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BOUNDARY VALUE PROBLEMS FOR A CLASS OF ELLIPTIC OPERATOR PENCILS

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In this paper operator pencils $A(x, D, \lambda)$ are studied which act on a manifold with boundary and satisfy the condition of N-ellipticity with parameter, a generalization of the notion of ellipticity with parameter as introduced by Agmon and Agranovich–Vishik. Sobolev spaces corresponding to the Newton polygon are defined and investigated; in particular it is possible to describe their trace spaces. With respect to these spaces, an a priori estimate is proved for the Dirichlet boundary value problem connected with an N-elliptic pencil.

1. Introduction

In this paper we consider operator pencils of the form

$$A(x, D, \lambda) = A_{2m}(x, D) + \lambda A_{2m-1}(x, D) + \cdots + \lambda^{2m-2\mu} A_{2\mu}(x, D) \tag{1.1}$$

acting on a smooth manifold M with smooth boundary ∂M. Here m and μ are integer numbers with $m > \mu \geq 0$, $A_{2\mu}, \ldots, A_{2m}$ are partial differential operators in M with infinitely smooth coefficients and λ is a complex parameter. We assume that

$$A_j(x, D) = \sum_{|\alpha| \leq j} a_{\alpha j}(x) D^\alpha \quad (j = 2\mu,\, 2\mu + 1, \ldots, 2m)$$

is a differential operator of order j with scalar coefficients $a_{\alpha j}(x) \in C^\infty(\overline{M})$. As usual, we use for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ the notation $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = -i\frac{\partial}{\partial x_j}$ and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

The operator pencil (1.1), supplemented with Dirichlet boundary conditions, serves as an example of a polynomial operator pencil in the sense of [14]. The aim of the present paper is to develop some ellipticity theory for such pencils using the so-called Newton polygon

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method. In particular, we will obtain a priori estimates in appropriately defined Sobolev spaces.

The Newton polygon has proved to be an important tool in the theory of general parabolic and elliptic problems. There is a close connection between such type of problems and pencils of the form (1.1) which we want to describe briefly. For a given polynomial

$$P(\xi,\lambda)=\sum_{\alpha,k} p_{\alpha k}\xi^\alpha\lambda^k\,, \tag{1.2}$$

where $\xi\in\mathbb{R}^n$ and $\lambda\in\mathbb{C}$, let $\nu(P)$ be the set of all integer points (i,k) such that an α exists with $|\alpha|=i$ and $p_{\alpha k}\neq 0$. Then the Newton polygon $N(P)$ is defined as the convex hull of all points in $\nu(P)$, their projections on the coordinate axes and the origin. The polynomial $P(\xi,\lambda)$ is called N-parabolic (see [9], Chapter 2) if $N(P)$ has no edges parallel to the coordinate axes and if the inequality

$$|P(\xi,\lambda)|\geq\delta\sum_{(i,k)\in N(P)\cap\mathbb{Z}^2}|\xi|^i\,|\lambda|^k \tag{1.3}$$

holds for all $\lambda\in\mathbb{C}$ with $\operatorname{Im}\lambda<\lambda_0$ where $\delta>0$ and λ_0 are constants. Similarly (see [5]), the polynomial $P(\xi,\lambda)$ is called N-elliptic with parameter along some ray $\mathcal{L}$ of the complex plane if (1.3) holds for all $\xi\in\mathbb{R}^n$ and all $\lambda\in\mathcal{L}\,,\ |\lambda|\geq R$, with large enough R. This type of polynomials appears, for instance, if one considers Douglis–Nirenberg systems $A(x,D)-\lambda I$. Such systems were investigated by Kozhevnikov in [11],[12] and by the authors in [5]. It turned out that an equivalent condition for unique solvability of a Douglis–Nirenberg system $A(x,D)-\lambda I$ on a closed manifold and sharp a priori estimates is the condition that for every x the determinant

$$P(x,\xi,\lambda)=\det(A(x,\xi)-\lambda I)$$

satisfies inequality (1.3).

The basic idea of the Newton polygon method for the problems mentioned above is to assign to λ various weights r_j which are defined by the Newton polygon. For each of these weights we obtain a different principal part of $P(\xi,\lambda)$ which we denote by $P_{r_j}(\xi,\lambda)$. On a manifold without boundary there is a finite open covering $\{U_j\}_j$ of the set of all (ξ,λ) and a corresponding partition of unity $\sum_j\psi_j(\xi,\lambda)\equiv 1$ such that $P(D,\lambda)\psi_j(D,\lambda)$ differs from the corresponding principal part $P_{r_j}(D,\lambda)\psi_j(D,\lambda)$ only by a small regular perturbation. This allows estimates and existence results for the operators $P(D,\lambda)$, cf. [5] for N-elliptic systems and [9] for parabolic problems.

As an example, let us consider an operator $P(D,\lambda)$ being the product of two operators which are parabolic in the sense of Petrovskii, i.e.

$$P(D,\lambda)=(\lambda+A_{2p}(D))\,(\lambda+A_{2q}(D))\,,$$

where $\lambda+A_{2p}(D)$ and $\lambda+A_{2q}(D)$ are $2p$- and $2q$-parabolic operators, respectively, with $p>q$. For each weight r which we assign to the parameter λ we obtain the principal part

$P_r(D,\lambda)$ which is given by

$$P_r(D,\lambda) = \begin{cases} A_{2p}(D)\ A_{2q}(D) & \text{if } r < 2q\,, \\ A_{2p}(D)\ A_{2q}(D) + \lambda\ A_2p(D) & \text{if } r = 2q\,, \\ \lambda\ A_{2p}(D) & \text{if } 2q < r < 2p\,, \\ \lambda\ A_{2p}(D) + \lambda^2 & \text{if } r = 2p\,, \\ \lambda^2 & \text{if } r > 2p\,. \end{cases}$$

Note that for $r = r_2 = 2q$ the principal part $P_{r_2}(D,\lambda)$ is of the form (1.1). Now let us consider a weight r_3 with $2q < r_3 < 2p$. On a manifold without boundary the operator $P(D,\lambda)\psi_3(D,\lambda)$ is a small regular perturbation of the operator $P_{r_3}(D,\lambda)\psi_3(D,\lambda) = \lambda\, A_{2p}(D)\, \psi_3(D,\lambda)$.

On a manifold with boundary, however, the situation is different. The operator $P_{r_3}(D,\lambda)$ has to be supplied with p boundary conditions while the operator $P(D,\lambda)$ needs $p+q$ boundary conditions. Thus we can see that now $P(D,\lambda)\psi_3(D,\lambda)$ is (after division by λ) a singular perturbation of the principal part $P_{r_3}(D,\lambda)\psi_3(D,\lambda)$. A similar situation occurs if the weight of λ is larger than $2p+2q$.

So we can see that, apart from its own importance as a singularly perturbed problem, operator pencils of the form (1.1) may serve as a model problem in the theory of general N-parabolic and N-elliptic boundary value problems, including Douglis–Nirenberg systems.

Replacing λ by ε^{-1}, we obtain a boundary value problem with small parameter as studied by Vishik–Lyusternik [17], Nazarov, Frank and others. Nazarov obtained in [15] a priori estimates under the assumption that the solutions of the model ODE problem fulfill some estimates which are similar to those proved in Section 4 below. (The norms used in [15] differ slightly from the norms used in the present paper.) In several papers Frank and other authors investigated singular perturbed problems and corresponding a priori estimates, cf. [7] and the references therein. The use of the Newton polygon method which gives the connection to general parabolic problems as described above seems to be new even in the context of singularly perturbed problems.

Finally, we want to mention another reason for studying pencils of the form (1.1). Apart from the general connection to parabolic theory, these pencils arise directly in stationary problems corresponding to parabolic operators which are not resolved with respect to the time derivative.

The present paper contains basic results on N-ellipticity for pencils of the form (1.1), Sobolev spaces connected with the Newton polygon and the proof of an a priori estimate for the Dirichlet boundary value problem connected with (1.1). These investigations are continued in the forthcoming paper [6] where general boundary conditions are treated (in particular we define in [6] the analogue of the Shapiro–Lopatinskii condition), the parametrix construction is described and the necessity of the N-ellipticity conditions is proved.

We now turn to a more detailed exposition of the results of the present paper. We will assume pencil (1.1) to be elliptic with parameter along the ray $[0,\infty)$ in the following sense: denote by

$$A_j^{(0)}(x,\xi) := \sum_{|\alpha|=j} a_{\alpha j}(x)\xi^\alpha \quad (j = 2\mu,\dots,2m)$$

the principal symbol of A_j, where $\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$ for $\xi = (\xi_1,\dots,\xi_n) = (\xi',\xi_n)$, and by

$$A^{(0)}(x,\xi,\lambda) := A_{2m}^{(0)}(x,\xi) + \lambda A_{2m-1}^{(0)}(x,\xi) + \ldots + \lambda^{2m-2\mu}A_{2\mu}^{(0)}(x,\xi)$$

the principal symbol of $A(x,D,\lambda)$. Then our main assumption is that the estimate

$$|A^{(0)}(x,\xi,\lambda)| \geq C|\xi|^{2\mu}\,(\lambda+|\xi|)^{2m-2\mu} \quad (\xi\in\mathbb{R}^n, \lambda\in[0,\infty), x\in\overline{M}) \tag{1.4}$$

holds where the constant C does not depend on x,ξ or λ. Note that this inequality may be considered as a particular case of (1.3) where now the Newton polygon associated to A is a trapezoid (see Figure 2 below). However, in the present case one edge of the polygon is parallel to one of the coordinate axes, which is excluded in the definition of N-parabolicity.

In the case $\mu = 0$ the inequality (1.3) is the usual definition of ellipticity with parameter which was introduced by Agmon [1] and Agranovich–Vishik [3]. Therefore we may assume in the following that $\mu > 0$. In this case even for $\lambda \neq 0$ the principal symbol $A^{(0)}(x,\xi,\lambda)$ vanishes for $\xi = 0$ which causes the main difficulties in proving existence results and estimates. Note that the symbol $A^{(0)}(x,\xi,\lambda)$ is homogeneous in ξ and λ of degree $2m$, as it is the case for the problems treated in [3].

The norms appearing in the a priori estimate will be parameter-dependent norms connected with the Newton polygon. For this, we assign to each Newton polygon $N(P)$ a weight function $\Xi_P(\xi,\lambda)$ and a Sobolev space $H^\Xi(\mathbb{R}^n)$. On the half-space $\mathbb{R}^n_+ := \{(x',x_n)\in\mathbb{R}^n : x_n > 0\}$ and on the manifold M we can define H^Ξ in a standard way. Of particular interest for the theory of N-elliptic boundary value problems is to describe the space of all traces of functions $u \in H^\Xi(\mathbb{R}^n_+)$, i.e. the space $\{D_n^{j-1}u(x',0) : u\in H^\Xi(\mathbb{R}^n_+)\}$. It turn out that this trace space is given by $H^{\Xi^{(-j+\frac12)}}(\mathbb{R}^{n-1})$ where $\Xi^{(-j+\frac12)}(\xi',\lambda)$ denotes the weight function corresponding to the Newton polygon which is constructed from $N(P)$ by a shift of length $j-\frac12$ to the left parallel to the abscissa.

The description of the trace spaces on the boundary by a shifted Newton polygon is an important part of this theory and holds for general Newton polygons. For future purposes, we derive this result in this generality in Section 2, not restricting ourselves to the case where the Newton polygon is a trapezoid.

The first step for proving estimates for the solutions is to obtain precise knowledge of the zeros of the principal symbol $A^{(0)}(x,\xi,\lambda)$ considered as a polynomial in ξ_n. These zeros can (for large λ) be arranged in two groups, one group remaining bounded for $\lambda\to\infty$, the other group of zeros being exactly of order $O(\lambda)$ for $\lambda\to\infty$. To obtain this result we have to impose an additional condition on the principal symbol $A^{(0)}(x,\xi,\lambda)$ which is the analogue of the condition of regular degeneration which is known from the theory of singular perturbations (cf. Vishik-Lyusternik [17]). See Section 3 for details.

In Sections 4 and 5 we turn to the Dirichlet boundary value problem connected with (1.1). The proof of the a priori estimate for this boundary value problem is based on estimates of the solution of an ordinary differential equation which arises from the boundary value problem by fixing $x\in\partial M$, rewriting the boundary value problem in coordinates corresponding to x and taking the partial Fourier transform with respect to the first $n-1$ variables. Estimates for the system of fundamental solutions of the resulting ordinary differential equation can be found in Section 4, and the a priori estimate is proved in Section 5.

Boundary value problems corresponding to operators of the form (1.1) can also be treated using a combination of the parameter-independent Boutet de Monvel calculus and its parameter-dependent version. This approach is described in the book of Grubb [10], Section 4.7, see also the references therein. Here the degeneracy of the symbol of (1.1) which appears for $\xi = 0$ is "divided out" by use of the parameter-independent calculus. Roughly speaking (and ignoring several reductions and modifications), to find a solution u of the Dirichlet boundary value problem connected with (1.1) one considers the parameter-independent boundary value problem

$$\begin{aligned} &A_{2\mu}(x,D)u = v \quad \text{ in } M,\\ &(\partial/\partial\nu)^{j-1}u = \psi_j \quad (j=1,\dots,\mu) \quad \text{ on } \partial M, \end{aligned}$$

where $\partial/\partial\nu$ stands for the normal derivative. Inserting its solution (or parametrix) into the original problem, one obtains for v a parameter-elliptic problem in the sense of [10]. For a detailed realization of this approach many additional questions arise, and therefore in the present paper we prefer the more elementary way which is based on the traditional formulation of the ellipticity conditions and which directly leads to the desired a priori estimates in terms of the Newton polygon.

2. Newton's polygon and functional spaces corresponding to it

In this section we consider a polynomial $P(\xi,\lambda)$ of the form (1.2) and its Newton polygon $N(P)$ which was defined in the Introduction. For a detailed discussion of the Newton polygon, we refer the reader to Gindikin-Volevich [9], Chapters 1 and 2, and to [5].

To construct function spaces corresponding to the Newton polygon, we consider the weight function

$$\Xi_P(\xi,\lambda) := \sum_{(i,k)\in N(P)} |\xi|^i\,|\lambda|^k\,, \tag{2.1}$$

where the summation on the right-hand side is extended over all integer points of $N(P)$. The Sobolev space H^{Ξ} will arise as a special case of the following more general definition which is taken from Volevich-Paneah [18]. It can be seen directly that the function $\sigma(\xi) := \Xi_P(\xi,\lambda)$ satisfies the condition which appears in this definition (cf. also Remark 2.4 below). In the following, the Fourier transform F is defined by

$$Fu(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, the definition is extended in the usual way to distributions $u \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 2.1. Let $\sigma(\xi)$ be a continuous function on $\mathbb{R}^n$ with values in $\mathbb{R}_+$ and assume that $\sigma(\xi)\sigma^{-1}(\eta) \le C(1+|\xi-\eta|^N)$ holds for all $\xi,\eta \in \mathbb{R}^n$ with constants C and N not depending on ξ or η. Then H^σ is defined as the space of all distributions u in $\mathcal{S}'(\mathbb{R}^n)$ such that $\sigma(\xi)Fu(\xi) \in L_2(\mathbb{R}^n)$. The space H^σ is endowed with the norm

$$\|u\|_{\sigma,\mathbb{R}^n} := \Big(\int_{\mathbb{R}^n} \sigma^2(\xi)|Fu(\xi)|^2 \, d\xi \Big)^{1/2} .$$

Proposition 2.2. (See Volevich-Paneah [18].) *Let $\sigma(\xi,\lambda)$ be a continuous function of ξ and assume that*

$$\sigma(\xi,\lambda)\sigma^{-1}(\eta,\lambda) \le C_1(1+|\xi-\eta|^N)$$

holds with a constant C_1 not depending on ξ, η or λ. Let

$$\sigma_l'(\xi',\lambda) := \Big(\int_{-\infty}^{\infty} \frac{\xi_n^{2l}}{\sigma^2(\xi',\xi_n,\lambda)} \, d\xi_n \Big)^{-1/2} < \infty \, .$$

Then $D_n^l u(x',0)$ is well-defined as an element of $H^{\sigma_l'}(\mathbb{R}^{n-1})$ for every $u \in H^\sigma(\mathbb{R}^n)$, and there exists a constant C, independent of u and λ, such that

$$\|D_n^l u(x',0)\|_{\sigma_l',\mathbb{R}^{n-1}} \le C\|u\|_{\sigma,\mathbb{R}^n} \, .$$

We will apply Proposition 2.2 to the case where $\sigma(\xi,\lambda)$ is given by $\Xi_P(\xi,\lambda)$ (see (2.1)). Let one of the functions $\sigma(\xi,\lambda)$ or $\sigma_1(\xi,\lambda)$ for each λ satisfy the condition of Definition 2.1 and $\sigma(\xi,\lambda) \approx \sigma_1(\xi,\lambda)$. The symbol $\approx$ means that there exist positive constants C_1 and C_2, independent of ξ and λ, such that

$$C_1\sigma(\xi,\lambda) \le \sigma_1(\xi,\lambda) \le C_2\sigma(\xi,\lambda) \, .$$

Then the other function also satisfies the condition of Definition 2.1 and, evidently, the statement of Proposition 2.2 remains valid, if we replace σ by the equivalent function σ_1. In the following we will construct an equivalent function for $\Xi_P(\xi,\lambda)$ (cf. [5], Section 2). For this purpose we introduce some simple geometric notions connected with the Newton polygon (see, e.g., [9], Chapter 1).

Let $\Gamma_1,\dots,\Gamma_S$ be the edges of the Newton polygon not lying on the coordinate axes and indexed in the clockwise direction (cf. Fig. 1). Suppose that

$$(0,0),\ (a_1,b_1),\dots,(a_{S+1},b_{S+1}) , \quad a_1 = 0, \quad b_{S+1} = 0 , \tag{2.2}$$

are the vertices of the polygon $N(P)$. Then the edge Γ_s is given by

$$\Gamma_s = \{(a,b) \in \mathbb{R}^2 : 1 \cdot a + r_s \cdot b = d_s\} \quad (s = 1,\dots,S)$$

where $r_s = (a_{s+1} - a_s)/(b_s - b_{s+1})$. The vector $(1, r_s)$ is an exterior normal to Γ_s, where we admit $r_1 = \infty$ if Γ_1 is horizontal. Let us assume in the following that the edge Γ_S is not vertical, i.e. that we have $r_S > 0$. Since $N(P)$ is convex, we have

$$\infty \ge r_1 > \ldots > r_S > 0 \, .$$

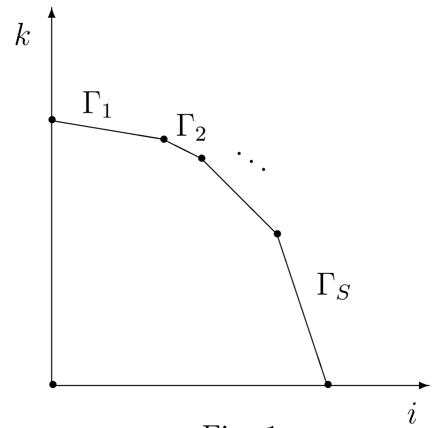

Fig. 1

The r_s-principal part of P is defined by

$$P_{r_s}(\xi,\lambda) := \sum_{|\alpha|+r_s k=d_s} a_{\alpha k}\xi^\alpha\lambda^k\,. \tag{2.3}$$

Here d_s is the so-called r_s-degree of P which may be defined by

$$d_s := \max_{(a,b)\in N(P)}(1\cdot a + r_s\cdot b)\,. \tag{2.4}$$

Now we set

$$\Xi_{(s)}(\xi,\lambda) = |\xi|^{-a_s}\;|\lambda|^{-b_{s+1}}\sum_{i+r_s k=d_s}|\xi|^i|\lambda|^k\,.$$

This function will be a polynomial of $|\xi|$ and $|\lambda|$.

Repeating the argument in [9], Theorem 1.1.3, we can prove that

$$\prod_{s=1}^{S}\Xi_{(s)}(\xi,\lambda) = \sum_{s=1}^{S}|\xi|^{a_s}|\lambda|^{b_s} + \dots\ , \tag{2.5}$$

where the dots denote the sum of monomials $|\xi|^i|\lambda|^k$ with $(i,k)\in N(P)$. For $|\lambda|\geq 1$ the right-hand side can be estimated from below by

$$1+\sum_{s=1}^{S}|\xi|^{a_s}|\lambda|^{b_s}\,.$$

This function can be estimated from below by $\Xi_P(\xi,\lambda)$ (see [5], Subsection 3.2). From this it follows that the left-hand side of (2.5) is equivalent to Ξ_P. Denote by $2m_s$ the largest degree of $|\xi|$ in $\Xi_{(s)}$. It is obvious that $\Xi_{(s)}$ is equivalent to $(|\xi|+|\lambda|^{\frac{1}{r_s}})^{2m_s}$, and consequently

$$\Xi_P(\xi,\lambda)\approx\prod_{s=1}^{S}\left(|\xi|^2+|\lambda|^{\frac{2}{r_s}}\right)^{m_s}\ . \tag{2.6}$$

We will suppose further, as in the case of parabolic polynomials (cf. [9], Chapter 2), that $m_1,\dots,m_S$ are integers.

Remark 2.3. For $r_1 = \infty$ (i.e. Γ_1 is horizontal) (2.3) and (2.4) have no sense and (2.3) should be replaced by

$$P_{r_1}(\xi,\lambda) := \sum_{|\alpha|=a_2} a_{\alpha b_1}\xi^\alpha\lambda^{b_1} .$$

As for the equivalence (2.6), it will be valid for $|\lambda| > \lambda_0$ with arbitrary $\lambda_0 > 0$ and the equivalence constants, of course, depend on λ_0.

Remark 2.4. The fact that $\Xi(\xi,\lambda)$ satisfies the condition of Definition 2.1 is an immediate consequence of (2.6) as this condition is fulfilled for each factor on the right-hand side.

Remark 2.5. From (2.6) it follows that the r_s-degree d_s (cf. (2.4)) is given by

$$d_s = 2\Big(\sum_{j=1}^{s} m_j + \sum_{j=s+1}^{S} \frac{r_s}{r_j} m_s\Big) . \tag{2.7}$$

To see this, we use the relation

$$\Xi_P(t\xi, t^{r_s}\lambda) = t^{d_s}\Xi_{P_{r_s}}(\xi,\lambda) + o(t^{d_s}), \quad t\to+\infty ,$$

cf. [9], Section 1.1.2. In our case we obtain, denoting the right-hand side of (2.7) by d_s',

$$\begin{aligned}
\Xi_P(t\xi, t^{r_s}\lambda) &= \prod_{j=1}^{S}\Big(t^2|\xi|^2 + t^{2\frac{r_s}{r_j}}|\lambda|^{\frac{2}{r_j}}\Big)^{m_j} \\
&= t^{d_s'}\prod_{j=1}^{s}\Big(|\xi|^2 + t^{2(\frac{r_s}{r_j}-1)}|\lambda|^{\frac{2}{r_j}}\Big)^{m_j}\prod_{j=s+1}^{S}\Big(t^{2(1-\frac{r_s}{r_j})}|\xi|^2 + |\lambda|^{\frac{2}{r_j}}\Big)^{m_j} \\
&= t^{d_s'}\Xi_{P_{r_s}}(\xi,\lambda) + o(t^{d_s'}) ,
\end{aligned}$$

which shows $d_s = d_s'$.

Now we will describe the trace spaces of the spaces H^Ξ. For this we use the following lemma:

Lemma 2.6. *Let $1 \le a_1 < a_2 < \ldots < a_S < \infty$ and $m_1,\ldots,m_S \in \mathbb{N}$. For $l \in \mathbb{N}$ with $0 \le l < 2(m_1+\ldots+m_S)$ define the index κ by*

$$2m_1 + \ldots + 2m_{\kappa-1} \le l < 2m_1 + \ldots + 2m_\kappa . \tag{2.8}$$

Then there exists a constant $C > 0$, independent of $a_1,\ldots,a_S$, such that

$$\begin{aligned}
C^{-1}a_\kappa^{2l+1-4m_1-\ldots-4m_\kappa}\prod_{s=\kappa+1}^{S} a_s^{-4m_s} &\le \int_{-\infty}^{\infty}\frac{t^{2l}}{\prod_{s=1}^{S}(t^2+a_s^2)^{2m_s}}\,dt \\
&\le C a_\kappa^{2l+1-4m_1-\ldots-4m_\kappa}\prod_{s=\kappa+1}^{S} a_s^{-4m_s} .
\end{aligned}$$

In the case $0 \le l < 2m_1$, we set $m_0 = 0$ in (2.8).

Proof. Substituting in the integral $t = a_S\tau$, we obtain

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} t^{2l} \prod_{s=1}^{S} (t^2 + a_s^2)^{-2m_s} dt \\ &= 2a_S^{2l+1-4m_1-\ldots-4m_S} \int_0^{\infty} t^{2l} \prod_{s=1}^{S} \left(t^2 + \left(\frac{a_s}{a_S}\right)^2\right)^{-2m_s} dt. \end{aligned}$$

For $t \geq 1$ we use

$$t^{2l}(1+t^2)^{-2m_1-\ldots-2m_S} \leq t^{2l} \prod_{s=1}^{S} \left(t^2 + \left(\frac{a_s}{a_S}\right)^2\right)^{-2m_s} \leq t^{2l-4m_1-\ldots-4m_S}.$$

As $l < 2\sum_{s=1}^{S} m_s$, the left-hand and right-hand side of this inequality are integrable functions over $[1,\infty)$, and we obtain

$$C_1^{-1} \leq \int_1^{\infty} t^{2l} \prod_{s=1}^{S} \left(t^2 + \left(\frac{a_s}{a_S}\right)^2\right)^{-2m_s} dt \leq C_1$$

for some $C_1 > 0$.

For $0 \leq t \leq 1$ we have $1 \leq 1 + t^2 \leq 2$, and therefore

$$\int_0^1 t^{2l} \prod_{s=1}^{S} \left(t^2 + \frac{a_s^2}{a_S^2}\right)^{-2m_s} dt \;\approx\; \int_0^1 t^{2l} \prod_{s=1}^{S-1} \left(t^2 + \frac{a_s^2}{a_S^2}\right)^{-2m_s} dt.$$

Now we substitute $t = \frac{a_{S-1}}{a_S}\tau$ and see that the last integral is equivalent to

$$\left(\frac{a_{S-1}}{a_S}\right)^{2l+1-4m_1-\ldots-4m_{S-1}} \int_0^{\frac{a_S}{a_{S-1}}} t^{2l} \prod_{s=1}^{S-1} \left(t^2 + \frac{a_s^2}{a_{S-1}^2}\right)^{-2m_s} dt.$$

Again we split up $\int_0^{\frac{a_S}{a_{S-1}}} \ldots = \int_0^1 \ldots + \int_1^{\frac{a_S}{a_{S-1}}} \ldots$ and use an estimate of the form $C_2^{-1} \leq \int_1^{\frac{a_S}{a_{S-1}}} \ldots \leq C_2$ for the second integral.

Proceeding in this way, we receive

$$\begin{aligned} I \;&\approx\; a_S^{2l+1-4m_1-\ldots-4m_S} \left(\frac{a_{S-1}}{a_S}\right)^{2l+1-4m_1-\ldots-4m_{S-1}} \cdot \ldots \\ &\quad \cdot \left(\frac{a_\kappa}{a_{\kappa+1}}\right)^{2l+1-4m_1-\ldots-4m_\kappa} \int_0^{\frac{a_{\kappa+1}}{a_\kappa}} t^{2l} \prod_{s=1}^{\kappa} \left(t^2 + \frac{a_s^2}{a_\kappa^2}\right)^{-2m_s} dt. \end{aligned}$$

For the last integral we use

$$\begin{aligned} t^{2l}(t^2+1)^{-2m_1-\ldots-2m_\kappa} &\leq t^{2l} \prod_{s=1}^{\kappa} \left(t^2 + \frac{a_s^2}{a_\kappa^2}\right)^{-2m_s} \\ &\leq t^{2l-4m_1-\ldots-4m_{\kappa-1}}(t^2+1)^{-2m_\kappa}. \end{aligned}$$

As $2m_1 + \ldots + 2m_{\kappa-1} \le l < 2m_1 + \ldots + 2m_\kappa$, the left-hand and the right-hand side of this inequality are integrable functions on $[0,\infty)$. Therefore

$$I \approx a_\kappa^{2l+1-4m_1-\ldots-4m_\kappa} a_{\kappa+1}^{-4m_{\kappa+1}} \cdot \ldots \cdot a_S^{-4m_S} .$$

$\square$

Remark 2.7. Using the substitution $t = a_1\tau$, it is easily seen that the condition $a_1 \ge 1$ in Lemma 2.6 may be replaced by $a_1 > 0$.

As in the Introduction, we denote by $\Xi_P^{(-l)}(\xi,\lambda)$ the function corresponding to the Newton polygon which is constructed from $N(P)$ by a shift of length l to the left parallel to the abscissa. More explicitly, if the vertices of $N(P)$ are given by (2.2) and if

$$a_{\kappa-1} \le l < a_\kappa ,$$

then an easy calculation shows that the vertices of the shifted Newton polygon are

$$(0,0), \Big(0, \frac{b_\kappa(l-a_{\kappa-1}) + b_{\kappa-1}(a_\kappa - l)}{a_\kappa - a_{\kappa-1}}\Big), (a_\kappa - l, b_\kappa), \ldots, (a_{S+1} - l, b_{S+1}) .$$

We preserve the notation $H^{\Xi_P^{(-l)}}(\mathbb{R}^{n-1})$ for the spaces in $\mathbb{R}^{n-1}$ corresponding to the weight functions $\Xi_P^{(-l)}(\xi',\lambda) := \Xi_P^{(-l)}(\xi',0,\lambda)$.

Lemma 2.8. *Let $\lambda_0 > 0$. Then for $|\lambda| \ge \lambda_0$ we have*

$$\sigma_l'(\xi',\lambda) \approx \Xi^{(-l-\frac{1}{2})}(\xi',\lambda) ,$$

where σ_l' is defined by

$$\sigma_l'(\xi',\lambda) := \Big(\int_{-\infty}^{\infty} \frac{\xi_n^{2l}}{\Xi_P^2(\xi,\lambda)}\, d\xi_n \Big)^{-\frac{1}{2}} .$$

Proof. Instead of Ξ_P we use the right-hand side of (2.6). From Lemma 2.6 with $a_s^2 = |\xi'|^2 + |\lambda|^{\frac{2}{r_s}}$ we obtain (see Remark 2.7) that

$$\sigma_l'(\xi',\lambda) \approx \Big(|\xi'|^2 + |\lambda|^{\frac{2}{r_\kappa}}\Big)^{m_1+\ldots+m_\kappa - \frac{l}{2} - \frac{1}{4}} \prod_{s=\kappa+1}^{S} \Big(|\xi'|^2 + |\lambda|^{\frac{2}{r_s}}\Big)^{m_s} , \tag{2.9}$$

where κ is chosen according to Lemma 2.6. From Remark 2.5 applied to $\sigma_l'(\xi',\lambda)$ we see that the edges of the Newton polygon corresponding to the weight function (2.9) are given by

$$\Gamma_j = \{(a,b) \in \mathbb{R}^2 : a + r_j b = d_j'\}$$

with $d_j' = d_j - l - \frac{l}{2}$ $(j = \kappa, \ldots, S)$. But this means that the Newton polygon for σ_l' is constructed from $N(P)$ by a shift of $l + \frac{1}{2}$ to the left, i.e. we have $\sigma_l'(\xi',\lambda) \approx \Xi_P^{(-l-\frac{1}{2})}(\xi',\lambda)$. $\square$

As an immediate consequence of Proposition 2.2 and Lemma 2.8, we obtain the following theorem.

Theorem 2.9. *For every $\lambda_0 > 0$ there exists a constant $C > 0$, independent of u and λ, such that*

$$\|D_n^l u(x',0)\|_{\Xi_P^{(-l-\frac{1}{2})},\mathbb{R}^{n-1}} \leq C\|u\|_{\Xi_P,\mathbb{R}^n} \quad (l = 0,\dots,2m_1+\dots+2m_S-1)$$

holds for $u \in H^{\Xi_P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$ with $|\lambda| \geq \lambda_0$.

In the following, we will also consider the function spaces in the half space $\mathbb{R}^n_+$ which correspond to Newton polygons. Using the binomial formula, it is easily seen that

$$\Xi_P^2(\xi,\lambda) \approx \sum_{l=0}^{M} \xi_n^{2l} \left(\Xi_P^{(-l)}(\xi',\lambda)\right)^2$$

where $M = 2m_1 + \cdots + 2m_S$. From this it follows that we can take

$$\Big(\sum_{l=0}^{M} \int_{-\infty}^{\infty} \|(D_n^l u)(\cdot,x_n)\|^2_{\Xi_P^{(-l)},\mathbb{R}^{n-1}}\, dx_n\Big)^{1/2} \tag{2.10}$$

as an equivalent norm in $H^{\Xi_P}(\mathbb{R}^n)$. Replacing the integral over $\mathbb{R}$ by the integral over $x_n \geq 0$ we define a norm in $H^{\Xi_P}(\mathbb{R}^n_+)$.

To define the space $H^{\frac{1}{\Xi_P}}(\mathbb{R}^n_+)$, we use the more general approach which can be found, e.g., in [18]. Let $\sigma(\xi)$ be a weight function fulfilling the condition in Definition 2.1. Denote by $H^\sigma(\mathbb{R}^n)_\pm$ the subspace of $H^\sigma(\mathbb{R}^n)$ consisting of elements with supports in the closure of $\mathbb{R}^n_\pm$. Then we set

$$H^\sigma(\mathbb{R}^n_+) = H^\sigma(\mathbb{R}^n)/H^\sigma(\mathbb{R}^n)_-$$

endowed with the natural quotient norm

$$\|f\|_{\sigma,\mathbb{R}^n_+} = \inf_{f_- \in H^\sigma(\mathbb{R}^n)_-} \|f_0 + f_-\|_{\sigma,\mathbb{R}^n}\,,$$

where f_0 is an arbitrary representative of the conjugacy class of f.

Suppose that $\sigma(\xi',\xi_n)$ for fixed $\xi' \in \mathbb{R}^{n-1}$ can be extended as a holomorphic function in ξ_n of polynomial growth in the lower half-plane $\operatorname{Im}\xi_n < 0$. In this case the quotient norm of $f \in H^\sigma(\mathbb{R}^n_+)$ coincides with the norm

$$\|\sigma(D',D_n)f_0\|_{L_2(\mathbb{R}^n_+)} \tag{2.11}$$

which does not depend on the choice of the element f_0 in the conjugacy class. In (2.11) the pseudo-differential operator (ps.d.o.) $\sigma(D',D_n) = \sigma(D)$ is defined by

$$\sigma(D)f := F^{-1}\sigma(\xi)(Ff)(\xi)\,.$$

In the case when

$$\sigma \approx \prod_{j=1}^{S} (|\xi|^2 + |\lambda|^{2/r_j})^{m_j}$$

we replace σ in the definition of $H^\sigma(\mathbb{R}^n_+)$ by

$$\prod_{s=1}^{S}\Big(i\xi_n+(|\xi'|^2+|\lambda|^{2/r_s})^{1/2}\Big)^{2m_s}. \tag{2.12}$$

In particular (cf. (2.6)), this gives us another equivalent description of $H^{\Xi_P}(\mathbb{R}^n_+)$. Replacing in (2.12) the exponent m_s by $-m_s$, we obtain the space $H^{1/\Xi_P}(\mathbb{R}^n_+)$.

3. The zeros of the symbol

Now we come back to the operator pencil (1.1) and consider the corresponding model problem with constant coefficients and without lower order terms. Let $A(\xi,\lambda)$ be a polynomial in $\xi\in\mathbb{R}^n$ and $\lambda\in\mathbb{C}$ of the form

$$A(\xi,\lambda)=A_{2m}(\xi)+\lambda A_{2m-1}(\xi)+\ldots+\lambda^{2m-2\mu}A_{2\mu}(\xi)\,, \tag{3.1}$$

where $A_j(\xi)$ is a homogeneous polynomial in ξ of degree j. The Newton polygon corresponding to A has the shape indicated in Figure 2 with $r=2m$ and $s=2\mu$.

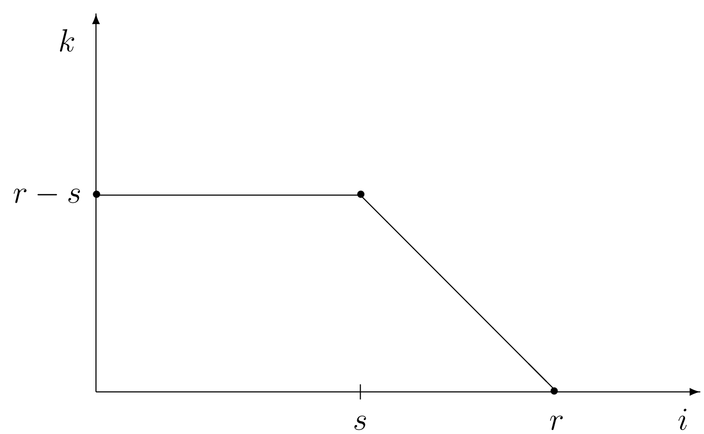

Fig. 2. The Newton polygon $N_{r,s}$.

Definition 3.1. The polynomial $A(\xi,\lambda)$ is called N-elliptic with parameter in $[0,\infty)$ if the estimate

$$|A(\xi,\lambda)|\geq C\,|\xi|^{2\mu}\,(\lambda+|\xi|)^{2m-2\mu}\quad(\xi\in\mathbb{R}^n,\lambda\in[0,\infty)) \tag{3.2}$$

holds with a constant C independent of ξ and λ.

Lemma 3.2. *The polynomial $A(\xi,\lambda)$ is N-elliptic with parameter in $[0,\infty)$ if and only if the following conditions are satisfied:*

(i) *$A_{2m}(\xi)$ is elliptic, i.e. $A_{2m}(\xi)\neq 0$ for $\xi\in\mathbb{R}^n\backslash\{0\}$.*

(ii) *$A_{2\mu}(\xi)$ is elliptic.*

(iii) *$A(\xi,\lambda)\neq 0$ for $\xi\in\mathbb{R}^n\backslash\{0\}$ and $\lambda\in[0,\infty)$.*

Proof. From (3.2) we trivially obtain condition (iii) and, setting $\lambda = 0$, condition (i). Taking $\varepsilon = \frac{1}{\lambda}$ and dividing (3.2) by $\varepsilon^{2\mu-2m}$, we receive

$$|A_{2\mu}(\xi) + \varepsilon A_{2\mu+1}(\xi) + \ldots + \varepsilon^{2m-2\mu} A_{2m}(\xi)| \geq C|\xi|^{2\mu}(1 + \varepsilon|\xi|)^{2m-2\mu}\,.$$

Taking the limit for $\varepsilon \to 0$, we obtain (ii).

Now let conditions (i)–(iii) be fulfilled. For $\xi \in \mathbb{R}^n \backslash \{0\}$ we write $A(\xi, \lambda)$ in the form

$$A(\xi, \lambda) = A_{2\mu}(\xi) B_{2m-2\mu}(\xi, \lambda)$$

with

$$B_{2m-2\mu}(\xi, \lambda) = \frac{A_{2m}(\xi)}{A_{2\mu}(\xi)} + \lambda \frac{A_{2m-1}(\xi)}{A_{2\mu}(\xi)} + \ldots + \lambda^{2m-2\mu}\,.$$

The coefficients of $B_{2m-2\mu}(\xi, \lambda)$ (considered as a polynomial in λ) are homogeneous functions in $\xi \in \mathbb{R}^n \backslash \{0\}$, and therefore $B(\xi, \lambda)$ is a homogeneous function in (ξ, λ) of degree $2m - 2\mu$. From this and from conditions (ii) and (iii) it follows that

$$|A_{2\mu}(\xi)| \geq C|\xi|^{2\mu}\,, \quad |B_{2m-2\mu}(\xi, \lambda)| \geq C(\lambda + |\xi|)^{2m-2\mu}\,.$$

Multiplying these estimates, we see that A is N-elliptic with parameter in $[0, \infty)$. □

Denote by $\tau_j(\xi', \lambda)$ $(j = 1, \ldots, 2m)$ the zeros of the algebraic equation

$$A(\xi', \tau, \lambda) = 0 \quad \big(\xi' \in \mathbb{R}^{n-1} \backslash \{0\}, \lambda \in [0, \infty)\big)\,.$$

Due to Lemma 3.2 (iii), this equation has no real roots. The number m_+ of roots with positive imaginary part is independent of (ξ', λ) and therefore coincides with the corresponding number for $\lambda = 0$. It is easily seen (cf. [4], Section 1.2) that in the case $n > 2$ the set $\{(\xi', \lambda) : \xi' \in \mathbb{R}^{n-1} \backslash \{0\},\ \lambda \in [0, \infty)\}$ is connected, and therefore we have $m_+ = m$. In the case $n \leq 2$ the relation $m_+ = m$ is an additional condition which will be assumed to hold in the following. We denote the roots of $A(\xi', \tau, \lambda)$ with positive imaginary part by $\tau_1(\xi', \lambda), \ldots, \tau_m(\xi', \lambda)$.

To investigate the elliptic pencil corresponding to $A(\xi', \tau, \lambda)$ we will need an additional assumption which is closely related to the condition of regularity of degeneration in the theory of singular perturbations (cf. Vishik-Lyusternik [17], Section 1.1). To formulate this assumption we consider the auxiliary polynomial of degree $2m - 2\mu$ given by

$$Q(\tau) := \tau^{-2\mu} A(0, \tau, 1)\,. \tag{3.3}$$

From inequality (3.2) with $\xi' = 0$ and $\lambda = 1$ we obtain for $\tau \neq 0$ the estimate

$$|Q(\tau)| \geq C(|\tau| + 1)^{2m-2\mu} \tag{3.4}$$

with a constant independent of τ. By continuity we obtain that $Q(0) \neq 0$, and thus $Q(\tau)$ has no real roots.

Definition 3.3. The polynomial $A(\xi', \tau, \lambda)$ is said to degenerate regularly for $\lambda \to \infty$ if the polynomial $Q(\tau)$ defined in (3.4) has exactly $m - \mu$ roots with positive imaginary part (counted according to their multiplicities).

Remark 3.4. a) Suppose that the polynomial $A(\xi,\lambda)$ contains only terms of even order, i.e.

$$A(\xi,\lambda)=A_{2m}(\xi)+\lambda^2A_{2m-2}(\xi)+\ldots+\ \lambda^{2m-2\mu-2}A_{2\mu+2}(\xi)+\lambda^{2m-2\mu}A_{2\mu}(\xi)\,.$$

Then the polynomial $Q(\tau)$ is a polynomial of degree $m-\mu$ in the variable τ^2 and $A(\xi,\lambda)$ degenerates regularly for $\lambda\to\infty$.

b) (Cf. [17], Lemma 3.4.) Assume that $A(\xi,\lambda)$ is the symbol of a differential operator $\tilde{A}(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n},\lambda)$ with real coefficients. Then the polynomials of even order $A_{2m-2j}(\xi)$ ($j=0,\ldots,m-\mu$) are real and the polynomials of odd order $A_{2m-2j-1}(\xi)$ ($j=0,\ldots,m-\mu-1$) are purely imaginary. Assume that $\tilde{A}$ is strongly elliptic, i.e. we have

$$\operatorname{Re}A(\xi,\lambda)\geq C|\xi|^{2\mu}(\lambda+|\xi|)^{2m-2\mu}\,.$$

Then we obtain that $\operatorname{Re}A=A_{2m}+\lambda^2A_{2m-2}+\ldots+\lambda^{2m-2\mu}A_{2\mu}$ satisfies (3.2), and due to part a) the polynomial $\operatorname{Re}Q(\tau)$ has $m-\mu$ roots with positive imaginary part and $m-\mu$ roots with negative imaginary part. Since the polynomial

$$Q_\delta(\tau):=\ \operatorname{Re}Q(\tau)+\delta i\operatorname{Im}Q(\tau)\quad(0\leq\delta\leq1)$$

satisfies

$$\operatorname{Re}Q_\delta(\tau)\geq C(|\tau|+1)^{2m-2\mu}\quad(0\leq\delta\leq1)\,,$$

the number of roots of Q_δ in the upper half complex plane does not depend on $\delta\in[0,1]$, and $A(\xi,\lambda)$ degenerates regularly for $\lambda\to\infty$.

Lemma 3.5. *Let the polynomial $A(\xi,\lambda)$ in* (3.1) *be N-elliptic with parameter in $[0,\infty)$ and assume that A degenerates regularly for $\lambda\to\infty$. Then, with a suitable numbering of the roots $\tau_j(\xi',\lambda)$ of $A(\xi',\tau,\lambda)$ with positive imaginary part, we have:*
(i) *Let $S(\xi')=\{\tau_1^0(\xi'),\ldots,\tau_\mu^0(\xi')\}$ be the set of all zeros of $A_{2\mu}(\xi',\tau)$ with positive imaginary part. Then for all $r>0$ there exists a $\lambda_0>0$ such that the distance between the sets $\{\tau_1(\xi',\lambda),\ldots,\tau_\mu(\xi',\lambda)\}$ and $S(\xi')$ is less than r for all ξ' with $|\xi'|=1$ and all $\lambda\geq\lambda_0$.*
(ii) *Let $\tau_{\mu+1}^1,\ldots,\tau_m^1$ be the roots of the polynomial $Q(\tau)$ (cf.* (3.3)*) with positive imaginary part. Then*

$$\tau_j(\xi',\lambda)=\lambda\tau_j^1+\tilde{\tau}_j^1(\xi',\lambda)\quad(j=\mu+1,\ldots,m)\,,$$

and there exist constants K_j and λ_1, independent of ξ' and λ, such that for $\lambda\geq\lambda_1$ the inequality

$$|\tilde{\tau}_j^1(\xi',\lambda)|\leq K_j|\xi'|^{\frac{1}{k_1}}\,\lambda^{1-\frac{1}{k_1}}\quad(|\xi'|\leq\lambda)$$

holds, where k_1 is the maximal multiplicity of the roots of $Q(\tau)$.

Proof. (i) We write $\xi'=\rho\omega$ with $|\omega|=1$ and set $\tilde{\tau}=\frac{\tau}{\rho}$, $\varepsilon=\frac{\rho}{\lambda}$. After division of $A(\xi',\tau,\lambda)$ by $\lambda^{2m-2\mu}\rho^{2\mu}$ we obtain the equation

$$B(\omega,\tilde{\tau},\varepsilon):=A_{2\mu}(\omega,\tilde{\tau})+\varepsilon A_{2\mu+1}(\omega,\tilde{\tau})+\ldots+\varepsilon^{2m-2\mu}A_{2m}(\omega,\tilde{\tau})=0\,.\tag{3.5}$$

For fixed ω let $\tilde{\tau}_j=\ldots=\tilde{\tau}_{j+p-1}$ be a zero of $B(\omega,\tilde{\tau},0)=A_{2\mu}(\omega,\tilde{\tau})$ of multiplicity p. Then there exists an $\alpha=\alpha(\omega)>0$ such that

$$\frac{1}{2\pi i}\int_{|z-\tilde{\tau}_j|=\alpha}\frac{\frac{d}{dz}B(\omega,z,\varepsilon)}{B(\omega,z,\varepsilon)}\,dz=\frac{1}{2\pi i}\int_{|z-\tilde{\tau}_j|=\alpha}\frac{\frac{d}{dz}B(\omega,z,0)}{B(\omega,z,0)}\,dz=p$$

holds for all $\varepsilon < \varepsilon_0 = \varepsilon_0(\omega)$. Therefore, for every $\varepsilon < \varepsilon_0$ the equation (3.5) has exactly p roots in $\{z \in \mathbb{C} : |z - \tilde{\tau}_j| < \alpha\}$ which we denote by $\tilde{\tau}_j(\omega, \varepsilon), \ldots, \tilde{\tau}_{j+p-1}(\omega, \varepsilon)$. Proceeding in this way for all zeros of $A_{2\mu}(\omega, \tilde{\tau})$, we obtain the set $S(\omega, \varepsilon) := \{\tilde{\tau}_1(\omega, \varepsilon), \ldots, \tilde{\tau}_\mu(\omega, \varepsilon)\}$ of zeros of $B(\omega, \tilde{\tau}, \varepsilon)$.

Now we assume that the statement in (i) is false. Then there exists a sequence $(\omega_n)_{n\geq 1}$ with $|\omega_n| = 1$ and a constant $C > 0$ such that $\mathrm{dist}(S(\omega_n), S(\omega_n, \varepsilon_n)) \geq C$ for all $n \geq 1$ where we have set $\varepsilon_n = \frac{1}{n}$. Due to compactness, we may assume that ω_n converges to ω_0. As the zeros of $A_{2\mu}(\omega, \tilde{\tau})$ depend continuously on ω, we obtain for large n that

$$\mathrm{dist}(S(\omega_0), S(\omega_n, \varepsilon_n)) \geq \frac{C}{2}\,. \tag{3.6}$$

But from the same considerations as above we see that for every sufficiently small $\alpha > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\omega_0)$ and an $s > 0$ such that $B(\omega, \tilde{\tau}, \varepsilon)$ has exactly μ roots in $\bigcup_j \{z \in \mathbb{C} : |z - \tilde{\tau}_j(\omega_0)| < \alpha\}$ for all $|\omega - \omega_0| < s$ and $0 < \varepsilon < \varepsilon_0$. Taking $\alpha < \frac{C}{2}$, we obtain a contradiction to (3.6).

(ii) We set $\xi' = \rho\omega$ with $|\omega| = 1$, $\tau = \lambda\tilde{\tau}$ and $\lambda = \rho/\varepsilon$ and obtain the equation

$$0 = A(\varepsilon\omega, \tilde{\tau}, 1) = \tilde{\tau}^{2\mu}\tilde{Q(\tau)} + \sum_{k=1}^{2m} \varepsilon^k a_k(\omega, \tilde{\tau})$$

where $a_k(\omega, \tilde{\tau}) := \frac{1}{k!}(\frac{\partial}{\partial\varepsilon})^k A(\varepsilon\omega, \tilde{\tau}, 1)|_{\varepsilon=0}$.

Let $\tau_j^1 = \ldots = \tau_{j+p-1}^1$ be a zero of $Q(\tau)$ of multiplicity p. Then we know from the theory of algebraic functions that there exist p roots $\tilde{\tau}_j(\omega, \varepsilon), \ldots, \tilde{\tau}_{j+p-1}(\omega, \varepsilon)$ of $A(\varepsilon\omega, \tilde{\tau}, 1)$ for which we have an expansion (Puiseux series) of the form

$$\tilde{\tau}_s(\omega, \varepsilon) = \tau_j^1 + \sum_{k=1}^{\infty} c_{jk}(\omega)\varepsilon^{k/p} \quad (s = j, \ldots, j + p - 1) \tag{3.7}$$

(cf., e.g., [8], Section 7). In formula (3.7) we have to take the p different branches of the function $\varepsilon^{\frac{1}{p}}$ to obtain the zeros $\tilde{\tau}_j(\varepsilon), \ldots, \tilde{\tau}_{j+p-1}(\varepsilon)$. The series on the right-hand side is a holomorphic function in $\varepsilon^{\frac{1}{p}}$ for $|\varepsilon| \leq \varepsilon_1(\omega)$ for some $\varepsilon_1(\omega) > 0$.

From the construction of the Puiseux series (cf. [8], Section 8) we know that the coefficients $c_{jk}(\omega)$ in the series (3.7) depend continuously on the coefficients of the polynomial $B(\omega, \tilde{\tau}, \varepsilon)$ and therefore on ω. Thus there exists an $\varepsilon_1 > 0$, independent of ω, such that the right-hand side of (3.7) is a holomorphic function in $\varepsilon^{\frac{1}{p}}$ for $|\varepsilon| \leq \varepsilon_1$. As the function

$$(\tilde{\tau}_j(\omega, \varepsilon) - \tau_j^1)\varepsilon^{-\frac{1}{p}} = \sum_{k=1}^{\infty} c_{jk}(\omega)\varepsilon^{\frac{k-1}{p}}$$

is continuous in ω and ε for $|\omega| = 1$ and $0 \leq \varepsilon \leq \varepsilon_0$, it is bounded by some constant K_1, independent of ω and ε, which finishes the proof of part (ii). $\square$

4. An estimate for the basic ordinary differential equation

In this section we consider the polynomial $A(\xi, \lambda)$ given by (3.1) and assume that this polynomial is N-elliptic with parameter in $[0, \infty)$ and degenerates regularly for $\lambda \to \infty$. For fixed $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in [0, \infty)$ and $j = 1, \dots, m$ we consider the ordinary differential equation on the half-line

$$
\begin{aligned}
A(\xi', D_t, \lambda)\, w_j(t) &= 0 \qquad (t > 0)\,, && (4.1)\\
D_t^{k-1}\, w_j(t)|_{t=0} &= \delta_{jk} \quad (k = 1, \dots, m)\,, && (4.2)\\
w_j(t) &\to 0 \qquad (t \to +\infty)\,.
\end{aligned}
$$

Here D_t stands for $-i\frac{\partial}{\partial t}$.

Theorem 4.1. *For every $\xi' \in \mathbb{R}^{n-1} \backslash \{0\}$ and $\lambda \in [0, \infty)$ the ordinary differential equation (4.1)–(4.2) has a unique solution $w_j(\xi', t, \lambda)$, and for $l = 0, 1, \dots$ the estimate*

$$
\|D_t^l w_j(\xi', \cdot, \lambda)\|_{L_2(\mathbb{R}_+)} \le C \begin{cases} |\xi'|^{l-j+\frac{1}{2}}, & j \le \mu, \quad l \le \mu, \\ |\xi'|^{1+\mu-j}(\lambda + |\xi'|)^{l-\mu-\frac{1}{2}}, & j \le \mu, \quad l > \mu \\ |\xi'|^{l-\mu}(\lambda + |\xi'|)^{\mu-j+\frac{1}{2}}, & j > \mu, \quad l \le \mu, \\ (\lambda + |\xi'|)^{l-j+\frac{1}{2}}, & j > \mu, \quad l > \mu, \end{cases}
$$

holds with a constant C not depending on ξ' and λ.

Proof. The existence and the uniqueness of the solution follows immediately from the fact that $A(\xi', \tau, \lambda)$ (considered as a polynomial in τ) has exactly m roots with positive imaginary part. Let $\gamma(\xi', \lambda)$ be a closed contour in the upper half of the complex plane enclosing all roots $\tau_1(\xi', \lambda), \dots, \tau_m(\xi', \lambda)$ with positive imaginary part. Then $w_j(\xi', t, \lambda)$ is given by

$$
w_j(\xi', t, \lambda) = \frac{1}{2\pi i} \int_{\gamma(\xi', \lambda)} \frac{M_j(\xi', \tau, \lambda)}{A_+(\xi', \tau, \lambda)} e^{it\tau}\, d\tau \tag{4.3}
$$

where

$$
A_+(\xi', \tau, \lambda) = \prod_{k=1}^{m} \Big(\tau - \tau_k(\xi', \lambda)\Big) =: \sum_{k=0}^{m} a_k(\xi', \lambda)\tau^k
$$

and

$$
M_j(\xi', \tau, \lambda) = \sum_{k=0}^{m-j} a_k(\xi', \lambda)\tau^{m-j-k}\,.
$$

(Cf., e.g., [2], Section 1.) The coefficients are given by the formula of Vieta,

$$
a_k(\xi', \lambda) = \sum_{1 \le l_1 < \dots < l_k \le m} (-1)^k \tau_{l_1}(\xi', \lambda) \cdot \ldots \cdot \tau_{l_k}(\xi', \lambda)\,. \tag{4.4}
$$

From (4.3) we see, substituting $\tau = r\tilde{\tau}$, that

$$
D_t^l w_j(\xi', t, \lambda) = r^{1-j+l}(D_t^l w_j)\Big(r\xi', \frac{t}{r}, r\lambda\Big)\,,
$$

and therefore

$$\|D_t^l w_j(\xi',\cdot,\lambda)\|_{L_2(\mathbb{R}_+)} = r^{\frac{1}{2}-j+l}\Big\|D_t^l w_j\Big(\frac{\xi'}{r},\cdot,\frac{\lambda}{r}\Big)\Big\|_{L_2(\mathbb{R}_+)}.$$

If we set $r = |\xi'|$ and $\omega' = \frac{\xi'}{|\xi'|}$ we obtain

$$\|D_t^l w_j(\xi',\cdot,\lambda)\|_{L_2(\mathbb{R}_+)} = |\xi'|^{\frac{1}{2}-j+l}\Big\|D_t^l w_j\Big(\omega',\cdot,\frac{\lambda}{|\xi'|}\Big)\Big\|_{L_2(\mathbb{R}_+)}.$$

The theorem will be proved if we show that for $|\omega'| = 1$ we have

$$\|(D_t^l w_j)(\omega',\cdot,\Lambda)\|_{L_2(\mathbb{R}_+)} \le \begin{cases} C\,, & j \le \mu\,,\ l \le \mu\,, \\ C\,\Lambda^{l-\mu-\frac{1}{2}}, & j \le \mu\,,\ l > \mu \\ C\,\Lambda^{\mu-j+\frac{1}{2}}, & j > \mu\,,\ l \le \mu\,, \\ C\,\Lambda^{l-j+\frac{1}{2}}, & j > \mu\,,\ l > \mu\,, \end{cases} \tag{4.5}$$

for $\Lambda \ge 1$ and that the left-hand side is bounded by a constant for $\Lambda \le 1$.

The boundedness for $\Lambda \le 1$ easily follows from the ellipticity of $A(\omega',\tau,\Lambda)$ and the continuity of A and thus of w_j with respect to Λ.

For large Λ we write

$$\gamma(\omega',\Lambda) = \gamma^{(1)}(\omega',\Lambda) \cup \gamma^{(2)}(\omega',\Lambda)$$

where $\gamma^{(1)}(\omega',\Lambda)$ encloses the zeros $\tau_1(\omega',\Lambda),\dots,\tau_\mu(\omega',\Lambda)$ and $\gamma^{(2)}(\omega',\Lambda)$ encloses the zeros $\tau_{\mu+1}(\omega',\Lambda),\dots,\tau_m(\omega',\Lambda)$. Here we assume that the zeros are numbered according to Lemma 3.5. According to this splitting of the contour γ, we write $w_j(\omega',t,\Lambda) = w_j^{(1)}(\omega',t,\Lambda) + w_j^{(2)}(\omega',t,\Lambda)$ with

$$w_j^{(k)}(\omega',t,\Lambda) := \frac{1}{2\pi i}\int\limits_{\gamma^{(k)}(\omega',t,\Lambda)} \frac{M_j(\omega',\tau,\Lambda)}{A_+(\omega',\tau,\Lambda)}\,e^{it\tau}\,d\tau \quad (k=1,2)\,.$$

From Lemma 3.5 we know that

$$\begin{aligned} |\tau_j(\omega',\Lambda)| &\le C \qquad (|\omega'|=1,\ \Lambda \ge \Lambda_0, \qquad j=1,\dots,\mu) \\ |\tau_j(\omega',\Lambda)| &\le C\Lambda \qquad (|\omega'|=1,\ \Lambda \ge \Lambda_0, \quad j=\mu+1,\dots,m)\,. \end{aligned}$$

As $A_{2\mu}$ is elliptic we have, with the notation of Lemma 3.5, $|\tau_j(\omega',\Lambda)| \ge C$ for $j=1,\dots,\mu$ and $|\omega'|=1,\ \Lambda \ge \Lambda_0$. With our additional assumption we also have

$$|\tau_j(\omega',\Lambda)| \ge C\Lambda \quad (|\omega'|=1,\ \Lambda \ge \Lambda_0, \quad j=\mu+1,\dots,m),$$

as $\frac{\tau_j(\omega',\Lambda)}{\Lambda} \to \tau_j^1$ and $\operatorname{Im}\tau_j^1 > 0$, cf. Lemma 3.5 (ii). Therefore

$$|A_+(\omega',\tau,\Lambda)| = \prod_{k=1}^m |\tau - \tau_k(\omega',\Lambda)| \ge \begin{cases} C\Lambda^{m-\mu} & \text{on } \gamma^{(1)}\,, \\ C\Lambda^m & \text{on } \gamma^{(2)} \end{cases}$$

(note that $|\tau| \approx C$ on $\gamma^{(1)}$ and $|\tau| \approx C\Lambda$ on $\gamma^{(2)}$). Now we have to estimate $|M_j(\omega', \tau, \Lambda)|$ in (4.3). For this we use the fact that according to (4.4)

$$|a_k(\omega', \Lambda)| \leq \sum_{l_1 < \ldots < l_k} |\tau_{l_1}| \cdot \ldots \cdot |\tau_{l_k}| \leq \begin{cases} C\Lambda^k, & k \leq m - \mu\,, \\ C\Lambda^{m-\mu}, & k \geq m - \mu\,. \end{cases}$$

On $\gamma^{(1)}$ we have

$$M_j(\omega', \tau, \Lambda)| \leq \begin{cases} C\Lambda^{m-\mu}, & j \leq \mu\,, \\ C\Lambda^{m-j}, & j \geq \mu\,. \end{cases}$$

As length$(\gamma^{(1)}) \leq C$ we obtain

$$\left| \int_{\gamma^{(1)}} (i\tau)^l \frac{M_j(\omega', \tau, \Lambda)}{A_+(\omega', \tau, \Lambda)} e^{it\tau}\, d\tau \right| \leq \begin{cases} C\exp(-Ct)\,, & j \leq \mu\,, \\ C\Lambda^{\mu-j}\exp(-Ct), & j \geq \mu\,, \end{cases}$$

and therefore

$$\|(D_t^l w_j^{(1)})(\omega', \cdot, \Lambda)\|_{L_2(\mathbb{R}_+)} \leq \begin{cases} C\,, & j \leq \mu\,, \\ C\Lambda^{\mu-j}, & j \geq \mu\,, \end{cases} \quad (|\omega'| = 1,\ \Lambda \geq \Lambda_0)\,. \tag{4.6}$$

For an estimation on $\gamma^{(2)}$ we first remark that for every $l \geq 0$ we have

$$|\tau^l M_j(\omega', \tau, \Lambda)| \leq \sum_{k=0}^{m-j} |a_k|\, |\tau^{m-j+l-k}| \leq C\Lambda^{m-j+l}\,.$$

Therefore the inequalities

$$|D_t^l w_j^{(2)}(\omega', t, \Lambda)| \leq C\Lambda^{l-j+1}\exp(-C\Lambda t)$$

and

$$\|D_t^l w_j^{(2)}(\omega', \cdot, \Lambda)\|_{L_2(\mathbb{R}_+)} \leq C\Lambda^{l-j+\frac{1}{2}} \quad (l \geq 0) \tag{4.7}$$

hold. To find a sharper estimate in the case $j \leq \mu$ we use the relation

$$\begin{aligned} \tau^l M_j(\omega', \tau, \Lambda) &= \tau^{l-j} \sum_{k=0}^{m-j} a_k(\omega', \Lambda)\tau^{m-k} \\ &= \tau^{l-j}\Big(A_+(\omega', \tau, \Lambda) - \sum_{k=m-j+1}^{m} a_k(\omega', \Lambda)\tau^{m-k}\Big) \end{aligned}$$

which yields

$$D_t^l w_j^{(2)}(\omega', t, \Lambda) = -\frac{1}{2\pi i} \int_{\gamma^{(2)}} \frac{\sum_{k=m-j+1}^m a_k(\omega', \Lambda)\tau^{m-k+l-j}}{A_+(\omega', \tau, \Lambda)} e^{it\tau}\, dt\,.$$

Here we used the fact that the contour $\gamma^{(2)}$ does not enclose the origin, and therefore $\tau^{l-j}e^{it\tau}$ is holomorphic inside $\gamma^{(2)}$.

We obtain for the case $j \le \mu$ and for every $l \ge 0$ that

$$\Big| \sum_{k=m-j+1}^{m} a_k \tau^{m-k+l-j} \Big| \le C\Lambda^{m-\mu}\Lambda^{m-(m-j+1)+l-j} = C\Lambda^{m-\mu+l-1}$$

and

$$\|D_t^l w_j^{(2)}(\omega', \cdot, \Lambda)\|_{L_2(\mathbb{R}_+)} \le C\Lambda^{l-\mu-\frac{1}{2}} \quad (j \le \mu,\ l \ge 0) \tag{4.8}$$

in view of Remark 2.3 for, say, $\Lambda \ge 1$. Now we compare the right-hand sides of (4.6)–(4.8) with the right-hand side of (4.5).

a) For $j,\, l \le \mu$ the norm of $D_t^l w^{(1)}$ is $O(1)$ and the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-\mu-\frac{1}{2}} \le \Lambda^{-\frac{1}{2}}$.

b) For $j \le \mu$ and $l > \mu$ according to (4.8) the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-\mu-\frac{1}{2}} \ge \Lambda^{\frac{1}{2}}$ and the norm of $D_t^l w^{(1)}$ is estimated by a constant.

c) For $j > \mu$ and $l \le \mu$ according to (4.6) and (4.8) the norm of $D_t^l w^{(1)}$ is estimated by $\Lambda^{\mu-j}$ and the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-j+\frac{1}{2}} \le \Lambda^{\mu-j+\frac{1}{2}}$.

d) For $j,\, l > \mu$ the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-j+\frac{1}{2}}$ and the norm of $D_t^l w^{(1)}$ is estimated by $\Lambda^{\mu-j} < \Lambda^{l-j+\frac{1}{2}}$.

Thus the inequality (4.5) holds, which finishes the proof of the theorem. □

5. A priori estimates

Now we want to prove an a priori estimate for the Dirichlet boundary value problem corresponding to the elliptic pencil $A(x, D, \lambda)$ defined in (1.1). First we consider model problems in $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Let A be a polynomial of the form (3.1). As it was already mentioned at the beginning of Section 4, the Newton polygon $N_{2m,2\mu}$ of $A(\xi, \lambda)$ has the form indicated in Figure 2 with $r = 2m$ and $s = 2\mu$. The a priori estimates which we will obtain below, however, do not use the Sobolev spaces corresponding to this Newton polygon but the "energy spaces" which are defined as the Sobolev spaces corresponding to the Newton polygon $N_{m,\mu}$. For this Newton polygon we have

$$\Xi(\xi, \lambda) := \Xi_{N_{m,\mu}}(\xi, \lambda) \approx (1 + |\xi|)^{\mu}(\lambda + |\xi|)^{m-\mu}\,.$$

As in Section 2, we will denote by $\Xi^{(-l)}(\xi, \lambda)$ the weight function corresponding to the shifted Newton polygon (with a shift of length l to the left). The edges of the shifted polygon are given by

$$\begin{array}{ll} (0,0),\ (0, m-\mu),\ (\mu - l, m-\mu),\ (m-l, 0) & \text{if } l \le \mu\,, \\ (0,0),\ (0, m-l),\ (m-l, 0) & \text{if } \mu < l < m\,. \end{array}$$

It is easy to see the following continuity results, using Theorem 2.9 for part b). Here the continuity of the operator means that the norm of this operator can be estimated by a constant independent of λ. Note also for the Sobolev spaces on the boundary that we have the equivalence

$$\Xi^{(-j+\frac{1}{2})}(\xi',\lambda) \approx \begin{cases} (1+|\xi'|)^{\mu-j+1/2}(\lambda+|\xi'|)^{m-\mu} & \text{if } j \le \mu\,, \\ (\lambda+|\xi'|)^{m-j+1/2} & \text{if } j > \mu\,. \end{cases}$$

Lemma 5.1. a) *The operator $A(D,\lambda)$ acts continuously from $H^{\Xi}(\mathbb{R}^n)$ to $H^{\frac{1}{\Xi}}(\mathbb{R}^n)$.*
b) *The boundary operator D_n^{j-1} ($j \le m$) acts continuously from $H^{\Xi}(\mathbb{R}^n)$ to $H^{\Xi^{(-j+\frac{1}{2})}}(\mathbb{R}^{n-1})$.*

Proposition 5.2. (A priori estimate in $\mathbb{R}^n$.) *Let $A(\xi,\lambda)$ be N-elliptic with parameter in $[0,\infty)$. Then for every $\lambda_0 > 0$ the inequality*

$$\|u\|_{\Xi,\mathbb{R}^n} \le C\Big(\|A(D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big) \tag{5.1}$$

holds for all $\lambda \ge \lambda_0$ with a constant $C = C(\lambda_0)$ independent of u and λ.

Proof. By changing the constant in (3.2) we can rewrite the N-ellipticity condition in the form

$$\lambda^{2m-2\mu} + C_1^{-1}\,\frac{|A(\xi,\lambda)|^2}{(1+|\xi|^2)^{\mu}(\lambda^2+|\xi|^2)^{m-\mu}} \ge \lambda^{2m-2\mu} + |\xi|^{4\mu}\,(1+|\xi|^2)^{-\mu}(\lambda^2+|\xi|^2)^{m-\mu}\,.$$

For $|\xi| \ge 1$ the right-hand side can be estimated from below by

$$2^{-2\mu}(1+|\xi|^2)^{\mu}(\lambda^2+|\xi|^2)^{m-\mu}$$

For $|\xi| \le 1$ and $\lambda \ge \lambda_0$ the right-hand side can be estimated from below by

$$\begin{aligned} \lambda^{2m-2\mu} &= (1+\lambda^{-2})^{-m+\mu}\,(1+\lambda^2)^{m-\mu} \\ &\ge (1+\lambda_0^{-2})^{-m+\mu}\,2^{-\mu}\,(1+|\xi|^2)^{\mu}\,(\lambda^2+|\xi|^2)^{m-\mu}\,. \end{aligned}$$

Combining these estimates we obtain for $\lambda \ge \lambda_0$

$$(1+|\xi|^2)^{\mu}(\lambda^2+|\xi|^2)^{m-\mu} \le C(\lambda_0)\Big(\frac{|A(\xi,\lambda)|^2}{(1+|\xi|^2)^{\mu}(\lambda^2+|\xi|^2)^{m-\mu}} + \lambda^{2m-2\mu}\Big)\,.$$

Multiplying both sides by $|Fu(\xi)|^2$ and integrating with respect to ξ we obtain the inequality

$$\|u\|^2_{\Xi,\mathbb{R}^n} \le C(\lambda_0)\Big(\|A(D,\lambda)u\|^2_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{2m-2\mu}\,\|u\|^2_{L^2(\mathbb{R}^n)}\Big)$$

equivalent to (5.1). □

Now we turn to estimates in the half space $\mathbb{R}^n_+$.

Theorem 5.3. (A priori estimate in $\mathbb{R}^n_+$.) *Let $A(\xi,\lambda)$ be N-elliptic with parameter in $[0,\infty)$ and degenerate regularly for $\lambda \to \infty$. Then for every $\lambda_0 > 0$ there exists a constant $C = C(\lambda_0)$ such that for all $\lambda \ge \lambda_0$ and all $u \in H^{\Xi}(\mathbb{R}^n_+)$ the estimate*

$$\begin{aligned} \|u\|_{\Xi,\mathbb{R}^n_+} \le\ & C\Big(\|A(D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n_+} \\ &+ \sum_{j=1}^{m}\|D_n^{j-1}u\|_{\Xi^{(-j+\frac{1}{2})},\mathbb{R}^{n-1}} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n_+)}\Big) \end{aligned} \tag{5.2}$$

holds.

Proof. We will follow a standard plan in elliptic theory. In the first part of the proof we reduce (5.2) to the case $f := Au = 0$. Then using Theorem 4.1, we treat the case of the homogeneous equation.

1) Denote by E a linear operator of extension of functions defined in $\mathbb{R}^n_+$ to functions in $\mathbb{R}^n$. If we use the well-known Hestenes construction then the operator $E : L_2(\mathbb{R}^n_+) \to L_2(\mathbb{R}^n)$ and its restriction $E : H^\Xi(\mathbb{R}^n_+) \to H^\Xi(\mathbb{R}^n)$ are bounded operators. We will denote by R the operator of restriction of functions on $\mathbb{R}^n$ onto $\mathbb{R}^n_+$.

2) Let $\psi(\xi) \in C^\infty(\mathbb{R}^n)$ be a cut-off function, i.e. $\psi(\xi) = 1$ for $|\xi| \leq 1$ and $\psi(\xi) = 0$ for $|\xi| \geq 2$. We write

$$u = u_1 + u_2 + v = R\psi(D)Eu + R(1 - \psi(D))A^{-1}(D, \lambda)Ef + v \tag{5.3}$$

where we have set $Ef = A(D, \lambda)Eu$.

First of all we show that u_1 and u_2 belong to $H^\Xi(\mathbb{R}^n_+)$ and their norms in this space can be estimated by a constant times

$$\|f\|_{\frac{1}{\Xi},\mathbb{R}^n_+} + \lambda^{m-\mu}\,\|u\|_{L_2(\mathbb{R}^n_+)}\,. \tag{5.4}$$

3) Since the operator $\psi(D)$ is infinitely smoothing we get for $\lambda \geq \lambda_0$ that

$$\|u_1\|_{\Xi,\mathbb{R}^n_+} \leq \|\psi(D)Eu\|_{\Xi,\mathbb{R}^n} \leq C\lambda^{m-\mu}\|Eu\|_{L_2(\mathbb{R}^n)} \leq C_1\lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n_+)}\,.$$

4) Using the Fourier transform we obtain

$$\begin{aligned} \|u_2\|_{\Xi,\mathbb{R}^n_+} &\leq \|(1 - \psi(D))A^{-1}(D, \lambda)Ef\|_{\Xi,\mathbb{R}^n} \\ &= \|\Xi(\xi, \lambda)(1 - \psi(\xi))A^{-1}(\xi, \lambda)(FEf)(\xi)\|_{L_2(\mathbb{R}^n)}\,. \end{aligned}$$

Since $1 - \psi(\xi) = 0$ for $|\xi| \leq 1$, we obtain from the N-ellipticity condition that

$$\Xi(\xi, \lambda)\ |1 - \psi(\xi)|\ |A^{-1}(\xi, \lambda)| \leq C\,\Xi^{-1}(\xi, \lambda)$$

and

$$\|u_2\|_{\Xi,\mathbb{R}^n_+} \leq\ \text{const}\ \|Ef\|_{\frac{1}{\Xi},\mathbb{R}^n}\,.$$

If the norm in $H^{\Xi^{-1}}(\mathbb{R}^n)$ is defined by means of the pseudodifferential operator

$$\Big((1 + |D'|^2)^{1/2} + iD_n\Big)^{-\mu}\Big((\lambda^2 + |D'|^2)^{1/2} + iD_n\Big)^{-m+\mu},$$

then according to Section 2

$$\|Ef\|_{\frac{1}{\Xi},\mathbb{R}^n} = \|f\|_{\frac{1}{\Xi},\mathbb{R}^n_+}\,.$$

5) Now we begin the estimation of v defined in (5.3). We have $v = u - u_1 - u_2 \in H^\Xi(\mathbb{R}^n_+)$ and

$$\begin{aligned} A(D, \lambda)\,v &= 0\,, && (5.5)\\ D_n^{j-1}v(x)|_{x_n=0} &= h_j(x')\,, && (5.6) \end{aligned}$$

where we set $h_j(x') := D_n^{j-1}u(x',0) - D_n^{j-1}u_1(x',0) - D_n^{j-1}u_2(x',0)$. We shall prove the inequality

$$\|v\|_{\Xi,\mathbb{R}^n_+} \leq \text{ const } \Big(\sum_{j=1}^m \|h_j\|_{\Xi(-j+1/2),\mathbb{R}^{n-1}} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)} \Big) \tag{5.7}$$

The a priori estimate (5.2) follows from this inequality because, due to Theorem 2.9,

$$\|D_n^{j-1}u_i\|_{\Xi(-j+1/2),\mathbb{R}^{n-1}} \leq \text{ const } \|u_i\|_{\Xi,\mathbb{R}^n_+} \quad (i=1,2)\,.$$

The right-hand side of this inequality is already estimated by the right-hand side of (5.2).

6) We define

$$\Phi(\xi,\lambda) := \sum_{i,k} |\xi|^i \lambda^k\,,$$

where the sum extends over all integer points (i,k) belonging to the side of $N_{m,\mu}$ which is not parallel to the coordinate lines. From this definition it follows that

$$\Phi(\xi,\lambda) \approx |\xi|^\mu (\lambda + |\xi|)^{m-\mu}\,.$$

and $\|v\|_{\Xi,\mathbb{R}^n_+}$ is equivalent to

$$\|v\|_{\Phi,\mathbb{R}^n_+} + \lambda^{m-\mu}\|v\|_{L_2(\mathbb{R}^n_+)}\,.$$

The second term can be estimated by $\lambda^{m-\mu}(\|u\|_{L_2(\mathbb{R}^n_+)} + \|u_1\|_{L_2(\mathbb{R}^n_+)} + \|u_2\|_{L_2(\mathbb{R}^n_+)})$. Due to steps 3) and 4), these terms are not greater than a constant times the expression (5.4). Therefore, it is enough to estimate $\|v\|_{\Phi,\mathbb{R}^n_+}$ by the right-hand side of (5.7). Repeating the argument in Section 2 (see (2.10)) we reduce our problem to the estimation of

$$\int_0^\infty \|(D_n^l v)(\cdot,x_n)\|^2_{\Phi(-l),\mathbb{R}^{n-1}}\, dx_n \quad (l=0,\dots,m)$$

or after the Fourier transform with respect to x'

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} |\Phi^{(-l)}(\xi',\lambda)(D_n^l F'v)(\xi',x_n)|^2\, d\xi' dx_n \quad (l=0,\dots,m)\,.$$

The function $F'v(\xi',x_n) =: w(\xi',x_n)$ is (for almost every $\xi' \in \mathbb{R}^{n-1}$) a solution of

$$A(\xi',D_n,\lambda)w(x_n) = 0\,, \tag{5.8}$$

$$D_n^{j-1}w(x_n)|_{x_n=0} = (F'h_j)(\xi')\,. \tag{5.9}$$

Due to Theorem 4.1, this solution is unique and given by

$$w(\xi',x_n) = \sum_{j=1}^m w_j(\xi',x_n,\lambda)(F'h_j)(\xi') \tag{5.10}$$

with $w_j(\xi',x_n,\lambda)$ being the solution of (4.1)–(4.2).

7) To obtain the estimate for $w = F'v$ we reformulate Theorem 4.1. It follows from the definition of $N_{m,\mu}$ that

$$\Phi^{(-r)}(\xi,\lambda) \le \begin{cases} |\xi|^{\mu-r}(\lambda+|\xi|)^{m-\mu}, & r \le \mu, \\ (\lambda+|\xi|)^{m-r}, & r > \mu. \end{cases}$$

From this it follows that

$$\frac{\Phi^{(-j+1/2)}(\xi',\lambda)}{\Phi^{(-l)}(\xi',\lambda)} \le \begin{cases} C|\xi'|^{l-j+\frac{1}{2}}, & l \le \mu, \quad j \le \mu, \\ C|\xi'|^{\mu-j+\frac{1}{2}}(\lambda+|\xi'|)^{l-\mu}, & l > \mu, \quad j \le \mu, \\ C|\xi'|^{l-\mu}(\lambda+|\xi'|)^{\mu-j+\frac{1}{2}}, & l \le \mu, \quad j > \mu, \\ C(\lambda+|\xi'|)^{l-j+\frac{1}{2}}, & l > \mu, \quad j > \mu. \end{cases}$$

Comparing the right-hand sides of these inequalities with the right-hand side of (4.1) we see that

$$\|D_n^l w_j(\xi',x_n,\lambda)\|_{L_2(\mathbb{R}_+)} \le C\frac{\Phi^{(-j+1/2)}(\xi',\lambda)}{\Phi^{(-l)}(\xi',\lambda)}\,.$$

From (5.10) and the last inequality it follows that

$$(\Phi^{(-l)}(\xi',\lambda))^2\int_0^\infty |D_n^l w(\xi',x_n,\lambda)|^2\,dx_n \le C\,\sum|\Xi^{(-j+\frac{1}{2})}(\xi',\lambda)(F'h_j)(\xi')|^2\,.$$

Integrating this inequality with respect to ξ' we obtain the desired estimate. □

Now we consider the Dirichlet boundary value problem for differential operators with parameter acting on a smooth compact manifold M with smooth boundary ∂M. In this case we can choose a finite number of coordinate systems. In each of these systems the operator is of the form (1.1). The principal part of the operator is invariantly defined at each of these systems and at every fixed point $x^0 \in M$ it is of the form

$$A^{(0)}(x^0,D,\lambda) = A_{2m}^{(0)}(x^0,D) + \ldots + \lambda^{2m-2\mu}A_{2\mu}^{(0)}(x^0,D)$$

(here $A_j^{(0)}$ denotes the principal part of A_j). We suppose that for each $x^0 \in \overline{M}$ our operator is N-elliptic with parameter. From the reason of continuity and compactness the constant C in inequality (3.2) can be chosen independent of x^0.

We can suppose without loss of generality that the coefficients of $A(x,D,\lambda)$ are of the form

$$a_{\alpha j}(x) = a_{\alpha j} + a'_{\alpha j}(x), \quad a_{\alpha j} \in \mathcal{D}(\mathbb{R}^n)\,.$$

Now we fix a point $x^0 \in \partial M$ and a coordinate system in the neighborhood of x^0 such that in this system locally the boundary ∂M is given by the equation $x_n = 0$. In this case we can define an analog of the polynomial (3.3):

$$Q(x^0,\tau) = \tau^{-2\mu}A^{(0)}(x^0,0,\tau,1) \tag{5.11}$$

Suppose that at a point $x^0 \in \partial M$ and in a fixed coordinate system this polynomial has $m-\mu$ roots in the upper half-plane of the complex plane. It easily follows from this fact that every polynomial (5.11) corresponding to an arbitrary $x^0 \in \partial M$ has the same property. In this case we say that the operator $A(x,D,\lambda)$ degenerates regularly at the boundary ∂M.

Lemma 5.4. *For $a(x) = a + a'(x)$ with $a' \in \mathcal{D}(\mathbb{R}^n)$ and $f \in H^{\frac{1}{\Xi}}(R^n)$ we have $af \in H^{\frac{1}{\Xi}}(R^n)$, and the following statements hold:*
a) *There exists a constant $C(a)$ depending on a but not on f or λ such that*

$$\|af\|_{\frac{1}{\Xi},\mathbb{R}^n} \leq C(a)\|u\|_{\frac{1}{\Xi},\mathbb{R}^n} .$$

b) *There exists a constant $C'(a)$ depending only on a such that the inequality*

$$\|af\|_{\frac{1}{\Xi},\mathbb{R}^n} \leq \sup_{x\in\mathbb{R}^n} |a(x)|\|f\|_{\frac{1}{\Xi},\mathbb{R}^n} + C'(a)\|f\|_{\Psi,\mathbb{R}^n}$$

holds, where we have set

$$\|f\|_{\Psi,\mathbb{R}^n} := \left(\int (1+|\xi|)^{-2\mu-2}(\lambda+|\xi|)^{-2m+2\mu}|\hat{f}(\xi)|^2\, d\xi\right)^{\frac{1}{2}} .$$

Proof. Part a) is a special case of the following more general result which is taken from [18], Section I.2.4. Let σ be a weight function which satisfies

$$\sigma(\xi)\sigma^{-1}(\eta) \leq C(1 + |\xi - \eta|^m) .$$

Then we have for $a' \in \mathcal{D}(\mathbb{R}^n)$ the inequality

$$\|a'f\|_{H^\sigma(\mathbb{R}^n)} \leq c(a')\|f\|_{H^\sigma(\mathbb{R}^n)}$$

with $c(a') := C\int(1 + |\xi|^m)|(Fa')(\xi)|d\xi$.

Part b) can be shown by standard arguments similar to those used in [13], Section 1.7.1, and [9], Lemma 1.4.5. □

Using the above mentioned covering of M by local coordinate systems and a partition of unity subordinated to this covering we can define the spaces $H^{\Xi}(M), H^{\frac{1}{\Xi}}(M)$ and $H^{\Xi^{(-j+3/2)}}(\partial M)$. From Lemma 5.4 and the trace results for model problems in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ we immediately obtain

Lemma 5.5. *The operator $\left(A(x,D,\lambda),\ u|_{\partial M},\ \frac{\partial}{\partial\nu}u|_{\partial M}, \ldots, (\frac{\partial}{\partial\nu})^{m-1}u|_{\partial M}\right)$ as an operator from $H^{\Xi}(M)$ to $H^{\frac{1}{\Xi}}(M) \times \prod_{j=1}^m H^{\Xi^{(-j+\frac{1}{2})}}(\partial M)$ is continuous with norm bounded by a constant independent of λ. Here $\frac{\partial}{\partial\nu}$ stands for the derivative in the direction of the inner normal to the boundary.*

Theorem 5.6. *Let $A(x,D,\lambda)$ be an operator pencil of the form (1.1), acting on the manifold M with boundary ∂M. Let A be N-elliptic with parameter in $[0,\infty)$ and assume that A degenerates regularly at the boundary ∂M. Then for $\lambda \geq \lambda_0$ there exists a constant $C = C(\lambda_0)$, independent of u and λ, such that*

$$\|u\|_{\Xi,M} \leq C\Big(\|A(x,D,\lambda)u\|_{\frac{1}{\Xi},M} + \sum_{j=1}^m \Big\|\Big(\frac{\partial}{\partial\nu}\Big)^{j-1}u\Big\|_{\Xi^{(-j+\frac{1}{2})},\partial M} + \quad \lambda^{m-\mu}\|u\|_{L_2(M)}\Big) . \quad (5.12)$$

Proof. For the proof we use the standard technique of localization ("freezing the coefficients"). We only indicate the main steps. By means of a partition of unity it is sufficient to prove (5.12) for $u \in H^{\Xi}(M)$ with small support $\operatorname{supp} u \subset U$. In the case $U \cap \partial M = \emptyset$, we fix $x_0 \in U$ and use local coordinates. We obtain from the a priori estimate for the model problem in $\mathbb{R}^n$ that

$$\begin{aligned} \|u\|_{\Xi,\mathbb{R}^n} &\leq C_1\Big(\|A^{(0)}(x_0,D)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{2m-2\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big) \\ &\leq C_1\Big(\|A(x,D)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{2m-2\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big) \\ &\quad + C_1\|(A(x,D) - A^{(0)}(x_0,D))u\|_{\frac{1}{\Xi},\mathbb{R}^n} \end{aligned} \tag{5.13}$$

with a constant C_1 independent of u and λ.

We fix $\varepsilon > 0$. From Lemma 5.4 b) we obtain if the support of u is sufficiently small that

$$\|(A(x,D) - A^{(0)}(x_0,D))u\|_{\frac{1}{\Xi},\mathbb{R}^n} \leq \varepsilon\|u\|_{\Xi,\mathbb{R}^n} + C\|u\|_{\Xi(-1),\mathbb{R}^n}\,.$$

Now we use the interpolation inequality

$$\|u\|_{\Xi(-1),\mathbb{R}^n} \leq \varepsilon\|u\|_{\Xi,\mathbb{R}^n} + C\lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}$$

which is a consequence of the interpolation inequality for the Sobolev spaces $H^s(\mathbb{R}^n)$ because of

$$\|u\|_{\Xi(-1),\mathbb{R}^n} \approx \|u\|_{H^{m-1}(\mathbb{R}^n)} + \lambda^{m-\mu}\|u\|_{H^{\mu-1}(\mathbb{R}^n)}\,.$$

If we choose ε with $C_1\varepsilon < 1$ we obtain

$$\|u\|_{\Xi,\mathbb{R}^n} \leq C\Big(\|A(x,D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big)\,.$$

In the case $U \cap \partial M \neq \emptyset$ we choose $x_0 \in U \cap \partial M$, use local coordinates, and obtain in the same way as above

$$\|u\|_{\Xi,\mathbb{R}^n_+} \leq C\Big(\|A(x,D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n_+} + \sum_{j=1}^{m}\|D_n^{j-1}u\|_{\Xi(-j+\frac{1}{2}),\mathbb{R}^{n-1}} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big)\,,$$

where we used the a priori estimate for $(A^{(0)}(x_0,D),(D_n^{j-1})_{j=1}^m)$. □

Remark 5.7. The a priori estimate above deals with functions $u \in H^{\Xi}(M)$. In the forthcoming paper [6] estimates in spaces of functions of arbitrary smoothness will be obtained. Additionally, in [6] the right parametrix for boundary value problems connected with pencils of the form (1.1) will be constructed.

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