Global Adapted Solution of One-Dimensional Backward Stochastic Riccati Equations, with Application to the Mean-Variance Hedging*

Michael Kohlmann[†] Shanjian Tang [‡]

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Abstract

We obtain the global existence and uniqueness result for a one-dimensional backward stochastic Riccati equation, whose generator contains a quadratic term of L (the second unknown component). This solves the one-dimensional case of Bismut-Peng's problem which was initially proposed by Bismut (1978) in the Springer yellow book LNM 649. We use an approximation technique by constructing a sequence of monotone generators and then passing to the limit. We make full use of the special structure of the underlying Riccati equation. The singular case is also discussed. Finally, the above results are applied to solve the mean-variance hedging problem with stochastic market conditions.

Key words: backward stochastic Riccati equation, stochastic linear-quadratic control problem, approximation, mean-variance hedging, Feynmann-Kac formula

AMS Subject Classifications. 93E20, 60H10, 91B28

Abbreviated title: Global solvability of backward stochastic Riccati equation

1 Introduction

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a fixed complete probability space on which is defined a standard d-dimensional $\mathcal{F}_t$-adapted Brownian motion $w(t) \equiv (w_1(t), \cdots, w_d(t))^*$. Assume that

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[†]Department of Mathematics and Statistics, University of Konstanz, D-78457, Konstanz, Germany

[‡]Department of Mathematics and the Laboratory of Mathematics for Nonlinear Sciences at Fudan University, Fudan University, Shanghai 200433, China. **This author is supported by a Research Fellowship from the Alexander von Humboldt Foundation and by the National Natural Science Foundation of China under Grant No. 79790130.**

$\mathcal{F}_t$ is the completion, by the totality $\mathcal{N}$ of all null sets of $\mathcal{F}$, of the natural filtration $\{\mathcal{F}_t^w\}$ generated by w. Denote by $\{\mathcal{F}_t^2, 0 \le t \le T\}$ the P-augmented natural filtration generated by the $(d-d_0)$-dimensional Brownian motion $(w_{d_0+1}, \ldots, w_d)$. Assume that all the coefficients A, B, C_i, D_i are $\mathcal{F}_t$-progressively measurable bounded matrix-valued processes, defined on $\Omega \times [0,T]$, of dimensions $n\times n, n\times m, n\times n, n\times m$ respectively. Also assume that M is an $\mathcal{F}_T$-measurable, nonnegative, and bounded $n \times n$ random matrix. Assume that Q and N are $\mathcal{F}_t$-progressively measurable, bounded, nonnegative and uniformly positive $n\times n$ matrix processes, respectively.

Consider the following **backward stochastic Riccati differential equation** (BSRDE in short):

$$\left\{\begin{array}{rcl} dK &=& -[A^*K + KA + \sum_{i=1}^d C_i^*KC_i + Q + \sum_{i=1}^d (C_i^*L_i + L_iC_i) \\ && -(KB + \sum_{i=1}^d C_i^*KD_i + \sum_{i=1}^d L_iD_i)(N + \sum_{i=1}^d D_i^*KD_i)^{-1} \\ && \times(KB + \sum_{i=1}^d C_i^*KD_i + \sum_{i=1}^d L_iD_i)^*]\,dt + \sum_{i=1}^d L_i\,dw_i, \qquad 0 \le t < T, \\ K(T) &=& M. \end{array}\right. \tag{1}$$

When the coefficients A, B, C_i, D_i, Q, N, M are all deterministic, then $L_1 = \cdots = L_d = 0$ and the BSRDE (1) reduces to the following ordinary nonlinear matrix differential equation:

$$\left\{\begin{array}{rcl} dK &=& -[A^*K + KA + \sum_{i=1}^d C_i^*KC_i + Q - (KB + \sum_{i=1}^d C_i^*KD_i) \\ && \times(N + \sum_{i=1}^d D_i^*KD_i)^{-1}(KB + \sum_{i=1}^d C_i^*KD_i + \sum_{i=1}^d L_iD_i)^*]\,dt, \\ && \quad 0 \le t < T, \\ K(T) &=& M, \end{array}\right. \tag{2}$$

which was completely solved by Wonham [36] by applying Bellman's quasilinear principle and a monotone convergence approach. Bismut [2, 3] initially studied the case of random coefficients, but he solved only some special simple cases. He always assumed that the randomness of the coefficients only comes from a smaller filtration $\{\mathcal{F}_t^2\}$, which leads to $L_1 = \cdots = L_{d_0} = 0$. He further assumed in his paper [2] that

$$C_{d_0+1} = \cdots = C_d = 0, \quad D_{d_0+1} = \cdots = D_d = 0, \tag{3}$$

under which the BSRDE (1) becomes the following one:

$$\left\{\begin{array}{rcl} dK &=& -[A^*K + KA + \sum_{i=1}^{d_0} C_i^*KC_i + Q \\ && -(KB + \sum_{i=1}^{d_0} C_i^*KD_i)(N + \sum_{i=1}^{d_0} D_i^*KD_i)^{-1}(KB + \sum_{i=1}^{d_0} C_i^*KD_i)^*]\,dt \\ && + \sum_{i=d_0+1}^{d} L_i\,dw_i, \qquad 0 \le t < T, \\ K(T) &=& M, \end{array}\right. \tag{4}$$

and the generator does not involve L at all. In his work [3] he assumed that

$$D_{d_0+1} = \cdots = D_d = 0, \tag{5}$$

under which the BSRDE (1) becomes the following one

$$\left\{\begin{array}{rcl} dK & = & -[A^*K + KA + \sum_{i=1}^{d} C_i^* K C_i + Q + \sum_{i=d_0+1}^{d} (C_i^* L_i + L_i C_i) \\ & & -(KB + \sum_{i=1}^{d_0} C_i^* K D_i)(N + \sum_{i=1}^{d_0} D_i^* K D_i)^{-1}(KB + \sum_{i=1}^{d_0} C_i^* K D_i)^*]\, dt \\ & & + \sum_{i=d_0+1}^{d} L_i\, dw_i, \qquad 0 \le t < T, \\ K(T) & = & M, \end{array}\right. \tag{6}$$

and the generator depends on the second unknown variable $(L_{d_0+1}, \ldots, L_d)^*$ in a linear way. Moreover his method was rather complicated. Later, Peng [27] gave a nice treatment on the proof of existence and uniqueness for the BSRDE (6), by using Bellman's quasi-linear principle and a method of monotone convergence—a generalization of Wonham's approach to the random situation.

As early as in 1978, Bismut [3] commented on page 220 that:"Nous ne pourrons pas démontrer l'existence de solution pour l'équation (2.49) dans le cas général." (We could not prove the existence of solution for equation (2.49) for the general case.) On page 238, he pointed out that the essential difficulty for solution of the general BSRDE (1) lies in the integrand of the martingale term which appears in the generator in a quadratic way. Two decades later in 1998, Peng [30] included the above problem in his list of open problems on BSDEs.

In this paper, we prove the global existence and uniqueness result for the one-dimensional case of BSRDE (1), that is

$$\left\{\begin{array}{rcl} dK & = & -[aK + \sum_{i=1}^{d} c_i L_i + Q + F(t, K, L)]\, dt + \sum_{i=1}^{d} L_i\, dw_i, \\ K(T) & = & M, \quad K \in \mathcal{L}_{\mathcal{F}}^{\infty}(0, T; R_+) \cap L^{\infty}(\Omega, \mathcal{F}_T, P; C([0, T]; R_+)) \end{array}\right. \tag{7}$$

with

$$\begin{array}{rcl} F(t, K, L) & := & -[B(t)K + \sum_{i=1}^{d} C_i D_i(t) K + \sum_{i=1}^{d} D_i(t) L_i][N(t) + |K| \sum_{i=1}^{d} D_i^* D_i(t)]^{-1} \\ & & \times [B(t)K + \sum_{i=1}^{d} C_i D_i(t) K + \sum_{i=1}^{d} D_i(t) L_i]^*, \quad 0 \le t \le T; \\ a(t) & := & 2A(t) + \sum_{i=1}^{d} C_i^2(t), \quad 0 \le t \le T; \\ c_i(t) & := & 2C_i(t), \quad 0 \le t \le T, i = 1, \ldots, d. \end{array} \tag{8}$$

The arguments given here are based on the following new observation that

$$F(t, K, L) \le 0, \quad \forall K \in R, \forall L \in R^d, 0 \le t \le T. \tag{9}$$

We make full use of this special structure for BSRDE (7). We apply an approximation technique, which is inspired by the works of Kobylanski [16] and Lepeltier and San Martin [20, 21].

Consider then the case where the control weight matrix N reduces to zero. Kohlmann and Zhou [18] discussed such a case. However, their context is rather restricted, as they make the following assumptions: (a) all the coefficients involved are deterministic; (b) $C_1 = \cdots = C_d = 0, D_1 = \cdots = D_d = I_{m\times m}$, and $M = I$;(c) $A + A^* \geq BB^*$. Their arguments are based on a result of Chen, Li and Zhou [4]. Kohlmann and Tang [17] considered a general framework along those analogues of Bismut [3] and Peng [27], which has the following features: (a) the coefficients A, B, C, D, N, Q, M are allowed to be random, but are only $\mathcal{F}_t^2$-progressively measurable processes or $\mathcal{F}_T^2$-measurable random variable; (b) the assumptions in Kohlmann and Zhou [18] are dispensed with or generalised; (c) the condition (5) is assumed to be satisfied. Kohlmann and Tang [17] obtained a general result and generalised Bismut's previous result on existence and uniqueness of a solution of BSRDE (6) to the singular case under the following additional two assumptions:

$$M \geq \varepsilon I, \quad \sum_{i=1}^{d} D_i^* D_i(t) \geq \varepsilon I. \tag{10}$$

In this paper the existence and uniqueness result is also obtained for the singular case $N = 0$ under the assumption (10), but for a more general framework of the following features: the coefficients A, B, C, D, N, Q, M are allowed to be $\mathcal{F}_t$-progressively measurable processes or $\mathcal{F}_T$-measurable random variable, and the coefficient D is not necessarily zero.

The BSRDE (1) arises from solution of the optimal control problem

$$\inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(u;0,x) \tag{11}$$

where for $t \in [0, T]$ and $x \in R^n$,

$$J(u;t,x) := E^{\mathcal{F}_t}[\int_t^T [(Nu,u) + (QX^{t,x;u}, X^{t,x;u})]\, ds + (MX^{t,x;u}(T), X^{t,x;u}(T))] \tag{12}$$

and $X^{t,x;u}(\cdot)$ solves the following stochastic differential equation

$$\begin{cases} dX &= (AX + Bu)\, ds + \sum_{i=1}^{d} (C_i X + D_i u)\, dw_i, \quad t \leq s \leq T, \\ X(t) &= x. \end{cases} \tag{13}$$

The following connection is well known: if the BSRDE (1) has a solution (K, L), the solution for the above **linear-quadratic optimal control problem** (LQ problem in short) has the following closed form (also called the feedback form):

$$u(t) := -(N + \sum_{i=1}^{d} D_i^* K D_i)^{-1} [B^* K + \sum_{i=1}^{d} D_i^* K C_i + \sum_{i=1}^{d} D_i^* L_i] X(t) \tag{14}$$

and the associated value function V is the following quadratic form

$$V(t,x) := \inf_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x) = (K(t)x, x), \qquad 0 \leq t \leq T, x \in R^n. \tag{15}$$

In this way, on the one hand, the solution of the above LQ problem is reduced to solving the BSRDE (1). On the other hand, the formula (15) actually provides a representation— of Feynman-Kac type— for the solution of BSRDE (1). The reader will see that the proofs given here for Theorems 2.1 and 2.2 depend heavily on this kind of representation.

As an application of the above results, the mean-variance hedging problem with random market conditions is considered. The mean-variance hedging problem was initially introduced by Föllmer and Sondermann [9], and later widely studied by Duffie and Richardson [7], Föllmer and Schweizer [10], Schweizer [32, 33, 34], Hipp [14], Monat and Stricker [23], Pham, Rheinländer and Schweizer [31], Gourieroux, Laurent and Pham [12], and Laurent and Pham [19]. All of these works are based on a projection argument. Recently, Kohlmann and Zhou [18] used a natural LQ theory approach to solve the case of deterministic market conditions. Kohlmann and Tang (10) used a natural LQ theory approach to solve the case of stochastic market conditions, but the market conditions are only allowed to involve a smaller filtration $\{\mathcal{F}_t^2\}$. In this paper, the case of random market conditions is completely solved by using the above results, and the optimal hedging portfolio and the variance-optimal martingale measure are characterized by the solution of the associated BSRDE.

The rest of the paper is organised as follows. Section 2 contains a list of notations and the statement of the main results which consist of Theorems 2.1 and 2.2. In Sections 3 and 4 the proofs of Theorems 2.1 and 2.2 are given respectively. Section 5 provides a straightforward application of the main results to the regular and singular stochastic LQ problems. Section 6 presents an application to solution of the mean-variance hedging problem in finance.

2 Notation and the Main Results: Global Existence and Uniqueness

Notation. Throughout this paper, the following additional notation will be used:

M^*	:	the transpose of any vector or matrix M;
$\|M\|$	:	$= \sqrt{\sum_{ij} m_{ij}^2}$ for any vector or matrix $M = (m_{ij})$;
(M_1, M_2)	:	the inner product of the two vectors M_1 and M_2;
R^n	:	the n-dimensional Euclidean space;
R_+	:	the set of all nonnegative real numbers;
$C([0,T];H)$	:	the Banach space of H-valued continuous functions on $[0,T]$, endowed with the maximum norm for a given Hilbert space H;
$\mathcal{L}^2_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued $\mathcal{F}_t$-adapted square-integrable stochastic processes f on $[0,T]$, endowed with the norm $(E\int_0^T \|f(t)\|^2\,dt)^{1/2}$ for a given Euclidean space H;
$\mathcal{L}^\infty_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued, $\mathcal{F}_t$-adapted, essentially bounded stochastic processes f on $[0,T]$, endowed with the norm $\operatorname{ess\,sup}_{t,\omega} \|f(t)\|$ for a given Euclidean space H;
$L^2(\Omega, \mathcal{F}, P; H)$	:	the Banach space of H-valued norm-square-integrable random variables on the probability space $(\Omega, \mathcal{F}, P)$ for a given Banach space H;

and $L^\infty(\Omega, \mathcal{F}, P; C([0,T];R^n))$ is the Banach space of $C([0,T];R^n)$-valued, essentially maximum-norm-bounded random variables f on the probability space $(\Omega, \mathcal{F}, P)$, endowed with the norm $\operatorname{ess\,sup}_{\omega\in\Omega} \max_{0\le t\le T} |f(t,\omega)|$.

The main results of this paper are stated by the following two theorems.

Theorem 2.1. (the regular case) *Assume that* $M \ge 0, Q(t) \ge 0$ *and* $N(t) \ge \varepsilon I_{m\times m}$ *for some positive constant* $\varepsilon > 0$*. Then, the* BSRDE (7) *has a unique* $\mathcal{F}_t$*-adapted global solution* (K, L) *with*

$$K \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;R_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];R_+)), \quad L \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d).$$

Theorem 2.2. (the singular case) *Assume that* $N(t) \ge 0$ *and* $Q(t) \ge 0$*. Also assume that*

$$M \ge \varepsilon \tag{16}$$

and

$$\sum_{i=1}^d D_i^* D_i(t) \ge \varepsilon I_{m\times m} \tag{17}$$

for some positive constant $\varepsilon > 0$*. Then, the* BSRDE (7) *has a unique* $\mathcal{F}_t$*-adapted global solution* (K, L) *with*

$$K \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;R_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];R_+)), \quad L \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d),$$

and $K(t,\omega)$ *being uniformly positive w.r.t.* (t,ω).

3 The Proof of Theorem 2.1.

This section gives the proof of Theorem 2.1.

3.1 Construction of a sequence of decreasing uniformly Lipschitz generators

Define for $j = 0, 1, \ldots,$

$$F_j(t, K, L) := \sup_{\tilde{K} \in R, \tilde{L} \in R^d} \left[F(t, \tilde{K}, \tilde{L}) - j|K - \tilde{K}| - j|L - \tilde{L}| \right], \quad \forall K \in R, L \in R^d. \tag{18}$$

Then, we have the following assertions. (i) The quadratic growth in (K, L): there is a deterministic positive constant ε_0 which is independent of j, such that for each $j = 0, 1, \ldots$, $|F_j(t, K, L)| \leq \varepsilon_0(1 + |K|^2 + |L|^2), \forall (t, K, L) \in [0, T] \times R \times R^d$. (ii)Monotonicity in j: $\{F_j; j = 0, 1, \ldots\}$ is decreasingly convergent to F, that is

$$F_0 \geq F_1 \geq \cdots \geq F_j \geq F_{j+1} \geq \cdots \geq F, \quad F_j \downarrow F. \tag{19}$$

(iii) The uniform Lipschitz property: for each $j = 0, 1, \ldots$, F_j is uniformly Lipschitz in (K, L). (iv) The strong convergence: if $\lim_{j \to \infty} K^j = K$ and $\lim_{j \to \infty} L^j = L$, then $\lim_{j \to \infty} F_j(t, K^j, L^j) = F(t, K, L)$. The proof of these four assertions is an easy adaptation to that of Lepeltier and San Martin [20]. Note that

$$F_0(t, K, L) \equiv 0. \tag{20}$$

Then consider the following approximating **backward stochastic differential equation** (BSDE in short)

$$\begin{cases} dK = -[aK + \sum_{i=1}^{d} c_i L_i + Q + F_j(t, K, L)]\, dt + \sum_{i=1}^{d} L_i\, dw_i, \\ K(T) = M + \frac{1}{j+1}. \end{cases} \tag{21}$$

The generator of the BSDE (21) is given by

$$G_j(t, K, L) := a(t)K + \sum_{i=1}^{d} c_i(t) L_i + Q(t) + F_j(t, K, L), \quad K \in R, L \in R^d. \tag{22}$$

In the following, we state Pardoux and Peng's fundamental result on the existence and uniqueness of a nonlinear BSDE under the assumption of uniform Lipschitz on the generator. The reader is referred to Pardoux and Peng [24] for details of the proof.

Lemma 3.1. (Pardoux and Peng (1990)) *Assume that $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ and the real valued function f defined on $\Omega \times [0, T] \times R \times R^d$ satisfies the following conditions:* (1) *The stochastic process $f(\cdot, y, z)$ is $\mathcal{F}_t$-adapted for each fixed pair (y, z);* (2) *$f(t, \cdot, \cdot)$ is uniformly Lipschitz, i.e. there is a constant $\lambda > 0$ such that*

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq \lambda \left(|y_1 - y_2| + |z_1 - z_2| \right), \quad \forall (y_i, z_i) \in R^{d+1}, i = 1, 2;$$

and (3) *$f(\cdot, 0, 0) \in \mathcal{L}^2_{\mathcal{F}}(0, T)$. Then, the following* BSDE

$$\begin{cases} dy = -f(t, y, z)\, dt + \sum_{i=1}^{d} z_i\, dw_i, \\ y(T) = \xi \end{cases} \tag{23}$$

has a unique solution (y,z) with $y \in \mathcal{L}^2_{\mathcal{F}}(0,T) \cap L^2(\Omega, \mathcal{F}, P; C[0,T])$ and $z \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$.

The next lemma states a comparison result due to Peng [29].

Lemma 3.2. (Peng (1992)) *Suppose that* $(f^i, \xi^i), i = 1,2$ *satisfy the assumptions made in* Lemma 3.1 *for* (f, ξ). *Assume that*

$$f^1(t,y,z) \geq f^2(t,y,z), \forall (y,z) \in R \times R^d; \qquad \xi^1 \geq \xi^2.$$

Let $(y^i, z^i), i = 1,2$ *denote the solutions of* BSDE (23) *with* (f, ξ) *being replaced with* $(f^i, \xi^i), i = 1,2,$ *respectively. Then, the following holds:*

$$y^1(t) \geq y^2(t), \quad a.s.a.e.$$

By applying Lemma 3.1, we see that for each $j = 0, 1, \ldots,$ the BSDE (21) has a unique $\mathcal{F}_t$-adapted global solution, denoted by (K^j, L^j). In view of the comparison result Lemma 3.2, we obtain

$$K_0 \geq K_1 \geq \cdots \geq K_j \geq K_{j+1} \geq \cdots, \quad a.s.a.e. \tag{24}$$

3.2 The positivity of K^j

Proposition 3.1. *For each* $j = 0, 1, \ldots,$ *we have*

$$K^j(t) > 0 \quad a.s.a.e.$$

Proof of Proposition 3.2. Define

$$\tau_j := \sup\{t \in [0,T] : K^j(t) \leq 0\}. \tag{25}$$

Since $K^j(T) = M + \frac{1}{j+1} > 0$ *a.s.*, we have

$$\tau_j < T, \quad a.s. \tag{26}$$

We assert that

$$\tau_j = -\infty, \qquad i.e. \qquad K^j(t) > 0, \quad a.s. \forall t \in [0,T]. \tag{27}$$

For this purpose, define

$$\sigma_{jl} := T \wedge \inf\{t \in [0,T] : \int_0^t |L^j|^2 \, ds \geq l\}. \tag{28}$$

Since $L^j \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$, we see that

$$\int_0^T |L^j|^2 \, ds < \infty, \quad a.s.; \qquad \lim_{l \to \infty} \sigma_{jl} = T, \quad a.s.$$

Define the following feedback control

$$u_j := -(N + K^j \sum_{i=1}^d D_i^* D_i)^{-1}(BK^j + \sum_{i=1}^d C_i D_i K^j + \sum_{i=1}^d D_i L_i^j)^* X. \tag{29}$$

Applying the existence and uniqueness result of Gal'Chuk [11], the stochastic equation has a unique solution $X^{t,x;u_j}$ corresponding to the above feedback control starting from arbitrary initial data (t,x). It is easily seen that $X^{t\vee\tau_j,1;u_j}$ is well defined on the stochastic time interval $[t\vee\tau_j, \sigma_{jl}]$ for $l = 1, 2, \ldots$. Using Itô's formula, we can check out that

$$\begin{aligned} & K^j(0\vee\tau_j) \\ = \ & E^{\mathcal{F}_{0\vee\tau_j}}\left[K(\sigma_{jl})|X^{0\vee\tau_j,1;u_j}(\sigma_{jl})|^2 + \int_{0\vee\tau_j}^{\sigma_{jl}} Q|X^{0\vee\tau_j,1;u_j}|^2\,ds\right] \\ & + E^{\mathcal{F}_{0\vee\tau_j}}\int_{0\vee\tau_j}^{\sigma_{jl}} N|u_j|^2\,ds + E^{\mathcal{F}_{0\vee\tau_j}}\int_{0\vee\tau_j}^{\sigma_{jl}} (F_j - F)(s, K^j, L^j)\,ds \\ \geq \ & E^{\mathcal{F}_{0\vee\tau_j}}\left[K^j(\sigma_{jl})|X^{0\vee\tau_j,1;u_j}(\sigma_{jl})|^2 + \int_{0\vee\tau_j}^{\sigma_{jl}} Q|X^{0\vee\tau_j,1;u_j}|^2\,ds\right] \\ & + E^{\mathcal{F}_{0\vee\tau_j}}\int_{0\vee\tau_j}^{\sigma_{jl}} N|u_j|^2\,ds. \end{aligned} \tag{30}$$

Letting $l \to \infty$ and passing to the limit, we get

$$\begin{aligned} & E^{\mathcal{F}_{0\vee\tau_j}}\int_{0\vee\tau_j}^{T} N|u_j|^2\,ds < \infty, \quad E^{\mathcal{F}_{0\vee\tau_j}}\int_{0\vee\tau_j}^{T} Q|X^{0\vee\tau_j,1;u_j}|^2\,ds < \infty, \\ & E^{\mathcal{F}_{0\vee\tau_j}}(M + \tfrac{1}{j+1})|X^{0\vee\tau_j,1;u_j}(T)|^2 < \infty, \\ K^j(0\vee\tau_j) \geq \ & E^{\mathcal{F}_{0\vee\tau_j}}\left[K^j(T)|X^{0\vee\tau_j,1;u_j}(T)|^2 + \int_{0\vee\tau_j}^{T} Q|X^{0\vee\tau_j,1;u_j}|^2\,ds\right] > 0. \end{aligned} \tag{31}$$

The last inequality implies that $\tau_j < 0, a.s.$, $i.e.$ $\tau_j = -\infty$.

3.3 The uniform boundedness of (K^j, L^j)

First we prove the following fact.

Proposition 3.2. *K^0 has the following Feynman-Kac representation:*

$$K^0(t) = E^{\mathcal{F}_t}[\int_t^T Q|X^{t,1;0}|^2\,ds + (M+1)|X^{t,1;0}(T)|^2], \qquad 0 \leq t \leq T. \tag{32}$$

It is uniformly bounded.

Proof of Proposition 3.2. The first assertion results from computing $|K^0 X^{t,1;0}|^2(s)$ with Itô's formula. The second assertion is obtained by applying Theorem 2.1 of Peng [27].

The uniform boundedness of (K^j, L^j) is stated by

Proposition 3.3. *The sequence $\{(K^j, L^j); j = 0, 1, \ldots\}$ is uniformly bounded in the Banach space $\mathcal{L}^\infty_{\mathcal{F}}(0,T) \times \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$. That is*

$$\operatorname*{ess\,sup}_{(t,\omega)} K^j(t) + E\int_0^T |L^j|^2\,ds \leq \beta_0 \tag{33}$$

where β_0 is a positive constant and is independent of j.

Proof of Proposition 3.3. The uniform boundedness of K^j is obvious from the following inequality

$$K^0(t) \geq K^j(t) \geq 0, \quad 0 \leq t \leq T$$

and Proposition 3.2. We show the uniform boundedness for L^j in the following.

In view of the BSDE (21), using Itô's formula to compute $|K^j|^2(t)$, we get

$$\begin{cases} d|K^j|^2(t) &= -2K^j[aK^j + (c, L^j) + Q + F_j(t, K^j, L^j)]\, dt \\ & \quad +|L^j|^2\, dt + 2K^j(L^j, dw), \quad 0 \leq t \leq T, \\ (K^j)^2(T) &= \left(M + \frac{1}{j+1}\right)^2. \end{cases} \tag{34}$$

Taking expectation on both sides, we have

$$\begin{aligned} & E|K^j|^2(t) + E\int_t^T |L^j|^2\, ds \\ = \;& E\left(M + \tfrac{1}{j+1}\right)^2 + 2E\int_t^T K^j[aK^j + (c, L^j) + Q + F_j(s, K^j, L^j)]\, ds. \end{aligned} \tag{35}$$

Our new observation is that

$$2K_jF_j(s, K^j, L^j) \leq 0, \tag{36}$$

(since $K^j \geq 0$ and $F_j \leq 0$) and so the following straightforward calculations hold:

$$\begin{aligned} & E|K^j|^2(t) + E\int_t^T |L^j|^2\, ds \\ \leq \;& E(M+1)^2 + 2E\int_t^T K^j[aK^j + (c, L^j) + Q]\, ds \\ \leq \;& E(M+1)^2 + E\int_t^T [2a|K^j|^2 + 2|c|^2|K^j|^2 + \frac{1}{2}|L^j|^2 + \frac{1}{2}|K^j|^2 + \frac{1}{2}Q^2]\, ds. \end{aligned} \tag{37}$$

Since the coefficients $a(s), c_i(s), Q(s)$ are uniformly bounded, there is a positive constant λ which is independent of j such that

$$E(K^j)^2(t) + \frac{1}{2}E\int_t^T |L^j|^2\, ds \leq \lambda + \lambda E\int_t^T |K^j|^2\, ds. \tag{38}$$

Using Gronwall's inequality, we get

$$\sup_{0\leq t\leq T} E|K^j|^2(t) + \frac{1}{2}E\int_0^T |L^j|^2\, ds \leq \lambda \exp(\lambda T). \tag{39}$$

3.4 The strong convergence result and the existence

Proposition 3.4. We have the following convergence result:

$$\lim_{l,r\to\infty} E\int_0^T |K^l - K^r|^2\, ds = 0. \tag{40}$$

Proof of Proposition 3.4 Since the sequence $\{K^j; j = 0, 1, \ldots\}$ is decreasing and uniformly bounded, we have by the dominated convergence theorem of Lebesgue:

$$\lim_{l,r\to\infty} E\int_0^T |K^l - K^r|^2\, ds = 0. \tag{41}$$

Since L^j is bounded in $\mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$, assume without loss of generality that as $j \to \infty$,

$$L^j \to L \qquad \text{weakly in } \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$$

for some $L \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$. We also assume that $l < r$.

Set

$$K^{lr} := K^l - K^r, \quad L^{lr} := L^l - L^r, \quad K^{l\infty} := K^l - K, \quad L^{l\infty} := L^l - L.$$

We have

$$\begin{cases} dK^{lr} = -[aK^{lr} + (c, L^{lr}) + F_l(t, K^l, L^l) - F_r(t, K^r, L^r)]\,dt + (L^{lr},\, dw), \\ K^{lr}(T) = \frac{1}{1+l} - \frac{1}{1+r}. \end{cases} \tag{42}$$

We now use a technique developed by Kobylanski [16] (see also Lepeltier and San Martin [21] in pages 236-237). Applying Itô's formula with the following function (with the positive constant λ being specified later)

$$\Psi(x) := \lambda_1^{-1}[\exp(\lambda_1 x) - 1] - x, \tag{43}$$

we have

$$\begin{aligned} &E\Psi\left(K^{lr}(t)\right) + \frac{1}{2}E\int_t^T \Psi''(K^{lr})|L^{lr}|^2\, ds \\ =\ &\Psi\left(\tfrac{1}{1+l} - \tfrac{1}{1+r}\right) + 2E\int_t^T \Psi'(K^{lr})[aK^{lr} + (c, L^{lr}) + F_l(s, K^l, L^l) - F_r(s, K^r, L^r)]\, ds. \end{aligned}$$

Noting the following facts:

$$K^{lr} \geq 0, \quad \Psi'(K^{lr}) = \exp(\lambda_1 K^{lr}) - 1 \geq 0, \quad F_l \leq 0, \quad F_r \geq F, \tag{44}$$

we obtain

$$\begin{aligned} &E\Psi\left(K^{lr}(t)\right) + \frac{1}{2}E\int_t^T \Psi''\left(K^{lr}\right)|L^{lr}|^2\, ds \\ \leq\ &\Psi\left(\tfrac{1}{1+l} - \tfrac{1}{1+r}\right) + 2E\int_t^T \Psi'(K^{lr})[aK^{lr} + (c, L^{lr}) - F(s, K^r, L^r)]\, ds. \end{aligned} \tag{45}$$

Note the following estimation

$$\begin{aligned} -2F(s, K^r, L^r) \leq\ & 2\varepsilon^{-1}|BK^r + \sum_{i=1}^d C_i D_i K^r + \sum_{i=1}^d D_i L_i^r|^2 \\ \leq\ & \lambda + \lambda|L^r|^2 \leq \lambda + 3\lambda(|L^{lr}|^2 + |L^{l\infty}|^2 + |L|^2). \end{aligned} \tag{46}$$

where λ is a positive constant and depends on ε and the bounds of $K^0(s), B(s), C(s), D(s)$ only (in view of Proposition 3.3), but independent of the integer r. Then we have

$$\begin{aligned} & E\Psi\left(K^{lr}(t)\right) + E\int_t^T (\frac{1}{2}\Psi'' - 3\lambda\Psi')(K^{lr})|L^{lr}|^2\, ds \\ \leq\ & \Psi\left(\tfrac{1}{1+l} - \tfrac{1}{1+r}\right) + 2E\int_t^T \Psi'(K^{lr})[aK^{lr} + (c, L^{lr})]\, ds \\ & +\lambda E\int_t^T \Psi'(K^{lr})(1 + 3|L^{l\infty}|^2 + 3|L|^2)\, ds. \end{aligned} \tag{47}$$

Take $\lambda_1 = 12\lambda$. Since

$$\frac{1}{2}\Psi''(x) - 3\lambda\Psi'(x) = 3\lambda\exp(12\lambda x) + 3\lambda,$$

we have that the term

$$\sqrt{\frac{1}{2}\Psi''(K^{lr}) - 3\lambda\Psi'(K^{lr})}$$

converges strongly to

$$\sqrt{\frac{1}{2}\Psi''(K^{l\infty}) - 3\lambda\Psi'(K^{l\infty})}$$

as $r \to \infty$, and it is uniformly bounded in view of Proposition 3.3. Therefore,

$$\sqrt{\frac{1}{2}\Psi''(K^{lr}) - 3\lambda\Psi'(K^{lr})}\, L^{lr}$$

converges weakly to

$$\sqrt{\frac{1}{2}\Psi''(K^{lr}) - 3\lambda\Psi'(K^{lr})}\, L^{l\infty}.$$

From the last weak convergence, we get

$$\begin{aligned} & E\int_t^T (\frac{1}{2}\Psi'' - 3\lambda\Psi')(K^{l\infty})|L^{l\infty}|^2\, ds \\ \leq\ & \underline{\lim}_{r\to\infty} E\int_t^T (\frac{1}{2}\Psi'' - 3\lambda\Psi')(K^{lr})|L^{lr}|^2\, ds \\ \leq\ & \Psi\left(\tfrac{1}{1+l}\right) + 2E\int_t^T \Psi'(K^{l\infty})[aK^{l\infty} + (c, L^{l\infty})]\, ds \\ & +\lambda E\int_t^T \Psi'(K^{l\infty})(1 + 3|L^{l\infty}|^2 + 3|L|^2)\, ds. \end{aligned} \tag{48}$$

Hence we have

$$\begin{aligned} & E\int_t^T (\frac{1}{2}\Psi'' - 6\lambda\Psi')(K^{l\infty})|L^{l\infty}|^2\, ds \\ \leq\ & \Psi\left(\tfrac{1}{1+l}\right) + 2E\int_t^T \Psi'(K^{l\infty})[aK^{l\infty} + (c, L^{l\infty})]\, ds \\ & +\lambda E\int_t^T \Psi'(K^{l\infty})(1 + 3|L|^2)\, ds. \end{aligned} \tag{49}$$

Since

$$(\frac{1}{2}\Psi'' - 6\lambda\Psi')(K^{l\infty}) = 6\lambda,$$

we have by passing to the limit $l \to \infty$ and applying the dominated convergence theorem of Lebesgue the following

$$\lim_{l\to\infty} E\int_0^T |L^{l\infty}|^2\, ds = 0. \tag{50}$$

At this stage, we can show that almost surely K^j converges to K uniformly in t. The proof is standard, and the reader is referred to Lepeltier and San Martin [20] for details.

With the uniform convergence in the time variable t of K^j and the strong convergence of L^j, we can pass to the limit by letting $j \to \infty$ in the BSDE (21), and conclude that the limit (K, L) is a solution.

3.5 A Feynman-Kac representation result and the uniqueness

Consider the optimal control problem

$$\textbf{Problem } \mathcal{P}_0 \qquad \inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(u;0,x) \tag{51}$$

where for $t \in [0,T]$ and $x \in R$,

$$J(u;t,x) := E^{\mathcal{F}_t}[\int_t^T (N|u|^2 + Q|X^{t,x;u}|^2)\, ds + M|X^{t,x;u}(T)|^2] \tag{52}$$

and $X^{t,x;u}(\cdot)$ solves the following stochastic differential equation

$$\begin{cases} dX &= (AX+Bu)\, ds + \sum_{i=1}^d (C_iX + D_iu)\, dw_i, \quad t \le s \le T, \\ X(t) &= x. \end{cases} \tag{53}$$

The associated value function is defined as

$$V(t,x) := \inf_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x), \qquad 0 \le t \le T, x \in R. \tag{54}$$

The following connection is straightforward.

Proposition 3.5. *Let (K, L) be an $\mathcal{F}_t$-adapted solution of the* BSRDE (7) *with $K \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;R_+)\cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];R_+))$ and $L \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$. Then, the solution for the LQ problem $\mathcal{P}_0$ has the following closed form (also called the feedback form):*

$$\hat{u} = -(N + \sum_{i=1}^d D_i^* K D_i)^{-1}[B^*K + \sum_{i=1}^d D_i^*KC_i + \sum_{i=1}^d D_i^* L_i]\widehat{X} \tag{55}$$

and the associated value function V is the following quadratic form

$$V(t,x) = K(t)x^2. \tag{56}$$

Remark 3.1. Although the proof of Proposition 3.5 is straightforward (use Itô's formula to do some calculations), we need to be careful about the solution of the optimal closed system: the coefficients of the closed system corresponding to the feedback control (55) involve the quantity L and might not be bounded. The reader is referred to Gal'chuk [11] for a rigorous argument on this respect.

Using Proposition 3.5, we get the representation of K (as the first part of solution of BSRDE (7)) as

$$K(t) = V(t,1) = \inf_{u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} E^{\mathcal{F}_t}[M|X^{t,1;u}(T)|^2 + \int_t^T (N|u|^2 + Q|X^{t,1;u}|^2)\,ds], \quad 0 \le t \le T. \tag{57}$$

The uniqueness is a consequence of the representation result. In fact, assume that (K,L) and $(\widetilde{K},\widetilde{L})$ are two $\mathcal{F}_t$-adapted solutions of the BSRDE (7) with $K,\widetilde{K} \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;R_+) \cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];R_+))$ and $L,\widetilde{L} \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$. Then, we have

$$\begin{cases} d\delta K &= -[a\delta K + \sum_{i=1}^d c_i \delta L_i + \delta F]\,dt + \sum_{i=1}^d \delta L_i\,dw_i, \\ \delta K(T) &= 0. \end{cases} \tag{58}$$

Here, we use the notation:

$$\delta K := K - \widetilde{K}, \quad \delta L_i := L_i - \widetilde{L}_i, \quad \delta F := F(\cdot,K,L) - F(\cdot,\widetilde{K},\widetilde{L}).$$

Applying Itô's formula, we have

$$E|\delta K(t)|^2 + E\int_t^T |\delta L|^2\,ds = 2E\int_t^T \delta K(a\delta K + \sum_{i=1}^d c_i\delta L_i + \delta F)\,ds. \tag{59}$$

Noting that K and $\widetilde{K}$ has the same representation (57), we have $\delta K = 0$. Putting this equality into (59), we have

$$E\int_0^T |\delta L|^2\,ds = 0.$$

This implies that $L = \widetilde{L}$.

3.6 A remark

Theorem 2.1 can also be proved by nontrivially employing the result of Kobylanski [16]. However, the proof given here avoids doing an exponential transformation of the unknown variable of the BSDE under discussion, instead it makes full use of the special structure of the stochastic Riccati equation. Therefore we preferred this approach.

4 The Proof of Theorem 2.2

This section gives the proof of Theorem 2.2. The regular approximation method proposed by Kohlmann and Tang [17] is adapted to the present case.

We begin with the citation of an a priori estimate for $X^{t,x;u}$, which was established by Kohlmann and Tang [17].

Lemma 4.1. (a priori estimate) *Assume that the assumption* (17) *is satisfied. Let* $u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m)$*. Then, there is* $\beta > 0$ *which only depends on the bounds of the coefficients* A, B, C, D *and* ε*, such that*

$$\frac{\varepsilon}{2} E^{\mathcal{F}_t} \int_t^T |u|^2\, ds + |x|^2 \leq \exp\left(\beta(T-t)\right) E^{\mathcal{F}_t} |X^{t,x;u}(T)|^2, \quad 0 \leq t \leq T. \tag{60}$$

Proof of Lemma 4.1. Using Itô's formula, we have from (53)

$$\begin{aligned}
& E^{\mathcal{F}_t}|X(T)|^2 \\
= \ & E^{\mathcal{F}_t}|X(r)|^2 + 2E^{\mathcal{F}_t}\int_r^T (AX+Bu, X)\, ds + E^{\mathcal{F}_t}\int_r^T \sum_{i=1}^d |C_iX + D_iu|^2\, ds \\
= \ & E^{\mathcal{F}_t}|X(r)|^2 + 2E^{\mathcal{F}_t}\int_r^T ((A + \sum_{i=1}^d C_i^* C_i)X, X)\, ds \\
& +2E^{\mathcal{F}_t}\int_r^T ((B + \sum_{i=1}^d C_i^* D_i)u, X)\, ds + E^{\mathcal{F}_t}\int_r^T u^*(\sum_{i=1}^d D_i^* D_i)u\, ds \\
\geq \ & E^{\mathcal{F}_t}|X(r)|^2 + \frac{\varepsilon}{2}E^{\mathcal{F}_t}\int_r^T |u|^2\, ds - \beta E^{\mathcal{F}_t}\int_r^T |X|^2\, ds
\end{aligned} \tag{61}$$

for some positive constant β. Write

$$\rho_r := E^{\mathcal{F}_t}|X(r)|^2, \quad t \leq r \leq T. \tag{62}$$

Then, the above reads

$$\rho_t + \frac{\varepsilon}{2}E^{\mathcal{F}_t}\int_t^T |u|^2\, ds \leq \rho_T + \beta \int_t^T \rho_s\, ds. \tag{63}$$

By Gronwall's inequality, we have

$$\rho_r \leq \exp(\beta(T-r))\rho_T, \tag{64}$$

$$\rho_t + \frac{\varepsilon}{2}E^{\mathcal{F}_t}\int_t^T |u|^2\, ds \leq \exp(\beta(T-t))\rho_T. \tag{65}$$

This concludes the proof.

Consider the following regular approximation of the original control problem $\mathcal{P}_0$

$$\textbf{Problem } \mathcal{P}_\alpha \qquad \min_{u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J_\alpha(u;t,x) \tag{66}$$

with

$$J_\alpha(u;t,x) = J(u;t,x) + \alpha E^{\mathcal{F}_t}\int_t^T |u|^2\, ds, \qquad \alpha > 0. \tag{67}$$

It is associated with the following **BSRDE**

$$
\begin{cases}
dK = -[A^*K + KA + \sum_{i=1}^{d} C_i^* K C_i + Q + \sum_{i=1}^{d}(C_i^* L_i + L_i C_i) \\
\quad -(KB + \sum_{i=1}^{d} C_i^* K D_i + \sum_{i=1}^{d} L_i D_i)(\alpha I + N + \sum_{i=1}^{d} D_i^* K D_i)^{-1} \\
\quad \times(KB + \sum_{i=1}^{d} C_i^* K D_i + \sum_{i=1}^{d} L_i D_i)^*]\, dt + \sum_{i=1}^{d} L_i\, dw_i, \quad 0 \le t < T, \\
K(T) = M.
\end{cases}
\tag{68}
$$

The value function of the problem $\mathcal{P}_\alpha$ is denoted by $V_\alpha(t,x)$.

Proposition 3.5 allows us to express the value function

$$V_\alpha(t,x) = K_\alpha(t)x^2. \tag{69}$$

Here, (K_α, L_α) is the unique $\mathcal{F}_t$-adapted solution of the BSRDE (68) with

$$K_\alpha \in \mathcal{L}^\infty_\mathcal{F}(0,T;R_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];R_+)) \text{ and } L_\alpha \in \mathcal{L}^2_\mathcal{F}(0,T;R^d).$$

From Lemma 4.1, we immediately have

Lemma 4.2. *Suppose that the assumptions of* Theorem 2.2 *hold. Then, we have*

$$V_\alpha(t,x) \ge V(t,x) \ge \varepsilon \exp\left(-\beta(T-t)\right)x^2. \tag{70}$$

This implies that

$$K_\alpha(t) \ge \varepsilon \exp\left(-\beta(T-t)\right). \tag{71}$$

The relationship between the original problem $\mathcal{P}_0$ and the approximating problem $\mathcal{P}_\alpha$ is given in the next lemma.

Lemma 4.3. *Assume that the conditions* (16) *and* (17) *are satisfied. Then, for fixed $x \in R$, as $\alpha \to 0+$, $V_\alpha(t,x)$ converges in a decreasing way to $V(t,x)$ strongly both in $\mathcal{L}^\infty_\mathcal{F}(0,T;R)$ and in $L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];R))$.*

Proof of Lemma 4.3. It is obvious that $V_\alpha(t,x)$ is decreasing in α.

Denote by $\widehat{u}$ the optimal control of the original problem, i.e. $V(t,x) = J(\widehat{u};t,x)$. Then,

$$
\begin{aligned}
& V(t,x) \le V_\alpha(t,x) \le J_\alpha(\widehat{u};t,x) \\
= \; & J(\widehat{u};t,x) + \alpha E^{\mathcal{F}_t} \int_t^T |\widehat{u}|^2\, ds = V(t,x) + \alpha E^{\mathcal{F}_t} \int_t^T |\widehat{u}|^2\, ds.
\end{aligned}
\tag{72}
$$

It is easy to show that there is a constant $\beta_1 > 0$ such that

$$J(0;t,x) \le |x|^2 \exp\left(\beta_1(T-t)\right). \tag{73}$$

Noting the positivity of M and Lemma 4.1, we have

$$J(\widehat{u};t,x) \geq \varepsilon E^{\mathcal{F}_t}|X^{t,x;\widehat{u}}(T)|^2 \geq \frac{\varepsilon^2}{2}\exp(-\beta(T-t))E^{\mathcal{F}_t}\int_t^T |\widehat{u}|^2\,ds. \tag{74}$$

Since

$$J(\widehat{u};t,x) = V(t,x) \leq J(0;t,x),$$

we have

$$\frac{\varepsilon^2}{2}\exp(-\beta(T-t))E^{\mathcal{F}_t}\int_t^T |\widehat{u}|^2\,ds \leq |x|^2\exp(\beta_1(T-t)). \tag{75}$$

Concluding the above, we have

$$V(t,x) \leq V_\alpha(t,x) \leq V(t,x) + 2\alpha\varepsilon^{-2}|x|^2\exp((\beta_1+\beta)(T-t)).$$

This completes the proof of this lemma.

With Lemma 4.3, the following is obvious:

Lemma 4.4. *Suppose that the assumptions of* Theorem 2.2 *are satisfied. Then, the value function V is a quadratic form. More precisely, there is an $\mathcal{F}_t$-adapted stochastic process $K(\cdot) \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;R_+) \cap L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];R_+))$ such that*

$$V(t,x) = (K(t)x,x), \quad \forall (t,x) \in [0,T]\times R, P-a.s. \tag{76}$$

Moreover, K_α converges to K strongly in the two Banach spaces

$$\mathcal{L}^\infty_{\mathcal{F}}(0,T;R_+) \quad \text{and} \quad L^\infty(\Omega,\mathcal{F}_T,P;C([0,T];R_+)),$$

and $K(t)$ is uniformly positive: $K(t) \geq \varepsilon\exp(-\beta(T-t))$.

Lemma 4.5. *Suppose that the assumptions of* Theorem 2.2 *are satisfied. Then, $\{L_\alpha\}$ is a Cauchy sequence in $\mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$.*

Proof First, we show that $\{L_\alpha\}$ is bounded in $\mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$. The arguments are similar to those in Section 3. Use Itô's formula to compute $|K_\alpha(t)|^2$. Then since

$$K_\alpha F(\cdot,K_\alpha,L_\alpha) \leq 0,$$

it can be left out in our estimation. The remainder is standard to show that $\{L_\alpha\}$ is bounded in $\mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$.

Now we return to show that $\{L_\alpha\}$ is a Cauchy sequence in $\mathcal{L}^2_{\mathcal{F}}(0,T;R^d)$. For this purpose, use Itô's formula to compute $|K_\alpha(t)-K_\gamma(t)|^2$. We get the following

$$\begin{aligned} &E|K_\alpha-K_\gamma|^2(t) + E\int_t^T |L_\alpha-L_\gamma|^2\,ds \\ = \;& 2E\int_t^T (K_\alpha-K_\gamma)[a(K_\alpha-K_\gamma)+(c,L_\alpha-L_\gamma)+F(s,K_\alpha,L_\alpha)-F(s,K_\gamma,L_\gamma)]\,ds. \end{aligned}$$

Since K_α is uniformly bounded and uniformly positive (in view of Lemma 4.2) and L_α is uniforly bounded, we have that the right hand side of the last equality is less than the term

$$\|K_\alpha - K_\gamma\|_{\mathcal{L}^\infty_\mathcal{F}(0,T;R)}$$

times the integral

$$2E\int_0^T [|a||K_\alpha - K_\gamma| + |c||L_\alpha - L_\gamma| + |F(s, K_\alpha, L_\alpha)| + |F(s, K_\gamma, L_\gamma)|]\, ds$$

which is bounded uniformly in (α, γ) (more precisely, it is less than a positive constant times the term $(1 + \|K_\alpha\|^2_{\mathcal{L}^\infty_\mathcal{F}} + \|K_\gamma\|^2_{\mathcal{L}^\infty_\mathcal{F}} + \|L_\alpha\|^2_{\mathcal{L}^2_\mathcal{F}} + \|L_\gamma\|^2_{\mathcal{L}^2_\mathcal{F}})$). While

$$\lim_{\alpha,\gamma\to 0+} \|K_\alpha - K_\gamma\|_{\mathcal{L}^\infty_\mathcal{F}(0,T;R)} = 0,$$

we then have the desired result.

Proof of Theorem 2.2 Let L be the strong limit in $\mathcal{L}^2_\mathcal{F}(0,T;R^d)$ of the Cauchy sequence $\{L_\alpha\}$. Lemma 4.4 shows that K^α uniformly converges to K. Moreover, $K \in \mathcal{L}^\infty_\mathcal{F}(0,T;R_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];R_+))$ is uniformly positive. Therefore, it is meaningful to take the limit in the approximating BSRDEs (68) by letting $\alpha \to 0$. As a result, (K, L) is shown to be an $\mathcal{F}_t$-adapted solution to the BSRDE (7).

The proof of the uniqueness assertion is similar as in the proof of Theorem 2.1, and is omitted here.

5 Application to the Stochastic LQ Problem

Consider the one-dimensional non-homogeneous stochastic LQ problem.

Assume that

$$\xi \in L^2(\Omega, \mathcal{F}_T, P), \quad q, f, g \in \mathcal{L}^2_\mathcal{F}(0,T;R). \tag{77}$$

Consider the optimal control problem (denoted by $\mathcal{P}_0$):

$$\min_{u\in\mathcal{L}^2_\mathcal{F}(0,T;R^m)} J(u;0,x) \tag{78}$$

with

$$J(u;t,x) = E^{\mathcal{F}_t}[M|X^{t,x;u}(T) - \xi|^2 + \int_t^T (Q|X^{t,x;u} - q|^2 + N|u|^2)\, ds] \tag{79}$$

and $X^{t,x;u}$ solving the following linear stochastic system

$$\begin{cases} dX &= (AX + Bu + f)\, ds + \sum_{i=1}^d (C_iX + D_iu + g_i)\, dw_i, \quad t < s \le T, \\ X(t) &= x, \quad u \in \mathcal{L}^2_\mathcal{F}(t,T;R^m). \end{cases} \tag{80}$$

The value function V is defined as

$$V(t,x) := \min_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x), \quad (t,x)\in[0,T]\times R. \tag{81}$$

Define $\Gamma : [0,T]\times R_+ \times R^d \to R^m$ by

$$\Gamma(\cdot,S,L) = -(N+\sum_{i=1}^d D_i^* S D_i)^{-1}(B^*S + \sum_{i=1}^d D_i^* S C_i + \sum_{i=1}^d D_i^* L_i). \tag{82}$$

and

$$\widehat{A} := A + B\Gamma(\cdot,K,L), \widehat{C}_i := C_i + D_i\Gamma(\cdot,K,L), \widehat{a} := 2\widehat{A} + \sum_{i=1}^d \widehat{C}_i^2, \widehat{c}_i := 2\widehat{C}_i. \tag{83}$$

Let (ψ,ϕ) be the $\mathcal{F}_t$-adapted solution of the following **BSDE**

$$\begin{cases} d\psi(t) &= -[\widehat{A}^*\psi + \sum_{i=1}^d \widehat{C}_i^*(\phi_i - Kg_i) - Kf - \sum_{i=1}^d L_i g_i + Qq]\,dt + \sum_{i=1}^d \phi_i\,dw_i, \\ \psi(T) &= M\xi \end{cases} \tag{84}$$

where (K,L) is the unique $\mathcal{F}_t$-adapted solution of the BSRDE (7). The following can be verified by a pure completion of squares.

Theorem 5.1 *Suppose that the assumptions of* Theorem 2.1 *or* Theorem 2.2 *are satisfied. Let (K,L) be the unique $\mathcal{F}_t$-adapted solution of* BSRDE (7). *Then, the optimal control $\widehat{u}$ for the non-homogeneous stochastic LQ problem $\mathcal{P}_0$ exists uniquely and has the following feedback law*

$$\begin{aligned} \widehat{u} \;=\; & -(N+\sum_{i=1}^d D_i^* K D_i)^{-1}[(B^*K + \sum_{i=1}^d D_i^* K C_i + \sum_{i=1}^d D_i^* L_i)\widehat{X} \\ & -B^*\psi + \sum_{i=1}^d D_i^*(Kg_i - \phi_i)]. \end{aligned} \tag{85}$$

The value function $V(t,x), (t,x)\in[0,T]\times R$ has the following explicit formula

$$V(t,x) = K(t)x^2 - 2\psi(t)x + V^0(t), \quad (t,x)\in[0,T]\times R \tag{86}$$

with

$$\begin{aligned} V^0(t) := \;& E^{\mathcal{F}_t} M|\xi|^2 + E^{\mathcal{F}_t}\int_t^T Q|q|^2\,ds - 2E^{\mathcal{F}_t}\int_t^T \psi f\,ds \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d (K|g_i|^2 - 2\phi_i g_i)\,ds \\ & - E^{\mathcal{F}_t}\int_t^T ((N+\sum_{i=1}^d D_i^* K D_i)u^0, u^0)\,ds. \end{aligned} \tag{87}$$

and

$$u^0 := (N+\sum_{i=1}^d D_i^* K D_i)^{-1}[B^*\psi + \sum_{i=1}^d D_i^*(\phi_i - Kg_i)], \quad t\le s\le T. \tag{88}$$

Proof Set

$$\widetilde{u} = u - \Gamma(\cdot, K, L)X. \tag{89}$$

Then the system (80) reads

$$\begin{cases} dX &= (\widehat{A}X + B\widetilde{u} + f)\,ds + \sum_{i=1}^d (\widehat{C}_i X + D_i\widetilde{u} + g_i)\,dw_i, \quad t < s \le T, \\ X(t) &= x, \quad u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m). \end{cases} \tag{90}$$

Applying Itô's formula, we have the equation for $\mathcal{X} =: X^2$:

$$\begin{cases} d\mathcal{X} &= [\widehat{a}\mathcal{X} + 2X(B\widetilde{u} + f)]\,ds + \sum_{i=1}^d [2\widehat{C}_i X(D_i\widetilde{u} + g_i) + |D_i\widetilde{u} + g_i|^2]\,ds \\ &\quad + \sum_{i=1}^d [\widehat{c}_i\mathcal{X} + 2X(D_i\widetilde{u} + g_i)]\,dw_i, \quad t < s \le T, \\ \mathcal{X}(t) &= x^2, \quad u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m). \end{cases} \tag{91}$$

Note that the BSRDE (7) can be rewritten as

$$\begin{cases} -dK &= (\widehat{a}K + \sum_{i=1}^d \widehat{c}_i L_i + Q + \Gamma^* N \Gamma)\,dt - \sum_{i=1}^d L_i\,dw_i, \\ K(T) &= M. \end{cases} \tag{92}$$

So, application of Itô's formula gives

$$\begin{aligned} & E^{\mathcal{F}_t} M|X(T)|^2 + E^{\mathcal{F}_t}\int_t^T Q|X|^2\,ds + E^{\mathcal{F}_t}\int_t^T \Gamma^* N\Gamma|X|^2\,ds \\ = \; & K(t)X^2(t) + 2E^{\mathcal{F}_t}\int_t^T KX(B\widetilde{u} + f)\,ds + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d 2K(D_i\widetilde{u} + g_i)\widehat{C}_i X\,ds \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d K|D_i\widetilde{u} + g_i|^2\,ds + 2E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d L_i(D_i\widetilde{u} + g_i)X\,ds, \end{aligned}$$

and

$$\begin{aligned} & E^{\mathcal{F}_t}[M\xi X(T) + \textstyle\int_t^T QqX\,ds] = E^{\mathcal{F}_t}[\psi(T)X(T) + \textstyle\int_t^T QqX\,ds] \\ = \; & (\psi(t), X(t)) + E^{\mathcal{F}_t}\int_t^T \psi(B\widetilde{u} + f)\,ds + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d \phi_i(D_i\widetilde{u} + g_i)\,ds \\ & + E^{\mathcal{F}_t}\int_t^T (\sum_{i=1}^d \widehat{C}_i^* K g_i + Kf + \sum_{i=1}^d L_i g_i)X\,ds. \end{aligned}$$

Combining the last two equations, we get

$$
\begin{aligned}
& E^{\mathcal{F}_t}[M|X(T)-\xi|^2 + \int_t^T Q|X-q|^2\,ds + \int_t^T (Nu,u)\,ds] \\
= \; & E^{\mathcal{F}_t}[M|X(T)|^2 + \int_t^T QX^2\,ds + \int_t^T \Gamma^* N\Gamma X^2\,ds] \\
& -2E^{\mathcal{F}_t}[M\xi X(T) + \int_t^T QqX\,ds] \\
& +E^{\mathcal{F}_t}[M|\xi|^2 + \int_t^T Qq^2\,ds] + E^{\mathcal{F}_t}\int_t^T [(N\tilde{u},\tilde{u}) + 2(N\Gamma X,\tilde{u})]\,ds \\
= \; & (KX(t),X(t)) - 2(\psi(t),X(t)) + E^{\mathcal{F}_t}[M\xi^2 + \int_t^T Qq^2\,ds] \\
& +E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d K|D_i\tilde{u}+g_i|^2\,ds - 2E^{\mathcal{F}_t}\int_t^T \psi(B\tilde{u}+f)\,ds \\
& -2E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d \phi_i(D_i\tilde{u}+g_i)\,ds + E^{\mathcal{F}_t}\int_t^T (N\tilde{u},\tilde{u})\,ds \\
= \; & K(t)x^2 - 2x\psi(t) + E^{\mathcal{F}_t}[M\xi^2 + \int_t^T Qq^2\,ds] \\
& -2E^{\mathcal{F}_t}\int_t^T \psi f\,ds + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^d [Kg_i^2 - 2\phi_i g_i]\,ds \\
& +E^{\mathcal{F}_t}\int_t^T ((N+\sum_{i=1}^d D_i^* K D_i)(\tilde{u}-u^0),\tilde{u}-u^0)\,ds \\
& -E^{\mathcal{F}_t}\int_t^T ((N+\sum_{i=1}^d D_i^* K D_i)u^0,u^0)\,ds.
\end{aligned}
$$

This completes the proof.

6 Application to the Mean-Variance Hedging Problem

In this section, we consider the mean-variance hedging problem when asset prices follow Itô's processes in an incomplete market framework. The market conditions are allowed to be random, but are assumed to be uniformly bounded which implies by Novikov's condition that there is an equivalent martingale measure. It will be shown that the mean-variance hedging problem in finance of this context is a special case of the linear quadratic optimal stochastic control problem discussed in Section 5, and therefore can be solved completely, by using the above results.

6.1 The financial market model

Consider the financial market in which there are $m+1$ primitive assets: one nonrisky asset (the bond) of price process

$$
S_0(t) = \exp\left(\int_0^t r(s)\,ds\right), \quad 0 \le t \le T, \tag{93}
$$

and m risky assets (the stocks)

$$dS(t) = \operatorname{diag}(S(t))(\mu(t)\,dt + \sigma(t)\,dW(t)), \quad 0 \le t \le T. \tag{94}$$

Here $W = (w_1, \ldots, w_d)^*$ is a d-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{F}_t, 0 \le t \le T\}$ is the P-augmentation of the natural filtration generated by the d-dimensional Brownian motion W. Assume that the instantaneous interest rate r, the m-dimensional appreciation vector process μ and the volatility $m \times d$ matrix process σ are progressively measurable with respect to $\{\mathcal{F}_t, 0 \le t \le T\}$. For simplicity of exposing the main ideas, assume that they are uniformly bounded and there exists a positive constant ε such that

$$\sigma\sigma^*(t) \ge \varepsilon I_{m\times m}, \quad 0 \le t \le T, a.s. \tag{95}$$

The risk premium process is given by

$$\lambda(t) = \sigma^*(\sigma\sigma^*)^{-1}\tilde{\mu}(t), \quad 0 \le t \le T \tag{96}$$

where $e_m = (1, \ldots, 1)^* \in R^m$, and $\tilde{\mu} := \mu - re_m$.

6.2 Formulation of the problem

For any $x \in R$ and $\pi \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$, define the self-financed wealth process X with initial capital x and with quantity π invested in the risky asset S by

$$\begin{cases} dX &= [rX + (\tilde{\mu}, \pi)]\,dt + \pi^*\sigma\,dW, \quad 0 < t \le T, \\ X(0) &= x, \quad \pi \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m). \end{cases} \tag{97}$$

Given a random variable $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, consider the quadratic optimal control problem:

$$\textbf{Problem } \mathcal{P}_{0,x}(\xi) \qquad \min_{\pi \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} E|X^{0,x;\pi}(T) - \xi|^2 \tag{98}$$

where $X^{0,x;\pi}$ is the solution to the wealth equation (97). The associated value function is denoted by $V(t,x), (t,x) \in [0,T] \times R$. The minimum point of $V(t,x)$ over $x \in R$ for given time t is defined to be the approximate price for the contingent claim ξ at time t.

The problem $\mathcal{P}_{0,x}(\xi)$ is the so-called mean-variance hedging problem in mathematical finance. It is a one-dimensional singular stochastic LQ problem $\mathcal{P}_0$. In the next subsection, Theorem 5.1 will be used to give a complete solution of the mean-variance hedging problem $\mathcal{P}_{0,x}(\xi)$.

6.3 A general case of random market conditions: a complete solution

For the case of the mean-variance hedging problem, we have

$$\begin{aligned} A(t) &= r(t), & B(t) &= \tilde{\mu}^*(t), & C_i(t) &= 0, \\ D_i(t) &= \sigma_i^*, i = 1, \ldots, d, & u(t) &= \pi(t), \\ M &= 1, & n &= 1, & \sum_{i=1}^d D_i^* D_i &= \sum_{i=1}^d \sigma_i \sigma_i^* = \sigma\sigma^* \end{aligned}$$

where σ_i is the i-th column of the volatility matrix σ. The associated Riccati equation is a non-linear singular **BSDE**:

$$
\begin{aligned}
dK &= -[2rK - (\tilde{\mu}^* K + \textstyle\sum_{i=1}^d L_i \sigma_i^*)(K\sigma\sigma^*)^{-1}(K\tilde{\mu} + \sum_{i=1}^d L_i\sigma_i)]\,dt + \sum_{i=1}^d L_i\,dw_i \\
&= -[(2r - |\lambda|^2)K - 2(\lambda, L) - K^{-1}L^*\sigma^*(\sigma\sigma^*)^{-1}\sigma L]\,dt + (L,\,dW),\ 0 \le t < T \\
K(T) &= 1.
\end{aligned}
\tag{99}
$$

Let (ψ, ϕ) is the $\mathcal{F}_t$-adapted solution of the following **BSDE**

$$
\begin{aligned}
d\psi &= -\{[r - |\lambda|^2 - (\lambda, K^{-1}L)]\psi \\
&\qquad - \textstyle\sum_{i=1}^d [\lambda_i + K^{-1}\sigma_i^*(\sigma\sigma^*)^{-1}\sigma L]\phi_i\}\,dt + \sum_{i=1}^d \phi_i\,dw_i, \\
&= -\{[r - |\lambda|^2 - (\lambda, K^{-1}L)]\psi - (\lambda + K^{-1}\sigma^*(\sigma\sigma^*)^{-1}\sigma L, \phi)\}\,dt + (\phi,\,dW), \\
\psi(T) &= \xi
\end{aligned}
\tag{100}
$$

An immediate application of Theorem 5.1 provides an explicit formula for the optimal hedging portfolio:

$$
\begin{aligned}
\pi &= -(\sum_{i=1}^d \sigma_i K\sigma_i^*)^{-1}[(\tilde{\mu}K + \sum_{i=1}^d \sigma_i L_i)X - \tilde{\mu}\psi - \sum_{i=1}^d \sigma_i\phi_i] \\
&= -(K\sigma\sigma^*)^{-1}[(\tilde{\mu}K + \sigma L)X - \tilde{\mu}\psi - \sigma\phi] \\
&= -(\sigma\sigma^*)^{-1}[(\tilde{\mu} + \sigma K^{-1}L)X - \tilde{\mu}K^{-1}\psi - \sigma K^{-1}\phi]
\end{aligned}
\tag{101}
$$

where (K, L) is the $\mathcal{F}_t$-adapted solution to the Riccati equation (99). The value function V is also given by

$$
V(t,x) = K(t)x^2 - 2\psi(t)x + E^{\mathcal{F}_t}\xi^2 - E^{\mathcal{F}_t}\int_t^T (\tilde{\mu}\psi + \sigma\phi)^*(\sigma K\sigma^*)(\tilde{\mu}\psi + \sigma\phi)\,ds \tag{102}
$$

where $\phi := (\phi_1, \ldots, \phi_n)^*$. So, the approximate price $p(t)$ at time t for the contingent claim ξ is given by

$$
p(t) = K^{-1}(t)\psi(t). \tag{103}
$$

The above solution need not introduce the additional concepts of the so-called *hedging numeraire* and *variance-optimal martingale measure,* and therefore is simpler than that of Gourieroux et al [12], and Laurent and Pham [19]. To be connected to the latter, the optimal hedging portfolio (101) is rewritten as

$$
\pi = -(\sigma\sigma^*)^{-1}[(\tilde{\mu} + \sigma\tilde{L})(X - \tilde{\psi}) - \sigma\tilde{\phi}]. \tag{104}
$$

Here,

$$
\tilde{L} := LK^{-1}, \quad \tilde{\psi} := \psi K^{-1}, \quad \tilde{\phi} := \phi K^{-1} - L\psi K^{-2}. \tag{105}
$$

and the pair $(\tilde{\psi}, \tilde{\phi})$ solves the following **BSDE**:

$$
\left\{
\begin{aligned}
d\tilde{\psi} &= \{r\tilde{\psi} + (\tilde{\lambda}, \tilde{\phi})\}\,dt + (\tilde{\phi},\,dW), \quad 0 \le t < T, \\
\tilde{\psi}(T) &= \xi
\end{aligned}
\right.
\tag{106}
$$

with

$$\widetilde{\lambda} := \lambda - [I - \sigma^*(\sigma\sigma^*)^{-1}\sigma]LK^{-1}. \tag{107}$$

The process $\widetilde{\psi}$ is just the *approximate price process*, and the **BSDE** (106) is the *approximate pricing equation.*

Note that the optimal hedging portfolio (101) consists of the following two parts:

$$\pi^1 := -(\sigma\sigma^*)^{-1}(\widetilde{\mu} + \sigma\widetilde{L})X \tag{108}$$

and

$$\pi^0 := (\sigma\sigma^*)^{-1}[(\widetilde{\mu} + \sigma\widetilde{L})\widetilde{\psi} + \sigma\widetilde{\phi}], \tag{109}$$

and satisfies

$$\pi = \pi^1 + \pi^0. \tag{110}$$

The first part π^1 is the optimal solution of the *homogeneous* mean-variance hedging problem $\mathcal{P}_{0,x}(0)$ (that is the case of $\xi = 0$ for the problem $\mathcal{P}_{0,x}(\xi)$). The corresponding optimal wealth process $X^{0,1;\pi^1}$ is the solution to the following *optimal closed system*

$$\begin{cases} dX = X[(r - |\lambda|^2 - (\lambda, \widetilde{L}))\, dt - (\lambda + \sigma^*(\sigma\sigma^*)^{-1}\sigma L,\, dW)], & 0 < t \le T, \\ X(0) = 1, \end{cases} \tag{111}$$

and is just the *hedging numéraire*. So, the *hedging numéraire* is just *the state (wealth) transition process* of the optimal closed system (111) from time 0, or it is just *the fundamental solution* of the optimal closed system (111).

To understand the quantity $\widetilde{\lambda}$, consider the **BSDE** satisfied by $(\mathcal{K}, \mathcal{L})$

$$\begin{cases} d\mathcal{K} = \{(2r - |\lambda|^2)\mathcal{K} + 2(\lambda, \mathcal{L}) + \mathcal{K}^{-1}\mathcal{L}^*[I - \sigma^*(\sigma\sigma^*)^{-1}\sigma]\mathcal{L}\}\, dt + (\mathcal{L},\, dW), \\ \mathcal{K}(T) = 1 \end{cases} \tag{112}$$

with $\mathcal{K} := K^{-1}$ and $\mathcal{L} := -LK^{-2}$. It is the **BSRDE** for the following singular stochastic LQ problem (denoted by $\mathcal{P}^*_{0,x}$):

$$\textbf{Problem } \mathcal{P}^*_{0,x} \qquad \min_{\theta \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)} E|\mathcal{X}^{0,x;\theta}(T)|^2 \tag{113}$$

where $\mathcal{X}^{0,x;\theta}$ is the solution to the following stochastic differential equation

$$\begin{cases} d\mathcal{X} = \mathcal{X}[-r\, dt - (\lambda,\, dW)] + ([I - \sigma^*(\sigma\sigma^*)^{-1}\sigma]\theta,\, dW), & 0 \le t \le T, \\ \mathcal{X}(0) = x, \quad \theta \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^d). \end{cases} \tag{114}$$

Its optimal control $\widehat{\theta}$ has the following feedback form

$$\widehat{\theta} = -\mathcal{K}^{-1}\mathcal{L}\mathcal{X} = LK^{-1}\mathcal{X}. \tag{115}$$

The problem $\mathcal{P}_{0,1}^*$ is just the so-called *dual problem* of the problem $\mathcal{P}_{0,1}(0)$ in [12, 19], and so the variance-optimal martingale measure is P_* defined as

$$dP_* := \exp\left\{-\int_0^T (\tilde{\lambda},\, dW) - \frac{1}{2}\int_0^T |\tilde{\lambda}|^2\, dt\right\} dP. \tag{116}$$

P_* is an equivalent martingale measure.

Note that $\tilde{\psi}$ has the following explicit formula:

$$\tilde{\psi}(t) = E_*^{\mathcal{F}_t}\xi \exp\left(-\int_t^T r(s)\, ds\right), \quad 0 \le t \le T. \tag{117}$$

Here, the notation $E_*^{\mathcal{F}_t}$ stands for the expectation operator conditioning on the σ-algebra $\mathcal{F}_t$ with respect to the probability P_*. The discounted $\tilde{\phi}$ is just the integrand of the stochastic-integral-representation of the P^*-martingale $\{E_*^{\mathcal{F}_t}\xi \exp(-\int_0^T r(s)\, ds), 0 \le t \le T\}$ (w.r.t. the P^*-martingale $W + \int_0 \tilde{\lambda}\, dt$).

As in Kohlmann and Zhou [18], again, the formula (104) has an interesting interpretation in terms of mathematical finance. The optimal hedging portfolio π in (104) consists of the two components: (a) $(\sigma\sigma^*)^{-1}\sigma\tilde{\phi}$—it may be interpreted as the perfect hedging portfolio for the contingent claim ξ with the risk premium process $\tilde{\lambda}$ (that is, under the variance-optimal martingale measure), (b) $(\sigma\sigma^*)^{-1}(\tilde{\mu} + \sigma\tilde{L})(\tilde{\psi} - X)$—it is a generalized Merton-type portfolio for a terminal utility function $c(x) = x^2$ (see Merton [22]), which invests the capital $(\tilde{\psi} - X)$ left over after fulfilling the obligation from the perfect hedge under the variance-optimal martingale measure.

6.4 The case of Markovian market conditions

Assume the following Markovian structure for the randomness of the market conditions:

$$r(t,\omega) := r(t, Y_t), \quad \mu(t,\omega) := \mu(t, Y_t), \quad \sigma(t,\omega) := \sigma(t, Y_t) \tag{118}$$

with $\{Y_t, 0 \le t \le T\}$ defined by the stochastic differential equation

$$\begin{cases} dY &= \eta(t,Y)\, dt + \gamma(t,Y)\, dW, \quad 0 \le t \le T, \\ Y_0 &= y \in R^d. \end{cases} \tag{119}$$

In this case, the risk premium process $\{\lambda(t,\omega), 0 \le t \le T\}$ reads

$$\lambda(t,\omega) = \sigma^*(\sigma\sigma^*)^{-1}(t,Y_t)[\mu(t,Y_t) - r(t,Y_t)e_m], \quad 0 \le t \le T. \tag{120}$$

This context includes the stochastic volatility models usually studied in the literature (Hull and White [15], Stein and Stein [35], Heston [13]).

Under the above assumption, the Riccati equation (99) and the stochastic differential equation (119) constitute *a forward-backward stochastic differential equation.* Define the function h as the generator of BSDE (99), that is

$$\begin{aligned} h(t,y,z,v) := \;& z(2r - |\lambda|^2)(t,y) - 2v^*\lambda(t,y) - z^{-1}v^*\sigma^*(\sigma\sigma^*)^{-1}\sigma(t,y)v, \\ & \forall (t,y,v) \in [0,T] \times R \times R^d \text{ and } z \neq 0. \end{aligned} \tag{121}$$

Then, it is straightforward in the literature that the solution to the Riccati equation (99) can be characterized by the parabolic partial differential equation:

$$\begin{cases} Z_t + (\eta(t,y), Z_y) + \frac{1}{2}\text{tr } (\gamma\gamma^*(t,y)Z_{yy}) + h(t,y,Z,Z_y\gamma(t,y)) = 0, \\ \qquad\qquad y \in R^d, 0 \le t < T, \\ Z(T,y) = 1, \quad y \in R^d \end{cases} \tag{122}$$

through the relation

$$K(t) = Z(t,Y_t), \qquad L(t) = [Z_y\gamma(t,Y_t)]^*. \tag{123}$$

The reader is referred to Peng [28], Pardoux and Peng [25], and Pardoux and Tang [26] for details.

6.5 On a modified model

Consider the optimal control problem:

$$\textbf{Problem } \mathcal{MP}_{0,x}(\xi) \qquad \min_{\pi\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} E\left[\int_0^T |X^{0,x;\pi}(s) - q_s|^2\, ds + |X^{0,x;\pi}(T) - \xi|^2\right] \tag{124}$$

where $q_s := E^{\mathcal{F}_s}\xi$ and $X^{0,x;\pi}$ is the solution to the wealth equation (97). Identically as before, we use Theorem 5.1 to solve it.

The associated Riccati equation is a non-linear singular **BSDE**:

$$\begin{array}{l} dK = -[(2r - |\lambda|^2)K + 1 - 2(\lambda, L) - K^{-1}L^*\sigma^*(\sigma\sigma^*)^{-1}\sigma L]\, dt + (L,\, dW), \\ K(T) = 1. \end{array} \tag{125}$$

Let (ψ, ϕ) is the $\mathcal{F}_t$-adapted solution of the following **BSDE**

$$\begin{array}{l} d\psi = -\{[r - |\lambda|^2 - (\lambda, K^{-1}L)]\psi - (\lambda + K^{-1}\sigma^*(\sigma\sigma^*)^{-1}\sigma L, \phi) + q\}\, dt + (\phi,\, dW), \\ \psi(T) = \xi \end{array} \tag{126}$$

An immediate application of Theorem 5.1 provides an explicit formula for the optimal hedging portfolio:

$$\pi = -(\sigma\sigma^*)^{-1}[(\tilde{\mu} + \sigma K^{-1}L)X - \tilde{\mu}K^{-1}\psi - \sigma K^{-1}\phi] \tag{127}$$

where (K, L) is the $\mathcal{F}_t$-adapted solution to the Riccati equation (125). The value function V is also given by

$$V(t,x) = K(t)x^2 - 2\psi(t)x + E^{\mathcal{F}_t}\left[\xi^2 + \int_t^T q_s^2\, ds\right] - E^{\mathcal{F}_t}\int_t^T (\tilde{\mu}\psi + \sigma\phi)^*(\sigma K\sigma^*)(\tilde{\mu}\psi + \sigma\phi)\, ds$$

where $\phi := (\phi_1, \ldots, \phi_n)^*$. So, the approximate price $p(t)$ at time t for the contingent claim ξ is given by

$$p(t) = K^{-1}(t)\psi(t). \tag{128}$$

The optimal hedging portfolio (127) is rewritten as

$$\pi = -(\sigma\sigma^*)^{-1}[(\tilde{\mu} + \sigma\widetilde{L})(X - \widetilde{\psi}) - \sigma\widetilde{\phi}]. \tag{129}$$

Here,

$$\widetilde{L} := LK^{-1}, \quad \widetilde{\psi} := \psi K^{-1}, \quad \widetilde{\phi} := \phi K^{-1} - L\psi K^{-2}. \tag{130}$$

and the pair $(\widetilde{\psi}, \widetilde{\phi})$ solves the following **BSDE**:

$$\begin{cases} d\widetilde{\psi} &= \{r\widetilde{\psi} + (\widetilde{\lambda}, \widetilde{\phi}) + K^{-1}(\widetilde{\psi} - q)\}\, dt + (\widetilde{\phi}, dW), \quad 0 \le t < T, \\ \widetilde{\psi}(T) &= \xi \end{cases} \tag{131}$$

with

$$\widetilde{\lambda} := \lambda - [I - \sigma^*(\sigma\sigma^*)^{-1}\sigma]LK^{-1}. \tag{132}$$

The process $\widetilde{\psi}$ is just the *approximate price process*, and the **BSDE** (131) is the *approximate pricing equation*.

Similarly as in Kohlmann and Zhou [18], the economic interpretation for the *approximate pricing equation* (131) can also be given.

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References

[1] Bismut, J. M., *Conjugate convex functions in optimal stochastic control,* J. Math. Anal. Appl., 44 (1973), 384–404

[2] Bismut, J. M., *Linear quadratic optimal stochastic control with random coefficients,* SIAM J. Control Optim., 14 (1976), 419-444

[3] Bismut, J. M., *Controle des systems lineares quadratiques: applications de l'integrale stochastique,* Séminaire de Probabilités XII, eds : C. Dellacherie, P. A. Meyer et M. Weil, LNM 649, Springer-Verlag, Berlin 1978

[4] Chen, S., Li, X. and Zhou, X., *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim. **36 (1998)**, 1685-1702

[5] Delbaen, F., Monat P., Schachermayer, W., Schweizer, M., and Stricker, C., *Weighted norm inequalities and hedging in incomplete markets,* Finance & Stochastics, 1 (1997), 181–227

[6] Delbaen, F. and Schachermayer, W., *Attainable claims with p'th moments,* Annales de l'Institut Henri Poincaré, 32 (1996), 743–763

[7] Duffie, D. and Richardson, H. R., *Mean-variance hedging in continuous time,* Ann. Appl. Prob., 1 (1991), 1-15

[8] Föllmer, H. and Leukert, P., *Efficient hedging: cost versus shortfall risk,* Finance & Stochastics, 4 (2000), 117–146

[9] Föllmer, H. and Sonderman, D., *Hedging of non-redundant contingent claims.* In: Mas-Colell, A., Hildebrand, W. (eds.) Contributions to Mathematical Economics. Amsterdam: North Holland 1986, pp. 205-223

[10] Föllmer, H. and Schweizer, M., *Hedging of contingent claims under incomplete information.* In: Davis, M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis. (Stochastics Monographs vol. 5) London-New York: Gordon & Breach, 1991, pp. 389-414

[11] Gal'chuk, L. I., *Existence and uniqueness of a solution for stochastic equations with respect to semimartingales,* Theory of Probability and Its Applications, 23 (1978), 751–763

[12] Gourieroux, C., Laurent, J. P. and Pham, H., *Mean-variance hedging and numéraire,* Mathematical Finance, 8 (1998), 179-200.

[13] Heston, S., *A closed-form solution for option with stochastic volatility with applications to bond and currency options.* Rev. Financial Studies, 6 (1993), 327–343

[14] Hipp, C., *Hedging general claims,* Proceedings of the 3rd AFIR colloqium, Rome, Vol. 2, pp. 603-613 (1993)

[15] Hull, J. and White, A., *The pricing of options on assets with stochastic volatilities.* J. Finance, 42 (1987), 281-300

[16] Kobylanski, M., *Résultats d'existence et d'unicité pour des équations différentielles stochastiques rétrogrades avec des générateurs á croissance quadratique,* C. R. Acad. Sci. Paris, 324 (1997), Série I, 81-86

[17] Kohlmann, M. and Tang, S., *Optimal control of linear stochastic systems with singular costs, and the mean-variance hedging problem with stochastic market conditions.* submitted

[18] Kohlmann, M. and Zhou, X., *Relationship between backward stochastic differential equations and stochastic controls: a linear-quadratic approach,* SIAM J. Control & Optim., 38 (2000), 1392-1407

[19] Laurent, J. P. and Pham, H., *Dynamic programming and mean-variance hedging,* Finance and Stochastics, 3 (1999), 83-110

[20] Lepeltier, J. P. and San Martin, J., *Backward stochastic differential equations with continuous coefficient,* Statistics & Probability Letters, 32 (1997), 425-430

[21] Lepeltier, J. P. and San Martin, J., *Existence for BSDE with superlinear-quadratic coefficient,* Stochastics & Stochastics Reports, 63 (1998), 227-240

[22] Merton, R., *Optimum consumption and portfolio rules in a continuous time model,* J. Econ. Theory, 3 (1971), pp. 373-413; Erratum 6 (1973), pp. 213-214.

[23] Monat, P. and Stricker, C., *Föllmer-Schweizer decomposition and mean-variance hedging of general claims,* Ann. Probab. 23, 605–628

[24] Pardoux, E. and Peng, S., *Adapted solution of backward stochastic equation,* Systems Control Lett., 14 (1990), 55–61

[25] Pardoux, E. and Peng, S., *Backward stochastic differential equations and quasi-linear parabolic partial differential equations,* in: Rozovskii, B. L., Sowers, R. S. (eds.) Stochastic Partial Differential Equations and Their Applications, Lecture Notes in Control and Information Science 176, Springer, Berlin, Heidelberg, New York 1992, pp. 200–217

[26] Pardoux, E. and Tang, S., *Forward-backward stochastic differential equations with application to quasi-linear partial differential equations of second-order*, Probability Theory and Related Fields, 114 (1999), 123-150

[27] Peng, S., *Stochastic Hamilton-Jacobi-Bellman equations*, SIAM J. Control and Optimization **30 (1992)** , 284-304

[28] Peng, S., *Probabilistic interpretation for systems of semilinear parabolic PDEs,* Stochastics & Stochastic Reports, 37 (1991), 61-74

[29] Peng, S., *A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation,* Stochastics & Stochastics Reports, 38 (1992), 119–134

[30] Peng, S., *Some open problems on backward stochastic differential equations,* Control of distributed parameter and stochastic systems, proceedings of the IFIP WG 7.2 international conference, June 19-22, 1998, Hangzhou, China

[31] Pham, H., Rheinländer, T., and Schweizer, M., *Mean-variance hedging for continuous processes: New proofs and examples,* Finance and Stochastics, 2 (1998), 173–198

[32] Schweizer, M., *Mean-variance hedging for general claims,* Ann. Appl. Prob., 2 (1992), 171-179

[33] Schweizer, M., *Approaching random variables by stochastic integrals,* Ann. Probab., 22 (1994), 1536–1575

[34] Schweizer, M., *Approximation pricing and the variance-optimal martingale measure,* Ann. Probab., 24 (1996), 206–236

[35] Stein, E. M. and Stein, J. C., *Stock price distributions with stochastic volatility: an analytic approach,* Rev. Financial Studies, 4 (1991), 727–752

[36] Wonham, W. M., *On a matrix Riccati equation of stochastic control,* SIAM J. Control Optim., 6 (1968), 312–326