Recent Advances in Backward Stochastic Riccati Equations and Their Applications*

Michael Kohlmann[†] Shanjian Tang [‡]

October 25, 2000

Abstract

The following backward stochastic Riccati differential equation (BSRDE in short)

$$\begin{cases} dK &= -[A'K + KA + \sum_{i=1}^{d} C_i' K C_i + Q + \sum_{i=1}^{d} (C_i' L_i + L_i C_i) \\ &\quad -(KB + \sum_{i=1}^{d} C_i' K D_i + \sum_{i=1}^{d} L_i D_i)(N + \sum_{i=1}^{d} D_i' K D_i)^{-1} \\ &\quad \times (KB + \sum_{i=1}^{d} C_i' K D_i + \sum_{i=1}^{d} L_i D_i)']\, dt + \sum_{i=1}^{d} L_i \, dw_i, \qquad 0 \le t < T, \\ K(T) &= M. \end{cases}$$

is motivated, and is then studied. Some properties are presented. The existence and uniqueness of a global adapted solution to a BSRDE has been open for the case $D_i \neq 0$ for more than two decades. Our recent results on this topic are summarized. Finally, applications are addressed, both in finance and control.

Key words: backward stochastic Riccati equation, stochastic linear-quadratic control problem, mean-variance hedging, variance-optimal martingale measure

AMS Subject Classifications. 93E20, 60H10, 91B28

Abbreviated title: Backward stochastic Riccati equations and applications

1 Introduction

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a fixed complete probability space on which is defined a standard d-dimensional $\mathcal{F}_t$-adapted Brownian motion $w(t) \equiv (w_1(t), \cdots, w_d(t))'$. Denote by $\mathcal{F}_t$ the

*Both authors gratefully acknowledge the support by the Center of Finance and Econometrics, University of Konstanz.**

[†]Department of Mathematics and Statistics, University of Konstanz, D-78457, Konstanz, Germany

[‡]Department of Mathematics, Fudan University, Shanghai 200433, China. **This author is supported by a Research Fellowship from the Alexander von Humboldt Foundation and by the National Natural Science Foundation of China under Grant No. 79790130.**

completion, by the totality $\mathcal{N}$ of all null sets of $\mathcal{F}$, of the natural filtration $\{\mathcal{F}_t^w\}$ generated by w.

Consider the following BSRDE:

$$\begin{cases} dK &= -G(t,K,L)\,dt + \sum_{i=1}^d L_i\,dw_i, \qquad 0 \le t < T, \\ K(T) &= M \end{cases} \tag{1}$$

where the generator G is given by

$$G(t,K,L) \;:=\; A'K + KA + \sum_{i=1}^d C_i'KC_i + Q + \sum_{i=1}^d (C_i'L_i + L_iC_i) + F(t,K,L) \tag{2}$$

with

$$\begin{aligned} F(t,K,L) := \; & -[KB(t) + \sum_{i=1}^d C_i(t)'KD_i(t) + \sum_{i=1}^d L_iD_i(t)][N(t) + \sum_{i=1}^d D_i(t)'KD_i(t)]^{-1} \\ & \times[KB(t) + \sum_{i=1}^d C_i(t)'KD_i(t) + \sum_{i=1}^d L_iD_i(t)]', \\ & \quad \forall (t,K,L) \in [0,T] \times \mathcal{S}_+^n \times (\mathcal{S}^n)^d. \end{aligned} \tag{3}$$

It will be called the BSRDE $(A,B;C_i,D_i,i=1,\ldots,d;Q,N,M)$ in the following for convenience of indicating the associated coefficients. The coefficients appearing here will be defined in Section 2.

BSRDEs have at least the following two motivations.

(1) The control-theoretic motivation. The BSRDE (1) arises from solution of the optimal control problem

$$\inf_{u \in \mathcal{L}_{\mathcal{F}}^2(0,T;R^m)} J(u;0,x) \tag{4}$$

where for $t \in [0,T]$ and $x \in R^n$,

$$J(u;t,x) := E^{\mathcal{F}_t}[\int_t^T [(Nu_s,u_s) + (QX_s^{t,x;u}, X_s^{t,x;u})]\,ds + (MX_T^{t,x;u}, X_T^{t,x;u})] \tag{5}$$

and $X^{t,x;u}$ solves the following stochastic differential equation

$$\begin{cases} dX_s &= (AX_s + Bu_s)\,ds + \sum_{i=1}^d (C_iX_s + D_iu_s)\,dw_i, \quad t \le s \le T, \\ X_t &= x. \end{cases} \tag{6}$$

We have the following connection: if the BSRDE (1) has a solution (K,L), the solution for the above *linear-quadratic optimal control problem* (LQ problem in short) has the following closed form (also called the feedback form):

$$u_s = -(N + \sum_{i=1}^d D_i'KD_i)^{-1}[B'K + \sum_{i=1}^d D_i'KC_i + \sum_{i=1}^d D_i'L_i]X_s \tag{7}$$

and the associated value function V is the following quadratic form

$$V(t,x) := \inf_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x) = (K(t)x,x), \qquad 0\le t\le T, x\in R^n. \tag{8}$$

In this way, solution of the above LQ problem is reduced to solving the BSRDE (1). The LQ problem with a terminal expected constraint ($EX(T)=x_T$ for some fixed $x_T\in R^n$, for example), is also reduced to solution of a BSRDE.

(2) The financial motivation. A mean-variance hedging problem is a one-dimensional, nonhomogeneous, singular stochastic LQ problem. A mean-variance portfolio selection problem is a one-dimensional, nonhomogeneous, singular stochastic LQ problem with an expected terminal state constraint. Solution of these two classes of mathematical financial problems is reduced to solution of the associated one-dimensional BSRDEs.

The rest of the paper is organized as follows. Preliminaries are done in Section 2 where the notation is listed and a solution of a BSRDE is defined. In Section 3, a historical review is given on BSRDEs, and the known existence and uniqueness result due to Bismut [5] and Peng [29] is stated. Section 4 collects various properties of BSRDEs. Section 5 summarizes our recent results on the existence and uniqueness of a global adapted solution of BSRDE (1). Finally in section 6, BSRDEs are applied to control and finance.

2 Preliminaries

Notation. Throughout this paper, the following additional notation will be used:

M'	:	the transpose of any vector or matrix M;
$\lvert M\rvert$	:	$=\sqrt{\sum_{ij} m_{ij}^2}$ for any vector or matrix $M=(m_{ij})$;
(M_1,M_2)	:	the inner product of the two vectors M_1 and M_2;
R^n	:	the n-dimensional Euclidean space;
R_+	:	the set of all nonnegative real numbers;
$\mathcal{S}^n$	:	the Euclidean space of all $n\times n$ symmetric matrices;
$\mathcal{S}^n_+$	:	the set of all $n\times n$ nonnegative definite matrices;
$C([0,T];H)$	:	the Banach space of H-valued continuous functions on $[0,T]$, endowed with the maximum norm for a given Hilbert space H;
$\mathcal{L}^2_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued $\mathcal{F}_t$-adapted square-integrable stochastic processes f on $[0,T]$, endowed with the norm $(E\int_0^T \lvert f(t)\rvert^2\,dt)^{1/2}$ for a given Euclidean space H;
$\mathcal{L}^\infty_{\mathcal{F}}(0,T;H)$	:	the Banach space of H-valued, $\mathcal{F}_t$-adapted, essentially bounded stochastic processes f on $[0,T]$, endowed with the norm $\operatorname{ess\,sup}_{t,\omega}\lvert f(t)\rvert$ for a given Euclidean space H;
$L^2(\Omega,\mathcal{F},P;H)$	:	the Banach space of H-valued norm-square-integrable random variables on the probability space $(\Omega,\mathcal{F},P)$ for a given Banach space H;

and $L^\infty(\Omega,\mathcal{F},P;C([0,T];R^n))$ is the Banach space of $C([0,T];R^n)$-valued, essentially maximum-norm-bounded random variables f on the probability space $(\Omega, \mathcal{F}, P)$, endowed with the norm $\operatorname{ess\,sup}_{\omega\in\Omega} \max_{0\le t\le T} |f(t,\omega)|$.

We make the following two basic assumptions.

($\mathcal{A}1$) *The coefficients A, B, C_i, and D_i are $\mathcal{F}_t$-progressively measurable bounded matrix-valued processes, defined on $\Omega\times[0,T]$, of dimensions $n\times n, n\times m, n\times n, n\times m$ respectively. M is an $\mathcal{F}_T$-measurable, nonnegative, and bounded $n\times n$ random matrix, and Q and N are $\mathcal{F}_t$-progressively measurable, bounded, and nonnegative $n\times n$ and $m\times m$ matrix processes, respectively.*

($\mathcal{A}2$) *N is uniformly positive.* Or

($\mathcal{A}3$) *M and $\sum_{i=1}^{d} D_i' D_i$ are uniformly positive.*

Definition 2.1. A solution of the BSRDE (1) is a pair (K, L) of processes such that

(i) $K \in \mathcal{L}^{\infty}_{\mathcal{F}}(0,T;\mathcal{S}^n) \cap L^{\infty}(\Omega, \mathcal{F}_T, P; C([0,T];\mathcal{S}^n))$, $L \in \left(\mathcal{L}^2_{\mathcal{F}}(0,T;\mathcal{S}^n)\right)^d$,

(ii) $N(t) + \sum_{i=1}^{d} D_i(t)' K D_i(t)$ is uniformly positive with respect to (t,ω),

(iii) $K(t) = M + \int_t^T G(s, K(s), L(s))\, ds - \int_t^T L(s)\, dw(s),\ 0 \le t \le T.$

When the pair (K, L) is a solution of the BSRDE (1), we also say that it solves the BSRDE (1).

3 A Historical Review

First, consider the regular case, i.e., N is assumed to be uniformly positive. When the coefficients A, B, C_i, D_i, Q, N, M are all deterministic, then $L_1 = \cdots = L_d = 0$ and the BSRDE (1) reduces to the following nonlinear matrix ordinary differential equation:

$$\begin{cases} dK &= -[A'K + KA + \sum_{i=1}^{d} C_i' K C_i + Q - (KB + \sum_{i=1}^{d} C_i' K D_i) \\ & \quad \times (N + \sum_{i=1}^{d} D_i' K D_i)^{-1} (KB + \sum_{i=1}^{d} C_i' K D_i)']\, dt,\ 0 \le t < T, \\ K(T) &= M, \end{cases} \tag{9}$$

which was completely solved by Wonham [41] by applying Bellman's principle of quasi-linearization and a monotone convergence approach.

The attention to the randomness of the coefficients A, B, C, D, Q, N, M is due to Bismut. Bismut [4, 5] initially studied the case of random coefficients, but he could solve only some special simple cases at that time. Let the integer $d_0 \ge 0$, and denote by $\{\mathcal{F}_t^2, 0 \le t \le T\}$ the P-augmented natural filtration generated by the $(d-d_0)$-dimensional Brownian motion $(w_{d_0+1}, \ldots, w_d)$. He assumed that the randomness of the coefficients only comes from the smaller filtration $\{\mathcal{F}_t^2\}$, which leads to $L_1 = \cdots = L_{d_0} = 0$. He further assumed in the paper [4] that

$$C_{d_0+1} = \cdots = C_d = 0, \quad D_{d_0+1} = \cdots = D_d = 0, \tag{10}$$

under which the BSRDE (1) becomes the following one:

$$\left\{\begin{array}{rcl} dK & = & -[A'K+KA+\sum_{i=1}^{d_0} C_i'KC_i+Q \\ & & -(KB+\sum_{i=1}^{d_0} C_i'KD_i)(N+\sum_{i=1}^{d_0} D_i'KD_i)^{-1}(KB+\sum_{i=1}^{d_0} C_i'KD_i)']\, dt \\ & & +\sum_{i=d_0+1}^{d} L_i\, dw_i, \qquad 0\le t<T, \\ K(T) & = & M, \end{array}\right. \tag{11}$$

and the generator does not involve L at all. In the work [5], he assumed that

$$D_{d_0+1}=\cdots=D_d=0, \tag{12}$$

under which the BSRDE (1) becomes the following one

$$\left\{\begin{array}{rcl} dK & = & -[A'K+KA+\sum_{i=1}^{d} C_i'KC_i+Q+\sum_{i=d_0+1}^{d}(C_i'L_i+L_iC_i) \\ & & -(KB+\sum_{i=1}^{d_0} C_i'KD_i)(N+\sum_{i=1}^{d_0} D_i'KD_i)^{-1}(KB+\sum_{i=1}^{d_0} C_i'KD_i)']\, dt \\ & & +\sum_{i=d_0+1}^{d} L_i\, dw_i, \qquad 0\le t<T, \\ K(T) & = & M, \end{array}\right. \tag{13}$$

and the generator depends on the second unknown variable $(L_{d_0+1},\ldots,L_d)'$ in a linear way. Moreover, his method was rather complicated.

Later, Peng [29] gave a nice treatment on the proof of existence and uniqueness for the BSRDE (13), by using Bellman's quasilinear principle and a method of monotone convergence—a generalization of Wonham's approach to the random situation.

The following proposition states the above-mentioned result on existence and uniqueness of a global adapted solution to the BSRDE (1), due to Bismut [5] and Peng [29].

Theorem 3.1. *Let the assumptions* $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$ *be satisfied. Assume that all the coefficients* A, B, C_i, D_i, Q *and* N *are* $\mathcal{F}_t^2$*-progressively measurable and that* M *is* $\mathcal{F}_T^2$*-measurable. Then, the* BSRDE (13) *has a unique* $\mathcal{F}_t^2$*-adapted global solution* (K,L) *with* $K(t)\ge 0, t\in[0,T]$.

The general case where the generator of BSRDE (1) is allowed to contain a quadratic term of L, turns out to become a long-standing problem. As early as in 1978, Bismut [5] commented on page 220 that:"Nous ne pourrons pas démontrer l'existence de solution pour l'équation (2.49) dans le cas général." (We could not prove the existence of solution for equation (2.49) for the general case.) On page 238, he pointed out that the essential difficulty for solution of the general BSRDE (1) lies in the integrand of the martingale term which appears in the generator in a quadratic way. Two decades later in 1998, Peng [32] included the above problem in his list of open problems on BSDEs.

Second, consider the singular case $N=0$. Kohlmann and Zhou [20] studied the following case:

$$C=0, M=I_{n\times n}, D_i=I_{m\times m},\ i=1,\cdots,d.$$

They made the two rather restrictive assumptions: (a) all the coefficients are deterministic and (b) $A + A' \geq BB'$. Under the above-described framework, they obtain the existence and uniqueness of a global solution. Their method is based on an existence criterion of Chen, Li and Zhou [6].

4 Fundamentals of BSRDEs

This section collects various properties of BSRDEs, most of which turn out to be helpful to the proof of the existence and uniqueness of a global adapted solution. They are consequences of the special structure of BSRDE (1) from different points of view.

For convenience of following exposition, we introduce the following notation. Define $\Gamma : [0,T] \times \mathcal{S}^n \times R^{n\times d} \to R^{m\times n}$ by

$$\Gamma(\cdot, K, L) = -(N + \sum_{i=1}^{d} D_i' K D_i)^{-1}(KB + \sum_{i=1}^{d} C_i' K D_i + \sum_{i=1}^{d} L_i D_i)' \tag{14}$$

and

$$\widehat{A} := A + B\Gamma(\cdot, K, L), \widehat{C}_i := C_i + D_i\Gamma(\cdot, K, L), \quad i = 1, \ldots, d. \tag{15}$$

Consider the following SDE

$$\begin{cases} dY_s &= \widehat{A}(s)Y_s\, ds + \sum_{i=1}^{d} \widehat{C}_i Y_s\, dw_i(s), \quad t \leq s \leq T, \\ Y_t &= I_{n\times n}. \end{cases} \tag{16}$$

In view of Gal'chuk [11], this equation has a unique strong solution (denoted by $\Phi(\cdot, t)$) when $L \in (\mathcal{L}^2_{\mathcal{F}}(0, T; \mathcal{S}^n))^d$.

4.1 Regularity

Theorem 4.1. *Let (K, L) solve the BSRDE (1). Then,*

$$\Gamma(\cdot, K, L)\Phi(\cdot, t) \in \mathcal{L}^2_{\mathcal{F}}(t, T; R^{m\times n}),\ \ \Phi^* := \max_{t\leq s\leq T} |\Phi(s, t)| \in L^2(\Omega, \mathcal{F}, P). \tag{17}$$

Consider the optimal control problem

$$\inf_{u\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(u; 0, x) \tag{18}$$

where for $t \in [0, T]$ and $x \in R^n$,

$$J(u; t, x) := E^{\mathcal{F}_t}[\int_t^T [(Nu_s, u_s) + (QX_s^{t,x;u}, X_s^{t,x;u})]\, ds + (MX_T^{t,x;u}, X_T^{t,x;u})] \tag{19}$$

and $X_\cdot^{t,x;u}$ solves the SDE

$$\begin{cases} dX_s &= (AX_s + Bu_s)\, ds + \sum_{i=1}^{d}(C_iX_s + D_iu_s)\, dw_i, \quad t \leq s \leq T, \\ X_t &= x. \end{cases} \tag{20}$$

Using a method of functional analysis as in Bismut [5] and Kohlmann and Tang [17], we can show that the assumption ($\mathcal{A}2$) or ($\mathcal{A}3$) implies the existence and uniqueness of the optimal control $\widehat{u} \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$. From the optimality conditions, we have

$$\widehat{u} = \Gamma(\cdot, K, L)\Phi(\cdot, t)x, \ \ X^{t,x;\widehat{u}} = \Phi(\cdot, t)x.$$

Then, Theorem 4.1 follows. The reader is referred to Subsection 6.1 for more details.

4.2 The Feynman-Kac Representation

Theorem 4.2. *Let (K, L) solve the BSRDE (1). Then,*

$$(K(t)x, x) = V(t,x) := \inf_{u \in L^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x), \forall t \in [0,T], \ x \in R^n. \tag{21}$$

In view of Theorem 4.1, the proof is straightforward. This theorem plays a crucial role in the proof of uniqueness: it immediately gives the uniqueness of K as the first part of the solution. It also plays an important role in the proof of the closed property of the solutions (See Theorem 4.9).

4.3 Nonnegativity

Theorem 4.3. *If (K, L) solves the BSRDE (1), then $K(t) \geq 0, \ 0 \leq t \leq T$.*
Noticing the nonnegativity of M, Q, N, we have $K(t) \geq 0$ from Theorem 4.2.

4.4 Monotonicity

From Theorem 4.2, we immediately obtain the following monotone property of the solution to the BSRDE (1).

Theorem 4.4. *Let (K, L) and $(\widetilde{K}, \widetilde{L})$ be the solutions to the BSRDEs $(A, B; C_i, D_i, i = 1, \cdots, d; Q, N, M)$ and $(A, B; C_i, D_i, i = 1, \cdots, d; \widetilde{Q}, \widetilde{N}, \widetilde{M})$, respectively. If*

$$\widetilde{Q} \geq Q \geq 0, \widetilde{N} \geq N > 0, \widetilde{M} \geq M \geq 0,$$

then $\widetilde{K} \geq K$, a.s.a.e..

4.5 Minimality

Theorem 4.5. *Let $\widetilde{\Gamma} \in \mathcal{L}^\infty_{\mathcal{F}}(0,T;R^{m\times n})$ and let $(\widetilde{K}, \widetilde{L})$ be the solution of the following linear BSDE*

$$\left\{\begin{array}{rcl} dK &=& -[\widetilde{A}'K + K\widetilde{A} + \sum_{i=1}^d \widetilde{C}_i'K\widetilde{C}_i + Q + \sum_{i=1}^d (\widetilde{C}_i'L_i + L_i\widetilde{C}_i) \\ && +\widetilde{\Gamma}N\widetilde{\Gamma}]\, dt + \sum_{i=1}^d L_i\, dw_i, \qquad 0 \leq t < T, \\ K(T) &=& M \end{array}\right. \tag{22}$$

where

$$\widetilde{A} := A + B\widetilde{\Gamma}, \quad \widetilde{C}_i := C_i + D_i\widetilde{\Gamma},\ I = 1, \ldots, d. \tag{23}$$

If (K, L) *is the solution to the BSRDE* (1)*, then* $K \le \widetilde{K}$*, a.s.a.e..*

Note that a deterministic version of Theorem 4.5 was given by Wonham [41].

For the one-dimensional case, in view of the minimality of the generator:

$$G(t, K, L) \le [\widetilde{A}'K + K\widetilde{A} + \sum_{i=1}^{d} \widetilde{C}_i'K\widetilde{C}_i + Q + \sum_{i=1}^{d}(\widetilde{C}_i'L_i + L_i\widetilde{C}_i) + \widetilde{\Gamma}N\widetilde{\Gamma}],\ \forall \widetilde{\Gamma} \in R^{m\times n},$$

this theorem is an immediate consequence of the existing comparison theory for one-dimensional BSDEs. For the general multi-dimensional case, it is an immediate consequence of the above monotone theorem 4.4 for the BSRDE $(A, 0; C, 0; Q, N, M)$.

4.6 *A priori* estimates and the BMO-property.

We have the following *a priori* estimate of boundedness.

Theorem 4.6. *Let* (K, L) *solve the BSRDE* $(A, B; C_i, D_i, i = 1, \ldots, d; Q, N, M)$*. Then, there is a deterministic positive constant* ε_0 *such that the following estimates hold:*

$$0 \le K(t) \le \varepsilon_0 I_{n\times n}, \quad E^{\mathcal{F}_t}\left(\int_t^T |L|^2\, ds\right)^p \le \varepsilon_0, \quad \forall p \ge 1. \tag{24}$$

Here ε_0 *depends on the uniform upper bound of all the coefficients.*

Proof of Theorem 4.6. From Theorem 4.3, we have $K \ge 0$. Note that (K, L) satisfies the BSRDE:

$$\left\{\begin{array}{rcl} dK &=& -\Big[A'K + KA + \sum_{i=1}^{d} C_i'KC_i + Q + \sum_{i=1}^{d}(C_i'L_i + L_iC_i) \\ && +F(t, K, L)\Big]\, dt + \sum_{i=1}^{d} L_i\, dw_i, \qquad 0 \le t < T, \\ K(T) &=& M. \end{array}\right. \tag{25}$$

Using Itô's formula, we get

$$\left\{\begin{array}{rcl} d|K|^2 &=& -\Big[4\,\mathrm{tr}\ \left(K^2A\right) + \sum_{i=1}^{d} 2\,\mathrm{tr}\ (KC_i'KC_i) + 2\,\mathrm{tr}\ (KQ) \\ && + \sum_{i=1}^{d} 4\,\mathrm{tr}\ (KL_iC_i) + 2\,\mathrm{tr}\ [KF(t, K, L)] - |L|^2\Big]\, dt \\ && + \sum_{i=1}^{d} 2\,\mathrm{tr}\ (KL_i)\, dw_i, \qquad 0 \le t < T, \\ |K|^2(T) &=& |M|^2. \end{array}\right. \tag{26}$$

We observe that since

$$F(t, K, L) \le 0, \quad K \ge 0,$$

we have

$$2\,\mathrm{tr}\ [KF(t,K,L)] = 2\,\mathrm{tr}\ \left[K^{\frac{1}{2}}F(t,K,L)K^{\frac{1}{2}}\right] \le 0. \tag{27}$$

Hence,

$$\begin{aligned} |K|^2(t) + \int_t^T |L|^2\,ds \ \le\ & |M|^2 + \int_t^T \Big[4\,\mathrm{tr}\ \left(K^2A\right) + \sum_{i=1}^d 2\,\mathrm{tr}\ (KC_i'KC_i) \\ & +2\,\mathrm{tr}\ (KQ) + \sum_{i=1}^d 4\,\mathrm{tr}\ (KL_iC_i)\Big]\,ds \\ & - \int_t^T \sum_{i=1}^d 2\,\mathrm{tr}\ (KL_i)\ dw_i, \qquad 0 \le t < T. \end{aligned} \tag{28}$$

Using the elementary inequality

$$2ab \le a^2 + b^2$$

and taking the expectation on both sides with respect to $\mathcal{F}_r$ for $r \ge t$, we obtain that

$$E^{\mathcal{F}_r}|K|^2(t) + \frac{1}{2}E^{\mathcal{F}_r}\int_t^T |L|^2\,ds \le \varepsilon_1 + \varepsilon_1\int_t^T E^{\mathcal{F}_r}|K|^2(s)\,ds, \quad 0 \le r \le t < T. \tag{29}$$

Using Gronwall's inequality, we derive from the last inequality the first one of the estimates (24). In return, we derive from the second last inequality that

$$\int_t^T |L|^2\,ds \le \varepsilon_2 + \varepsilon_2\int_t^T |L|\,ds - \int_t^T \sum_{i=1}^d 2\,\mathrm{tr}\ (KL_i)\ dw_i. \tag{30}$$

Therefore,

$$E^{\mathcal{F}_t}\left(\int_t^T |L|^2\,ds\right)^p \le 3^p\left[\varepsilon_2^p + \varepsilon_2^p E^{\mathcal{F}_t}\left(\int_t^T |L|\,ds\right)^p + E^{\mathcal{F}_t}\left|\int_t^T \sum_{i=1}^d 2\mathrm{tr}KL_i\,dw_i\right|^p\right]. \tag{31}$$

We have from the Burkholder-Davis-Gundy inequality the following

$$E^{\mathcal{F}_t}\left|\int_t^T \sum_{i=1}^d 2\,\mathrm{tr}\ (KL_i)\ dw_i\right|^p \le 2^p E^{\mathcal{F}_t}\left|\int_t^T |K|^2|L|^2\,ds\right|^{p/2},$$

while from the Cauchy-Schwarz inequality, we have

$$E^{\mathcal{F}_t}\left(\int_t^T |L|\,ds\right)^p \le T^{p/2}E^{\mathcal{F}_t}\left(\int_t^T |L|^2\,ds\right)^{p/2}.$$

Finally, we get

$$E^{\mathcal{F}_t}\left(\int_t^T |L|^2\,ds\right)^p \le 3^p\varepsilon_2^p + [3^pT^{p/2}\varepsilon_2^p + 6^pn^{p/2}\varepsilon_0^p]E^{\mathcal{F}_t}\left(\int_t^T |L|^2\,ds\right)^{p/2}, \tag{32}$$

which implies the last estimate of the lemma.

Theorem 4.7. *Let* (K, L) *solve the BSRDE* (1). *Then,* $\int_0^\cdot L(s)\, dw(s)$ *is a* BMO(P) *martingale.*

Proof of Theorem 4.7 From the inequality (30), we get that for any stopping time $\tau \leq T$,

$$\int_\tau^T |L|^2\, ds \leq \varepsilon_5 + \varepsilon_5 \int_\tau^T |L|\, ds - \int_\tau^T \sum_{i=1}^d 2\,\mathrm{tr}\ (KL_i)\ dw_i. \tag{33}$$

From this it follows that

$$E^{\mathcal{F}_\tau} \int_\tau^T |L(s)|^2\, ds \leq \varepsilon.$$

Then, Theorem 4.7 follows.

Note that a simple case (where the generator of BSRDE (1) depends on the martingale term L in a linear way) of Theorem 4.7 has been obtained by Bismut [5] by a quite different argument. This theorem plays an important role in solving a general nonhomogeneous stochastic LQ problem and in solving a general mean-variance hedging problem. The reader is referred to Section 6 for details.

We have the following *a priori* estimate of the uniform positivity for the first part K of the solution.

Theorem 4.8. *Let the assumption* $(\mathcal{A}3)$ *be satisfied, and* (K, L) *solve the BSRDE* (1). *Then, there is a deterministic positive constant* ε_0 *such that*

$$K(t) \geq \varepsilon_0 I_{n\times n}. \tag{34}$$

The proof is an immediate consequence of a combination of Theorem 4.2 and the following estimate.

Lemma 4.1. *Let the assumption* $(\mathcal{A}3)$ *be satisfied. Let* $X^{t,x;u}$ *solve the SDE* (20). *Then, there is a deterministic positive constant* ε_0*, which is independent of the control* u*, such that*

$$|x|^2 + \frac{1}{2} E^{\mathcal{F}_t} \int_t^T |u_s|^2\, ds \leq \exp\left(\varepsilon_0 (T - t)\right) E^{\mathcal{F}_t} |X_T^{t,x;u}|^2,\ \forall u \in \mathcal{L}^2_{\mathcal{F}}(t, T; R^m). \tag{35}$$

4.7 Closedness of solution.

Theorem 4.9. *Assume that* $\forall \gamma \geq 0$ *the coefficients* $A^\gamma, B^\gamma, C_i^\gamma, D_i^\gamma, Q^\gamma$*, and* N^γ *are* $\mathcal{F}_t$*-progressively measurable matrix-valued processes, defined on* $\Omega \times [0, T]$*, of dimensions* $n \times n, n \times m, n \times n, n \times m, n \times n$*, and* $m \times m$*, respectively. Assume that* M^γ *is an* $\mathcal{F}_T$*-measurable and nonnegative* $n \times n$ *random matrix. Assume that* Q^γ *is a.s.a.e. nonnegative. Assume that there are two deterministic positive constants* ε_1 *and* ε_2 *which are independent of the parameter* γ*, such that*

$$|A^\gamma(t)|, |B^\gamma(t)|, |C_i^\gamma(t)|, |D_i^\gamma(t)|, |Q^\gamma(t)|, |N^\gamma(t)|, |M^\gamma| \leq \varepsilon_1$$

and

$$N^\gamma \geq \varepsilon_2 I_{m\times m}.$$

Assume that as $\gamma \to 0$, $A^\gamma(t), B^\gamma(t), C_i^\gamma(t), D_i^\gamma(t), Q^\gamma(t)$, and $N^\gamma(t)$ converge uniformly in (t,ω) to $A^0(t), B^0(t), C_i^0(t), D_i^0(t), Q^0(t)$ and $N^0(t)$, respectively. Assume that M^γ uniformly converges to M^0 as $\gamma \to 0$. Assume that $\forall \gamma > 0$ the BSRDE $(A^\gamma, B^\gamma; C_i^\gamma, D_i^\gamma, i = 1, \dots, d; Q^\gamma, N^\gamma, M^\gamma)$ has a unique solution (K^γ, L^γ) with $K^\gamma(t) \geq 0, t \in [0,T]$. Then, there is a pair of processes (K, L) with

$$K \in \mathcal{L}^\infty_\mathcal{F}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];\mathcal{S}^n_+)), \quad L \in \mathcal{L}^2_\mathcal{F}(0,T;\mathcal{S}^n),$$

such that

$$\begin{aligned} &\lim_{\gamma\to 0} K^\gamma = K \quad \text{strongly in } \mathcal{L}^\infty_\mathcal{F}(0,T;\mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0,T];\mathcal{S}^n_+)), \\ &\lim_{\gamma\to 0} L^\gamma = L \quad \text{strongly in } \mathcal{L}^2_\mathcal{F}(0,T;\mathcal{S}^n), \end{aligned} \tag{36}$$

and such that (K, L) solves the BSRDE $(A^0, B^0, C^0, D^0, Q^0, N^0, M^0)$.

If the above assumption of uniform convergence of $(A^\gamma, C^\gamma, Q^\gamma, M^\gamma)$ is replaced with the following one:

$$\lim_{\gamma\to 0} \operatorname*{esssup}_{\omega\in\Omega} \int_0^T (|A^\gamma - A^0| + |C^\gamma - C^0|^2 + |Q^\gamma - Q^0|)\, ds + |M^\gamma - M^0| \to 0. \tag{37}$$

then the above assertions still hold.

The proof is referred to Kohlmann and Tang [19].

Remark 2.1. *When the assumption of uniform positivity on the control weight matrix N is relaxed to nonnegativity, Theorem 4.9 still holds with the additional assumption that there is a deterministic positive constant ε_3 such that*

$$\sum_{i=1}^d (D_i^\gamma)' D_i^\gamma \geq \varepsilon_3 I_{m\times m}, \quad M^\gamma \geq \varepsilon_3 I_{n\times n}.$$

4.8 Transformation of BSRDEs

Let Φ solve the differential equation:

$$\left\{ \begin{array}{rcl} \dfrac{d\Phi}{dt}(t) & = & A(t)\Phi(t), \quad t \in (0,T], \\ \Phi(0) & = & I_{n\times n}. \end{array} \right.$$

Consider the following transformation

$$\widetilde{K} := \Phi' K \Phi, \quad \widetilde{L} := \Phi' L \Phi \tag{38}$$

of the BSRDE (1).

Using Itô's formula, we verify that the BSRDE $(A, B; C_i, D_i, i = 1, \dots, d; Q, N, M)$ becomes the new BSRDE $(0, \widetilde{B}; \widetilde{C}_i, \widetilde{D}_i, i = 1, \cdots, d; \widetilde{Q}, N, \widetilde{M})$, which is satisfied by $(\widetilde{K}, \widetilde{L})$. Here,

$$\begin{aligned} &\widetilde{B} := \Phi^{-1} B, \quad \widetilde{C} := \Phi^{-1} C \Phi, \quad \widetilde{D} := \Phi^{-1} D \\ &\widetilde{Q} := \Phi' Q \Phi, \quad \widetilde{M} := \Phi(T)' M \Phi(T). \end{aligned} \tag{39}$$

The advantage of doing the above transformation is that it makes A disappear. Since A is bounded, the resulting transformation Φ is also bouned.

It is interesting to discuss the more general transformation Ψ which solves the SDE:

$$\left\{\begin{array}{rcl} d\Psi(t) &=& A(t)\Psi(t)\,dt+\sum_{i=1}^{d}C_i(t)\Psi(t)\,dw_i(t), \quad t\in(0,T],\\ \Psi(0) &=& I_{n\times n}. \end{array}\right. \tag{40}$$

This transformation can anneal both A and C in the BSRDE (1). Note that it is not necessarily bounded though A and $C=(C_1,\ldots,C_d)$ are assumed to be bounded.

4.9 The Markov case : a PDE characterization

Since optimal controls or optimal hedging/investing strategies are characterized in terms of the solutions of associated BSRDEs, it is important to characterize the solutions of BSRDEs. When the coefficients of BSRDEs are Markovian functions of an Itô's process, it is natural to connect the solution of a BSRDE with a system of parabolic PDEs. Note that in general the associated system of parabolic PDEs contains a quadratic term of the gradient. We are not going into the details due to the limitation of space. The reader is referred to among others, Peng [30, 31], Pardoux and Peng [27], and Pardoux and Tang [28] for this direction.

5 Recent Advances on Existence and Uniqueness.

Recently, Kohlmann and Tang [17] extended Theorem 3.1 to the singular case under the assumption $(\mathcal{A}3)$.

Theorem 5.1. *Let the assumption* $(\mathcal{A}3)$ *be satisfied. Then, Theorem* 3.1 *still holds even if* $N=0$. *Moreover,* $K(t)$ *is uniformly positive.*

Kohlmann and Tang [18] solved the one dimensional case of the above Bismut-Peng's problem.

Theorem 5.2. *Let the assumptions* $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$ *be satisfied, and* $n=1$. *Then, BSRDE* (1) *has a unique solution. Further, assume that* $(\mathcal{A}3)$ *is satisfied. Then, BSRDE* (1) *still has a unique solution* (K,L) *even if* $N=0$, *and moreover,* $K(t)$ *is uniformly positive.*

This theorem meets the needs for BSRDEs to be applied to finance. An approximation technique is used, which is motivated by the works of Kobylansky [16], and Lepeltier and San Martin [22, 23].

Kohlmann and Tang [19] proved the global existence and uniqueness result for BSRDE (1) for some multi-dimensional cases: They are special but typical, for the generator contains a quadratic term on L. The results are stated by the following two theorems.

Theorem 5.3. (the singular case) *Let the assumptions* $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$ *be satisfied, and* $d=1$. *Assume that there is a deterministic positive constant* ε *such that*

$$M\geq\varepsilon I_{n\times n} \tag{41}$$

and

$$D'D(t) \geq \varepsilon I_{m\times m}. \tag{42}$$

Then, the BSRDE:

$$\left\{\begin{array}{rcl} dK &=& -[A'K+KA+C'KC+Q+C'L+LC \\ && -(KB+C'KD+LD)(D'KD)^{-1}(KB+C'KD+LD)']\,dt+L\,dw, \\ && \quad 0\leq t<T, \\ K(T) &=& M. \end{array}\right. \tag{43}$$

has a unique solution (K, L) *with* $K(t, \omega)$ *being uniformly positive w.r.t.* (t, ω).

The main idea for the proof of Theorem 5.3 is to do the inverse transformation:

$$\widetilde{K} := K^{-1}, \tag{44}$$

which turns out to satisfy a Riccati equation whose generator depends on the martingale term in a linear way.

First, since D is inversable, we can rewrite the BSRDE (43) as

$$\left\{\begin{array}{rcl} dK &=& -[-\widetilde{A}'K-K\widetilde{A}+Q-K\widetilde{B}K^{-1}\widetilde{B}'K-LK^{-1}L \\ && +K\widetilde{B}K^{-1}L+LK^{-1}\widetilde{B}'K]\,dt+L\,dw, \\ K(T) &=& M, \end{array}\right. \tag{45}$$

where

$$\widetilde{A} := -A+BD^{-1}C, \quad \widetilde{B} := -BD^{-1}.$$

Note that we have the following rule for the first and the second differentials of the inverse of a positive matrix as a matrix-valued function:

$$d\left(K^{-1}\right) = -K^{-1}(dK)K^{-1}, \quad d^2\left(K^{-1}\right) = 2K^{-1}(dK)K^{-1}(dK)K^{-1}. \tag{46}$$

Using Itô's formula, we can write the equation for the inverse $\widetilde{K}$ of K:

$$\left\{\begin{array}{rcl} d\widetilde{K} &=& -[\widetilde{K}\widetilde{A}'+\widetilde{A}\widetilde{K}-\widetilde{K}Q\widetilde{K}+\widetilde{B}\widetilde{K}\widetilde{B}'+\widetilde{B}\widetilde{L}+\widetilde{L}\widetilde{B}']\,dt+\widetilde{L}\,dw, \\ \widetilde{K}(T) &=& M^{-1}, \end{array}\right. \tag{47}$$

where

$$\widetilde{L} := -K^{-1}LK^{-1}.$$

From Theorem 3.1, the above BSRDE $\left(\widetilde{A}, Q^{1/2}; \widetilde{B}, 0; 0, I_{m\times m}, M^{-1}\right)$ has a unique solution $(\widetilde{K}, \widetilde{L})$ with $\widetilde{K} \geq 0$, which implies that $\widetilde{K}^{-1}(t)$ is uniformly positive in (t, ω). Moreover, from the fact that $\widetilde{K}(T) = M^{-1} \geq \varepsilon_1^{-1} I_{n\times n}$, we derive that $\widetilde{K}$ is uniformly positive. This shows that $\widetilde{K}^{-1}(t)$ is uniformly bounded. Therefore $(\widetilde{K}^{-1}, -\widetilde{K}^{-1}\widetilde{L}\widetilde{K}^{-1})$ solves BSRDE (43).

The uniqueness results from the Feynman-Kac representation result Theorem 4.2. In fact, assume that $(\widehat{K}, \widehat{L})$ also solves the BSRDE (43). Then, from Theorem 4.2, we see that

$$(K(t)x, x) = V(t, x) = (\widehat{K}(t)x, x), \quad a.s., \quad \forall (t, x) \in [0, T] \times R^n.$$

So, we have $K(t) = \widehat{K}(t)$ almost surely for $\forall (t,x) \in [0,T] \times R^n$. Set

$$\delta K := K - \widehat{K}, \quad \delta L_i := L_i - \widehat{L}_i, \quad \delta G := G(t,K,L) - G(t,\widehat{K},\widehat{L}).$$

Then, we have $\delta K = 0$. Note that $(\delta K, \delta L)$ satisfies the following BSDE:

$$\left\{ \begin{array}{rcl} d\delta K(t) & = & -\delta G\, dt + \sum_{i=1}^{d} \delta L_i(t)\, dw_i(t), \quad 0 \leq t < T, \\ \delta K(T) & = & 0. \end{array} \right. \tag{48}$$

From this, proceeding identically as in the proof of Theorem 4.6, we have

$$E\int_t^T |\delta L|^2(s)\, ds \leq E|\delta K(T)|^2 + \varepsilon \operatorname*{esssup}_{s,\omega} |\delta K(s)| E\int_t^T (1 + |L|^2 + |\widehat{L}|^2)\, ds = 0. \tag{49}$$

Hence, $\delta L = L - \widehat{L} = 0$.

Theorem 5.4. (the regular case) *Let the assumptions* $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$ *be satisfied. Further assume that* $d = 1, B = C = 0$, *and* D *and* N *satisfy the following*

$$\begin{array}{rcl} \lim_{h\to 0+} \operatorname*{esssup}_{\omega\in\Omega} \max_{t_1,t_2\in[0,T];\,|t_1-t_2|\leq h} |D(t_1) - D(t_2)| & = & 0, \\ \lim_{h\to 0+} \operatorname*{esssup}_{\omega\in\Omega} \max_{t_1,t_2\in[0,T];\,|t_1-t_2|\leq h} |N(t_1) - N(t_2)| & = & 0. \end{array} \tag{50}$$

Then, the BSRDE:

$$\left\{ \begin{array}{rcl} dK & = & -[A'K + KA + Q - LD(N + D'KD)^{-1}D'L]\, dt + L\, dw, \\ & & \quad 0 \leq t < T, \\ K(T) & = & M. \end{array} \right. \tag{51}$$

has a unique solution (K, L) *with* $K(t) \geq 0, \forall t \in [0,T]$.

For the regular case, the situation is a little complex: we easily see that the above inverse transformation on the first unknown variable can not eliminate the quadratic term of the second unknown variable. However, we can still solve some classes of BSRDEs with the help of doing some appropriate transformation. The whole proof is divided into several propositions.

Proposition 5.1. *Assume that* $Q \geq A'(D^{-1})'ND^{-1} + (D^{-1})'ND^{-1}A, m = n$, *and* D *and* N *are positive constant matrices. Then, Theorem* 5.4 *holds.*

Proof of Proposition 5.1. Write

$$\widehat{N} := (D^{-1})'ND^{-1}. \tag{52}$$

Then, the BSRDE (51) reads

$$\left\{ \begin{array}{rcl} dK & = & -[A'K + KA + Q - L(\widehat{N} + K)^{-1}L]\, dt + L\, dw, \\ & & \quad 0 \leq t < T, \\ K(T) & = & M. \end{array} \right. \tag{53}$$

The equation for $\widehat{K} := \widehat{N} + K$ is

$$\begin{cases} d\widehat{K} &= -[A'\widehat{K} + \widehat{K}A + Q - A'\widehat{N} - \widehat{N}A - \widehat{L}\widehat{K}^{-1}\widehat{L}]\,dt + \widehat{L}\,dw, \\ & \quad 0 \leq t < T, \\ \widehat{K}(T) &= \widehat{N} + M. \end{cases} \tag{54}$$

Note that $\widehat{N}+M$ is uniformly positive. From Theorem 5.3, it follows that the BSRDE (54) has a solution $(\widehat{K}, \widehat{L})$. Therefore $(\widehat{K} - \widehat{N}, \widehat{L})$ solves BSRDE (51).

Proposition 5.2. *Assume that $A = 0$ and D and N are constant matrices. Then, Theorem 5.4 holds.*

Proof of Proposition 5.2. First assume $m = n$. Consider the following approximating BSRDEs:

$$\begin{cases} dK &= -[Q - LD_\alpha(N + D'_\alpha K D_\alpha)^{-1} D'_\alpha L]\,dt + L\,dw, \\ K(T) &= M \end{cases} \tag{55}$$

where

$$D_\alpha := D + \alpha I_{m\times m} > 0, \alpha > 0.$$

From Proposition 5.1, we see that the BSRDE (55) has a solution (K_α, L_α) for every $\alpha > 0$. From Theorem 4.9, it follows that K_α uniformly converges to some $K \in \mathcal{L}^\infty_{\mathcal{F}}(0, T; \mathcal{S}^n_+) \cap L^\infty(\Omega, \mathcal{F}_T, P; C([0, T]; \mathcal{S}^n_+))$ and L_α strongly converges to some $L \in \mathcal{L}^2_{\mathcal{F}}(0, T; \mathcal{S}^n)$, and that (K, L) solves BSRDE (51) when $A = 0$.

Consider the case $n > m$. Then consider the $n \times n$ matrices $\widetilde{D}$ whose first m columns are D and whose last $(n - m)$ columns are zero column vectors, and $\widetilde{N}$ which is defined as

$$\widetilde{N} := \begin{pmatrix} R & 0 \\ 0 & I \end{pmatrix}.$$

The BSRDE (51) when $A = 0$ is rewritten as

$$\begin{cases} dK &= -[Q - L\widetilde{D}(\widetilde{N} + \widetilde{D}'K\widetilde{D})^{-1}\widetilde{D}'L]\,dt + L\,dw, \\ K(T) &= M \end{cases}$$

From the preceding result, we obtain the desired existence result.

Consider the case $n < m$. Then, there is a $m \times m$ orthogonal transformation matrix T such that

$$D = [\widehat{D}, 0]T, \quad \widehat{D} \in R^{n\times n} \text{ and is non-singular.}$$

Write

$$\widetilde{N} := (T^{-1})'NT^{-1} := \begin{pmatrix} \widehat{N}_{11} & \widehat{N}_{12} \\ \widehat{N}'_{12} & \widehat{N}_{22} \end{pmatrix} > 0.$$

Then, $\widehat{N}_{11} > 0$. The BSRDE (51) when $A = 0$ is rewritten as

$$\begin{cases} dK &= -[Q - L\widehat{D}(\widetilde{N}_{11} + \widehat{D}'K\widehat{D})^{-1}\widehat{D}'L]\,dt + L\,dw, \\ K(T) &= M \end{cases}$$

From the preceding result, we obtain the desired existence result.

Proposition 5.3. *Assume that $A = 0$, and D and N are piece-wisely constant $\mathcal{F}_t$-adapted bounded matrix processes. Then, Theorem 5.4 holds.*

Proof of Proposition 5.3. Since D and N are piece-wisely constant $\mathcal{F}_t$-adapted bounded matrix processes, there is a finite partion:

$$0 =: t_0 < t_1 < \cdots < t_J := T$$

such that on each interval $[t_i, t_{i+1}] \subset [0, T]$, D and N are constant $\mathcal{F}_{t_i}$-measurable bounded random matrices. Then, we can solve the BSRDE one interval by one, in an inductive and backward way.

Proposition 5.4. *Assume that $A = 0$. Then, Theorem 5.4 holds.*

Proof of Proposition 5.4. For an arbitrary positive integer k, consider the 2^k-partion of the time interval. Define

$$D^k(t) = D\left(\frac{i-1}{2^k}T\right), \quad \forall t \in \left[\frac{i-1}{2^k}T, \frac{i}{2^k}T\right), i = 1, 2, \ldots, 2^k;$$

and

$$N^k(t) = N\left(\frac{i-1}{2^k}T\right), \quad \forall t \in \left[\frac{i-1}{2^k}T, \frac{i}{2^k}T\right), i = 1, 2, \ldots, 2^k.$$

For each k, D^k and N^k are are piece-wisely constant, $\mathcal{F}_t$-adapted, bounded matrix processes. Further, since the trajectories of D and N are uniformly continuous in ω, $D^k(t)$ and $N^k(t)$ converge respectively to D and N, uniformly in (t, ω). That is, we have

$$\lim_{k\to\infty} \operatorname*{esssup}_{\omega\in\Omega} \max_{t\in[0,T]} |D^k(t) - D(t)| = 0, \quad \lim_{k\to\infty} \operatorname*{esssup}_{\omega\in\Omega} \max_{t\in[0,T]} |N^k(t) - N(t)| = 0.$$

From Proposition 5.3, we see that the BSRDE $(0, 0, 0, D^k; Q, N^k; M)$ has a global solution (K^k, L^k), and then from Theorem 4.9, it follows that Theorem 5.4 holds.

Proof of Theorem 5.4. From Proposition 5.4, we see that the BSRDE $(0, 0, 0, \widetilde{D}; \widetilde{Q}, N, \widetilde{M})$ has a global adapted solution $(\widetilde{K}, \widetilde{L})$, and thus the pair

$$((\Phi')^{-1}\widetilde{K}\Phi^{-1}, (\Phi')^{-1}\widetilde{L}\Phi^{-1})$$

solves the original BSRDE $(A, 0, 0, D; Q, N, M)$. Here, $\widetilde{D}, \widetilde{Q}$, and $\widetilde{M}$ are defined by (39).

The uniqueness can be proved in the same way as in the proof of Theorem 5.3.

6 Applications to Control and Finance

6.1 Control-theoretic application

Assume that

$$\xi \in L^2(\Omega, \mathcal{F}_T, P; R^n), \quad q, f, g_i \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^n). \tag{56}$$

Consider the following optimal control problem (denoted by $\mathcal{P}_0$):

$$\min_{u\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} J(u; 0, x) \tag{57}$$

with

$$J(u;t,x) = E^{\mathcal{F}_t}(M(X_T^{t,x;u}-\xi), X_T^{t,x;u}-\xi) + E^{\mathcal{F}_t}\int_t^T [(Q(X^{t,x;u}-q), X^{t,x;u}-q) + (Nu,u)]\,ds \tag{58}$$

and $X^{t,x;u}$ solving the following linear SDE

$$\begin{cases} dX_s = (AX_s + Bu_s + f(s))\,ds + \sum_{i=1}^{d}(C_iX_s + D_iu_s + g_i(s))\,dw_i, \quad t < s \le T, \\ X_t = x, \quad u \in \mathcal{L}^2_{\mathcal{F}}(t,T;R^m). \end{cases} \tag{59}$$

The value function V is defined as

$$V(t,x) := \min_{u\in\mathcal{L}^2_{\mathcal{F}}(t,T;R^m)} J(u;t,x), \quad (t,x)\in[0,T]\times R^n. \tag{60}$$

Theorem 6.1 *Let the two assumptions* $(\mathcal{A}1)$ *and* $(\mathcal{A}2)$*, or* $(\mathcal{A}1)$ *and* $(\mathcal{A}3)$ *be satisfied. Let* (K,L) *solve the BSRDE* (1)*. Then, the BSDE*

$$\begin{cases} d\psi(t) = -[\widehat{A}'\psi + \sum_{i=1}^{d}\widehat{C}_i'(\phi_i - Kg_i) - Kf - \sum_{i=1}^{d}L_ig_i + Qq]\,dt + \sum_{i=1}^{d}\phi_i\,dw_i, \\ \psi(T) = M\xi \end{cases} \tag{61}$$

where $\widehat{A}$ *and* $\widehat{C}_i$ *are defined by* (15)*, has a unique* $\mathcal{F}_t$*-adapted solution* (ψ,ϕ) *with*

$$\psi\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^n)\cap L^2(\Omega,\mathcal{F}_T,P;C([0,T];R^n)),\ \phi\in\left(\mathcal{L}^2_{\mathcal{F}}(0,T;R^n)\right)^d. \tag{62}$$

Moreover, the optimal control $\widehat{u}$ *for the non-homogeneous stochastic LQ problem* $\mathcal{P}_0$ *exists uniquely and has the following feedback law*

$$\widehat{u} = -(N+\sum_{i=1}^{d}D_i'KD_i)^{-1}[(B'K+\sum_{i=1}^{d}D_i'KC_i+\sum_{i=1}^{d}D_i'L_i)\widehat{X} - B'\psi + \sum_{i=1}^{d}D_i'(Kg_i-\phi_i)] \tag{63}$$

where $\widehat{X} := X^{0,x;\widehat{u}}$.

Remark 6.1. *Note that* $\widehat{A}$ *and* $\widehat{C}_i$ *depend on L in general, and thus they might not be uniformly bounded. In this case, we have no available—to the authors' best knowledge—theorem to guarantee the existence and the uniqueness of a global adapted solution, though the BSDE* (61) *is linear.*

Proof of Theorem 6.1. Our assumptions guarantee that there is a unique optimal control $\widehat{u}\in\mathcal{L}^2_{\mathcal{F}}(0,T;R^m)$. The optimality condition implies

$$B'\widetilde{p} + \sum_{i=1} D_i'\widetilde{q}_i + N\widehat{u} = 0$$

where $(\widetilde{p}, \widetilde{q})$ solves the BSDE (called the adjoint equation)

$$\begin{cases} d\widetilde{p}(t) &= -[A'\widetilde{p}(t) + \sum_{i=1}^{d} C_i'\widetilde{q}_i(t) + Q(\widehat{X}_t - q(t))]\,dt + \sum_{i=1}^{d} \widetilde{q}_i(t)\,dw_i(t), \\ \widetilde{p}(T) &= M(\widehat{X}_T - \xi) \end{cases} \tag{64}$$

with

$$\widetilde{p} \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^n) \cap L^2(\Omega, \mathcal{F}_T, P; C([0, T]; R^n)), \ \widetilde{q} \in \left(\mathcal{L}^2_{\mathcal{F}}(0, T; R^n)\right)^d. \tag{65}$$

Via Itô's formula, we check out that the pair (ψ, ϕ) defined by the following

$$\psi(t) := K(t)\widehat{X}_t - \widetilde{p}(t), \ \phi_i(t) := K(t)[C_i\widehat{X}_t + D_i\widehat{u}_t + g_i(t)] + L_i\widehat{X}_t - \widetilde{q}_i(t)$$

solves the BSDE (61). It is obvious that $\psi \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^n) \cap L^2(\Omega, \mathcal{F}_T, P; C([0, T]; R^n))$. Since $\int_0^{\cdot} L(s)\,dw(s)$ is a BMO(P)-martingale, it follows from Theorem 1.1 (i) and (iii) of Bañuelos and Bennett [1] that $\int_0^T L_i(s)\widehat{X}_s\,dw_i(s)$ is square-integrable. Therefore, $L_i\widehat{X} \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^n)$, and $\phi_i \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^n)$.

It is standard to get the explicit formula (63) of $\widehat{u}$ from the optimality condition. The proof is complete.

The following can be verified by a pure completion of squares.

Theorem 6.2 *Suppose that the two assumptions $(\mathcal{A}1)$ and $(\mathcal{A}2)$, or $(\mathcal{A}1)$ and $(\mathcal{A}3)$ are satisfied. Let (K, L) solve BSRDE (1). Then, the value function $V(t, x), (t, x) \in [0, T] \times R^n$ has the following explicit formula*

$$V(t, x) = (K(t)x, x) - 2(\psi(t), x) + V^0(t), \quad (t, x) \in [0, T] \times R^n \tag{66}$$

with

$$\begin{aligned} V^0(t) := \quad & E^{\mathcal{F}_t}(M\xi, \xi) + E^{\mathcal{F}_t}\int_t^T (Qq, q)\,ds - 2E^{\mathcal{F}_t}\int_t^T (\psi, f)\,ds \\ & + E^{\mathcal{F}_t}\int_t^T \sum_{i=1}^{d} [(Kg_i, g_i) - 2(\phi_i g_i)]\,ds \\ & - E^{\mathcal{F}_t}\int_t^T ((N + \sum_{i=1}^{d} D_i'KD_i)u^0, u^0)\,ds \end{aligned} \tag{67}$$

and

$$u^0 := (N + \sum_{i=1}^{d} D_i'KD_i)^{-1}[B'\psi + \sum_{i=1}^{d} D_i'(\phi_i - Kg_i)], \quad t \leq s \leq T. \tag{68}$$

6.2 Financial application.

As an application of the above results, the mean-variance hedging problem with random market conditions is considered. The mean-variance hedging problem was initially introduced by Föllmer and Sondermann [9], and later widely studied by Duffie and Richardson

[7], Föllmer and Schweizer [10], Schweizer [36, 37, 38], Hipp [14], Monat and Stricker [25], Pham, Rheinländer and Schweizer [34], Gourieroux, Laurent and Pham [12], and Laurent and Pham [21]. All of these works are based on a projection argument. Recently, Kohlmann and Zhou [20] used a natural LQ theory approach to solve the case of deterministic market conditions. Kohlmann and Tang [17] used a natural LQ theory approach to solve the case of stochastic market conditions, but the market conditions are only allowed to involve a smaller filtration $\{\mathcal{F}_t^2\}$. Kohlmann and Tang [18] completely solved the case of random market conditions by using Theorems 6.1 and 6.2, and the optimal hedging portfolio and the variance-optimal martingale measure are characterized by the solution of the associated BSRDE.

Consider the financial market in which there are $m+1$ primitive assets: one nonrisky asset (the bond) of price process

$$S_0(t) = \exp\,(\int_0^t r(s)\,ds), \quad 0 \le t \le T, \tag{69}$$

and m risky assets (the stocks)

$$dS(t) = \mathrm{diag}(S(t))(\mu(t)\,dt + \sigma(t)\,dw(t)), \quad 0 \le t \le T. \tag{70}$$

Here $w = (w_1, \ldots, w_d)'$ is a d-dimensional standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$, and $\{\mathcal{F}_t, 0 \le t \le T\}$ is the P-augmentation of the natural filtration generated by the d-dimensional Brownian motion w. Assume that the instantaneous interest rate r, the m-dimensional appreciation vector process μ and the volatility $m \times d$ matrix process σ are progressively measurable with respect to $\{\mathcal{F}_t, 0 \le t \le T\}$. For simplicity of exposing the main ideas, assume that they are uniformly bounded and there exists a positive constant ε such that

$$\sigma\sigma'(t) \ge \varepsilon I_{m\times m}, \quad 0 \le t \le T, a.s. \tag{71}$$

The risk premium process is given by

$$\lambda(t) = \sigma'(\sigma\sigma')^{-1}\tilde{\mu}(t), \quad 0 \le t \le T \tag{72}$$

where $e_m = (1, \ldots, 1)' \in R^m$, and $\tilde{\mu} := \mu - re_m$.

For any $x \in R$ and $\pi \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^m)$, define the self-financed wealth process X with initial capital x and with quantity π invested in the risky asset S by

$$\left\{\begin{array}{rcl} dX_t &=& [rX_t + (\tilde{\mu}, \pi)]\,dt + \pi'\sigma\,dw, \quad 0 < t \le T, \\ X_0 &=& x, \quad \pi \in \mathcal{L}^2_{\mathcal{F}}(0, T; R^m). \end{array}\right. \tag{73}$$

Given a random variable $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, consider the quadratic optimal control problem:

$$\textbf{Problem } \mathcal{P}_{0,x}(\xi) \qquad \min_{\pi \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^m)} E|X_T^{0,x;\pi} - \xi|^2 \tag{74}$$

where $X^{0,x;\pi}$ is the solution to the wealth equation (73). The associated value function is denoted by $V(t, x), (t, x) \in [0, T] \times R$. The minimum point of $V(t, x)$ over $x \in R$ for given time t is defined to be the approximate price for the contingent claim ξ at time t.

The problem $\mathcal{P}_{0,x}(\xi)$ is the so-called mean-variance hedging problem in mathematical finance. It is a one-dimensional singular stochastic LQ problem $\mathcal{P}_0$.

Denote by σ_i the i-th column of the volatility matrix σ. The associated Riccati equation is a non-linear singular BSDE:

$$\begin{aligned} dK &= -[2rK - (\tilde{\mu}'K + \textstyle\sum_{i=1}^d L_i\sigma_i')(K\sigma\sigma')^{-1}(K\tilde{\mu} + \sum_{i=1}^d L_i\sigma_i)]\,dt + \sum_{i=1}^d L_i\,dw_i \\ &= -[(2r - |\lambda|^2)K - 2(\lambda, L) - K^{-1}L'\sigma'(\sigma\sigma')^{-1}\sigma L]\,dt + (L,\,dw),\ 0 \le t < T \\ K(T) &= 1. \end{aligned} \tag{75}$$

Let (ψ, ϕ) is the $\mathcal{F}_t$-adapted solution of the BSDE

$$\begin{aligned} d\psi &= -\{[r - |\lambda|^2 - (\lambda, K^{-1}L)]\psi \\ &\quad - \textstyle\sum_{i=1}^d[\lambda_i + K^{-1}\sigma_i'(\sigma\sigma')^{-1}\sigma L]\phi_i\}\,dt + \sum_{i=1}^d \phi_i\,dw_i, \\ &= -\{[r - |\lambda|^2 - (\lambda, K^{-1}L)]\psi - (\lambda + K^{-1}\sigma'(\sigma\sigma')^{-1}\sigma L, \phi)\}\,dt + (\phi,\,dw), \\ \psi(T) &= \xi \end{aligned} \tag{76}$$

Theorem 6.1 provides an explicit formula for the optimal hedging portfolio:

$$\begin{aligned} \pi &= -(\sum_{i=1}^d \sigma_i K\sigma_i')^{-1}[(\tilde{\mu}K + \sum_{i=1}^d \sigma_i L_i)X - \tilde{\mu}\psi - \sum_{i=1}^d \sigma_i\phi_i] \\ &= -(K\sigma\sigma')^{-1}[(\tilde{\mu}K + \sigma L)X - \tilde{\mu}\psi - \sigma\phi] \\ &= -(\sigma\sigma')^{-1}[(\tilde{\mu} + \sigma K^{-1}L)X - \tilde{\mu}K^{-1}\psi - \sigma K^{-1}\phi] \end{aligned} \tag{77}$$

where (K, L) is the $\mathcal{F}_t$-adapted solution to the Riccati equation (75). The value function V is also given by

$$V(t,x) = K(t)x^2 - 2\psi(t)x + E^{\mathcal{F}_t}\xi^2 - E^{\mathcal{F}_t}\int_t^T (\tilde{\mu}\psi + \sigma\phi)'(\sigma K\sigma')(\tilde{\mu}\psi + \sigma\phi)\,ds \tag{78}$$

where $\phi := (\phi_1, \ldots, \phi_n)'$. So, the approximate price $p(t)$ at time t for the contingent claim ξ is given by

$$p(t) = K^{-1}(t)\psi(t). \tag{79}$$

The above solution need not introduce the additional concepts of the so-called *hedging numeraire* and *variance-optimal martingale measure,* and therefore is simpler than that of Gourieroux et al [12], and Laurent and Pham [21]. To be connected to the latter, the optimal hedging portfolio (77) is rewritten as

$$\pi = -(\sigma\sigma')^{-1}[(\tilde{\mu} + \sigma\widetilde{L})(X - \widetilde{\psi}) - \sigma\widetilde{\phi}]. \tag{80}$$

Here,

$$\widetilde{L} := LK^{-1}, \quad \widetilde{\psi} := \psi K^{-1}, \quad \widetilde{\phi} := \phi K^{-1} - L\psi K^{-2}. \tag{81}$$

and the pair $(\widetilde{\psi}, \widetilde{\phi})$ solves the BSDE:

$$\left\{ \begin{aligned} d\widetilde{\psi} &= \{r\widetilde{\psi} + (\widetilde{\lambda}, \widetilde{\phi})\}\,dt + (\widetilde{\phi},\,dw), \quad 0 \le t < T, \\ \widetilde{\psi}(T) &= \xi \end{aligned} \right. \tag{82}$$

with

$$\widetilde{\lambda} := \lambda - [I - \sigma'(\sigma\sigma')^{-1}\sigma]LK^{-1}. \tag{83}$$

The process $\widetilde{\psi}$ is just the *approximate price process*, and the BSDE (82) is the *approximate pricing equation.*

In view of Theorem 6.1, it follows from Theorem 1.1 (i) and (iii) of Bañuelos and Bennett [1] that

$$\widetilde{\psi} \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^n) \cap L^2(\Omega, \mathcal{F}_T, P; C([0,T];R)),\ \widetilde{\phi} \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d). \tag{84}$$

Note that the optimal hedging portfolio (77) consists of the following two parts:

$$\pi^1 := -(\sigma\sigma')^{-1}(\widetilde{\mu} + \sigma\widetilde{L})X \tag{85}$$

and

$$\pi^0 := (\sigma\sigma')^{-1}[(\widetilde{\mu} + \sigma\widetilde{L})\widetilde{\psi} + \sigma\widetilde{\phi}], \tag{86}$$

and satisfies

$$\pi = \pi^1 + \pi^0. \tag{87}$$

The first part π^1 is the optimal solution of the *homogeneous* mean-variance hedging problem $\mathcal{P}_{0,x}(0)$ (that is the case of $\xi = 0$ for the problem $\mathcal{P}_{0,x}(\xi)$). The corresponding optimal wealth process $X^{0,1;\pi^1}$ is the solution to the following *optimal closed system*

$$\left\{\begin{array}{rcl} dX_t &=& X_t[(r - |\lambda|^2 - (\lambda, \widetilde{L}))\,dt - (\lambda + \sigma'(\sigma\sigma')^{-1}\sigma\widetilde{L},\, dw)], \quad 0 < t \le T, \\ X_0 &=& 1, \end{array}\right. \tag{88}$$

and is just the *hedging numéraire.* So, the *hedging numéraire* is just *the state (wealth) transition process* of the optimal closed system (88) from time 0, or it is just *the fundamental solution* of the optimal closed system (88).

To understand the quantity $\widetilde{\lambda}$, consider the BSDE satisfied by $(\mathcal{K}, \mathcal{L})$

$$\left\{\begin{array}{rcl} d\mathcal{K} &=& \{(2r - |\lambda|^2)\mathcal{K} + 2(\lambda, \mathcal{L}) + \mathcal{K}^{-1}\mathcal{L}'[I - \sigma'(\sigma\sigma')^{-1}\sigma]\mathcal{L}\}\,dt + (\mathcal{L},\, dw), \\ \mathcal{K}(T) &=& 1 \end{array}\right. \tag{89}$$

with $\mathcal{K} := K^{-1}$ and $\mathcal{L} := -LK^{-2}$. It is the BSRDE for the following singular stochastic LQ problem (denoted by $\mathcal{P}'_{0,x}$):

$$\textbf{Problem } \mathcal{P}'_{0,x} \qquad \min_{\theta \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d)} E|\mathcal{X}_T^{0,x;\theta}|^2 \tag{90}$$

where $\mathcal{X}^{0,x;\theta}$ is the solution to the following stochastic differential equation

$$\left\{\begin{array}{rcl} d\mathcal{X}_t &=& \mathcal{X}_t[-r\,dt - (\lambda,\, dw)] + ([I - \sigma'(\sigma\sigma')^{-1}\sigma]\theta,\, dw), \quad 0 \le t \le T, \\ \mathcal{X}_0 &=& x, \quad \theta \in \mathcal{L}^2_{\mathcal{F}}(0,T;R^d). \end{array}\right. \tag{91}$$

Its optimal control $\widehat{\theta}$ has the following feedback form

$$\widehat{\theta} = -\mathcal{K}^{-1}\mathcal{L}\mathcal{X} = LK^{-1}\mathcal{X}. \tag{92}$$

The problem $\mathcal{P}'_{0,1}$ is just the so-called *dual problem* of the problem $\mathcal{P}_{0,1}(0)$ in [12, 21], and so the variance-optimal martingale measure is P_* defined as

$$dP_* := \exp\left\{-\int_0^T (\tilde{\lambda},\, dW) - \frac{1}{2}\int_0^T |\tilde{\lambda}|^2\, dt\right\} dP. \tag{93}$$

P_* is an equivalent martingale measure.

Note that $\tilde{\psi}$ has the following explicit formula:

$$\tilde{\psi}(t) = E_*^{\mathcal{F}_t}\xi \exp\left(-\int_t^T r(s)\, ds\right), \quad 0 \le t \le T. \tag{94}$$

Here, the notation $E_*^{\mathcal{F}_t}$ stands for the expectation operator conditioning on the σ-algebra $\mathcal{F}_t$ with respect to the probability P_*. The discounted $\tilde{\phi}$ is just the integrand of the stochastic-integral-representation of the P_*-martingale $\{E_*^{\mathcal{F}_t}\xi \exp(-\int_0^T r(s)\, ds), 0 \le t \le T\}$ (w.r.t. the P_*-martingale $W + \int_0 \tilde{\lambda}\, dt$).

As in Kohlmann and Zhou [20], again, the formula (80) has an interesting interpretation in terms of mathematical finance. The optimal hedging portfolio π in (80) consists of the two components: (a) $(\sigma\sigma')^{-1}\sigma\tilde{\phi}$—it may be interpreted as the perfect hedging portfolio for the contingent claim ξ with the risk premium process $\tilde{\lambda}$ (that is, under the variance-optimal martingale measure), (b) $(\sigma\sigma')^{-1}(\tilde{\mu} + \sigma\tilde{L})(\tilde{\psi} - X)$—it is a generalized Merton-type portfolio for a terminal utility function $c(x) = x^2$ (see Merton [24]), which invests the capital $(\tilde{\psi} - X)$ left over after fulfilling the obligation from the perfect hedge under the variance-optimal martingale measure.

Acknowledgement The second author would like to thank the hospitality of Department of Mathematics and Statistics, and the Center of Finance and Econometrics, Universität Konstanz, Germany.

References

[1] Bañuelos, R. and Bennett, A., *Paraproducts and commutators of martingale transforms.* Proceedings of the American Mathematical Society, 103 (1988), 1226–1234

[2] Bellman, R., *Functional equations in the theory of dynamic programming, positivity and quasilinearity,* Proc. Nat. Acad. Sci. U.S.A., 41 (1955), 743–746

[3] Bismut, J. M., *Conjugate convex functions in optimal stochastic control,* J. Math. Anal. Appl., 44 (1973), 384–404

[4] Bismut, J. M., *Linear quadratic optimal stochastic control with random coefficients,* SIAM J. Control Optim., 14 (1976), 419-444

[5] Bismut, J. M., *Controle des systems lineares quadratiques: applications de l'integrale stochastique,* Séminaire de Probabilités XII, eds : C. Dellacherie, P. A. Meyer et M. Weil, LNM 649, Springer-Verlag, Berlin 1978

[6] Chen, S., Li, X. and Zhou, X., *Stochastic linear quadratic regulators with indefinite control weight costs*, SIAM J. Control Optim. **36 (1998)**, 1685-1702

[7] Duffie, D. and Richardson, H. R., *Mean-variance hedging in continuous time,* Ann. Appl. Prob., 1 (1991), 1-15

[8] Föllmer, H. and Leukert, P., *Efficient hedging: cost versus shortfall risk,* Finance & Stochastics, 4 (2000), 117–146

[9] Föllmer, H. and Sonderman, D., *Hedging of non-redundant contingent claims.* In: Mas-Colell, A., Hildebrand, W. (eds.) Contributions to Mathematical Economics. Amsterdam: North Holland 1986, pp. 205-223

[10] Föllmer, H. and Schweizer, M., *Hedging of contingent claims under incomplete information.* In: Davis, M.H.A., Elliott, R.J. (eds.) Applied Stochastic Analysis. (Stochastics Monographs vol. 5) London-New York: Gordon & Breach, 1991, pp. 389-414

[11] Gal'chuk, L. I., *Existence and uniqueness of a solution for stochastic equations with respect to semimartingales,* Theory of Probability and Its Applications, 23 (1978), 751–763

[12] Gourieroux, C., Laurent, J. P. and Pham, H., *Mean-variance hedging and numéraire,* Mathematical Finance, 8 (1998), 179-200.

[13] Heston, S., *A closed-form solution for option with stochastic volatility with applications to bond and currency options.* Rev. Financial Studies, 6 (1993), 327–343

[14] Hipp, C., *Hedging general claims,* Proceedings of the 3rd AFIR colloqium, Rome, Vol. 2, pp. 603-613 (1993)

[15] Hull, J. and White, A., *The pricing of options on assets with stochastic volatilities.* J. Finance, 42 (1987), 281-300

[16] Kobylanski, M., *Résultats d'existence et d'unicité pour des équations différentielles stochastiques rétrogrades avec des générateurs á croissance quadratique,* C. R. Acad. Sci. Paris, 324 (1997), Série I, 81-86

[17] Kohlmann, M. and Tang, S., *Optimal control of linear stochastic systems with singular costs, and the mean-variance hedging problem with stochastic market conditions.* submitted

[18] Kohlmann, M. and Tang, S., *Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging.* submitted

[19] Kohlmann, M. and Tang, S., *Multi-dimensional backward stochastic Riccati equations and applications.* submitted

[20] Kohlmann, M. and Zhou, X., *Relationship between backward stochastic differential equations and stochastic controls: a linear-quadratic approach,* SIAM J. Control & Optim., 38 (2000), 1392-1407

[21] Laurent, J. P. and Pham, H., *Dynamic programming and mean-variance hedging,* Finance and Stochastics, 3 (1999), 83-110

[22] Lepeltier, J. P. and San Martin, J., *Backward stochastic differential equations with continuous coefficient,* Statistics & Probability Letters, 32 (1997), 425-430

[23] Lepeltier, J. P. and San Martin, J., *Existence for BSDE with superlinear-quadratic coefficient,* Stochastics & Stochastics Reports, 63 (1998), 227-240

[24] Merton, R., *Optimum consumption and portfolio rules in a continuous time model,* J. Econ. Theory, 3 (1971), pp. 373-413;Erratum 6 (1973), 213-214.

[25] Monat, P. and Stricker, C., *Föllmer-Schweizer decomposition and mean-variance hedging of general claims,* Ann. Probab. 23, 605–628

[26] Pardoux, E. and Peng, S., *Adapted solution of backward stochastic equation,* Systems Control Lett., 14 (1990), 55–61

[27] Pardoux, E. and Peng, S., *Backward stochastic differential equations and quasi-linear parabolic partial differential equations,* in: Rozovskii, B. L., Sowers, R. S. (eds.) Stochastic Partial Differential Equations and Their Applications, Lecture Notes in Control and Information Science 176, Springer, Berlin, Heidelberg, New York 1992, pp. 200–217

[28] Pardoux, E. and Tang, S., *Forward-backward stochastic differential equations with application to quasi-linear partial differential equations of second-order*, Probability Theory and Related Fields, 114 (1999), 123-150

[29] Peng, S., *Stochastic Hamilton-Jacobi-Bellman equations*, SIAM J. Control and Optimization **30 (1992)** , 284-304

[30] Peng, S., *Probabilistic interpretation for systems of semilinear parabolic PDEs,* Stochastics & Stochastic Reports, 37 (1991), 61-74

[31] Peng, S., *A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation,* Stochastics & Stochastics Reports, 38 (1992), 119–134

[32] Peng, S., *Some open problems on backward stochastic differential equations,* Control of distributed parameter and stochastic systems, proceedings of the IFIP WG 7.2 international conference, June 19-22, 1998, Hangzhou, China

[33] Peng, S., *A general stochastic maximum principle for optimal control problems,* SIAM J. Control Optim., 28 (1990), 966-979

[34] Pham, H., Rheinländer, T., and Schweizer, M., *Mean-variance hedging for continuous processes: New proofs and examples,* Finance and Stochastics, 2 (1998), 173-198

[35] Richardson, H, *A minimum result in continuous trading portfolio optimization.* Management Science, 35 (1989), 1045-1055

[36] Schweizer, M., *Mean-variance hedging for general claims,* Ann. Appl. Prob., 2 (1992), 171-179

[37] Schweizer, M., *Approaching random variables by stochastic integrals,* Ann. Probab., 22 (1994), 1536–1575

[38] Schweizer, M., *Approximation pricing and the variance-optimal martingale measure,* Ann. Probab., 24 (1996), 206–236

[39] Stein, E. M. and Stein, J. C., *Stock price distributions with stochastic volatility: an analytic approach,* Rev. Financial Studies, 4 (1991), 727-752

[40] Tang, S. and Li, X., *Necessary conditions for optimal control of stochastic systems with random jumps,* SIAM J. Control Optim., 32 (1994), 1447–1475

[41] Wonham, W. M., *On a matrix Riccati equation of stochastic control,* SIAM J. Control Optim., 6 (1968), 312-326

[42] Zhou, X. and Li, D., *The continuous time mean-variance selection problem,* Applied Mathematics and Optimization, (2000),