results in [9] directly to the multidimensional case, turned out to be too difficult. To overcome these difficulties we will rather use an appropriate combination of the techniques presented in [12], and [9], respectively.

2 Radial symmetry and local well-posedness

To begin with, we will characterize the term of *radial symmetry*:

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a* radially symmetric *domain, i. e. for all $x \in \Omega$ and $R \in O(n)$, the orthogonal group of $\mathbb{R}^n$, we get $Rx \in \Omega$.*

(i) *A function $f\colon \Omega \longrightarrow \mathbb{R}$ is called* radially symmetric*, if for all $R \in O(n)$ it holds*

$$f \circ R = f.$$

(ii) *A vector field $v\colon\Omega\longrightarrow\mathbb{R}^n$ is called* radially symmetric*, if for all $R \in O(n)$ it holds*

$$R^T \circ v \circ R = v.$$

(iii) *Let F, and V be a set of functions, and vector fields, respectively. Then*

$$\mathring{F} := \big\{ f \in F \colon f \text{ radially symmetric} \big\},$$
$$\mathring{V} := \big\{ v \in V \colon v \text{ radially symmetric} \big\}.$$

A subset $\Omega \subset \mathbb{R}^n$ is obviously radially symmetric, if and only if so is the appropriate characteristic function $\chi_\Omega\colon \mathbb{R}^n \longrightarrow \{0,1\}$. We get the following characterization (folklore, cf. [8], p. 64):

Lemma 2.2. *Let $I := \big\{ |x| \colon x \in \Omega \big\}$.*

(i) *A function $f\colon \Omega \longrightarrow \mathbb{R}$ is radially symmetric, if and only if there is a function $\varphi_f\colon I \longrightarrow \mathbb{R}$ with $f(x) = \varphi_f\big(|x|\big)$ for all $x \in \Omega$.*

(ii) *A vector field $v\colon \Omega \longrightarrow \mathbb{R}^n$ is radially symmetric, if and only if $v(0) = 0$ (provided that $0 \in \Omega$) and there is a function $\Phi_v\colon I \longrightarrow \mathbb{R}$ with $v(x) = \Phi_v\big(|x|\big)\frac{x}{|x|}$ for all $x \in \Omega \setminus \{0\}$.*

Proof. (i) Take $v \in \mathbb{R}^n$ with $|v| = 1$. The function $\varphi_f\colon I \longrightarrow \mathbb{R}$ is now declared by

$$\varphi_f(r) := f(rv).$$

For $x \in \Omega$ there is $R \in O(n)$ with $Rx = |x|v$, and we get as asserted

$$f(x) = f(Rx) = f\big(|x|v\big) = \varphi_f\big(|x|\big).$$

(ii) Let $x \in \Omega \setminus \{0\}$. Then choose $x^\perp \in \Omega$ with $x \cdot x^\perp = 0$, and it exists $R \in O(n)$ with $Rx = x$ and $Rx^\perp = -x^\perp$. It follows

$$v(x) \cdot x^\perp = R^T v(Rx) \cdot x^\perp = v(Rx) \cdot Rx^\perp = -v(x) \cdot x^\perp,$$

hence $v(x)\cdot x^\perp = 0$. Since $x^\perp$ is an arbitrary element of the orthogonal complement of $\operatorname{lin}_{\mathbb{R}}\{x\}$, a function $f\colon \Omega \longrightarrow \mathbb{R}$ with

$$v(x) = f(x)\frac{x}{|x|}$$

for $x \neq 0$ must exist. Thus $f(x) = v(x)\cdot \frac{x}{|x|}$ holds, and for all $R \in O(n)$ we get

$$f(Rx) = v(Rx)\cdot\frac{Rx}{|x|} = R^T v(Rx)\cdot\frac{x}{|x|} = v(x)\cdot\frac{x}{|x|} = f(x).$$

For this reason f is a radially symmetric function, and according to (i) we have $f(x) = \Phi_v\big(|x|\big)$.

Obviously, the inversion holds in both cases. □

To develope a theory of radially symmetric solutions of (1.4), the appearing coefficients must transform as follows:

Assumption 2.3. *For all $R \in O(n)$, $\zeta \in \mathbb{R}^{n\times n}$, $\eta \in \mathbb{R}$, and $\chi \in \mathbb{R}^n$ we have*

$$M(R^T\zeta R, \eta, R^T\chi) = R^T M(\zeta,\eta,\chi)R \quad \textit{for} \quad M \in \{A, B\}, \tag{2.1}$$

and

$$f(R^T\zeta R, \eta, R^T\chi) = f(\zeta,\eta,\chi) \quad \textit{for} \quad f \in \{c, g\}, \tag{2.2}$$

as well as

$$N(R^T\zeta R, \eta) = R^T N(\zeta,\eta)R \quad \textit{for} \quad N \in \{T, K\}. \tag{2.3}$$

This guarantees that the operators in (1.4) preserve the radial symmetry. For example let $v := A(\nabla u, \vartheta, q)\Delta u$. If u, ϑ, and q are radially symmetric, then so is v:

$$\begin{aligned} R^T v(Rx) &= R^T A\big((\nabla u)(Rx), \vartheta(Rx), q(Rx)\big) R R^T (\Delta u)(Rx) \\ &= A\big(R^T(\nabla u)(Rx)R, \vartheta(Rx), R^T q(Rx)\big) R^T(\Delta u)(Rx) \\ &= A(\nabla u(x), \vartheta(x), q(x))\Delta u(x) \\ &= v(x). \end{aligned}$$

Before formulating the local well-posedness theorem, we recall the following Sobolev embedding therorem which is important for the proof an can be found for example in [1]:

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a domain satisfying the cone condition. For $s \in \mathbb{N}_0$ with $s \geqslant [n/2] + 1$ the following embeddings hold:*

$$H^s(\Omega,\mathbb{R}) \hookrightarrow C_b^0(\Omega,\mathbb{R}) \quad \textit{and} \quad H^s(\Omega,\mathbb{R}^n) \hookrightarrow C_b^0(\Omega,\mathbb{R}^n).$$

Thus we need at least $s = 2$ for $n = 2, 3$. Even in the case of radial symmetry this cannot be improved in general. But if we consider a spherical shell, then we can deduce from Lemma 2.2 and the embedding theorem in one space dimension the following proposition for radially symmetric functions, and vector fields, respectively. It says that we get the same embedding theorem for a spherical shell in higher space dimensions as for a bounded interval:

Proposition 2.5 (Embedding theorem for the spherical shell). *For the spherical shell* $S := B(0, r, R) := \{x \in \mathbb{R}^n : r < |x| < R\}$, $0 < r < R$, *we get*

$$\mathring{H}^1(S, \mathbb{R}) \hookrightarrow \mathring{C}_b^0(S, \mathbb{R}) \quad \text{and} \quad \mathring{H}^1(S, \mathbb{R}^n) \hookrightarrow \mathring{C}_b^0(S, \mathbb{R}^n).$$

Using spherical coordinates we can now transform the multidimensional, radially symmetric problem to a one-dimensional one with additionally local dependent coefficients. Hence, we can directly transfer the local well-posedness theorem, which is given for $n = 1$ in [11], to the case of a spherical shell. Note that due to the radial symmetry the existence of $\operatorname{div} q$, for example, immediately gives the optimal regularity $q \in H^1(S, \mathbb{R}^n)$.

Theorem 2.6. *Let* $s \geqslant 3$ *and* $S := B(0, r, R) \subset \mathbb{R}^n$ *with* $0 < r < R$, $n \in \mathbb{N}$. *Assume the following compatibility conditions:*

$$u^k \in \mathring{H}^{s-k}(S, \mathbb{R}^n) \cap \mathring{H}_0^1(S, \mathbb{R}^n) \quad \text{for} \quad k = 0, 1, ..., s-1 \quad \text{and} \quad u^s \in \mathring{L}^2(S, \mathbb{R}^n),$$
$$\vartheta^l \in \mathring{H}^{s-1-l}(S, \mathbb{R}) \cap \mathring{H}_0^1(S, \mathbb{R}) \quad \text{for} \quad l = 0, ..., s-2 \quad \text{and} \quad \vartheta^{s-1} \in \mathring{L}^2(S, \mathbb{R}),$$

as well as $q^0 \in \mathring{H}^{s-1}(S, \mathbb{R}^n)$. *Then, for sufficiently small* $T > 0$, *the initial boundary value problem (1.4), (1.5), (1.6) has a unique solution* (u, ϑ, q) *on* $[0, T]$ *with*

$$u \in \bigcap_{k=0}^{s-1} C^k\big([0,T], \mathring{H}^{s-k}(S, \mathbb{R}^n) \cap \mathring{H}_0^1(S, \mathbb{R}^n)\big), \qquad \partial_t^s u \in C^0\big([0,T], \mathring{L}^2(S, \mathbb{R}^n)\big),$$
$$\vartheta \in \bigcap_{l=0}^{s-2} C^l\big([0,T], \mathring{H}^{s-1-l}(K, \mathbb{R}) \cap \mathring{H}_0^1(S, \mathbb{R})\big), \qquad \partial_t^{s-1} \vartheta \in C^0\big([0,T], \mathring{L}^2(S, \mathbb{R})\big),$$
$$q \in \bigcap_{m=0}^{s-2} C^m\big([0,T], \mathring{H}^{s-1-m}(S, \mathbb{R}^n)\big), \qquad \partial_t^{s-1} q \in C^0\big([0,T], \mathring{L}^2(S, \mathbb{R}^n)\big).$$

In addition to the one derivative from Proposition 2.5 two more are required to get a local existence result. If we don't want to restrict ourselves to spherical shells we may formulate for arbitrary, radially symmetric domains – in particular for a ball – the following theorem of local well-posedness:

Theorem 2.7. *Let* $s \geqslant [n/2] + 3$. *Furthermore, assume that the compatibility conditions in theorem 2.6 hold. Then, for sufficiently small* $T > 0$, *the initial boundary value problem (1.4), (1.5), (1.6) has a unique solution* (u, ϑ, q) *on* $[0, T]$ *with*

$$u \in \bigcap_{k=0}^{s} C^k\big([0,T], \mathring{H}^{s-k}(\Omega, \mathbb{R}^n)\big), \quad (\vartheta, q) \in \bigcap_{l=0}^{s-1} C^l\big([0,T], \mathring{H}^{s-1-l}(\Omega, \mathbb{R}) \times \mathring{H}^{s-1-l}(\Omega, \mathbb{R}^n)\big),$$

Thus the treatment of the two- ore threedimensional case requires at least $s = 4$. For the expansion to a global existence theorem of small solutions we will therefore take all the derivatives up to order four into account. Before beginning this we still need one more preliminary lemma.

In analogy to lemma 2.3 from [9] we verify the following

Lemma 2.8. *Let v with $v_{i|j} = v_{j|i}$ solve the equation of elasticity*

$$v_{i|t|t} - A_{ij} v_{j|k|k} = h_i \quad \textit{in} \quad [0,\infty) \times \Omega,$$
$$v|_{\partial\Omega} = 0 \quad \textit{für} \quad t \geqslant 0$$

with $A = A(t,x)$, $h = h(t,x)$. Then we get

$$\oint_{\partial\Omega} A_{ij} v_{j|l} v_{i|l} \,\mathrm{d}S = 2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_{i|t}\sigma_k v_{i|k}\,\mathrm{d}x + \int_\Omega v_{i|t} v_{i|t}\sigma_{k|k}\,\mathrm{d}x + 2\int_\Omega A_{ij} v_{j|l}\sigma_{k|l} v_{i|k}\,\mathrm{d}x$$
$$- \int_\Omega A_{ij} v_{j|l} v_{i|l}\sigma_{k|k}\,\mathrm{d}x - 2\int_\Omega h_i \sigma_k v_{i|k}\,\mathrm{d}x + R,$$

where $\sigma \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with $\sigma|_{\partial\Omega} = \nu$ is a smooth continuation of the normal into the interior, and

$$R := 2\int_\Omega A_{ij|l} v_{j|l}\sigma_k v_{i|k}\,\mathrm{d}x - \int_\Omega A_{ij|k} v_{j|l}\sigma_k v_{i|l}\,\mathrm{d}x.$$

Proof. Multiplying the equation by $\sigma_k v_{i|k}$ and integrating, we obtain

$$\int_\Omega v_{i|t|t}\sigma_k v_{i|k}\,\mathrm{d}x - \int_\Omega A_{ij} v_{i|l|l}\sigma_k v_{i|k}\,\mathrm{d}x = \int_\Omega h_i\sigma_k v_{i|k}\,\mathrm{d}x. \tag{2.4}$$

The first term can be written as

$$\int_\Omega v_{i|t|t}\sigma_k v_{i|k}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v_{i|t}\sigma_k v_{i|k}\,\mathrm{d}x + \frac{1}{2}\int_\Omega v_{i|t} v_{i|t}\sigma_{k|k}\,\mathrm{d}x. \tag{2.5}$$

For the second integral we get

$$\int_\Omega A_{ij} v_{j|l|l}\sigma_k v_{i|k}\,\mathrm{d}x = \oint_{\partial\Omega} A_{ij} v_{j|l}\sigma_k\sigma_l v_{i|k}\,\mathrm{d}S - \int_\Omega A_{ij} v_{j|l}\sigma_{k|l} v_{i|k}\,\mathrm{d}x$$
$$- \int_\Omega A_{ij} v_{j|l}\sigma_k v_{i|l|k}\,\mathrm{d}x - \int_\Omega A_{ij|l} v_{j|l}\sigma_k v_{i|k}\,\mathrm{d}x, \tag{2.6}$$

and after another partial integration it follows

$$\int_\Omega A_{ij} v_{j|l|l}\sigma_k v_{i|k}\,\mathrm{d}x = \oint_{\partial\Omega} A_{ij} v_{j|l}\sigma_k\sigma_l v_{i|k}\,\mathrm{d}S - \int_\Omega A_{ij} v_{j|l}\sigma_{k|l} v_{i|k}\,\mathrm{d}x - \oint_{\partial\Omega} A_{ij} v_{j|l}\sigma_k\sigma_k v_{i|l}\,\mathrm{d}S$$
$$+ \int_\Omega A_{ij} v_{j|l|k}\sigma_k v_{i|l}\,\mathrm{d}x + \int_\Omega A_{ij} v_{j|l}\sigma_{k|k} v_{i|l}\,\mathrm{d}x$$
$$- \int_\Omega A_{ij|l} v_{j|l}\sigma_k v_{i|k}\,\mathrm{d}x + \int_\Omega A_{ij|k} v_{j|l}\sigma_k v_{i|l}\,\mathrm{d}x. \tag{2.7}$$

Due to the assumed symmetry of A

$$A_{ij} v_{j|l|k}\sigma_k v_{i|l} = A_{ji} v_{i|l}\sigma_k v_{j|l|k} = A_{ij} v_{j|l}\sigma_k v_{i|l|k}$$

holds. Note, that for the last equality the indizes of summation have been renamed. Therefore we obtain from (2.6) and (2.7)

$$\int_\Omega A_{ij} v_{j|l|l} \sigma_k v_{i|k}\,\mathrm{d}x = 2\oint_{\partial\Omega} A_{ij} v_{j|l} \sigma_k \sigma_l v_{i|k}\,\mathrm{d}S - \oint_{\partial\Omega} A_{ij} v_{j|l} v_{i|l}\,\mathrm{d}S - \int_\Omega A_{ij} v_{j|l|l} \sigma_k v_{i|k}\,\mathrm{d}x \\ - 2\int_\Omega A_{ij} v_{j|l} \sigma_{k|l} v_{i|k}\,\mathrm{d}x + \int_\Omega A_{ij} v_{j|l} \sigma_{k|k} v_{i|l}\,\mathrm{d}x - R. \tag{2.8}$$

From $v|_{\partial\Omega} = 0$ we conclude

$$\sigma_k \sigma_l v_{i|k} = \sigma_k \sigma_k v_{i|l} = v_{i|l}. \tag{2.9}$$

The combination of (2.4), (2.5), (2.8), and (2.9) yields the assertion. □

3 Global existence and exponential stability

In the following we assume Ω and the initial data to be radially symmetric. Let (2.1), (2.2), and (2.3) be satisfied. Furthermore we make the following assumption:

Assumption 3.1. *For all $\zeta \in \mathbb{R}^{n\times n}$ and $\eta \in \mathbb{R}$ it holds*

$$K(\zeta,\eta)T(\zeta,\eta) = T(\zeta,\eta)K(\zeta,\eta). \tag{3.1}$$

Note that in the standard case, i. e. $T_{ij} = \tau\delta_{ij}$, this does not mean any restriction.

Now we can prove that for sufficiently small initial date the local, radially symmetric solution according to Theorem 2.7 is a global one:

Theorem 3.2. *There is $\varepsilon > 0$ such that if*

$$\|u^0\|_{H^4}^2 + \|u^1\|_{H^3}^2 + \|\vartheta^0\|_{H^3}^2 + \|q^0\|_{H^3}^2 < \varepsilon$$

then (1.4) has an unique solution

$$u \in \bigcap_{k=0}^{4} C^k\big([0,\infty), \mathring{H}^{4-k}(\Omega,\mathbb{R}^n)\big), \quad (\vartheta, q) \in \bigcap_{l=0}^{3} C^l\big([0,\infty), \mathring{H}^{3-l}(\Omega,\mathbb{R}) \times \mathring{H}^{3-l}(\Omega,\mathbb{R}^n)\big).$$

Moreover, the system is exponentially stable, i. e. there are constants $C, d > 0$ such that for all $t \geqslant 0$

$$\Lambda(t) := \sum_{k=0}^{4} \big\|(\partial_t,\nabla)^k u(t,\,\cdot\,)\big\|^2 + \sum_{l=0}^{3} \big\|(\partial_t,\nabla)^l (\vartheta,q)(t,\,\cdot\,)\big\|^2 \leqslant Ce^{-dt}\Lambda(0).$$

Proof. The following proof combines the techniques from [11] and [12]: According to the first paper we will use nonlinear multipliers, while the structure is more similar to that one in the latter work. In essence, we will deduce an *a priori* estimate which simultaneously gives a uniform bound on the highest norms of the local solution allowing a continuation-argument, as well as shows the exponential decay.

First we define

$$
\begin{aligned}
E_1(t) &:= E[u,\vartheta,q](t) &\text{(3.2a)}\\
&:= \frac{1}{2}\int_\Omega \big(u_{i|t}u_{i|t} + u_{i|k}A_{kl}u_{l|i} + c\vartheta^2 + q_i K^{\#}_{il} T_{lm} q_m\big)(t,x)\,\mathrm{d}x, &\text{(3.2b)}\\
E_2(t) &:= E[u_{|t},\vartheta_{|t},q_{|t}](t) &\text{(3.2c)}
\end{aligned}
$$

with $K^{\#} := gK^{-1}$, as well as

$$E(t) := E_1(t) + E_2(t). \tag{3.3}$$

According to the assumptions $K^{\#}T$ is symmetric and positive definite, and from the equations (1.4) – (1.6)

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1 = -\int_\Omega q_i K^{\#}_{il} q_l \,\mathrm{d}x + R_\mathrm{c} \tag{3.4}$$

follows, where

$$
\begin{aligned}
R_\mathrm{c} = \int_\Omega \Big(&\tfrac{1}{2}\big(u_{i|k}A_{kl|t}u_{l|i} + c_{|t}\vartheta^2 + q_i(K^{\#}_{il}T_{lm})_{|t}q_m\big) &\text{(3.5)}\\
&- u_{i|k}A_{kl|i}u_{l|t} + \vartheta B_{ik|k}u_{i|t} + \vartheta g_{|i}q_i\Big)\,\mathrm{d}x. &\text{(3.6)}
\end{aligned}
$$

The nonlinear term R_c contains only summands of at least third order in $\sqrt{\Lambda}$ as we will prove in the subsequent section. In the following we will refer to all these arising perturbations of cubic type as R_c.

Differentiating (1.4) with respect to t we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2 = -\int_\Omega q_{i|t} K^{\#}_{il} q_{l|t} \,\mathrm{d}x + R_\mathrm{c}. \tag{3.7}$$

The equation (1.4c) gives

$$\|\nabla\vartheta\|^2 \leqslant C_0\big(\|q\|^2 + \|q_{|t}\|^2\big), \tag{3.8}$$

where C_0 – as well as C_i, $i \in \mathbb{N}$, throughout the rest of the proof – is a positive constant independent of (u,ϑ,q).

We multiplicate (1.8a) by $\frac{1}{\alpha}u_{i|k|k} = \frac{1}{\alpha}u_{k|k|i}$ and get after integration

$$
\begin{aligned}
\|\Delta u\|^2 &= \int_\Omega (\tfrac{1}{\alpha}u_{i|t|t}u_{k|k|i} + \tfrac{\beta}{\alpha}\vartheta_{|k}u_{i|j|j})\,\mathrm{d}x + R_\mathrm{c}\\
&\leqslant \int_\Omega -\tfrac{1}{\alpha}u_{i|i|t|t}u_{k|k}\,\mathrm{d}x + \frac{1}{3}\|\Delta u\|^2 + \frac{3}{4}\frac{\beta^2}{\alpha^2}\|\nabla\vartheta\|^2 + R_\mathrm{c}\\
&\leqslant \int_\Omega \big((-\tfrac{1}{\alpha}u_{i|i|t}u_{k|k})_{|t} + \tfrac{1}{\alpha}u_{i|i|t}u_{k|k|t}\big)\,\mathrm{d}x + \frac{1}{3}\|\Delta u\|^2 + \frac{3}{4}\frac{\beta^2}{\alpha^2}\|\nabla\vartheta\|^2 + R_\mathrm{c},
\end{aligned}
$$

hence

$$\frac{2}{3}\|\Delta u\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{1}{\alpha}\mathrm{div}\, u_{|t}\mathrm{div}\, u\,\mathrm{d}x \leqslant \int_\Omega \tfrac{1}{\alpha}(\mathrm{div}\, u_{|t})^2\,\mathrm{d}x + \frac{3}{4}\frac{\beta^2}{\alpha^2}\|\nabla\vartheta\|^2 + R_\mathrm{c}. \tag{3.9}$$

Multiplication of (1.8b) by $\frac{3}{\alpha\beta}u_{k|k|t}$ yields

$$\begin{aligned}\int_\Omega \tfrac{3}{\alpha}(\mathrm{div}\, u_{|t})^2\,\mathrm{d}x &= -\int_\Omega (\tfrac{3\gamma}{\alpha\beta}q_{i|i}u_{k|k|t} + \tfrac{3\zeta}{\alpha\beta}\vartheta_{|t}u_{k|k|t})\,\mathrm{d}x + R_\mathrm{c}\\ &= \int_\Omega \tfrac{3\gamma}{\alpha\beta}q_i u_{i|k|k|t}\,\mathrm{d}x - \oint_{\partial\Omega}\tfrac{3\gamma}{\alpha\beta}\nu_i q_i u_{k|k|t}\,\mathrm{d}S + \int_\Omega \tfrac{3\zeta}{\alpha\beta}\vartheta_{|k|t}u_{k|t}\,\mathrm{d}x + R_\mathrm{c}\\ &= \int_\Omega \big((\tfrac{3\gamma}{\alpha\beta}q_i u_{i|k|k})_{|t} - \tfrac{3\gamma}{\alpha\beta}q_{i|t}u_{i|k|k}\big)\,\mathrm{d}x\\ &\quad + \int_\Omega \big((\tfrac{3\zeta}{\alpha\beta}\vartheta_{|k}u_{k|t})_{|t} - \tfrac{3\zeta}{\alpha\beta}\vartheta_{|k}u_{k|t|t}\big)\,\mathrm{d}x - \oint_{\partial\Omega}\tfrac{3\gamma}{\alpha\beta}\nu_i q_i u_{k|k|t}\,\mathrm{d}S + R_\mathrm{c},\end{aligned}$$

thus from (1.4a) follows

$$\begin{aligned}\int_\Omega \tfrac{3}{\alpha}(\mathrm{div}\, u_{|t})^2\,\mathrm{d}x \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}G_1 + \frac{1}{6}\|\Delta u\|^2 + C_1\|q_{|t}\|^2 + C_2\|\nabla\vartheta\|^2\\ - \oint_{\partial\Omega}\tfrac{3\gamma}{\alpha\beta}\nu_i q_i u_{k|k|t}\,\mathrm{d}S + R_\mathrm{c}\end{aligned} \tag{3.10}$$

with the energy functional

$$G_1(t) := -\int_\Omega (\tfrac{3\gamma}{\alpha\beta}q_i u_{i|k|k} + \tfrac{3\zeta}{\alpha\beta}\vartheta_{|k}u_{k|t})(t,x)\,\mathrm{d}x.$$

From (3.9) and (3.10) we deduce

$$\begin{aligned}\int_\Omega \tfrac{2}{\alpha}(\mathrm{div}\, u_{|t})^2\,\mathrm{d}x + \frac{1}{2}\|\Delta u\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}G_2\\ \leqslant C_3\|\nabla\vartheta\|^2 + C_4\|q_{|t}\|^2 - \oint_{\partial\Omega}\tfrac{3\gamma}{\alpha\beta}\nu_i q_i u_{k|k|t}\,\mathrm{d}S + R_\mathrm{c}\end{aligned} \tag{3.11}$$

with

$$G_2(t) := G_1(t) + \int_\Omega \tfrac{1}{\alpha}u_{i|i|t}u_{k|k}(t,x)\,\mathrm{d}x. \tag{3.12}$$

Poincaré's inequality for $u_{|t}$ and ϑ, as well as (1.4a) give

$$\|u_{|t|t}\|^2 + \|u_{|t}\|^2 + \|\vartheta\|^2 \leqslant C_5\big(\|\Delta u\|^2 + \|\nabla\vartheta\|^2 + \|\nabla u_{|t}\|^2\big). \tag{3.13}$$

From (1.8a) follows (multiply by u)

$$-\int_\Omega \alpha\Delta u\cdot u\,\mathrm{d}x = -\int_\Omega u_{|t|t}\cdot u\,\mathrm{d}x + \int_\Omega \beta\nabla\vartheta\cdot u\,\mathrm{d}x + R_\mathrm{c},$$

hence Poincaré's inequalitity for u gives

$$\frac{1}{2}\int_\Omega \alpha|\nabla u|^2\,\mathrm{d}x \leqslant C_6\big(\|u_{|t|t}\|^2 + \|\nabla\vartheta\|^2\big) + R_\mathrm{c}. \tag{3.14}$$

Multiplying (1.8b) by $\frac{1}{\zeta}\vartheta_{|t}$ we easily can see

$$\begin{aligned}\|\vartheta_{|t}\|^2 &\leqslant \int_\Omega \tfrac{\gamma}{\zeta} q_i \vartheta_{|i|t}\,\mathrm{d}x + C_7\|\mathrm{div}\,u_{|t}\|^2 + \frac{1}{2}\|\vartheta_{|t}\|^2 + R_\mathrm{c} \\ &= \int_\Omega \big((\tfrac{\gamma}{\zeta} q_i \vartheta_{|i})_{|t} - \tfrac{\gamma}{\zeta} q_{i|t}\vartheta_{|i}\big)\,\mathrm{d}x + C_8\|\mathrm{div}\,u_{|t}\|^2 + \frac{1}{2}\|\vartheta_{|t}\|^2 + R_\mathrm{c},\end{aligned}$$

and for arbitrary $\varepsilon_1 > 0$ we get

$$\varepsilon_1\|\vartheta_{|t}\|^2 - 2\varepsilon_1\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tfrac{\gamma}{\zeta} q_i\vartheta_{|i}\,\mathrm{d}x \leqslant \varepsilon_1 C_9\|q_{|t}\|^2 + \varepsilon_1 C_{10}\|\nabla\vartheta\|^2 + \varepsilon_1 C_{11}\|\mathrm{div}\,u_{|t}\|^2 + R_\mathrm{c}. \tag{3.15}$$

For some $\varepsilon_2 > 0$ the boundary term appearing in (3.11) can be eastimated as follows:

$$\left|\oint_{\partial\Omega} \tfrac{3\gamma}{\alpha\beta}\nu_i q_i u_{k|k|t}\,\mathrm{d}S\right| \leqslant \frac{C_{12}}{\varepsilon_2}\|\nu\cdot q\|^2_{\partial\Omega} + \varepsilon_2\|\nabla u_{|t}\|^2_{\partial\Omega} \tag{3.16}$$

Now let $\sigma \in C^1(\overline{\Omega}, \mathbb{R}^n)$ be a smooth continuation of the normal into the interior (cf. lemma 2.8). Then the multiplication of (1.4b) by $\sigma_k\vartheta_{|k|t}$ gives

$$\begin{aligned}0 &= -\int_\Omega c\vartheta_{|t}\sigma_k\vartheta_{|k|t}\,\mathrm{d}x - \int_\Omega g q_{i|i}\sigma_k\vartheta_{|k|t}\,\mathrm{d}x - \int_\Omega B_{ij}u_{i|j|t}\sigma_k\vartheta_{|k|t}\,\mathrm{d}x \\ &= -\frac{1}{2}\oint_{\partial\Omega} c\vartheta_{|t}\vartheta_{|t}\,\mathrm{d}S + \frac{1}{2}\int_\Omega c\vartheta_{|t}\sigma_{k|k}\vartheta_{|t}\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g q_{i|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x \\ &\quad + \int_\Omega g q_{i|t|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x - \int_\Omega B_{ij}u_{i|j|t}\sigma_k\vartheta_{|k|t}\,\mathrm{d}x + R_\mathrm{c},\end{aligned} \tag{3.17}$$

thus with (1.4c)

$$\begin{aligned}0 = &-\frac{1}{2}\oint_{\partial\Omega} c\vartheta_{|t}\vartheta_{|t}\,\mathrm{d}S + \frac{1}{2}\int_\Omega c\vartheta_{|t}\sigma_{k|k}\vartheta_{|t}\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g q_{i|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x \\ &- \int_\Omega gT^{-1}_{ij}q_{j|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x - \int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x - \int_\Omega B_{ij}u_{i|j|t}\sigma_k\vartheta_{|k|t}\,\mathrm{d}x + R_\mathrm{c}.\end{aligned} \tag{3.18}$$

For we have $\sigma_i\vartheta_{|k} = \sigma_k\vartheta_{|i}$ on $\partial\Omega$ as a result of the radial symmetry of ϑ, and due to the symmetry of $T^{-1}K$, the following equality holds:

$$\begin{aligned}\int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x &= \oint_{\partial\Omega} gT^{-1}_{ij}K_{jl}\vartheta_{|l}\sigma_i\sigma_k\vartheta_{|k}\,\mathrm{d}S - \int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l}\sigma_{k|i}\vartheta_{|k}\,\mathrm{d}x \\ &\quad - \int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l}\sigma_k\vartheta_{|i|k}\,\mathrm{d}x + R_\mathrm{c} \\ &= \oint_{\partial\Omega} gT^{-1}_{ij}K_{jl}\vartheta_{|l}\vartheta_{|i}\,\mathrm{d}S - \int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l}\sigma_{k|i}\vartheta_{|k}\,\mathrm{d}x \\ &\quad - \frac{1}{2}\oint_{\partial\Omega} gT^{-1}_{ij}K_{jl}\vartheta_{|l}\vartheta_{|i}\,\mathrm{d}S + \frac{1}{2}\int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l}\sigma_{k|k}\vartheta_{|i}\,\mathrm{d}x + R_\mathrm{c}.\end{aligned}$$

This gives in combination with (3.18)

$$\begin{aligned}0 = &\oint_{\partial\Omega} c\vartheta_{|t}\vartheta_{|t}\,\mathrm{d}S - \int_\Omega c\vartheta_{|t}\sigma_{k|k}\vartheta_{|t}\,\mathrm{d}x + 2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega g q_{i|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x \\ &+ 2\int_\Omega gT^{-1}_{ij}q_{j|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x + \oint_{\partial\Omega} gT^{-1}_{ij}K_{jl}\vartheta_{|l}\vartheta_{|i}\,\mathrm{d}S - 2\int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l}\sigma_{k|i}\vartheta_{|k}\,\mathrm{d}x \\ &+ \int_\Omega gT^{-1}_{ij}K_{jl}\vartheta_{|l}\sigma_{k|k}\vartheta_{|i}\,\mathrm{d}x + 2\int_\Omega B_{ij}u_{i|j|t}\sigma_k\vartheta_{|k|t}\,\mathrm{d}x + R_\mathrm{c}.\end{aligned} \tag{3.19}$$

Defining

$$
\begin{aligned}
v_i &:= u_{i|t},\\
h_i &:= -B_{ij}\vartheta_{|j|t} - B_{ij|t}\vartheta_{|j} + A_{ij|t}u_{j|l|l},
\end{aligned}
$$

the time-differentiated equality (1.4a) is equivalent to

$$
\begin{aligned}
v_{i|t|t} - A_{ij}v_{j|l|l} &= h_i,\\
v|_{\partial\Omega} &= 0,
\end{aligned}
$$

and lemma 2.8 yields

$$
\begin{aligned}
\oint_{\partial\Omega} A_{ij}u_{j|l|t}u_{i|l|t}\mathrm{d}S = 2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u_{i|t|t}\sigma_k u_{i|k|t}\mathrm{d}x + \int_\Omega |u_{|t|t}|^2\sigma_{k|k}\mathrm{d}x + 2\int_\Omega A_{ij}u_{j|l|t}\sigma_{k|l}u_{i|k|t}\mathrm{d}x\\
- \int_\Omega A_{ij}u_{j|l|t}\sigma_{k|k}u_{i|l|t}\,\mathrm{d}x + 2\int_\Omega B_{ij}\vartheta_{|j|t}\sigma_k u_{i|k|t}\,\mathrm{d}x + R_\mathrm{c}. \qquad (3.20)
\end{aligned}
$$

By partial integration we get

$$
\begin{aligned}
\int_\Omega B_{ij}\vartheta_{|j|t}\sigma_k u_{i|k|t}\,\mathrm{d}x &= -\int_\Omega B_{ij}\vartheta_{|t}\sigma_{k|j}u_{i|k|t}\,\mathrm{d}x - \int_\Omega B_{ij}\vartheta_{|t}\sigma_k u_{i|j|k|t}\,\mathrm{d}x + R_\mathrm{c}\\
&= -\int_\Omega B_{ij}\vartheta_{|t}\sigma_{k|j}u_{i|k|t}\,\mathrm{d}x + \int_\Omega B_{ij}\vartheta_{|k|t}\sigma_k u_{i|j|t}\,\mathrm{d}x\\
&\quad + \int_\Omega B_{ij}\vartheta_{|t}\sigma_{k|k}u_{i|j|t}\,\mathrm{d}x + R_\mathrm{c},
\end{aligned}
$$

thus

$$
\int_\Omega B_{ij}\vartheta_{|j|t}\sigma_k u_{i|k|t}\,\mathrm{d}x = \int_\Omega B_{ij}u_{i|j|t}\sigma_k\vartheta_{|k|t}\,\mathrm{d}x + \int_\Omega B_{ij}\vartheta_{|t}(\sigma_{k|k}u_{i|j|t} - \sigma_{k|j}u_{i|k|t})\,\mathrm{d}x. \qquad (3.21)
$$

From (1.7) and (3.19) – (3.21) follows

$$
\begin{aligned}
&\oint_{\partial\Omega} c(\vartheta_{|t})^2\,\mathrm{d}S + \oint_{\partial\Omega} gT^{-1}_{ij}K_{jl}\vartheta_{|l}\vartheta_{|i}\,\mathrm{d}S + A_0\|\nabla u_{|t}\|^2_{\partial\Omega}\\
&+ \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega 2(gq_{i|i}\sigma_k\vartheta_{|k} - u_{i|t|t}\sigma_k u_{i|k|t})\,\mathrm{d}x\\
&\qquad\leqslant C_{13}\big(\|q\|^2 + \|\vartheta_{|t}\|^2 + \|\nabla\vartheta\|^2 + \|\nabla u_{|t}\|^2 + \|\Delta u\|^2\big) + R_\mathrm{c}. \qquad (3.22)
\end{aligned}
$$

Thereto one has to note that (1.4b) gives

$$
\begin{aligned}
\|\mathrm{div}\, q\|^2 &= \|g^{-1}c\vartheta_{|t} + g^{-1}B\cdot\nabla u_{|t}\|^2\\
&\leqslant C_{14}\big(\|\vartheta_{|t}\|^2 + \|\nabla u_{|t}\|^2\big),
\end{aligned}
$$

and that theorem 8.6 from [10] can be applied in the case of radial symmetry, i. e. there is a positive constant C with $\|\nabla q\| \leqslant C\big(\|\mathrm{div}\, q\| + \|q\|\big)$, hence

$$
\left|\int_\Omega gT^{-1}_{ij}q_{j|i}\sigma_k\vartheta_{|k}\,\mathrm{d}x\right| \leqslant C_{15}\big(\|q\|^2 + \|\vartheta_{|t}\|^2 + \|\nabla u_{|t}\|^2 + \|\nabla\vartheta\|^2\big).
$$

Multiplication of (1.4b) by $\sigma_k q_k$ yields

$$\int_\Omega c\vartheta_{|t}\sigma_k q_k \,\mathrm{d}x + \int_\Omega g q_{i|i}\sigma_k q_k \,\mathrm{d}x + \int_\Omega B_{ij} u_{i|j|t}\sigma_k q_k \,\mathrm{d}x = 0. \tag{3.23}$$

We will treat these three terms separately. For $\varepsilon_3 > 0$ we estimate

$$\left|\int_\Omega c\vartheta_{|t}\sigma_k q_k \,\mathrm{d}x\right| \leqslant \varepsilon_3 \|\vartheta_{|t}\|^2 + \frac{C_{16}}{\varepsilon_3}\|q\|^2 \tag{3.24}$$

and

$$\left|\int_\Omega B_{ij} u_{i|j|t}\sigma_k q_k \,\mathrm{d}x\right| \leqslant \varepsilon_3 \|\nabla u_{|t}\|^2 + \frac{C_{17}}{\varepsilon_3}\|q\|^2. \tag{3.25}$$

It is easy to verify

$$\begin{aligned} \int_\Omega g q_{i|i}\sigma_k q_k \,\mathrm{d}x &= \oint_{\partial\Omega} g\sigma_i q_i \sigma_k q_k \,\mathrm{d}S - \int_\Omega g q_i \sigma_{k|i} q_k \,\mathrm{d}x - \int_\Omega g q_i \sigma_k q_{i|k} \,\mathrm{d}x + R_\mathrm{c} \\ &= \frac{1}{2}\oint_{\partial\Omega} g q_i q_i \,\mathrm{d}S - \int_\Omega g q_i \sigma_{k|i} q_k \,\mathrm{d}x + \frac{1}{2}\int_\Omega g q_i \sigma_{k|k} q_i \,\mathrm{d}x + R_\mathrm{c}, \end{aligned} \tag{3.26}$$

(note $q_i\sigma_k = q_k\sigma_i$ and $q_{k|i} = q_{i|k}$ because of the symmetry), hence the combination of the equations (1.7), and (3.23) – (3.26) gives

$$g_0\|q\|^2_{\partial\Omega} \leqslant \varepsilon_3\big(\|\vartheta_{|t}\|^2 + \|\nabla u_{|t}\|^2\big) + \frac{C_{18}}{\varepsilon_3}\|q\|^2 + R_\mathrm{c}. \tag{3.27}$$

Finally, for sufficiently small ε_1, ε_2, and ε_3, we can deduce from (3.8), (3.11), (3.15), (3.16), (3.22), and (3.27)

$$\frac{1}{\alpha}\|\nabla u_{|t}\|^2 + \frac{1}{4}\|\Delta u\|^2 + C_{19}\|\vartheta_{|t}\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}H \leqslant C_{20}\big(\|q\|^2 + \|q_{|t}\|^2\big) + R_\mathrm{c} \tag{3.28}$$

with

$$H(t) := G_2(t) + \frac{2\varepsilon_2}{A_0}\int_\Omega (g q_{i|i}\sigma_k\vartheta_{|k} - u_{i|t|t}\sigma_k u_{i|k|t})\,\mathrm{d}x - 2\varepsilon_1\int_\Omega \tfrac{g}{c} q_i\vartheta_i \,\mathrm{d}x. \tag{3.29}$$

An appropriate Lyapunov functional is given by

$$\tilde{\lambda}[u,\vartheta,q](t) := \frac{1}{\varepsilon_4}E(t) + H(t)$$

with some $\varepsilon_4 > 0$, because the combination of (3.7), (3.13), (3.14), and (3.28) yields (choose ε_4 sufficiently small)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\lambda}[u,\vartheta,q](t) \leqslant -C_{21}E(t) + R_\mathrm{c}. \tag{3.30}$$

By construction $\tilde{\lambda}[u,\vartheta,q]$ and E are equivalent, i. e. for ε_4 small enough there are constants C_{22} and C_{23} such that

$$C_{22}E(t) \leqslant \tilde{\lambda}[u,\vartheta,q](t) \leqslant C_{23}E(t) \tag{3.31}$$

for all $t \geqslant 0$. Therefore, (3.30) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\lambda}[u,\vartheta,q](t) \leqslant -C_{24}\tilde{\lambda}[u,\vartheta,q](t) + C_{25}\Lambda^{3/2}(t). \tag{3.32}$$

Repeating all the calculations also for higher time derivatives, we get for $k = 1, 2$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\lambda}[\partial_t^k u, \partial_t^k \vartheta, \partial_t^k q](t) \leqslant -C_{24}\tilde{\lambda}[\partial_t^k u, \partial_t^k \vartheta, \partial_t^k q](t) + C_{26}\Lambda^{3/2}(t), \tag{3.33}$$

since all appearing perturbation terms are cubic in $\sqrt{\Lambda}$. Now let

$$\tilde{\Lambda}(t) := \sum_{k=0}^{2} \tilde{\lambda}[\partial_t^k u, \partial_t^k \vartheta, \partial_t^k q](t),$$

then

$$\tilde{\Lambda}(t) \leqslant C_{27}\Lambda(t). \tag{3.34}$$

Differentiating (1.4) adequately many times with respect to t one can see that there are C_{28} and C_{29} such that

$$\Lambda(t) \leqslant C_{28}\tilde{\Lambda}(t) + C_{29}\Lambda^{3/2}(t). \tag{3.35}$$

(Thereto one may also consider the argumentation in the following section about the cubic terms.)

Now we make a first *a priori* assumption:

$$\Lambda(0) < \left(\frac{1}{2C_{29}}\right)^2. \tag{3.36}$$

Due to the continuity of Λ there is $t_0 > 0$ such that

$$\Lambda(t) \leqslant \left(\frac{1}{2C_{29}}\right)^2$$

for all $t \in [0, t_0]$, hence

$$C_{29}\Lambda^{3/2}(t) \leqslant \frac{1}{2}\Lambda(t),$$

and together with (3.35) this gives

$$\Lambda(t) \leqslant 2C_{28}\tilde{\Lambda}(t). \tag{3.37}$$

Noting (3.34) we get the equivalence

$$\frac{1}{2C_{28}}\Lambda(t) \leqslant \tilde{\Lambda}(t) \leqslant C_{27}\Lambda(t) \tag{3.38}$$

for all $t \in [0, t_0]$. All in all, from (3.32), (3.33), and (3.38)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Lambda}(t) \leqslant -C_{30}\tilde{\Lambda}(t) + C_{31}\tilde{\Lambda}^{3/2}(t) \tag{3.39}$$

follows for the mentioned time-interval.
Secondary we assume now

$$\tilde{\Lambda}(0) < \left(\frac{C_{30}}{2C_{31}}\right)^2.$$

Then there is $t_1 \in (0, t_0]$ with

$$\tilde{\Lambda}(t) \leqslant \left(\frac{C_{30}}{2C_{31}}\right)^2$$

for all $t \in [0, t_1]$, and in this interval we conclude from (3.39)

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Lambda}(t) \leqslant -\frac{C_{30}}{2}\tilde{\Lambda}(t),$$

and so we get there the estimate

$$\tilde{\Lambda}(t) \leqslant e^{-\frac{C_{30}}{2}t}\tilde{\Lambda}(0). \tag{3.40}$$

In particular one has due to (3.37) for $t \in [0, t_1]$

$$\Lambda(t) \leqslant 2C_{28}e^{-\frac{C_{30}}{2}t}\tilde{\Lambda}(0).$$

Let us now make the third *a priori* assumption

$$\tilde{\Lambda}(0) < \frac{1}{4C_{28}}\left(\frac{1}{2C_{29}}\right)^2, \tag{3.41}$$

and

$$\Lambda(t) < \frac{1}{2}\left(\frac{1}{2C_{29}}\right)^2 \tag{3.42}$$

follows for $t \in [0, t_1]$. Note that since we have (3.38), the condition (3.41) implies (3.36). Now we can proceed at $t = t_1$ with the same arguments and finally obtain for all $t \geqslant 0$ the inequality (3.42). Thus, (3.38) as well as (3.40) hold for all times, and we get for $t \geqslant 0$

$$\Lambda(t) \leqslant 2C_{27}C_{28}e^{-\frac{C_{30}}{2}t}\Lambda(0),$$

if $\Lambda(0) < \varepsilon_0$ where

$$\varepsilon_0 := \min\left\{\frac{1}{C_{27}}\left(\frac{C_{30}}{2C_{31}}\right)^2, \frac{1}{4C_{27}C_{28}}\left(\frac{1}{2C_{29}}\right)^2\right\}.$$

Finally note that there is $\varepsilon > 0$ such that $\Lambda(0) < \varepsilon_0$ as far as

$$\|u^0\|_{H^4}^2 + \|u^1\|_{H^3}^2 + \|\vartheta^0\|_{H^3}^2 + \|q^0\|_{H^3}^2 < \varepsilon.$$

To conclude the proof it remains to show that the appearing terms R_{c} are at least cubic.

4 Cubic terms

In addition to the embedding Theorem 2.4 we still need one more, cf. [1]:

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the cone property. Then the following embeddings hold:*

$$H^1(\Omega,\mathbb{R}) \hookrightarrow L^4(\Omega,\mathbb{R}) \quad \textit{and} \quad H^1(\Omega,\mathbb{R}^n) \hookrightarrow L^4(\Omega,\mathbb{R}^n).$$

Combining the theorems 2.4 and 4.1 there are constants $C_{L^\infty}, C_{L^4} > 0$ such that the local solution (u,ϑ,q) satisfies

$$\|\partial_t^2 u\|_\infty + \|\partial_t u\|_{W^{1,\infty}} + \|u\|_{W^{2,\infty}} + \big\|\partial_t(\vartheta,q)\big\|_\infty + \big\|(\vartheta,q)\big\|_{W^{1,\infty}} \leqslant C_{L^\infty}\Lambda^{1/2}$$

and

$$\begin{aligned}&\|\partial_t^3 u\|_{L^4} + \|\partial_t^2 u\|_{W^{1,4}} + \|\partial_t u\|_{W^{2,4}} + \|u\|_{W^{3,4}}\\ &+ \big\|\partial_t^2(\vartheta,q)\big\|_{L^4} + \big\|\partial_t(\vartheta,q)\big\|_{W^{1,4}} + \big\|(\vartheta,q)\big\|_{W^{2,4}} \leqslant C_{L^4}\Lambda^{1/2}.\end{aligned}$$

That the arising perturbations R_c are at least cubic in $\sqrt{\Lambda}$ will be shown with three examples of the most problematic terms. All the others can be treated entirely in the same way.

As mentioned before a calculation analogue to that presented for $\tilde\lambda[u,\vartheta,q]$ gives the equivalent to (3.32) for $\tilde\lambda[\partial_t^2 u, \partial_t^2\vartheta, \partial_t^2 q]$. In (3.7) then the following summand appears in R_c due to the both additional time-derivatives:

$$\begin{aligned}&\left|\int_\Omega u_{i|t|t|t|t}(A_{ij|t|t|t}\Delta u_j + A_{ij|t|t}\Delta u_{j|t} + A_{ij|t}\Delta u_{j|t|t})\,\mathrm{d}x\right|\\ &\leqslant C\big(\Lambda^{1/2}\|\partial_t^3 A\|\,\|\Delta u_j\|_\infty + \Lambda^{1/2}\|\partial_t^2 A\|_{L^4}\|\partial_t\Delta u\|_{L^4} + \Lambda\|\partial_t A\|_\infty\big)\\ &\leqslant C\Big(\Lambda\big\|\partial_t^3(\nabla u,\vartheta,q)\big\| + \Lambda\big\|\partial_t^2(\nabla u,\vartheta,q)\big\|\,\big\|\partial_t(\nabla u,\vartheta,q)\big\|_\infty + \Lambda\big\|\partial_t(\nabla u,\vartheta,q)\big\|_\infty^3\\ &\qquad + \Lambda\big\|\partial_t^2(\nabla u,\vartheta,q)\big\|_{L^4} + \Lambda\big\|\partial_t(\nabla u,\vartheta,q)\big\|_\infty^2 + \Lambda\big\|\partial_t(\nabla u,\vartheta,q)\big\|_\infty\Big)\\ &\leqslant C\big(\Lambda^{3/2} + \Lambda^2 + \Lambda^{5/2}\big).\end{aligned} \tag{4.1}$$

Thereby $C > 0$ is a constant independent of (u,ϑ,q) which may change from time to time.

Now let

$$\tilde F := A_{|t|t}\Delta u + 2A_{|t}\Delta u_{|t} + (A-\alpha E_n)\Delta u_{|t|t} - B_{|t|t}\nabla\vartheta - 2B_{|t}\nabla\vartheta_{|t} - (B-\beta E_n)\nabla\vartheta_{|t|t}.$$

Than we have to conider a term similar to

$$\int_\Omega \tilde F \Delta u_{|t|t}\,\mathrm{d}x$$

in the derivation of (3.9) for $\tilde{\lambda}[\partial_t^2 u, \partial_t^2\vartheta, \partial_t^2 q]$. Only the summands containing on of the factors $(A - \alpha E_n)$ and $(B - \beta E_n)$ cannot be treated as done in (4.1). But since (A, B) is assumed to be smooth enough one can apply the mean value theorem and conclude

$$\big|A(\nabla u, \vartheta, q) - \alpha E_n\big| = \big|A(\nabla u, \vartheta, q) - A(0,0,0)\big| \leqslant C\big|(\nabla u, \vartheta, q)\big|,$$

and $\big|B - \beta E_n\big| \leqslant C\big|(\nabla u, \vartheta, q)\big|$, respectively.
Thus, we obtain again

$$\left|\int_\Omega \tilde{F}\Delta u_{|t|t}\,\mathrm{d}x\right| \leqslant C\big(\Lambda^{3/2} + \Lambda^2\big).$$

As a last example we want to estimate a summand of R_c in (3.17) by at least $\sqrt{\Lambda}^3$. For this purpose we will use explicitely the boundary condition $\vartheta|_{\partial\Omega} = 0$ and succeed with partial integration:

$$\begin{aligned}
&\left|\int_\Omega \Big(c_{|t|t}\vartheta_{|t} + c_{|t}\vartheta_{|t|t} + g_{|t|t}\mathrm{div}\, q + g_{|t}\mathrm{div}\, q_{|t} + \mathrm{tr}\big(B_{|t|t}\nabla u_{|t} + B_{|t}\nabla u_{|t|t}\big)\Big)\nabla\vartheta_{|t|t|t}\,\mathrm{d}x\right| \\
&= \left|\int_\Omega \nabla\Big(c_{|t|t}\vartheta_{|t} + c_{|t}\vartheta_{|t|t} + g_{|t|t}\mathrm{div}\, q + g_{|t}\mathrm{div}\, q_{|t} + \mathrm{tr}\big(B_{|t|t}\nabla u_{|t} + B_{|t}\nabla u_{|t|t}\big)\Big)\vartheta_{|t|t|t}\,\mathrm{d}x\right| \\
&\leqslant C\Big(\|\partial_t^2\nabla c\|\,\|\partial_t\vartheta\|_\infty + \|\partial_t^2 c\|_{L^4}\|\partial_t\nabla\vartheta\|_{L^4} + \|\partial_t\nabla c\|_{L^4}\|\partial_t^2\vartheta\|_{L^4} + \Lambda^{1/2}\|\partial_t c\|_\infty \\
&\qquad + \|\partial_t^2\nabla g\|\,\|\mathrm{div}\, q\|_\infty + \|\partial_t^2 g\|_{L^4}\|\Delta q\|_{L^4} + \|\partial_t\nabla g\|_{L^4}\|\partial_t\mathrm{div}\, q\|_{L^4} + \Lambda^{1/2}\|\partial_t g\|_\infty \\
&\qquad + \|\partial_t^2\nabla B\|\,\|\partial_t\nabla u\|_\infty + \|\partial_t^2 B\|_{L^4}\|\partial_t\nabla^2 u\|_{L^4} + \|\partial_t\nabla B\|_{L^4}\|\partial_t^2\nabla u\|_{L^4} \\
&\qquad + \Lambda^{1/2}\|\partial_t B\|_\infty\Big)\Lambda^{1/2} \\
&\leqslant C\big(\Lambda^{3/2} + \Lambda^2\big).
\end{aligned}$$

As mentioned before all the appearing perturbation terms can be bounded by $\Lambda^{3/2}$ or higher powers of Λ. Thereto, no other techniques than those used above are required.

Finally, all arguments are presented and we can conclude the proof of theorem 3.2. □

Remark 4.2. Actually we did not apply the radial symmetry in the sence of Lemma 2.2 which suggests a transformation of the system to spherical coordinates as done for example in [7] for the parabolic system of thermoelasticity. In fact we just used that the appearing vector fields have symmetric gradients in Ω and are orthogonal to $\partial\Omega$, i. e. for $n = 3$

$$\mathrm{rot}\, u = \mathrm{rot}\, q = 0 \quad \text{in} \quad [0,\infty) \times \Omega, \tag{4.2}$$

$$\nu \times q = 0 \quad \text{in} \quad [0,\infty) \times \partial\Omega. \tag{4.3}$$

This are exactly those conditions which are made in [12]. In this mentioned paper it suffices to assume that only the initial data satisfy (4.2) and (4.3), for in the linear case then both hold automatically for all times. For the nonlinear equations this conclusion is not possible. Therefore we rather made the assumption 2.3 on the behavior of transformation. Nevertheless, we presented the proof as generally as possible.

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References

[1] Adams, R. A.: *Sobolev spaces.* Academic Press, New York (1978).

[2] Carlson, D. E.: *Linear thermoelasticity.* Handbuch der Physik VIa/2, Springer-Verlag, New York (1972).

[3] Cattaneo, C.: *Sulla conduzione del calore.* Atti Sem. Mat. Fis. Univ. Modena 3, 83-101 (1948).

[4] Chandrasekharaiah, D. S.: *Hyperbolic thermoelasticity: a review of recent literature.* Appl. Mech. Rev.

[5] Irmscher, T.: *Rate of stability in hyperbolic thermoelasticity.* Konstanzer Schriften in Mathematik und Informatik Nr. 214 (2006, submitted).

[6] Irmscher, T. und Racke, R.: *Sharp decay rates in parabolic and hyperbolic thermoelasticity.* IMA J. Appl. Math. 71, 459-478 (2006).

[7] Jiang, S.: *Exponential decay and global existence of spherically symmetric solutions in thermoelasticity.* Chin. Ann. Math. 19A, 629-640 (1998, in Chinese).

[8] Jiang, S. und Racke, R.: *Evolution equations in thermoelasticity.* π Monographs Surveys Pure Appl. Math. 112, Chapman & Hall/CRC, Boca Raton (2000).

[9] Jiang, S., Muñoz Rivera, J. E. und Racke, R.: *Asymptotic stability and global existence in thermoelasticity with symmetry.* Quart. Appl. Math. 56, 259-275 (1998).

[10] Leis, R.: *Initial boundary value problems in mathematical physics.* B. G. Teubner-Verlag, Stuttgart; John Wiley & Sons, Chichester (1986).

[11] Racke, R.: *Thermoelasticity with second sound – exponential stability in linear and non-linear 1-d.* Math. Meth. Appl. Sci. 25, 409-441 (2002).

[12] Racke, R.: *Asymptotic behavior of solutions in linear 2- or 3-d thermoelasticity with second sound.* Quart. Appl. Math. 61, Nr. 2, 315-328 (2003).

[13] Rieger, M. O.: *Exponential stability and global existence in thermoelasticity with radial symmetry.* Quart. Appl. Math. 62, 1-25 (2004).

[14] Tarabek, M. A.: *On the existence of smooth solutions in one-dimensional nonlinear thermoelasticity with second sound.* Quart. Appl. Math. 50, 727-742 (1992).