First publ. in: Zeitschrift für angewandte Mathematik und Mechanik 75 (1995), pp. 463-470

Convergence improvement for the infinite determinants of Hill systems

Robert Denk

Dedicated to Professor R. Mennicken on the occasion of his 60th birthday

Die Floquet-Exponenten der matrizenwertigen Version der endlichen Hillschen Differentialgleichung können als Nullstellen einer unendlichen Determinante berechnet werden. In dieser Arbeit wird die Konvergenz dieser Determinante durch Abspaltung geeigneter unendlicher Produkte verbessert. Die Definition dieser Produkte verwendet dabei die Kenntnis des asymptotischen Verhaltens der endlichen Abschnittsdeterminanten. Verschiedene Methoden der Konvergenzverbesserung werden sowohl für den symmetrischen als auch für den nicht-symmetrischen Fall der endlichen Hillschen Differentialgleichung vorgestellt. Numerische Beispiele belegen, daß diese Methoden zu einer effizienten Berechnung der unendlichen Determinante führen.

The Floquet exponents of the matrix-valued version of finite Hill's equation can be calculated as the zeros of an infinite determinant. In this paper the convergence of this determinant is improved by splitting up suitable infinite products where the definition of such products is based on the knowledge of the asymptotic behaviour of the finite section determinants. Both for the symmetric and the non-symmetric case of finite Hill's equation several methods of convergence acceleration are presented. Numerical examples show that these methods lead to an efficient evaluation of the infinite determinant.

MCC (1991): 34B30, 65L05, 65B05.

1. Introduction

Infinite determinants corresponding to periodic differential equations of second order have a long history, starting with the famous work of Hill [7]. With the availibility of computers the so-called determinantal method was investigated under numerical aspects. Mennicken [9] seems to be the first to consider the order of convergence of the infinite determinants which he called Hill-type determinants. His results and related papers of him and Wagenführer ([10], [13], [15]) deal with one scalar equation.

For applications in mechanics, however, it is often necessary to consider ODE *systems* of Hill's type. Such systems arise, for instance, if the Galerkin method is used to solve parametric resonance problems for mechanical systems (see [16], VI.7). In [16] several physical examples are given which lead to finite Hill systems or Mathieu systems. We just want to mention vibrations of suspension bridges ([16], VI.1) and parametric resonance of rotating disks ([16], VI.2).

Motivated by investigations of Weidenhammer (cf. [11]) on resonance effects in gearing systems, the author developed a determinantal method for Hill systems in [4]. The numerical evaluation of the corresponding infinite determinant, however, is slower than numerical integration of the ODE-system which is the standard method to compute the Floquet exponents. For a discussion of numerical integration see e.g. [14] for the scalar case and [1] for a treatment of systems using enclosure methods. For a numerically useful determinantal method it is necessary to accelerate the convergence. In this paper we want to present some methods of acceleration which lead in effect to a computation of the Floquet exponents which is comparable or even faster than numerical integration. A typical example of computing times can be found in Section 5 below.

To accelerate the convergence rate of the infinite determinants, we need

- a description of the asymptotic behaviour of the sequence of the finite section determinants,
- a suitable choice for modifying factors which leads to a new sequence with faster convergence.

The first point is answered for finite Hill systems in Section 2. In Sections 3 and 4 we define appropriate factors for the general and the symmetric case of finite Hill's equation (see the definitions below). In Section 5 numerical results are discussed. Some formulas for infinite products which are needed in Sections 3 and 4 are summarized in the appendix. Theorem 2.1 and Method 4B are taken from the author's thesis [2].

We now want to give a precise description of the problem and summarize the main results about the Floquet exponents and the determinantal method for Hill-type systems.

The ODE-system we consider is the matrix-valued version of the finite Hill's equation:

$$y''(x) + A(x) \cdot y(x) = 0 \quad (y : \mathbb{K} \to \mathbb{K}^n) \tag{1}$$

where $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$ and

$$A(x) = \sum_{k=-b}^{b} \exp(2\pi i k x) A_k$$

with $A_k \in \mathbb{K}^{n\times n}$. In the case $\mathbb{K} = \mathbb{R}$ we assume that A_{-k} is the conjugate complex of A_k. As a special case we obtain the *symmetric case* where $\mathbb{K} = \mathbb{R}$ and $A_k = A_{-k}$. This class of ODE-systems contains the classical equation of Hill [7] and the matrix-valued version of Mathieu's equation

$$y''(x) + (A_0 + \cos(2\pi x)\, A_1) \cdot y(x) = 0\ .$$

Due to the theory of Floquet-Lyapunov (see [16], for instance), the stability of (1) depends on the *Floquet exponents* (*characteristic exponents*) defined as the complex numbers ν where a solution y of (1) exists with $y(x+1) = \exp(i\nu)y(x)$. Another description of ν is given using the *matrizant* of (1), i.e. the matrix solution Y of the corresponding first order system with the initial value $Y(0) = I_{2n}$, the unit matrix in $\mathbb{K}^{2n\times 2n}$. The Floquet exponents are exactly the zeros of $\det(Y(1) - \exp(i\nu)I_{2n})$.

On the other hand, the Floquet exponents can be computed using infinite determinants (see, e.g. [7], [8], [9] for the case $n = 1$, [2], [4] for arbitrary n). We summarize some results of [4]:

Theorem 1.1 *Let the infinite block matrix $B(\nu)$ be defined by*

$$B(\nu) = (B_{kl}(\nu))_{k,l\in\mathbb{Z}} = \left(\frac{(2\pi k+\nu)^2 \delta_{kl} I_n - A_{k-l}}{(2\pi k)^2 - \delta_{0,k}} \right)_{k,l\in\mathbb{Z}} .$$

Then for all $\nu \in \mathbb{C}$ the infinite determinant $\det B(\nu)$ *exists and we get:*

a) *The zeros of* $\det B(\nu)$ *are precisely the Floquet exponents of* (1).

b) $\exp(i\nu n)\cdot \det B(\nu)$ *is a trigonometric polynomial in ν with degree* $2n$.

Due to this theorem, it is necessary to evaluate $\det B(\nu)$ for different values of ν in order to obtain the Floquet exponents of (1). Here $\det B(\nu)$ is defined as the limit of the finite section determinants $\det B_m(\nu)$ where

$$B_m(\nu) := (B_{kl}(\nu))_{k,l=-m}^{m} .$$

Previous papers ([9], [10], [15]) on the convergence properties of the sequence $(\det B_m(\nu))_{m\in\mathbb{N}}$ deal with the case $n = 1$ (the scalar case) and with the symmetric form of (1). In [13] very fast methods for the case of Mathieu's equation can be found. In all these investigations ν is restricted to $\nu = 0$ or $\nu = 1$. In this case the twosided infinite determinant $\det B(\nu)$ can be written as a product of two onesided infinite determinants. In the present paper the values of ν and n are allowed to be arbitrary. The results for general ν and for the non-symmetric case seem to be new even for $n = 1$.

2. Asymptotic properties of the determinant

As ν will be fixed, we write B instead of $B(\nu)$. Following an idea given in [9], we define the onesided infinite block matrix $C = (C_{kl})_{k,l=0}^{\infty}$ by

$$C_{kl} := \begin{pmatrix} B_{-k,-l} & B_{-k,l} \\ B_{k,-l} & B_{k,l} \end{pmatrix} \in \mathbb{K}^{2n\times 2n} \quad (k,l > 0) . \tag{2}$$

If k is equal to zero, we replace the first n rows of the right side of (2) by $(\delta_{kl} I_n \; 0)$. Similarly, for $l = 0$ we replace the first n columns by $\begin{pmatrix} \delta_{kl} I_n \\ 0 \end{pmatrix}$.

For all $m \in \mathbb{N}$ we have $\det C_m = \det B_m$ where $C_m := (C_{kl})_{k,l=0}^{m}$. To get some information about the convergence of the sequence $(\det C_m)_m$, we define a partition of C_m where b, lb and 1 are the numbers of $(2n \times 2n)$-blocks in the corresponding columns:

$$C_m = \begin{pmatrix} \multicolumn{2}{c}{C_{m-lb-1}} & \begin{matrix} 0 \\ R_m \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 & L_m & D_m & s_m \\ 0 & \underbrace{0}_{b} & \underbrace{z_m}_{lb} & \underbrace{C_{m,m}}_{1} \end{pmatrix} .$$

Theorem 2.1 *The following expressions are defined for large values of m and have the stated convergence orders for $m \to \infty$.*

a) $\det C_m - \det \Delta_m \cdot \det C_{m-1} = \mathrm{O}(m^{-4l-4})$ *with*

$$\Delta_m := C_{mm} - z_m\, D_m^{-1}\, s_m \,.$$

b) $\det \Delta_m - \det \left(C_{mm} - \sum_{p=1}^{b} C_{m,m-p} C_{m-p,m-p}^{-1} C_{m-p,m} \right) = \mathrm{O}(m^{-6})$.

c) $$\det \Delta_m - \det \Big(C_{mm} - \sum_{p=1}^{b} C_{m,m-p} C_{m-p,m-p}^{-1} C_{m-p,m} + \\ + \sum_{\substack{p,q=1 \\ p \neq q}}^{b} C_{m,m-p} C_{m-p,m-p}^{-1} C_{m-p,m-q} C_{m-q,m-q}^{-1} C_{m-q,m} \Big) = \mathrm{O}(m^{-8})\,.$$

Proof. As the proof of this theorem is quite technical, we only want to give a short sketch of it. For details the reader is referred to [2], Satz IV.2.3 and Lemma IV.3.4.

a) We write

$$\det C_m = \det \begin{pmatrix} \multicolumn{2}{c}{C_{m-lb-1}} & \begin{matrix} 0 \\ R_m \end{matrix} & \begin{matrix} 0 \\ -R_m D_m^{-1} s_m \end{matrix} \\ 0 & L_m & D_m & 0 \\ 0 & -z_m D_m^{-1} L_m & 0 & \Delta_m \end{pmatrix} . \qquad (3)$$

The norms of $R_m D_m^{-1} s_m$ and $z_m D_m^{-1} L_m$ can be estimated with an expression of order $\mathrm{O}(m^{-2l-2})$. The Laplace expansion of the right determinant in (3) corresponding to the last $2n$ rows and columns leads to the desired estimation.

b), c) The linear matrix equation

$$\begin{pmatrix} D_m & 0 \\ z_m & I_{2n} \end{pmatrix} X = \begin{pmatrix} s_m \\ C_{mm} \end{pmatrix}, \quad X \in \mathbb{K}^{2n(lb+1)\times 2n}, \tag{4}$$

is solved by $X = \begin{pmatrix} D_m^{-1} s_m \\ \Delta_m \end{pmatrix}$. If we apply the block Jacobi method for the iterative solution of a linear equation system as described in [12], (8.5.2), we obtain the expressions stated in b) and c) after 2 and 3 steps, respectively.

□

In the parts b) and c) of the preceding theorem we can find the expressions that have to be approximated if we want to obtain a method of convergence acceleration. In the following corollary the expression for C_{kl} is replaced by the corresponding expression for B_{kl} modulo $\mathrm{O}(m^{-8})$. We state here only the analogue of 2.1 c) where αI_n is abbreviated by α. For an approximation of order $\mathrm{O}(m^{-6})$ we just have to omit the last sum.

Corollary 2.2 *We have* $\det B_m - \gamma_m \det B_{m-1} = \mathrm{O}(m^{-8})$ $(m \to \infty)$ *for*

$$\begin{aligned}
\gamma_m := \det & \left[\frac{(2\pi m - \nu)^2 - A_0}{(2\pi m)^2)} \quad \frac{(2\pi m + \nu)^2 - A_0}{(2\pi m)^2} \right] \cdot \\
\cdot \quad \det & \Bigg[I_n - \sum_{p=1}^{b} \big((2\pi m - \nu)^2 - A_0\big)^{-1} A_{-p} \big((2\pi(m-p) - \nu)^2 - A_0\big)^{-1} A_p - \\
& - \sum_{p=1}^{b} \big((2\pi m + \nu)^2 - A_0\big)^{-1} A_p \big((2\pi(m-p) + \nu)^2 - A_0\big)^{-1} A_{-p} - \\
& - \sum_{\substack{p,q=1 \\ p \neq q}}^{b} \left(1 + \frac{3\nu}{\pi m}\right) \frac{A_{-p} A_{-q+p} A_q}{(2\pi m)^2 (2\pi(m-p))^2 (2\pi(m-q))^2} - \\
& - \sum_{\substack{p,q=1 \\ p \neq q}}^{b} \left(1 - \frac{3\nu}{\pi m}\right) \frac{A_p A_{q-p} A_{-q}}{(2\pi m)^2 (2\pi(m-p))^2 (2\pi(m-q))^2} \Bigg].
\end{aligned}$$

To improve the order of convergence of the sequence $(\det B_m)_m$, we choose $\tilde{\gamma}_m \in \mathbb{C}\backslash\{0\}$ with $\tilde{\gamma}_m - \gamma_m = \mathrm{O}(m^{-N})$ for some $N \leq 8$. Then the sequence $(\beta_m)_{m=0}^{\infty}$ with

$$\beta_m := \left(\prod_{j=1}^{m} \tilde{\gamma}_j \right)^{-1} \cdot \det B_m$$

fulfills $\beta_m - \beta_{m-1} = \mathrm{O}(m^{-N})$ and

$$\det B = \left(\lim_{m\to\infty} \beta_m\right) \cdot \prod_{m=1}^{\infty} \tilde{\gamma}_m \; .$$

From the last equation one can see that the infinite product over the factors $\tilde{\gamma}_m$ must be known explicitly. In Section 3 we will give two examples for $\tilde{\gamma}_m$ and the corresponding infinite products for the general case of the finite Hill's equation (1) with $N = 6$, in Section 4 we will define $\tilde{\gamma}_m$ for the symmetric case with $N = 8$.

3. Acceleration of convergence: General case

For a convergence order $\mathrm{O}(m^{-6})$ the last two sums in Corollary 2.2 can be omitted. We present two examples for a definition of $\tilde{\gamma}_m$. In the following, $\lambda_j(M)$ denotes the j-th eigenvalue of a matrix M, counted with multiplicity.

Theorem 3.1 (Method 3A)
We suppose

$$\det((2\pi m + \nu)^2 - A_0) \neq 0 \qquad (m \in \mathbb{Z}) \; ,$$

$$\det\left(\left(2\pi\left(m - \frac{p}{2}\right)\right)^4 - A_{-p}A_p - A_pA_{-p}\right) \neq 0 \quad \left(p = 1, \ldots, b;\; m > \frac{p}{2}\right) \; .$$

For $\tilde{\gamma}_m$ defined by

$$\tilde{\gamma}_m := \det\left[\frac{(2\pi m - \nu)^2 - A_0}{(2\pi m)^2} \quad \frac{(2\pi m + \nu)^2 - A_0}{(2\pi m)^2}\right] \cdot$$

$$\cdot \prod_{\substack{p=1 \\ p/2<m}}^{b} \det\left[I_n - \frac{A_{-p}A_p + A_pA_{-p}}{(2\pi(m - p/2))^4}\right]$$

$(m \in \mathbb{N})$ *we obtain* $\tilde{\gamma}_m - \gamma_m = \mathrm{O}(m^{-6})$.

Proof. For $p = 0, \ldots, b$ we have

$$\begin{aligned}
&\left((2\pi(m-p) \pm \nu)^2 - A_0\right)^{-1} = \\
&= \frac{1}{(2\pi(m-p))^2}\left(I_n \pm \frac{\nu}{\pi(m-p)} + \frac{\nu^2 - A_0}{(2\pi(m-p))^2}\right)^{-1} = \\
&= \frac{1}{(2\pi(m-p))^2}\left(I_n \mp \frac{\nu}{\pi(m-p)}\right) + \mathrm{O}(m^{-4}) \; .
\end{aligned}$$

Therefore the second factor in Corollary 2.2 is equal to

$$\begin{aligned}
&\det\Bigg[I_n - \sum_{p=1}^{b} \frac{A_{-p}A_p + A_pA_{-p}}{(2\pi m)^2(2\pi(m-p))^2} - \\
&\quad - \sum_{p=1}^{b}\left(\frac{\nu}{\pi m} + \frac{\nu}{\pi(m-p)}\right)\frac{A_{-p}A_p - A_pA_{-p}}{(2\pi m)^2(2\pi(m-p))^2}\Bigg] + \mathrm{O}(m^{-6}) = \\
&= 1 - \sum_{p=1}^{b}\frac{2\,\mathrm{tr}\,(A_pA_{-p})}{(2\pi m)^2(2\pi(m-p))^2} + \mathrm{O}(m^{-6}) = \\
&= \prod_{p=1}^{b}\left(1 - \frac{2\,\mathrm{tr}\,(A_pA_{-p})}{(2\pi(m-p/2))^4}\right) + \mathrm{O}(m^{-6}) = \\
&= \prod_{p=1}^{b}\det\left(I_n - \frac{A_{-p}A_p + A_pA_{-p}}{(2\pi(m-p/2))^4}\right) + \mathrm{O}(m^{-6})
\end{aligned}$$

where we used

$$\frac{1}{m^2(m-p)^2} = \frac{1}{(m-p/2)^4} + \mathrm{O}(m^{-6})$$

and the fact that for matrices $M_m = \mathrm{O}(m^{-4})$ the estimation

$$\det(I_n - M_m) = 1 - \,\mathrm{tr}\, M_m + \mathrm{O}(m^{-8}) \tag{5}$$

holds. (Equation (5) can be seen using Taylor's formula for the determinant $\det : \mathbb{K}^{n\times n} \to \mathbb{K}$ at the point I_n.) □

The infinite product which belongs to method 3A is stated in the following lemma which is proved using formulas (8) and (10) of the appendix:

Lemma 3.2 *For $\tilde\gamma_m$ as defined in Theorem 3.1 the infinite product has the value*

$$\prod_{m=1}^{\infty}\tilde\gamma_m = \det\begin{bmatrix}\dfrac{\sin\frac12(\nu+\sqrt{A_0})}{\frac12(\nu+\sqrt{A_0})} & \dfrac{\sin\frac12(\nu-\sqrt{A_0})}{\frac12(\nu-\sqrt{A_0})}\end{bmatrix}\cdot$$
$$\cdot\prod_{p=1}^{b}\prod_{j=1}^{n} f_p\Big(\tfrac12[\lambda_j(A_{-p}A_p + A_pA_{-p})]^{1/4}\Big)$$

where

$$f_p(\alpha) := \begin{cases}\dfrac{\sin\alpha}{\alpha}\cdot\dfrac{\sinh\alpha}{\alpha} & \textit{if } p \textit{ even,}\\ \cos\alpha\cdot\cosh\alpha & \textit{if } p \textit{ odd.}\end{cases}$$

As the proof of Theorem 3.1 shows, it is not necessary to calculate the eigenvalues of the matrices $A_{-p}A_p + A_pA_{-p}$. Theorem 3.3 defines the method which uses the traces of these matrices (f_p is defined above).

Theorem 3.3 (Method 3B)
Let $\det((2\pi m+\nu)^2 - A_0) \neq 0$ *for all* $m \in \mathbb{Z}$ *and* $\operatorname{tr}(A_pA_{-p}) \neq \frac{1}{2}(2\pi(m-p/2))^4$ *for all* $p = 1,\dots,b$ *and* $m > \frac{p}{2}$. *Define for* $m \in \mathbb{N}$

$$\tilde{\gamma}_m := \det\left[\frac{(2\pi m-\nu)^2 - A_0}{(2\pi m)^2} \quad \frac{(2\pi m+\nu)^2 - A_0}{(2\pi m)^2}\right] \cdot$$

$$\cdot \prod_{\substack{p=1\\ p/2<m}}^{b} \left(1 - \frac{2\operatorname{tr}(A_pA_{-p})}{(2\pi(m-p/2))^4}\right) .$$

Then $\tilde{\gamma}_m - \gamma_m = \mathrm{O}(m^{-6})$ *and*

$$\prod_{m=1}^{\infty} \tilde{\gamma}_m = \det\left[\frac{\sin\frac{1}{2}(\nu+\sqrt{A_0})}{\frac{1}{2}(\nu+\sqrt{A_0})} \quad \frac{\sin\frac{1}{2}(\nu-\sqrt{A_0})}{\frac{1}{2}(\nu-\sqrt{A_0})}\right] \cdot$$

$$\cdot \prod_{p=1}^{b}\prod_{j=1}^{n} f_p\left(\tfrac{1}{2}[2\operatorname{tr}(A_pA_{-p})]^{1/4}\right) .$$

4. Acceleration of convergence: Symmetric case

In addition to the methods developed in the preceding section (which have convergence order $\mathrm{O}(m^{-6})$), we will present a definition for $\tilde{\gamma}_m$ which works for the symmetric case and leads to a convergence of order $\mathrm{O}(m^{-8})$. The corresponding infinite products can be found in Lemma 4.3 and Theorem 4.5. We suppose in the following that all factors $\tilde{\gamma}_m$ are well-defined and different from zero. The sequence $(\gamma_m)_{m\in\mathbb{N}}$ that has to be approximated can be found in Corollary 2.2.

Theorem 4.1 (Method 4A)
In the symmetric case ($A_k = A_{-k}$) *we have* $\tilde{\gamma}_m - \gamma_m = \mathrm{O}(m^{-8})$ *with*

$$\tilde{\gamma}_m := \left(\prod_{p=0}^{b} \gamma_{m,p}(\nu)\cdot\gamma_{m,p}(-\nu)\right)\cdot\left(\prod_{p,q=1}^{b}\gamma_{m,p,q}\right) \qquad (6)$$

where we define for $m \in \mathbb{N}$ *and* $p,q = 1,\dots,b$:

$$\gamma_{m,0}(\nu) := \det\frac{(2\pi m+\nu)^2 - A_0}{(2\pi m)^2} ,$$

$$\gamma_{m,p}(\nu) := \begin{cases} \det\left(I_n - [((2\pi m+\nu)(2\pi(m-p)+\nu) - A_0)^{-1}A_p]^2\right) & \textit{if } m > \frac{p}{2} , \\ 1 & \textit{else,} \end{cases}$$

$$\gamma_{m,p,q} \;:=\; \begin{cases} 1-\dfrac{2\,\mathrm{tr}\,(A_pA_{q-p}A_q)}{(2\pi(m-(p+q)/3))^6} & \textit{if } p\neq q \textit{ and } m>\frac{p+q}{3}\,, \\ 1 & \textit{else.} \end{cases}$$

Proof. Direct calculation shows for $p=1,\dots,b$ and $m>p/2$

$$\det\Big[I_n-((2\pi m+\nu)^2-A_0)^{-1}A_p((2\pi(m-p)+\nu)^2-A_0)^{-1}A_p\Big]= \tag{7}$$

$$= \gamma_{m,p}(\nu)\cdot\det\Bigg[I_n-\frac{1}{(2\pi m+\nu)^2(2\pi(m-p)+\nu)^2}\bigg(\frac{A_0A_p^2}{(2\pi m+\nu)^2}+$$

$$+\frac{A_pA_0A_p}{(2\pi(m-p)+\nu)^2}-\frac{A_0A_p^2+A_pA_0A_p}{(2\pi m+\nu)(2\pi(m-p)+\nu)}\bigg)\Bigg]+\mathrm{O}(m^{-8})\,.$$

Using (5) and

$$\frac{1}{(2\pi m+\nu)^2}+\frac{1}{(2\pi(m-p)+\nu)^2}=\frac{2}{(2\pi m+\nu)(2\pi(m-p)+\nu)}+\mathrm{O}(m^{-4})$$

we see that the second determinant in (7) is equal to $1+\mathrm{O}(m^{-8})$. The expression for $\gamma_{m,p,q}$ is obtained from the corresponding term in Corollary 2.2 in connection with (5) and

$$\frac{1}{m^2(m-p)^2(m-q)^2}=\frac{1}{(m-(p+q)/3)^6}+\mathrm{O}(m^{-8})\,.$$

□

Remark 4.2 If we omit the second product in the definition (6) of $\tilde\gamma_m$, we obtain a method with convergence order $\mathrm{O}(m^{-6})$. This method which we call method 4B can be found in [2], [3]. Of course there are many cases where the second product is equal to 1, for example if $b=1$. In these cases both methods are identical, in particular method 4B has convergence order $\mathrm{O}(m^{-8})$, too.

We first calculate the infinite products of $\gamma_{m,p}(\nu)$.

Lemma 4.3 *For $M\in\mathbb{K}^{n\times n}$ we define*

$$g_p(M) := \begin{cases} \det\left(\dfrac{\sin\frac12(\nu+\sqrt{M})}{\frac12(\nu+\sqrt{M})}\quad\dfrac{\sin\frac12(\nu-\sqrt{M})}{\frac12(\nu-\sqrt{M})}\right) & \textit{if } p \textit{ even,} \\ \det\left(\cos\frac12(\nu+\sqrt{M})\cdot\cos\frac12(\nu-\sqrt{M})\right) & \textit{if } p \textit{ odd.} \end{cases}$$

a) $\displaystyle\prod_{m=1}^{\infty}\gamma_{m,0}(\nu)\cdot\gamma_{m,0}(-\nu)=g_0(A_0)\,.$

b) *For* $p = 1, \ldots, b$ *we have*

$$\prod_{m=1}^{\infty} \gamma_{m,p}(\nu) \cdot \gamma_{m,p}(-\nu) = \frac{g_p(A_0 + A_p + \pi^2 p^2) g_p(A_0 - A_p + \pi^2 p^2)}{(g_0(A_0 + \pi^2 p^2))^2} \ .$$

Proof. a) follows immediately from formula (8) of the appendix. To obtain b), we write with $c := (2\pi m \pm \nu)(2\pi(m-p) \pm \nu)$:

$$I_n - [(c - A_0)^{-1} A_p]^2 = (c - A_0)^{-2} (c - A_0 - A_p)(c - A_0 + A_p) \ .$$

For $M \in \{A_0, A_0 + A_p, A_0 - A_p\}$ we see

$$c - M = (2\pi(m - \tfrac{p}{2}) \pm \nu)^2 - \pi^2 p^2 - M \ .$$

Now b) follows from formula (8) of the appendix for even p and from formula (9) for odd p. □

Remark 4.4 The following considerations will be needed for the calculation of the product $\prod_m \gamma_{m,p,q}$: The function $(p,q) \mapsto (r,s) := (q, q-p)$ is a bijection from $\{(p,q) : 1 \le p < q \le b\}$ onto $\{(p,q) : 1 \le q < p \le b\}$ with $\mathrm{tr}\,(A_p A_{q-p} A_q) = \mathrm{tr}\,(A_r A_{r-s} A_s)$ and $p + q \equiv -(r+s) \bmod 3$ (cf. also [15], pp. 416 - 417).

Theorem 4.5

$$\prod_{m=1}^{\infty} \prod_{p,q=1}^{b} \gamma_{m,p,q} = \prod_{\substack{p,q=1 \\ p<q}}^{b} \prod_{l=0}^{2} h_{p,q,l}$$

$$\textit{with } h_{p,q,l} := \begin{cases} \left(\dfrac{\sin \frac{1}{2} z_{p,q,l}}{\frac{1}{2} z_{p,q,l}} \right)^2 & \textit{if } p+q \equiv 0 \bmod 3 \,, \\[2ex] \frac{1}{3} + \frac{2}{3} \cos z_{p,q,l} & \textit{if } p+q \not\equiv 0 \bmod 3 \end{cases}$$

$$\textit{and } z_{p,q,l} := [2\, \mathrm{tr}\,(A_p A_{q-p} A_q)]^{1/6} \exp(\tfrac{\pi i l}{3}) \ .$$

Proof. For $p + q \equiv 0 \bmod 3$ we get using Remark 4.4

$$\prod_{m=1}^{\infty} \gamma_{m,p,q} = \prod_{m=1}^{\infty} \gamma_{m,q,q-p} = \prod_{m=1}^{\infty} \left(1 - \frac{2\, \mathrm{tr}\,(A_p A_{q-p} A_q)}{(2\pi m)^6} \right) \ .$$

If $p + q \not\equiv 0 \bmod 3$, again with Remark 4.4 we obtain

$$\prod_{m=1}^{\infty} \gamma_{m,p,q}\, \gamma_{m,q,q-p} = \prod_{m=1}^{\infty} \left(1 - \frac{2\, \mathrm{tr}\,(A_p A_{q-p} A_q)}{(2\pi(m-1/3))^6} \right) \left(1 - \frac{2\, \mathrm{tr}\,(A_p A_{q-p} A_q)}{(2\pi(m-2/3))^6} \right) \ .$$

The expressions for these infinite products can be found in equations (11) and (12) of the appendix, respectively. □

5. Numerical results

To calculate the Floquet exponents of the finite Hill's equation (1) it is necessary to evaluate $\det B(\nu)$ for $2n-1$ different values of ν in the general case and for n different values of ν in the symmetric case, see [5], Theorem 5, and [4], Theorem 3.7, respectively. Instead of $\det B = \det B(\nu)$ we use the onesided infinite determinant $\det C$ which allows us to apply a suitably chosen stop condition.

The determinants $(\det C_m)_{m=0,1,\ldots}$ can be calculated iteratively using the Gauß elimination method with partial pivoting. Due to the band structure of C_m we have only $2n(b+1)\cdot 2n(2b+1)$ relevant elements of this matrix in each step. We don't want to go into the details of the implementation. Some ideas for the implementation – including the stop condition – can be transfered from [15], Section 3, where the scalar case is treated.

We remark that the conditions of Theorems 3.1, 3.2 and Section 4 concerning the nonvanishing of the factors $\tilde{\gamma}_m$ are not crucial. The exceptional cases can be handled analogously to the considerations in [13], pp. 58 - 59. To calculate terms like $\det(\cos\frac{1}{2}(\nu+\sqrt{M}))$ we first compute the eigenvalues $(\lambda_j(M))_{j=1,\ldots,n}$ of the matrix M and then the expression $\prod_{j=1}^{n}\cos\frac{1}{2}(\nu+\lambda_j(M))$. In the same manner the factors $\tilde{\gamma}_m$ are obtained.

As a model problem we take two coupled differential equations of Hill's type:

$$
\begin{aligned}
y_1''(x)+\lambda_1 y_1(x)+2\sum_{k=1}^{b}\cos(2\pi kx)(\gamma_{11}^k y_1(x)+\gamma_{12}^k y_2(x)) &= 0\,,\\
y_2''(x)+\lambda_2 y_2(x)+2\sum_{k=1}^{b}\cos(2\pi kx)(\gamma_{21}^k y_1(x)+\gamma_{22}^k y_2(x)) &= 0\,.
\end{aligned}
$$

The computation was done on a SUN-Sparc station using double precision with machine precision eps $\approx 1.1\cdot 10^{-16}$. As an estimation for the relative error we took $(b_{m-1}-b_m)/b_m$ where b_m is the estimation for $\det B(\nu)$ in the m-th step. The iteration was stopped if the absolute value of the estimated relative error was smaller than the given tolerance TOL. In Table 1 we give an example for the numbers of iterations needed by the different methods of acceleration. The input data for this example were $b=4$, $\nu=0.2$, $\lambda_1=1.2\,,\lambda_2=1.3$, $\gamma_{11}^k=1.1\cdot k^{-2}$, $\gamma_{12}^k=\gamma_{21}^k=1.2\cdot k^{-2}$ and $\gamma_{22}^k=1.3\cdot k^{-2}$. The exact value for $\det B(\nu)$ is 1.4789503933908.

We see that the implementation of the determinantal method without convergence acceleration cannot be used for numerical purposes. This becomes even more clear if we consider the relative error $(b_m-\det B(\nu))/\det B(\nu)$ and the computing time (CPU-time in seconds), see Table 2. Between the methods 3A and 3B there is no significant difference.

The method without acceleration cannot achieve the necessary precision – due to rounding errors and the stop condition which is useful only if $|b_m-\det B(\nu)|$ is small compared to $|b_{m-1}-\det B(\nu)|$. Without acceleration this is not the case for large

TOL	Method of acceleration none	3A	3B	4A	4B
10^{-6}	362	6	6	5	5
10^{-7}	1144	8	8	5	5
10^{-8}	3616	11	11	6	7
10^{-9}	11433	15	15	8	10
10^{-10}	36154	21	21	10	14

Table 1: Numbers of iterations

TOL	Method of acceleration none		3A/3B		4A		4B	
10^{-6}	$3.6 \cdot 10^{-4}$	4.42	$2.1 \cdot 10^{-7}$	0.07	$9.8 \cdot 10^{-9}$	0.07	$4.8 \cdot 10^{-8}$	0.06
10^{-7}	$1.1 \cdot 10^{-4}$	14.30	$4.6 \cdot 10^{-8}$	0.09	$9.8 \cdot 10^{-9}$	0.07	$4.8 \cdot 10^{-8}$	0.06
10^{-8}	$3.6 \cdot 10^{-5}$	47.61	$8.7 \cdot 10^{-9}$	0.12	$2.0 \cdot 10^{-9}$	0.08	$6.2 \cdot 10^{-9}$	0.08
10^{-9}	$1.1 \cdot 10^{-5}$	161.23	$1.8 \cdot 10^{-9}$	0.17	$1.8 \cdot 10^{-10}$	0.10	$8.5 \cdot 10^{-10}$	0.12
10^{-10}	$3.6 \cdot 10^{-6}$	518.15	$3.2 \cdot 10^{-10}$	0.23	$3.1 \cdot 10^{-11}$	0.12	$1.4 \cdot 10^{-10}$	0.16

Table 2: Relative error (left) and CPU-time in seconds (right)

values of m. The methods with acceleration, however, obtain the results much faster and with the necessary precision. We see that method 4A gives us the best results (as expected from the asymptotic convergence properties).

The computation time is approximately linear in the number of iterations, even for the $O(m^{-8})$-method 4A where the corresponding formulas seem to be quite complicated. Because of the small number of iterations, the computation time for the accelerated methods is often smaller than numerical integration, i.e. evaluation of $\det(Y(1) - \exp(i\nu)I_{2n})$ (cf. Table 3). As a numerical integrator we chose a variable-order variable-step Adams method.

TOL	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$b = 5$	$b = 6$	$b = 7$	$b = 8$
10^{-6}	0.09	0.20	0.27	0.44	0.58	0.77	0.97	1.21
10^{-7}	0.09	0.18	0.25	0.32	0.47	0.61	0.81	0.98
10^{-8}	0.08	0.18	0.27	0.33	0.44	0.52	0.64	0.82
10^{-9}	0.08	0.16	0.23	0.34	0.42	0.58	0.63	0.78
10^{-10}	0.08	0.16	0.25	0.33	0.41	0.51	0.64	0.73

Table 3: CPU-time of method 4A compared to the time of numerical integration which is set to 1

For a computation of the Floquet exponents we have to evaluate $\det B(\nu)$ for n $[2n-1]$ different values of ν for the symmetric [general] case of finite Hill's equation, as mentioned above. Therefore the determinantal method seems to be not recommendable for large values of n. In many cases, however, it is enough to know the value for one ν (for example, $\nu = 0$ corresponds to the existence of periodic solutions). For questions of this type or for small values of n the determinantal method with the appropriate acceleration of convergence leads to a precise and fast computation.

Appendix: Infinite product formulas

We state some formulas for infinite products which were used in the preceding sections. Although we take here the scalar case ($\alpha \in \mathbb{C}$), it is clear that most formulas can be used if α is a matrix because both sides are holomorphic functions of α.

$$\prod_{m=1}^{\infty} \frac{(2\pi m+\nu)^2-\alpha^2}{(2\pi m)^2}\;\frac{(2\pi m-\nu)^2-\alpha^2}{(2\pi m)^2} = \frac{\sin\frac{1}{2}(\nu-\alpha)}{\frac{1}{2}(\nu-\alpha)}\;\frac{\sin\frac{1}{2}(\nu+\alpha)}{\frac{1}{2}(\nu+\alpha)}\;. \tag{8}$$

$$\prod_{m=1}^{\infty} \frac{(\pi(2m-1)+\nu)^2-\alpha^2}{(\pi(2m-1))^2}\;\frac{(\pi(2m-1)-\nu)^2-\alpha^2}{(\pi(2m-1))^2} = \cos\tfrac{1}{2}(\nu-\alpha)\;\cos\tfrac{1}{2}(\nu+\alpha)\;. \tag{9}$$

$$\prod_{\substack{m=1\\ m>p/2}}^{\infty} \left(1-\frac{\alpha^4}{(2\pi(m-p/2))^4}\right) = \begin{cases} \dfrac{\sin(\alpha/2)}{\alpha/2}\cdot\dfrac{\sinh(\alpha/2)}{\alpha/2} & \text{if } p \text{ even,} \\[2ex] \cos(\alpha/2)\cdot\cosh(\alpha/2) & \text{if } p \text{ odd.} \end{cases} \tag{10}$$

For (11) and (12), z_l is defined by $z_l := \alpha\exp(\frac{\pi i l}{3})$.

$$\prod_{m=1}^{\infty}\left(1-\frac{\alpha^6}{(2\pi m)^6}\right) = \prod_{l=0}^{2}\frac{\sin(z_l/2)}{z_l/2}\;. \tag{11}$$

$$\prod_{m=1}^{\infty}\left(1-\frac{\alpha^6}{(2\pi(m-1/3))^6}\right)\left(1-\frac{\alpha^6}{(2\pi(m-2/3))^6}\right) = \prod_{l=0}^{2}\left(\tfrac{1}{3}+\tfrac{2}{3}\cos z_l\right)\;. \tag{12}$$

For a proof of (8) we use

$$\frac{(2\pi m+\nu)^2-\alpha^2}{(2\pi m)^2}\;\frac{(2\pi m-\nu)^2-\alpha^2}{(2\pi m)^2} = \left(1-\frac{(\nu+\alpha)^2}{(2\pi m)^2}\right)\left(1-\frac{(\nu-\alpha)^2}{(2\pi m)^2}\right)$$

and the product formula for the sinus function ([6], (89.5.11)). (9) is proved in the same way. (10) follows from [6], (89.7.1) for even p and [6], (89.7.3) for odd p. The decomposition

$$1 - \frac{\alpha^6}{(2\pi m)^6} = \prod_{l=0}^{2} \left(1 - \frac{z_l^2}{(2\pi m)^2}\right)$$

leads to (11). Similarly, the left side of (12) is equal to

$$\prod_{l=0}^{2} \left[\left(1 - \frac{(3z_l)^2}{(2\pi)^2}\right) \prod_{m=1}^{\infty} \left(1 - \frac{z_l^2}{(2\pi(m-1/3))^2}\right) \left(1 - \frac{z_l^2}{(2\pi(m+1/3))^2}\right) \right]$$

and [6], (89.10.3) leads to equation (12). For (11) and (12) cf. also [15], (4.4) and (4.5) (i).

Acknowledgement. The author wants to express his gratitude to Prof. R. Mennicken and Prof. E. Wagenführer who drew his attention to the field of infinite determinants corresponding to Hill systems and gave valuable hints and comments.

References

[1] E. Adams, D. Cordes, H. Keppler: Enclosure methods as applied to linear periodic ODE's and matrices. Z. angew. Math. Mech. **70** (1990), 565 - 578.

[2] R. Denk: Die Determinantenmethode zur Bestimmung der charakteristischen Exponenten von Hillschen Differentialgleichungs-Systemen. Thesis Universität Regensburg, 1993.

[3] R. Denk: The application of the determinantal method to the finite Hill's equation. Z. angew. Math. Mech. **74** (1994), T642 - T644.

[4] R. Denk: Hill's equation systems and infinite determinants. To appear in Math. Nachr.

[5] R. Denk: Infinite determinants of periodic ODE-systems. Submitted.

[6] E. R. Hansen: A table of series and products. Prentice-Hall, Engelwood Cliffs, N. J. 1975.

[7] G. W. Hill: On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon. Acta Math. **8** (1886), 1 - 36.

[8] W. Magnus, S. Winkler: Hill's equation. Interscience Publishers, New York 1966.

[9] R. Mennicken: On the convergence of infinite Hill-type determinants. Arch. Rational Mech. Anal. **30** (1968), 12 - 37.

[10] R. Mennicken, E. Wagenführer: Über die Konvergenz verallgemeinerter Hillscher Determinanten. Math. Nachr. **72** (1976), 21 - 49.

[11] G. Schmidt, F. Weidenhammer: Gedämpfte erzwungene Schwingungen in rheolinearen Systemen. Math. Nachr. **27** (1963/64), 215 - 228.

[12] J. Stoer, R. Bulirsch: Introduction to numerical analysis. Springer-Verlag New York 1980.

[13] E. Wagenführer: Ein Verfahren höherer Konvergenzordnung zur Berechnung des charakteristischen Exponenten der Mathieuschen Differentialgleichung. Numer. Math. **27** (1976), 53 - 65.

[14] E. Wagenführer: Berechnung des charakteristischen Exponenten der endlichen Hillschen Differentialgleichung durch numerische Integration. Numer. Math. **32** (1979), 31 - 50.

[15] E. Wagenführer: Die Determinantenmethode zur Berechnung des charakteristischen Exponenten der endlichen Hillschen Differentialgleichung. Numer. Math. **35** (1980), 405 - 420.

[16] V. A. Yakubovich, V. M. Starzhinskii: Linear differential equations with periodic coefficients. J. Wiley & Sons New York 1975.

Address: Dr. Robert Denk, NWF I - Mathematik, Universität Regensburg, D-93040 Regensburg, Germany